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Well-posedness for coupled bulk-interface diffusion with mixed boundary conditions

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Abstract

In this paper, we consider a quasilinear parabolic system of equations describing coupled bulk and interface diffusion, including mixed boundary conditions. The setting naturally includes non-smooth domains Ω. We show local well-posedness using maximal L^s-regularity in dual Sobolev spaces of type $W^{-1,q}(\Omega)$ for the associated abstract Cauchy problem.

1. Introduction

We consider a parabolic system of equations describing coupled bulk and interface diffusion. The equations are related to the classical Stefan problem, and they are derived for a number of different dissipative processes taking place on both a bulk domain and part of its boundary in [20] and [9]. In [5, Examples 3.4 and 3.5], for a large class of problems of this type on smooth domains, maximal L^s-regularity has been shown. We give an extension of these results for a specific setting, where the main point is that the interface cuts the bulk into two parts, naturally creating non-smooth domains, cf. Figure 1.

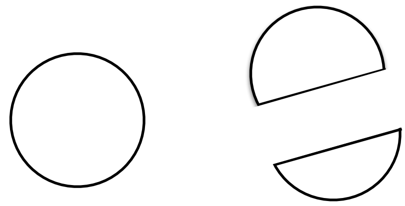

Figure 1. Cutting a smooth domain by a hyperplane does not create two smooth domains in general, but, in many cases, two Lipschitz domains.

More precisely, the situation is the following. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain, divided into two open disjoint subdomains Ω_+ and Ω_- by an $\mathbb{R}^{d-1}$-plane. The intersection of Ω and the plane is denoted by Γ. We choose coordinates such that $\Gamma \subset \{(y, 0) \in \mathbb{R}^d : y \in \mathbb{R}^{d-1}\}$ and write $x = (y, x')$ for $x \in \Omega$, $x' \in \mathbb{R}$ and $y \in \mathbb{R}^{d-1}$. We consider the evolution of the quantities $u_+ : (0,T) \times \Omega_+ \to \mathbb{R}$, $u_- : (0,T) \times \Omega_- \to \mathbb{R}$ and $u_\Gamma : (0,T) \times \Gamma \to \mathbb{R}$ satisfying

$$\left\{ \begin{array}{rcll} \partial_t u_+ - \operatorname{div}(k_+ \nabla u_+) &=& f_+, & \text{in } (0,T) \times \Omega_+, \\ (k_+ \nabla u_+)\nu_+ - m_+(u_+ - u_\Gamma) - m_\Gamma(u_+ - u_-) &=& 0, & \text{on } (0,T) \times \Gamma, \end{array} \right. \tag{1.1}$$

on the upper bulk part and interface,

$$\left\{ \begin{array}{rcll} \partial_t u_- - \operatorname{div}(k_- \nabla u_-) &=& f_-, & \text{in } (0,T) \times \Omega_-, \\ (k_- \nabla u_-)\nu_- - m_-(u_- - u_\Gamma) - m_\Gamma(u_- - u_+) &=& 0, & \text{on } (0,T) \times \Gamma, \end{array} \right. \tag{1.2}$$

on the lower bulk part and interface, coupled with the evolution

$$\partial_t u_\Gamma - \operatorname{div}(k_\Gamma \nabla u_\Gamma) - m_+(u_+ - u_\Gamma) - m_-(u_- - u_\Gamma) = f_\Gamma, \quad \text{in } (0,T) \times \Gamma, \tag{1.3}$$

on the interface Γ, where ν_+, ν_-, ν_Γ denote the outer normal vector fields of Ω_+, Ω_- and Γ and f_+, f_- and f_Γ are given external forces. The system is complemented with mixed boundary conditions on $\partial\Omega$ in the following way. There is a Dirichlet part of the boundary $D \subset \partial\Omega$ which splits into three parts $D_+ = D \cap \partial\Omega_+$, $D_- = D \cap \partial\Omega_-$ and $D_\Gamma = D \cap \partial\Gamma$ and there are Neumann boundary parts for each subdomain, $N_+ = \partial\Omega_+ \setminus D_+$, $N_- = \partial\Omega_- \setminus D_-$ and $N_\Gamma = \partial\Gamma \setminus D_\Gamma$. Since the conditions on Γ as a boundary of Ω_+ and Ω_- have already been fixed, it remains to set

