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REVIEW OF THE METHODS OF REFLECTIONS*

G. CIARAMELLA[†], M. J. GANDER[‡], L. HALPERN[§], AND J. SALOMON[¶]

Abstract. The methods of reflections were invented to obtain approximate solutions of the motion of more than one particle in a given environment, provided that one can represent the solution for one particle rather easily. This motivation is quite similar to the motivation of the Schwarz domain decomposition method, which was invented to prove existence and uniqueness of solutions of the Laplace equation on complicated domains, which are composed of simpler ones, for which existence and uniqueness of solutions was known. Like for Schwarz methods, there is also an alternating and a parallel method of reflections, but interestingly, the parallel method is not always convergent. We carefully trace in this paper the historical development of these methods of reflections, give several precise mathematical formulations, an equivalence result with the alternating Schwarz method for two particles, and also an analysis for a one dimensional model problem with three particles of the alternating, parallel, and a recent averaged parallel method of reflections.

Key words. Alternating method of reflections; Parallel method of reflections; Averaged parallel method of reflections; alternating Schwarz method; stationary iterative methods, Laplace's equation.

AMS subject classifications. 65N55, 65F10, 35J05, 35J57.

1. Historical Introduction.

The idea underlying the method of reflections is very old; it can be found already in the book of Murphy [21, page 93] from 1833 under the name "principle of successive influences". Murphy uses the principle as an iterative method to obtain more and more accurate approximations for the interaction between two objects, and he says that the principle can be used for two purposes:

> "**First.** To obtain numerical approximations to the state of electrisised bodies influencing each other, by calculating the effects of 4 or 5 successive acts of influence.
> **Second.** To obtain the analytical expression for that state; for the consideration of a few successive influences will show what the form of the quantity V is, and assuming a corresponding form with indeterminate coefficients, we may get the form for the state of B due to the influence of A, and then the state due to the influence of B or its own *reflected* influence, comparing the form thus obtained with that assumed, the indeterminate coefficients may be found."

We see that the term 'reflection' already appears here. Murphy then used this principle to compute the solution for the concrete example of two spheres in form of an infinite series expansion.

This same approach was also used by Lamb [16, page 122] for Laplace's equation in 1906:

> "The motion of a liquid bounded by *two* spherical surfaces can be found by successive approximations in certain cases".

He first solves the problem with one surface only, using the method of images to obtain a zero normal velocity at its surface. The velocity field so obtained does however not satisfy the zero normal velocity condition on the second surface, and he thus needs to compute a correction, again using the method of images, which will then however

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when added to the first approximation lead to a violation of the zero normal velocity condition on the first surface. Continuing this process of correction, Lamb notices:

> "The images continually diminish in intensity, and this very rapidly if the radius of either sphere is small compared with the shortest distance between the two surfaces."

A method which was later called "Spiegelungsmethode" (method of reflections), see [10, page 928], is attributed to the work of Lorentz, who describes such a method in [19, page 29]:

> "Der allgemeine Fall, dass in der Flüssigkeit eine Anzahl von Körpern mit gegebener Bewegung liegen, lässt sich wegen der linearen Gestalt der Gleichungen auf die Spezialfälle zurückführen, in welchen jedesmal nur ein Körper eine vorgeschriebene Bewegung hat, die übrigen aber in Ruhe erhalten werden [...] Indem wir in dieser Weise fortfahren und uns alle die Zustände 1, 2, 3 usw. superponiert denken, erhalten wir für die Komponenten der Geschwindigkeit unendliche Reihen, die, wenn sie konvergent sind, offenbar allen Bedingungen der Aufgabe genügen. In einfachen Fällen könnte man die Berechnung wirklich durchführen; z.B. kann man die Störung oder, wie man auch sagen kann, die 'Zurückwerfung" eines bekannten Bewegungszustandes durch eine ruhende Kugel immer bestimmen."

Here, 'Zurückwerfung' can be understood as 'reflection', and we thus also find the term in the name of the method.

The method of reflections was then presented in concrete mathematical notation by Smoluchowski in 1911 with the goal to understand how the motion of a sphere in a viscous fluid is influenced by the presence or motion of one or several other spheres [23]:

> "Die nachfolgende Untersuchung bezweckt die Beantwortung der Frage, inwieweit die Bewegung einer in einem zähen Medium befindlichen Kugel durch die Anwesenheit (oder Bewegung) einer oder mehrerer anderer Kugeln modifiziert wird."

Smoluchowski used the Stokes equations for the fluid, and started first investigating the case of two spheres, where he states:

> "Um die Wechselwirkung zweier Kugeln zu studieren, kann man eine Annäherungsmethode anwenden, welche auf sukzessiver Superponierung partikulärer Lösungen beruht, analog den Spiegelungsmethoden zur Lösung verschiedener Probleme der mathematischen Physik."

In the case of the Stokes equation, the analytical solution for one sphere is already a series expansion, and thus Smoluchowski assumes that the radii of the two spheres are small compared to their distance, and then uses the method of reflections to compute a series expansion of the coupled solution up to some order in the inverse distance of the spheres, and says:

> "Wollte man in der Entwicklung auch noch höhere Glieder berücksichtigen, so müsste man natürlich eine entsprechend grössere Anzahl von 'vielfachen Reflexionen' in Rechnung ziehen."

In the case of Smoluchowski, the method is thus not considered as an iterative method where an infinite number of iterations are needed to converge, but a direct method to obtain a series solution in the inverse of the distance, up to some order. Smoluchowski then generalizes the method of reflections to the case of more than two spheres, where he explains:

> "Beschränkt man sich auf die Glieder derselben Grössenordnung wie

> vorhin, so ist in jedem dieser Bewegungszustände nicht nur der direkte [...] Einfluss der bewegten Kugel auf die betrachtete Kugel zu berücksichtigen, sondern auch die einmaligen 'Relfexionen' desselben an den übrigen ruhenden Kugeln."

So in the case of more than two spheres, the method also seems to be applicable, leading to similar series approximations in the inverse of the distance like in the case of two spheres.

In 1934, Golusin introduced a parallel method of reflections for Laplace's equation for J objects [8, 7], and derived a condition for its convergence. He then says in [7, page 280]

> "For $J = 2$ this condition is always satisfied. For $J > 2$ this is however not always the case. It would be possible, changing the equation as this was done in the previous work, to increase the area of applicability of the preceding results; but we confine ourselves here to the above simplest case."

However in the previous paper [8] in the same volume, Golusin just says:

> "For $J = 2$ for a doubly connected region, this condition is always satisfied. For $J > 2$ however this is not always the case. For a 5-connected region, it is probably satisfied if the range of C is separated from the rest by more than its radius."

So we see that the parallel method of reflections in the case of more than two objects seems to converge only under certain additional conditions, which Golusin conjectures to depend on the distance between objects and their radius, an issue we will more closely investigate in Section 3.

In 1942, Burgers [3, 4] investigated the influence of the concentration of spherical particles on the sedimentation velocity for the Stokes equation, mentioning the work of Smoluchowski, but without describing precisely an algorithm, and using to a large extend physical intuition.

The idea of simply summing two solutions corresponding to two particles alone can be found in the work of Kinch [15, page 197] from 1959, under the assumption that the distance between their centers is large, again for the Stokes equation. This could be interpreted as a parallel method of reflections, where the separate contributions are also summed, but again, no algorithm is given.

