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Fractional elliptic problems with critical growth in the whole of $\mathbb{R}^n$

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Abstract

We study the following nonlinear and nonlocal elliptic equation in $\mathbb{R}^n$

$$(-\Delta)^s u = \varepsilon\, h\, u^q + u^p \quad \text{in } \mathbb{R}^n,$$

where $s \in (0,1)$, $n > 2s$, $\varepsilon > 0$ is a small parameter, $p = \frac{n+2s}{n-2s}$, $q \in (0,1)$, and $h \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. The problem has a variational structure, and this allows us to find a positive solution by looking at critical points of a suitable energy functional. In particular, in this paper, we find a local minimum and a mountain pass solution of this functional. One of the crucial ingredient is a Concentration-Compactness principle.

Some difficulties arise from the nonlocal structure of the problem and from the fact that we deal with an equation in the whole of $\mathbb{R}^n$ (and this causes lack of compactness of some embeddings). We overcome these difficulties by looking at an equivalent extended problem.

CHAPTER 1

Introduction

1.1. Fractional critical problems

A classical topic in nonlinear analysis is the study of the existence and multiplicity of solutions for nonlinear equations. Typically, the equations under consideration possess some kind of ellipticity, which translates into additional regularity and compactness properties at a functional level.

In this framework, an important distinction arises between "subcritical" problems and "critical" ones. Namely, in subcritical problems the exponent of the nonlinearity is smaller than the Sobolev exponent, and this gives that any reasonable bound on the Sobolev seminorm implies convergence in some L^p-spaces: for instance, minimizing sequences, or Palais-Smale sequences, usually possess naturally a uniform bound in the Sobolev seminorm, and this endows the subcritical problems with additional compactness properties that lead to existence results via purely functional analytic methods.

The situation of critical problems is different, since in this case the exponent of the nonlinearity coincides with the Sobolev exponent and therefore no additional L^p-convergence may be obtained only from bounds in Sobolev spaces. As a matter of fact, many critical problems do not possess any solution. Nevertheless, as discovered in [**12**], critical problems do possess solutions once suitable lower order perturbations are taken into account. Roughly speaking, these perturbations are capable to modify the geometry of the energy functional associated to the problem, avoiding the critical points to "drift towards infinity", at least at some appropriate energy level. Of course, to make such argument work, a careful analysis of the variational structure of the problem is in order, joint with an appropriate use of topological methods that detect the existence of the critical points of the functional via its geometric features.

Recently, a great attention has also been devoted to problems driven by nonlocal operators. In this case, the "classical" ellipticity (usually modeled by the Laplace operator) is replaced by a "long range, ferromagnetic interaction", which penalizes the oscillation of the function (roughly speaking, the function is seen as a state parameter, whose value at a given point of the space influences the values at all the other points, in order to avoid sharp fluctuations). The ellipticity condition in this cases reduces to the validity of some sort of maximum principle, and the prototype nonlocal operators studied in the literature are the fractional powers of the Laplacian.

In this paper we deal with the problem

$$(-\Delta)^s u = \varepsilon\, h\, u^q + u^p \quad \text{in } \mathbb{R}^n, \tag{1.1.1}$$

where $s \in (0,1)$ and $(-\Delta)^s$ is the so-called fractional Laplacian, that is

$$(-\Delta)^s u(x) = c_{n,s}\, PV \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy \quad \text{for } x \in \mathbb{R}^n, \tag{1.1.2}$$

where $c_{n,s}$ is a suitable positive constant (see [**21, 37**] for the definition and the basic properties). Moreover, $n > 2s$, $\varepsilon > 0$ is a small parameter, $0 < q < 1$, $p = \frac{n+2s}{n-2s}$ is the fractional critical Sobolev exponent, and h satisfies

$$h \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \tag{1.1.3}$$

$$\text{and} \qquad \text{there exists a ball } B \subset \mathbb{R}^n \text{ such that } \inf_B h > 0. \tag{1.1.4}$$

Notice that condition (1.1.3) implies that

$$h \in L^r(\mathbb{R}^n) \qquad \text{for any } r \in (1, +\infty). \tag{1.1.5}$$

In the classical case, that is when $s = 1$ and the fractional Laplacian boils down to the classical Laplacian, there is an intense literature regarding this type of problems, see [**1, 2, 3, 4, 5, 6, 9, 12, 17, 18, 20, 30, 31**], and references therein. See also [**26**], where the concave term appears for the first time.

In a nonlocal setting, in [**8**] the authors deal with problem (1.1.1) in a bounded domain with Dirichlet boundary condition. Problems related to ours have also been studied in [**33, 34, 36**].

Furthermore, in [**22**], we find solutions to (1.1.1) by considering the equation as a perturbation of the problem with the fractional critical Sobolev exponent, that is

$$(-\Delta)^s u = u^{\frac{n+2s}{n-2s}} \qquad \text{in } \mathbb{R}^n.$$

Indeed, it is known that the minimizers of the Sobolev embedding in $\mathbb{R}^n$ are unique, up to translations and positive dilations, and non-degenerate (see [**22**] and references therein). In particular, in [**22**] we used perturbation methods and Lyapunov-Schmidt reduction to find solutions to (1.1.1) that bifurcate from these minimizers. The explicit form of the fractional Sobolev minimizers was found in [**19**] and it is given by

$$z(x) := \frac{c_\star}{\left(1+|x|^2\right)^{\frac{n-2s}{2}}}, \tag{1.1.6}$$

for a suitable $c_\star > 0$, depending on n and s.

In order to state our main results, we introduce some notation. We set

$$[u]^2_{\dot H^s(\mathbb{R}^n)} := c_{n,s} \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy,$$

and we define the space $\dot{H}^s(\mathbb{R}^n)$ as the completion of the space of smooth and rapidly decreasing functions (the so-called Schwartz space) with respect to the norm $[u]_{\dot{H}^s(\mathbb{R}^n)} + \|u\|_{L^{2^*_s}(\mathbb{R}^n)}$, where

$$2^*_s = \frac{2n}{n-2s}$$

is the fractional critical exponent. Notice that we can also define $\dot{H}^s(\mathbb{R}^n)$ as the space of measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that the norm $[u]_{\dot{H}^s(\mathbb{R}^n)} + \|u\|_{L^{2^*_s}(\mathbb{R}^n)}$ is finite, thanks to a density result, see e.g. [**23**].

Given $f \in L^\beta(\mathbb{R}^n)$, where $\beta := \frac{2n}{n+2s}$, we say that $u \in \dot{H}^s(\mathbb{R}^n)$ is a (weak) solution to $(-\Delta)^s u = f$ in $\mathbb{R}^n$ if

$$c_{n,s} \iint_{\mathbb{R}^{2n}} \frac{\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy = \int_{\mathbb{R}^n} f\,\varphi\,dx,$$

for any $\varphi \in \dot{H}^s(\mathbb{R}^n)$.

Thus, we can state the following

THEOREM 1.1.1. *Let $0 < q < 1$. Suppose that h satisfies* (1.1.3) *and* (1.1.4). *Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ problem* (1.1.1) *has at least two nonnegative solutions. Furthermore, if $h \geqslant 0$ then the solutions are strictly positive.*

This result can be seen as the nonlocal counterpart of Theorem 1.3 in [**3**]. To prove it we will take advantage of the variational structure of the problem. The idea is first to "localize" the problem, via the extension introduced in [**16**] and consider a functional in the extended variables. More precisely, this extended functional will be introduced in the forthcoming formula (1.2.10). It turns out that the existence of critical points of the "extended" functional implies the existence of critical points for the functional on the trace, that is related to problem (1.1.1). The functional in the original variables will be introduced in (1.2.2), see Section 1.2 for the precise framework.

The proof of Theorem 1.1.1 is divided in two parts. More precisely, in the first part we obtain the existence of the first solution, that turns out to be a minimum for the extended functional introduced in the forthcoming Section 1.2. Then in the second part we will find a mountain pass solution, by applying the Mountain Pass Theorem introduced in [**7**].

Notice that in [**22**] we have proved that if h changes sign then there exist two distinct solutions of (1.1.1) that bifurcate from a non trivial critical manifold. Here we also show that there exists a third solution that bifurcates from $u = 0$. This means that when h changes sign, problem (1.1.1) admits at least three different solutions.

Let us point out that, when h changes sign, the solution $u_{1,\varepsilon}$ found in [**22**] can possibly coincide with the mountain pass solution that we construct in this paper. One additional information produced by Theorem 1.1.1 is that this solution is of mountain pass type. It would be an interesting open problem to investigate on the Morse index of the solutions found.

So the main point is to show that the extended functional satisfies a compactness property. In particular, for the existence of the minimum, we will prove that a Palais-Smale condition holds true below a certain level, see Proposition 3.2.1. Then the existence of the minimum will be ensured by the fact that the critical level lies below this threshold.

In order to show the Palais-Smale condition we will use a version of the Concentration-Compactness Principle, see Section 2.2, and for this we will borrow some ideas from [**28, 29**]. Differently from [**8**], here we are dealing with a problem in the whole of $\mathbb{R}^n$, therefore, in order to apply the Concentration-Compactness Principle, we also need to show a tightness property (see Definition 2.2.1). Of course, fractional problems may, in principle, complicate the tightness issues, since the nonlocal interaction could produce (or send) additional mass from (or to) infinity.

As customary in many fractional problems, see [**16**], we will work in an extended space, which reduce the fractional operator to a local (but possibly singular and degenerate) one, confining the nonlocal feature to a boundary reaction problem. This functional simplification (in terms of nonlocality) creates additional difficulties coming from the fact that the extended functional is not homogeneous. Hence, we will have to deal with weighted Sobolev spaces, and so we have to prove some weighted embedding to obtain some convergences needed throughout the paper, see Section 2.1.

A further source of difficulty is that the exponent q in (1.1.1) is below 1, hence the associated energy is not convex and not smooth.

In the forthcoming Section 1.2 we present the variational setting of the problem, both in the original and in the extended variables, and we state the main results of this paper. In particular, we first introduce the material that we are going to use in order to construct the first solution, that is the minimum solution. Then, starting from this minimum, we introduce a translated functional, that we will exploit to obtain the existence of the mountain pass solution.

1.2. An extended problem and statement of the main results

In this section we introduce the variational setting of the problem, we present a related extended problem, and we state the main results of this paper.

Since we are looking for positive solutions, we will consider the following problem:

$$(-\Delta)^s u = \varepsilon h u_+^q + u_+^p \quad \text{in } \mathbb{R}^n. \tag{1.2.1}$$

Hence, we say that $u \in \dot{H}^s(\mathbb{R}^n)$ is a (weak) solution to (1.2.1) if for every $v \in \dot{H}^s(\mathbb{R}^n)$ we have

$$\iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_{\mathbb{R}^n} h(x)u_+^q(x)v(x)\,dx + \int_{\mathbb{R}^n} u_+^p(x)v(x).$$

It turns out that if u is a solution to (1.2.1), then it is nonnegative in $\mathbb{R}^n$ (see the forthcoming Proposition 1.2.3, and also Section 4.2 for the discussion about the positivity of the solutions). Therefore, u is also a solution of (1.1.1).

Notice that problem 1.2.1 has a variational structure. Namely, solutions to (1.2.1) can be found as critical points of the functional $f_\varepsilon : \dot{H}^s(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$\begin{aligned} f_\varepsilon(u) := {} & \frac{c_{n,s}}{2}\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ & - \frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\,u_+^{q+1}(x)\,dx - \frac{1}{p+1}\int_{\mathbb{R}^n} u_+^{p+1}(x)\,dx. \end{aligned} \tag{1.2.2}$$

However, instead of working with this framework derived from Definition 1.1.2 of the Laplacian, we will consider the extended operator given by [**16**], that allows us to transform a nonlocal problem into a local one by adding one variable.

For this, we will denote by $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,+\infty)$. Also, for a point $X \in \mathbb{R}^{n+1}_+$, we will use the notation $X = (x, y)$, with $x \in \mathbb{R}^n$ and $y > 0$.

Moreover, for $x \in \mathbb{R}^n$ and $r > 0$, we will denote by $B_r(x)$ the ball in $\mathbb{R}^n$ centered at x with radius r, i.e.

$$B_r(x) := \{x' \in \mathbb{R}^n : |x - x'| < r\},$$

and, for $X \in \mathbb{R}^{n+1}_+$ and $r > 0$, $B_r^+(X)$ will be the ball in $\mathbb{R}^{n+1}_+$ centered at X with radius r, that is

$$B_r^+(X) := \{X' \in \mathbb{R}^{n+1}_+ : |X - X'| < r\}.$$

Now, given a function $u : \mathbb{R}^n \to \mathbb{R}$, we associate a function U defined in $\mathbb{R}^{n+1}_+$ as

$$U(\cdot, z) = u * P_s(\cdot, z), \qquad \text{where } P_s(x, z) := c_{n,s}\frac{z^{2s}}{(|x|^2+z^2)^{(n+2s)/2}}. \tag{1.2.3}$$

Here $c_{n,s}$ is a normalizing constant depending on n and s.

Set also $a := 1 - 2s$, and

$$[U]_a^* := \left(\kappa_s \int_{\mathbb{R}^{n+1}_+} y^a |\nabla U|^2\,dX\right)^{1/2}, \tag{1.2.4}$$

where κ_s is a normalization constant. We define the spaces

$$\dot{H}^s_a(\mathbb{R}^{n+1}_+) := \overline{C_0^\infty(\mathbb{R}^{n+1}_+)}^{[\cdot]_a^*},$$

and

$$\dot{H}^s_a(\mathbb{R}^{n+1}_+) := \{U := \tilde{U}|_{\mathbb{R}^{n+1}_+} \text{ s.t. } \tilde{U} \in \dot{H}^s_a(\mathbb{R}^{n+1}), \quad \tilde{U}(x,y) = \tilde{U}(x,-y) \text{ a.e. in } \mathbb{R}^n \times \mathbb{R}\}, \tag{1.2.5}$$

endowed with the norm

$$[U]_a := \left(\kappa_s \int_{\mathbb{R}^{n+1}_+} y^a |\nabla U|^2 \, dX \right)^{1/2}.$$

From now on, for simplicity, we will neglect the dimensional constants $c_{n,s}$ and κ_s. It is known that finding a solution $u \in \dot{H}^s(\mathbb{R}^n)$ to a problem

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^n$$

is equivalent to find $U \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$ that solves the local problem

$$\begin{cases} \operatorname{div}(y^a \nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\lim\limits_{y\to 0^+} y^a \dfrac{\partial U}{\partial \nu} = f(u). \end{cases}$$

and that this extension is an isometry between $\dot{H}^s(\mathbb{R}^n)$ and $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$ (again up to constants), that is,

$$[U]_a = [u]_{\dot{H}^s(\mathbb{R}^n)}, \tag{1.2.6}$$

where we make the identification $u(x) = U(x,0)$, with $U(x,0)$ understood in the sense of traces (see e.g. [**16**] and [**13**]).

Also, we recall that the Sobolev embedding in $\dot{H}^s(\mathbb{R}^n)$ gives that

$$S \, \|u\|^2_{L^{2^*_s}(\mathbb{R}^n)} \leqslant [u]^2_{\dot{H}^s(\mathbb{R}^n)},$$

where S is the usual constant of the Sobolev embedding of $\dot{H}^s(\mathbb{R}^n)$, see for instance Theorem 6.5 in [**21**]. As a consequence of this and (1.2.6) we have the following result.

Proposition 1.2.1 (Trace inequality). *Let $U \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$. Then,*

$$S \, \|U(\cdot,0)\|^2_{L^{2^*_s}(\mathbb{R}^n)} \leqslant [U]^2_a. \tag{1.2.7}$$

Therefore, we can reformulate problem (1.2.1) as

$$\begin{cases} \operatorname{div}(y^a \nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\lim\limits_{y\to 0^+} y^a \dfrac{\partial U}{\partial \nu} = \varepsilon h u^q_+ + u^p_+. \end{cases} \tag{1.2.8}$$

In particular, we will say that $U \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$ is a (weak) solution of problem (1.2.8) if

$$\int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U, \nabla \varphi \rangle \, dX = \int_{\mathbb{R}^n} (\varepsilon h(x) U^q_+(x,0) + U^p_+(x,0)) \varphi(x,0) \, dx, \tag{1.2.9}$$

for every $\varphi \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$. Likewise, the associated energy functional to the problem (1.2.8) is

$$
\begin{aligned}
\mathcal{F}_\varepsilon(U) :=& \frac{1}{2}\int_{\mathbb{R}^{n+1}_+} y^a|\nabla U|^2\,dX - \frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)U_+^{q+1}(x,0)\,dx \\
&- \frac{1}{p+1}\int_{\mathbb{R}^n} U_+^{p+1}(x,0)\,dx.
\end{aligned} \tag{1.2.10}
$$

Notice that for any $U, V \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$ we have

$$
\begin{aligned}
\langle \mathcal{F}'_\varepsilon(U), V\rangle =& \int_{\mathbb{R}^{n+1}_+} y^a\langle\nabla U, \nabla V\rangle\,dX \\
&- \varepsilon\int_{\mathbb{R}^n} h(x)U_+^q(x,0)\,V(x,0)\,dx - \int_{\mathbb{R}^n} U_+^p(x,0)\,V(x,0)\,dx.
\end{aligned} \tag{1.2.11}
$$

Hence, if U is a critical point of $\mathcal{F}_\varepsilon$, then it is a weak solution of (1.2.8), according to (1.2.9). Therefore $u := U(\cdot,0)$ is a solution to (1.2.1).

Moreover, if U is a minimum of $\mathcal{F}_\varepsilon$, then $u(x) := U(x,0)$ is a minimum of f_ε, thanks to (1.2.6), and so u is a solution to problem (1.2.1).

In this setting, we can prove the existence of a first solution of problem (1.2.8), and consequently of problem (1.2.1).

Theorem 1.2.2. *Let $0 < q < 1$ and suppose that h satisfies* (1.1.3) *and* (1.1.4). *Then, there exists $\varepsilon_0 > 0$ such that $\mathcal{F}_\varepsilon$ has a local minimum $U_\varepsilon \neq 0$, for any $\varepsilon < \varepsilon_0$. Moreover, $U_\varepsilon \to 0$ in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$ when $\varepsilon \to 0$.*

We now set $u_\varepsilon := U_\varepsilon(\cdot,0)$, where U_ε is the local minimum of $\mathcal{F}_\varepsilon$ found in Theorem 1.2.2. Then, according to (1.2.6), u_ε is a local minimum for f_ε, and so a solution to (1.2.1).

Notice that, again by (1.2.6),

$$
[u_\varepsilon]_{\dot{H}^s(\mathbb{R}^n)} = [U_\varepsilon]_a \to 0 \qquad \text{as } \varepsilon \to 0.
$$

In this sense, the solution u_ε obtained by minimizing the functional bifurcates from the solution $u = 0$.

Furthermore, u_ε is nonnegative, and thus u_ε is a true solution of (1.1.1). Indeed, we can prove the following:

Proposition 1.2.3. *Let $u \in \dot{H}^s(\mathbb{R}^n)$ be a nontrivial solution of* (1.2.1) *and let U be its extension, according to* (1.2.3). *Then, $u \geqslant 0$ and $U > 0$.*

Proof. We set $u_-(x) := -\min\{u(x), 0\}$, namely u_- is the negative part of u, and we claim that

$$
u_- = 0. \tag{1.2.12}
$$

For this, we multiply (1.2.1) by u_- and we integrate over $\mathbb{R}^n$: we obtain

$$
\int_{\mathbb{R}^n} (-\Delta)^s u\, u_-\,dx = \int_{\mathbb{R}^n} \left(\varepsilon h(x)u_+^q + u_+^p\right) u_-\,dx = 0.
$$

Hence, by an integration by parts we get

$$\iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(u_-(x)-u_-(y))}{|x-y|^{n+2s}}\,dx\,dy = 0. \tag{1.2.13}$$

Now, we observe that

$$(u(x)-u(y))(u_-(x)-u_-(y)) \geqslant |u_-(x)-u_-(y)|^2. \tag{1.2.14}$$

Indeed, if both $u(x) \geqslant 0$ and $u(y) \geqslant 0$ and if both $u(x) \leqslant 0$ and $u(y) \leqslant 0$ then the claim trivially follows. Therefore, we suppose that $u(x) \geqslant 0$ and $u(y) \leqslant 0$ (the symmetric situation is analogous). In this case

$$\begin{aligned}(u(x)-u(y))(u_-(x)-u_-(y)) &= -(u(x)-u(y))u(y)\\ &= -u(x)u(y)+|u(y)|^2 \geqslant |u(y)|^2 = |u_-(x)-u_-(y)|^2,\end{aligned}$$

which implies (1.2.14).

