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ABSTRACT. A class of nonlinear Fokker–Planck equations with superlinear drift is investigated in the L^1-supercritical regime, which exhibits a finite critical mass. The equations have a formal Wasserstein-like gradient-flow structure with a convex mobility and a free energy functional whose minimising measure has a singular component if above the critical mass. Singularities and concentrations also arise in the evolutionary problem and their finite-time appearance constitutes a primary technical difficulty. This paper aims at a global-in-time qualitative analysis—the main focus being on isotropic solutions, in which case the unique minimiser of the free energy will be shown to be the global attractor. A key step in the analysis consists in properly controlling the singularity profiles during the evolution. Our study covers the 3D Kaniadakis–Quarati model for Bose–Einstein particles, and thus provides a first rigorous result on the continuation beyond blow-up and long-time asymptotic behaviour for this model.

1. Introduction

This manuscript is concerned with a class of Fokker–Planck equations with superlinear drift taking the form

$$\begin{aligned} \partial_t f &= \nabla\cdot(\nabla f + v h(f)), & t>0, v\in\mathbb{R}^d, \\ f(0,v) &= f_{\mathrm{in}}(v)\geq 0, & v\in\mathbb{R}^d, \end{aligned} \tag{FP$_\gamma$}$$

where $h(f)=f(1+\sigma|f|^\gamma)$ for some $\gamma\geq 1$ and $\sigma=1$. For $\gamma=1$ and $\sigma\in\{\pm1\}$ this equation has been introduced in the 1990s by Kandiadakis and Quarati [KaQ93, KaQ94] as a model for the relaxation to equilibrium of quantum particles of Fermi–Dirac ($\sigma=-1$) and Bose–Einstein ($\sigma=1$) type. We refer to [CHW20, Fra05] and references therein for more background on the physical model. The interest of the mathematics community in problems of the form (FP$_\gamma$) mainly stems from their variational structure: for densities $f\geq 0$ eq. (FP$_\gamma$) can formally be written as a continuity equation

$$\partial_t f = \nabla\cdot(h(f)\nabla\delta\mathcal{H}(f)) \tag{1.1}$$

with $\delta\mathcal{H}$ denoting the variational derivative of the convex integral functional

$$\mathcal{H}(f) := \int_{\mathbb{R}^d}\left(\frac{|v|^2}{2}f+\Phi(f)\right)\mathrm{d}v,$$

where $\Phi(f) := \frac{1}{\gamma}\int_0^f \log\left(\frac{s^\gamma}{1+\sigma s^\gamma}\right)\mathrm{d}s$ and thus $\Phi''(f)=1/h(f)$. (If $\sigma=-1$ one should restrict to $0\leq f\leq 1$.) Thus, the *free energy* $\mathcal{H}(f)$ is formally dissipated along solutions $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f) = -\int_{\mathbb{R}^d} h(f)\,|\nabla\delta\mathcal{H}(f)|^2\,\mathrm{d}v\leq 0$. Let us note that for $\sigma=1$ the function Φ is sublinear at infinity, and the natural extension of $\mathcal{H}$ to finite, nonnegative measures (cf. [DeT84]) vanishes on Dirac deltas centred at the origin. We further observe that for $\sigma=1$ the nonlinear mobility $h(f)=f(1+\sigma f^\gamma)$ in (1.1) is convex, while it is concave if $\sigma=-1$.

The equation for fermions, where $\sigma=-1$ and $\gamma=1$, is mathematically well-understood. Here, in any dimension, solutions emanating from suitably regular initial data $0\leq f_{\mathrm{in}}\leq 1$ remain bounded between zero and one, i.e. satisfy $0\leq f\leq 1$, consistent with the well-known Pauli exclusion principle. In the long-time limit they converge to the unique minimiser of $\mathcal{H}$ of the given mass [CLR09, CRS08], namely to the corresponding (smooth) Fermi–Dirac distribution. The concavity of the mobility even allows to give a rigorous meaning to the above gradient-flow structure with respect to generalised Wasserstein distances [DNS09, CL*10], which fails for the convex/non-concave mobilities associated with $\sigma=1$.

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The bosonic case, where $\sigma = 1$ (and $\gamma = 1$), is more challenging. Then, eq. (FP$_\gamma$) becomes L^1-supercritical in dimension $d > 2$, in which case the large-data long-time analysis has remained open for quite a while. In fact, a first global-in-time rigorous study of the L^1-supercritical regime has only recently been obtained in [CHR20] for a 1D analogue, that is for (FP$_\gamma$) with $\sigma = 1$, $d = 1$ and $\gamma > 2$, and is based on a Lagrangian approach and viscosity solution techniques. In the physically most interesting case $d = 3$ and $\gamma = 1$, which will be the main focus of this manuscript, no rigorous long-time analysis exists when $\sigma = 1$, except for the ref. [Tos12] showing finite-time blow-up for large data by a virial-type contradiction argument. In the L^1-critical case, in contrast, solutions are globally regular [CnC*16]. For numerical studies on the singularity formation in the supercritical case, we refer to [CHW20, SSC06]. The qualitative properties obtained in the present manuscript are in agreement with the simulations in [CHW20], although our approximation scheme is different and not restricted to the isotropic case. Let us mention that the uniqueness and stability properties of the present scheme in the isotropic setting may also be of interest numerically.

In this paper we perform a rigorous global-in-time existence and qualitative analysis of (FP$_\gamma$) with $\sigma = 1$ in the L^1-supercritical regime in higher dimensions $d \geq 1$ our main interest being the bosonic 3D Kaniadakis–Quarati model ($\sigma = 1$ and $d = 3, \gamma = 1$); thus, hereafter $\sigma = 1$. Preservation of the variational structure beyond finite-time blow-up being a primary concern, we build our analysis on a suitably chosen approximation scheme that respects the basic mass conservation and structural properties of the continuity equation (1.1). To begin with, we note that the static mass-constrained minimisation problem for $\mathcal{H}$ is well-understood. The minimisers of $\mathcal{H}$ for a given mass have been characterised in [BAGT11] and are in fact explicit:

Theorem ([BAGT11], Theorem 3.1)**.** *For every $m \in (0, \infty)$ the functional $\mathcal{H}$ has a unique minimiser $\mu_{\min} = \mu_{\min}^{(m)}$ on the manifold $\{\mu \in \mathcal{M}_+(\mathbb{R}^d) : \int \mathrm{d}\mu = m\}$.*

This minimiser takes the form

$$\mu_{\min} = \begin{cases} f_{\infty,\theta}\,\mathcal{L}^d & \text{if } m \leq m_c, \text{ where } \theta \in \mathbb{R}_{\geq 0} \text{ obeys } \int f_{\infty,\theta} = m, \\ f_c\,\mathcal{L}^d + (m - m_c)\,\delta_0 & \text{if } m > m_c. \end{cases} \tag{1.2}$$

Here

$$f_{\infty,\theta}(v) = \left(\mathrm{e}^{\gamma\left(\frac{|v|^2}{2}+\theta\right)} - 1\right)^{-\frac{1}{\gamma}}, \quad \theta \in \mathbb{R}_{\geq 0}, \tag{1.3}$$

and we abbreviated $f_c := f_{\infty,0}$ as well as $m_c := \int_{\mathbb{R}^d} f_c(v)\,\mathrm{d}v \in (0, \infty]$.

For general $\gamma \geq 1$, the L^1-supercritical regime as determined by a dimensional analysis is given by $d - \frac{2}{\gamma} > 0$. Observe that this is exactly the regime, where the *critical mass* m_c appearing in the above theorem is finite and where minimisers with singular parts concentrated at velocity zero emerge. Such singular components are termed Bose–Einstein condensates in the physics literature (at least when $\gamma = 1$).

Let us now put the analysis of the present work into context with existing literature and discuss the main new difficulties. Naturally, several aspects of our approach have their roots in the work [CHR20]. This is particularly true for the fact that our fundamental a priori bound consists in a space-uniform *temporal* Lipschitz estimate (of an integral quantity) that is propagated in time. Both, in [CHR20] and in the present paper, such estimates are derived by means of suitable comparison principles. However, the approach in [CHR20] relies on a Lagrangian reformulation of the problem in terms of the pseudo-inverse cumulative distribution function giving access to the powerful instrument of viscosity solution theory [CIL92]. While in higher dimensions such a reformulation is, in principle, still possible [CHW20, CRW16, ESG05], the structural properties of the resulting PDE (system) greatly deteriorate—even in the isotropic case.

The new challenges we encounter in higher dimensions are thus mainly of a technical nature. Especially the derivation of the universal space profile at $\{v = 0\}$ for unbounded densities in Section 3 (applying to isotropic flows) is significantly more delicate than in the 1D case and requires several intermediate steps. Determining the profile at the *first* blow-up time is still quite feasible and, as in the 1D case, amounts to solving an ordinary differential equation—in higher dimensions to be combined with a bootstrap argument. However, in eq. (FP$_\gamma$) solutions may regularise after a first blow-up, and such successions of „blow-ups“ and „blow-downs“ could in principle be highly oscillatory. Thus, for a global-in-time analysis a particular challenge lies in gaining information at general points in time. We should emphasize that the space profile, while of interest in its own right, encodes a certain time-uniform continuity-at-infinity property that appears to be vital for proving relaxation to the minimiser $\mu_{\min}$ in the long-time limit. (Observe that when only looking at the equation (FP$_\gamma$) from a PDE point of view, other stationary „solutions“ consisting of a smooth steady state $f_{\infty,\theta}$ for some $\theta > 0$ plus a suitably weighted non-trivial Dirac measure at zero are conceivable, though unphysical.) Let us finally point out that, in contrast to [CHR20] where the mass of the condensate component (i.e. of the singular part of the measure solution, which turns out to be supported in $\{v = 0\}$) has only been shown to be a continuous function of time, the present approach allows us to infer Lipschitz continuity in the isotropic case and thus refines [CHR20] (cf. [Hop19]). Some of the basic ideas of this manuscript have been sketched for the 1D model in the author's PhD Thesis [Hop19, Chapter 5]. As indicated in Chapter 5.3 of [Hop19], when $d = 1$, the solutions to be constructed below coincide with those obtained from the viscosity solution approach in [CHR20].

1.1. **Main results.** In the subsequent analysis, unless specified otherwise, we assume the following general hypotheses:

(H1) L^1-supercriticality: $\frac{\gamma d}{2} > 1$, where $\gamma \in [1, \infty)$, $d \in \mathbb{N}_+$ are fixed parameters.
(H2) Initial data: $f_{\rm in} \in (L^\infty_{\ell_1} \cap L^1_{\ell_2})(\mathbb{R}^d)$, where $\ell_1 \geq d, \ell_1 > 2$ and $\ell_2 > \max\{d, 2\}$; $f_{\rm in} \geq 0$ a.e. in $\mathbb{R}^d$;
if $f_{\rm in}$ is not isotropic, assume the stronger decay property that $f_{\rm in} \in L^\infty_{\ell_3}(\mathbb{R}^d)$ for some $\ell_3 > 2d$.

The spaces $L^p_\ell(\mathbb{R}^d)$ in (H2) are weighted L^p spaces, see def. (2.4).

Our results for the nonlinear Fokker–Planck equations (FP$_\gamma$) rely on a careful analysis of the proposed approximation scheme, which is devised in such a way as to preserve the Fokker–Planck-type gradient-flow structure (1.1). Approximation schemes for continuation beyond blow-up have been employed in the literature for various other PDE problems. Closest to the present situation are perhaps the constructions in [LSV12, Vel04] for the 2D Patlak–Keller–Segel model.

Approximation scheme. Pick $\eta \in C^{0,1}(\mathbb{R}) \cap C^\infty(\mathbb{R}\setminus\{0\})$ satisfying $\operatorname{sign}(s)\eta'(s) \geq 0, \eta(s) = |s|^\gamma$ for $|s| \leq 1, \eta'(s) = 0$ for $|s| \geq 2$, and moreover $0 \leq \eta(s) \leq |s|^\gamma$ for all $s \geq 0$.

For $\varepsilon \in (0, 1]$ we then define $\eta_\varepsilon(s) = \varepsilon^{-\gamma}\eta(\varepsilon s)$, let

$$\begin{aligned} h_\varepsilon(s) &:= s(1 + \eta_\varepsilon(s)) \\ &=: s + \vartheta_\varepsilon(s), \quad \text{where } \vartheta_\varepsilon(s) := s\eta_\varepsilon(s), \end{aligned} \tag{1.4}$$

and consider the associated Cauchy problem

$$\begin{aligned} \partial_t f_\varepsilon &= \nabla \cdot (\nabla f_\varepsilon + v h_\varepsilon(f_\varepsilon)), \quad t > 0, v \in \mathbb{R}^d, \\ f_\varepsilon(0, v) &= f_{\rm in}(v) \geq 0, \qquad\qquad v \in \mathbb{R}^d. \end{aligned} \tag{FP$_{\gamma.\rm reg}$}$$

For details on the variational structure of eq. (FP$_{\gamma.\rm reg}$) we refer to Section 4. Since h_ε has linear growth at infinity, global existence of nonnegative mild solutions for suitably regular data is obtained from standard arguments (cf. Prop. 2.2 below). The relatively strong decay hypotheses in (H2) are needed to establish estimates that are independent of ε (cf. Prop. 2.6).

Proposition 1.1 (Limiting measure for (FP$_\gamma$)). *Suppose (H1), (H2). Then there exists a locally finite nonnegative measure μ on $[0,\infty)\times\mathbb{R}^d$ with the following properties:*

(i) Mass-conserving curve: we have $\mathrm{d}\mu = \mathrm{d}\mu_t\mathrm{d}t$ for a family of measures $\{\mu_t\}_{t\ge 0}\subset\mathcal{M}_+(\mathbb{R}^d)$ with the property that $t\mapsto\mu_t$ is a weakly-$*$ continuous curve in $\mathcal{M}_+(\mathbb{R}^d)$ that preserves mass, and more precisely satisfies $\mu_t(\mathbb{R}^d)=\|f_{\mathrm{in}}\|_{L^1}=: m$ for all $t\ge 0$.

(ii) Decomposition: there exists a measurable function $a:[0,\infty)\to[0,m]$ and a nonnegative function $f\in L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^d)\cap C^{1,2}((0,\infty)\times U)$, $U:=\mathbb{R}^d\setminus\{0\}$, such that for all $t\ge 0$

$$\mu_t = a(t)\delta_0 + f(t,\cdot)\mathcal{L}^d,$$

where δ_0 denotes the Dirac measure concentrated at the origin.
The function f is a classical solution of (FP$_\gamma$) in $(0,\infty)\times U$.

(iii) Approximation property: denote by $f_\varepsilon\in C([0,\infty);L^\infty\cap L^1_2)$ the unique mild solution[1] of (FP$_{\gamma.\mathrm{reg}}$) (cf. Sec. 2.1) and let $\mu^{(\varepsilon)}=f_\varepsilon\mathcal{L}^{1+d}_+$, where $\mathcal{L}^{1+d}_+$ denotes the $(1+d)$-dimensional Lebesgue measure on $[0,\infty)\times\mathbb{R}^d$.
Then, along a subsequence $\varepsilon\downarrow 0$

$$\begin{aligned}&\mu^{(\varepsilon)}\overset{*}{\rightharpoonup}\mu \text{ in } \mathcal{M}([0,T]\times\mathbb{R}^d), \quad \text{for all } T<\infty,\\ &f_\varepsilon\to f \text{ in } C^{1,2}_{\mathrm{loc}}((0,\infty)\times U),\end{aligned}$$

where $U:=\mathbb{R}^d\setminus\{0\}$.

(iv) Unique limit: if f_{in} is isotropic, the convergence in (iii) is true along any sequence $\varepsilon\downarrow 0$.

(v) Lipschitz continuity of point mass: if f_{in} is isotropic[2], the map $t\mapsto\mu_t(\{0\})$ is Lipschitz continuous.

See Section 2.3 for the proof of Proposition 1.1. Later on we show for the isotropic case that the limiting measure μ satisfies (FP$_\gamma$) is the sense of renormalised solutions. One of the technical difficulties of problem (FP$_\gamma$) is related to the fact that the function $t\mapsto\mu_t(\{0\})$ in general fails to be monotonic (cf. Sec. 5.2).

Prop. 1.1 (ii) implies that $\operatorname{supp}\mu_t^{\mathrm{sing}}\subset\{v=0\}$. Hence, recalling the sublinearity of $\Phi(s)$ as $s\to\infty$, we infer that for every $t\ge 0$

$$\mathcal{H}(\mu_t)=\mathcal{H}(f(t)).$$

Since all relevant measures in this work will have singular parts supported at the origin, we (continue to) denote by the symbol $\mathcal{H}$ both the functional acting on densities as well as the extended functional acting on nonnegative finite measures.

The following result provides a sharp characterisation of the space profile of isotropic solutions, and moreover it is a key ingredient for uniquely identifying the long-time asymptotic limit. It will be established in Section 3.

Theorem 1.2 (Universal space profile). *In addition to (H1), (H2) suppose that*

$$\tfrac{2}{\gamma}+2-d>0. \tag{1.5}$$

[1] The approximate solutions f_ε enjoy further regularity properties, which will be needed in the analysis; see Sec. 2 for details.

[2] In the anisotropic case, we will see in Section 3 that $t\mapsto\mu_t(\{0\})$ is at least continuous, see Cor. 3.3.