$$\left\{ \begin{array}{rcll} (k_+ \nabla u_+)\nu_+ &=& 0, & \text{on } (0,T) \times \{\partial\Omega_+ \setminus (\Gamma \cup D_+)\}, \\ u_+ &=& 0, & \text{on } (0,T) \times D_+, \end{array} \right. \tag{1.4}$$

on $\Omega_+ \setminus \Gamma$,

$$\begin{cases} (k_-\nabla u_-)\nu_- &= 0, \quad \text{on } (0,T) \times \{\partial\Omega_- \backslash (\Gamma \cup D_-)\}, \\ u_- &= 0, \quad \text{on } (0,T) \times D_-, \end{cases} \tag{1.5}$$

on $\Omega_- \setminus \Gamma$, and

$$\begin{cases} (k_\Gamma \nabla u_\Gamma)\nu_\Gamma &= 0, \quad \text{on } (0,T) \times \{\partial\Gamma \backslash D_\Gamma\}, \\ u_\Gamma &= 0, \quad \text{on } (0,T) \times D_\Gamma, \end{cases} \tag{1.6}$$

on $\partial\Gamma$. In the following, we often use the shorthands

$$u = (u_+, u_-, u_\Gamma),\ f = (f_+, f_-, f_\Gamma), \ldots$$

The matrix-valued coefficient functions k and scalar transmission coefficients m may depend on the space variable and the solution u. More precise assumptions on k and m as well as on the domain Ω and on D and N are given in Subsection 2.1 below.

Note here that the system is essentially scalar, except on the interface Γ where the three unknowns (u_+, u_-, u_Γ) interact.

As an example from applications, this system may describe non-linear heat conduction in a bulk material which is divided into two parts by a thin metal plate and heated or cooled from the outside only on specific parts of its boundary, cf. Figure 2.

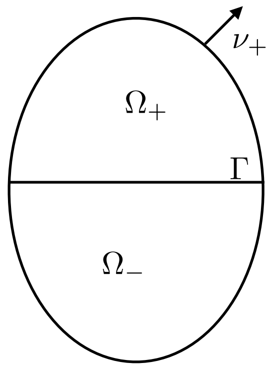

FIGURE 2. An example domain.

In [6], a similar system is being studied, with a focus on its gradient or Onsager structure, cf. [20] and [9], leading to more complex but also more specific nonlinearities and to particular questions concerning long-term behaviour. In contrast, in this work, we do not require the system to be isolated but allow for general exterior forcing f and mixed Dirichlet and Neumann boundary conditions, but only show local-in-time well-posedness. Mathematically, the question of how non-smooth paroblic and elliptic problems can be treated is strongly connected to the question of how to treat these problems with mixed boundary conditions. It thus seems natural to use recent result in this direction and see how they can be made to apply in the present situation. We focus on the two- and three-dimensional problems and use the results derived in [14] and [7]. The results of these works are conditions on a domain $\omega \subset \mathbb{R}^d$ and coefficient function $k : \omega \to \mathbb{R}^{d\times d}$to guarantee that the elliptic operator

$$\operatorname{div}(k\nabla\cdot) : W^{1,q}_\delta(\omega) \to W^{-1,q}_\delta(\omega) \tag{1.7}$$

is an isomorphism for some $q \geq d$ and $W^{1,q}_\delta(\omega)$ the Sobolev space realizing mixed boundary conditions. This isomorphism property is particularly important in the study of quasilinear problems as it turns out to be stable for many k, cf. the discussion of Assumption 4.1, and yield sufficient regularity on u to allow for unique solutions given by a Banach fixed point argument. The main point is that since the problem is assumed to be quasilinear, it requires particular regularity properties to establish well-posedness, which are non-trivial to establish for non-smooth domains.

The paper is organized as follows. In the next Section, we fix some notation, precise assumptions on the domain and on coefficients and establish a functional analytic framework in which equations (1.1)-(1.6) can be recast as a quasilinear abstract Cauchy problem. In particular, we give a precise definition of the spaces in (1.7). In Section 3, it is shown that the linear problem has the property of maximal L^s-regularity in the dual Sobolev space $\mathrm{W}_D^{-1,q}$. In Section 4, the linear result is used to show local well-posedness of the system and remark on simple extensions following from the theory of maximal L^s-regularity.

2. Notation, assumptions and an abstract framework for the system

The aim of this Section is to introduce notation and precise assumptions on Ω, k and m and to set up a functional analytic framework in which the problem can be recast as a quasilinear abstract Cauchy problem.