From (1.2.13) and (1.2.14), we obtain that

$$\iint_{\mathbb{R}^{2n}} \frac{|u_-(x)-u_-(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant 0,$$

and this implies (1.2.12), since $u \in \dot{H}^s(\mathbb{R}^n)$. Hence $u \geqslant 0$. This implies that $U > 0$, being a convolution of u with a positive kernel. □

We can also prove the existence of a second solution of problem (1.2.8), and consequently of problem (1.1.1).

Theorem 1.2.4. *Let $0 < q < 1$ and suppose that h satisfies* (1.1.3) *and* (1.1.4). *Then, there exists $\varepsilon_0 > 0$ such that $\mathcal{F}_\varepsilon$ has a mountain pass solution $\overline{U}_\varepsilon \neq 0$, for any $\varepsilon < \varepsilon_0$.*

To prove the existence of a second solution of problem (1.2.8) we consider a translated functional. Namely, we let U_ε be the local minimum of the functional (1.2.10) (already found in Theorem 1.2.2), and we consider the functional $\mathcal{I}_\varepsilon : \dot{H}^s_a(\mathbb{R}^{n+1}_+) \to \mathbb{R}$ defined as

$$\mathcal{I}_\varepsilon(U) = \frac{1}{2}\int_{\mathbb{R}^{n+1}_+} y^a|\nabla U|^2\,dX - \int_{\mathbb{R}^n} G(x,U(x,0))\,dx, \tag{1.2.15}$$

where

$$G(x,U) := \int_0^U g(x,t)\,dt,$$

and

$$g(x,t) := \begin{cases}\varepsilon h(x)((U_\varepsilon+t)^q - U_\varepsilon^q) + (U_\varepsilon+t)^p - U_\varepsilon^p, & \text{if } t \geqslant 0,\\ 0 \ \text{ if } t<0.\end{cases} \tag{1.2.16}$$

Explicitly,

$$\begin{aligned}G(x,U) = {}& \frac{\varepsilon\, h(x)}{q+1}\left((U_\varepsilon+U_+)^{q+1} - U_\varepsilon^{q+1}\right) - \varepsilon\, h(x)\, U_\varepsilon^q U_+\\ &+ \frac{1}{p+1}\left((U_\varepsilon+U_+)^{p+1} - U_\varepsilon^{p+1}\right) - U_\varepsilon^p U_+.\end{aligned} \tag{1.2.17}$$

Moreover, for any $U, V \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$, we have that

$$\langle \mathcal{J}'_\varepsilon(U), V\rangle = \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U, \nabla V\rangle \, dX - \int_{\mathbb{R}^n} g(x, U(x,0))\, V(x,0)\, dx. \tag{1.2.18}$$

Notice that a critical point of (1.2.15) is a solution to the following problem

$$\begin{cases} \operatorname{div}(y^a \nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\lim\limits_{y\to 0^+} y^a \dfrac{\partial U}{\partial \nu} = g(x, U(x,0)). \end{cases} \tag{1.2.19}$$

One can prove that a solution U to this problem is positive, as stated in the forthcoming Lemma 1.2.5. Therefore, $\overline{U} := U_\varepsilon + U > 0$, thanks to Proposition 1.2.3. Also, $\overline{U}$ will be the second solution of (1.2.9), and so $\overline{u} := \overline{U}(\cdot, 0)$ will be the second solution to (1.1.1).

LEMMA 1.2.5. *Let $U \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$, $U \neq 0$, be a solution to* (1.2.19). *Then U is positive.*

PROOF. We first observe that, if U is a solution to (1.2.19), then $u := U(\cdot, 0)$ is a solution of

$$(-\Delta)^s u = g(x,u) \quad \text{in } \mathbb{R}^n. \tag{1.2.20}$$

Now, we set $u_-(x) := -\min\{u(x), 0\}$, namely u_- is the negative part of u, and we claim that

$$u_- = 0. \tag{1.2.21}$$

For this, we multiply (1.2.20) by u_- and we integrate over $\mathbb{R}^n$: we obtain

$$\int_{\mathbb{R}^n} (-\Delta)^s u\, u_-\, dx = \int_{\mathbb{R}^n} g(x,u)\, u_-\, dx.$$

Recalling the definition of g in (1.2.16), we have that

$$\int_{\mathbb{R}^n} g(x,u)\, u_-\, dx = 0.$$

Hence, by an integration by parts we get

$$\iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(u_-(x)-u_-(y))}{|x-y|^{n+2s}}\, dx\, dy = 0. \tag{1.2.22}$$

Now, we observe that

$$(u(x)-u(y))(u_-(x)-u_-(y)) \geqslant |u_-(x)-u_-(y)|^2. \tag{1.2.23}$$

Indeed, if both $u(x) \geqslant 0$ and $u(y) \geqslant 0$ and if both $u(x) \leqslant 0$ and $u(y) \leqslant 0$ then the claim trivially follows. Therefore, we suppose that $u(x) \geqslant 0$ and $u(y) \leqslant 0$ (the symmetric situation is analogous). In this case

$$\begin{aligned} (u(x)-u(y))(u_-(x)-u_-(y)) &= -(u(x)-u(y))u(y) \\ &= -u(x)u(y) + |u(y)|^2 \geqslant |u(y)|^2 = |u_-(x)-u_-(y)|^2, \end{aligned}$$

which implies (1.2.23).

From (1.2.22) and (1.2.23), we obtain that

$$\iint_{\mathbb{R}^{2n}} \frac{|u_-(x) - u_-(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \leqslant 0,$$

and this implies (1.2.21), since $u \in H^s(\mathbb{R}^n)$. Hence $u \geqslant 0$. This implies that $U > 0$, being a convolution of u with a positive kernel. □

The next sections will be devoted to the proof of Theorems 1.2.2 and 1.2.4.

More precisely, for this goal some preliminary material from functional analysis is needed. The main analytic tools are contained in Chapter 2. Namely, since we will work with an extended functional (that contains also terms with weighted Sobolev norms), we devote Section 2.1 to show some weighted Sobolev embeddings and Section 2.2 to prove a suitable Concentration-Compactness Principle.

The existence of a minimal solution is discussed in Chapter 3. In particular, in Section 3.1 we deal with some convergence results, that we need in the subsequent Section 3.2, where we show that under a given level the Palais-Smale condition holds true for the extended functional. Then, in Section 3.3 we complete the proof of Theorem 1.2.2.

In Chapter 4, we discuss some regularity and positivity issues about the solution that we constructed. More precisely, in Section 4.1 we show some regularity results, and in Section 4.2 we prove the positivity of the solutions to (1.1.1), making use of a strong maximum principle for weak solutions.

Then, in Chapter 5 we deal with the existence of the mountain pass solution. We first show, in Section 5.1, that the translated functional introduced in Section 1.2 has $U = 0$ as a local minimum (notice that this is a consequence of the fact that we are translating the original functional with respect to its local minimum).

Sections 5.2 and 5.3 are devoted to some preliminary results. We will exploit these basic lemmata in the subsequent Section 5.4, where we prove that the abovementioned translated functional satisfies a Palais-Smale condition.

In Section 5.5 we estimate the minimax value along a suitable path (roughly speaking, the linear path constructed along a suitably cut-off minimizer of the fractional Sobolev inequality). This estimate is needed to exploit the Mountain Pass Theorem via the convergence of the Palais-Smale sequences at appropriate energy levels. With this, in Section 5.6 we finish the proof of Theorem 1.2.4.

CHAPTER 2

Functional analytical setting

2.1. Weighted Sobolev embeddings

For any $r \in (1, +\infty)$, we denote by $L^r(\mathbb{R}^{n+1}_+, y^a)$ the weighted Lebesgue space, endowed with the norm

$$\|U\|_{L^r(\mathbb{R}^{n+1}_+, y^a)} := \left(\int_{\mathbb{R}^{n+1}_+} y^a |U|^r \, dX \right)^{1/r}.$$

The following result shows that $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$ is continuously embedded in $L^{2\gamma}(\mathbb{R}^{n+1}_+, y^a)$.

Proposition 2.1.1 (Sobolev embedding). *There exists a constant $\hat{S} > 0$ such that for all $U \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$ it holds*

$$\left(\int_{\mathbb{R}^{n+1}_+} y^a |U|^{2\gamma} \, dX \right)^{1/2\gamma} \leqslant \hat{S} \left(\int_{\mathbb{R}^{n+1}_+} y^a |\nabla U|^2 \, dX \right)^{1/2}, \tag{2.1.1}$$

where $\gamma = 1 + \dfrac{2}{n-2s}$.

Proof. Let us first prove the result for $U \in C^\infty_0(\mathbb{R}^{n+1})$. If $s \in (0, 1/2)$, inequality (2.1.1) is easily deduced from Theorem 1.3 of [**14**]. By a density argument, we obtain that inequality (2.1.1) holds for any function $U \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$. Indeed, if $U \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$, then there exists a sequence of functions $\{U_k\}_{k\in\mathbb{N}} \in C^\infty_0(\mathbb{R}^{n+1})$ such that U_k converges to some $\tilde{U}$ in $\dot{H}^s_a(\mathbb{R}^{n+1})$ as $k \to \infty$, where $U = \tilde{U}$ in $\mathbb{R}^{n+1}_+$ and $\tilde{U}$ is even with respect to the $(n+1)$-th variable. Hence, for any k, we have

$$\begin{aligned} \left(\int_{\mathbb{R}^{n+1}_+} y^a |U_k|^{2\gamma} \, dX \right)^{1/2\gamma} &\leqslant \hat{S} \left(\int_{\mathbb{R}^{n+1}_+} y^a |\nabla U_k|^2 \, dX \right)^{1/2} \\ &\leqslant \hat{S} \left(\int_{\mathbb{R}^{n+1}} y^a |\nabla U_k|^2 \, dX \right)^{1/2}. \end{aligned} \tag{2.1.2}$$

Moreover, given two functions of the approximating sequence, there holds

$$\left(\int_{\mathbb{R}^{n+1}_+} y^a |U_k - U_m|^{2\gamma} \, dX \right)^{1/2\gamma} \leqslant \hat{S} \left(\int_{\mathbb{R}^{n+1}} y^a |\nabla (U_k - U_m)|^2 \, dX \right)^{1/2} \to 0,$$

and thus, up to a subsequence,

$$\begin{aligned} &U_k \to \tilde U \text{ in } L^{2\gamma}(\mathbb{R}^{n+1}_+, y^a),\\ &U_k \to \tilde U \text{ a.e. in } \mathbb{R}^{n+1}_+. \end{aligned}$$

Hence, by Fatou's Lemma and (2.1.2) we get

$$\begin{aligned} \left(\int_{\mathbb{R}^{n+1}_+} y^a |U|^{2\gamma}\, dX\right)^{1/2\gamma} &= \left(\int_{\mathbb{R}^{n+1}_+} y^a |\tilde U|^{2\gamma}\, dX\right)^{1/2\gamma}\\ &\leqslant \lim_{k\to+\infty} \left(\int_{\mathbb{R}^{n+1}_+} y^a |U_k|^{2\gamma}\, dX\right)^{1/2\gamma}\\ &\leqslant \lim_{k\to+\infty} \hat S \left(\int_{\mathbb{R}^{n+1}_+} y^a |\nabla U_k|^{2}\, dX\right)^{1/2}\\ &\leqslant \lim_{k\to+\infty} \hat S \left(\int_{\mathbb{R}^{n+1}} |y|^a |\nabla U_k|^{2}\, dX\right)^{1/2}\\ &= \hat S \left(\int_{\mathbb{R}^{n+1}} |y|^a |\nabla \tilde U|^{2}\, dX\right)^{1/2}\\ &= \hat S \left(2\int_{\mathbb{R}^{n+1}_+} y^a |\nabla U|^{2}\, dX\right)^{1/2}, \end{aligned} \tag{2.1.3}$$

which shows that Proposition 2.1.1 holds true for any function $U \in \dot H^s_a(\mathbb{R}^{n+1}_+)$, up to renaming $\hat S$.

On the other hand, the case $s = \frac{1}{2}$ corresponds to the classical Sobolev inequality, so we can now concentrate on the range $s \in (1/2, 1)$, that can be derived from Theorem 1.2 of [**25**] by arguing as follows.

Let us denote

$$w(X) := |y|^a.$$

Thus, it can be checked that

$$w \in A_q \text{ for every } q \in (2-2s, 2], \tag{2.1.4}$$

where A_q denotes the class of Muckenhoupt weights of order q. Since in particular $w \in A_2$, by Theorem 1.2 of [**25**], we know that there exist positive constants C and δ such that for all balls $B_R \subset \mathbb{R}^{n+1}$, all $u \in C^\infty_0(B_R)$ and all γ satisfying $1 \leqslant \gamma \leqslant \frac{n+1}{n} + \delta$, one has

$$\left(\frac{1}{w(B_R)}\int_{B_R} |U|^{2\gamma} w\, dX\right)^{1/2\gamma} \leqslant CR \left(\frac{1}{w(B_R)}\int_{B_R} |\nabla U|^{2} w\, dX\right)^{1/2}. \tag{2.1.5}$$

In particular, it yields

$$\begin{aligned} w(B_R) &= \int_{B_R} |y|^a \, dX = \int_{|x|^2+y^2 \leqslant R^2} |y|^a \, dx \, dy \\ &= \int_{|\eta|^2+\xi^2 \leqslant 1} |\xi|^a R^{n+1+a} \, d\xi \, d\eta = CR^{a+n+1} = CR^{2-2s+n}, \end{aligned}$$

with C independent of R. Thus,

$$Rw(B_R)^{\frac{1}{2\gamma}-\frac{1}{2}} = CR^{1+\frac{(1-\gamma)(2-2s+n)}{2\gamma}},$$

and plugging this into (2.1.5) we get

$$\left(\int_{B_R} |U|^{2\gamma} w \, dX\right)^{1/2\gamma} \leqslant CR^{1+\frac{(1-\gamma)(2-2s+n)}{2\gamma}} \left(\int_{B_R} |\nabla U|^2 w \, dX\right)^{1/2},$$

where C is a constant independent of R. In particular, if we set $\gamma = 1+\frac{2}{n-2s}$, then

$$1 + \frac{(1-\gamma)(2-2s+n)}{2\gamma} = 0,$$

and the inequality holds for every ball with the same constant. It remains to check that this value of γ is under the hypotheses of Theorem 1.2 of [**25**], that is, $1 \leqslant \gamma \leqslant \frac{n+1}{n} + \delta$. Keeping track of δ in [**25**], this condition actually becomes

$$1 \leqslant \gamma \leqslant \frac{n+1}{n+1-\frac{2}{q}},$$

for every $q < 2$ such that $w \in A_q$. Thus, by (2.1.4), we can choose any $q \in (2-2s, 2)$. Since γ is clearly greater than 1, we have to prove the upper bound, that is,

$$1 + \frac{2}{n-2s} \leqslant \frac{n+1}{n+1-\frac{2}{q}},$$

but this is equivalent to ask

$$q \leqslant \frac{n-2s+2}{n+1}.$$

Since we can choose q as close as we want to $2-2s$, this inequality will be true whenever

$$2 - 2s < \frac{n-2s+2}{n+1},$$

which holds if and only if $s > \frac{1}{2}$. Summarizing, we have that

$$\left(\int_{B_R} |y|^a |U|^{2\gamma} \, dX\right)^{1/2\gamma} \leqslant C \left(\int_{B_R} |y|^a |\nabla U|^2 \, dX\right)^{1/2}, \tag{2.1.6}$$

where $\gamma = 1+\frac{2}{n-2s}$ and C is a constant independent of the domain. Choosing R large enough, it yields

$$\left(\int_{\mathbb{R}^{n+1}} |y|^a |U|^{2\gamma} \, dX\right)^{1/2\gamma} \leqslant C \left(\int_{\mathbb{R}^{n+1}} |y|^a |\nabla U|^2 \, dX\right)^{1/2}, \tag{2.1.7}$$

Consider now $U \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$. We perform the same density argument as in the case $s \in (0,1/2)$, with the only difference that instead of (2.1.3) we have

$$\begin{aligned}\left(\int_{\mathbb{R}^{n+1}_+} y^a |U|^{2\gamma}\, dX\right)^{1/2\gamma} &= \left(\int_{\mathbb{R}^{n+1}_+} y^a |\tilde{U}|^{2\gamma}\, dX\right)^{1/2\gamma} \\ &\leqslant \lim_{k\to+\infty} \left(\int_{\mathbb{R}^{n+1}_+} y^a |U_k|^{2\gamma}\, dX\right)^{1/2\gamma} \\ &\leqslant \lim_{k\to+\infty} \left(\int_{\mathbb{R}^{n+1}} |y|^a |U_k|^{2\gamma}\, dX\right)^{1/2\gamma} \\ &\leqslant \lim_{k\to+\infty} \hat{S} \left(\int_{\mathbb{R}^{n+1}} |y|^a |\nabla U_k|^{2}\, dX\right)^{1/2} \\ &= \hat{S} \left(\int_{\mathbb{R}^{n+1}} |y|^a |\nabla \tilde{U}|^{2}\, dX\right)^{1/2} \\ &= \hat{S} \left(2\int_{\mathbb{R}^{n+1}_+} y^a |\nabla U|^{2}\, dX\right)^{1/2}. \qquad \square\end{aligned}$$

We also show a compactness result that we will need in the sequel. More precisely, we prove that $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$ is locally compactly embedded in $L^2(\mathbb{R}^{n+1}_+, y^a)$. The precise statement goes as follows:

Lemma 2.1.2. *Let $R > 0$ and let $\mathcal{J}$ be a subset of $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$ such that*

$$\sup_{U\in\mathcal{J}} \int_{\mathbb{R}^{n+1}_+} y^a |\nabla U|^2\, dX < +\infty.$$

Then $\mathcal{J}$ is precompact in $L^2(B^+_R, y^a)$.

Proof. We will prove that $\mathcal{J}$ is totally bounded in $L^2(B^+_R, y^a)$, i.e. for any $\varepsilon > 0$ there exist M and $U_1, \dots, U_M \in L^2(B^+_R, y^a)$ such that for any $U \in \mathcal{J}$ there exists $i \in \{1, \dots, M\}$ such that

$$\|U_i - U\|_{L^2(B^+_R, y^a)} \leqslant \varepsilon. \tag{2.1.8}$$

For this, we fix $\varepsilon > 0$, we set

$$A := \sup_{U\in\mathcal{J}} \int_{\mathbb{R}^{n+1}_+} y^a |\nabla U|^2\, dX < +\infty \tag{2.1.9}$$

and we let

$$\eta := \left[\frac{\varepsilon^2}{2\hat{S}^2 A}\left(\frac{a+1}{|B_R|}\right)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{\gamma}{(\gamma-1)(a+1)}}, \tag{2.1.10}$$

where γ and $\hat{S}$ are the constants introduced in the statement of Proposition 2.1.1, and $|B_R|$ is the Lebesgue measure of the ball B_R in $\mathbb{R}^n$.