Happel and Brenner explain in 1983 a different parallel version of the method of reflections which alternates between one fixed object and the group of all the others treated in parallel, see [9]. The goal of their method is to increase the order of approximation of the expansion of the solution in a neighborhood of a given object. Therefore, if one wants a good approximation for all the objects, the method has to be applied (independently) for each particle.

A first convergence analysis for the alternating method of reflections was given by Luke in 1989 for the Stokes equation, see [20], and Luke states:

> "This paper considers a reflection method in the spirit of Schwarz's alternating procedure that reduces the calculation of a Stokes flow in a complex geometry of a suspension to a sequence of calculations of flows around single particles."

The analysis is using a variational characterization of the method based on projections, similar in style of one of the classical convergence analyses of the Schwarz method given by Lions in the first of his three seminal papers [18].

The entire Chapter 8 in the book from 1991 by Kim and Karrila [14] is dedicated to the parallel method of reflections for the Stokes problem, and the method is already

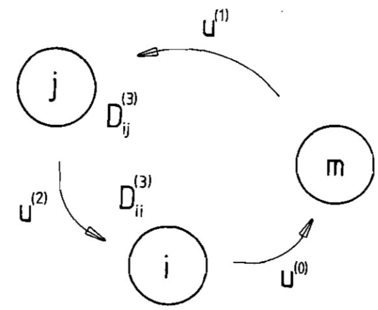

Figure 5.5:
Three body hydrodynamic interaction.

FIG. 1. *Figure from the book of Dhont [6] for the alternating method of reflections with three objects.*

mentioned in Chapter 7:

> "For widely separated particles, a general asymptotic method know as the method of reflections is available. The solution can be expressed analytically as a series in terms of particle size over separation."

The method is first motivated like in [15] by just summing two one-particle solutions:

> "In the zeroth order approximation, the solution for two widely separated particles is formed by superposition of the fields produced by the isolated particle solutions. In other words, we neglect hydrodynamic interactions between particles. [...] The method of reflections is based on the idea that the ambient field about each particle consists of the original ambient field plus the disturbance field produced by the other particle(s). The method is iterative, since a correction of the ambient field about a given particle generates a new disturbance solution for that particle, which in turn modifies the ambient field about another particle."

The authors also start first by explaining the parallel method of reflections for two objects, and then generalize it to the case of three objects. Just the first terms in the series expansion are obtained, and no convergence of the method is discussed.

A special section is also dedicated to the alternating method of reflections for the Stokes equation in the 1996 book of Dhont [6, Section 5.12]. The case of two objects is described on page 258, where Dhont says for the coupled boundary value problem of two objects:

> "This boundary value problem is too complicated to solve in closed analytical form. Instead the problem is solved by iteration [...] resulting in a series expansion representation of the flow field in powers of [...] the distance between the spheres.

The method is then also described for three objects on page 274, where Dhont goes cyclically through the three object in the algorithm, see Figure 1.

Balabane proved in 1997 convergence of the alternating method of reflections for the Helmholtz equation in unbounded domains in [2], and generalized his results to the parallel method of reflections in [1]. These convergence results are valid however only in low frequency regimes.

In 2001, Ichiki and Brady presented the parallel method of reflections [12] for

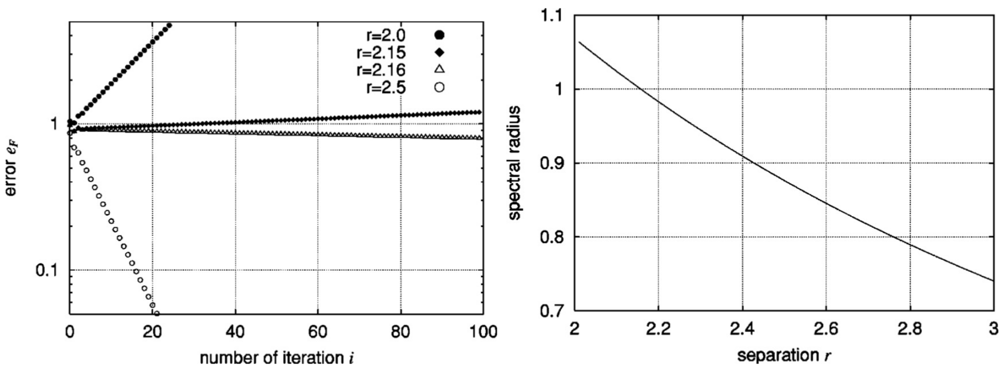

FIG. 2. *Figures from the paper by Ichiki and Brady [12] for the alternating method of reflections with three objects arranged on the vertices of an equilateral triangle. Left: error as a function of iteration for different distances between the object. Right: spectral radius of the iteration operator of the parallel method of reflections as a function of the distance of the objects.*

Stokes type problems. They start with the two particle case and then state:

> "It is easy to extend this procedure to the N-body problem by superposing disturbances by the other particles."

They are thus summing all contributions that were computed in parallel, and present this iterative approach also in matrix form, relating it to a stationary iteration based on a matrix splitting. They then show by numerical experiments that the method does not converge for three particles, if the separation distance of the particles is not large enough, and thus the parallel method of reflections might diverge in that case, see Figure 2. To alleviate this situation, they use their matrix formulation of the algorithm to consider the method at the fixed point, and then propose solving the fixed point equation by Krylov methods, which is equivalent to using the method of reflections as a preconditioner.

Traytak poses in 2006 in a short note directly the parallel method of reflections for N objects, written in PDE form for Laplace's equation [24, Section 2], and then uses a theorem proved by Golusin [7] to derive sufficient conditions for the convergence based on the distances between the objects.

More recently, Höfer and Velázquez used the parallel method of reflections as an analytic tool to prove homogenization results [11] (see also [13]), and they modified the usual parallel method by adding different weighting coefficients. They were interested in the theoretical case of an infinite number of objects, and thus an alternating method can not be considered. Laurent, Legendre and Salomon studied the alternating and parallel methods of reflections in [17], introducing also an averaged version of the parallel method. They proved convergence based on the alternating projection method in Hilbert spaces, see for example [22], and also using techniques like in [2, 1].

We have seen that there are two main variants of the method of reflections: the alternating one and the parallel one. There are also two different approaches to analyze the convergence of the method of reflections: first people worked on direct estimates performed on the single/double layer formulation of the boundary value problems involved in the iterations, see [7, 24, 2, 1]. There is however also the interpretation of the method as alternating projections in Hilbert spaces, see [20, 17]. In the case of orthogonal projections this interpretation leads to convergence estimates.

Our goal here is to formulate the different variants of the method of reflections as

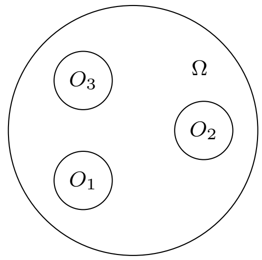

Fig. 3. *Example of a domain Ω (a disk) with three objects O_j (holes).*

iterative methods for the simple model involving the Laplacian. We start by presenting in Section 2 a class of Laplace problems on perforated domains. We then present the different forms of the methods of reflections: in Subsection 2.1 the alternating method of reflections; in Subsection 2.2 the parallel method of reflections; in Subsection 2.3 an averaged version of the parallel method introduced in [17]; and finally in Subsection 2.4 a variant introduced by Happel and Brenner in [9]. In Section 3, we give a convergence analysis of the alternating and parallel methods of reflections in the case of one space dimension and three objects.