Further assume that the initial value f_{in} is isotropic and let $g(t,|v|) := f(t,v)$, where f denotes the density of the regular part of the limiting measure obtained in Prop. 1.1. There exists $r^* > 0$ and a bounded function $A \in C_b((0,\infty)\times\mathbb{R}_+)$ such that for each $\hat{t} > 0$ either $g(\hat{t},\cdot) \in L^\infty(\mathbb{R}_+)$ or

$$g(\hat{t},r) = g_c(r) + A(\hat{t},r)r^{2-d} \qquad \text{for } r \in (0,r^*), \tag{1.6}$$

where $g_c(|v|) := f_c(v) = f_{\infty,0}(v)$ (cf. eq. (1.3)). The upper bound „$\leq$“ in (1.6) is true for all $\hat{t} \in (0,\infty)$.

If $\mu_{\hat{t}}(\{0\}) > 0$, the second option, i.e. (1.6), must hold true.

The proof of Theorem 1.2 (see Section 3.3) combines a smoothing estimate for mild solutions with a bootstrap argument based on the temporal Lipschitz continuity of the partial mass function. Observe that $g_c(r) = \left(\frac{2}{\gamma}\right)^{\frac{1}{\gamma}} r^{-\frac{2}{\gamma}} + O(r^{-\frac{2}{\gamma}+2})$ for $0 < r \ll 1$, so that the remainder $O(r^{2-d})$ is indeed of lower order under condition (1.5). Moreover, in the expansion for g one can replace the limiting steady state $g_c(r)$ by the power law $\left(\frac{2}{\gamma}\right)^{\frac{1}{\gamma}} r^{-\frac{2}{\gamma}}$ since $d > \frac{2}{\gamma}$. In the present work, we focus on regime (1.5) as it covers the most interesting case of the 3D Kaniadakis–Quarati model for bosons ($\gamma = 1, d = 3$).

Owing to the strong nonlinearity in the drift one cannot expect the limiting density f to be a distributional solution of (FP$_\gamma$) in $(0,\infty)\times\mathbb{R}^d$. Our analysis leading to Theorem 1.2 allows to show that the limiting measure satisfies (FP$_\gamma$) in the sense of renormalised solutions.

Definition 1.3 (Renormalised solution of (FP$_\gamma$))**.** Let μ be a locally finite nonnegative measure on $[0,\infty)\times\mathbb{R}^d$ and denote by $\mu = \mu^{\mathrm{reg}} + \mu^{\mathrm{sing}} = f(t,v)\mathcal{L}_+^{1+d} + \mu^{\mathrm{sing}}$ its Lebesgue decomposition into regular part μ^{reg} with density $f \in L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^d)$ and singular part μ^{sing}. We call μ a renormalised solution of (FP$_\gamma$) in $(0,\infty)\times\mathbb{R}^d$ with initial data f_{in} if $\mathrm{d}\mu = \mathrm{d}\mu_t\mathrm{d}t$ for some weakly-$*$ continuous curve $[0,\infty) \ni t \mapsto \mu_t$ in $\mathcal{M}(\mathbb{R}^d)$ with preserved mass $\int \mathrm{d}\mu_t \equiv \|f_{\mathrm{in}}\|_{L^1(\mathbb{R}^d)}$, if $\mathcal{T}_k(f) := \min\{f,k\} \in L^2_{\mathrm{loc}}([0,\infty); H^1_{\mathrm{loc}}(\mathbb{R}^d))$ for every $k > 0$, and if for all $\xi \in C^\infty([0,\infty))$ with compactly supported derivative ξ', for a.a. $T \in (0,\infty)$ and all $\psi \in C_c^\infty([0,T]\times\mathbb{R}^d)$:

$$\begin{aligned}\int_{\mathbb{R}^d} \xi(f(T,\cdot))\psi(T,\cdot)\,\mathrm{d}v - \int_{\mathbb{R}^d} \xi(f_{\mathrm{in}})\psi(0,\cdot)\,\mathrm{d}v - \int_0^T\int_{\mathbb{R}^d} \xi(f)\partial_t\psi\,\mathrm{d}v\mathrm{d}t \\ = -\int_0^T\int_{\mathbb{R}^d} (\nabla f + h(f)v)\cdot\nabla(\xi'(f)\psi)\,\mathrm{d}v\mathrm{d}t.\end{aligned} \tag{1.7}$$

As usual, the gradients of f on the RHS of (1.7) are to be understood as $\nabla\mathcal{T}_k(f)$ for $k = k(\xi)$ large enough such that $\xi'(s) = 0$ for $s \geq k$ (cf. [BB*95, DMM*99]).

Theorem 1.4 (The limit μ is a renormalised solution)**.** *Assume the hypotheses of Theorem 1.2. Then the limiting measure μ constructed in Proposition 1.1 satisfies eq. (FP$_\gamma$) in the renormalised sense as specified in Definition 1.3.*

The proof of this theorem is given in Section 4.2 and makes use, among others, of a local and truncated version of the energy dissipation estimate. The following energy dissipation identity is crucial for deducing the long-time asymptotic behaviour.

Proposition 1.5 (Energy dissipation (in)equality)**.** *Assume (H1), (H2) and use the notations of Prop. 1.1. Then for all $t > 0$*

$$\mathcal{H}(f(t)) - \mathcal{H}(f_{\mathrm{in}}) \leq -\int_0^t\int_{\mathbb{R}^d} \frac{1}{h(f)}|\nabla f + h(f)v|^2\,\mathrm{d}v\mathrm{d}\tau.$$

When supposing in addition the hypotheses of Theorem 1.2, the stronger balance law holds true: for all $t \geq s \geq 0$

$$\mathcal{H}(f(t)) - \mathcal{H}(f(s)) = -\int_s^t \int_{\mathbb{R}^d} \frac{1}{h(f)} |\nabla f + h(f)v|^2 \, \mathrm{d}v \mathrm{d}\tau. \tag{1.8}$$

See Section 4.3 for the proof of Proposition 1.5.

The long-time asymptotic behaviour and further transient dynamical properties can be seen as corollaries of the above results (cf. Section 5 for details). Let us here only highlight the long-time asymptotics.

Theorem 1.6 (Convergence to minimiser)**.** *Assume the hypotheses of Theorem 1.2 and denote by* $m = \int f_{\mathrm{in}} > 0$ *the total mass of the initial data. Further let* $\mu_{\min} := \mu_{\min}^{(m)}$ *denote the unique minimising measure of* $\mathcal{H}$ *for the given mass* m *(cf. eq.* (1.2)*). Then, as* $t \to \infty$*,* $\mathcal{H}(\mu_t) \to \mathcal{H}(\mu_{\min})$*, and moreover*

$$\begin{aligned} &\mu_t \overset{*}{\rightharpoonup} \mu_{\min} \quad \text{in } \mathcal{M}(\mathbb{R}^d) \qquad \text{and} \qquad \mu_t(\{0\}) \to \mu_{\min}(\{0\}), \\ &f(t) \to f_{\min} \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\}), \\ &f(t) \to f_{\min} \quad \text{in } L^p(\mathbb{R}^d) \quad \text{for any } p \in [1, \tfrac{\gamma}{2}), \end{aligned}$$

where $f_{\min}$ *denotes the density of the regular part of* $\mu_{\min}$ *with respect to the Lebesgue measure. In addition,* $\lim_{t\to\infty} \int |v|^2 \, \mathrm{d}\mu_t = \int |v|^2 \, \mathrm{d}\mu_{\min}$.

The proof of this result will be completed in Section 5.1.

1.2. **Outline.** The remaining part of this paper is structured as follows. In Section 2 we establish global existence and uniform estimates for the approximate problem ($\mathrm{FP}_{\gamma.\mathrm{reg}}$). An important ingredient is the uniform bound in Prop. 2.6, which is obtained using a comparison technique. This allows us to pass to the limit $\varepsilon \to 0$ in Section 2.3. Section 3 lies at the heart of our analysis. Its main purpose is to establish the universal profile asserted in Theorem 1.2 (see Sec. 3.3). In Section 4 we introduce variational tools and use the results from Section 3 to show, for the isotropic case, the renormalised solution property of (FP_γ) as well as the energy dissipation identity. Section 5 concludes with a characterisation of the long-time asymptotics and some additional remarks.

1.3. **Notations.** Unless specified otherwise, the following notations will henceforth be adopted:

- $\mathcal{M}_+(G)$: space of nonnegative finite measures on $G \subset \mathbb{R}^N$.
- $L^p_\ell(\mathbb{R}^d)$: weighted L^p space with norm $\|f\|_{L^p_\ell} := \|(1+|v|^\ell)f\|_{L^p(\mathbb{R}^d)}$, cf. (2.4).
- $C^{1,2}((0,\infty)\times\mathbb{R}^d)$: space of continuously differentiable functions $f = f(t,v)$ that are twice continuously differentiable with respect to $v \in \mathbb{R}^d$.
- $\mathcal{L}^d$: d-dimensional Lebesgue measure.
- $\mathcal{L}^{1+d}_+$: $(1+d)$-dimensional Lebesgue measure on $[0,\infty)\times\mathbb{R}^d$.
- $s_+ := \max\{s, 0\}$ for $s \in \mathbb{R}$.
- $B_r := B_r(0) := \{v \in \mathbb{R}^d : |v| < r\}$.
- $g_c(r) = f_c(v)$ for $r = |v|$, where $f_c = f_{\infty,0}$ as defined in (1.3).

2. Approximation scheme

As pointed out in the introduction, local-in-time classical solutions of (FP_γ) emanating from initial data that are large in a suitable sense may cease to exist in $L^\infty(\mathbb{R}^d)$ after a finite time. The main purpose of this section is to establish global existence for the approximation scheme ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) in spaces of suitable regularity as well as certain compactness and convergence properties for the corresponding approximate solutions. In the isotropic case, our scheme obeys a monotonicity property and, as a consequence, gives rise to a unique limiting measure. Note that this feature may also be of interest from a numerics point of view. A key ingredient in the analysis is a uniform temporal Lipschitz bound for the partial mass function of isotropic solutions (see Prop. 2.6).

2.1. **Mild solutions.** The local-in-time wellposedness of equations (FP_γ) and ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) in suitably weighted spaces can conveniently be obtained in the framework of mild solutions using the Duhamel integral formulation of (FP_γ) resp. of ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) given by

$$f(t,v)=\int_{\mathbb{R}^d}\mathcal{F}(t,v,w)f_{\mathrm{in}}(w)\,\mathrm{d}w+\int_0^t\int_{\mathbb{R}^d}\mathcal{F}(t-s,v,w)\big(\mathrm{div}_w(w\,|f|^\gamma f)\big)|_{(s,w)}\,\mathrm{d}w\mathrm{d}s, \tag{2.1}$$

$$f_\varepsilon(t,v)=\int_{\mathbb{R}^d}\mathcal{F}(t,v,w)f_{\mathrm{in}}(w)\,\mathrm{d}w+\int_0^t\int_{\mathbb{R}^d}\mathcal{F}(t-s,v,w)\big(\mathrm{div}_w(w\,\vartheta_\varepsilon(f_\varepsilon))\big)|_{(s,w)}\,\mathrm{d}w\mathrm{d}s, \tag{2.2}$$

where $\mathcal{F}=\mathcal{F}(t,v,w)$ denotes the fundamental solution of the linear Fokker–Planck equation, that is

$$\mathcal{F}(t,v,w)=\mathrm{e}^{dt}G_{\nu(t)}(\mathrm{e}^t v-w)$$

with

$$\nu(t)=\mathrm{e}^{2t}-1,\qquad G_\lambda(\xi)=(2\pi\lambda)^{-\frac{d}{2}}\mathrm{e}^{-\frac{|\xi|^2}{2\lambda}}.$$

In this subsection, we collect several auxiliary results for mild solutions, most of which can be obtained as in [CLR09]. The reasoning is therefore kept brief.

Using integration by parts, the last integral in (2.1) can formally be rewritten as

$$\int_0^t \mathrm{e}^{-(t-s)}\int_{\mathbb{R}^d}\nabla_v\mathcal{F}(t-s,v,w)\cdot w\,|f|^\gamma f|_{(s,w)}\,\mathrm{d}w\,\mathrm{d}s. \tag{2.3}$$

Hereafter, eq. (2.1) will always be understood in this way, i.e. with the last integral replaced by (2.3). Analogously, we rewrite the last term in (2.2). For estimating integrals of the form (2.3) we use the semigroup estimates in [CLR09, Appendix A]. To state these estimates we define for $p\in[1,\infty]$ and $\ell\geq 0$

$$L^p_\ell(\mathbb{R}^d):=\{f\in L^p(\mathbb{R}^d):\|f\|_{L^p_\ell}<\infty\},\qquad \|f\|_{L^p_\ell}:=\|(1+|v|^\ell)f\|_{L^p(\mathbb{R}^d)}. \tag{2.4}$$

By [CLR09, Proposition A.1] the linear operator

$$\mathscr{F}[f](t,v):=\int_{\mathbb{R}^d}\mathcal{F}(t,v,w)f(w)\,\mathrm{d}w$$

enjoys the following smoothing estimates for all $t\in(0,T]$

$$\|\nabla^k\mathscr{F}[f](t)\|_{L^q_\ell}\leq C_T\nu(t)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-\frac{k}{2}}\|f\|_{L^p_\ell} \tag{2.5}$$

for any $1\leq p\leq q\leq\infty$, $\ell\geq 0$ and $k\in\mathbb{N}_0$, where the constant $C_T=C_T(d,q,k)$ is given by

$$C_T=C\exp((\tfrac{d}{q'}+k)T)$$

with $\frac{1}{q'}+\frac{1}{q}=1$ and $C<\infty$ a universal constant.

We begin with a uniqueness result, which implies among others consistency of the approximation scheme (cf. Corollary 2.5 below).

Lemma 2.1 (Uniqueness of mild solutions for (FP_γ) and ($\mathrm{FP}_{\gamma.\mathrm{reg}}$))**.** *Let $p>d$. There exists at most one mild solution $f\in C([0,T];(L^\infty\cap L^p_1)(\mathbb{R}^d))$ of eq. (FP_γ). An analogous result holds for eq. ($\mathrm{FP}_{\gamma.\mathrm{reg}}$).*

Proof. Let $f, \tilde{f} \in C([0,T]; (L^\infty \cap L_1^p)(\mathbb{R}^d))$ both satisfy eq. (2.1) for $t \in [0,T]$ — as always, with the last term on the RHS replaced by (2.3). Then, given $T' \in (0,T]$ we estimate for $t \in [0,T']$, using the bound (2.5) and recalling that $\nu(t) = \mathrm{e}^{2t} - 1$,

$$\begin{aligned}\|f(t) - \tilde{f}(t)\|_{L^\infty} &\le C_{T'} \int_0^t \nu(t{-}s)^{-\frac{1}{2}-\frac{d}{2}\frac{1}{p}} \|w(|f|^\gamma f(s,w) - |\tilde{f}|^\gamma \tilde{f}(s,w))\|_{L^p}\,\mathrm{d}s \\ &\le \kappa(T')\||f| + |\tilde{f}|\|^\gamma_{C([0,T];L_1^p\cap L^\infty)} \|f - \tilde{f}\|_{C([0,T'];L^\infty)}\end{aligned}$$

for some function $\kappa \in C([0,T])$ which only depends on p, d, γ and satisfies $\kappa(0) = 0$ (this uses hp. $p > d$). Hence for $T' \in (0,T]$ small enough such that $\kappa(T')\||f| + |\tilde{f}|\|^\gamma_{C([0,T];L_1^p\cap L^\infty)} \le \frac{1}{2}$, we infer that $f(t) \equiv \tilde{f}(t)$ for all $t \in [0,T']$. Iterating this argument for a finite number of times, we conclude that $f(t) \equiv \tilde{f}(t)$ for all $t \in [0,T]$. □

For global existence of ($\mathsf{FP}_{\gamma.\mathrm{reg}}$) let us introduce the Banach space

$$X = (L^\infty \cap L_\ell^p \cap L_n^1)(\mathbb{R}^d) \tag{2.6}$$

for parameters p, ℓ, n obeying the conditions

$$\begin{aligned} &p \in (d,\infty),\ p \ge 2, \\ &\ell \ge n \ge 2. \end{aligned} \tag{2.7}$$

Proposition 2.2 (Global existence for eq. ($\mathsf{FP}_{\gamma.\mathrm{reg}}$))**.** *Let $\varepsilon \in (0,1]$. Let X be as in* (2.6) *with* (2.7) *being satisfied, and suppose that $f_{\mathrm{in}} \in X$ is nonnegative. There exists a unique global-in-time mild solution $f_\varepsilon \in C([0,\infty); X)$ of the Cauchy problem* ($\mathsf{FP}_{\gamma.\mathrm{reg}}$).