Now, notice that

$$\text{if } X \in B_R^+ \cap \{y \geqslant \eta\} \text{ then } y^a \geqslant \min\{\eta^a, R^a\}. \tag{2.1.11}$$

Indeed, if $a \geqslant 0$ (that is $s \in (0, 1/2]$) then $y^a \geqslant \eta^a$, while if $a < 0$ (that is $s \in (1/2, 1)$) then we use that $y \leqslant R$, and so $y^a \geqslant R^a$, thus proving (2.1.11). Analogously, one can prove that

$$\text{if } X \in B_R^+ \cap \{y \geqslant \eta\} \text{ then } y^a \leqslant \max\{\eta^a, R^a\}. \tag{2.1.12}$$

Therefore, using (2.1.11), we have that, for any $U \in \mathcal{J}$,

$$A \geqslant \int_{B_R^+ \cap \{y \geqslant \eta\}} y^a |\nabla U|^2 \, dX \geqslant \min\{\eta^a, R^a\} \int_{B_R^+ \cap \{y \geqslant \eta\}} |\nabla U|^2 \, dX.$$

Hence,

$$\int_{B_R^+ \cap \{y \geqslant \eta\}} |\nabla U|^2 \, dX < +\infty$$

for any $U \in \mathcal{J}$. So by the Rellich-Kondrachov theorem we have that $\mathcal{J}$ is totally bounded in $L^2(B_R^+ \cap \{y \geqslant \eta\})$. Namely, there exist $\tilde{U}_1, \dots, \tilde{U}_M \in L^2(B_R^+ \cap \{y \geqslant \eta\})$ such that for any $U \in \mathcal{J}$ there exists $i \in \{1, \dots, M\}$ such that

$$\|U_i - U\|_{L^2(B_R^+ \cap \{y \geqslant \eta\})} \leqslant \frac{\varepsilon^2}{2 \max\{\eta^a, R^a\}}. \tag{2.1.13}$$

Now for any $i \in \{1, \dots, M\}$ we set

$$U_i := \begin{cases} \tilde{U}_i & \text{if } y \geqslant \eta, \\ 0 & \text{if } y < \eta. \end{cases}$$

Notice that $U_i \in L^2(B_R^+, y^a)$ for any $i \in \{1, \dots, M\}$. Indeed, fixed $i \in \{1, \dots, M\}$, we have that

$$\begin{aligned} \int_{B_R^+} y^a |U_i|^2 \, dX &= \int_{B_R^+ \cap \{y < \eta\}} y^a |U_i|^2 \, dX + \int_{B_R^+ \cap \{y \geqslant \eta\}} y^a |U_i|^2 \, dX \\ &= 0 + \int_{B_R^+ \cap \{y \geqslant \eta\}} y^a |\tilde{U}_i|^2 \, dX \\ &\leqslant \max\{\eta^a, R^a\} \int_{B_R^+ \cap \{y \geqslant \eta\}} |U_i|^2 \, dX < +\infty, \end{aligned}$$

thanks to (2.1.12) and the fact that $\tilde{U}_i \in L^2(B_R^+ \cap \{y \geqslant \eta\})$ for any $i \in \{1, \dots, M\}$.

It remains to show (2.1.8). For this, we first observe that

$$\|U_i - U\|^2_{L^2(B_R^+, y^a)} = \int_{B_R^+ \cap \{y < \eta\}} y^a |U|^2 \, dX + \int_{B_R^+ \cap \{y \geqslant \eta\}} y^a |\tilde{U}_i - U|^2 \, dX. \tag{2.1.14}$$

Using the Hölder inequality with exponents γ and $\frac{\gamma}{\gamma-1}$ and Proposition 2.1.1 and recalling (2.1.9) and (2.1.10), we obtain that

$$\begin{aligned}\int_{B_R^+\cap\{y<\eta\}} y^a|U|^2\,dX &= \int_{B_R^+\cap\{y<\eta\}} y^{\frac{a}{\gamma}}|U|^2 y^{\frac{a(\gamma-1)}{\gamma}}\,dX\\ &\leqslant \left(\int_{B_R^+\cap\{y<\eta\}} y^a|U|^{2\gamma}\,dX\right)^{\frac{1}{\gamma}}\left(\int_{B_R^+\cap\{y<\eta\}} y^a\,dX\right)^{\frac{\gamma-1}{\gamma}}\\ &\leqslant \hat{S}^2\int_{\mathbb{R}^{n+1}_+} y^a|\nabla U|^2\,dX\left(\frac{|B_R|}{a+1}\right)^{\frac{\gamma-1}{\gamma}}\eta^{\frac{(a+1)(\gamma-1)}{\gamma}}\\ &\leqslant \hat{S}^2 A\left(\frac{|B_R|}{a+1}\right)^{\frac{\gamma-1}{\gamma}}\eta^{\frac{(a+1)(\gamma-1)}{\gamma}}\\ &= \frac{\varepsilon^2}{2}.\end{aligned}$$

Moreover, making use of (2.1.12) and (2.1.13), we have that

$$\int_{B_R^+\cap\{y\geqslant\eta\}} y^a|\tilde{U}_i-U|^2\,dX \leqslant \max\{\eta^a,R^a\}\int_{B_R^+\cap\{y\geqslant\eta\}} |\tilde{U}_i-U|^2\,dX \leqslant \frac{\varepsilon^2}{2}.$$

Plugging the last two formulas into (2.1.14), we get

$$\|U_i-U\|^2_{L^2(B_R^+,y^a)} \leqslant \frac{\varepsilon^2}{2}+\frac{\varepsilon^2}{2}=\varepsilon^2,$$

which implies (2.1.8) and thus concludes the proof of Lemma 2.1.2. □

2.2. A Concentration-Compactness Principle

In this section we show a Concentration-Compactness Principle, in the spirit of the original result proved by P. L. Lions in [**28**] and [**29**]. In particular, we want to adapt Lemma 2.3 of [**29**]. See also, [**3, 32**], where this principle was proved for different problems.

For this, we recall the following definitions:

Definition 2.2.1. We say that a sequence $\{U_k\}_{k\in\mathbb{N}}$ is tight if for every $\eta>0$ there exists $\rho>0$ such that

$$\int_{\mathbb{R}^{n+1}_+\setminus B_\rho^+} y^a|\nabla U_k|^2\,dX \leqslant \eta \quad \text{for any } k.$$

Definition 2.2.2. Let $\{\mu_k\}_{k\in\mathbb{N}}$ be a sequence of measures on a topological space X. We say that μ_k converges to μ in X if and only if

$$\lim_{k\to+\infty}\int_X \varphi\,d\mu_k = \int_X \varphi\,d\mu,$$

for every $\varphi\in C_0(X)$.

This definition is standard, see for instance Definition 1.1.2 in [**24**]. In particular, we will consider measures on $\mathbb{R}^n$ and $\mathbb{R}^{n+1}_+$.

PROPOSITION 2.2.3 (Concentration-Compactness Principle). *Let* $\{U_k\}_{k\in\mathbb{N}}$ *be a bounded tight sequence in* $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$*, such that* U_k *converges weakly to* U *in* $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$*. Let* μ,ν *be two nonnegative measures on* $\mathbb{R}^{n+1}_+$ *and* $\mathbb{R}^n$ *respectively and such that*

$$\lim_{k\to+\infty} y^a|\nabla U_k|^2=\mu \tag{2.2.1}$$

and

$$\lim_{k\to+\infty} |U_k(x,0)|^{2^*_s}=\nu \tag{2.2.2}$$

in the sense of Definition 2.2.2.

Then, there exist an at most countable set J *and three families* $\{x_j\}_{j\in J}\in\mathbb{R}^n$*,* $\{\nu_j\}_{j\in J}$*,* $\{\mu_j\}_{j\in J}$*,* $\nu_j,\mu_j\geqslant 0$ *such that*

(i) $\nu=|U(x,0)|^{2^*_s}+\sum_{j\in J}\nu_j\delta_{x_j}$,

(ii) $\mu\geqslant y^a|\nabla U|^2+\sum_{j\in J}\mu_j\delta_{(x_j,0)}$,

(iii) $\mu_j\geqslant S\nu_j^{2/2^*_s}$ *for all* $j\in J$.

PROOF. We first suppose that $U\equiv 0$. We claim that

$$\left(\int_{\mathbb{R}^n}|\varphi(x,0)|^{2^*_s}\,d\nu\right)^{2/2^*_s}\leqslant C\int_{\mathbb{R}^{n+1}_+}\varphi^2\,d\mu,\qquad \text{for any } \varphi\in C^\infty_0(\mathbb{R}^{n+1}_+), \tag{2.2.3}$$

for some $C>0$. For this, let $\varphi\in C^\infty_0(\mathbb{R}^{n+1}_+)$ and $K:=\mathrm{supp}(\varphi)$. By Proposition 1.2.1, we have that

$$\left(\int_{\mathbb{R}^n}|(\varphi U_k)(x,0)|^{2^*_s}\,dx\right)^{2/2^*_s}\leqslant C\int_{\mathbb{R}^{n+1}_+}y^a|\nabla(\varphi U_k)|^2\,dX, \tag{2.2.4}$$

for a suitable positive constant C. By (2.2.2), we deduce

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}|(\varphi U_k)(x,0)|^{2^*_s}\,dx=\int_{\mathbb{R}^n}|\varphi(x,0)|^{2^*_s}\,d\nu. \tag{2.2.5}$$

On the other hand, the right hand side in (2.2.4) can be written as

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+}y^a|\nabla(\varphi U_k)|^2\,dX \;&=\; \int_{\mathbb{R}^{n+1}_+}y^a\varphi^2|\nabla U_k|^2\,dX+\int_{\mathbb{R}^{n+1}_+}y^aU_k^2|\nabla\varphi|^2\,dX\\ &\quad +2\int_{\mathbb{R}^{n+1}_+}y^a\varphi\,U_k\,\langle\nabla\varphi,\nabla U_k\rangle\,dX.\end{aligned} \tag{2.2.6}$$

Now we observe that

$$[U_k]_a\leqslant C \tag{2.2.7}$$

for some $C>0$ independent of k, and so, by Lemma 2.1.2, we have that, up to a subsequence,

$$U_k \text{ converges to } U=0 \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}_+,y^a) \text{ as } k\to+\infty. \tag{2.2.8}$$

Therefore,

$$\lim_{k\to+\infty}\int_{\mathbb{R}^{n+1}_+} y^a U_k^2|\nabla\varphi|^2\,dX \leqslant C\lim_{k\to+\infty}\int_K y^a U_k^2\,dX = 0. \tag{2.2.9}$$

Also, by the Hölder inequality and (2.2.7),

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{n+1}_+} y^a\varphi\, U_k\,\langle\nabla\varphi,\nabla U_k\rangle\,dX\right| \\
\leqslant\;& \left(\int_{\mathbb{R}^{n+1}_+} y^a|\varphi|^2\,|\nabla U_k|^2\,dX\right)^{1/2}\left(\int_{\mathbb{R}^{n+1}_+} y^a|\nabla\varphi|^2\,|U_k|^2\,dX\right)^{1/2} \\
\leqslant\;& C\left(\int_{\mathbb{R}^{n+1}_+} y^a|\nabla U_k|^2\,dX\right)^{1/2}\left(\int_K y^a|U_k|^2\,dX\right)^{1/2} \\
\leqslant\;& C\left(\int_K y^a|U_k|^2\,dX\right)^{1/2},
\end{aligned}$$

where C may change from line to line. Hence, from (2.2.8) we have that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^{n+1}_+} y^a\varphi\, U_k\,\langle\nabla\varphi,\nabla U_k\rangle\,dX = 0.$$

Thus, plugging this and (2.2.9) into (2.2.6), and using (2.2.1), we obtain

$$\lim_{k\to+\infty}\int_{\mathbb{R}^{n+1}_+} y^a|\nabla(\varphi U_k)|^2\,dX = \int_{\mathbb{R}^{n+1}_+}\varphi^2\,d\mu.$$

Therefore, taking the limit in (2.2.4) as $k\to+\infty$, and using (2.2.5), we get

$$\left(\int_{\mathbb{R}^n}|\varphi(x,0)|^{2^*_s}\,d\nu\right)^{2/2^*_s} \leqslant C\int_{\mathbb{R}^{n+1}_+}\varphi^2\,d\mu, \qquad \text{for all } \varphi\in C^\infty_0(\mathbb{R}^{n+1}_+),$$

which shows (2.2.3) in the case $U\equiv 0$.

Let us consider now the case $U\not\equiv 0$. First, we define a function $V_k := U_k - U$, and we observe that $V_k\in\dot H^s_a(\mathbb{R}^{n+1}_+)$, and

$$V_k \text{ converges weakly to } 0 \text{ in } \dot H^s_a(\mathbb{R}^{n+1}_+) \text{ as } k\to+\infty. \tag{2.2.10}$$

Also, we denote by

$$\tilde\nu := \lim_{k\to\infty}|V_k(x,0)|^{2^*_s} \qquad\text{and}\qquad \tilde\mu := \lim_{k\to\infty} y^a|\nabla V_k|^2, \tag{2.2.11}$$

where both limits are understood in the sense of Definition 2.2.2. Then, we are in the previous case, and so we can apply (2.2.3), that is

$$\left(\int_{\mathbb{R}^n}|\varphi(x,0)|^{2^*_s}\,d\tilde\nu\right)^{2/2^*_s} \leqslant C\int_{\mathbb{R}^{n+1}_+}\varphi^2\,d\tilde\mu, \qquad \text{for all } \varphi\in C^\infty_0(\mathbb{R}^{n+1}_+). \tag{2.2.12}$$

Furthermore, by [**11**], we know that

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}|(\varphi V_k)(x,0)|^{2^*_s}\,dx=\lim_{k\to\infty}\int_{\mathbb{R}^n}|(\varphi U_k)(x,0)|^{2^*_s}\,dx-\int_{\mathbb{R}^n}|(\varphi U)(x,0)|^{2^*_s}\,dx,$$

that is, recalling (2.2.11),

$$\int_{\mathbb{R}^n}|\varphi(x,0)|^{2^*_s}\,d\tilde{\nu}=\int_{\mathbb{R}^n}|\varphi(x,0)|^{2^*_s}\,d\nu-\int_{\mathbb{R}^n}|(\varphi U)(x,0)|^{2^*_s}\,dx.$$

Therefore

$$\nu=\tilde{\nu}+|U(x,0)|^{2^*_s}. \tag{2.2.13}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+}y^a\varphi^2|\nabla U_k|^2\,dX &= \int_{\mathbb{R}^{n+1}_+}y^a\varphi^2|\nabla U|^2\,dX+\int_{\mathbb{R}^{n+1}_+}y^a\varphi^2|\nabla V_k|^2\,dX\\ &\quad +2\int_{\mathbb{R}^{n+1}_+}y^a\varphi^2\langle\nabla V_k,\nabla U\rangle\,dX.\end{aligned}$$

Now we take the limit as $k\to+\infty$, we use (2.2.2), (2.2.11) and (2.2.10), and we obtain

$$\int_{\mathbb{R}^{n+1}_+}\varphi^2\,d\mu=\int_{\mathbb{R}^{n+1}_+}y^a\varphi^2|\nabla U|^2\,dX+\int_{\mathbb{R}^{n+1}_+}\varphi^2\,d\tilde{\mu},$$

i.e.,

$$\mu=\tilde{\mu}+y^a|\nabla U|^2. \tag{2.2.14}$$

Now, since inequality (2.2.12) is satisfied, we can apply Lemma 1.2 in [**28**] to $\tilde{\nu}$ and $\tilde{\mu}$ (see also Lemma 2.3 in [**29**]). Therefore, there exist an at most countable set J and families $\{x_j\}_{j\in J}\in\mathbb{R}^n$, $\{\nu_j\}_{j\in J}$, $\{\mu_j\}_{j\in J}$, with $\nu_j\geqslant 0$ and $\mu_j>0$, such that

$$\tilde{\nu}=\sum_{j\in J}\nu_j\delta_{x_j}\quad\text{ and }\quad\tilde{\mu}\geqslant\sum_{j\in J}\mu_j\delta_{(x_j,0)}.$$

So the proof is finished, thanks to (2.2.13) and (2.2.14). □

CHAPTER 3

Existence of a minimal solution and proof of Theorem 1.2.2

3.1. Some convergence results in view of Theorem 1.2.2

In this section we collect some results about the convergence of sequences of functions in suitable $L^r(\mathbb{R}^n)$ spaces. We will exploit the following lemmata in the forthcoming Section 3.2, see in particular the proof of Proposition 3.2.1.

The first result that we prove is the following:

LEMMA 3.1.1. *Let $v_k \in L^{2^*_s}(\mathbb{R}^n, [0, +\infty))$ be a sequence converging to some v in $L^{2^*_s}(\mathbb{R}^n)$. Then*

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} |v_k^q(x) - v^q(x)|^{\frac{2^*_s}{q}}\, dx = 0 \tag{3.1.1}$$

$$\text{and} \qquad \lim_{k\to+\infty} \int_{\mathbb{R}^n} |v_k^p(x) - v^p(x)|^{\frac{2n}{n+2s}}\, dx = 0. \tag{3.1.2}$$

PROOF. For any $t \geqslant -1$, let

$$f(t) := \frac{|(1+t)^q - 1|}{|t|^q}.$$

We have that $(1+t)^q = 1 + qt + o(t)$ for t close to 0, and therefore

$$f(t) = \frac{|qt + o(t)|}{|t|^q} \to 0$$

as $t \to 0$. In addition, $f(-1) = 1$ and

$$\lim_{t\to+\infty} f(t) = 1.$$

As a consequence, we can define

$$L := \sup_{t\geqslant -1} f(t)$$

and we have that $L \in [1, +\infty)$. Now we show that

$$|a^q - b^q| \leqslant L|a - b|^q \tag{3.1.3}$$

for any a, $b \geqslant 0$. To prove this, we can suppose that $b \neq 0$, otherwise we are done, and we write $t := \frac{a}{b} - 1$. Then we have that

$$|a^q - b^q| = b^q|(1+t)^q - 1| \leqslant Lb^q|t|^q = L|a-b|^q,$$

which proves (3.1.3).

As a consequence of this and of the convergence of v_k, we have that

$$\int_{\mathbb{R}^n} |v_k^q(x) - v^q(x)|^{\frac{2_s^*}{q}}\,dx \leqslant L \int_{\mathbb{R}^n} |v_k(x) - v(x)|^{2_s^*}\,dx \to 0,$$

as $k \to +\infty$, which establishes (3.1.1). Now we prove (3.1.2). For this, given $a \geqslant b \geqslant 0$, we notice that

$$a^p - b^p = p \int_b^a t^{p-1}\,dt \leqslant pa^{p-1}(a-b) \leqslant p(a+b)^{p-1}(a-b).$$

By possibly exchanging the roles of a and b, we conclude that, for any a, $b \geqslant 0$,

$$|a^p - b^p| \leqslant p(a+b)^{p-1}|a-b|.$$

Accordingly, for any a, $b \geqslant 0$,

$$|a^p - b^p|^{\frac{2n}{n+2s}} \leqslant p^{\frac{2n}{n+2s}}(a+b)^{\frac{2n(p-1)}{n+2s}}|a-b|^{\frac{2n}{n+2s}} = p^{\frac{2n}{n+2s}}(a+b)^{\frac{8sn}{(n-2s)(n+2s)}}|a-b|^{\frac{2n}{n+2s}}.$$

We use this and the Hölder inequality with exponents $\frac{n+2s}{4s}$ and $\frac{n+2s}{n-2s}$ to deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \big|v_k^p(x) - v^p(x)\big|^{\frac{2n}{n+2s}}\,dx \\
\leqslant\;& p^{\frac{2n}{n+2s}} \int_{\mathbb{R}^n} \big(v_k(x) + v(x)\big)^{\frac{8sn}{(n-2s)(n+2s)}} \big|v_k(x) - v(x)\big|^{\frac{2n}{n+2s}}\,dx \\
\leqslant\;& p^{\frac{2n}{n+2s}} \left(\int_{\mathbb{R}^n} \big(v_k(x) + v(x)\big)^{\frac{2n}{n-2s}}\,dx\right)^{\frac{4s}{n+2s}} \left(\int_{\mathbb{R}^n} \big|v_k(x) - v(x)\big|^{\frac{2n}{n-2s}}\,dx\right)^{\frac{n-2s}{n+2s}} \\
=\;& p^{\frac{2n}{n+2s}} \|v_k + v\|_{L^{2_s^*}(\mathbb{R}^n)}^{\frac{8sn}{(n-2s)(n+2s)}} \|v_k - v\|_{L^{2_s^*}(\mathbb{R}^n)}^{\frac{2n}{n+2s}}.
\end{aligned}$$

From the convergence of v_k, we have that $\|v_k + v\|_{L^{2_s^*}(\mathbb{R}^n)} \leqslant \|v_k\|_{L^{2_s^*}(\mathbb{R}^n)} + \|v\|_{L^{2_s^*}(\mathbb{R}^n)}$ is bounded uniformly in k, while $\|v_k - v\|_{L^{2_s^*}(\mathbb{R}^n)}$ in infinitesimal as $k \to +\infty$, therefore (3.1.2) now plainly follows. □

Next result shows that we can deduce strong convergence in $L^{2_s^*}(\mathbb{R}^n)$ from the convergence in the sense of Definition 2.2.2.