2. Methods of reflections. Consider a simply connected domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and assume that J objects O_j (holes) are present in Ω. The objects $O_j \subset \Omega$ are simply connected open sets with smooth boundaries ∂O_j, and they are not allowed to overlap or touch,

$$\overline{O}_j \cap \overline{O}_k = \emptyset \text{ for any } j \neq k, \tag{1}$$

for an example, see Figure 3. They are also not allowed to touch the boundary, $\partial\Omega \cap \partial O_j = \emptyset$. Our goal is to compute an approximate solution u to the Laplace problem

$$\begin{aligned} \Delta u = 0 \text{ in } \Omega \setminus \cup_j \overline{O_j},\ u = 0 \text{ on } \partial\Omega, \\ u = g_j \text{ on } \partial O_j \text{ for } j = 1, \dots, J, \end{aligned} \tag{2}$$

where we assume that the g_j are sufficiently smooth functions on ∂O_j. We now present the various methods of reflections for the model problem (2). They all use as a common building block solutions of the problem with only one of the objects.

2.1. The alternating method of reflections. We formalize now the natural descriptions of the different methods of reflections from Section 1. The alternating method of reflections (AMR) for J objects starts with some u^0 that satisfies[1]

$$\Delta u^0 = 0 \text{ in } \Omega \setminus \cup_j \partial O_j,\ u^0 = 0 \text{ on } \partial\Omega,$$

but not necessarily the boundary conditions on the holes

$$u^0 = g_j \text{ on } \partial O_j \text{ for } j = 1, \dots, J.$$

[1]Historically the method starts with a u^0 solution for the first object, neglecting all the other objects. Such a solution is often available in closed form or as a series expansion for spherical objects. We want to formulate the methods here for arbitrary initial guesses, and the equivalent to the historical case would be obtained by starting with $u^0 = 0$, see also the comment before Algorithm 1.

The sequence of approximate solutions $\{u^k\}_k$ of the AMR is then defined for $k = 1, 2, \dots$ by

$$\begin{aligned} u^{k-1+\frac{1}{J}} &= u^{k-1} + d_1^k, \\ u^{k-1+\frac{2}{J}} &= u^{k-1} + d_1^k + d_2^k, \\ u^{k-1+\frac{3}{J}} &= u^{k-1} + d_1^k + d_2^k + d_3^k, \\ &\vdots \\ u^k &= u^{k-1} + \sum_{j=1}^{J} d_j^k, \end{aligned}$$

where the correction d_j^k for the j-th object is computed in such a way that, when added to $u^{k-1+\frac{j-1}{J}}$, the new approximation $u^{k-1+\frac{j}{J}}$ equals on the boundary ∂O_j of the j-th object the correct boundary value g_j specified in the underlying problem (2), that is

$$u^{k-1+\frac{j}{J}} = u^{k-1+\frac{j-1}{J}} + d_j^k = g_j \text{ on } \partial O_j.$$

This means that d_j^k must be the solution to

$$\begin{aligned} &\Delta d_j^k = 0 \text{ in } \Omega \setminus \partial O_j, \; d_j^k = 0 \text{ on } \partial\Omega, \\ &d_j^k = g_j - u^{k-1+\frac{j-1}{J}} = g_j - u^{k-1} - \sum_{\ell=1}^{j-1} d_\ell^k \text{ on } \partial O_j. \end{aligned} \tag{3}$$

Note that when adding the correction d_j^k to the current approximation $u^{k-1+\frac{j-1}{J}}$, d_j^k is defined in all of the domain Ω except the hole O_j, whereas $u^{k-1+\frac{j-1}{J}}$ is not defined in any of the holes O_l, $l = 1, \dots, J$, so the sum is to be understood only in the domain of definition of the overall approximation $u^{k-1+\frac{j-1}{J}}$.

Problem (3) can be written in another form, by rewriting the boundary condition of (3) on ∂O_j as follows (see for example [2, 1, 17]):

$$\begin{aligned} d_j^k &= g_j - u^{k-1} - \sum_{\ell=1}^{j-1} d_\ell^k = g_j - u^{k-2} - \sum_{\ell=1}^{J} d_\ell^{k-1} - \sum_{\ell=1}^{j-1} d_\ell^k \\ &= g_j - u^{k-2} - \sum_{\ell=1}^{j} d_\ell^{k-1} - \sum_{\ell=j+1}^{J} d_\ell^{k-1} - \sum_{\ell=1}^{j-1} d_\ell^k. \end{aligned}$$

Notice that $g_j - u^{k-2} - \sum_{\ell=1}^{j} d_\ell^{k-1} = 0$, since d_j^{k-1} solves problem (3) at the iteration $k-1$, that is

$$\begin{aligned} &\Delta d_j^{k-1} = 0 \text{ in } \Omega \setminus \partial O_j, \; d_j^{k-1} = 0 \text{ on } \partial\Omega, \\ &d_j^{k-1} = g_j - u^{k-2} - \sum_{\ell=1}^{j-1} d_\ell^{k-1} \text{ on } \partial O_j. \end{aligned}$$

Therefore, we have obtained that d_j^k can be expressed on ∂O_j only as combination of other differences d_ℓ^k and d_ℓ^{k-1},

$$d_j^k = -\sum_{\ell=j+1}^{J} d_\ell^{k-1} - \sum_{\ell=1}^{j-1} d_\ell^k, \tag{4}$$

and the explicit dependence on g_j and u^{k-1} disappeared. Hence, problem (3) becomes

$$\begin{aligned} &\Delta d_j^k = 0 \text{ in } \Omega \setminus \partial O_j,\ d_j^k = 0 \text{ on } \partial\Omega,\\ &d_j^k = -\sum_{\ell=1}^{j-1} d_\ell^k - \sum_{\ell=j+1}^{J} d_\ell^{k-1} \text{ on } \partial O_j, \end{aligned} \tag{5}$$

which is the form of the alternating method of reflections presented in [17], see also [2, 1]. Obviously, the sequences $\{d_j^k\}_k$ have to be initialized for all j. To do so, it is sufficient, for example, to set $u^0 = 0$ and to consider the initialization problems

$$\begin{aligned} &\Delta d_j^1 = 0 \text{ in } \Omega \setminus \partial O_j,\ d_j^1 = 0 \text{ on } \partial\Omega,\\ &d_j^1 = g_j - \sum_{\ell=1}^{j-1} d_\ell^1 \text{ on } \partial O_j. \end{aligned} \tag{6}$$

We show in Algorithm 1 how the AMR can be implemented using pseudo-code:

Algorithm 1 Alternating Method of Reflections (AMR)

Input: K (maximum number of iterations), *tol* (tolerance).

```
Set u^0 = 0 and k = 1;
for j = 1:J do
    Initialize d_j^1 solving the problem (6);
end for
Compute u^1 = \sum_{j=1}^J d_j^1;
while k < K and ||u^k - u^{k-1}|| > tol do
    Update k = k + 1;
    for j = 1:J do
        Compute d_j^k solving (5);
        Compute the approximation u^{k+1-j/J} = \sum_{n=1}^k \sum_{\ell=1}^j d_\ell^n;
    end for
end while
```