Moreover, the following additional properties hold true:

(i) *Positivity: $f_\varepsilon \ge 0$ in $(0,\infty) \times \mathbb{R}^d$.*
(ii) *Smoothness: $f_\varepsilon \in C^{1,2}((0,\infty) \times \mathbb{R}^d)$ and* ($\mathsf{FP}_{\gamma.\mathrm{reg}}$) *holds in the classical sense.*
(iii) *Mass conservation: $\|f_\varepsilon(t)\|_{L^1} = \|f_{\mathrm{in}}\|_{L^1}$ for all $t > 0$.*
(iv) *Preservation of radial symmetry: if f_{in} is isotropic, so is $f_\varepsilon(t)$ for all $t > 0$.*

Given nonnegative initial data $f_{\mathrm{in}}^{(i)} \in X$, $i = 1,2$, denote by $f_\varepsilon^{(i)}$, $i = 1,2$, the solution emanating from $f_{\mathrm{in}}^{(i)}$. Then

(v) *L^1-Contractivity: $\|f_\varepsilon^{(1)}(t) - f_\varepsilon^{(2)}(t)\|_{L^1} \le \|f_{\mathrm{in}}^{(1)} - f_{\mathrm{in}}^{(2)}\|_{L^1}$ for all $t > 0$.*
(vi) *Comparison: if $f_{\mathrm{in}}^{(1)} \le f_{\mathrm{in}}^{(2)}$, then $f_\varepsilon^{(1)} \le f_\varepsilon^{(2)}$ in $(0,\infty) \times \mathbb{R}^d$.*

Proof. Uniqueness is a direct consequence of Lemma 2.1 and the embedding $X \hookrightarrow (L^\infty \cap L_1^p)(\mathbb{R}^d)$. Local existence (using the contraction mapping principle) and versions of the properties (i)–(vi) have either been established in the literature or can be deduced from classical arguments. See [CLR09] as well as [LSU68, QuS19]. The local existence result is equally valid for (FP_γ). Since [CLR09, Section 2.1] only considers the case $n = 1$, let us briefly indicate how to deal with the L_n^1 part for $n \ge 1$. Denoting by $\mathcal{T}[f](t)$ the RHS of (2.1) (reformulated as in (2.3)), we have the bounds

$$\|\mathcal{T}[f](t)\|_{L_n^1} \le C_T \|f_{\mathrm{in}}\|_{L_n^1} + C_T \int_0^t \nu(t-s)^{-\frac{1}{2}} \||w||f(s)|^{\gamma+1}\|_{L_n^1}\,\mathrm{d}s$$

and

$$\begin{aligned}\||w||f|^{\gamma+1}\|_{L_n^1(\mathbb{R}^d)} &\le \int_{\{|f|\ge 1\}} (1+|w|)^{n+1}|f|^p|f|^{\gamma+1-p}\,\mathrm{d}w + \int_{\{|f|<1\}} (1+|w|)^{n+1}|f|^2\,\mathrm{d}w \\ &\le \int_{\mathbb{R}^d} (1+|w|)^{\ell p}|f|^p\,\mathrm{d}w \cdot (1 + \|f\|_{L^\infty(\mathbb{R}^d)})^{(\gamma+1-p)_+} + \|f\|_{L_n^1}^\theta \|f\|_{L_\ell^p}^{2-\theta},\end{aligned}$$

where $\theta \in [0,1)$ is determined by $1+(1-\theta)^{-1} = p$ and where we used the relations (2.7). For ($\mathsf{FP}_{\gamma.\mathrm{reg}}$) one argues similarly.

For proving global existence it remains to show that $\|f(t)\|_X$ cannot blow up in finite time $T<\infty$. To see this, we recall that $p \geq 2$ and $p > d$ and estimate for some $r \in [\frac{p}{2}, p-1]$ and $t \in [0,T']$, where $0 < T' < T < \infty$,

$$\begin{aligned}\|f_\varepsilon(t)\|_{L^p_\ell} &\leq C_T\|f_{\mathrm{in}}\|_{L^p_\ell} + C_T\int_0^t \nu(t-s)^{-\frac12-\frac{d}{2}(\frac1r-\frac1p)}\|wf_\varepsilon\eta_\varepsilon(f_\varepsilon)\|_{L^r_\ell}\,\mathrm{d}s \\ &\leq C_T\|f_{\mathrm{in}}\|_{L^p_\ell} + (\tfrac{2}{\varepsilon})^\gamma\kappa_T(T')\|f_\varepsilon\|_{C([0,T'];L^p_\ell)}\end{aligned} \tag{2.8}$$

for suitable $\kappa_T \in C([0,T])$ with $\kappa_T(0)=0$ only depending on fixed parameters. In the second step we used the fact that $\gamma \geq 1$, $p\ell \geq r\ell+1$ and $2r \geq p \geq r$ to find that

$$\begin{aligned}\|wf_\varepsilon\eta_\varepsilon(f_\varepsilon)\|_{L^r_\ell(\mathbb{R}^d)} &\leq (\tfrac{2}{\varepsilon})^\gamma\int_{\mathbb{R}^d}(1+|w|)^{r\ell+1}|f_\varepsilon|^r\min(|f_\varepsilon|^r,1)\,\mathrm{d}w \\ &\leq (\tfrac{2}{\varepsilon})^\gamma\int_{\mathbb{R}^d}(1+|w|)^{p\ell}|f_\varepsilon|^p\,\mathrm{d}w.\end{aligned}$$

Since the RHS of (2.8) has at most linear growth in $F(T') := \|f_\varepsilon\|_{C([0,T'];L^p_\ell)}$, one easily finds that $\sup_{t\in(0,T)}F(t) < \infty$. With this in hand, the L^∞-bound on $[0,T)$, $T<\infty$, is obtained as follows: using the bound $0 \leq \eta_\varepsilon \leq (\frac{2}{\varepsilon})^\gamma$ we estimate for $0 \leq t \leq T' < T$

$$\begin{aligned}\|f_\varepsilon(t)\|_{L^\infty} &\leq C_T\|f_{\mathrm{in}}\|_{L^\infty} + C_T\int_0^t \nu(t-s)^{-\frac12-\frac{d}{2p}}\|wf_\varepsilon\eta_\varepsilon(f_\varepsilon)\|_{L^p}\,\mathrm{d}s \\ &\leq C_T\|f_{\mathrm{in}}\|_{L^\infty} + (\tfrac{2}{\varepsilon})^\gamma\kappa_1(T')\|f_\varepsilon\|_{C([0,T'];L^p_1)}\end{aligned}$$

for suitable $\kappa_1 \in C([0,T])$ with $\kappa_1(0)=0$ only depending on fixed parameters.

For the $L^1_n(\mathbb{R}^d)$-norm we even obtain a bound that is uniform in time, see Lemma 2.3 below. ☐

Lemma 2.3 (Uniform moment bound)**.** *Assume the hypotheses of Prop. 2.2 and let $f_\varepsilon \in C([0,T);X)$ denote the (local-in-time) mild solution from the proof of Prop. 2.2. Then, for all $t \in [0,T)$*

$$\int_{\mathbb{R}^d} f_\varepsilon(t,v)(1+|v|^n)\,\mathrm{d}v \leq C(\|f_{\mathrm{in}}\|_{L^1_n}) \tag{2.9}$$

for a finite constant $C(\|f_{\mathrm{in}}\|_{L^1_n}) < \infty$ that is independent of T and ε.

Proof. This result follows from an estimate on the time derivative of the $\tilde n$-th moment $\int_{\mathbb{R}^d} f_\varepsilon(t,v)|v|^{\tilde n}\,\mathrm{d}v$ obtained first for $\tilde n = 2$, which thanks to mass conservation and interpolation implies a uniform bound for all $n' \in (0,2]$ with a constant $C(\|f_{\mathrm{in}}\|_{L^1_2})$. A subsequent induction step from n' to $n'+2$ allows to deduce ineq. (2.9). Similar arguments can be found in [CHR20, Lemma 4.11] (in different coordinates) and in [CLR09, Lemma 2.9]. Observe that in the bosonic case the nonlinear drift occurs along a vector field pointing towards the origin. Thus, moment bounds are more easily available than in the fermionic case treated in [CLR09]. ☐

For technical reasons, we further need a local-in-time existence result for (FP_γ) in a functional setting that provides a uniform spatial decay property of the local mild solution on some short time interval. For this purpose we introduce for $\ell \geq 1$ the space

$$Y := (L^\infty_\ell \cap L^1_1)(\mathbb{R}^d).$$

Note that by interpolation $Y \hookrightarrow L^p_1(\mathbb{R}^d)$ for any $p \in (1,\infty)$. Local existence (and uniqueness) of (FP_γ) and ($\mathsf{FP}_{\gamma.\mathrm{reg}}$) in Y together with a gradient control can be shown as in [CLR09, Section 2] using the contraction mapping principle.

Lemma 2.4 (Local existence for (FP_γ) and ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) in Y)**.** *Let $\ell \geq 1$. For any $L \in (0,\infty)$ there exists $T = T(L) > 0$ such that for every $f_{\mathrm{in}} \in Y$ with $\|f_{\mathrm{in}}\|_Y \leq L$ there exists a unique mild solution $f \in C([0,T];Y)$ of eq.* (FP_γ). *Moreover, one has the additional regularity $t \mapsto \nu(t)^{\frac{1}{2}}|\nabla f(t)| \in C_b((0,T);(L^\infty_\ell \cap L^1_1)(\mathbb{R}^d))$.*
An analogous result holds for eq. ($\mathrm{FP}_{\gamma.\mathrm{reg}}$).

Assume for the moment the hypotheses of Prop. 2.2 and L. 2.4, and let $f^{(X)}(t) \in X$ and $f^{(Y)}(t) \in Y$ denote the local solutions emanating from some $f_{\mathrm{in}} \in X \cap Y$, as obtained in the proof of Prop. 2.2 and in L. 2.4. Then, since both spaces X and Y embed into $(L^\infty \cap L^p_1)(\mathbb{R}^d)$ for some $p > d$, we can invoke Lemma 2.1 to infer that $f^{(Y)}$ and $f^{(X)}$ coincide as long as they exist.

The following remark is another application of the uniqueness property in L. 2.1.

Corollary 2.5 (Short-time consistency)**.** Let $f \in C([0,T];(L^\infty \cap L^p_1)(\mathbb{R}^d))$ be a local-in-time mild solution of (FP_γ), let $0 < \epsilon_* < (\|f\|_{C([0,T];L^\infty)})^{-1}$ and $\varepsilon \in (0,\epsilon_*]$. Then, since $h_\varepsilon(s) = h(s)$ for $s \leq \varepsilon^{-1}$, the function f is also the unique mild solution $f = f_\varepsilon \in C([0,T];(L^\infty \cap L^p_1)(\mathbb{R}^d))$ of ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) in $[0,T]$. In particular, as long as a suitably regular mild solution of the original problem (FP_γ) exists, the scheme trivially converges to this solution.

Suppose now that $f_{\mathrm{in}} \in Y \cap X$ and let $f, f_\varepsilon \in C([0,T];Y \cap X)$ denote the local-in-time mild solutions of (FP_γ) resp. ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) obtained as a consequence of L. 2.4 and Prop. 2.2. Further pick $0 < \epsilon_* < (\|f\|_{C([0,T];L^\infty)})^{-1}$. Then, arguing as in Corollary 2.5, $f = f_\varepsilon$ in $[0,T] \times \mathbb{R}^d$ for all $\varepsilon \in (0,\epsilon_*]$. Since $f \in C^{1,2}((0,T) \times \mathbb{R}^d)$, we may henceforth assume without loss of generality the additional regularity $f_{\mathrm{in}} \in C^2(\mathbb{R}^d)$ and $f \in C^{1,2}([0,T) \times \mathbb{R}^d)$ (otherwise, we consider the solution emanating from $f(t_0)$ for some small $t_0 \in (0,T)$). Thus, from now on we assume the following stronger version of hp. (H2)

$$
\text{(H2')}\quad\begin{cases} f_{\mathrm{in}} \in (L^\infty_{\ell_1} \cap L^1_{\ell_2})(\mathbb{R}^d), \quad \text{where } \ell_1 \geq d, \ell_1 > 2 \text{ and } \ell_2 > \max\{d,2\}, \\ f_{\mathrm{in}} \geq 0 \text{ a.e. in } \mathbb{R}^d; \\ \text{if } f_{\mathrm{in}} \text{ is not isotropic, assume in addition } f_{\mathrm{in}} \in L^\infty_{\ell_3}(\mathbb{R}^d) \text{ for some } \ell_3 > 2d. \\ \text{Without loss of generality, we further suppose that } f \in C^{1,2}([0,\tau_*] \times \mathbb{R}^d), \\ \qquad \text{where } f \text{ denotes the local regular solution of } (\mathrm{FP}_\gamma) \text{ with } f(0) = f_{\mathrm{in}}. \end{cases}
$$

In the following subsection it will become clear that some extra decay hypothesis on the initial data as compared to the space X used in Prop. 2.2 is needed. The specific choice in hp. (H2) (resp. (H2')) has been made for convenience, and we have not attempted to optimise the regularity conditions.

We henceforth denote by f_ε the global mild solution of the regularised equation ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) as obtained in Prop. 2.2. In view of Lemma 2.4 and the subsequent remarks, this solution enjoys some additional decay properties on a short time interval. Besides, the family $\{f_\varepsilon\}$ has an (ε,t)-uniformly bounded moment of order $\min\{\ell_1,\ell_2\} > 2$ (cf. Lemma 2.3).

2.2. **Uniform bounds.**

2.2.1. *Isotropic solutions.* In this subsection we assume f_{in} to be isotropic and write $g_{\mathrm{in}}(r) = f_{\mathrm{in}}(v)$, $|v| = r$. By Prop. 2.2 (iv), the global mild solution f_ε of ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) is isotropic, allowing us to write $g_\varepsilon(t,r) := f_\varepsilon(t,v)$ for $r = |v| \geq 0$. Observe that g_ε satisfies the equation

$$
\begin{aligned} \partial_t g_\varepsilon &= r^{-(d-1)}\partial_r\Big(r^{d-1}\partial_r g_\varepsilon + r^d h_\varepsilon(g_\varepsilon)\Big) \text{ in } \mathbb{R}_+ \times \mathbb{R}_+, \\ 0 &= \lim_{r\to 0}\Big(r^{d-1}\partial_r g_\varepsilon + r^d h_\varepsilon(g_\varepsilon)\Big), \end{aligned} \tag{2.10}
$$

where the limit in the last line holds locally uniformly in $t \in [0,\infty)$.

Our fundamental a priori bound for (FP_γ) relies on the fact that in the isotropic case, equation ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) can be expressed as an evolution equation for the partial mass function

$$M_\varepsilon(t,r) := \int_0^r g_\varepsilon(t,\rho)\rho^{d-1}\mathrm{d}\rho = c_d^{-1}\int_{B_r} f_\varepsilon(t,v)\,\mathrm{d}v, \tag{2.11}$$

where c_d denotes the area of the unit sphere ∂B_1 in $\mathbb{R}^d$. The equation for M_ε is obtained by multiplying (2.10) by r^{d-1} and integrating in r

$$\partial_t M_\varepsilon = r^{d-1}\partial_r g_\varepsilon + r^d h_\varepsilon(g_\varepsilon). \tag{2.12}$$

Using the relations

$$\begin{aligned}\partial_r M_\varepsilon &= r^{d-1}g_\varepsilon,\\ \partial_r^2 M_\varepsilon &= r^{d-1}\partial_r g_\varepsilon + \tfrac{(d-1)}{r}\partial_r M_\varepsilon,\end{aligned}$$

one arrives at

$$\begin{cases}\partial_t M_\varepsilon = \partial_r^2 M_\varepsilon - \frac{(d-1)}{r}\partial_r M_\varepsilon + r^d h_\varepsilon(r^{1-d}\partial_r M_\varepsilon), & t>0,\ r\in\mathbb{R}_+,\\ M_\varepsilon(t,0)=0, & t>0,\\ M_\varepsilon(0,r)=M_{\mathrm{in}}(r), & r\in\mathbb{R}_+.\end{cases} \tag{2.13}$$

As a consequence of Corollary 2.5, there exists $\tau_*>0$ and $\epsilon_*>0$ such that

$$M_\varepsilon \equiv M \text{ in } [0,\tau_*]\times[0,\infty) \text{ for all } \varepsilon\in(0,\epsilon_*], \tag{2.14}$$

where $M(t,r) = c_d^{-1}\int_{B_r} f(t,v)\,\mathrm{d}v$ with $f\in C([0,\tau_*];L^\infty_{\ell_1}(\mathbb{R}^d)\cap X)$ denoting the local-in-time mild solution of eq. (FP_γ). Hence, thanks to the regularity established in Prop. 2.2 and L. 2.4, we can ensure that after a small time shift (replacing the initial time $t=0$ by a small positive time $t=t_0>0$, cf. hp. (H2'))

$$M\in C^{1,2}([0,\tau_*]\times[0,\infty)) \text{ with } \sup_{\tau\in[0,\tau_*]}\|\partial_t M(\tau,\cdot)\|_{L^\infty([0,\infty))}\le K<\infty, \tag{2.15}$$

where the last estimate follows from (2.12) and the regularity $f\in C([0,\tau_*];L^\infty_{\ell_1}(\mathbb{R}^d))$, $\nabla f\in C([0,\tau_*];L^\infty_{\ell_1}(\mathbb{R}^d))$ and the hypothesis that $\ell_1\ge d$.

Proposition 2.6 (Lipschitz regularity in time)**.** *Suppose that f_{in} is isotropic and satisfies the hypotheses in (H2'). Denote by M_ε the partial mass function* (2.11) *of the global solution f_ε of* ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) *obtained in Prop. 2.2. In particular, M_ε is a classical solution of eq.* (2.13) *satisfying* (2.14), (2.15) *and is such that f_ε enjoys the uniform moment bound* (2.9) *for $n:=\min\{\ell_1,\ell_2\}>2$. Then*

$$\sup_{\varepsilon\in(0,\epsilon_*]}\sup_{t,r>0}|\partial_t M_\varepsilon(t,r)|\le K_*, \tag{2.16}$$

where

$$K_* := \max\{K,\tfrac{\tilde m}{\tau_*}\}<\infty \tag{2.17}$$

with K,τ_* as in (2.15) *and $\tilde m := c_d^{-1}m = c_d^{-1}\|f_{\mathrm{in}}\|_{L^1(\mathbb{R}^d)}$.*

Proof. Let K_* be as in (2.17). We will show by contradiction that

$$\sup_{\varepsilon\in(0,\epsilon_*]}\big(M_\varepsilon(t,r)-M_\varepsilon(s,r)\big)\le K_*|t-s|$$

for all $t,s,r>0$.