2.1. **Assumptions on Ω, D and N .** Following the ideas in [14] and e.g. [3], [7, 16, 18, 15, 12, 11], we pose assumptions on the domains Ω_+, Ω_- and Γ which will guarantee the property in (1.7).

Assumption 2.1 ($d = 2, 3$)**.** *We suppose that Ω_+, Ω_- and Γ are bounded Lipschitz domains, cf. [13, Def. 1.2.12], and that their Neumann boundary parts $N_+ := \partial\Omega_+ \setminus D_+$, $N_- := \partial\Omega_- \setminus D_-$ and $\partial\Gamma \setminus D_\Gamma$ are relatively open subsets of their respective boundaries $\partial\Omega_+$, $\partial\Omega_-$ and $\partial\Gamma$. Moreover, Γ must be contained in both N_+ and N_-.*

Remark 2.2. *Note that we do not need to assume that Ω_+, Ω_- or Γ are* strong*, or equivalently,* graph *Lipschitz domains, cf. [1, Def. 4.5].*

If $d = 2$, Assumption 2.1 is sufficient for the remainder of the paper. If $d = 3$, we must put additional restrictions on Ω_+ and Ω_-, following the approach in [7].

Assumption 2.3 ($d = 3$)**.** *If $d = 3$, then in addition to Assumption 2.1, we ask that Ω_+, with given N_+ and D_+ must locally at the boundary be bi-Lipschitz diffeomorphic to one of the model constellations given in [7, Sect. 3]. In particular, this includes conditions on the discontinuities of the coefficient matrix κ_+ near the boundary of Ω_+. The same must hold for Ω_-, N_-, D_- and κ_-.*

Remark 2.4. *We refer to [7] for the precise formulation and an extensive discussion of Assumption 2, including its necessity in many cases for proving the property (1.7). We note here that Assumption 2 is (trivially) satisfied if κ_+ is continuous and $D_+ = \emptyset$ or if $\partial\Omega_+$ is smooth and $D_+ = \emptyset$, i.e. it mainly concerns the problems of how N_+ and D_+ are allowed to meet and of how non-smoothness of κ_+ and non-smoothness of $\partial\Omega_+$ are allowed to meet.*

2.2. **Function spaces.** For $q \in [1, \infty]$ and $\omega \in \{\Omega_+, \Omega_-, \Gamma\}$, we denote by $L^q(\omega)$ the usual real Lebesgue space of q-integrable functions and denote the norm by $\|\cdot\|_q$ if the domain is known. For $\delta \in \{D_+, D_-, D_\Gamma\}$ corresponding to ω, we define

$$C_\delta^\infty(\omega) := \{v \in C_c^\infty(\overline{\omega}) : \operatorname{supp} v \cap \delta = \emptyset\}$$

and denote by

$$W_\delta^{1,q}(\omega) := \overline{C_\delta^\infty(\omega)}^{W^{1,q}(\omega)}$$

the closure of $C_\delta^\infty(\omega)$ with respect to the usual Sobolev norm, which is denoted by $\|\cdot\|_{1,q}$. Note that ω is assumed to be sufficiently regular to guarantee that $W_\emptyset^{1,q}(\omega) = W^{1,q}(\omega)$, where $W^{1,q}(\omega)$ denotes the usual Sobolev space, cf. [14, Def. 3].

For the full unknown function u, we define the spaces

$$\mathrm{L}^q := L^q(\Omega_+) \times L^q(\Omega_-) \times L^q(\Gamma) \simeq L^q(\Omega, \mathrm{d}\Omega + \mathrm{d}\Gamma)$$

and

$$\mathrm{W}^{1,q}_D := W^{1,q}_{D_+}(\Omega_+) \times W^{1,q}_{D_-}(\Omega_-) \times W^{1,q}_{D_\Gamma}(\Gamma).$$

Note also that since Γ is a smooth part of the boundaries of Ω_+, the trace operator

$$\mathrm{tr}_\Gamma : W^{1,q}_{D_+}(\Omega_+) \to L^q(\Gamma) \tag{2.1}$$

is well-defined and continuous (likewise for Ω_-). We write

$$\mathrm{tr}_\Gamma\, u = (\mathrm{tr}_\Gamma\, u_+, \mathrm{tr}_\Gamma\, u_-, u_\Gamma) \tag{2.2}$$

for the trace components of u on the interface Γ.