To illustrate the behavior of the AMR, we consider the one-dimensional problem

$$\begin{aligned} -\Delta u &= 0 \text{ in } (0,1) \setminus (\partial O_1 \cup \partial O_2),\\ u(0) &= u(1) = 0,\\ u(a_j) &= g_j^a, \text{ for } j = 1, 2,\\ u(b_j) &= g_j^b, \text{ for } j = 1, 2, \end{aligned}$$

where $O_j = (a_j, b_j)$, and g_j^a and g_j^b are given boundary data in $\mathbb{R}$, $j = 1, 2$. We assume that $0 < a_1 < b_1 < a_2 < b_2 < 1$, such that the two objects O_1 and O_2 are disjoint. In this special case of two objects, Algorithm 1 becomes Algorithm 2. To get an understanding how the AMR functions, we now discuss the first four iterations performed by Algorithm 2 in detail, and also provide their graphical representation.

Algorithm 2 Alternating Method of Reflections (AMR) for two objects

Input: K (maximum number of iterations), tol (tolerance);

Set $u^0 = 0$ and $k = 1$;

Initialize d_1^1 and d_2^1 solving sequentially the problems

$$(7)\qquad \begin{aligned} -\Delta d_1^1 &= 0 \text{ in } (0,1)\setminus \partial O_1,\\ d_1^1(0) &= d_1^1(1) = 0,\\ d_1^1(a_1) &= g_1^a,\\ d_1^1(b_1) &= g_1^b, \end{aligned} \quad \text{and} \quad \begin{aligned} -\Delta d_2^1 &= 0 \text{ in } (0,1)\setminus \partial O_2,\\ d_2^1(0) &= d_2^1(1) = 0,\\ d_2^1(a_2) &= g_2^a - d_1^1(a_2),\\ d_2^1(b_2) &= g_2^b - d_1^1(b_2). \end{aligned}$$

Compute the approximation $u^1 = d_1^1 + d_2^1$;

while $k < K$ and $\|u^k - u^{k-1}\| > tol$ **do**

  Update $k = k + 1$;

  Compute d_1^k and d_2^k solving (sequentially) the problems

$$(8)\qquad \begin{aligned} -\Delta d_1^k &= 0 \text{ in } (0,1)\setminus \partial O_1,\\ d_1^k(0) &= d_1^k(1) = 0,\\ d_1^k(a_1) &= -d_2^{k-1}(a_1),\\ d_1^k(b_1) &= -d_2^{k-1}(b_1), \end{aligned} \quad \text{and} \quad \begin{aligned} -\Delta d_2^k &= 0 \text{ in } (0,1)\setminus \partial O_2,\\ d_2^k(0) &= d_2^k(1) = 0,\\ d_2^k(a_2) &= -d_1^k(a_2),\\ d_2^k(b_2) &= -d_1^k(b_2). \end{aligned}$$

  Compute the approximation $u^k = u^{k-1} + d_1^k + d_2^k$;

end while

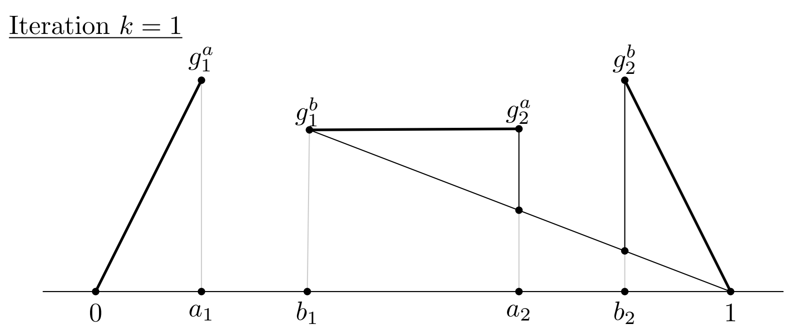

FIG. 4. *Iteration $k = 1$: The black thick line is the exact solution u. The first approximation u^1 is computed setting the exact boundary values for the object O_1. At the first iteration, u^1 coincides with the difference d_1^1 that is represented by the thin black line. The two black vertical segments correspond to the errors $e^1(a_2)$ and $e^1(b_2)$, which are used in iteration $k = 2$ to compute d_2^2.*

The first approximation u^1 is computed setting the exact boundary values on ∂O_1 and solving (7) (recall that $u^1 = d_1^1$ because $u^0 = 0$). The solution u is represented by the black thick line in Figure 4. The approximation u^1 (thin black line) allows us to compute the errors at a_2 and b_2, which are $e^1(a_2) = g_2^a - u^1(a_2)$ and $e^1(b_2) = g_2^b - u^1(b_2)$. These errors are used to compute d_2^2 (solving (7) (right)) represented by the dashed line in Figure 5. We can then compute the approximation $u^2 = u^1 + d_2^2$ (thin line in Figure 5), which is exact on $\partial O_2 = \{a_2, b_2\}$. Once again, we compute the error on ∂O_1: $e^2(a_1) = g_1^a - u^2(a_1)$ and $e^2(b_1) = g_1^b - u^2(b_1)$ that we use to solve (8) to obtain d_1^3 represented by the dashed line in Figure 6. The approximation

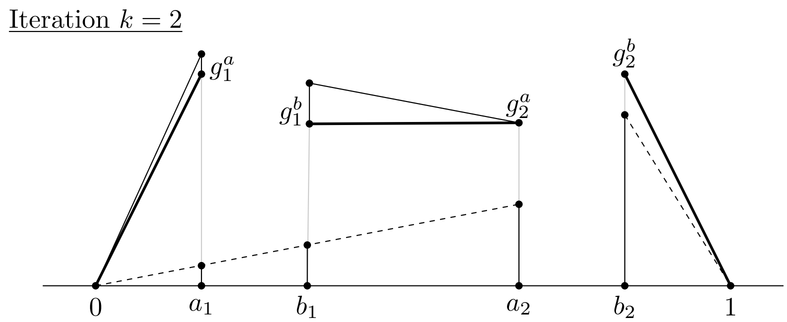

Fig. 5. *Iteration $k = 2$: The thick solid line is the exact solution u. The difference d_2^2 (black dashed line) is computed using the errors $e^1(a_2)$ and $e^1(b_2)$ as boundary condition on ∂O_2. The current approximation u^2 (black thin line) is obtained by summing d_2^2 to u^1. Notice that $u^2 = u^1 + d_2^2$ is exact on ∂O_2. The two vertical segments correspond to the errors $e^2(a_1)$ and $e^2(b_1)$, which are used in Step 3 to compute d_1^3.*