Suppose the last inequality is false for some $\varepsilon>0$. Then there exist t,s,r such that

$$M_\varepsilon(t,r)-M_\varepsilon(s,r)-K_*|t-s|>0.$$

Pick some $T > \max\{t,s\}$. Without loss of generality we further assume that $T > \tau_*$ with τ_* being as in (2.14), (2.15). Then, for $\delta > 0$ small enough, we have

$$M_\varepsilon(t,r) - \frac{\delta}{T-t} - \frac{\delta}{T-s} - M_\varepsilon(s,r) - K_*|t-s| > 0$$

and hence

$$\sup_{(t,s,r)\in Q}\Big(M_\varepsilon(t,r) - M_\varepsilon(s,r) - K_*|t-s| - \frac{\delta}{T-t} - \frac{\delta}{T-s}\Big) > 0,$$

where $Q = (0,T)\times(0,T)\times(0,\infty)$.

We assert that the function

$$U(t,s,r) := M_\varepsilon(t,r) - M_\varepsilon(s,r) - K_*|t-s| - \frac{\delta}{T-t} - \frac{\delta}{T-s}$$

attains its (positive) supremum in the interior of Q. This can be seen as follows: by the uniform continuity of M_ε on $[0,T]\times[0,1]$ and the fact that $M_\varepsilon(\cdot,0)\equiv 0$, there exists $r'>0$ such that $U<0$ in $[0,T]\times[0,T]\times[0,r']$. Moreover, by (2.14) and (2.15) one has $U<0$ in $[0,\tau_*]\times[0,\tau_*]\times[0,\infty)$. The bound $M_\varepsilon \le \tilde m$ further shows that $U<0$ in $[0,T]\times[T-\epsilon,T]\times[0,\infty)$ and in $[T-\epsilon,T]\times[0,T]\times[0,\infty)$ for some $\epsilon=\epsilon(\delta,\tilde m)>0$. Next, for all $\bar s\in[\tau^*,T]$ and $r\in[0,\infty)$, we have $U(0,\bar s,r)\le \tilde m - K_*\tau_* - \frac{2\delta}{T} < 0$ thanks to the choice of K_*. Likewise, $U(\bar t,0,r)\le -\frac{2\delta}{T}$ for all $\bar t\in[t^*,T]$ and $r\in[0,\infty)$. Hence, it remains to rule out the existence of a maximising sequence (t_n,s_n,r_n) with $r_n\to\infty$. To this end, we take advantage of mass conservation and the moment bound (2.9) (for $n=1$) to estimate

$$\begin{aligned}
U(t,s,r) &\le c_d^{-1}\int_{\mathbb{R}^d\setminus B_r} f_\varepsilon(s,v)\,\mathrm{d}v - \frac{2\delta}{T}\\
&\le \frac{1}{c_d(1+r)}\int_{\mathbb{R}^d\setminus B_r} f_\varepsilon(s,v)\,(1+|v|)\,\mathrm{d}v - \frac{2\delta}{T}\\
&\le \frac{1}{c_d(1+r)}\|f_\varepsilon\|_{C([0,T];L^1_1)} - \frac{2\delta}{T}.
\end{aligned}$$

Observe that the right-hand side is negative whenever $r\ge R_*$ for a finite radius $R_* = R_*(\|f_{\mathrm{in}}\|_{L^1_2},T,\delta)$ large enough. Hence, the same is true for $U(t,s,r)$.

Thus, the supremum of U must be attained at some interior point $p^*=(t,s,r)\in Q$. At the point p^* we have the optimality conditions

$$\begin{aligned}
\partial_t M_\varepsilon(t,r) - K_*\frac{t-s}{|t-s|} &= \frac{\delta}{(T-t)^2},\\
-\partial_s M_\varepsilon(s,r) + K_*\frac{t-s}{|t-s|} &= \frac{\delta}{(T-s)^2},
\end{aligned}$$

and hence

$$\partial_t M_\varepsilon(t,r) - \partial_s M_\varepsilon(s,r) = \frac{\delta}{(T-t)^2} + \frac{\delta}{(T-s)^2}.$$

Moreover,

$$\partial_r M_\varepsilon(t,r) = \partial_r M_\varepsilon(s,r)$$

and thus

$$h_\varepsilon(r^{1-d}\partial_r M_\varepsilon(t,r)) - h_\varepsilon(r^{1-d}\partial_r M_\varepsilon(s,r)) = 0. \tag{2.18}$$

Further note that $0\ge \partial_r^2 U(t,s,r) = \partial_r^2 M_\varepsilon(t,r) - \partial_r^2 M_\varepsilon(s,r)$.

In combination with equation (2.13) we deduce at the point $(t,s,r)=p^*$:

$$0=\partial_t M_\varepsilon(t,r)-\partial_s M_\varepsilon(s,r)-(\partial_r^2 M_\varepsilon(t,r)-\partial_r^2 M_\varepsilon(s,r)) \\ \geq \frac{\delta}{(T-t)^2}+\frac{\delta}{(T-s)^2}>0,$$

which is a contradiction. This completes the proof of Proposition 2.6. □

The comparison principle underlying the proof of Prop. 2.6 can further be used to deduce monotonicity in ε of $M_\varepsilon(t,r)$.

Proposition 2.7 (Monotonicity of the scheme)**.** *Let the hypotheses of Prop. 2.6 hold. For any* $0<\varepsilon'\leq\varepsilon\leq\epsilon_*$

$$M_{\varepsilon'}(t,r)\geq M_\varepsilon(t,r),\qquad t,r>0.$$

Proof. The reasoning is similar to the proof of Prop. 2.6. By contradiction, one assumes that $M_\varepsilon(t,r)-M_{\varepsilon'}(t,r)$ has a positive supremum on $(0,T)\times(0,\infty)$ for some $T<\infty$, and then considers for $\delta>0$ small enough the function

$$U(t,r)=M_\varepsilon(t,r)-M_{\varepsilon'}(t,r)-\tfrac{\delta}{T-t}.$$

At an interior maximum point, one uses elementary calculus as before, where the main difference is that instead of line (2.18), we have now an inequality

$$h_\varepsilon(r^{1-d}\partial_r M_\varepsilon(t,r))-h_{\varepsilon'}(r^{1-d}\partial_r M_{\varepsilon'}(t,r))\leq 0,$$

since by definition $h_\varepsilon\leq h_{\varepsilon'}$. The conclusion is then obtained by conceptually following the proof of Prop. 2.6. □

The bound in Proposition 2.6 combined with the conservation of mass allows us to infer a uniform pointwise bound of the family $\{f_\varepsilon\}_\varepsilon$ away from the origin.

Lemma 2.8 (Bound away from origin: isotropic case)**.** *Assume the hypotheses of Proposition 2.6 and let* K_* *be as in* (2.16)*. Then for all* $\varepsilon\in(0,\epsilon_*]$

$$g_\varepsilon(t,r)\leq\max\{2K_*,2d\tilde{m}\}r_0^{-d}\quad\text{for all } t>0, r\geq r_0, \tag{2.19}$$

where as before we let $g_\varepsilon(t,|v|):=f_\varepsilon(t,v)$ *for* $f_\varepsilon(t,\cdot)$ *isotropic.*

Proof. Recall that in the isotropic case (cf. (2.12))

$$\partial_t M_\varepsilon=r^{d-1}\partial_r g_\varepsilon+r^d h_\varepsilon(g_\varepsilon),$$

where, by Prop. 2.6, $|\partial_t M_\varepsilon|\leq K_*$ for some finite constant $K_*>0$ that is independent of ε. In the following, we let $r_0>0$ be fixed but arbitrary. Then for all $t>0$ and $r\geq r_0>0$

$$h_\varepsilon(g_\varepsilon(t,r))\leq\frac{K_*}{r_0^d}-\frac{\partial_r g_\varepsilon(t,r)}{r}.$$

If $\partial_r g_\varepsilon(t,r)\geq 0$, the bound $s\leq h_\varepsilon(s)$ immediately yields

$$g_\varepsilon(t,r)\leq\frac{K_*}{r_0^d}. \tag{2.20}$$

It remains to consider the case $\partial_r g_\varepsilon(t,r)<0$. An elementary argument based on monotonicity and the bound obtained in (2.20) (with r replaced by $\rho\in[r_0,r)$) shows that

$$g_\varepsilon(t,r)\leq\max\{K_* r_0^{-d},g_\varepsilon(t,r_0)\}.$$

In order to estimate $g_\varepsilon(t,r_0)$ we rely on mass conservation. We assert that unless $g_\varepsilon(t,r_0) \le 2d\tilde{m}r_0^{-d}$, there exists $\lambda \in [2^{-\frac{1}{d}},1)$ such that $\partial_r g_\varepsilon(t,\lambda r_0) \ge 0$. Indeed, if $g_\varepsilon(t,r_0) > 2d\tilde{m}r_0^{-d}$ and $\partial_r g_\varepsilon(t,\rho) < 0$ for all $\rho \in [2^{-\frac{1}{d}}r_0, r_0]$, then $g_\varepsilon(t,\rho) \ge g_\varepsilon(t,r_0)$ and hence

$$\tilde{m} \ge \int_{2^{-\frac{1}{d}}r_0}^{r_0} g_\varepsilon(t,\rho)\rho^{d-1}\mathrm{d}\rho > 2d\tilde{m}r_0^{-d}\cdot \tfrac{1}{d}r_0^d\big(1-2^{-1}\big) = \tilde{m},$$

a contradiction. Taking the largest such λ and using again (2.20) (with r_0 replaced by λr_0), we find

$$g(t,r_0) \le 2K_* r_0^{-d}.$$

In conclusion, we obtain the bound (2.19). □

2.2.2. *Anisotropic case.* For general anisotropic initial data $f_{\rm in}$ satisfying (H2') we consider as in [CnC*16] an isotropic envelope $\hat{f}_{\rm in}(v) \ge f_{\rm in}(v)$ given by

$$\hat{f}_{\rm in}(v) = \frac{\|f_{\rm in}\|_{L^\infty_{\ell_3}}}{(1+|v|^{\ell_3})}.$$

The hypothesis that $\ell_3 > 2d$ ensures that $\hat{f}_{\rm in}$ satisfies hp. (H2') and thus in particular the hypotheses of Prop. 2.2. Invoking Prop. 2.2, we obtain global-in-time (mild) solutions f_ε and $\hat{f}_\varepsilon$ of ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) emanating from $f_{\rm in}$ resp. $\hat{f}_{\rm in}$, where by the comparison property, Prop. 2.2 (vi), $f_\varepsilon \le \hat{f}_\varepsilon$ in $[0,\infty)\times\mathbb{R}^d$. Thus, the uniform bound away from zero in the isotropic case (cf. Lemma 2.8) implies a similar result for anisotropic solutions:

Corollary 2.9 (Bound away from origin: anisotropic case)**.** Assume (H2'). There exists a finite (non-explicit) constant $\hat{K}_*$ only depending on $\|f_{\rm in}\|_{L^\infty_{\ell_3}}$ and ℓ_3 such that for all $r_0 > 0$ and all $|v| \ge r_0$

$$f_\varepsilon(t,v) \le \max\big\{2\hat{K}_*, 2d\hat{m}\big\}r_0^{-d} \quad \text{for all } t>0,$$

where $\hat{m} = c_d^{-1}\|\hat{f}_{\rm in}\|_{L^1}$.

2.3. **Passage to the limit.**

Proof of Proposition 1.1. For $\varepsilon \in (0,\epsilon_*]$ let f_ε be the global-in-time mild solution of ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) emanating from $f_{\rm in}$ as constructed in Prop. 2.2. In the rest of this proof we abbreviate $U := \mathbb{R}^d\setminus\{0\}$.

- *Approximation property.* We first assert that for every $G \subset\subset (0,\infty)\times U$, we have an ε-uniform bound of the form

$$\|f_\varepsilon\|_{H^{1+\frac{\alpha}{2},2+\alpha}(G)} \le C_G \tag{2.21}$$

for some $\alpha \in (0,1)$, where $H^{1+\frac{\alpha}{2},2+\alpha}(G)$ denotes the parabolic Hölder space with $\frac{\alpha}{2}$-Hölder continuous first order temporal and α-Hölder continuous second order spatial derivatives. Ineq. (2.21) can be shown using standard results on parabolic regularity [LSU68, Lie96]. To sketch the main points, we first observe that each f_ε is strictly positive and smooth in $(0,\infty)\times\mathbb{R}^d$. Moreover, as a consequence of L. 2.8 resp. Cor. 2.9, the family $\{f_\varepsilon\}_\varepsilon$ is ε-uniformly bounded in $L^\infty(G)$. Hence, rewriting ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) as $\partial_t f_\varepsilon = \Delta f_\varepsilon + h_\varepsilon'(f_\varepsilon)v\cdot\nabla f_\varepsilon + dh_\varepsilon(f_\varepsilon)$, Theorem 1.11 in [LSU68, Chapter III] on linear parabolic equations provides us with an ε-uniform gradient bound $\|\nabla f_\varepsilon\|_{C^0(G)} \le C_G$. For higher-order spatial derivatives, ε-uniform bounds on G are obtained by applying a similar reasoning to the equation satisfied by $\partial_{v_i}f_\varepsilon$ etc., and time regularity follows from the equation itself.

Hence, there exists $f \in C^{1,2}((0,\infty)\times U)$ such that, upon passing to a subsequence $\varepsilon\to 0$ (not relabelled),

$$f_\varepsilon \to f \quad \text{in } C^{1,2}(G) \text{ for every } G \subset\subset (0,\infty)\times U, \tag{2.22}$$

and f is a classical solution of (FP_γ) in $(0,\infty)\times U$.

The uniform moment bound in Lemma 2.3 further yields

$$\lim_{\varepsilon\to 0}\|f_\varepsilon(t)-f(t)\|_{L^1(\mathbb{R}^d\setminus B_\rho(0))}=0$$

for all $\rho>0$ and locally uniformly in $t\in[0,\infty)$.

The family of measures $\{\mu^{(\varepsilon)}\}_\varepsilon$, $\mu^{(\varepsilon)}:=f_\varepsilon\mathcal{L}_+^{1+d}$, is tight on any finite time horizon as ensured by Lemma 2.3. Hence, there exists a locally finite measure μ on $[0,\infty)\times\mathbb{R}^d$ such that after passing to another subsequence $\varepsilon\downarrow 0$

$$\mu^{(\varepsilon)}\overset{*}{\rightharpoonup}\mu \text{ in } \mathcal{M}([0,T]\times\mathbb{R}^d)$$

for any $T<\infty$.

- *Mass-conserving curve and decomposition.* Combining the above convergence results we infer the existence of a family of finite measures $\mu_t\in\mathcal{M}_+(\mathbb{R}^d)$ of mass $\mu_t(\mathbb{R}^d)=m$ for all $t\geq 0$ satisfying $\mathrm{d}\mu=\mathrm{d}\mu_t\mathrm{d}t$ and taking the form

$$\mu_t=a(t)\delta_0+f(t,\cdot)\mathcal{L}^d,\quad t\geq 0,$$

with a measurable function $a:[0,\infty)\to[0,m]$ given by $a(t)=m-\|f(t,\cdot)\|_{L^1(\mathbb{R}^d)}$.

We further note that $\mu_t^{(\varepsilon)}:=f_\varepsilon(t,\cdot)\mathcal{L}^d$ satisfies

$$\mu_t^{(\varepsilon)}\overset{*}{\rightharpoonup}\mu_t\quad\text{in } \mathcal{M}(\mathbb{R}^d). \tag{2.23}$$

To prove the asserted weak-$*$ continuity of the mapping $[0,\infty)\ni t\mapsto\mu_t$, it suffices (by the Portmanteau theorem) to show for every open subset $O\subset\mathbb{R}^d$ the estimate

$$\int_O \mathrm{d}\mu_{\hat t}\leq\liminf_{t_j\to\hat t}\int_O\mathrm{d}\mu_{t_j}. \tag{2.24}$$

If $0\in O$, then $B_\rho(0)\subset O$ for $\rho>0$ small enough, and (2.24) holds with an equality. If $0\notin O$, ineq. (2.24) is equivalent to $\int_O f(\hat t,v)\,\mathrm{d}v\leq\liminf_{t_j\to\hat t}\int_O f(t_j,v)\,\mathrm{d}v$, and this bound is a consequence of Fatou's lemma. This establishes (2.24).

- *Unique limit.* We now prove (iv). In the isotropic case, Proposition 2.7 ensures that the limit $M(t,r):=\lim_{\varepsilon\to 0}M_\varepsilon(t,r)=c_d^{-1}\lim_{\varepsilon\to 0}\mu_t^{(\varepsilon)}(B_r)$ is well-defined for all $t,r>0$. Thus, in this case, the limiting density f in (2.22) and hence μ can be uniquely recovered from M, which is independent of the choice of the sequence $\varepsilon\downarrow 0$. In view of the above compactness properties, this implies the asserted uniqueness of the limit.

- *Lipschitz continuity of point mass.* Restricting to isotropic data, we have for $r>0$,

$$\begin{aligned} c_dM_\varepsilon(t,r)=\mu_t^{(\varepsilon)}(B_r)&\to\mu_t(B_r) &&\text{as } \varepsilon\to 0,\\ \mu_t(B_r)&\to a(t) &&\text{as } r\to 0,\end{aligned}$$

where the first line follows from (2.23) and the fact that $\operatorname{supp}\mu_t^{\mathrm{sing}}\subseteq\{0\}$. Thus, the Lipschitz bound (2.16) implies that $|a(t)-a(s)|\leq K_*|t-s|$, hence part (v). □

3. Universal space profile

Equipped with the uniform control (2.16), we will now combine ODE and bootstrap arguments with localised semigroup estimates to study the regularity and the space profile of the density f near the origin. A rigorous analysis is achieved by working with the family of approximate solutions f_ε constructed in Section 2.1. We will show that for isotropic data the solution at any fixed positive time is either regular and smooth, or the density of the regular part follows, up to a lower order term, a universal profile at the origin that is uniquely determined by the limiting steady state f_c. This even slightly improves the profile obtained in [CHR20] for $d=1$.