Deviating slightly from the usual notation for $\delta = \partial\omega$, the dual spaces of $W^{1,q}_\delta(\omega)$ and $\mathrm{W}^{1,q}_D$ are denoted by

$$W^{-1,q'}_\delta(\omega) := (W^{1,q}_\delta(\omega))' \text{ and } \mathrm{W}^{-1,q'}_D := (\mathrm{W}^{1,q}_D)'$$

with $\frac{1}{q} + \frac{1}{q'} = 1$.

2.3. **Assumptions on k and m.** The coefficient functions $k_\pm$ may depend on $x \in \Omega_\pm$ as well as on $u_\pm(x)$ and $k_\Gamma, m_\pm, m_\Gamma$ may depend on $y \in \Gamma$ and $\mathrm{tr}_\Gamma\, u(y) \in \mathbb{R}^3$. We assume that k is given by positive semi-definite matrices,

$$k_\pm : \Omega_\pm \times \mathbb{R} \to \mathbb{R}^{d\times d}_{\geq 0} \qquad \text{and} \qquad k_\Gamma : \Gamma \times \mathbb{R}^3 \to \mathbb{R}^{(d-1)\times(d-1)}_{\geq 0}, \tag{2.3}$$

and that the transmission coefficients are also positive, satisfying

$$m_\Gamma, m_+, m_- : \Gamma \times \mathbb{R}^3 \to \mathbb{R}_+. \tag{2.4}$$

Let κ and μ denote coefficients of this type but independent of u. Then with slight abuse of notation, by μ we denote the transmission coefficent matrix

$$\mu = \begin{pmatrix} \mu_+ + \mu_\Gamma & -\mu_\Gamma & -\mu_+ \\ -\mu_\Gamma & \mu_- + \mu_\Gamma & -\mu_- \\ -\mu_+ & -\mu_- & \mu_+ + \mu_- \end{pmatrix}$$

where $\mu_\pm, \mu_\Gamma : \Gamma \to \mathbb{R}_+$ are as in (2.4) but independent of u.

Assumption 2.5 (Assumptions on κ). *The coefficients κ are bounded, measurable and uniformly elliptic, i.e. there is a constant $\kappa_0 > 0$ such that for all $x \in \mathbb{R}^d, y \in \mathbb{R}^{d-1}$, almost everywhere in $\Omega_\pm, \Gamma$,*

$$\frac{1}{\kappa_0}|x|^2 \leq x^T \kappa_\pm x \leq \kappa_0 |x|^2 \quad \textit{and} \quad \frac{1}{\kappa_0}|y|^2 \leq y^T \kappa_\Gamma y \leq \kappa_0 |y|^2.$$

Assumption 2.6 (Assumptions on μ). *The coefficients $\mu_\pm$,μ_Γ are bounded, measurable and there is a constant $\mu_0 > 0$ such that almost everywhere in Γ,*

$$\mu_0 \leq \mu_\pm, \mu_\Gamma.$$

Note that under Assumption 2.6, the matrix μ is positive semi-definite and for $r = (r_+, r_-, r_\Gamma)^T \in \mathbb{R}^3$,

$$r^T \mu r = 0 \text{ a.e., iff } r_+ = r_- = r_\Gamma. \tag{2.5}$$

2.4. **A bilinear form and an elliptic operator associated to this problem.** We define the bilinear form $\mathfrak{a}_{\kappa,\mu} : \mathrm{W}^{1,2}_D \times \mathrm{W}^{1,2}_D \to \mathbb{R}$ by

$$\mathfrak{a}_{\kappa,\mu}(u,\varphi) = \mathfrak{l}_\kappa(u,\varphi) + \mathfrak{m}_\mu(u,\varphi),$$

where

$$\begin{aligned}\mathfrak{l}_\kappa(u,\varphi) &= \textstyle\int_{\Omega_+}(\kappa_+\nabla u_+, \nabla\varphi_+)_{\mathbb{R}^d}\,\mathrm{d}x + \int_{\Omega_-}(\kappa_-\nabla u_-, \nabla\varphi_-)_{\mathbb{R}^d}\,\mathrm{d}x \\ &\quad + \textstyle\int_\Gamma(\kappa_\Gamma\nabla u_\Gamma, \nabla\varphi_\Gamma)_{\mathbb{R}^d}\mathrm{d}y,\end{aligned}$$

and

$$\mathfrak{m}_\mu(u,\varphi) = \int_\Gamma (\mu \operatorname{tr}_\Gamma u, \operatorname{tr}_\Gamma \varphi)_{\mathbb{R}^3}\mathrm{d}y.$$