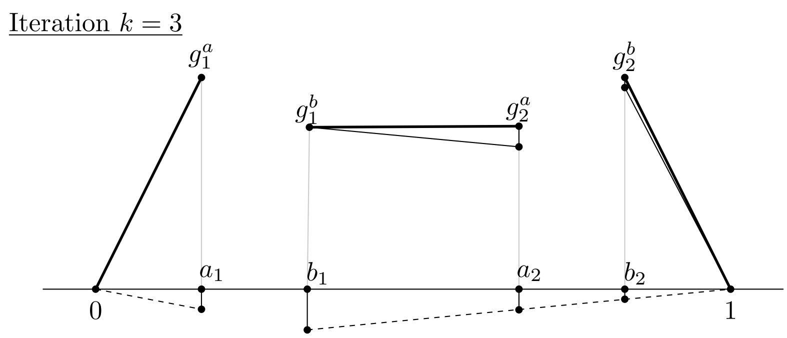

Fig. 6. *Iteration $k = 3$. The black thick line is the exact solution u. The difference d_3^1 (black dashed line) is computed using the errors $e^2(a_1)$ and $e^2(b_1)$ as boundary condition on ∂O_1. The current approximation u^3 (thin black line) is obtained by summing d_1^3 to u^2. Notice that $u^3 = u^2 + d_1^3$ is exact on ∂O_1. The two vertical segments correspond to the errors $e^3(a_2)$ and $e^3(b_2)$, which will be used in iteration $k = 4$ to compute d_2^4.*

u^3 is then obtained as $u^3 = u^2 + d_1^3$ (thin line in Figure 6). Similar arguments are used to obtain d_2^4 and $u^4 = u^3 + d_2^4$ (thin line in Figure 7). The iterative process continues in this way until convergence is reached. In order to show the contracting behavior of the AMR, we show the absolute value of the differences in Figure 8: $|d_2^2|$ (coarse dashed line), $|d_1^3|$ (fine dashed line), and $|d_2^4|$ (solid line). The figure shows that the maximum of the absolute value of the differences d_j^k decreases in the iterations, showing the contracting behavior of the AMR. Figure 8 also shows a close relation of the contraction of the absolute values of the differences $|d_j^k|$ with the convergence of an overlapping Schwarz method: if the overlapping Schwarz method was using as subdomains $\Omega_1 := (0, a_2)$ and $\Omega_1 := (b_1, 1)$ and was used to solve simply a Poisson equation on the interval $(0, 1)$,

$$\partial_{xx} v = f, \quad v(0) = v(1) = 0, \tag{9}$$

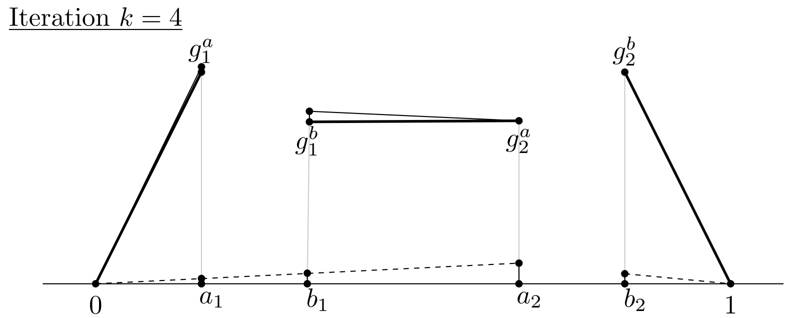

FIG. 7. *Iteration $k=4$. The black thick line is the exact solution u. The difference d_4^2 (black dashed line) is computed using the errors $e^3(a_2)$ and $e^3(b_2)$ as boundary condition on ∂O_2. The current approximation u^4 (thin black line) is obtained by summing d_2^4 to u^3. Notice that $u^4=u^3+d_2^4$ is exact on ∂O_2. The errors $e^4(a_1)$ and $e^4(b_1)$ will be used in iteration $k=5$ to compute d_1^5.*

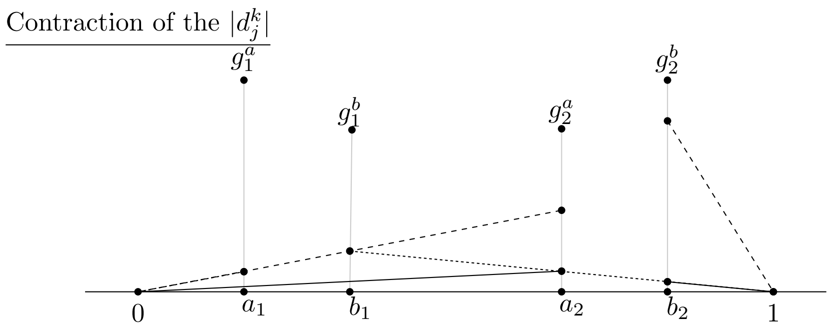

FIG. 8. *Absolute value of the differences: $|d_2^2|$ (coarse dashed line), $|d_1^3|$ (fine dashed line), and $|d_2^4|$ (solid line). It is clear that the maximum of the absolute value of the differences d_j^k decreases in the iterations, showing the contracting behavior of the AMR.*

the Schwarz iteration would be

$$\begin{array}{rclrrcll} \partial_{xx}v_1^k &=& f & \text{in } \Omega_1, & \partial_{xx}v_2^k &=& f & \text{in } \Omega_2, \\ v_1^k(a_2) &=& v_2^{k-1}(a_2), & & v_2^k(b_1) &=& v_1^k(b_1), & \\ v_1^k(0) &=& 0, & & v_2^k(1) &=& 0. & \end{array} \tag{10}$$

The equations for the errors $e_j^k := v - v_j^k$ of this alternating Schwarz method are by linearity just the homogeneous counterpart of (10),

$$\begin{array}{rclrrcll} \partial_{xx}e_1^k &=& 0 & \text{in } \Omega_1, & \partial_{xx}e_2^k &=& 0 & \text{in } \Omega_2, \\ e_1^k(a_2) &=& e_2^{k-1}(a_2), & & e_2^k(b_1) &=& e_1^k(b_1), & \\ e_1^k(0) &=& 0, & & e_2^k(1) &=& 0, & \end{array}$$

and we see that the errors of the alternating Schwarz method, which are just affine functions, coincide with the modulus of the differences of the alternating method of reflections, $|d_j^k| = e_{3-j}^k$, provided the second error satisfies $e_1^2(a_2) = |d_2^2(a_2)|$. So for the two-object case, the AMR converges at the same rate as the alternating Schwarz method. Note that the overlapping subdomains Ω_1 and Ω_2 could also be replaced with $\tilde{\Omega}_1 := \Omega\backslash O_2$ and $\tilde{\Omega}_2 := \Omega\backslash O_1$ without changing the contraction, so in the two object case, the AMR converges like an alternating overlapping Schwarz method with domains representing the entire domain, just with the corresponding object removed, a result which also holds in higher dimensions for two objects, for more details see [5].

2.2. The parallel method of reflections. The parallel method of reflections is obtained by replacing on the right-hand side of the boundary condition (4) the differences at the iteration k with the corresponding differences at the iteration $k-1$. Hence, problem (5) becomes

$$\begin{aligned} &\Delta d_j^k = 0 \text{ in } \Omega \setminus \partial O_j,\ d_j^k = 0 \text{ on } \partial\Omega, \\ &d_j^k = -\sum_{\ell=1,\ell\neq j}^{J} d_\ell^{k-1} \text{ on } \partial O_j. \end{aligned} \tag{11}$$

The sequences $\{d_j^k\}_k$ are initialized by solving for each $j = 1, \dots, J$ the problem

$$\Delta d_j^1 = 0 \text{ in } \Omega \setminus \partial O_j,\ d_j^1 = 0 \text{ on } \partial\Omega,\ d_j^1 = g_j \text{ on } \partial O_j, \tag{12}$$

and the approximate solution at the kth iteration is defined by

$$u^k = u^{k-1} + \sum_{j=1}^{J} d_j^k,$$

where $u^0 = 0$. The PMR (11)-(12) leads to Algorithm 3.