LEMMA 3.1.2. *Let $v_k \in L^{2_s^*}(\mathbb{R}^n, [0, +\infty))$ be a sequence converging to some v a.e. in $\mathbb{R}^n$. Assume also that $v_k^{2_s^*}$ converges to $v^{2_s^*}$ in the measure sense given in Definition 2.2.2, i.e.*

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} v_k^{2_s^*} \varphi\,dx = \int_{\mathbb{R}^n} v^{2_s^*} \varphi\,dx \tag{3.1.4}$$

for any $\varphi \in C_0(\mathbb{R}^n)$.

In addition, assume that for any $\eta > 0$ there exists $\rho > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_\rho} v_k^{2_s^*}(x)\,dx < \eta. \tag{3.1.5}$$

Then, $v_k \to v$ in $L^{2_s^*}(\mathbb{R}^n, [0, +\infty))$ as $k \to +\infty$.

Proof. First of all, by Fatou's lemma,

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} v_k^{2_s^*}\,dx \geqslant \int_{\mathbb{R}^n} v^{2_s^*}\,dx. \tag{3.1.6}$$

Now we fix $\eta>0$ and we take $\rho=\rho(\eta)$ such that (3.1.5) holds true. Let $\varphi_\rho \in C_0^\infty(B_{\rho+1},[0,1])$ such that $\varphi_\rho=1$ in B_ρ. Then, by (3.1.5)

$$\int_{\mathbb{R}^n} v_k^{2_s^*}\,dx < \int_{B_\rho} v_k^{2_s^*}\,dx+\eta \leqslant \int_{\mathbb{R}^n} v_k^{2_s^*}\varphi_\rho\,dx+\eta.$$

Hence, exploiting (3.1.4),

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} v_k^{2_s^*}\,dx \leqslant \lim_{k\to+\infty}\int_{\mathbb{R}^n} v_k^{2_s^*}\varphi_\rho\,dx+\eta = \int_{\mathbb{R}^n} v^{2_s^*}\varphi_\rho\,dx+\eta.$$

Since $\varphi_\rho\leqslant 1$, this gives that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} v_k^{2_s^*}\,dx \leqslant \int_{\mathbb{R}^n} v^{2_s^*}\,dx+\eta.$$

Since η can be taken arbitrarily small, we obtain that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} v_k^{2_s^*}\,dx \leqslant \int_{\mathbb{R}^n} v^{2_s^*}\,dx.$$

This, together with (3.1.6), proves that

$$\lim_{k\to+\infty}\|v_k\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^n)} = \lim_{k\to+\infty}\int_{\mathbb{R}^n} v_k^{2_s^*}\,dx = \int_{\mathbb{R}^n} v^{2_s^*}\,dx = \|v\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^n)}.$$

This and the Brezis-Lieb lemma (see e.g. formula (1) in [**11**]) implies the desired result. □

3.2. Palais-Smale condition for $\mathcal{F}_\varepsilon$

In this section we show that the functional $\mathcal{F}_\varepsilon$ introduced in (1.2.10) satisfies a Palais-Smale condition. The precise statement is contained in the following proposition.

Proposition 3.2.1 (Palais-Smale condition). *There exists $\bar{C},c_1>0$, depending on h, q, n and s, such that the following statement holds true.*

Let $\{U_k\}_{k\in\mathbb{N}}\subset \dot{H}^s_a(\mathbb{R}^{n+1}_+)$ be a sequence satisfying

(i) $\lim_{k\to+\infty}\mathcal{F}_\varepsilon(U_k)=c_\varepsilon$, *with*

$$c_\varepsilon + c_1\varepsilon^{1/\gamma} + \bar{C}\varepsilon^{\frac{p+1}{p-q}} < \frac{s}{n}S^{\frac{n}{2s}}, \tag{3.2.1}$$

where $\gamma = 1+\frac{2}{n-2s}$ and S is the Sobolev constant appearing in Proposition 1.2.1,

(ii) $\lim_{k\to+\infty}\mathcal{F}'_\varepsilon(U_k)=0$.

Then there exists a subsequence, still denoted by $\{U_k\}_{k\in\mathbb{N}}$, which is strongly convergent in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$ as $k\to+\infty$.

REMARK 3.2.2. The limit in ii) is intended in the following way

$$\begin{aligned}&\lim_{k\to+\infty}\|\mathcal{F}'_\varepsilon(U_k)\|_{\mathcal{L}(\dot H^s_a(\mathbb{R}^{n+1}_+),\dot H^s_a(\mathbb{R}^{n+1}_+))}\\&\quad=\lim_{k\to+\infty}\sup_{\substack{V\in\dot H^s_a(\mathbb{R}^{n+1}_+)\\ [V]_a=1}}\left|\langle\mathcal{F}'_\varepsilon(U_k),V\rangle\right|=0,\end{aligned}$$

where $\mathcal{L}(\dot H^s_a(\mathbb{R}^{n+1}_+),\dot H^s_a(\mathbb{R}^{n+1}_+))$ consists of all the linear functional from $\dot H^s_a(\mathbb{R}^{n+1}_+)$ in $\dot H^s_a(\mathbb{R}^{n+1}_+)$.

First we show that a sequence that satisfies the assumptions in Proposition 3.2.1 is bounded.

LEMMA 3.2.3. *Let* $\varepsilon,\ \kappa>0$. *Let* $\{U_k\}_{k\in\mathbb{N}}\subset\dot H^s_a(\mathbb{R}^{n+1}_+)$ *be a sequence satisfying*

$$|\mathcal{F}_\varepsilon(U_k)|+\sup_{\substack{V\in\dot H^s_a(\mathbb{R}^{n+1}_+)\\ [V]_a=1}}\left|\langle\mathcal{F}'_\varepsilon(U_k),V\rangle\right|\leqslant\kappa, \tag{3.2.2}$$

for any $k\in\mathbb{N}$.

Then there exists $M>0$ *such that*

$$[U_k]_a\leqslant M. \tag{3.2.3}$$

PROOF. If $[U_k]_a=0$ we are done. So we can suppose that $[U_k]_a\neq0$ and use (3.2.2) to obtain

$$|\mathcal{F}_\varepsilon(U_k)|\leqslant\kappa,\ \text{ and }\ \left|\langle\mathcal{F}'_\varepsilon(U_k),U_k/[U_k]_a\rangle\right|\leqslant\kappa.$$

Therefore, we have that

$$\mathcal{F}_\varepsilon(U_k)-\frac{1}{p+1}\langle\mathcal{F}'_\varepsilon(U_k),U_k\rangle\leqslant\kappa\ (1+[U_k]_a)\,. \tag{3.2.4}$$

On the other hand, by the Hölder inequality and Proposition 1.2.1, we obtain

$$\begin{aligned}&\mathcal{F}_\varepsilon(U_k)-\frac{1}{p+1}\langle\mathcal{F}'_\varepsilon(U_k),U_k\rangle\\=&\left(\frac12-\frac{1}{p+1}\right)\int_{\mathbb{R}^{n+1}}y^a|\nabla U_k|^2\,dX-\varepsilon\left(\frac{1}{q+1}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n}h(x)(U_k)_+^{q+1}(x,0)\,dx\\\geqslant&\left(\frac12-\frac{1}{p+1}\right)[U_k]_a^2-\varepsilon C\left(\frac{1}{q+1}-\frac{1}{p+1}\right)\|h\|_{L^{\frac{p+1}{p-q}}(\mathbb{R}^n)}[U_k]_a^{q+1}.\end{aligned}$$

From this and (3.2.4) we conclude that $[U_k]_a$ must be bounded (recall also (1.1.5) and that $q+1<2$). So we obtain the desired result. □

In order to prove that $\mathcal{F}_\varepsilon$ satisfies the Palais-Smale condition, we need to show that the sequence of functions satisfying the hypotheses of Proposition 3.2.1 is tight, according to Definition 2.2.1.

First we make the following preliminary observation:

LEMMA 3.2.4. *Let $m := \frac{p+1}{p-q}$. Then there exists a constant $\bar{C} = \bar{C}(n, s, p, q, \|h\|_{L^m(\mathbb{R}^n)}) > 0$ such that, for any $\alpha > 0$,*

$$\frac{s}{n}\alpha^{p+1} - \varepsilon\left(\frac{1}{q+1} - \frac{1}{2}\right)\|h\|_{L^m(\mathbb{R}^n)}\alpha^{q+1} \geqslant -\bar{C}\varepsilon^{\frac{p+1}{p-q}}.$$

PROOF. Let us define the function $f : (0, +\infty) \to \mathbb{R}$ as

$$f(\alpha) := c_1\alpha^{p+1} - \varepsilon c_2\alpha^{q+1}, \quad c_1 := \frac{s}{n}, \quad c_2 := \left(\frac{1}{q+1} - \frac{1}{2}\right)\|h\|_{L^m(\mathbb{R}^n)}.$$

Differentiating, we obtain that

$$f'(\alpha) = \alpha^q((p+1)c_1\alpha^{p-q} - \varepsilon(q+1)c_2),$$

and thus, f has a local minimum at the point

$$\overline{\alpha} := c_3\varepsilon^{\frac{1}{p-q}}, \quad c_3 = c_3(n, s, p, q, \|h\|_{L^m(\mathbb{R}^n)}) := \left(\frac{c_2(q+1)}{c_1(p+1)}\right)^{\frac{1}{p-q}}.$$

Evaluating f at $\overline{\alpha}$, we obtain that the minimum value that f will reach is

$$f(\overline{\alpha}) = c_4\varepsilon^{\frac{p+1}{p-q}},$$

with c_4 a constant depending on n, s, p, q and $\|h\|_{L^m(\mathbb{R}^n)}$. Therefore, there exists $\bar{C} = \bar{C}(n, s, p, q, \|h\|_{L^m(\mathbb{R}^n)}) > 0$ such that

$$f(\alpha) \geqslant f(\overline{\alpha}) \geqslant -\bar{C}\varepsilon^{\frac{p+1}{p-q}},$$

for any $\alpha > 0$, and this concludes the proof. □

The tightness of the sequence in Proposition 3.2.1 is contained in the following lemma:

LEMMA 3.2.5 (Tightness). *Let $\{U_k\}_{k\in\mathbb{N}} \subset \dot{H}^s_a(\mathbb{R}^{n+1}_+)$ be a sequence satisfying the hypotheses of Proposition 3.2.1.*

Then for all $\eta > 0$ there exists $\rho > 0$ such that for every $k \in \mathbb{N}$ it holds

$$\int_{\mathbb{R}^{n+1}_+\setminus B^+_\rho} y^a|\nabla U_k|^2\,dX + \int_{\mathbb{R}^n\setminus\{B_\rho\cap\{y=0\}\}} (U_k)^{2^*_s}_+(x,0)\,dx < \eta.$$

In particular, the sequence $\{U_k\}_{k\in\mathbb{N}}$ is tight.

PROOF. First we notice that (3.2.2) holds in this case, due to conditions (i) and (ii) in Proposition 3.2.1. Hence, Lemma 3.2.3 gives that the sequence $\{U_k\}_{k\in\mathbb{N}}$ is bounded in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$, that is $[U_k]_a \leqslant M$. Thus,

$$\begin{aligned} U_k &\rightharpoonup U \quad \text{in } \dot{H}^s_a(\mathbb{R}^{n+1}_+) \quad \text{as } k\to+\infty \\ \text{and } U_k &\to U \quad \text{a.e. in } \mathbb{R}^{n+1}_+ \quad \text{as } k\to+\infty. \end{aligned} \tag{3.2.5}$$

Now, we proceed by contradiction. Suppose that there exists $\eta_0 > 0$ such that for all $\rho > 0$ there exists $k = k(\rho) \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^{n+1}_+\setminus B^+_\rho} y^a|\nabla U_k|^2\,dX + \int_{\mathbb{R}^n\setminus\{B_\rho\cap\{y=0\}\}} (U_k)^{2^*_s}_+(x,0)\,dx \geqslant \eta_0. \tag{3.2.6}$$

We observe that

$$k \to +\infty \quad \text{as } \rho \to +\infty. \tag{3.2.7}$$

Indeed, let us take a sequence $\{\rho_i\}_{i\in\mathbb{N}}$ such that $\rho_i \to +\infty$ as $i \to +\infty$, and suppose that $k_i := k(\rho_i)$ given by (3.2.6) is a bounded sequence. That is, the set $F := \{k_i :\ i \in \mathbb{N}\}$ is a finite set of integers.

Hence, there exists an integer $k^\star$ so that we can extract a subsequence $\{k_{i_j}\}_{j\in\mathbb{N}}$ satisfying $k_{i_j} = k^\star$ for any $j \in \mathbb{N}$. Therefore,

$$\int_{\mathbb{R}^{n+1}_+\setminus B^+_{\rho_{i_j}}} y^a|\nabla U_{k^\star}|^2\, dX + \int_{\mathbb{R}^n\setminus\{B_{\rho_{i_j}}\cap\{y=0\}\}} (U_{k^\star})_+^{2^*_s}(x,0)\, dx \geqslant \eta_0, \tag{3.2.8}$$

for any $j \in \mathbb{N}$.

But on the other hand, since $U_{k^\star}$ belongs to $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$ (and so $U_{k^\star}(\cdot, 0) \in L^{2^*_s}(\mathbb{R}^n)$ thanks to Proposition 1.2.1), for j large enough there holds

$$\int_{\mathbb{R}^{n+1}_+\setminus B^+_{\rho_{i_j}}} y^a|\nabla U_{k^\star}|^2\, dX + \int_{\mathbb{R}^n\setminus\{B_{\rho_{i_j}}\cap\{y=0\}\}} (U_{k^\star})_+^{2^*_s}(x,0)\, dx \leqslant \frac{\eta_0}{2},$$

which is a contradiction with (3.2.8). This shows (3.2.7).

Now, since U given in (3.2.5) belongs to $\in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$, by Propositions 2.1.1 and 1.2.1, we have that, for a fixed $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^{n+1}_+\setminus B^+_{r_\varepsilon}} y^a|\nabla U|^2\, dX + \int_{\mathbb{R}^{n+1}_+\setminus B^+_{r_\varepsilon}} y^a|U|^{2\gamma}\, dX + \int_{\mathbb{R}^n\setminus\{B_{r_\varepsilon}\cap\{y=0\}\}} |U(x,0)|^{2^*_s}\, dx < \varepsilon.$$

Notice that

$$r_\varepsilon \to +\infty \text{ as } \varepsilon \to 0. \tag{3.2.9}$$

Moreover, by (3.2.3) and again by Propositions 2.1.1 and 1.2.1, we obtain that there exists $\tilde{M} > 0$ such that

$$\int_{\mathbb{R}^{n+1}_+} y^a|\nabla U_k|^2\, dX + \int_{\mathbb{R}^{n+1}_+} y^a|U_k|^{2\gamma}\, dX + \int_{\mathbb{R}^n} |U_k(x,0)|^{2^*_s}\, dx \leqslant \tilde{M}. \tag{3.2.10}$$

Now let $j_\varepsilon \in \mathbb{N}$ be the integer part of $\frac{\tilde{M}}{\varepsilon}$. Notice that j_ε tends to $+\infty$ as ε tends to 0. We also set

$$I_l := \{(x, y) \in \mathbb{R}^{n+1}_+ : r + l \leqslant |(x, y)| \leqslant r + (l + 1)\},\ l = 0, 1, \cdots , j_\varepsilon.$$

Thus, from (3.2.10) we get

$$\begin{aligned}(j_\varepsilon + 1)\varepsilon &\geqslant \frac{\tilde{M}}{\varepsilon}\varepsilon \\ &\geqslant \sum_{l=0}^{j_\varepsilon} \left(\int_{I_l} y^a|\nabla U_k|^2\, dX + \int_{I_l} y^a|U_k|^{2\gamma}\, dX + \int_{I_l\cap\{y=0\}} |U_k(x,0)|^{2^*_s}\, dx \right).\end{aligned}$$

This implies that there exists $\bar{l} \in \{0, 1, \cdots, j_\varepsilon\}$ such that, up to a subsequence,

$$\int_{I_{\bar{l}}} y^a |\nabla U_k|^2 \, dX + \int_{I_{\bar{l}}} y^a |U_k|^{2\gamma} \, dX + \int_{I_{\bar{l}} \cap \{y=0\}} |U_k(x,0)|^{2_s^*} \, dx \leqslant \varepsilon. \tag{3.2.11}$$

Now we take a cut-off function $\chi \in C_0^\infty(\mathbb{R}_+^{n+1}, [0,1])$, such that

$$\chi(x,y) = \begin{cases} 1, & |(x,y)| \leqslant r + \bar{l} \\ 0, & |(x,y)| \geqslant r + (\bar{l}+1), \end{cases} \tag{3.2.12}$$

and

$$|\nabla \chi| \leqslant 2. \tag{3.2.13}$$

We also define

$$V_k := \chi U_k \quad \text{and} \quad W_k := (1-\chi) U_k. \tag{3.2.14}$$

We estimate

$$\begin{aligned} &|\langle \mathcal{F}'_\varepsilon(U_k) - \mathcal{F}'_\varepsilon(V_k), V_k \rangle| \\ &= \bigg| \int_{\mathbb{R}_+^{n+1}} y^a \langle \nabla U_k, \nabla V_k \rangle \, dX - \varepsilon \int_{\mathbb{R}^n} h(x) (U_k)_+^q(x,0) \, V_k(x,0) \, dx \\ &\quad - \int_{\mathbb{R}^n} (U_k)_+^p(x,0) \, V_k(x,0) \, dx - \int_{\mathbb{R}_+^{n+1}} y^a \langle \nabla V_k, \nabla V_k \rangle \, dX \\ &\quad + \varepsilon \int_{\mathbb{R}^n} h(x) (V_k)_+^{q+1}(x,0) \, dx + \int_{\mathbb{R}^n} (V_k)_+^{p+1}(x,0) \, dx \bigg|. \end{aligned} \tag{3.2.15}$$

First, we observe that

$$\begin{aligned} &\left| \int_{\mathbb{R}_+^{n+1}} y^a \langle \nabla U_k, \nabla V_k \rangle \, dX - \int_{\mathbb{R}_+^{n+1}} y^a \langle \nabla V_k, \nabla V_k \rangle \, dX \right| \\ &\leqslant \int_{I_{\bar{l}}} y^a |\nabla U_k|^2 |\chi| |1-\chi| \, dX + \int_{I_{\bar{l}}} y^a |\nabla U_k| |U_k| |\nabla \chi| \, dX \\ &\quad + 2 \int_{I_{\bar{l}}} y^a |U_k| |\nabla U_k| |\nabla \chi| |\chi| \, dX + \int_{I_{\bar{l}}} y^a |U_k|^2 |\nabla \chi|^2 \, dX \\ &=: A_1 + A_2 + A_3 + A_4. \end{aligned} \tag{3.2.16}$$

By (3.2.11), we have that $A_1 \leqslant C\varepsilon$, for some $C > 0$. Furthermore, by the Hölder inequality, (3.2.13) and (3.2.11), we obtain

$$\begin{aligned} A_2 &\leqslant 2 \int_{I_{\bar{l}}} y^a |\nabla U_k| |U_k| \, dX \leqslant 2 \left(\int_{I_{\bar{l}}} y^a |\nabla U_k|^2 \, dX \right)^{1/2} \left(\int_{I_{\bar{l}}} y^a |U_k|^2 \, dX \right)^{1/2} \\ &\leqslant 2\varepsilon^{1/2} \left(\int_{I_{\bar{l}}} y^a |U_k|^{2\gamma} \, dX \right)^{1/2\gamma} \left(\int_{I_{\bar{l}}} y^{(a - \frac{a}{\gamma})m} \, dX \right)^{1/2m}, \end{aligned}$$

where $m = \dfrac{n+2-2s}{2}$. Since $\left(a - \dfrac{a}{\gamma}\right) m = a = (1-2s) > -1$, we have that the second integral is finite, and therefore, for $\varepsilon < 1$,

$$A_2 \leqslant \tilde{C}\varepsilon^{1/2} \left(\int_{I_{\bar{l}}} y^a |U_k|^{2\gamma} \, dX \right)^{1/2\gamma} \leqslant C\varepsilon^{1/2}\varepsilon^{1/2\gamma} \leqslant C\varepsilon^{1/\gamma},$$

where (3.2.11) was used once again. In the same way, we get that $A_3 \leqslant C\varepsilon^{1/\gamma}$. Finally, by (3.2.11),

$$A_4 \leqslant C \left(\int_{I_{\bar{l}}} y^a |U_k|^{2\gamma} \, dX \right)^{1/\gamma} \left(\int_{I_{\bar{l}}} y^{(a-\frac{a}{\gamma})m} \, dX \right)^{1/m} \leqslant C\varepsilon^{1/\gamma}.$$

Using these informations in (3.2.16), we obtain that

$$\left| \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_k, \nabla V_k \rangle \, dX - \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla V_k, \nabla V_k \rangle \, dX \right| \leqslant C\varepsilon^{1/\gamma},$$

up to renaming the constant C.