By (2.1), the trace operator $\operatorname{tr}_\Gamma : W^{1,q}_D(\Omega_\pm) \to L^q(\Gamma)$ is well-defined and bounded and thus, the form $\mathfrak{a}_{\kappa,\mu}$ is continuous and bounded from below by 0 by the assumptions on κ and μ. In particular,

$$\mathfrak{a}_{\kappa,\mu}(u,u) = 0 \text{ iff } \ u_+ = u_- = u_\Gamma \equiv c, \tag{2.6}$$

due to (2.5). The form $\mathfrak{a}_{\kappa,\mu}$ induces an operator $\mathcal{A}_{\kappa,\mu} : \mathrm{W}^{1,2}_D \to \mathrm{W}^{-1,2}_D$ by

$$\langle \mathcal{A}_{\kappa,\mu}u, \varphi\rangle_{\mathrm{W}^{-1,2}_D \times \mathrm{W}^{1,2}_D} := \mathfrak{a}_{\kappa,\mu}(u,\varphi).$$

By the Lax-Milgram theorem, for every $\lambda > 0$, $\mathcal{A}_{\kappa,\mu} + \lambda$ has a bounded inverse and for $q \in [2,\infty)$, let $\mathcal{A}_{q,\kappa,\mu}$ be the closed and densely defined restriction of $\mathcal{A}_{q,\kappa,\mu}$ to $\mathrm{W}^{-1,q}_D$. We write $\mathcal{L}_{q,\kappa}$ for the divergence operator in $\mathrm{W}^{-1,q}_D$ analogously induced by $\mathfrak{l}_\kappa$ and we write $\mathcal{M}_{q,\mu}$ for the bounded transmission operator given by

$$\langle \mathcal{M}_{q,\mu}u, \varphi\rangle_{\mathrm{W}^{-1,q}_D \times \mathrm{W}^{1,q'}_D} := \mathfrak{m}_\mu(u,\varphi), \qquad u \in \operatorname{dom}(\mathcal{L}_{q,\kappa}), \varphi \in \mathrm{W}^{1,q'}_D, \tag{2.7}$$

so that

$$\mathcal{A}_{q,\kappa,\mu} = \mathcal{L}_{q,\kappa} + \mathcal{M}_{q,\mu}.$$

In the quasilinear case, we use the notation $\mathfrak{a}_{k,m}$ instead of $\mathfrak{a}_{\kappa,\mu}$ to indicate that the form depends on $u(t)$ and we similarly write $\mathcal{A}_{q,k,m}, \mathcal{L}_{q,k}$ and $\mathcal{M}_{q,m}$ for the corresponding operators. The assumptions on κ and q will guarantee that $\operatorname{dom}(\mathcal{L}_{q,\kappa})$ and thus $\mathcal{M}_{q,\mu}$ are independent of κ. We interprete the set of equations (1.1), (1.2), (1.3) complemented by (1.4), (1.5) and (1.6) as the quasilinear problem

$$\dot{u} + \mathcal{A}_{q,k,m}u = f, \quad u(0) = u_0, \tag{2.8}$$

posed in $\mathrm{W}^{-1,q}_D$, $q \geq 2$. In the next section, we show that the linear problem

$$\dot{u} + \mathcal{A}_{q,\kappa,\mu}u = f, \quad u(0) = u_0, \tag{2.9}$$

is well-posed. In Section 4, we show that under additional assumptions on q and k, this result transfers to the quasilinear equation (2.8).

3. Well-posedness for the linear problem

The aim of this section is to prove the following result.

Theorem 3.1. *If all the assumptions in Section 2 hold, and $2 \leq q < \infty$, then $\mathcal{A}_{q,\kappa,\mu}$ has maximal $L^s(0,T;\mathrm{W}^{-1,q}_D)$-regularity.*

Let us first briefly recall the notion of maximal $L^s(0,T;X)$-regularity for a Banach space X.