Algorithm 3 Parallel Method of Reflections (PMR)

Input: K (maximum number of iterations), *tol* (tolerance);

Set $u^0 = 0$ and $k = 1$;
for $j = 1 : J$ **do**
  Compute d_j^1 solving the problem (12);
end for
Compute the approximation $u^1 = \sum_{j=1}^{J} d_j^1$;
while $k < K$ and $\|u^k - u^{k-1}\| > tol$ **do**
  Update $k = k + 1$;
  for $j = 2 : J$ (this loop is executed in parallel) **do**
    Compute d_j^k solving the problem (11);
  end for
  Compute the approximation $u^k = u^{k-1} + \sum_{j=1}^{J} d_j^k$;
end while

2.3. The averaged parallel method of reflections. The PMR discussed in Section 2.2 is not always convergent. This fact has been mentioned in several publications, see, e.g., [7, 12], and we will illustrate this in Section 3 with a concrete example. In order to improve the convergence behavior of the PMR, Laurent et al. proposed in [17] a modified version that is obtained (as mentioned by the authors) by averaging the different components d_j^k: the problem (11) is modified as

$$\begin{aligned} &\Delta d_j^k = 0 \text{ in } \Omega \setminus \partial O_j,\ d_j^k = 0 \text{ on } \partial\Omega, \\ &d_j^k = \Big(1 - \frac{1}{J}\Big) d_j^{k-1} - \frac{1}{J} \sum_{\ell=1,\ell\neq j}^{J} d_\ell^{k-1} \text{ on } \partial O_j, \end{aligned} \tag{13}$$

for $j = 1, \dots, J$, with the initialization problems

$$\Delta d_j^1 = 0 \text{ in } \Omega \setminus \partial O_j,\ d_j^1 = 0 \text{ on } \partial\Omega,\ d_j^1 = g_j \text{ on } \partial O_j, \tag{14}$$

for $j = 1, \dots, J$. The approximate u^k can then be obtained from u^{k-1} by

$$u^k = u^{k-1} + \frac{1}{J}\sum_{j=1}^{J} d_j^k, \tag{15}$$

assuming that $u^0 = 0$. As for the PMR, this new formulation of the method, which we call averaged parallel method of reflections (APMR), can be interpreted as an alternating projection procedure in Hilbert spaces. If the projections are orthogonal, the APMR is proved to be always convergent in [17]. A pseudo-algorithm of the APMR is given by Algorithm 4.

Algorithm 4 Averaged Parallel Method of Reflections (APMR)

Input: K (maximum number of iterations), *tol* (tolerance);
Set $u^0 = 0$ and $k = 1$;
for $j = 1 : J$ **do**
  Compute d_j^1 solving the problem (14);
end for
Compute the approximation $u^1 = \sum_{j=1}^{J} d_j^1$;
while $k < K$ and $\|u^k - u^{k-1}\| > tol$ **do**
  Update $k = k + 1$;
  for $j = 2 : J$ (this loop is executed in parallel) **do**
    Compute d_j^k solving the problem (13);
  end for
  Compute the approximation $u^k = u^{k-1} + \frac{1}{J}\sum_{j=1}^{J} d_j^k$;
end while

2.4. Variant of Happel and Brenner. The version appearing in Happel and Brenner [9] is different from the parallel method of reflections described in Sections 2.2 and 2.3. In this procedure, one focuses on one specific object, say the first, and the updates of the $J-1$ other objects are done in parallel. We give the concrete implementation of their variant, which we call HBMR, in Algorithm 5. This algorithm is the first parallelizable version of the method after the original one of Golusin [7]. However, it still (sequentially) alternates between one object and all the others that are treated in parallel.

3. One-dimensional convergence analysis. In this section, we present a convergence analysis of AMR, PMR, and APMR for the solution of a one-dimensional problem. In particular, we are interested in the influence of the distance between the objects on the convergence of the methods of reflections. We consider a domain $\Omega = (0, 1)$ with three holes $O_j = (a_j, b_j)$ for $j = 1, 2, 3$ with $0 < a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < 1$, and the Laplace problem

$$\begin{aligned} \Delta u = 0 \text{ in } (0,1) \setminus \cup_j \partial O_j,\ u(0) = u(1) = 0,\\ u(a_j) = g_j^a,\ u(b_j) = g_j^b \text{ for } j = 1, \dots, 3, \end{aligned} \tag{16}$$

where g_j^a and g_j^b are given real numbers. We fix the size of the objects to $h := b_j - a_j = \frac{1}{10}$ and denote by d the distance between their centers; see Figure 9. In

Algorithm 5 Happel-Brenner Method of Reflections (HBMR)

Input: K (maximum number of iteration), *tol* (tolerance);

Set $u^0 = 0$, $d_j^0 = 0$ for $j = 1, \dots, J$, and $k = 1$;

Compute d_1^1 solving

$$\begin{aligned} &\Delta d_1^1 = 0 \text{ in } \Omega \setminus \overline{O_1},\ d_1^1 = 0 \text{ on } \partial\Omega, \\ &d_1^1 = g_1 \text{ on } \partial O_1. \end{aligned}$$

for $j = 2 \colon J$ **do**

  Compute d_j^1 in $\Omega \setminus O_j$ solving the problem

$$\begin{aligned} &\Delta d_j^1 = 0 \text{ in } \Omega \setminus \overline{O_j},\ d_j^1 = 0 \text{ on } \partial\Omega, \\ &d_j^1 = g_j - d_1^1 \text{ on } \partial O_j. \end{aligned}$$

end for

Compute the approximation $u^1 = \sum_{j=1}^J d_j^1$ in $\Omega \setminus \cup_j O_j$;

while $k < K$ and $\|u^k - u^{k-1}\| > tol$ **do**

  Update $k = k + 1$;

  Compute d_1^k solving

$$\begin{aligned} &\Delta d_1^k = 0 \text{ in } \Omega \setminus \overline{O_1},\ d_1^k = 0 \text{ on } \partial\Omega, \\ &d_1^k = g_1 - u^{k-1} \text{ on } \partial O_1. \end{aligned}$$

  for $j = 2 \colon J$ (this loop is executed in parallel) **do**

    Compute d_j^k in $\Omega \setminus O_j$ solving the problem

$$\begin{aligned} &\Delta d_j^k = 0 \text{ in } \Omega \setminus \overline{O_j},\ d_j^k = 0 \text{ on } \partial\Omega, \\ &d_j^k = g_j - (u^{k-1} + d_1^k) \text{ on } \partial O_j. \end{aligned}$$

  end for

  Compute the approximation $u^k = u^{k-1} + \sum_{j=1}^J d_j^k$ in $\Omega \setminus \cup_j O_j$;