On the other hand, since $p+1 = 2^*_s$, by (3.2.14) and (3.2.11),

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \left((U_k)^p_+(x,0)\, V_k(x,0) - (V_k)^{p+1}_+(x,0) \right) dx \right| &\leqslant \int_{\mathbb{R}^n} |1-\chi^p||\chi||U_k(x,0)|^{p+1} \, dx \\ &\leqslant C \int_{I_{\bar{l}} \cap \{y=0\}} |U_k(x,0)|^{2^*_s} \, dx \leqslant C\varepsilon. \end{aligned}$$

In the same way, applying the Hölder inequality, one obtains

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} h(x) \left((U_k)^q_+(x,0)\, V_k(x,0) - (V_k)^{q+1}_+(x,0) \right) dx \right| \\ &\leqslant \int_{\mathbb{R}^n} |h(x)|\, |1-\chi^q||\chi||U_k(x,0)|^{q+1} \, dx \\ &\leqslant C\, \|h\|_{L^\infty(\mathbb{R}^n)} \int_{I_{\bar{l}} \cap \{y=0\}} |U_k(x,0)|^{2^*_s} \, dx \leqslant C\varepsilon. \end{aligned}$$

All in all, plugging these observations in (3.2.15), we obtain that

$$|\langle \mathcal{F}'_\varepsilon(U_k) - \mathcal{F}'_\varepsilon(V_k), V_k \rangle| \leqslant C\varepsilon^{1/\gamma}. \tag{3.2.17}$$

Likewise, one can see that

$$|\langle \mathcal{F}'_\varepsilon(U_k) - \mathcal{F}'_\varepsilon(W_k), W_k \rangle| \leqslant C\varepsilon^{1/\gamma}. \tag{3.2.18}$$

Now we claim that

$$|\langle \mathcal{F}'_\varepsilon(V_k), V_k \rangle| \leqslant C\varepsilon^{1/\gamma} + o_k(1), \tag{3.2.19}$$

where $o_k(1)$ denotes (here and in the rest of this paper) a quantity that tends to 0 as k tends to $+\infty$. For this, we first observe that

$$[V_k]_a \leqslant C, \tag{3.2.20}$$

for some $C > 0$. Indeed, recalling (3.2.14) and using (3.2.12) and (3.2.13), we have

$$\begin{aligned}
[V_k]_a^2 &= \int_{\mathbb{R}^{n+1}_+} y^a |\nabla V_k|^2 \, dX \\
&= \int_{\mathbb{R}^{n+1}_+} y^a |\nabla \chi|^2 |U_k|^2 \, dX + \int_{\mathbb{R}^{n+1}_+} y^a \, \chi^2 |\nabla U_k|^2 \, dX + 2 \int_{\mathbb{R}^{n+1}_+} y^a \, \chi \, U_k \, \langle \nabla U_k, \nabla \chi \rangle \, dX \\
&\leqslant 4 \int_{I_{\bar{l}}} y^a |U_k|^2 \, dX + [U_k]_a^2 + C \left(\int_{I_{\bar{l}}} y^a |\nabla U_k|^2 \, dX \right)^{1/2} \left(\int_{I_{\bar{l}}} y^a |U_k|^2 \, dX \right)^{1/2} \\
&\leqslant C \left(\int_{I_{\bar{l}}} y^a |U_k|^{2\gamma} \, dX \right)^{1/\gamma} + [U_k]_a^2 + C \, [U_k]_a \left(\int_{I_{\bar{l}}} y^a |U_k|^{2\gamma} \, dX \right)^{1/2\gamma},
\end{aligned}$$

where the Hölder inequality was used in the last two lines. Hence, from Proposition 2.1.1 and (3.2.3), we obtain (3.2.20).

Now, we notice that

$$|\langle \mathcal{F}'_\varepsilon(V_k), V_k \rangle| \leqslant |\langle \mathcal{F}'_\varepsilon(V_k) - \mathcal{F}'_\varepsilon(U_k), V_k \rangle| + |\langle \mathcal{F}'_\varepsilon(U_k), V_k \rangle| \leqslant C \, \varepsilon^{1/\gamma} + |\langle \mathcal{F}'_\varepsilon(U_k), V_k \rangle|,$$

thanks to (3.2.17). Thus, from (3.2.20) and assumption (ii) in Proposition 3.2.1 we get the desired claim in (3.2.19).

Analogously (but making use of (3.2.18)), one can see that

$$|\langle \mathcal{F}'_\varepsilon(W_k), W_k \rangle| \leqslant C \varepsilon^{1/\gamma} + o_k(1), \tag{3.2.21}$$

From now on, we divide the proof in three main steps: we first show lower bounds for $\mathcal{F}_\varepsilon(V_k)$ and $\mathcal{F}_\varepsilon(W_k)$ (see Step 1 and Step 2, respectively), then in Step 3 we obtain a lower bound for $\mathcal{F}_\varepsilon(U_k)$, which will give a contradiction with the hypotheses on $\mathcal{F}_\varepsilon$, and so the conclusion of Lemma 3.2.5.

Step 1: Lower bound for $\mathcal{F}_\varepsilon(V_k)$. By (3.2.19) we obtain that

$$\mathcal{F}_\varepsilon(V_k) \geqslant \mathcal{F}_\varepsilon(V_k) - \frac{1}{2} \langle \mathcal{F}'_\varepsilon(V_k), V_k \rangle - \frac{1}{2} C \varepsilon^{1/\gamma} + o_k(1). \tag{3.2.22}$$

Using the Hölder inequality, it yields

$$\begin{aligned}
\mathcal{F}_\varepsilon(V_k) - \frac{1}{2} \langle \mathcal{F}'_\varepsilon(V_k), V_k \rangle &= \left(\frac{1}{2} - \frac{1}{p+1} \right) \|(V_k)_+(\cdot, 0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} \\
&\quad - \varepsilon \left(\frac{1}{q+1} - \frac{1}{2} \right) \int_{\mathbb{R}^n} h(x) (V_k)_+^{q+1}(x, 0) \, dx \\
&\geqslant \frac{s}{n} \|(V_k)_+(\cdot, 0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} - \varepsilon \left(\frac{1}{q+1} - \frac{1}{2} \right) \|h\|_{L^m(\mathbb{R}^n)} \|(V_k)_+(\cdot, 0)\|^{q+1}_{L^{p+1}(\mathbb{R}^n)},
\end{aligned}$$

with $m = \dfrac{p+1}{p-q}$ (recall (1.1.5)). Therefore, from Lemma 3.2.4 (applied here with $\alpha := \|(V_k)_+(\cdot, 0)\|_{L^{p+1}(\mathbb{R}^n)}$) we deduce that

$$\mathcal{F}_\varepsilon(V_k) - \frac{1}{2} \langle \mathcal{F}'_\varepsilon(V_k), V_k \rangle \geqslant -\bar{C} \, \varepsilon^{\frac{p+1}{p-q}}.$$

Going back to (3.2.22), this implies that

$$\mathcal{F}_\varepsilon(V_k) \geqslant -c_0\varepsilon^{1/\gamma} - \bar{C}\,\varepsilon^{\frac{p+1}{p-q}} + o_k(1). \tag{3.2.23}$$

Step 2: Lower bound for $\mathcal{F}_\varepsilon(W_k)$. First of all, by the definition of W_k in (3.2.14), Proposition 1.2.1 and (3.2.3), we have that

$$\begin{aligned}\left|\varepsilon\int_{\mathbb{R}^n} h(x)(W_k)_+^{q+1}(x,0)\,dx\right| &\leqslant \varepsilon\,\|h\|_{L^m(\mathbb{R}^n)}\|(W_k)_+(\cdot,0)\|^{q+1}_{L^{2^*_s}(\mathbb{R}^n)}\\ &\leqslant \varepsilon\,C\,\|h\|_{L^m(\mathbb{R}^n)}\|(U_k)_+(\cdot,0)\|^{q+1}_{L^{2^*_s}(\mathbb{R}^n)}\\ &\leqslant \varepsilon\,C\,\|h\|_{L^m(\mathbb{R}^n)}[U_k]_a^{q+1} \leqslant C\varepsilon.\end{aligned} \tag{3.2.24}$$

Thus, from (3.2.18) we get

$$\begin{aligned}&\left|\int_{\mathbb{R}^{n+1}_+} y^a|\nabla W_k|^2\,dX - \int_{\mathbb{R}^n}(W_k)_+^{p+1}(x,0)\,dx\right|\\ &\leqslant \left|\langle \mathcal{F}'_\varepsilon(W_k),W_k\rangle\right| + \left|\varepsilon\int_{\mathbb{R}^n} h(x)(W_k)_+^{q+1}(x,0)\,dx\right|\\ &\leqslant C\varepsilon^{1/\gamma} + o_k(1),\end{aligned} \tag{3.2.25}$$

where (3.2.21) was also used in the last passage. Moreover, notice that $W_k = U_k$ in $\mathbb{R}^{n+1}_+ \setminus B_{r+\bar{l}+1}$ (recall (3.2.12) and (3.2.14)). Hence, using (3.2.6) with $\rho := r+\bar{l}+1$, we get

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+\setminus B^+_{r+\bar{l}+1}} y^a|\nabla W_k|^2\,dX + \int_{\mathbb{R}^n\setminus\{B_{r+\bar{l}+1}\cap\{y=0\}\}}(W_k)_+^{2^*_s}(x,0)\,dx\\ &\quad= \int_{\mathbb{R}^{n+1}_+\setminus B^+_{r+\bar{l}+1}} y^a|\nabla U_k|^2\,dX + \int_{\mathbb{R}^n\setminus\{B_{r+\bar{l}+1}\cap\{y=0\}\}}(U_k)_+^{2^*_s}(x,0)\,dx \geqslant \eta_0,\end{aligned} \tag{3.2.26}$$

for $k = k(\rho)$. We observe that k tends to $+\infty$ as $\varepsilon \to 0$, thanks to (3.2.7) and (3.2.9).

From (3.2.26) we obtain that either

$$\int_{\mathbb{R}^n\setminus\{B_{r+\bar{l}+1}\cap\{y=0\}\}}(W_k)_+^{2^*_s}(x,0)\,dx \geqslant \frac{\eta_0}{2}$$

or

$$\int_{\mathbb{R}^{n+1}_+\setminus B^+_{r+\bar{l}+1}} y^a|\nabla W_k|^2\,dX \geqslant \frac{\eta_0}{2}.$$

In the first case, we get that

$$\int_{\mathbb{R}^n}(W_k)_+^{2^*_s}(x,0)\,dx(x,0) \geqslant \int_{\mathbb{R}^n\setminus\{B_{r+\bar{l}+1}\cap\{y=0\}\}}(W_k)_+^{2^*_s}(x,0)\,dx(x,0) \geqslant \frac{\eta_0}{2}.$$

In the second case, taking ε small (and so k large enough), by (3.2.25) we obtain that

$$\begin{aligned}\int_{\mathbb{R}^n} (W_k)_+^{p+1}(x,0)\,dx &\geqslant \int_{\mathbb{R}^{n+1}_+} y^a |\nabla W_k|^2\,dX - C\varepsilon^{1/\gamma} - o_k(1)\\ &\geqslant \int_{\mathbb{R}^{n+1}_+\setminus B^+_{r+\bar{l}+1}} y^a |\nabla W_k|^2\,dX - C\varepsilon^{1/\gamma} - o_k(1)\\ &> \frac{\eta_0}{4}.\end{aligned}$$

Hence, in both the cases we have that

$$\int_{\mathbb{R}^n} (W_k)_+^{p+1}(x,0)\,dx > \frac{\eta_0}{4}. \tag{3.2.27}$$

Now we define $\psi_k := \alpha_k W_k$, with

$$\alpha_k^{p-1} := \frac{[W_k]_a^2}{\|(W_k)_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}}.$$

Notice that from (3.2.21) we have that

$$\begin{aligned}[W_k]_a^2 &\leqslant \|(W_k)_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} + \left|\varepsilon\int_{\mathbb{R}^n} h(x)(W_k)_+^{q+1}(x,0)\,dx\right| + C\,\varepsilon^{1/\gamma} + o_k(1)\\ &\leqslant \|(W_k)_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} + C\,\varepsilon^{1/\gamma} + o_k(1)\end{aligned}$$

where (3.2.24) was used in the last line. Hence, thanks to (3.2.27), we get that

$$\alpha_k^{p-1} \leqslant 1 + C\varepsilon^{1/2\gamma} + o_k(1). \tag{3.2.28}$$

Also, we notice that for this value of α_k, we have the following:

$$[\psi_k]_a^2 = \alpha_k^2 [W_k]_a^2 = \alpha_k^{p+1}\|(W_k)_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} = \|(\psi_k)_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}.$$

Thus, by (1.2.6) and Proposition 1.2.1, we obtain

$$\begin{aligned}S &\leqslant \frac{[\psi_k(\cdot,0)]^2_{\dot{H}^s(\mathbb{R}^n)}}{\|(\psi_k)_+(\cdot,0)\|^2_{L^{p+1}(\mathbb{R}^n)}} = \frac{[\psi_k]_a^2}{\|(\psi_k)_+(\cdot,0)\|^2_{L^{p+1}(\mathbb{R}^n)}}\\ &= \frac{\|(\psi_k)_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}}{\|(\psi_k)_+(\cdot,0)\|^2_{L^{p+1}(\mathbb{R}^n)}} = \|(\psi_k)_+(\cdot,0)\|^{\frac{4s}{n-2s}}_{L^{p+1}(\mathbb{R}^n)}.\end{aligned}$$

In the last equality we have used the fact that $p+1 = 2^*_s$. Consequently,

$$\|(W_k)_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} = \frac{\|(\psi_k)_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}}{\alpha_k^{p+1}} \geqslant S^{n/2s}\frac{1}{\alpha_k^{p+1}}.$$

This together with (3.2.28) give that

$$\begin{aligned}S^{n/2s} &\leqslant (1 + C\varepsilon^{1/\gamma} + o_k(1))^{\frac{p+1}{p-1}}\|(W_k)_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}\\ &\leqslant \|(W_k)_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} + C\varepsilon^{1/\gamma} + o_k(1).\end{aligned}$$

Also, we observe that

$$\frac{1}{2}-\frac{1}{p+1}=\frac{s}{n}.$$

Hence,

$$\begin{aligned}\mathcal{F}_\varepsilon(W_k)-\frac{1}{2}\langle\mathcal{F}'_\varepsilon(W_k),W_k\rangle &= \frac{s}{n}\|(W_k)_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}\\ &\qquad -\varepsilon\left(\frac{1}{q+1}-\frac{1}{2}\right)\int_{\mathbb{R}^n}h(x)(W_k)_+^{q+1}(x,0)\,dx\\ &\geqslant \frac{s}{n}S^{n/2s}-C\varepsilon^{1/\gamma}+o_k(1).\end{aligned}$$

Finally, using also (3.2.21), we get

$$\mathcal{F}_\varepsilon(W_k)\geqslant\frac{s}{n}S^{n/2s}-C\varepsilon^{1/\gamma}+o_k(1). \tag{3.2.29}$$

Step 3: Lower bound for $\mathcal{F}_\varepsilon(U_k)$. We first observe that, thanks to (3.2.14), we can write

$$U_k=(1-\chi)U_k+\chi U_k=W_k+V_k. \tag{3.2.30}$$

Therefore

$$\begin{aligned}\mathcal{F}_\varepsilon(U_k)=&\,\mathcal{F}_\varepsilon(V_k)+\mathcal{F}_\varepsilon(W_k)+\int_{\mathbb{R}^{n+1}_+}y^a\langle\nabla V_k,\nabla W_k\rangle\,dX\\ &+\frac{1}{p+1}\int_{\mathbb{R}^n}(V_k)_+^{p+1}(x,0)\,dx\\ &+\frac{\varepsilon}{q+1}\int_{\mathbb{R}^n}h(x)(V_k)_+^{q+1}(x,0)\,dx\\ &+\frac{1}{p+1}\int_{\mathbb{R}^n}(W_k)_+^{p+1}(x,0)\,dx\\ &+\frac{\varepsilon}{q+1}\int_{\mathbb{R}^n}h(x)(W_k)_+^{q+1}(x,0)\,dx\\ &-\frac{1}{p+1}\int_{\mathbb{R}^n}(U_k)_+^{p+1}(x,0)\,dx\\ &-\frac{\varepsilon}{q+1}\int_{\mathbb{R}^n}h(x)(U_k)_+^{q+1}(x,0)\,dx.\end{aligned} \tag{3.2.31}$$

On the other hand,

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+}y^a\langle\nabla V_k,\nabla W_k\rangle\,dX\\ &\quad=\frac{1}{2}\int_{\mathbb{R}^{n+1}_+}y^a\langle\nabla U_k-\nabla V_k,\nabla V_k\rangle\,dX+\frac{1}{2}\int_{\mathbb{R}^{n+1}_+}y^a\langle\nabla U_k-\nabla W_k,\nabla W_k\rangle\,dX.\end{aligned}$$

Recall also that, according to (1.2.11),

$$\begin{aligned}
&\langle \mathcal{F}'_\varepsilon(U_k) - \mathcal{F}'_\varepsilon(V_k), V_k\rangle \\
= \; & \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_k - \nabla V_k, \nabla V_k\rangle \, dX \\
& \quad -\varepsilon \int_{\mathbb{R}^n} h(x)(U_k)^q_+(x,0)\, V_k(x,0)\, dx - \int_{\mathbb{R}^n} (U_k(x,0))^p_+ \, V_k(x,0)\, dx \\
& \quad +\varepsilon \int_{\mathbb{R}^n} h(x)(V_k)^{q+1}_+(x,0)\, dx + \int_{\mathbb{R}^n} (V_k)^{p+1}_+(x,0)\, dx,
\end{aligned}$$

and

$$\begin{aligned}
&\langle \mathcal{F}'_\varepsilon(U_k) - \mathcal{F}'_\varepsilon(W_k), W_k\rangle \\
= \; & \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_k - \nabla W_k, \nabla W_k\rangle \, dX \\
& \quad -\varepsilon \int_{\mathbb{R}^n} h(x)(U_k)^q_+(x,0)\, W_k(x,0)\, dx - \int_{\mathbb{R}^n} (U_k)^p_+(x,0)\, W_k(x,0)\, dx \\
& \quad +\varepsilon \int_{\mathbb{R}^n} h(x)(W_k)^{q+1}_+(x,0)\, dx + \int_{\mathbb{R}^n} (W_k)^{p+1}_+(x,0)\, dx.
\end{aligned}$$