Definition 3.2. *Let $1 < s < \infty$, let X be a Banach space and let $J_T = [0, T]$, $T > 0$, be a bounded interval. Assume that B is a closed operator in X with dense domain* $\mathrm{dom}(B) \subset X$, *equipped with the graph norm. We say that B satisfies maximal $L^s(J_T; X)$-regularity if for all $u_0 \in (\mathrm{dom}(B), X)_{1-\frac{1}{s},s}$ and $f \in L^s(0, T; X)$ there is a unique solution*

$$u \in L^s(J_T; \mathrm{dom}(B)) \cap W^{1,s}(J_T; X) =: \mathcal{H}^s(J_T; \mathrm{dom}(B); X)$$

of the abstract Cauchy problem

$$\begin{cases} \dot{u} + Bu &= f, \\ u(0) &= u_0, \end{cases}$$

posed in X, satisfying

$$\|\dot{u}\|_{L^s(J_T;X)} + \|Bu\|_{L^s(J_T;X)} \leq C(\|u_0\|_{(\mathrm{dom}(B),X)_{1-\frac{1}{s},s}} + \|f\|_{L^s(J_T;X)})$$

with a constant $C > 0$ independent of u_0, f (see e.g. [2, Ch. III.1]).

Note that the wording in Theorem 3.1 is justified in the sense that the notion of maximal $L^s(J_T; X)$-regularity is independent of $1 < s < \infty$ and $T > 0$, cf. [8]. In particular, Theorem 3.1 shows that problem (2.9) is well-posed for $f \in L^s(J_T; \mathrm{W}^{-1,q}_D)$ and

$$u_0 \in X_{s,q} := (\mathrm{dom}(\mathcal{A}_{q,\kappa,\mu}), \mathrm{W}^{-1,q}_D)_{1-\frac{1}{s},s}.$$

We can consider Theorem 3.1 as a corollary to the results in [17] and [3], where it is shown that the operator $L_{q,\kappa} = \mathrm{div}(\kappa\nabla\cdot)$ corresponding to the form $t_\kappa(u, \varphi) = \int_\omega (\kappa\nabla u, \nabla\varphi)_{\mathbb{R}^d} \, \mathrm{d}x$ has maximal L^s-regularity in $W^{-1,q}_\delta(\omega)$ on domains ω with Dirichlet boundary part δ satisfying the Assumptions 2.1 and 2.3, i.e. if $L_{q,\kappa}$ provides an isomorphism of $W^{-1,q}_\delta(\omega)$ and $W^{1,q}_\delta(\omega)$, which is guaranteed by the results in [7]. Clearly, by [17, Remark 8.3], the result carries over from $L_{q,\kappa}$ to the diagonal "system" given by $\mathcal{L}_{q,\kappa}$, i.e. we have shown

Proposition 3.3. *Let $2 \leq q < \infty$, then $\mathcal{L}_{q,\kappa}$ has maximal $L^s(J_T; \mathrm{W}^{-1,q}_D)$-regularity.*

In order to prove Theorem 3.1, in the following lemma, it remains to treat the transmission part $\mathcal{M}_{q,\mu}$ by a perturbation argument, cf. [17, Lemma 5.15].

Lemma 3.4. *Let $2 \leq q < \infty$, then the operator $\mathcal{M}_{q,\mu}$ defined in (2.7) is relatively bounded with respect to $\mathcal{L}_{q,\kappa}$, where the relative bound can be taken arbitrarily small.*

Proof. For every $u \in \mathrm{dom}(\mathcal{L}_{q,\kappa}) \subset \mathrm{W}^{1,2}_D$,

$$\|\mathcal{M}_{q,\mu} u\|_{\mathrm{W}^{-1,q}_D} \leq \|\mu\|_{L^\infty(\Gamma)^{3\times 3}} \|\mathrm{tr}_\Gamma u\|_{L^q(\Gamma)^3} \sup_{\|\varphi\|_{\mathrm{W}^{1,q'}}=1} \|\mathrm{tr}_\Gamma \varphi\|_{L^{q'}(\Gamma)^3}.$$

For every $\varepsilon > 0$, by the trace theorem [17, Thm. 3.6], interpolation and Young's inequality,

$$\begin{aligned} & \|u\|_{L^q(\Gamma)^3} \\ \leq \; & C(\|u_+\|_{1,q} + \|u_-\|_{1,q})^{1-1/q}(\|u_+\|_q + \|u_-\|_q)^{1/q} + C\|u_\Gamma\|_{L^q(\Gamma)} \\ \leq \; & C(\|u_+\|_{1,q} + \|u_-\|_{1,q})^{1-1/2q}(\|u_+\|_{-1,q} + \|u_-\|_{-1,q})^{1/2q} \\ & + C\, \|u_\Gamma\|^{1/2}_{1,q} \|u_\Gamma\|^{1/2}_{-1,q} \\ \leq \; & \varepsilon\|u\|_{\mathrm{W}^{1,q}} + c(\varepsilon)\|u\|_{\mathrm{W}^{-1,q}}. \end{aligned}$$

□

From this lemma and Proposition 3.3, Theorem 3.1 follows for

$$\mathcal{A}_{q,\kappa,\mu} = \mathcal{L}_{q,\kappa} + \mathcal{M}_{q,\mu}$$

by an abstract perturbation argument for maximal regularity shown in [19].