Throughout this section, the constant K_* denotes the least upper bound such that inequality (2.16) holds true, that is

$$K_* := \sup_{\varepsilon\in(0,\epsilon_*]} \sup_{t>0,r>0} |\partial_t M_\varepsilon(t,r)|. \tag{3.1}$$

For initial data only satisfying (H2), but not (H2'), this bound holds with the supremum being taken over $t > t_0$ for some small $t_0 > 0$ chosen such that $f = f_\varepsilon$ in $[0, 2t_0]$ for all $\varepsilon \in (0, \epsilon_*]$.

3.1. **Lower and upper bounds.** The analysis in this subsection mostly concerns isotropic solutions, for which the uniform bound (2.16) is available. As introduced in Section 2.2.1, in the isotropic case we write $g_\varepsilon(t, r) := f_\varepsilon(t, v)$ whenever $r = |v| > 0$, and likewise $g(t, r) := f(t, v)$ for the pointwise limit obtained upon sending $\varepsilon \downarrow 0$.

Proposition 3.1 (Lower bound). *Abbreviate $\alpha_c = \frac{2}{\gamma}$. In addition to (H1), (H2) suppose that*

$$\alpha_c + 2 - d > 0.$$

Further assume that the initial value $f_{\rm in}$ is isotropic. Pick any $\underline{\alpha} \in ((d-2)_+, \alpha_c)$. For $\alpha \in [\underline{\alpha}, \alpha_c]$ let $\tilde g(r) = c_\gamma r^{-\alpha}$, where $c_\gamma = \big(2/\gamma\big)^{1/\gamma}$. For $\varepsilon \in (0, \epsilon_*]$ define

$$\tilde r_\varepsilon = \tilde r_\varepsilon(t) = \sup\{r > 0 : g_\varepsilon(t, \rho) < \tilde g(\rho) \text{ for all } \rho \in (0, r)\}.$$

There exists a constant $B < \infty$ and a radius $r^* > 0$ only depending on K_* (cf. (3.1)) and on $\gamma, d, \underline{\alpha}$ (but not on α or t) such that for every $\varepsilon \in (0, \epsilon_*]$ the following holds: whenever $\tilde r_\varepsilon(t) \in (0, r^*)$, then

$$g_\varepsilon(t, r) \geq \tilde g(r) - Br^{2-d} \qquad \text{for } r \in (\tilde r_\varepsilon, r^*).$$

Proof of Proposition 3.1. To begin with, we note that for any $\alpha \in [\underline{\alpha}, \alpha_c]$

$$-\alpha \leq -\underline{\alpha} < 2 - d \leq 4 - d - \alpha\gamma. \tag{3.2}$$

Defining $b_\varepsilon(t, r) := \partial_t M_\varepsilon(t, r)$, we have (cf. (2.12))

$$b_\varepsilon = r^{d-1}\partial_r g_\varepsilon + r^d h_\varepsilon(g_\varepsilon) = r^{d-1}\partial_r g_\varepsilon + r^d g_\varepsilon(1 + \eta_\varepsilon(g_\varepsilon)).$$

Letting $k_\varepsilon = g_\varepsilon^{-\gamma}$ this can be rewritten as

$$-\frac{1}{\gamma}\frac{\partial_r k_\varepsilon}{k_\varepsilon} + r\eta_\varepsilon(g_\varepsilon) + r = b_\varepsilon r^{1-d} k_\varepsilon^{\frac{1}{\gamma}},$$

i.e.

$$\partial_r k_\varepsilon + \left(\gamma b_\varepsilon r^{1-d} k_\varepsilon^{\frac{1}{\gamma}} - \gamma r\right) k_\varepsilon = \gamma r \eta_\varepsilon(g_\varepsilon) k_\varepsilon. \tag{3.3}$$

Observe that, in order to prove the assertion, it suffices to consider the case $\tilde r_\varepsilon(t) \leq \frac{1}{2}$, which will henceforth be assumed.

Since $k_\varepsilon(t, \tilde r_\varepsilon) = (\tilde g(\tilde r_\varepsilon))^{-\gamma}$, we may define, by continuity, a radius $\tilde r_{1,\varepsilon} > \tilde r_\varepsilon$ via

$$\tilde r_{1,\varepsilon}(t) := \sup\{r \in (\tilde r_\varepsilon(t), 1) : k_\varepsilon(t, \rho) \leq 2(\tilde g(\rho))^{-\gamma} \text{ for all } \rho \in (\tilde r_\varepsilon(t), r)\}.$$

Defining

$$a_\varepsilon(t, r) = \gamma b_\varepsilon r^{1-d} k_\varepsilon^{\frac{1}{\gamma}} - \gamma r,$$
$$q_\varepsilon(t, r) = \exp\left(\int_{\tilde r_\varepsilon}^r a_\varepsilon(t, s)\,\mathrm{d}s\right),$$

we infer from eq. (3.3)

$$k_\varepsilon(t, r) = \frac{1}{q_\varepsilon(t, r)}\left(k_\varepsilon(t, \tilde r_\varepsilon) + \gamma \int_{\tilde r_\varepsilon}^r s q_\varepsilon(t, s) w_\varepsilon(t, s)\,\mathrm{d}s\right), \tag{3.4}$$

where we abbreviated $w_\varepsilon := \eta_\varepsilon(g_\varepsilon)k_\varepsilon \le 1$.

For all $r \in [\tilde r_\varepsilon, \tilde r_{1,\varepsilon}]$, we can estimate for some constant $C_1 = C_1(\underline{\alpha}) < \infty$

$$\left|\int_{\tilde r_\varepsilon}^r a_\varepsilon(t,s)\,\mathrm{d}s\right| \le C_1 K_* r^{\alpha+2-d},$$

where we used the hypothesis that $\alpha \ge \underline{\alpha} > d-2$ as well as the fact that $\alpha+2-d \le \frac{2}{\gamma}+2-d < 2$. Next, we let $r_{K_*} = r_{K_*}(\underline{\alpha}) > 0$ be such that $C_1K_*r^{\alpha+2-d} \le 2^{-1}$ for all $r \in (0, r_{K_*}]$ and define $\tilde r_{2,\varepsilon} := \min\{\tilde r_{1,\varepsilon}, r_{K_*}\}$. Then for $r \in [\tilde r_\varepsilon, \tilde r_{2,\varepsilon}]$

$$|q_\varepsilon(t,r) - 1| \le C_2 K_* r^{\alpha+2-d}. \tag{3.5}$$

We now estimate for $r \in [\tilde r_\varepsilon, \tilde r_{2,\varepsilon}]$, using the last inequality, identity (3.4) and the bound $0 \le w_\varepsilon \le 1$,

$$\begin{aligned} k_\varepsilon(t,r) &\le \frac{1}{q_\varepsilon(t,r)}\left(k_\varepsilon(t,\tilde r_\varepsilon) + \gamma\int_{\tilde r_\varepsilon}^r s q_\varepsilon(t,s)\,\mathrm{d}s\right) \\ &= \frac{1}{q_\varepsilon(t,r)}\left(\frac{\gamma}{2}\tilde r_\varepsilon^{\alpha\gamma} + \frac{\gamma}{2}r^2 - \frac{\gamma}{2}\tilde r_\varepsilon^2 + O(r^{\alpha+4-d})\right). \end{aligned} \tag{3.6}$$

To estimate the last line, we let $\beta := \frac{\alpha\gamma}{2} \in (0,1]$. Then, by concavity,

$$\beta r^{\beta-1}(r-\rho) \le r^\beta - \rho^\beta \qquad \text{for all } 0<\rho<r<1.$$

Hence, since $r+\rho \le r^\beta + \rho^\beta$, we infer that

$$\beta r^{\beta-1}(r^2-\rho^2) \le r^{2\beta} - \rho^{2\beta} \qquad \text{for all } 0<\rho<r<1. \tag{3.7}$$

It is elementary to see that there exists $r_\circ = r_\circ(\underline{\alpha},\gamma) \in (0, \mathrm{e}^{-1}]$ such that $\beta r^{\beta-1} \ge 1$ for all $\beta \in [\frac{\underline{\alpha}\gamma}{2}, 1]$ and all $r \in (0, r_\circ]$. (For instance, note that $\beta r^{\beta-1} \ge 1$ is equivalent to $r \le \beta^{\frac{1}{1-\beta}}$, where $\beta^{\frac{1}{1-\beta}} \uparrow \mathrm{e}^{-1}$ as $\beta \uparrow 1$.) Thus, after possibly decreasing $r_{K_*} = r_{K_*}(\underline{\alpha}, d, \gamma) > 0$, ineq. (3.7) and (3.5) allow us to further estimate the RHS of (3.6) to obtain for all $r \in [\tilde r_\varepsilon, \tilde r_{2,\varepsilon}]$

$$\begin{aligned} k_\varepsilon(t,r) &\le \frac{1}{q_\varepsilon(t,r)}\left(\frac{\gamma}{2}r^{\alpha\gamma} + O(r^{\alpha+4-d})\right) \\ &\le \frac{\gamma}{2}r^{\alpha\gamma}(1+O(r^{\alpha-\alpha\gamma+4-d}))(1+O(r^{\alpha+2-d})) \\ &\le (\tilde g(r))^{-\gamma}\big(1+O(r^{\alpha+2-d})\big) \\ &\le (\tilde g(r))^{-\gamma}\big(1+\tfrac12\big) < 2(\tilde g(r))^{-\gamma}, \end{aligned}$$

where we used the inequalities $0 < \underline{\alpha}+2-d \le \alpha+2-d \le \alpha - \alpha\gamma + 4 - d$ (cf. (3.2)).

This implies that $\tilde r_{2,\varepsilon} < \tilde r_{1,\varepsilon}$ and hence $\tilde r_{1,\varepsilon} \ge r_{K_*}(\underline{\alpha}, d, \gamma) =: r^* > 0$ for all ε and all t.

In conclusion, the above estimates show that for all $r \in (\tilde r_\varepsilon(t), r^*]$

$$g_\varepsilon(t,r) \ge \tilde g(r) + O(r^{2-d}).$$

□

Proposition 3.2 (Upper bound)**.** *Use the notations and assume the hypotheses of Proposition 3.1. There exists a finite constant B and a radius r^* only depending on K_*, γ, d such that for all $t > 0$ and all $r \in (0, r^*)$*

$$g(t,r) \le g_c(r) + B|r|^{2-d}.$$

Proof. We adopt the notations of Prop. 3.1 and its proof, where here it will suffice to consider the choice $\alpha = \frac{2}{\gamma}$. Thus, we let $\tilde g(r) = c_\gamma r^{-\frac{2}{\gamma}}$ and set

$$r_\varepsilon = r_\varepsilon(t) = \sup\{r > 0 : g_\varepsilon(t,\rho) < \tilde g(\rho) \text{ for all } \rho \in (0,r)\}.$$

Then, by definition and continuity, the function $k_\varepsilon = g_\varepsilon^{-\gamma}$ satisfies $k_\varepsilon(t, r_\varepsilon(t)) = \frac{\gamma}{2} r_\varepsilon^2(t)$ (if $r_\varepsilon(t) < \infty$) and

$$k_\varepsilon(t,r) \geq \tfrac{\gamma}{2} r^2. \qquad \text{for all } r \in (0, r_\varepsilon(t)]. \tag{3.8}$$

Let r^* be the radius obtained in Prop. 3.1. Observing that $g_c(r) = \tilde{g}(r) + O(r^{2-\frac{2}{\gamma}})$ and recalling $d > \frac{2}{\gamma}$, it remains to consider the interval $(r_\varepsilon(t), r^*)$ assuming hereafter that $r_\varepsilon(t) < r^*$. As in the proof of Prop. 3.1 we have the formula

$$k_\varepsilon(t,r) = \frac{1}{q_\varepsilon(t,r)} \left(k_\varepsilon(t, r_\varepsilon) + \gamma \int_{r_\varepsilon}^{r} s q_\varepsilon(t,s) w_\varepsilon(t,s)\,\mathrm{d}s \right).$$

Our task amounts to obtaining a suitable lower bound on k_ε. For this purpose, we define the set

$$J_\varepsilon = J_\varepsilon(t) = \{\rho \in (r_\varepsilon(t), r^*) : g_\varepsilon(t,\rho) \geq \varepsilon^{-1}\}.$$

On $(r_\varepsilon, r^*) \setminus J_\varepsilon$ we have $w_\varepsilon \equiv 1$, while on J_ε we only know that $0 \leq w_\varepsilon \leq 1$. Hence, we estimate for $r \in (r_\varepsilon, r^*)$, using also the fact that $q_\varepsilon(t,r) = 1 + O(r^{\frac{2}{\gamma}+2-d})$ (cf. (3.5)),

$$\begin{aligned}
\int_{r_\varepsilon}^{r} s q(s) w_\varepsilon(s)\,\mathrm{d}s &= \int_{r_\varepsilon}^{r} s w_\varepsilon(s)\,\mathrm{d}s + O(r^{4+\frac{2}{\gamma}-d}) \\
&\geq \int_{(r_\varepsilon, r)\setminus J_\varepsilon} s\,\mathrm{d}s + O(r^{4+\frac{2}{\gamma}-d}) \\
&\geq \frac{1}{2}(r^2 - r_\varepsilon^2) - \mathcal{L}^1(J_\varepsilon) r^* + O(r^{4+\frac{2}{\gamma}-d}),
\end{aligned}$$

where the (fixed) time argument t has been omitted.

Insertion into the formula for k_ε yields for $r \in (r_\varepsilon, r^*)$

$$\begin{aligned}
k_\varepsilon(t,r) &\geq \frac{1}{q_\varepsilon(t,r)} \left(\frac{\gamma}{2} r^2 - \mathcal{L}^1(J_\varepsilon)\gamma r^* + O(r^{4+\frac{2}{\gamma}-d}) \right) \\
&= \left(\frac{\gamma}{2} r^2 - \mathcal{L}^1(J_\varepsilon)\gamma r^* + O(r^{4+\frac{2}{\gamma}-d}) \right) (1 + O(r^{2+\frac{2}{\gamma}-d})),
\end{aligned}$$

while for $r \in (0, r_\varepsilon]$ we recall the bound (3.8). Mass conservation, i.e. $\int_{\mathbb{R}^d} f_\varepsilon(t) \equiv \int_{\mathbb{R}^d} f_{\mathrm{in}}$, implies that $\lim_{\varepsilon\to 0} \mathcal{L}^1(J_\varepsilon(t)) = 0$. Hence, using also the pointwise convergence of g_ε to g for $r \neq 0$, we obtain in the limit $\varepsilon \to 0$

$$k(t,r) \geq \frac{\gamma}{2} r^2 + O(r^{4+\frac{2}{\gamma}-d}) = \frac{\gamma}{2} r^2 (1 + O(r^{2+\frac{2}{\gamma}-d})), \qquad r \in (0, r^*).$$

Thus, $g(t,r) \leq c_\gamma r^{-\frac{2}{\gamma}} + O(r^{2-d}) = g_c(r) + O(r^{2-d})$. □

For anisotropic data, the approximate solutions $\{f_\varepsilon\}$ are dominated by an isotropic scheme $\{\hat{f}_\varepsilon\}$ (cf. Section 2.2.2). Hence, the density $f(t,v)$ of the regular part of the limiting measure in Prop. 1.1 inherits the upper bound obtained above for the isotropic case.

Corollary 3.3 (Upper bound on space profile: anisotropic case)**.** In addition to (H1), (H2) suppose that $\frac{2}{\gamma} + 2 - d > 0$. There exists a finite constant $\hat{B}$ and a radius $\hat{r}^*$ only depending on f_{in} (non-explicitly) and on γ, d such that for all $t > 0$ and all v with $|v| \in (0, \hat{r}^*)$

$$f(t,v) \leq f_c(v) + \hat{B}|v|^{2-d}.$$

In particular, the point mass at the origin $t \mapsto \mu_t(\{0\}) = m - \int f(t,\cdot)$ is continuous (as a consequence of Lebesgue's dominated convergence theorem).

3.2. **Instantaneous regularisation.** For the nonlinear problem (FP_γ) the Lebesgue space $L^{p_c}(\mathbb{R}^d)$, $p_c := \frac{\gamma d}{2}$ is critical (as regards high values of the density). Thus, for $p > p_c$ one would expect eq. (FP_γ) to enjoy a smoothing property in L^p. The following result formalises these heuristics.

Proposition 3.4 (Smoothing out subcritical singularities)**.** *Let $\{f_\varepsilon\}$ be a family of (suitably regular) nonnegative mild solutions of the ε-regularised problems* ($\mathrm{FP}_{\gamma.\mathrm{reg}}$)[3] *with uniformly bounded mass $\|f_\varepsilon(t)\|_{L^1} \lesssim 1$. Let $p > p_c := \frac{\gamma d}{2}$, let $t_0 \geq 0$ and assume that there exists $L < \infty$ such that for all $\varepsilon \in (0, \epsilon_*]$*

$$\|f_\varepsilon(t_0, \cdot)\|_{L^p(\mathbb{R}^d)} \leq L.$$

Further suppose that there exists $t_1 \in (t_0, t_0+1]$, a constant $\tilde{L} < \infty$ and a radius $r^* \in (0,1]$ such that for all $t \in (t_0, t_1]$, all $v \in \mathbb{R}^d$ with $|v| \leq r^*$ and all $\varepsilon \in (0, \epsilon_*]$

$$f_\varepsilon(t, v) \leq \tilde{L}|v|^{-\frac{2}{\gamma}}.$$

Finally, assume that $f_\varepsilon(t, v) \lesssim 1$ on $\{|v| \geq r^*\}$ uniformly in ε and t.