end while

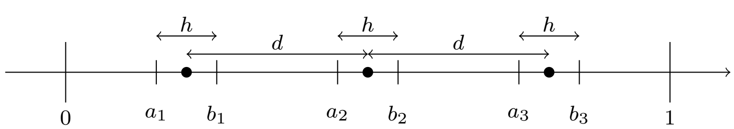

FIG. 9. *Geometry of the one-dimensional domain* $[0, 1]$ *with three equidistant holes of size* h. *The distance between the holes is denoted by* d. *The black dots are the centers of the objects.*

particular, the object O_2 is centered at the midpoint of the domain. Therefore, we have $a_1 = \frac{1}{2} - d - \frac{h}{2}$, $b_1 = \frac{1}{2} - d + \frac{h}{2}$, $a_2 = \frac{1}{2} - \frac{h}{2}$, $b_2 = \frac{1}{2} + \frac{h}{2}$, $a_3 = \frac{1}{2} + d - \frac{h}{2}$, and $b_3 = \frac{1}{2} + d + \frac{h}{2}$. Notice that to guarantee that hypothesis (1) is satisfied, that is $\overline{O}_j \cap \overline{O}_k = \emptyset$ for any $j \neq k$, the distance d has to satisfy the relation $d > h = \frac{1}{10}$. Moreover, to have that $a_1 > 0$ and $b_3 < 1$ it is required that $d < \frac{1}{2} - \frac{h}{2}$.

We begin with the AMR that, for the solution to (16), is given by

$$\begin{cases} \Delta d_1^k = 0 \text{ in } \Omega_1, \quad d_1^k(0) = d_1^k(1) = 0, \\ d_1^k(\tilde{x}) = -d_2^{k-1}(\tilde{x}) - d_3^{k-1}(\tilde{x}) \text{ for } \tilde{x} = a_1, b_1, \end{cases}$$

$$\begin{cases} \Delta d_2^k = 0 \text{ in } \Omega_2, & d_2^k(0) = d_2^k(1) = 0, \\ d_2^k(\tilde{x}) = -d_1^k(\tilde{x}) - \alpha_{2,3} d_3^{k-1}(\tilde{x}) \text{ for } \tilde{x} = a_2, b_2, \end{cases}$$

$$\begin{cases} \Delta d_3^k = 0 \text{ in } \Omega_3, & d_3^k(0) = d_3^k(1) = 0, \\ d_3^k(\tilde{x}) = -d_1^k(\tilde{x}) - d_2^{k-1}(\tilde{x}) \text{ for } \tilde{x} = a_3, b_3. \end{cases}$$

The general solutions of these three problems are

$$d_1^k(x) = \begin{cases} \frac{A_1^k x}{a_1} & x \in [0, a_1], \\ \frac{B_1^k(1-x)}{1-b_1} & x \in [b_1, 1], \end{cases} \quad d_2^k(x) = \begin{cases} \frac{A_2^k x}{a_2} & x \in [0, a_2], \\ \frac{B_2^k(1-x)}{1-b_2} & x \in [b_2, 1], \end{cases}$$

$$d_3^k(x) = \begin{cases} \frac{A_3^k x}{a_3} & x \in [0, a_3], \\ \frac{B_3^k(1-x)}{1-b_3} & x \in [b_3, 1], \end{cases}$$

where A_j^k and B_j^k are constants depending on the transmission conditions. Defining $\mathbf{v}^k := [A_1^k, B_1^k, A_2^k, B_2^k, A_3^k, B_3^k]^\top$ and using the transmission conditions, we obtain a stationary iteration in matrix form,

$$(I + \widetilde{L})\mathbf{v}^k = -\widetilde{U}\mathbf{v}^{k-1},$$

where

$$\widetilde{L} = -\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1-a_2}{(1-b_1)} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1-b_2}{(1-b_1)} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1-a_3}{(1-b_1)} & 0 & -\frac{1-a_3}{1-b_2} & 0 & 0 \\ 0 & -\frac{1-b_3}{(1-b_1)} & 0 & -\frac{1-b_3}{1-b_2} & 0 & 0 \end{bmatrix} \text{ and } \widetilde{U} = -\begin{bmatrix} 0 & 0 & -\frac{a_1}{a_2} & 0 & -\frac{a_1}{a_3} & 0 \\ 0 & 0 & -\frac{b_1}{a_2} & 0 & -\frac{b_1}{a_3} & 0 \\ 0 & 0 & 0 & 0 & -\frac{a_2}{a_3} & 0 \\ 0 & 0 & 0 & 0 & -\frac{b_2}{a_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The iteration matrix of the AMR is therefore

$$G_{\text{AMR}} := -(I + \widetilde{L})^{-1}\widetilde{U} = \begin{bmatrix} 0 & 0 & \frac{20d-9}{9} & 0 & \frac{20d-9}{20d+9} & 0 \\ 0 & 0 & \frac{20d-11}{9} & 0 & \frac{20d-11}{20d+9} & 0 \\ 0 & 0 & -\frac{220d-121}{180d+81} & 0 & -\frac{400d-40}{400d^2+360d+81} & 0 \\ 0 & 0 & -\frac{20d-11}{20d+9} & 0 & -\frac{400d}{400d^2+360d+81} & 0 \\ 0 & 0 & 0 & 0 & -\frac{220d-121}{180d+81} & 0 \\ 0 & 0 & 0 & 0 & -\frac{220d-99}{180d+81} & 0 \end{bmatrix},$$

where we used the expressions of a_j and b_j. The spectral radius of G_{AMR} can be explicitly calculated,

$$\rho(G_{\text{AMR}}) = \frac{121 - 220d}{180d + 81},$$

and one can show that $\rho(G_{\text{AMR}}) < 1$ for any $d > \frac{1}{10}$, i.e. the objects are not touching. Hence, the AMR is convergent. The spectral radius $\rho(G_{\text{AMR}})$ as a function of d is shown in Figure 10 (left).

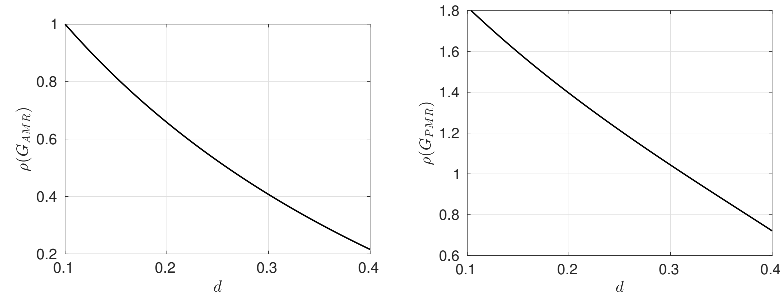

FIG. 10. *Spectral radii* $\rho(G_{\mathrm{AMR}})$ *(left) and* $\rho(G_{\mathrm{PMR}})$ *(left) as functions of the distance* d*.*