Hence, plugging the three formulas above into (3.2.31) we get

$$\begin{aligned}
\mathcal{F}_\varepsilon(U_k) = \; & \mathcal{F}_\varepsilon(V_k) + \mathcal{F}_\varepsilon(W_k) + \frac{1}{2}\langle \mathcal{F}'_\varepsilon(U_k) - \mathcal{F}'_\varepsilon(V_k), V_k\rangle + \frac{1}{2}\langle \mathcal{F}'_\varepsilon(U_k) - \mathcal{F}'_\varepsilon(W_k), W_k\rangle \\
& + \frac{1}{p+1}\int_{\mathbb{R}^n} (V_k)^{p+1}_+(x,0)\, dx + \frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)(V_k)^{q+1}_+(x,0)\, dx \\
& + \frac{1}{p+1}\int_{\mathbb{R}^n} (W_k)^{p+1}_+(x,0)\, dx + \frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)(W_k)^{q+1}_+(x,0)\, dx \\
& - \frac{1}{p+1}\int_{\mathbb{R}^n} (U_k)^{p+1}_+(x,0)\, dx - \frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)(U_k)^{q+1}_+(x,0)\, dx \\
& + \frac{\varepsilon}{2}\int_{\mathbb{R}^n} h(x)(U_k)^q_+(x,0)\, V_k(x,0)\, dx + \frac{1}{2}\int_{\mathbb{R}^n} (U_k)^p_+(x,0)\, V_k(x,0)\, dx \\
& - \frac{\varepsilon}{2}\int_{\mathbb{R}^n} h(x)(V_k)^{q+1}_+(x,0)\, dx - \frac{1}{2}\int_{\mathbb{R}^n} (V_k)^{p+1}_+(x,0)\, dx \\
& + \frac{\varepsilon}{2}\int_{\mathbb{R}^n} h(x)(U_k)^q_+(x,0)\, W_k(x,0)\, dx + \frac{1}{2}\int_{\mathbb{R}^n} (U_k)^p_+(x,0)\, W_k(x,0)\, dx \\
& - \frac{\varepsilon}{2}\int_{\mathbb{R}^n} h(x)(W_k)^{q+1}_+(x,0)\, dx - \frac{1}{2}\int_{\mathbb{R}^n} (W_k)^{p+1}_+(x,0)\, dx.
\end{aligned}$$

Notice that all the integrals with ε in front are bounded. Therefore, using this and (3.2.17) and (3.2.18) we obtain that

(3.2.32)

$$\begin{aligned}\mathcal{F}_\varepsilon(U_k) \geqslant\, & \mathcal{F}_\varepsilon(V_k) + \mathcal{F}_\varepsilon(W_k) \\ & + \frac{1}{p+1}\int_{\mathbb{R}^n} (V_k)_+^{p+1}(x,0)\,dx + \frac{1}{p+1}\int_{\mathbb{R}^n} (W_k)_+^{p+1}(x,0)\,dx \\ & - \frac{1}{p+1}\int_{\mathbb{R}^n} (U_k)_+^{p+1}(x,0)\,dx + \frac{1}{2}\int_{\mathbb{R}^n} (U_k)_+^{p}(x,0)\,V_k(x,0)\,dx \\ & - \frac{1}{2}\int_{\mathbb{R}^n} (V_k)_+^{p+1}(x,0)\,dx + \frac{1}{2}\int_{\mathbb{R}^n} (U_k)_+^{p}(x,0)\,W_k(x,0)\,dx \\ & - \frac{1}{2}\int_{\mathbb{R}^n} (W_k)_+^{p+1}(x,0)\,dx - C\varepsilon^{1/\gamma},\end{aligned}$$

for some positive C. We observe that, thanks to (3.2.30),

$$\begin{aligned} & \int_{\mathbb{R}^n} (U_k)_+^{p}(x,0)\,V_k(x,0)\,dx + \int_{\mathbb{R}^n} (U_k)_+^{p}(x,0)\,W_k(x,0)\,dx \\ = & \int_{\mathbb{R}^n} (U_k)_+^{p}(x,0)\big(V_k(x,0)+W_k(x,0)\big)\,dx \\ = & \int_{\mathbb{R}^n} (U_k)_+^{p+1}(x,0)\,dx.\end{aligned}$$

Therefore, (3.2.32) becomes

$$\begin{aligned}\mathcal{F}_\varepsilon(U_k) \geqslant\, & \mathcal{F}_\varepsilon(V_k) + \mathcal{F}_\varepsilon(W_k) \\ & + \frac{1}{p+1}\int_{\mathbb{R}^n} (V_k)_+^{p+1}(x,0)\,dx + \frac{1}{p+1}\int_{\mathbb{R}^n} (W_k)_+^{p+1}(x,0)\,dx \\ & - \frac{1}{p+1}\int_{\mathbb{R}^n} (U_k)_+^{p+1}(x,0)\,dx - \frac{1}{2}\int_{\mathbb{R}^n} (V_k)_+^{p+1}(x,0)\,dx \\ & + \frac{1}{2}\int_{\mathbb{R}^n} (U_k)_+^{p+1}(x,0)\,dx - \frac{1}{2}\int_{\mathbb{R}^n} (W_k)_+^{p+1}(x,0)^{p+1}\,dx - C\varepsilon^{1/\gamma} \\ =\, & \mathcal{F}_\varepsilon(V_k) + \mathcal{F}_\varepsilon(W_k) \\ & + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n} \Big((U_k)_+^{p+1}(x,0) - (V_k)_+^{p+1}(x,0) - (W_k)_+^{p+1}(x,0)\Big)\,dx - C\varepsilon^{1/\gamma} \\ =\, & \mathcal{F}_\varepsilon(V_k) + \mathcal{F}_\varepsilon(W_k) \\ & + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n} (U_k)_+^{p+1}(x,0)\left(1 - \chi^{p+1}(x,0) - (1-\chi(x,0))^{p+1}\right)\,dx - C\varepsilon^{1/\gamma},\end{aligned}$$

where (3.2.14) was used in the last line. Since $p+1>2$ and

$$1 - \chi^{p+1}(x,0) - (1-\chi(x,0))^{p+1} \geqslant 0 \quad \text{for any } x \in \mathbb{R}^n,$$

this implies that

$$\mathcal{F}_\varepsilon(U_k) \geqslant \mathcal{F}_\varepsilon(V_k) + \mathcal{F}_\varepsilon(W_k) - C\varepsilon^{1/\gamma}.$$

This, (3.2.23) and (3.2.29) imply that

$$\mathcal{F}_\varepsilon(U_k) \geqslant \frac{s}{n} S^{n/2s} - c_1 \varepsilon^{1/\gamma} - \bar{C} \varepsilon^{\frac{p+1}{p-q}} + o_k(1).$$

Hence, taking the limit as $k \to +\infty$ we obtain that

$$c_\varepsilon = \lim_{k\to+\infty} \mathcal{F}_\varepsilon(U_k) \geqslant \frac{s}{n} S^{n/2s} - c_1 \varepsilon^{1/\gamma} - \bar{C} \varepsilon^{\frac{p+1}{p-q}},$$

which is a contradiction with assumption (i) of Proposition 3.2.1. This concludes the proof of Lemma 3.2.5. □

Proof of Proposition 3.2.1. By Lemma 3.2.5, we know that $\{U_k\}_{k\in\mathbb{N}}$ is a tight sequence. Moreover, from Lemma 3.2.3 we have that $[U_k]_a \leqslant M$, for $M > 0$. Hence, also $\{(U_k)_+\}_{k\in\mathbb{N}}$ is a bounded tight sequence in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$. Therefore, there exists $\overline{U} \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$ such that

$$(U_k)_+ \rightharpoonup \overline{U} \qquad \text{in } \dot{H}^s_a(\mathbb{R}^{n+1}_+).$$

Also, we observe that Theorem 1.1.4 in [**24**] implies that there exist two measures on $\mathbb{R}^n$ and $\mathbb{R}^{n+1}_+$, ν and μ respectively, such that $(U_k)_+^{2^*_s}(x,0)$ converges to ν and $y^a|\nabla(U_k)_+|^2$ converges to μ as $k \to +\infty$, according to Definition 2.2.2.

Hence, we can apply Proposition 2.2.3 and we obtain that

(3.2.33)
$$(U_k)_+^{2^*_s}(\cdot,0) \text{ converges to } \nu = \overline{U}^{2^*_s}(\cdot,0) + \textstyle\sum_{j\in J} \nu_j \delta_{x_j} \text{ as } k \to +\infty, \text{ with } \nu_j \geqslant 0,$$

(3.2.34)
$$y^a|\nabla(U_k)_+|^2 \text{ converges to } \mu \geqslant y^a|\nabla\overline{U}|^2 + \textstyle\sum_{j\in J} \mu_j \delta_{(x_j,0)} \text{ as } k \to +\infty, \text{ with } \mu_j \geqslant 0,$$

and

$$\mu_j \geqslant S \nu_j^{2/2^*_s} \quad \text{for all } j \in J, \tag{3.2.35}$$

where J is an at most countable set.

We want to prove that $\mu_j = \nu_j = 0$ for any $j \in J$. For this, we suppose by contradiction that there exists $j \in J$ such that $\mu_j \neq 0$. We denote $X_j := (x_j, 0)$. Moreover, we fix $\delta > 0$ and we consider a cut-off function $\phi_\delta \in C^\infty(\mathbb{R}^{n+1}_+, [0,1])$, defined as

$$\phi_\delta(X) = \begin{cases} 1, & \text{if } X \in B^+_{\delta/2}(X_j), \\ 0, & \text{if } X \in (B^+_\delta(X_j))^c, \end{cases}$$

with $|\nabla\phi_\delta| \leqslant \frac{C}{\delta}$.

We claim that there exists a constant $C > 0$ such that

$$[\phi_\delta\,(U_k)_+]_a \leqslant C. \tag{3.2.36}$$

Indeed, we compute

$$\begin{aligned}
[\phi_\delta\,(U_k)_+]_a^2 &= \int_{\mathbb{R}^{n+1}_+} y^a |\nabla(\phi_\delta(U_k)_+)|^2\,dX\\
&= \int_{B_\delta^+(X_j)} y^a|\nabla\phi_\delta|^2 (U_k)_+^2\,dX + \int_{B_\delta^+(X_j)} y^a\,\phi_\delta^2|\nabla(U_k)_+|^2\,dX\\
&\qquad +2\int_{B_\delta^+(X_j)} y^a\,\phi_\delta\,(U_k)_+\langle\nabla\phi_\delta,\nabla(U_k)_+\rangle\,dX\\
&\leqslant \frac{C^2}{\delta^2}\int_{B_\delta^+(X_j)} y^a|U_k|^2\,dX + \int_{B_\delta^+(X_j)} y^a|\nabla U_k|^2\,dX\\
&\qquad +\frac{2C}{\delta}\int_{B_\delta^+(X_j)} y^a|U_k|\,|\nabla U_k|\,dX\\
&\leqslant C\left(\int_{B_\delta^+(X_j)} y^a|U_k|^{2\gamma}\,dX\right)^{1/\gamma} + \int_{B_\delta^+(X_j)} y^a|\nabla U_k|^2\,dX\\
&\qquad +C\left(\int_{B_\delta^+(X_j)} y^a|U_k|^{2\gamma}\,dX\right)^{1/2\gamma}\left(\int_{B_\delta^+(X_j)} y^a|\nabla U_k|^2\,dX\right)^{1/2}\\
&\leqslant C\,M^2,
\end{aligned}$$

up to renaming C, where we have used Proposition 2.1.1 and Lemma 3.2.3 in the last step. This shows (3.2.36).

Hence, from (1.2.11), (3.2.36) and (ii) in Proposition 3.2.1 we deduce that

(3.2.37)

$$\begin{aligned}
0 &= \lim_{k\to\infty}\langle \mathcal{F}_\varepsilon'(U_k),\phi_\delta(U_k)_+\rangle\\
&= \lim_{k\to\infty}\left(\int_{\mathbb{R}^{n+1}_+} y^a\langle\nabla U_k,\nabla(\phi_\delta(U_k)_+)\rangle\,dX\right.\\
&\qquad\left. -\,\varepsilon\int_{\mathbb{R}^n} h(x)\phi_\delta(x,0)(U_k)_+^{q+1}(x,0)\,dx - \int_{\mathbb{R}^n}\phi_\delta(x,0)(U_k)_+^{p+1}(x,0)\,dx\right)\\
&= \lim_{k\to\infty}\left(\int_{\mathbb{R}^{n+1}_+} y^a|\nabla(U_k)_+|^2\phi_\delta\,dX + \int_{\mathbb{R}^{n+1}_+} y^a\langle\nabla(U_k)_+,\nabla\phi_\delta\rangle(U_k)_+\,dX\right.\\
&\qquad\left. -\,\varepsilon\int_{\mathbb{R}^n} h(x)\phi_\delta(x,0)(U_k)_+^{q+1}(x,0)\,dx - \int_{\mathbb{R}^n}\phi_\delta(x,0)(U_k)_+^{p+1}(x,0)\,dx\right).
\end{aligned}$$

Now we recall that $p+1=2_s^*$, and so, using (3.2.33) and (3.2.34), we have that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}\phi_\delta(x,0)(U_k)_+^{p+1}(x,0)\,dx = \int_{\mathbb{R}^n}\phi_\delta(x,0)\,d\nu \tag{3.2.38}$$

$$\text{and}\quad \lim_{k\to+\infty}\int_{\mathbb{R}^{n+1}_+} y^a|\nabla(U_k)_+|^2\phi_\delta\,dX = \int_{\mathbb{R}^{n+1}_+}\phi_\delta\,d\mu. \tag{3.2.39}$$

Also, we observe that $\mathrm{supp}(\phi_\delta) \subseteq B_\delta^+(X_j)$. Moreover $[(U_k)_+(\cdot,0)]_{\dot{H}^s(\mathbb{R}^n)} = [(U_k)_+]_a \leqslant M$, thanks to (1.2.6) and Lemma 3.2.3. Finally, the Hölder inequality and Proposition 1.2.1 imply that $\|(U_k)_+(\cdot,0)\|_{L^2(B_\delta^+(x_j))} \leqslant C$, for a suitable positive constant C. Hence, we can apply Theorem 7.1 in [**21**] and we obtain that

$$(U_k)_+(\cdot,0) \text{ converges to } \overline{U}(\cdot,0) \text{ strongly in } L^r(B_\delta^+(x_j)) \text{ as } k\to+\infty, \text{ for any } r\in[1,2]. \tag{3.2.40}$$

Therefore,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} h(x)\phi_\delta(x,0)(U_k)_+^{q+1}(x,0)\,dx - \int_{\mathbb{R}^n} h(x)\phi_\delta(x,0)\overline{U}^{q+1}(x,0)\,dx\right| \\
&\quad= \left|\int_{B_\delta^+(X_j)\cap\{y=0\}} h(x)\,\phi_\delta(x,0)((U_k)_+^{q+1}(x,0)-\overline{U}^{q+1}(x,0))\,dx\right| \\
&\quad\leqslant \|h\|_{L^\infty(\mathbb{R}^n)}\left|\int_{B_\delta^+(X_j)\cap\{y=0\}} ((U_k)_+^{q+1}(x,0)-\overline{U}^{q+1}(x,0))\,dx\right|,
\end{aligned}$$

which together with (3.2.40) implies that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} h(x)\phi_\delta(x,0)(U_k)_+^{q+1}(x,0)\,dx = \int_{B_\delta^+(X_j)\cap\{y=0\}} h(x)\phi_\delta(x,0)\,\overline{U}^{q+1}(x,0)\,dx. \tag{3.2.41}$$

Finally, taking the limit as $\delta\to 0$ we get

$$\begin{aligned}
&\lim_{\delta\to0}\lim_{k\to+\infty}\int_{\mathbb{R}^n} h(x)\phi_\delta(x,0)(U_k)_+^{q+1}(x,0)\,dx \\
&\qquad= \lim_{\delta\to0}\int_{B_\delta^+(X_j)\cap\{y=0\}} h(x)\phi_\delta(x,0)\,\overline{U}^{q+1}(x,0)\,dx = 0.
\end{aligned} \tag{3.2.42}$$

Also, by the Hölder inequality and Lemma 3.2.3 we obtain that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{n+1}_+} y^a\langle\nabla(U_k)_+,\nabla\phi_\delta\rangle(U_k)_+\,dX\right| = \left|\int_{B_\delta^+(X_j)} y^a\langle\nabla(U_k)_+,\nabla\phi_\delta\rangle(U_k)_+\,dX\right| \\
&\quad\leqslant \left(\int_{B_\delta^+(X_j)} y^a|\nabla U_k|^2\,dX\right)^{1/2}\left(\int_{B_\delta^+(X_j)} y^a(U_k)_+^2|\nabla\phi_\delta|^2\,dX\right)^{1/2} \\
&\quad\leqslant M\left(\int_{B_\delta^+(X_j)} y^a(U_k)_+^2|\nabla\phi_\delta|^2\,dX\right)^{1/2}.
\end{aligned} \tag{3.2.43}$$

Notice that, since $\{(U_k)_+\}$ is a bounded sequence in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$, using Lemma 2.1.2, we have

$$\lim_{k\to+\infty}\int_{B_\delta^+(X_j)} y^a(U_k)_+^2|\nabla\phi_\delta|^2\,dX = \int_{B_\delta^+(X_j)} y^a\overline{U}^2|\nabla\phi_\delta|^2\,dX. \tag{3.2.44}$$

Moreover, by the Hölder inequality,

$$\int_{B_\delta^+(X_j)} y^a \overline{U}^2 |\nabla\phi_\delta|^2 \, dX \leqslant \left(\int_{B_\delta^+(X_j)} y^a \overline{U}^{2\gamma} \, dX\right)^{1/\gamma} \left(\int_{B_\delta^+(X_j)} y^a |\nabla\phi_\delta|^{2\gamma'} \, dX\right)^{1/\gamma'}, \tag{3.2.45}$$

where

$$\gamma' = \frac{n-2s+2}{2}. \tag{3.2.46}$$

Thus, taking into account that $|\nabla\phi_\delta| \leqslant \frac{C}{\delta}$, we have

$$\begin{aligned}\left(\int_{B_\delta^+(X_j)} y^a |\nabla\phi_\delta|^{2\gamma'} \, dX\right)^{1/\gamma'} &\leqslant \frac{C^2}{\delta^2} \left(\int_{B_\delta^+(X_j)} y^a \, dX\right)^{1/\gamma'} \\ &\leqslant \frac{C^2}{\delta^2} \, \delta^{\frac{n+1+a}{\gamma'}}.\end{aligned}$$

We recall (3.2.46) and that $a = 1-2s$, and we obtain that

$$\frac{n+1+a}{\gamma'} - 2 = 0,$$

and so

$$\left(\int_{B_\delta^+(X_j)} y^a |\nabla\phi_\delta|^{2\gamma'} \, dX\right)^{1/\gamma'} \leqslant C^2.$$

This and (3.2.45) give that

$$\int_{B_\delta^+(X_j)} y^a \overline{U}^2 |\nabla\phi_\delta|^2 \, dX \leqslant C^2 \left(\int_{B_\delta^+(X_j)} y^a \overline{U}^{2\gamma} \, dX\right)^{1/\gamma},$$