Then there exists $T = T(L, \tilde{L}, p, d, \gamma) \in (0, 1]$ such that for all (small) $\tau > 0$

$$\sup_{\varepsilon \in (0, \epsilon_*]} \sup_{t \in [t_0+\tau, \hat{t}_1]} \|f_\varepsilon(t, \cdot)\|_{L^\infty} < \infty \quad \textit{where } \hat{t}_1 = \min\{t_0 + T, t_1\}. \tag{3.9}$$

Proof. We proceed in two steps. In a first step, we derive smoothing estimates based on the mild formulation (2.2) satisfied by f_ε, where as in Section 2 we rewrite the nonlinear term analogously to (2.3).

Step 1: *localised smoothing estimate.* Let $\epsilon_0 > 0$ and let $\tilde{p}, \tilde{q}$ be exponents satisfying $p_c + \epsilon_0 \leq \tilde{p} \leq \tilde{q} \leq \infty$. Let further $a := \frac{d\gamma}{4}\frac{1}{\tilde{q}} + \frac{1}{2}$ and $b := \frac{d}{2}(\frac{1}{\tilde{p}} - \frac{1}{\tilde{q}})(\frac{\gamma}{2}+1)$, and assume that $0 \leq a \leq a_0 < 1$ and $0 \leq b < b_0 < 1$. We assert that there exists a function $\kappa \in C([0,1])$ only depending on ϵ_0, a_0, b_0 and d with $\kappa(0) = 0$, and a finite constant $C = C(d)$ such that for all $T \in (0, 1]$

$$\|\chi_{\{|v| \leq r^*\}} f_\varepsilon^{(t_0)}\|_{Z_T} \leq C_0 \|f_\varepsilon(t_0, \cdot)\|_{L^{\tilde{p}}(\mathbb{R}^d)} + \kappa(T)\|f_\varepsilon^{(t_0)}\|_{Z_T}\big(\|f_\varepsilon^{(t_0)}\|_{Z_T}^{\frac{\gamma}{2}} + 1\big), \tag{3.10}$$

where $f_\varepsilon^{(t_0)}(\tau, \cdot) := f_\varepsilon(t_0 + \tau, \cdot)$ and

$$\|\tilde{f}\|_{Z_T} := \sup_{s \in (0,T)} \nu(s)^{\frac{d}{2}(\frac{1}{\tilde{p}} - \frac{1}{\tilde{q}})} \|\tilde{f}(s, \cdot)\|_{L^{\tilde{q}}(\mathbb{R}^d)}.$$

Proof of Step 1. Let $\zeta \in C_c^\infty(\mathbb{R}^d)$ with $0 \leq \zeta \leq 1, \zeta = 1$ on $\{|v| \leq r^*\}, \operatorname{supp}\zeta \subset B_{2r^*}(0)$.

By the mild solution property of f_ε, we have (cf. Sec. 2.1)

$$\begin{aligned} f_\varepsilon^{(t_0)}(\tau, v) = &\int_{\mathbb{R}^d} \mathcal{F}(\tau, v, w) f_\varepsilon(t_0, w)\,\mathrm{d}w \\ &- \int_0^\tau \mathrm{e}^{-(\tau-s)} \int_{\mathbb{R}^d} \nabla_v \mathcal{F}(\tau - s, v, w) \cdot \big(w\,\vartheta_\varepsilon(f_\varepsilon^{(t_0)})\big)|_{(s,w)}\,\mathrm{d}w\mathrm{d}s. \end{aligned} \tag{3.11}$$

[3] The family $\{f_\varepsilon\}$ does not have to take the same initial data.

Using the bound $|\vartheta_\varepsilon(s)| \le |s|^{\gamma+1}$ (cf. def. (1.4)), the fact that $|w| f^{\frac{\gamma}{2}}(s,w) \le C(\tilde L,\gamma)$[4] for $|w| \le r^*$, and the uniform control $|f_\varepsilon(t,w)| \lesssim 1$ for $|w| \ge r^*$, we now estimate for $0 < s < \tau \le T$

$$\begin{aligned}
\Big| \int_{\mathbb{R}^d} \nabla_v \mathcal{F}(\tau{-}s,v,w) \cdot \big(w\, \vartheta_\varepsilon(f_\varepsilon^{(t_0)}(s,w))\big) | \,\mathrm{d}w \Big| \\
\le C(\tilde L) \int_{\{|w|\le r^*\}} |\nabla_v \mathcal{F}(\tau{-}s,v,w)| (f_\varepsilon^{(t_0)})^{\frac{\gamma}{2}+1}(s,w)\,\mathrm{d}w \\
+ C\mathrm{e}^{\tau-s} \int_{\{|w|>r^*\}} |\nabla_v \mathcal{F}(\tau{-}s,v,w)| \big(|v| + |\mathrm{e}^{-(\tau-s)}w - v|\big)\, f_\varepsilon^{(t_0)}(s,w)\,\mathrm{d}w.
\end{aligned}$$

The integrals on the RHS will be handled similarly as in the proof of [CLR09, Prop. A.1]. For estimating the $L^{\tilde q}(\mathbb{R}^d)$-norm, Young's convolution inequality is employed. For the first term on the RHS we simply invoke ineq. (2.5) and estimate

$$\begin{aligned}
\Big\| \int_{\mathbb{R}^d} |\nabla_v \mathcal{F}(\tau{-}s,v,w)| (f_\varepsilon^{(t_0)})^{\frac{\gamma}{2}+1}(s,w)\,\mathrm{d}w \Big\|_{L^{\tilde q}(\mathbb{R}^d)} \\
&\le \bar C \nu(\tau{-}s)^{-\frac12 - \frac d2\left(\frac{\frac{\gamma}{2}+1}{\tilde q} - \frac1{\tilde q}\right)} \|(f_\varepsilon^{(t_0)})^{\frac{\gamma}{2}+1}(s)\|_{L^{\tilde q/(\frac{\gamma}{2}+1)}(\mathbb{R}^d)} \\
&\le \bar C \nu(\tau{-}s)^{-\frac12 - \frac d2\left(\frac{\frac{\gamma}{2}+1}{\tilde q} - \frac1{\tilde q}\right)} \nu(s)^{-\frac d2(\frac1{\tilde p} - \frac1{\tilde q})(\frac{\gamma}{2}+1)} \|f_\varepsilon^{(t_0)}\|_{Z_T}^{\frac{\gamma}{2}+1} \\
&= \bar C \nu(\tau{-}s)^{-a} \nu(s)^{-b} \|f_\varepsilon^{(t_0)}\|_{Z_T}^{\frac{\gamma}{2}+1}.
\end{aligned}$$

Here and below, $\bar C$ denotes a constant that only depends on T and other fixed parameters, but which may change from line to line.

We next estimate

$$\begin{aligned}
\Big\| \zeta(v) \int_{\{|w|>r^*\}} |\nabla_v \mathcal{F}(\tau{-}s,v,w)| |v|\, f_\varepsilon^{(t_0)}(s,w)\,\mathrm{d}w \Big\|_{L^{\tilde q}(\mathbb{R}^d)} \\
&\le \bar C \Big\| \int_{\mathbb{R}^d} |\nabla_v \mathcal{F}(\tau{-}s,v,w)| f_\varepsilon^{(t_0)}(s,w)\,\mathrm{d}w \Big\|_{L^{\tilde q}(\mathbb{R}^d)} \\
&\le \bar C \nu(\tau{-}s)^{-\frac12} \nu(s)^{-\frac d2(\frac1{\tilde p} - \frac1{\tilde q})} \|f_\varepsilon^{(t_0)}\|_{Z_T},
\end{aligned}$$

where the second step makes directly use of the bound (2.5).

Finally, the rapid decay of the Fokker–Planck kernel further allows us to estimate

$$\begin{aligned}
\Big\| \int_{\mathbb{R}^d} |\nabla_v \mathcal{F}(\tau{-}s,v,w)| |\mathrm{e}^{-(\tau-s)}w - v|\, f_\varepsilon^{(t_0)}(s,w)\,\mathrm{d}w \Big\|_{L^{\tilde q}(\mathbb{R}^d)} \\
\le \bar C \nu(\tau{-}s)^{-\frac12} \nu(s)^{-\frac d2(\frac1{\tilde p} - \frac1{\tilde q})} \|f_\varepsilon^{(t_0)}\|_{Z_T},
\end{aligned}$$

see Lemma A.1 for details.

Inserting the above estimates into (3.11), we infer for $\tau \in (0,T]$, $T \in (0,1]$,

$$\begin{aligned}
\nu(\tau)^{\frac d2(\frac1{\tilde p} - \frac1{\tilde q})} \|f_\varepsilon^{(t_0)}(\tau)\zeta\|_{L^{\tilde q}(\mathbb{R}^d)} \le C_T \|f_\varepsilon(t_0)\|_{L^{\tilde p}(\mathbb{R}^d)} \\
+ \bar C \nu(\tau)^{\frac d2(\frac1{\tilde p} - \frac1{\tilde q})} \int_0^\tau \nu(\tau{-}s)^{-a} \nu(s)^{-b}\,\mathrm{d}s \|f_\varepsilon^{(t_0)}\|_{Z_T} \big(\|f_\varepsilon^{(t_0)}\|_{Z_T}^{\frac{\gamma}{2}} + 1\big),
\end{aligned}$$

where we used once more ineq. (2.5) as well as the fact that $a \ge \frac12$ and $b \ge \frac d2(\frac1{\tilde p} - \frac1{\tilde q})$.

[4] Any dependence on γ will henceforth not be explicitly indicated.

Hence,

$$\|f_\varepsilon^{(t_0)}(\tau)\zeta\|_{Z_T} \le \|f_\varepsilon(t_0)\|_{L^{\tilde p}(\mathbb{R}^d)} + \bar C\kappa(T)\|f_\varepsilon^{(t_0)}\|_{Z_T}\big(\|f_\varepsilon^{(t_0)}\|_{Z_T}^{\frac{\gamma}{2}}+1\big),$$

where

$$\begin{aligned}\kappa(T) &:= \sup_{\tau\in(0,T)} \nu(\tau)^{\frac{d}{2}(\frac{1}{\tilde p}-\frac{1}{\tilde q})}\int_0^\tau \nu(\tau-s)^{-a}\nu(s)^{-b}\,\mathrm{d}s \\ &\le CT^{\frac{d}{2}(\frac{1}{\tilde p}-\frac{1}{\tilde q})+1-a-b}\int_0^1 \nu(1-\tilde s)^{-a}\nu(\tilde s)^{-b}\,\mathrm{d}\tilde s\end{aligned}$$

and $\frac{d}{2}(\frac{1}{\tilde p}-\frac{1}{\tilde q})+1-a-b \ge c(\varepsilon_0) > 0$ as a consequence of the condition $\tilde p \ge p_c+\varepsilon_0$. This proves Step 1.

Step 2. We are now ready to *complete the proof of Proposition 3.4* using estimate (3.10) combined with the uniform bound on $\{|v| \ge r^*\}$. The idea is to perform a finite number of iterations in the integrability exponents to eventually upgrade the ε-uniform L^p bound on $f_\varepsilon(t_0)$ to an ε-uniform L^∞ bound on $f_\varepsilon(t_0+\tau)$ for given $\tau>0$ small. As this is a rather standard procedure in regularity theory, we only indicate the main points.

First, fix some $\Lambda > 1$ such that

$$(1-\tfrac{1}{\Lambda}) < \frac{\gamma}{(\frac{\gamma}{2}+1)}.$$

It is elementary to verify that for any $\tilde p \ge p > p_c$ and for $\tilde q := \Lambda\tilde p$ the couple $(\tilde p,\tilde q)$ satisfies the hypotheses of Step 1 with parameters ϵ_0, a_0, b_0 only depending on Λ and p. Next, choose $N \in \mathbb{N}_+$ large enough such that $\Lambda^N p > d$ and let $\tau_N := \frac{1}{N+1}\tau$. In the present situation, the function $\kappa \in C([0,1])$ in Step 1 only depends Λ and p and satisfies $\kappa(0)=0$. Hence, for any $L_1\in[1,\infty)$ there exists a time span $\bar T = \bar T(L_1,C_1,C_2,\Lambda,p,\gamma) \in (0,1]$ such that every continuous, non-decreasing function $F(t)$ obeying the estimate

$$F(t) \le L_1 + C_1\kappa(t)F(t)^{1+\frac{\gamma}{2}} + C_2, \qquad t\in[0,1],$$

satisfies $F(\bar T) \le 2L_1$. Letting $\tilde p = p$ and $F(t) := \|\chi_{\{|v|\le r^*\}}f_\varepsilon^{(t_0)}\|_{Z_t}$ and invoking estimate (3.10), the uniform bound of f_ε on $\{|v| \ge r^*\}$ as well as the global $L^1(\mathbb{R}^d)$-control yields $F(\bar T) \le 2C_0L$, which implies that

$$\|\chi_{\{|v|\le r^*\}}f_\varepsilon(t_0+\tau_N)\|_{L^{\Lambda p}} \lesssim \nu(\tau_N)^{-\frac{d}{2p}(1-\frac{1}{\Lambda})}2C_0L.$$

Iterating this argument N times provides an ε-uniform bound on $\|f_\varepsilon(t_0+\frac{N}{N+1}\tau)\|_{L^{\Lambda^N p}}$. Hence, we can take $\tilde p := \Lambda^N p > \max\{d,p_c\}$ in Step 1, in which case the choice $\tilde q = \infty$ is admissible. We thus infer (3.9).

□

3.3. **Space profile.** Finally, we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Fix some $\underline{\alpha} < \alpha_c := \frac{2}{\gamma}$ as in Prop. 3.1 and let $r^*>0$ denote the associated radius obtained in Prop. 3.1. Let $\hat t > 0$. We assert that the behaviour of $g(\hat t,\cdot)$ near zero is determined by the fact of whether or not the hypotheses of *Case 1* are fulfilled, where Case 1 is determined as follows:

Case 1: there exists $\alpha \in [\underline{\alpha}, \alpha_c)$, a time $t_0 < \hat{t}$, a radius $r_0 \in (0, r^*)$, and $\varepsilon_0 \in (0, \epsilon_*]$ such that for all $\varepsilon \in (0, \varepsilon_0]$, all $r \in (0, r_0]$ and all $t \in [t_0, \hat{t}]$

$$g_\varepsilon(t,r) \le \tilde{g}^{(\alpha)}(r) := c_\gamma r^{-\alpha}, \text{ where } c_\gamma := \left(\tfrac{2}{\gamma}\right)^{\frac{1}{\gamma}}. \tag{3.12}$$

Here, $\{g_\varepsilon\}$ denotes the family of isotropic approximate solutions in radial coordinates.

If Case 1 is fulfilled, Proposition 3.4 easily yields $g(\hat{t}, \cdot) \in L^\infty(\mathbb{R}_+)$. Indeed, since $\alpha < \alpha_c$, we can choose $p > p_c$ such that $f^{(\alpha)}(v) := c_\gamma |v|^{-\alpha} \in L^p(B_1)$, where $B_1 := \{v : |v| \le 1\}$. Combined with the moment control in Lemma 2.3 and the uniform bound away from the origin, the upper bound (3.12) on g_ε then ensures that

$$\sup_{\varepsilon \in (0,\varepsilon_0]} \|f_\varepsilon(t,\cdot)\|_{L^p} \le L$$

for all $t \in [t_0, \hat{t}]$ and some finite constant L. Invoking Proposition 3.4 we infer that $f(\hat{t}, \cdot)$ is bounded.

Case 2: it remains to consider the situation where the hypotheses of Case 1 are not satisfied. In this case, we can find sequences

$$\alpha_j \uparrow \alpha_c, \qquad t_j \uparrow \hat{t}, \qquad r_j \downarrow 0, \qquad \varepsilon_j \downarrow 0,$$

with $\underline{\alpha} \le \alpha_j$ and $r_j < r^*$ for all j, in such a way that

$$g_{\varepsilon_j}(t_j, r_j) \ge \tilde{g}^{(\alpha_j)}(r_j) \qquad \text{for all } j \in \mathbb{N}.$$

Thus, invoking Prop. 3.1, we infer

$$g_{\varepsilon_j}(t_j, r) \ge \tilde{g}^{(\alpha_j)}(r) - Cr^{2-d} \qquad \text{for all } r \in (r_j, r^*). \tag{3.13}$$

By construction, $\lim_{j\to\infty} g_{\varepsilon_j}(t_j, r) = g(\hat{t}, r)$ for every $r > 0$. Hence, sending $j \to \infty$ in ineq. (3.13) yields

$$g(\hat{t}, r) \ge \tilde{g}^{(\alpha_c)}(r) - Cr^{2-d} \qquad \text{for all } r \in (0, r^*),$$

which implies that

$$g(\hat{t}, r) \ge g_c(r) - Cr^{2-d} \qquad \text{for all } r \in (0, r^*).$$

In view of the upper bound in Prop. 3.2 this completes the proof of the main assertion in Theorem 1.2.

Let now $\alpha = \alpha_c$ in Prop. 3.1 and define $\tilde{r}_\varepsilon$ correspondingly. If $\hat{t}$ is such that $\mu_{\hat{t}}(\{0\}) > 0$, we must have $\lim_{\varepsilon\to 0} \tilde{r}_\varepsilon(\hat{t}) = 0$. Prop. 3.1 (combined with Prop. 3.2) thus implies the assertion concerning this case. □

4. Renormalised form

4.1. **Variational structure.** Our subsequent analysis relies on the following gradient-flow structure of the regularised Fokker–Planck equation (FP$_{\gamma,\mathrm{reg}}$). Such a structure has previously been used in [CHR20, Section 3.3] for the proof of an energy dissipation identity.