Next,we consider the PMR, that is

$$\begin{cases} \Delta d_1^k = 0 \ \text{ in } \Omega_1, \quad d_1^k(0) = d_1^k(1) = 0, \\ d_1^k(\tilde{x}) = -d_2^{k-1}(\tilde{x}) - d_3^{k-1}(\tilde{x}) \ \text{ for } \tilde{x} = a_1, b_1, \end{cases}$$

$$\begin{cases} \Delta d_2^k = 0 \ \text{ in } \Omega_2, \quad d_2^k(0) = d_2^k(1) = 0, \\ d_2^k(\tilde{x}) = -d_1^{k-1}(\tilde{x}) - \alpha_{2,3} d_3^{k-1}(\tilde{x}) \ \text{ for } \tilde{x} = a_2, b_2, \end{cases}$$

$$\begin{cases} \Delta d_3^k = 0 \ \text{ in } \Omega_3, \quad d_3^k(0) = d_3^k(1) = 0, \\ d_3^k(\tilde{x}) = -d_1^{k-1}(\tilde{x}) - d_2^{k-1}(\tilde{x}) \ \text{ for } \tilde{x} = a_3, b_3. \end{cases}$$

Proceeding as for the AMR, we obtain a stationary iteration in matrix form,

$$\mathbf{v}^k = G_{\mathrm{PMR}} \mathbf{v}^{k-1},$$

where the iteration matrix G_{PMR} is

$$G_{\mathrm{PMR}} = \begin{bmatrix} 0 & 0 & \frac{10\left(2d-\frac{9}{10}\right)}{9} & 0 & -\frac{2d-\frac{9}{10}}{-2d-\frac{9}{10}} & 0 \\ 0 & 0 & -\frac{10\left(\frac{11}{10}-2d\right)}{9} & 0 & \frac{\frac{11}{10}-2d}{-2d-\frac{9}{10}} & 0 \\ 0 & \frac{11}{10\left(-2d-\frac{9}{10}\right)} & 0 & 0 & \frac{9}{10\left(-2d-\frac{9}{10}\right)} & 0 \\ 0 & \frac{9}{10\left(-2d-\frac{9}{10}\right)} & 0 & 0 & \frac{11}{10\left(-2d-\frac{9}{10}\right)} & 0 \\ 0 & \frac{\frac{11}{10}-2d}{-2d-\frac{9}{10}} & 0 & -\frac{10\left(\frac{11}{10}-2d\right)}{9} & 0 & 0 \\ 0 & -\frac{2d-\frac{9}{10}}{-2d-\frac{9}{10}} & 0 & \frac{10\left(2d-\frac{9}{10}\right)}{9} & 0 & 0 \end{bmatrix},$$

whose spectral radius is

$$\rho(G_{\mathrm{PMR}}) = \frac{33 - 60d + \sqrt{-28400d^2 - 760d + 9009}}{120d + 54}.$$

As shown in Figure 10 (right), if the objects are close, the spectral radius $\rho(G_{\mathrm{PMR}})$ is bigger than 1, which means that the PMR does not always converge. This result is in agreement with the convergence analyses in [17, 24]. This loss of convergence motivated Laurent et al. to introduce the APMR that, for our one-dimensional example,

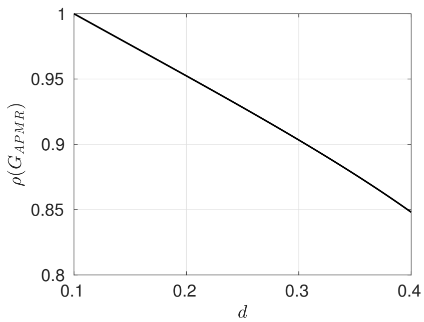

FIG. 11. *Spectral radius $\rho(G_{\mathrm{APMR}})$ as a function of the distance d.*

is given by

$$\begin{cases} \Delta d_1^k = 0 \text{ in } \Omega_1, \quad d_1^k(0) = d_1^k(1) = 0, \\ d_1^k(\tilde{x}) = \frac{2}{3} d_1^{k-1}(\tilde{x}) - \frac{1}{3} d_2^{k-1}(\tilde{x}) - \frac{1}{3} d_3^{k-1}(\tilde{x}) \text{ for } \tilde{x} = a_1, b_1, \end{cases}$$

$$\begin{cases} \Delta d_2^k = 0 \text{ in } \Omega_2, \quad d_2^k(0) = d_2^k(1) = 0, \\ d_2^k(\tilde{x}) = \frac{2}{3} d_2^{k-1}(\tilde{x}) - \frac{1}{3} d_1^{k-1}(\tilde{x}) - \frac{1}{3} \alpha_{2,3} d_3^{k-1}(\tilde{x}) \text{ for } \tilde{x} = a_2, b_2, \end{cases}$$

$$\begin{cases} \Delta d_3^k = 0 \text{ in } \Omega_3, \quad d_3^k(0) = d_3^k(1) = 0, \\ d_3^k(\tilde{x}) = \frac{2}{3} d_3^{k-1}(\tilde{x}) - \frac{1}{3} d_1^{k-1}(\tilde{x}) - \frac{1}{3} d_2^{k-1}(\tilde{x}) \text{ for } \tilde{x} = a_3, b_3. \end{cases}$$

In this case, the iteration matrix G_{APMR} is given by

$$G_{\mathrm{APMR}} = \begin{bmatrix} \frac{2}{3} & 0 & \frac{10\left(2d-\frac{9}{10}\right)}{27} & 0 & -\frac{2d-\frac{9}{10}}{-6d-\frac{27}{10}} & 0 \\ 0 & \frac{2}{3} & -\frac{10\left(\frac{11}{10}-2d\right)}{27} & 0 & \frac{\frac{11}{10}-2d}{-6d-\frac{27}{10}} & 0 \\ 0 & \frac{11}{10\left(-6d-\frac{27}{10}\right)} & \frac{2}{3} & 0 & \frac{9}{10\left(-6d-\frac{27}{10}\right)} & 0 \\ 0 & \frac{9}{10\left(-6d-\frac{27}{10}\right)} & 0 & \frac{2}{3} & \frac{11}{10\left(-6d-\frac{27}{10}\right)} & 0 \\ 0 & \frac{\frac{11}{10}-2d}{-6d-\frac{27}{10}} & 0 & -\frac{10\left(\frac{11}{10}-2d\right)}{27} & \frac{2}{3} & 0 \\ 0 & -\frac{2d-\frac{9}{10}}{-6d-\frac{27}{10}} & 0 & \frac{10\left(2d-\frac{9}{10}\right)}{27} & 0 & \frac{2}{3} \end{bmatrix},$$

and its spectral radius is

$$\rho(G_{\mathrm{APMR}}) = \frac{75 + 300d + \sqrt{-28400d^2 - 760d + 9009}}{360d + 162}.$$

It is possible to show that $\rho(G_{\mathrm{APMR}}) < 1$ for $d > h = \frac{1}{10}$, as shown in Figure 11. In particular, the convergence of the APMR deteriorates as d approaches $h = \frac{1}{10}$, but in contrast to the PMR the spectral radius remains bounded by 1, and thus the method is convergent.

4. Conclusion. We traced the history of the method of reflections, and showed that there are two main variants of it, the alternating method of reflections and the parallel method of reflections. For the parallel method of reflections there is also a very recent variant using averaging. We then gave a precise mathematical formulation of the methods, and indicated for the two object case a relation with the alternating Schwarz method. We finally studied the convergence properties of these methods for a one dimensional model problem and three objects. Our results show that the alternating method of reflections is always convergent, while the parallel method of reflections is only converging if the objects are far enough apart. This convergence problem can be alleviated using averaging in the parallel method of reflections. This manuscript is just the beginning of a more complete analysis of these methods of reflections, and improved variants which we are currently developing, see [5].

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