4. Local well-posedness for the quasilinear equations

The aim of this section is to use Theorem 3.1 and solve the quasilinear problem (2.8) under suitable assumptions on the coefficient functions k and m.

4.1. **Preliminary assumptions and results.** For $\alpha, \alpha' > 0$, we define the Hölder space

$$\mathrm{C}^{\alpha,\alpha'} := C^{\alpha}(\Omega_+) \times C^{\alpha}(\Omega_-) \times C^{\alpha'}(\Gamma),$$

where $C^{\alpha}(\omega)$ are the uniform Hölder spaces of exponent $\alpha \geq 0$ with $C^0(\omega) = C(\overline{\omega})$ for bounded domains ω.

We will make the following assumptions on the coefficient functions k, m as Carathéodory functions:

Assumption 4.1. *Let k and m be given as in (2.3) and (2.4).*

(1) Uniformly in $r \in \mathbb{R}^3$,$k(\cdot, r)$ satisfies Assumption 2.5 and $m(\cdot, r)$ satisfies Assumption 2.6.

(2) The functions $r \in \mathbb{R}^3 \mapsto m(y, r)$ and $r \in \mathbb{R}^3 \to k(x, r)$ are Lipschitz uniformly in $y \in \Gamma, x \in \Omega$.

(3) There exists a $q > d$ such that for all $u \in \mathrm{C}$,

$$\mathrm{dom}(\mathcal{L}_{q,k(\cdot,u(\cdot))}) = \mathrm{W}^{1,q}_D.$$

We discuss conditions on the validity of Assumption 4.13 in some more detail.

For $d = 2$, under Assumption 4.1(1), Assumption 4.1(3) is always satisfied, cf. [14]. If $d = 3$, Assumption (3) is satisfied if all $k_+(\cdot, u_+(\cdot))$ and $k_-(\cdot, u_-(\cdot))$ satisfy Assumption 2.3, providing a uniform value of $q > 3$. As an example, this is guaranteed to hold if $k_+$is uniformly continuous in $x \in \Omega_+$ and Ω_+, N_+ and D_+ are suitable, so that $k_+(\cdot, u(\cdot))$ satisfies Assumption 2.3 for *some* $u \in C^0(\Omega_+)$ and the same holds for k_- on Ω_-.

4.2. **Main result.** By the previous assumptions, $\mathrm{dom}(\mathcal{A}_{q,k,m}) = \mathrm{W}^{1,q}_D$ for all $u \in \mathrm{C}$ and suitable $q > d$. This implies the embedding

$$\mathrm{dom}(\mathcal{A}_{q,k,m}) \hookrightarrow \mathrm{C}^{\alpha,\alpha'} \hookrightarrow \mathrm{C} \tag{4.1}$$

for suitable $\alpha, \alpha' > 0$. The following lemma shows that Hölder regularity carries over from $\mathrm{dom}(\mathcal{A}_{q,k,m})$ to the time trace space

$$X_{s,q} = (\mathrm{W}^{1,q}_D, \mathrm{W}^{-1,q}_D)_{1-\frac{1}{s},s}$$

for sufficiently large s. In particular, this ensures $u \in C([0,T]; \mathrm{C})$ for $u \in \mathcal{H}^s(J_T; \mathrm{W}^{1,q}_D, \mathrm{W}^{-1,q}_D)$.

Lemma 4.2. *Let $d < q < q^*$, $\frac{2q}{q-d} < s < \infty$ and $T^* > T > 0$. Then for all $u \in \mathcal{H}^s(J_{T^*}; \mathrm{W}^{1,q}_D, \mathrm{W}^{-1,q}_D)$, we obtain*

$$u \in C([0,T]; \mathrm{C}^{\beta,\beta'}),$$

where $\beta = 1 - 2/s - d/q, \beta' = 1 - 2/s - (d-1)/q > 0$.