Hence

$$\lim_{\delta\to 0} \int_{B_\delta^+(X_j)} y^a \overline{U}^2 |\nabla\phi_\delta|^2 \, dX \leqslant C^2 \lim_{\delta\to 0} \left(\int_{B_\delta^+(X_j)} y^a \overline{U}^{2\gamma} \, dX\right)^{1/\gamma} = 0.$$

From this and (3.2.44) we obtain

$$\lim_{\delta\to 0} \lim_{k\to+\infty} \int_{B_\delta^+(X_j)} y^a (U_k)_+^2 |\nabla\phi_\delta|^2 \, dX = \lim_{\delta\to 0} \int_{B_\delta^+(X_j)} y^a \overline{U}^2 |\nabla\phi_\delta|^2 \, dX = 0. \tag{3.2.47}$$

Using (3.2.38), (3.2.39), (3.2.42) and (3.2.47) in (3.2.37), we obtain that

$$\begin{aligned} 0 &= \lim_{\delta\to 0} \lim_{k\to\infty} \langle \mathcal{F}_\varepsilon'(U_k), \phi_\delta (U_k)_+ \rangle \\ &= \lim_{\delta\to 0} \left(\int_{\mathbb{R}^{n+1}_+} \phi_\delta \, d\mu - \int_{\mathbb{R}^n} \phi_\delta(x,0) \, d\nu \right) \\ &= \lim_{\delta\to 0} \left(\int_{B_\delta^+(X_j)} \phi_\delta \, d\mu - \int_{B_\delta^+(X_j)\cap\{y=0\}} \phi_\delta(x,0) \, d\nu \right) \\ &\geqslant \mu_j - \nu_j, \end{aligned} \tag{3.2.48}$$

thanks to (3.2.33) and (3.2.34). Therefore, recalling (3.2.35), we obtain that

$$\nu_j \geqslant \mu_j \geqslant S\,\nu_j^{2/2^*_s}.$$

Hence, either $\nu_j = \mu_j = 0$ or $\nu_j^{1-2/2^*} \geqslant S$. Since we are assuming that $\mu_j \neq 0$, the first possibility cannot occur, and so, from the second one, we have that

$$\nu_j \geqslant S^{n/2s}. \tag{3.2.49}$$

Now, from Lemma 3.2.3 we know that $[(U_k)_+(\cdot,0)]_a \leqslant M$. Moreover we observe that $q+1 < 2 < 2^*_s$. Hence Proposition 1.2.1 and the compact embedding in Theorem 7.1 in [**21**] imply that

$$\begin{aligned} &\|(U_k)_+(\cdot,0) - \overline{U}(\cdot,0)\|_{L^{2^*_s}(\mathbb{R}^n)} \leqslant 2M \\ \text{and} \quad &(U_k)_+(\cdot,0) \to \overline{U}(\cdot,0) \text{ in } L^{q+1}_{loc}(\mathbb{R}^n) \text{ as } k \to +\infty. \end{aligned}$$

Therefore, recalling (1.1.5) and (1.1.3), and using the Hölder inequality, we obtain

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} h(x)\left((U_k)_+(x,0) - \overline{U}(x,0)\right)^{q+1}\, dx \right| \\ &\leqslant \int_{B_R} |h(x)|\,|(U_k)_+(x,0) - \overline{U}(x,0)|^{q+1}\, dx + \int_{\mathbb{R}^n\setminus B_R} |h(x)|\,|(U_k)_+(x,0) - \overline{U}(x,0)|^{q+1}\, dx \\ &\leqslant \|h\|_{L^\infty(\mathbb{R}^n)} \|((U_k)_+ - \overline{U})(\cdot,0)\|_{L^{q+1}(B_R)} + \|h\|_{L^\alpha(\mathbb{R}^n\setminus B_R)} \|((U_k)_+ - \overline{U})(\cdot,0)\|^{q+1}_{L^{2^*_s}(\mathbb{R}^n)} \\ &\leqslant C\,\|((U_k)_+ - \overline{U})(\cdot,0)\|_{L^{q+1}(B_R)} + (2M)^{q+1}\|h\|_{L^\alpha(\mathbb{R}^n\setminus B_R)}, \end{aligned}$$

where α satisfies $\dfrac{1}{\alpha} = 1 - \dfrac{q+1}{2^*_s}$. Hence, letting first $k \to +\infty$ and then $R \to +\infty$, we conclude that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} h(x)(U_k)_+^{q+1}(x,0)\, dx = \int_{\mathbb{R}^n} h(x)\overline{U}^{q+1}(x,0)\, dx. \tag{3.2.50}$$

On the other hand, let $\{\varphi_m\}_{m\in\mathbb{N}} \in C^\infty_0(\mathbb{R}^n)$ be a sequence such that $0 \leqslant \varphi_m \leqslant 1$ and $\lim_{m\to\infty} \varphi_m(x) = 1$ for all $x \in \mathbb{R}^n$. Thus, by (3.2.33), we have that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} (U_k)_+^{p+1}(x,0)\, dx \geqslant \lim_{k\to+\infty} \int_{\mathbb{R}^n} (U_k)_+^{p+1}(x,0)\varphi_m\, dx = \int_{\mathbb{R}^n} \varphi_m\, d\nu.$$

Furthermore, by Fatou's lemma and (3.2.49),

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} \varphi_m\, d\nu \geqslant \int_{\mathbb{R}^n} d\nu \geqslant S^{n/2s} + \int_{\mathbb{R}^n} \overline{U}^{p+1}(x,0)\, dx.$$

So, using the last two formulas we get

$$\begin{aligned}\lim_{k\to\infty}\int_{\mathbb{R}^n}(U_k)_+^{p+1}(x,0)\,dx &= \lim_{m\to+\infty}\lim_{k\to\infty}\int_{\mathbb{R}^n}(U_k)_+^{p+1}(x,0)\,dx\\ &\geqslant \lim_{m\to+\infty}\int_{\mathbb{R}^n}\varphi_m\,d\nu\\ &\geqslant S^{n/2s}+\int_{\mathbb{R}^n}\overline{U}^{p+1}(x,0)\,dx.\end{aligned}\tag{3.2.51}$$

Now, since $[U_k]_a \leqslant M$ (thanks to Lemma 3.2.3), from (ii) in Proposition 3.2.1 we have that

$$\lim_{k\to+\infty}\langle \mathcal{F}'_\varepsilon(U_k),U_k\rangle = 0,$$

and so, by hypothesis (i) we get

$$\lim_{k\to\infty}\left(\mathcal{F}_\varepsilon(U_k)-\frac{1}{2}\langle \mathcal{F}'_\varepsilon(U_k),U_k\rangle\right)=c_\varepsilon.\tag{3.2.52}$$

On the other hand,

$$\begin{aligned}&\lim_{k\to\infty}\left(\mathcal{F}_\varepsilon(U_k)-\frac{1}{2}\langle \mathcal{F}'_\varepsilon(U_k),U_k\rangle\right)\\ =\;&\lim_{k\to+\infty}\left(\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n}(U_k)_+^{p+1}(x,0)\,dx-\varepsilon\left(\frac{1}{q+1}-\frac{1}{2}\right)\int_{\mathbb{R}^n}h(x)(U_k)_+^{q+1}(x,0)\,dx\right).\end{aligned}$$

We notice that

$$\frac{1}{2}-\frac{1}{p+1}=\frac{s}{n},$$

and so from (3.2.50) and (3.2.51) we obtain that

$$\begin{aligned}&\lim_{k\to\infty}\left(\mathcal{F}_\varepsilon(U_k)-\frac{1}{2}\langle \mathcal{F}'_\varepsilon(U_k),U_k\rangle\right)\\ \geqslant\;&\frac{s}{n}S^{n/2s}+\frac{s}{n}\int_{\mathbb{R}^n}\overline{U}^{p+1}(x,0)\,dx-\varepsilon\left(\frac{1}{q+1}-\frac{1}{2}\right)\int_{\mathbb{R}^n}h(x)\overline{U}^{q+1}(x,0)\,dx\\ \geqslant\;&\frac{s}{n}S^{n/2s}+\frac{s}{n}\|\overline{U}(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}-\varepsilon\left(\frac{1}{q+1}-\frac{1}{2}\right)\|h\|_{L^m(\mathbb{R}^n)}\|\overline{U}(\cdot,0)\|^{q+1}_{L^{p+1}(\mathbb{R}^n)}\\ \geqslant\;&\frac{s}{n}S^{n/2s}-\bar{C}\varepsilon^{\frac{p+1}{p-q}},\end{aligned}$$

where we have applied Lemma 3.2.4 with $\alpha := \|\overline{U}(\cdot,0)\|_{L^{p+1}(\mathbb{R}^n)}$ in the last line. This and (3.2.52) imply that

$$c_\varepsilon \geqslant \frac{s}{n}S^{n/2s}-\bar{C}\varepsilon^{\frac{p+1}{p-q}},$$

which gives a contradiction with (3.2.1).

Therefore, necessarily $\mu_j=\nu_j=0$. Repeating this argument for every $j\in J$, we obtain that $\mu_j=\nu_j=0$ for any $j\in J$. Hence, by (3.2.33),

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}(U_k)_+^{2^*_s}(x,0)\varphi\,dx=\int_{\mathbb{R}^n}\overline{U}^{2^*_s}(x,0)\varphi\,dx,\tag{3.2.53}$$

for any $\varphi \in C_0(\mathbb{R}^n)$.

Then the desired result will follow. Indeed, we use Lemmata 3.1.1 and 3.1.2, with $v_k(x) := (U_k)_+(x,0)$ and $v(x) := \overline{U}(x,0)$. More precisely, condition (3.1.4) is guaranteed by (3.2.53), while condition (3.1.5) follows from Lemma 3.2.5. This says that we can use Lemma 3.1.2 and obtain that $(U_k)_+(\cdot,0) \to \overline{U}(\cdot,0)$ in $L^{2^*_s}(\mathbb{R}^n,[0,+\infty))$. With this, the assumptions of Lemma 3.1.1 are satisfied, which in turn gives that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} |(U_k)^q_+(x,0) - \overline{U}^q(x,0)|^{\frac{2^*_s}{q}}\,dx = 0$$
$$\text{and}\qquad \lim_{k\to+\infty}\int_{\mathbb{R}^n} |(U_k)^p_+(x,0) - \overline{U}^p(x,0)|^{\frac{2n}{n+2s}}\,dx = 0.$$

Therefore, we can fix $\delta \in (0,1)$, to be taken arbitrarily small in the sequel, and say that

$$\begin{aligned}\int_{\mathbb{R}^n} &|(U_k)^q_+(x,0) - (U_m)^q_+(x,0)|^{\frac{2^*_s}{q}}\,dx \\ &+ \int_{\mathbb{R}^n} |(U_k)^p_+(x,0) - (U_m)^p_+(x,0)|^{\frac{2n}{n+2s}}\,dx \leqslant \delta\end{aligned} \tag{3.2.54}$$

for any k, m large enough, say larger than some $k_\star(\delta)$.

Let us now take $\Phi \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$ with

$$[\Phi]_a = 1. \tag{3.2.55}$$

By assumption (ii) in Proposition 3.2.1 we know that for large k (again, say, up to renaming quantities, that $k \geqslant k_\star(\delta)$),

$$|\langle \mathcal{F}'_\varepsilon(U_k), \Phi\rangle| \leqslant \delta.$$

This and (1.2.11) say that

$$\begin{aligned}\Big|\int_{\mathbb{R}^{n+1}_+} &y^a \langle \nabla U_k(X), \nabla\Phi(X)\rangle\,dX \\ &-\varepsilon\int_{\mathbb{R}^n} h(x)\,(U_k)^q_+(x,0)\phi(x)\,dx - \int_{\mathbb{R}^n} (U_k)^p_+(x,0)\phi(x)\,dx\Big| \leqslant \delta,\end{aligned}$$

where $\phi(x) := \Phi(x,0)$. In particular, for k, $m \geqslant k_\star(\delta)$,

$$\begin{aligned}\Big|\int_{\mathbb{R}^{n+1}_+} &y^a \langle \nabla (U_k(X) - U_m(X)), \nabla\Phi(X)\rangle\,dX \\ &-\varepsilon\int_{\mathbb{R}^n} h(x)\,\big((U_k)^q_+(x,0) - (U_m)^q_+(x,0)\big)\phi(x)\,dx \\ &-\int_{\mathbb{R}^n} \big((U_k)^p_+(x,0) - (U_m)^p_+(x,0)\big)\phi(x)\,dx\Big| \leqslant 2\delta.\end{aligned}$$

So, using the Hölder inequality with exponents $\frac{2n}{n+2s-q(n-2s)}$ $\frac{2_s^*}{q} = \frac{2n}{q(n-2s)}$ and $2_s^* = \frac{2n}{n-2s}$, and with exponents $\frac{2_s^*}{p} = \frac{2n}{n+2s}$ and 2_s^*, we obtain

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla(U_k(X) - U_m(X)), \nabla\Phi(X)\rangle \, dX \right| \\
\leqslant\ & \left| \int_{\mathbb{R}^n} h(x) \big((U_k)_+^q(x,0) - (U_m)_+^q(x,0) \big) \phi(x) \, dx \right| \\
&\qquad + \left| \int_{\mathbb{R}^n} \big((U_k)_+^p(x,0) - (U_m)_+^p(x,0) \big) \phi(x) \, dx \right| + 2\delta \\
\leqslant\ & \left[\int_{\mathbb{R}^n} |h(x)|^{\frac{2n}{n+2s-q(n-2s)}} \, dx \right]^{\frac{n+2s-q(n-2s)}{2n}} \\
&\qquad \cdot \left[\int_{\mathbb{R}^n} \left| (U_k)_+^q(x,0) - (U_m)_+^q(x,0) \right|^{\frac{2_s^*}{q}} dx \right]^{\frac{q(n-2s)}{2n}} \left[\int_{\mathbb{R}^n} |\phi(x)|^{2_s^*} dx \right]^{\frac{1}{2_s^*}} \\
&\qquad + \left[\int_{\mathbb{R}^n} \left| (U_k)_+^p(x,0) - (U_m)_+^p(x,0) \right|^{\frac{2n}{n+2s}} dx \right]^{\frac{n+2s}{2n}} \left[\int_{\mathbb{R}^n} |\phi(x)|^{2_s^*} dx \right]^{\frac{1}{2_s^*}} + 2\delta.
\end{aligned}$$

Thus, by (1.1.5) and (3.2.54),

$$\left| \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla(U_k(X) - U_m(X)), \nabla\Phi(X)\rangle \, dX \right| \leqslant C\, \delta^{\frac{q(n-2s)}{2n}} \, \|\phi\|_{L^{2_s^*}(\mathbb{R}^n)} + C\, \delta^{\frac{n+2s}{2n}} \, \|\phi\|_{L^{2_s^*}(\mathbb{R}^n)} + 2\delta,$$

for some $C > 0$. Now, by (1.2.7) and (3.2.55), we have that $\|\phi\|_{L^{2_s^*}(\mathbb{R}^n)} \leqslant S^{-1/2}[\Phi]_a = S^{-1/2}$, therefore, up to renaming constants,

$$\left| \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla(U_k(X) - U_m(X)), \nabla\Phi(X)\rangle \, dX \right| \leqslant C\delta^\gamma,$$

for some C, $\gamma > 0$, as long as k, $m \geqslant k_\star(\delta)$. Since this inequality is valid for any Φ satisfying (3.2.55), we have proved that

$$[U_k - U_m]_a \leqslant C\delta^\gamma,$$

that says that U_k is a Cauchy sequence in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$, and then the desired result follows. □

3.3. Proof of Theorem 1.2.2

With all this, we are in the position to prove Theorem 1.2.2.

Proof of Theorem 1.2.2. We recall that (1.1.5) holds true. Thus, applying the Hölder inequality and Proposition 1.2.1, for $U \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$ we have

$$\begin{aligned}
\mathcal{F}_\varepsilon(U) &\geqslant \frac{1}{2}[U]_a^2 - c_1 \|U_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} - c_2\varepsilon \, \|h\|_{L^m(\mathbb{R}^n)} \|U_+(\cdot,0)\|^{q+1}_{L^{p+1}(\mathbb{R}^n)} \\
&\geqslant \frac{1}{2}[U]_a^2 - \tilde{c}_1 [U]_a^{p+1} - \varepsilon \tilde{c}_2 [U]_a^{q+1}.
\end{aligned}$$

We consider the function

$$\phi(t) = \frac{1}{2}t^2 - \tilde{c}_1 t^{p+1} - \varepsilon \tilde{c}_2 t^{q+1}, \qquad t \geqslant 0.$$

Since $q+1 < 2 < p+1$, we have that for every $\varepsilon > 0$ we can find $\rho = \rho(\varepsilon) > 0$ satisfying $\phi(\rho) = 0$ and $\phi(t) < 0$ for any $t \in (0, \rho)$. As a matter of fact, ρ is the first zero of the function ϕ.

Furthermore, it is not difficult to see that

$$\rho(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.3.1}$$

Thus, there exist $c_0 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,

$$\begin{cases} \mathcal{F}_\varepsilon(U) \geqslant -c_0 & \text{if } [U]_a < \rho(\varepsilon_0), \\ \mathcal{F}_\varepsilon(U) > 0 & \text{if } [U]_a = \rho(\varepsilon_0). \end{cases} \tag{3.3.2}$$

Now we take $\varphi \in C_0^\infty(\mathbb{R}^{n+1}_+)$, $\varphi \geqslant 0$, $[\varphi]_a = 1$, and such that $\operatorname{supp}(\varphi(\cdot, 0)) \subset B$, where B is given in condition (1.1.4). Hence, for any $t > 0$,

$$\begin{aligned} \mathcal{F}_\varepsilon(t\varphi) &= \frac{1}{2}t^2 - \frac{\varepsilon}{q+1} t^{q+1} \int_{\mathbb{R}^n} h(x)\varphi^{q+1}(x,0)\,dx - \frac{t^{p+1}}{p+1} \int_{\mathbb{R}^n} \varphi^{p+1}(x,0)\,dx \\ &\leqslant \frac{1}{2}t^2 - \frac{\varepsilon}{q+1} t^{q+1} \inf_B h \int_B \varphi^{q+1}(x,0)\,dx - \frac{t^{p+1}}{p+1} \int_B \varphi^{p+1}(x,0)\,dx. \end{aligned}$$

This inequality and condition (1.1.4) give that, for any $\varepsilon < \varepsilon_0$ (possibly taking ε_0 smaller) there exists $t_0 < \rho(\varepsilon_0)$ such that, for any $t < t_0$, we have

$$\mathcal{F}_\varepsilon(t\varphi) < 0.$$

This implies that

$$i_\varepsilon := \inf_{U \in \dot{H}^s_a(\mathbb{R}^{n+1}_+), [U]_a < \rho(\varepsilon_0)} \mathcal{F}_\varepsilon(U) < 0.$$

This and (3.3.2) give that, for $0 < \varepsilon < \varepsilon_0$,

$$-\infty < -c_0 \leqslant i_\varepsilon < 0.$$

Now we take a minimizing sequence $\{U_k\}$ and we observe that

$$\lim_{\varepsilon \to 0} \lim_{k \to +\infty} \mathcal{F}_\varepsilon(U_k) = \lim_{\varepsilon \to 0} i_\varepsilon \leqslant 0 < \frac{s}{n} S^{\frac{n}{2s}}.$$

Hence, condition (3.2.1) is satisfied with $c_\varepsilon := i_\varepsilon$, and so we can apply Proposition 3.2.1 and we conclude that i_ε is attained at some minimum U_ε.

Finally, since $[U_\varepsilon]_a \leqslant \rho(\varepsilon_0)$, (3.3.1) implies that U_ε converges to 0 in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$ as ε tends to 0. This concludes the proof of Theorem 1.2.2. $\square$

CHAPTER 4

Regularity and positivity of the solution

4.1. A regularity result

In this section we show a regularity result that allows us to say that a nonnegative solution to (1.1.1) is bounded.