Define the approximate free energy functional by

$$\mathcal{H}_\varepsilon(f) = \int_{\mathbb{R}^d} \left(\frac{|v|^2}{2} f(v) + \Phi_\varepsilon(f) \right) \mathrm{d}v,$$

where $\Phi_\varepsilon \in C([0,\infty)) \cap C^\infty((0,\infty))$ satisfies

$$\Phi_\varepsilon(s) = \Phi(s) \text{ for } s \in [0, \varepsilon^{-1}] \tag{4.1}$$

and

$$\Phi_\varepsilon'' = \frac{1}{h_\varepsilon}, \quad \Phi_\varepsilon \geq \Phi. \tag{4.2}$$

The function Φ_ε with the above properties is obtained by setting $\Phi_\varepsilon(s) = \int_0^s \Phi_\varepsilon'(\sigma)\,\mathrm{d}\sigma$, where $\Phi_\varepsilon'(s)$ is given by

$$\Phi_\varepsilon'(s) = -\int_s^{B_\varepsilon} \frac{1}{h_\varepsilon(\sigma)}\,\mathrm{d}\sigma$$

with the constant $B_\varepsilon > \frac{1}{\varepsilon}$ being such that

$$\int_{\frac{1}{\varepsilon}}^{B_\varepsilon} \frac{1}{h_\varepsilon(\sigma)}\,\mathrm{d}\sigma = \int_{\frac{1}{\varepsilon}}^{\infty} \frac{1}{h(\sigma)}\,\mathrm{d}\sigma.$$

Identity (4.1) is a consequence of the fact that $h_\varepsilon(s) = h(s)$ in $[0, \varepsilon^{-1}]$, while the second property in (4.2) follows from the inequality $h_\varepsilon \leq h$.

Observe that $\delta H_\varepsilon(f) = \frac{1}{2}|v|^2 + \Phi_\varepsilon'(f)$, allowing us to rewrite eq. ($\mathrm{FP}_{\gamma.\mathrm{reg}}$) as

$$\partial_t f_\varepsilon = \operatorname{div}\big(h_\varepsilon(f_\varepsilon)\nabla\delta\mathcal{H}_\varepsilon(f_\varepsilon)\big).$$

Lemma 4.1 (Energy dissipation balance for ($\mathrm{FP}_{\gamma.\mathrm{reg}}$))**.** *The approximate solutions f_ε satisfy for all $0 \leq s \leq t < \infty$*

$$\mathcal{H}_\varepsilon(f_\varepsilon(t)) + \int_s^t \int_{\mathbb{R}^d} \frac{1}{h_\varepsilon(f_\varepsilon)} |\nabla f_\varepsilon + v h_\varepsilon(f_\varepsilon)|^2 \mathrm{d}v\,\mathrm{d}\tau = \mathcal{H}_\varepsilon(f_\varepsilon(s)). \tag{4.3}$$

Proof. Recall that $f_\varepsilon \in C^{1,2}((0,\infty)\times\mathbb{R}^d)$ is a classical solution of ($\mathrm{FP}_{\gamma.\mathrm{reg}}$). Hence, the only task in deriving equation (4.3) lies in appropriately controlling the tails as $|v| \to \infty$. This is a consequence of the moment control of the bounded function f_ε and follows from classical arguments, see e.g. [CLR09]. □

Lemma 4.2. *For any $t \geq 0$*

$$\liminf_{\varepsilon\to0} \mathcal{H}_\varepsilon(f_\varepsilon(t)) \geq \mathcal{H}(f(t)).$$

Proof. Since $\Phi_\varepsilon(s) \geq \Phi(s)$ (cf. (4.2)), we can estimate

$$\int \Phi_\varepsilon(f_\varepsilon(t,v))\,\mathrm{d}v \geq \int \Phi(f_\varepsilon(t,v))\,\mathrm{d}v.$$

Given $\delta > 0$, let $L = L(\delta) > 0$ be large enough such that $|\Phi(s)| \leq \delta s$ for $s \geq L$. Then

$$\begin{aligned}\int \Phi(f_\varepsilon(t,v))\,\mathrm{d}v &\geq \int \Phi(f_\varepsilon(t,v))\chi_{\{f_\varepsilon\leq L\}}\,\mathrm{d}v + \int \Phi(f_\varepsilon(t,v))\chi_{\{f_\varepsilon> L\}}\,\mathrm{d}v \\ &\geq \int \Phi(f_\varepsilon(t,v))\chi_{\{f_\varepsilon\leq L\}}\,\mathrm{d}v - \delta m,\end{aligned}$$

where $m = \int f_{\mathrm{in}}$. Sending first $\varepsilon \to 0$ (using dominated convergence) and then $L \to \infty$, we infer

$$\liminf_{\varepsilon\to0} \int \Phi_\varepsilon(f_\varepsilon(t,v))\,\mathrm{d}v \geq \int \Phi(f(t,v))\,\mathrm{d}v - \delta m$$

and hence

$$\liminf_{\varepsilon\to0} \int \Phi_\varepsilon(f_\varepsilon(t,v))\,\mathrm{d}v \geq \int \Phi(f(t,v))\,\mathrm{d}v.$$

For the kinetic part, we let $\mathcal{A}_{\rho,R} := \{\rho \le |v| \le R\}$ for $0 < \rho < R < \infty$. Then

$$\lim_{\varepsilon\to 0}\int_{\mathcal{A}_{\rho,R}} |v|^2 f_\varepsilon(t,v)\,\mathrm{d}v = \int_{\mathcal{A}_{\rho,R}} |v|^2 f(t,v)\,\mathrm{d}v.$$

Next, by mass conservation $\int_{B_\rho(0)} |v|^2(f+f_\varepsilon)\,\mathrm{d}v \le 2\rho^2 m$, while the uniform n^{th}-moment bound for some $n > 2$ (cf. L. 2.3) is used to handle the tails

$$\int_{\mathbb{R}^d\setminus B_R(0)} |v|^2(f+f_\varepsilon)\,\mathrm{d}v = O(R^{-(n-2)}).$$

We hence conclude that $\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} |v|^2 f_\varepsilon(t,v)\,\mathrm{d}v = \int_{\mathbb{R}^d} |v|^2 f(t,v)\,\mathrm{d}v$. □

4.2. **The limiting measure is a renormalised solution.** The proof of the renormalised formulation (1.7) relies on the following main ingredients: an energy estimate (true for general anisotropic solutions) and the uniform bound (2.16) for isotropic solution, including its implications on the singularity profile.

Proof of Theorem 1.4. It suffices to verify eq. (1.7) as the remaining part of the assertion has been shown in Prop. 1.1.

Recall that $\mathcal{T}_k(f) = \min\{f,k\}$ for $k > 0$. We start by showing that $\mathcal{T}_k(f)$ has a distributional derivative $\nabla\mathcal{T}_k(f) \in L^2_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^d)$. For this purpose, we choose $s = 0$ and $t = T$ in estimate (4.3) and, letting $\epsilon_* > 0$ be small enough so that, by L. 4.2, $-\mathcal{H}_\varepsilon(f_\varepsilon(T)) \le -\mathcal{H}(f(T)) + 1$ for all $\varepsilon \in (0,\epsilon_*]$, we infer the ε-uniform bound

$$\int_0^T\int_{\mathbb{R}^d} \frac{1}{h_\varepsilon(f_\varepsilon)}|\nabla f_\varepsilon + v h_\varepsilon(f_\varepsilon)|^2 \mathrm{d}v\,\mathrm{d}\tau \le \mathcal{H}(f_{\mathrm{in}}) - \mathcal{H}(f(T)) + 1. \tag{4.4}$$

To deduce a bound on $\nabla\mathcal{T}_k(f_\varepsilon)$, we note that

$$|\nabla\mathcal{T}_k(f_\varepsilon)|^2 \le 2|\mathcal{T}_k'(f_\varepsilon)[\nabla f_\varepsilon + v h_\varepsilon(f_\varepsilon)]|^2 + 2|\mathcal{T}_k'(f_\varepsilon) v h_\varepsilon(f_\varepsilon)|^2.$$

Hence, using the fact that $|\mathcal{T}_k'| \le 1$ and $\mathcal{T}_k'(s) = 0$ for $s > k$, we deduce from (4.4) for any $R \in (0,\infty)$

$$\int_0^T\int_{\{|v|\le R\}} |\nabla\mathcal{T}_k(f_\varepsilon)|^2\,\mathrm{d}v\mathrm{d}t \le C(k)(\mathcal{H}(f_{\mathrm{in}}) - \mathcal{H}(f(T)) + 1) + C(k,R)T. \tag{4.5}$$

Thanks to the convergence (2.22),

$$\begin{aligned} &\mathcal{T}_k(f_\varepsilon) \to \mathcal{T}_k(f) \quad \text{a.e. in } [0,\infty)\times\mathbb{R}^d,\\ &\mathcal{T}_k(f_\varepsilon) \to \mathcal{T}_k(f) \quad \text{in } L^p_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^d), \quad \text{for all } p\in[1,\infty),\\ \text{and thus, by (4.5),}\quad &\nabla\mathcal{T}_k(f_\varepsilon) \rightharpoonup \nabla\mathcal{T}_k(f) \quad \text{in } L^2_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^d). \end{aligned}$$

As a consequence,

$$\int_0^T\int_{\{|v|\le R\}} |\nabla\mathcal{T}_k(f)|^2\,\mathrm{d}v\mathrm{d}t \le C(k)(\mathcal{H}(f_{\mathrm{in}}) - \mathcal{H}(f(T)) + 1) + C(k,R)T. \tag{4.6}$$

Let now $\xi \in C^\infty([0,\infty))$ have a compactly supported derivative ξ', let $T < \infty$ and let $\psi \in C_c^\infty([0,T]\times\mathbb{R}^d)$. Further let $\varphi \in C^\infty([0,\infty);[0,1])$ satisfy $\varphi(r) = 0$ for $r\in[0,1]$ and $\varphi(r) = 1$

for $r \geq 2$, and abbreviate $\varphi_\rho(r) = \varphi(r/\rho)$. Then, since f is a classical solution of (FP_γ) in $(0,\infty) \times \left(\mathbb{R}^d \setminus \{0\}\right)$, a direct calculation gives

$$\begin{aligned}\int_{\mathbb{R}^d} \xi(f(T,\cdot))\psi(T,\cdot)\varphi_\rho(|v|)\,\mathrm{d}v - \int_{\mathbb{R}^d} \xi(f_{\mathrm{in}})\psi(0,\cdot)\varphi_\rho(|v|)\,\mathrm{d}v - \int_0^T\int_{\mathbb{R}^d} \xi(f)\partial_t\psi\varphi_\rho(|v|)\,\mathrm{d}v\mathrm{d}t \\ = -\int_0^T\int_{\mathbb{R}^d} (\nabla f + h(f)v)\cdot[\xi''(f)\nabla f\psi\varphi_\rho(|v|) + \xi'(f)\nabla\psi\varphi_\rho(|v|)]\,\mathrm{d}v\mathrm{d}t \\ - \int_0^T\int_{\mathbb{R}^d} (\nabla f + h(f)v)\cdot[\xi'(f)\psi\varphi_\rho'(|v|)\cdot \tfrac{v}{|v|}]\,\mathrm{d}v\mathrm{d}t.\end{aligned} \tag{4.7}$$

By the dominated convergence theorem and since $\varphi_\rho(r) \overset{\rho\downarrow 0}{\to} 1$ for all $r > 0$, the LHS of (4.7) converges, as $\rho \to 0$, to

$$\int_{\mathbb{R}^d} \xi(f(T,\cdot))\psi(T,\cdot)\,\mathrm{d}v - \int_{\mathbb{R}^d} \xi(f_{\mathrm{in}})\psi(0,\cdot)\,\mathrm{d}v - \int_0^T\int_{\mathbb{R}^d} \xi(f)\partial_t\psi\,\mathrm{d}v\mathrm{d}t.$$

Likewise, thanks to the bound (4.6) and the compact support of ξ'', ξ' and of ψ, the dominated convergence theorem allows to pass to the limit in the first integral on the RHS of (4.7) giving the term

$$-\int_0^T\int_{\mathbb{R}^d} (\nabla f + h(f)v)\cdot[\xi''(f)\nabla f\psi + \xi'(f)\nabla\psi]\,\mathrm{d}v\mathrm{d}t.$$

We are left to show that the last integral in (4.7) vanishes in the limit $\rho \downarrow 0$. First, since $|h(f)\xi'(f)| \leq C(\operatorname{supp}\xi') < \infty$ and $\varphi_\rho'(|v|) = 0$ for $|v| \geq 2\rho$ as well as $|v\varphi_\rho'(|v|)| = |\rho^{-1}v\varphi'(|\rho^{-1}v|)| \lesssim 1$, the dominated convergence theorem yields

$$\begin{aligned}\Big|\int_0^T\int_{\mathbb{R}^d} h(f)v\cdot[\xi'(f)\psi\varphi_\rho'(|v|)\cdot\tfrac{v}{|v|}]\,\mathrm{d}v\mathrm{d}t\Big| \\ \leq \int_0^T\int_{\mathbb{R}^d} |h(f)\xi'(f)||\psi||v\varphi_\rho'(|v|)|\,\mathrm{d}v\mathrm{d}t \;\longrightarrow\; 0 \text{ as } \rho \to 0.\end{aligned}$$

The remaining part of the integral is more delicate. We estimate using the radial symmetry of $f(t,v)$ $(=: g(t,|v|))$

$$\begin{aligned}\Big|\int_0^T\int_{\mathbb{R}^d} \nabla f\cdot[\xi'(f)\psi\varphi_\rho'(|v|)\cdot\tfrac{v}{|v|}]\,\mathrm{d}v\mathrm{d}t\Big| \\ \leq C\int_0^T\int_0^{2\rho} |\xi'(g)\partial_r g|\rho^{-1}|\varphi'(\rho^{-1}r)|r^{d-1}\mathrm{d}r\mathrm{d}t \\ = C\int_0^T A(t,\rho)\,\mathrm{d}t,\end{aligned}$$

where we abbreviated

$$A(t,\rho) = \int_0^{2\rho} |\xi'(g)\partial_r g|\rho^{-1}|\varphi'(\rho^{-1}r)|r^{d-1}\mathrm{d}r.$$

As a consequence of the bound (2.16), we have $|\xi'(g)r^{d-1}\partial_r g| \leq CK_* + C(\operatorname{supp}\xi')r^d$. We hence infer the following (t,ρ)-uniform bound on $|A(t,\rho)|$:

$$|A(t,\rho)| \leq C\int_0^{2\rho} \rho^{-1}|\varphi'(\rho^{-1}r)|\,\mathrm{d}r = C\int_0^2 |\varphi'(\hat r)|\,\mathrm{d}\hat r.$$

Thus, to show that $\lim_{\rho\to 0}\int_0^T A(t,\rho)\,\mathrm{d}t = 0$ it suffices to prove the pointwise convergence $\lim_{\rho\to 0} A(t,\rho) = 0$ for (almost) all $t \in (0,T]$.

Thanks to Theorem 1.2, only the following two cases may occur.

Case 1: $g(t,0+) = +\infty$. In this case, there exists $r^* > 0$ such that $\xi'(g(t,r)) = 0$ for all $r \in (0,r^*)$. Hence, we trivially have $\lim_{\rho\to 0} A(t,\rho) = 0$.

Case 2: $g(t,\cdot) \in L^\infty$. In this case, there exists a neighbourhood J of t such that the approximate sequence $\{f_\varepsilon\}$ is uniformly bounded on $J \times \mathbb{R}^d$. Hence, arguing as in the first step of the proof of Proposition 1.1 (see page 14), the limiting density f must be smooth in a neighbourhood of $\{t\} \times \mathbb{R}^d$. If $d > 1$, the conclusion $\lim_{\rho\to 0} A(t,\rho) = 0$ then directly follows from the definition of $A(t,\rho)$, while for $d = 1$ we resort to the fact that $\sup_{r\in(0,\rho)} |\partial_r g(t,r)| \to 0$ as $\rho \to 0$. □

4.3. **Energy dissipation identity.** An argument similar to that in the proof of Theorem 1.4 shows that isotropic solutions satisfy the energy dissipation balance. In the anisotropic case, we obtain an inequality.