Proof. By [2, Section III.4.10], it holds that $W^{1,s}(\mathrm{W}^{-1,q}_D) \cap L^s(\mathrm{W}^{1,q}_D) \hookrightarrow C([0,T]; X_{s,q})$. By [10], interpolation and embedding results for $W^{1,q}_\delta(\omega)$ work "as usual" for Sobolev spaces and their duals. In particular,

$$(W^{1,q}_\delta(\omega), W^{-1,q'}_\delta(\omega))_{1-1/s,s} \subset B^{1-2/s}_{q,s}(\omega)$$

by [10] and [22, p. 186, (14)], where we do not give a precise definition of $B^r_{q,s}(\omega)$ here, but note that the definition may be based on the extension and retraction results also shown in [10] and we obtain $X_{s,q} \subset B^r_{q,s}(\Omega_+) \times B^r_{q,s}(\Omega_-) \times B^r_{q,s}(\Gamma)$. It only remains to use the embeddings $B^{d/q+\beta}_{q,s}(\Omega_\pm) \hookrightarrow C^\beta(\Omega_\pm)$, $B^{(d-1)/q+\beta'}_{q,s}(\Gamma) \hookrightarrow C^{\beta'}(\Gamma)$, cf. [22, 2.8.1]. □

Theorem 4.3. *Let Assumption 4.1 and the assumptions in Section 2 hold and let $d < q < q^*$, $s > \frac{2q}{q-d}$ and $u_0 \in X_{s,q}$ and $f \in L^s(0,\infty; \mathrm{W}_D^{-1,q})$. Then there is a unique solution*

$$u \in \mathcal{H}^s(J_T; \mathrm{W}_D^{1,q}; \mathrm{W}_D^{-1,q})$$

to (2.8) which depends continuously on u_0 and f in their respective norms. The regularity of u holds for all times T smaller than a maximal time $T_{\max} \in \mathbb{R}_+ \cup \{\infty\}$ with

$$\|u(t)\|_{X_{s,q}} \to \infty \quad \text{as } t \to T_{\max}. \tag{4.2}$$

Proof. We use the criteria in [4, Thm 2.1], cf. also [21], to show that the result follows from Theorem 3.1, i.e. it essentially remains to verify that for all $u \in \mathcal{H}^s(J_T; \mathrm{W}_D^{1,q}, \mathrm{W}_D^{-1,q})$,

$$u(t) \mapsto \mathcal{A}_{q,k,m} \in \mathcal{L}(\mathrm{W}_D^{1,q}; \mathrm{W}_D^{-1,q})$$

is a well-defined Lipschitz continuous map. Given any

$$u \in \mathcal{H}^s(J_T; \mathrm{W}_D^{1,q}, \mathrm{W}_D^{-1,q})$$

with $u(0) = u_0$, Assumptions 4.1 on k, m and Lemma 4.2 guarantee that for all $t \geq 0$, the operators $\mathcal{A}_{q,k,m}$ have maximal $L^s(J_T; \mathrm{W}_D^{-1,q})$-regularity by Theorem 3.1 and that $\mathrm{dom}(\mathcal{A}_{q,k,m}) = \mathrm{W}_D^{1,q}$ by Lemma 3.4. Moreover, by definition, Assumption 4.1(2) and 4.2, given $u_1, u_2 \in X_{s,q}$,

$$\|\mathcal{A}_{q,k_1,m_1} - \mathcal{A}_{q,k_2,m_2}\|_{\mathcal{L}(\mathrm{W}_D^{1,q}, \mathrm{W}_D^{-1,q})} \leq C\|u_1 - u_2\|_{\mathrm{L}^\infty} \leq C\|u_1 - u_2\|_{X_{s,q}},$$

with $C > 0$ independent of $u_1, u_2 \in X_{s,q}$. This proves Theorem 4.3, where the characterization of $T_{\max}$ and continuous dependence on the data follow as in [21]. □

Remark 4.4. *Also following [4, Thm 2.1] or [21], we can include the semilinear case of f suitably depending on u.*

Remark 4.5. *The main result relies on the fact that $\mathcal{A}_{q,k,m}$ has maximal L^s-regularity. By the result in [19], lower-order perturbations do not affect this property and may be included. In particular, the non-negativity of $\mathcal{M}_{q,m}$ was not explicitly used for this result and much more general transmission conditions may be considered. On the other hand, the particular choice of $\mathcal{M}_{q,m}$ ensures the gradient structure of the isolated system, cf. [20] and it is crucial for determining its long-time behaviour, cf. [6].*

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