Proof of Proposition 1.5. Combining L. 4.1 and L. 4.2 with the convergence properties of $\mu^{(\varepsilon)}$ to μ in Proposition 1.1, we readily infer for all $t > 0$ the inequality

$$\mathcal{H}(f(t)) + \int_0^t D(f(\tau))\,\mathrm{d}\tau \le \mathcal{H}(f_{\rm in}),$$

where

$$\mathcal{D}(f) := \int_{\mathbb{R}^d} \tfrac{1}{h(f)} |\nabla f + h(f)v|^2\,\mathrm{d}v.$$

It remains to prove that in the isotropic case the above inequality holds with an equality. Then, the asserted identity (1.8) follows by subtracting on both sides the quantity $\mathcal{H}(f(s))$, which is then known to equal $\mathcal{H}(f_{\rm in}) - \int_0^s D(f(\tau))\,\mathrm{d}\tau$. Thus, in the remainder, we assume that $f_{\rm in}$ is isotropic. As in the proof of Theorem 1.4 (cf. Section 4.2), we pick a nondecreasing function $\varphi \in C^\infty([0,\infty);[0,1])$ satisfying $\varphi(r) = 0$ for $r \in [0,1]$ and $\varphi(r) = 1$ for $r \ge 2$, and abbreviate $\varphi_\rho(r) = \varphi(r/\rho)$. Then, defining

$$\mathcal{H}^{(\rho)}(f) := \int_{\mathbb{R}^d} [\,\tfrac12 |v|^2 f + \Phi(f)\,]\varphi_\rho(|v|)\,\mathrm{d}v,$$

one has

$$\begin{aligned}\mathcal{H}^{(\rho)}(f(t)) - \mathcal{H}^{(\rho)}(f_{\rm in}) &= \int_0^t \int_{\mathbb{R}^d} [\,\tfrac12|v|^2 + \Phi'(f)\,]\,\mathrm{div}(\nabla f + h(f)v)\varphi_\rho(|v|)\,\mathrm{d}v\mathrm{d}\tau \\ &= -\int_0^t\int_{\mathbb{R}^d} \frac{1}{h(f)}|\nabla f + h(f)v|^2 \varphi_\rho(|v|)\,\mathrm{d}v\mathrm{d}\tau \\ &\quad - \int_0^t\int_{\mathbb{R}^d} [\,\tfrac12|v|^2 + \Phi'(f)\,]\,(\nabla f + h(f)v)\cdot \tfrac{v}{|v|}\varphi_\rho'(|v|)\,\mathrm{d}v\mathrm{d}\tau.\end{aligned}$$

We note that

$$\lim_{\rho\to 0} \mathcal{H}^{(\rho)}(f(t)) = \mathcal{H}(f(t)) \text{ for all } t \ge 0.$$

Furthermore, monotone convergence gives

$$\lim_{\rho\to 0}\int_0^t\int_{\mathbb{R}^d} \frac{1}{h(f)}|\nabla f + h(f)v|^2\varphi_\rho(|v|)\,\mathrm{d}v\mathrm{d}\tau = \int_0^t\int_{\mathbb{R}^d}\frac{1}{h(f)}|\nabla f + h(f)v|^2\,\mathrm{d}v\mathrm{d}\tau.$$

Hence, it remains to prove that the quantity

$$B(\tau,\rho) := \int_{\mathbb{R}^d} [\,\tfrac12|v|^2 + \Phi'(f)\,]\,(\nabla f + h(f)v)\cdot\tfrac{v}{|v|}\varphi_\rho'(|v|)\,\mathrm{d}v$$

satisfies

$$\lim_{\rho\to 0}\int_0^t B(\tau,\rho)\,\mathrm{d}\tau = 0. \tag{4.8}$$

Using the isotropy of $f(\tau,\cdot)$ we write

$$B(\tau,\rho) = c_d \int_0^{2\rho} \left[\, \tfrac{1}{2}r^2 + \Phi'(g) \,\right] (\partial_r g + h(g) r)\varphi_\rho'(r)\, r^{d-1}\mathrm{d}r,$$

where c_d denotes the area of the unit sphere. We can now argue similarly as in the proof of Theorem 1.4, see page 25. The function $|B(\tau,\rho)|$ is uniformly bounded, and hence identity (4.8) follows from the dominated convergence theorem provided we can prove the pointwise convergence $\lim_{\rho\to 0} B(\tau,\rho) = 0$ for a.e. $\tau > 0$.

Case 1: $g(\tau, 0+) = +\infty$. In this case, we recall the bound $|(\partial_r g + h(g) r)\, r^{d-1}| \le K_*$ and estimate

$$|B(\tau,\rho)| \le CK_* \int_0^{2\rho} |\varphi_\rho'(r)|\mathrm{d}r \cdot \sup_{r\in(0,\rho)} |\tfrac{1}{2}r^2 + \Phi'(g(\tau,r))|.$$

The integral $\int_0^{2\rho} |\varphi_\rho'(r)|\mathrm{d}r = \int_0^2 |\varphi'(\hat r)|\mathrm{d}\hat r$ is independent of ρ, while thanks to the sublinearity of Φ at infinity

$$\sup_{r\in(0,\rho)} |\tfrac{1}{2}r^2 + \Phi'(g(\tau,r))| \to 0 \quad \text{as } \rho \to 0$$

Hence, $\lim_{\rho\to 0} |B(\tau,\rho)| = 0$.

Case 2: $g(\tau,\cdot) \in L^\infty$. Here, the assertion $\lim_{\rho\to 0} |B(\tau,\rho)| = 0$ is obtained similarly as in Case 2 of the proof of Theorem 1.4, using parabolic regularity. □

5. Long-time behaviour

5.1. **Relaxation to equilibrium.**

Proof of Theorem 1.6. The proof takes advantage of the energy dissipation balance, the singularity profile and the compactness properties established above. Equipped with these observations, the general reasoning is quite standard (see e.g. [CnC*16]).

Let

$$\mathcal{D}(f) := \int_{\mathbb{R}^d} \frac{1}{h(f)} |\nabla f + h(f) v|^2 \,\mathrm{d}v = \int_{\mathbb{R}^d} h(f) |\nabla \delta\mathcal{H}(f)|^2 \,\mathrm{d}v.$$

Proposition 1.5 implies that $\int_0^\infty \mathcal{D}(f(t))\,\mathrm{d}t \le \mathcal{H}(f_{\mathrm{in}}) - \inf_{\mathcal{M}_+} \mathcal{H} < \infty$, and hence there exists a sequence $t_k \to \infty$ such that

$$\lim_{k\to\infty} \mathcal{D}(f(t_k)) = 0. \tag{5.1}$$

Since $\{\mu_{t_k}\}_k$ is bounded in $\mathcal{M}_+(\mathbb{R}^d)$ and has a bounded moment of order $n := \min\{\ell_1, \ell_2\} > 2$, there exists a subsequence (not relabelled) and a limiting measure $\mu_\infty \in \mathcal{M}_+(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \mathrm{d}\mu_\infty = m$ and $\int |v|^n \mathrm{d}\mu_\infty < \infty$ such that

$$\mu_{t_k} \overset{*}{\rightharpoonup} \mu_\infty \quad \text{in } \mathcal{M}(\mathbb{R}^d),$$
$$\int |v|^2 \mathrm{d}\mu_{t_k} \to \int |v|^2 \mathrm{d}\mu_\infty.$$

At the same time we know that $\mu_{t_k} = a(t_k)\delta_0 + f(t_k)\mathcal{L}^d$, and revoking parabolic regularity as in the proof of Prop. 1.1 and possibly passing to another subsequence, we infer

$$f(t_k) \to f_\infty \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\}).$$

Hence, $h(f(t_k))|\nabla\delta\mathcal{H}(f(t_k))|^2 \to h(f_\infty)|\nabla\delta\mathcal{H}(f_\infty)|^2$ locally uniformly in $\mathbb{R}^d \setminus \{0\}$ as $k \to \infty$. Since (5.1) means that $h(f(t_k))|\nabla\delta\mathcal{H}(f(t_k))|^2 \to 0$ in $L^1(\mathbb{R}^d)$ and hence a.e. along a subsequence, we deduce

$$h(f_\infty)|\nabla\delta\mathcal{H}(f_\infty)|^2 = 0. \tag{5.2}$$

The smoothness of the densities away from the origin allows us to apply a strong minimum principle giving that f_∞ is strictly positive. Hence, as a consequence of eq. (5.2), $f_\infty = f_{\infty,\theta}$ for some $\theta \geq 0$ (cf. (1.3) for the definition of $f_{\infty,\theta}$). If $\|f_{\infty,\theta}\|_{L^1} < m$, then $\mu_\infty(\{0\}) > 0$, and invoking Theorem 1.2 we conclude that $\theta = 0$. In any case, we infer that the measure μ_∞ coincides with the unique minimiser $\mu_{\min} = \mu^{(m)}_{\min}$ of mass m.

In view of the convergence properties established above, we infer $\lim_{k\to\infty} \mathcal{H}(\mu_{t_k}) = \mathcal{H}(\mu_{\min})$. Since $t \mapsto \mathcal{H}(\mu_t)$ is nonincreasing, this immediately yields

$$\lim_{t\to\infty} \mathcal{H}(\mu_t) = \mathcal{H}(\mu_{\min}).$$

Combining this result with the above compactness properties, mass conservation, and the uniqueness of the minimiser $\mu^{(m)}_{\min}$, one can easily deduce the remaining convergence properties along any sequence $t \to \infty$ as asserted in Theorem 1.6. □

5.2. **Long-time and transient properties.** For the reader less acquainted with this kind of problem, let us point out some implications of the above analysis on further qualitative dynamical properties, restricting for consistency to the isotropic case. If $m < m_c$, Theorem 1.6 implies the eventual regularity of μ_t after some sufficiently large time $T \gg 1$. However, using a contradiction argument, finite-time blow-up and the formation of a condensate (that is $\mu_t(\{0\}) > 0$ for some $t > 0$) can be shown to occur for any size of the mass $m > 0$ by choosing the smooth initial data sufficiently concentrated near the origin (cf. [CHR20, Tos12]). Hence, there exist flows exhibiting *transient condensates* with singular parts compactly supported in time. On the other hand, whenever $m > m_c$, the above theory implies the eventual formation of a condensate: $\exists T \gg 1$ such that $\mu_t(\{0\}) > 0$ for all $t \geq T$. This is a consequence of the convergence $\lim_{t\to\infty} \mu_t(\{0\}) = \mu_{\min}(\{0\})$. It is also possible to infer information on the spatio-temporal features of singularity formation and regularisation using rescaling methods. We refer to [Hop19, Chapter 5.2], where such dynamics have been shown to be of „type II“ for the 1D case.

5.3. **Concluding remark.** The comparison principle structure provides us with a priori bounds that allow for a detailed characterisation of the singularities which isotropic flows starting from regular data may exhibit (and even gives uniqueness in the 1D case [CHR20, Hop19] resp. convergence of the scheme to a unique limit in higher dimensions). However, one may not expect such a structure to persist in more complex situations. Particularly with regard to the study of uniqueness and stability properties in the presence of singularities, it would be interesting to see whether variational problems like (FP$_\gamma$) allow for more robust approaches.

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APPENDIX A.

Recall that $\mathcal{F}(t,v,w) = \mathrm{e}^{dt} G_{\nu(t)}(\mathrm{e}^t v - w)$, $\nu(t) = \mathrm{e}^{2t} - 1$, $G_\lambda(\xi) = (2\pi\lambda)^{-\frac{d}{2}} \mathrm{e}^{-\frac{|\xi|^2}{2\lambda}}$.

Lemma A.1. *Let $T < \infty$ and let $\tilde q \in [1,\infty]$. There exists $C_T = C_T(\tilde q, d) < \infty$ such that for all $t \in (0,T]$*

$$\begin{aligned}\left\| \int_{\mathbb{R}^d} |\nabla_v \mathcal{F}(t,v,w)||\mathrm{e}^{-t}w - v||f(w)|\,\mathrm{d}w \right\|_{L^{\tilde q}(\mathbb{R}^d)} &\le C_T \|f\|_{L^{\tilde q}(\mathbb{R}^d)} \\ &\le C_T \nu(t)^{-\frac12} \|f\|_{L^{\tilde q}(\mathbb{R}^d)}.\end{aligned} \tag{A.1}$$

Proof. The second bound in (A.1) is trivial.

To verify the first inequality, we compute for $t \in (0,T]$

$$\begin{aligned}&\int_{\mathbb{R}^d} |\nabla_v \mathcal{F}(t,v,w)||\mathrm{e}^{-t}w - v||f(w)|\,\mathrm{d}w \\ &\quad= (2\pi\nu(t))^{-\frac{d}{2}} \mathrm{e}^{dt} \mathrm{e}^t (2\nu(t))^{-\frac12} \int_{\mathbb{R}^d} 2\frac{|\mathrm{e}^t v - w|}{\sqrt{2\nu(t)}} \mathrm{e}^{-\frac{|\mathrm{e}^t v - w|^2}{2\nu(t)}} |\mathrm{e}^{-t}w - v||f(w)|\,\mathrm{d}w \\ &\quad= (2\pi\nu(t))^{-\frac{d}{2}} \mathrm{e}^{2dt} \mathrm{e}^t (2\nu(t))^{-\frac12} \int_{\mathbb{R}^d} 2\frac{|\mathrm{e}^t(v-\tilde w)|}{\sqrt{2\nu(t)}} \mathrm{e}^{-\frac{|\mathrm{e}^t(v-\tilde w)|^2}{2\nu(t)}} |\tilde w - v||f(\mathrm{e}^t\tilde w)|\,\mathrm{d}\tilde w \\ &\quad= (2\pi\nu(t))^{-\frac{d}{2}} \mathrm{e}^{2dt} \int_{\mathbb{R}^d} 2\frac{|\mathrm{e}^t(v-\tilde w)|^2}{2\nu(t)} \mathrm{e}^{-\frac{|\mathrm{e}^t(v-\tilde w)|^2}{2\nu(t)}} |f(\mathrm{e}^t\tilde w)|\,\mathrm{d}\tilde w.\end{aligned}$$

Now, the asserted inequality follows upon an application of Young's convolution inequality, $\|a*b\|_{L^{\tilde q}} \le \|a\|_{L^1}\|b\|_{L^{\tilde q}}$. ☐

REFERENCES

[BAGT11] N. Ben Abdallah, I. M. Gamba, and G. Toscani: *On the minimization problem of sub-linear convex functionals*. Kinet. Relat. Models **4**:4 (2011) 857–871.

[BB*95] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vázquez: *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **22**:2 (1995) 241–273.

[CHR20] J. A. Carrillo, K. Hopf, and J. L. Rodrigo: *On the singularity formation and relaxation to equilibrium in 1D Fokker-Planck model with superlinear drift*. Adv. Math. **360** (2020) 106883, 66.

[CHW20] J. A. Carrillo, K. Hopf, and M.-T. Wolfram: *Numerical study of Bose-Einstein condensation in the Kaniadakis-Quarati model for bosons*. Kinet. Relat. Models **13**:3 (2020) 507–529.

[CIL92] M. G. Crandall, H. Ishii, and P.-L. Lions: *User's guide to viscosity solutions of second order partial differential equations*. Bull. Amer. Math. Soc. (N.S.) **27**:1 (1992) 1–67.

[CL*10] J. A. Carrillo, S. Lisini, G. Savaré, and D. Slepčev: *Nonlinear mobility continuity equations and generalized displacement convexity*. J. Funct. Anal. **258**:4 (2010) 1273–1309.

[CLR09] J. A. Carrillo, P. Laurençot, and J. Rosado: *Fermi-Dirac-Fokker-Planck equation: well-posedness & long-time asymptotics*. J. Differential Equations **247**:8 (2009) 2209–2234.

[CnC*16] J. A. Cañizo, J. A. Carrillo, P. Laurençot, and J. Rosado: *The Fokker-Planck equation for bosons in 2D: well-posedness and asymptotic behavior*. Nonlinear Anal. **137** (2016) 291–305.

[CRS08] J. A. Carrillo, J. Rosado, and F. Salvarani: *1D nonlinear Fokker-Planck equations for fermions and bosons*. Appl. Math. Lett. **21**:2 (2008) 148–154.

[CRW16] J. A. Carrillo, H. Ranetbauer, and M.-T. Wolfram: *Numerical simulation of nonlinear continuity equations by evolving diffeomorphisms*. J. Comput. Phys. **327** (2016) 186–202.

[DeT84] F. Demengel and R. Temam: *Convex functions of a measure and applications*. Indiana Univ. Math. J. **33**:5 (1984) 673–709.

[DMM*99] G. Dal Maso, F. Murat, L. Orsina, and A. Prignet: *Renormalized solutions of elliptic equations with general measure data*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28**:4 (1999) 741–808.

[DNS09] J. Dolbeault, B. Nazaret, and G. Savaré: *A new class of transport distances between measures*. Calc. Var. Partial Differential Equations **34**:2 (2009) 193–231.

[ESG05] L. C. Evans, O. Savin, and W. Gangbo: *Diffeomorphisms and nonlinear heat flows*. SIAM J. Math. Anal. **37**:3 (2005) 737–751.

[Fra05] T. D. Frank, *Nonlinear Fokker-Planck equations*, Springer Series in Synergetics, Springer-Verlag, Berlin, 2005, Fundamentals and applications.

[Hop19] K. Hopf, *On the singularity formation and long-time asymptotics in a class of nonlinear Fokker–Planck equations*, Ph.D. thesis, University of Warwick, 2019, http://wrap.warwick.ac.uk/151487/.

[KaQ93] G. Kaniadakis and P. Quarati: *Kinetic equation for classical particles obeying an exclusion principle*. Phys. Rev. E **48**:6 (1993) 4263.

[KaQ94] G. Kaniadakis and P. Quarati: *Classical model of bosons and fermions*. Phys. Rev. E **49** (1994) 5103–5110.

[Lie96] G. M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[LSU68] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Uralceva, *Linear and quasilinear equations of parabolic type*, Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1968.

[LSV12] S. Luckhaus, Y. Sugiyama, and J. J. L. Velázquez: *Measure valued solutions of the 2D Keller-Segel system*. Arch. Ration. Mech. Anal. **206**:1 (2012) 31–80.

[QuS19] P. Quittner and P. Souplet, *Superlinear parabolic problems*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser/Springer, Cham, 2019.

[SSC06] J. Sopik, C. Sire, and P.-H. Chavanis: *Dynamics of the Bose–Einstein condensation: analogy with the collapse dynamics of a classical self-gravitating Brownian gas*. Phys. Rev. E (3) **74**:1 (2006) 011112, 15.

[Tos12] G. Toscani: *Finite time blow up in Kaniadakis-Quarati model of Bose-Einstein particles*. Comm. Partial Differential Equations **37**:1 (2012) 77–87.

[Vel04] J. J. L. Velázquez: *Point dynamics in a singular limit of the Keller-Segel model. I. Motion of the concentration regions*. SIAM J. Appl. Math. **64**:4 (2004) 1198–1223.