PROPOSITION 4.1.1. *Let* $u \in \dot{H}^s(\mathbb{R}^n)$ *be a nonnegative solution to the problem*

$$(-\Delta)^s u = f(x,u) \qquad \text{in } \mathbb{R}^n,$$

and assume that $|f(x,t)| \leqslant C(1+|t|^p)$*, for some* $1 \leqslant p \leqslant 2^*_s - 1$ *and* $C > 0$*. Then* $u \in L^\infty(\mathbb{R}^n)$*.*

PROOF. Let $\beta \geqslant 1$ and $T > 0$, and let us define

$$\varphi(t) = \begin{cases} 0, \text{ if } t \leqslant 0, \\ t^\beta, \text{ if } 0 < t < T, \\ \beta T^{\beta-1}(t-T) + T^\beta, \text{ if } t \geqslant T. \end{cases}$$

Since φ is convex and Lipschitz,

$$\varphi(u) \in \dot{H}^s(\mathbb{R}^n) \tag{4.1.1}$$

and

$$(-\Delta)^s \varphi(u) \leqslant \varphi'(u)(-\Delta)^s u \tag{4.1.2}$$

in the weak sense.

We recall that (1.2.6) and Proposition 1.2.1 imply that, for any $u \in \dot{H}^s(\mathbb{R}^n)$,

$$\|u\|_{L^{2^*_s}(\mathbb{R}^n)} \leqslant S^{-2}[u]_{\dot{H}^s(\mathbb{R}^n)}.$$

Moreover, by Proposition 3.6 in [**21**] we have that

$$[u]_{\dot{H}^s(\mathbb{R}^n)} = \|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^n)}.$$

Hence, from (4.1.1), an integration by parts and (4.1.2) we deduce that

$$\begin{aligned} \|\varphi(u)\|^2_{L^{2^*_s}(\mathbb{R}^n)} &\leqslant S^{-1}\int_{\mathbb{R}^n} |(-\Delta)^{s/2}\varphi(u)|^2\,dx \\ &= S^{-1}\int_{\mathbb{R}^n} \varphi(u)(-\Delta)^s\varphi(u)\,dx \leqslant S^{-1}\int_{\mathbb{R}^n}\varphi(u)\,\varphi'(u)(-\Delta)^s u\,dx. \end{aligned}$$

Therefore, from the assumption on u and f we obtain

$$\begin{aligned}\|\varphi(u)\|^2_{L^{2^*_s}(\mathbb{R}^n)} &\leqslant S^{-1}\int_{\mathbb{R}^n}\varphi(u)\,\varphi'(u)(1+u^{2^*_s-1})\,dx\\ &= S^{-1}\left(\int_{\mathbb{R}^n}\varphi(u)\,\varphi'(u)\,dx+\int_{\mathbb{R}^n}\varphi(u)\,\varphi'(u)\,u^{2^*_s-1}\,dx\right).\end{aligned}$$

Using that $\varphi(u)\varphi'(u)\leqslant \beta u^{2\beta-1}$ and $u\varphi'(u)\leqslant\beta\varphi(u)$, we have

$$\left(\int_{\mathbb{R}^n}(\varphi(u))^{2^*_s}\right)^{2/2^*_s}\leqslant C\beta\left(\int_{\mathbb{R}^n}u^{2\beta-1}\,dx+\int_{\mathbb{R}^n}(\varphi(u))^2u^{2^*_s-2}\,dx\right), \tag{4.1.3}$$

where C is a positive constant that does not depend on β. Notice that the last integral is well defined for every T in the definition of φ. Indeed,

$$\begin{aligned}\int_{\mathbb{R}^n}(\varphi(u))^2u^{2^*_s-2}\,dx &= \int_{\{u\leqslant T\}}(\varphi(u))^2u^{2^*_s-2}\,dx+\int_{\{u>T\}}(\varphi(u))^2u^{2^*_s-2}\,dx\\ &\leqslant T^{2\beta-2}\int_{\mathbb{R}^n}u^{2^*_s}\,dx+C\int_{\mathbb{R}^n}u^{2^*_s}\,dx<+\infty,\end{aligned}$$

where we have used that $\beta>1$ and that $\varphi(u)$ is linear when $u\geqslant T$. We choose now β in (4.1.3) such that $2\beta-1=2^*_s$, and we name it β_1, that is,

$$\beta_1:=\frac{2^*_s+1}{2}. \tag{4.1.4}$$

Let $R>0$ to be fixed later. Attending to the last integral in (4.1.3) and applying the Hölder's inequality with exponents $r:=2^*_s/2$ and $r':=2^*_s/(2^*_s-2)$,

(4.1.5)

$$\begin{aligned}\int_{\mathbb{R}^n}(\varphi(u))^2u^{2^*_s-2}\,dx &= \int_{\{u\leqslant R\}}(\varphi(u))^2u^{2^*_s-2}\,dx+\int_{\{u>R\}}(\varphi(u))^2u^{2^*_s-2}\,dx\\ &\leqslant \int_{\{u\leqslant R\}}\frac{(\varphi(u))^2}{u}R^{2^*_s-1}\,dx+\left(\int_{\mathbb{R}^n}(\varphi(u))^{2^*_s}\,dx\right)^{2/2^*_s}\left(\int_{\{u>R\}}u^{2^*_s}\,dx\right)^{\frac{2^*_s-2}{2^*_s}}.\end{aligned}$$

By the Monotone Convergence Theorem, we can choose R large enough so that

$$\left(\int_{\{u>R\}}u^{2^*_s}\,dx\right)^{\frac{2^*_s-2}{2^*_s}}\leqslant\frac{1}{2C\beta_1},$$

where C is the constant appearing in (4.1.3). Therefore, we can absorb the last term in (4.1.5) by the left hand side of (4.1.3) to get

$$\left(\int_{\mathbb{R}^n}(\varphi(u))^{2^*_s}\,dx\right)^{2/2^*_s}\leqslant 2C\beta_1\left(\int_{\mathbb{R}^n}u^{2^*_s}\,dx+R^{2^*_s-1}\int_{\mathbb{R}^n}\frac{(\varphi(u))^2}{u}\,dx\right),$$

where (4.1.4) is also used. Now we use that $\varphi(u) \leqslant u^{\beta_1}$ and (4.1.4) once again in the right hand side and we take $T \to \infty$: we obtain

$$\left(\int_{\mathbb{R}^n} u^{2_s^*\beta_1}\,dx\right)^{2/2_s^*} \leqslant 2C\beta_1 \left(\int_{\mathbb{R}^n} u^{2_s^*}\,dx + R^{2_s^*-1}\int_{\mathbb{R}^n} u^{2_s^*}\,dx\right) < +\infty,$$

and therefore

$$u \in L^{2_s^*\beta_1}(\mathbb{R}^n). \tag{4.1.6}$$

Let us suppose now $\beta > \beta_1$. Thus, using that $\varphi(u) \leqslant u^{\beta}$ in the right hand side of (4.1.3) and letting $T \to \infty$ we get

$$\left(\int_{\mathbb{R}^n} u^{2_s^*\beta}\,dx\right)^{2/2_s^*} \leqslant C\beta \left(\int_{\mathbb{R}^n} u^{2\beta-1}\,dx + \int_{\mathbb{R}^n} u^{2\beta+2_s^*-2}\,dx\right). \tag{4.1.7}$$

Furthermore, we can write

$$u^{2\beta-1} = u^a u^b,$$

with $a := \frac{2_s^*(2_s^*-1)}{2(\beta-1)}$ and $b := 2\beta - 1 - a$. Notice that, since $\beta > \beta_1$, then $0 < a, b < 2_s^*$. Hence, applying Young's inequality with exponents

$$r := 2_s^*/a \quad \text{and} \quad r' := 2_s^*/(2_s^* - a),$$

there holds

$$\begin{aligned}
\int_{\mathbb{R}^n} u^{2\beta-1}\,dx &\leqslant \frac{a}{2_s^*}\int_{\mathbb{R}^n} u^{2_s^*}\,dx + \frac{2_s^*-a}{2_s^*}\int_{\mathbb{R}^n} u^{\frac{2_s^* b}{2_s^*-a}}\,dx \\
&\leqslant \int_{\mathbb{R}^n} u^{2_s^*}\,dx + \int_{\mathbb{R}^n} u^{2\beta+2_s^*-2}\,dx \\
&\leqslant C\left(1 + \int_{\mathbb{R}^n} u^{2\beta+2_s^*-2}\,dx\right),
\end{aligned}$$

with $C > 0$ independent of β. Plugging this into (4.1.7),

$$\left(\int_{\mathbb{R}^n} u^{2_s^*\beta}\,dx\right)^{2/2_s^*} \leqslant C\beta\left(1 + \int_{\mathbb{R}^n} u^{2\beta+2_s^*-2}\,dx\right),$$

with C changing from line to line, but remaining independent of β. Therefore,

$$\left(1 + \int_{\mathbb{R}^n} u^{2_s^*\beta}\,dx\right)^{\frac{1}{2_s^*(\beta-1)}} \leqslant (C\beta)^{\frac{1}{2(\beta-1)}}\left(1 + \int_{\mathbb{R}^n} u^{2\beta+2_s^*-2}\,dx\right)^{\frac{1}{2(\beta-1)}}, \tag{4.1.8}$$

that is (2.6) in [**8**, Proposition 2.2]. From now on, we follow exactly their iterative argument. That is, we define β_{m+1}, $m \geqslant 1$, so that

$$2\beta_{m+1} + 2_s^* - 2 = 2_s^*\beta_m.$$

Thus,

$$\beta_{m+1} - 1 = \left(\frac{2_s^*}{2}\right)^m (\beta_1 - 1),$$

and replacing in (4.1.8) it yields

$$\left(1+\int_{\mathbb{R}^n} u^{2^*_s\beta_{m+1}}\,dx\right)^{\frac{1}{2^*_s(\beta_{m+1}-1)}} \leqslant (C\beta_{m+1})^{\frac{1}{2(\beta_{m+1}-1)}} \left(1+\int_{\mathbb{R}^n} u^{2^*_s\beta_m}\,dx\right)^{\frac{1}{2^*_s(\beta_m-1)}}.$$

Defining $C_{m+1} := C\beta_{m+1}$ and

$$A_m := \left(1+\int_{\mathbb{R}^n} u^{2^*_s\beta_m}\,dx\right)^{\frac{1}{2^*_s(\beta_m-1)}},$$

we conclude that there exists a constant $C_0 > 0$ independent of m, such that

$$A_{m+1} \leqslant \prod_{k=2}^{m+1} C_k^{\frac{1}{2(\beta_k-1)}} A_1 \leqslant C_0 A_1.$$

Thus,

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leqslant C_0 A_1 < +\infty,$$

thanks to (4.1.6). This finishes the proof of Proposition 4.1.1. □

COROLLARY 4.1.2. *Let $u \in \dot{H}^s(\mathbb{R}^n)$ be a solution of* (1.2.1) *and let U be its extension, according to* (1.2.3).

Then $u \in L^\infty(\mathbb{R}^n)$, and $U \in L^\infty(\mathbb{R}^{n+1}_+)$.

PROOF. First we observe that $u \geqslant 0$, thanks to Proposition 1.2.3. Moreover, since u is a solution to (1.1.1), it solves

$$(-\Delta)^s u = f(x,u) \quad \text{ in } \mathbb{R}^n,$$

where $f(x,t) := \varepsilon h(x) t_+^q + t_+^p$. It is easy to check that f satisfies the hypotheses of Proposition 4.1.1. Hence the boundedness of u simply follows from Proposition 4.1.1.

Let us now show the L^∞ estimate for U. According to (1.2.3), for any $(x,z) \in \mathbb{R}^{n+1}_+$,

$$U(x,z) = \int_{\mathbb{R}^n} u(x-y)\,P_s(y,z)\,dy.$$

Therefore,

$$|U(x,z)| \leqslant \|u\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} P_s(y,z)\,dy = \|u\|_{L^\infty(\mathbb{R}^n)},$$

for any $(x,z) \in \mathbb{R}^{n+1}_+$, which implies the L^∞-bound for U, and concludes the proof of the corollary. □

Finally, we can prove that a solution to (1.2.1) is continuous, as stated in the following:

COROLLARY 4.1.3. *Let $u \in \dot{H}^s(\mathbb{R}^n)$ be a solution of* (1.2.1) *and let U be its extension, according to* (1.2.3).

Then $u \in C^\alpha(\mathbb{R}^n)$, for any $\alpha \in (0, \min\{2s, 1\})$, and $U \in C(\overline{\mathbb{R}^{n+1}_+})$.

PROOF. The regularity of u follows from Corollary 4.1.2 and Proposition 5 in [**35**], being u a solution to (1.2.1). The continuity of U follows from Lemma 4.4 in [**15**]. □

4.2. A strong maximum principle and positivity of the solutions

In this section we deal with the problem of the positivity of the solutions to (1.2.1). We have shown in Proposition 1.2.3 that a solution of (1.2.1) is nonnegative. Here we prove that if $h \geqslant 0$ then the solution is strictly positive.

Following is the strong maximum principle for weak solutions needed for our purposes:

PROPOSITION 4.2.1. *Let u be a bounded, continuous function, with $u \geqslant 0$ in $\mathbb{R}^n$ and $(-\Delta)^s u \geqslant 0$ in the weak sense in Ω. If there exists $x_\star \in \Omega$ such that $u(x_\star) = 0$, then u vanishes identically in $\mathbb{R}^n$.*

PROOF. Let $R > 0$ such that $B_R(x_\star) \subset \Omega$. For any $r \in (0, R)$, we consider the solution of

$$\begin{cases} (-\Delta)^s v_r = 0 & \text{in } B_r(x_\star), \\ v_r = u & \text{in } \mathbb{R}^n \setminus B_r(x_\star) \end{cases} \tag{4.2.1}$$

Notice that v_r may be obtained by direct minimization and v_r is continuous in the whole of $\mathbb{R}^n$ (see e.g. Theorem 2 in [**35**]). Moreover, if $w_r := v_r - u$, we have that $(-\Delta)^s w_r \leqslant 0$ in the weak sense in $B_r(x_\star)$, and w_r vanishes outside $B_r(x_\star)$. Accordingly, by the weak maximum principle for weak solutions (see e.g. Lemma 6 in [**35**]), we have that $w_r \leqslant 0$ in the whole of $\mathbb{R}^n$, which gives that $v_r \leqslant u$. In particular,

$$v_r(x_\star) \leqslant u(x_\star) = 0. \tag{4.2.2}$$

The weak maximum principle for weak solutions and the fact that $v_r = u \geqslant 0$ outside $B_r(x_\star)$ also imply that $v_r \geqslant 0$ in $\mathbb{R}^n$. This and (4.2.2) say that

$$\min_{\mathbb{R}^n} v_r = v_r(x_\star) = 0. \tag{4.2.3}$$

In addition, v_r is also a solution of (4.2.1) in the viscosity sense (see e.g. Theorem 1 in [**35**]), hence it is smooth in the interior of $B_r(x_\star)$, and we can compute $(-\Delta)^s v_r(x_\star)$ and obtain from (4.2.3) that

$$0 = \int_{\mathbb{R}^n} \frac{v_r(x_\star + y) + v_r(x_\star - y) - 2v_r(x_\star)}{|y|^{n+2s}}\, dy \geqslant 0.$$

This implies that v_r is constant in $\mathbb{R}^n$, that is $v_r(x) = v_r(x_\star) = 0$ for any $x \in \mathbb{R}^n$. In particular $0 = v_r(x) = u(x)$ for any $x \in \mathbb{R}^n \setminus B_r(x_\star)$. By taking r arbitrarily small, we obtain that $u(x) = 0$ for any $x \in \mathbb{R}^n \setminus \{x_\star\}$, and the desired result plainly follows. □

Thanks to Proposition 4.2.1 we now show the positivity of solutions of (1.2.1).

Corollary 4.2.2. *Let $u \in \dot{H}^s(\mathbb{R}^n)$, $u \neq 0$, be a solution of* (1.2.1). *Suppose also that $h \geqslant 0$. Then, $u > 0$.*

Proof. First we observe that $u \in C^\alpha(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, for some $\alpha \in (0,1)$, thanks to Corollaries 4.1.2 and 4.1.3. Also, by Proposition 1.2.3 we have that $u \geqslant 0$. Moreover, since u is a solution to (1.2.1) with $h \geqslant 0$, then u satisfies

$$(-\Delta)^s u \geqslant 0 \quad \text{ in } \mathbb{R}^n.$$

This means that the hypotheses of Proposition 4.2.1 are satisfied, and so if u is equal to zero at some point then u must be identically zero in $\mathbb{R}^n$. This contradicts the fact that $u \neq 0$, and thus implies the desired result. □

CHAPTER 5

Existence of a mountain pass solution and proof of Theorem 1.2.4

5.1. Existence of a local minimum for $\mathcal{I}_\varepsilon$

In this section we show that $U=0$ is a local minimum for $\mathcal{I}_\varepsilon$.

PROPOSITION 5.1.1. *Let U_ε be a local positive minimum of $\mathcal{F}_\varepsilon$ in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$. Then $U=0$ is a local minimum of $\mathcal{I}_\varepsilon$ in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$.*

PROOF. Let U_ε be a local minimum of $\mathcal{F}_\varepsilon$ in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$. Then, there exists $\eta>0$ such that

$$\mathcal{F}_\varepsilon(U_\varepsilon+U)\geqslant \mathcal{F}_\varepsilon(U_\varepsilon), \quad \text{if } \ u\in \dot{H}^s_a(\mathbb{R}^{n+1}_+) \ \text{ s.t. } [U]_a\leqslant \eta. \tag{5.1.1}$$

Moreover, since U_ε is a positive critical point of $\mathcal{F}_\varepsilon$, we have that, for every $V\in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$,

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} y^a\langle\nabla U_\varepsilon,\nabla V\rangle\,dX-\int_{\mathbb{R}^n} U_\varepsilon(x,0)^pV(x,0)\,dx\\ -\varepsilon\int_{\mathbb{R}^n} h(x)U_\varepsilon(x,0)^qV(x,0)\,dx=0.\end{aligned} \tag{5.1.2}$$

Now, we take $U\in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$ such that

$$[U]_a\leqslant \eta. \tag{5.1.3}$$

From (1.2.15) and (1.2.17), we have that

$$\begin{aligned}\mathcal{I}_\varepsilon(U) \ = \ & \frac12\int_{\mathbb{R}^{n+1}_+} y^a|\nabla U|^2\,dX\\ & -\frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\,\left((U_\varepsilon+U_+)^{q+1}-U_\varepsilon^{q+1}\right)\,dx+\varepsilon\int_{\mathbb{R}^n} h(x)\,U_\varepsilon^qU_+\,dx\\ & -\frac{1}{p+1}\int_{\mathbb{R}^n}\left((U_\varepsilon+U_+)^{p+1}-U_\varepsilon^{p+1}\right)\,dx+\int_{\mathbb{R}^n} U_\varepsilon^pU_+\,dx.\end{aligned}$$

On the other hand, recalling the definition of $\mathcal{F}_\varepsilon$ in (1.2.10), we have that

$$\begin{aligned}
& \mathcal{F}_\varepsilon(U_\varepsilon+U_+)-\mathcal{F}_\varepsilon(U_\varepsilon) \\
= \; & \frac{1}{2}\int_{\mathbb{R}^{n+1}_+} y^a|\nabla U_+|^2\,dX+\int_{\mathbb{R}^{n+1}_+} y^a\langle\nabla U_\varepsilon,\nabla U_+\rangle\,dX \\
& -\frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\,\left((U_\varepsilon+U_+)^{q+1}-U_\varepsilon^{q+1}\right)\,dx \\
& -\frac{1}{p+1}\int_{\mathbb{R}^n}\left((U_\varepsilon+U_+)^{p+1}-U_\varepsilon^{p+1}\right)\,dx \\
= \; & \frac{1}{2}\int_{\mathbb{R}^{n+1}_+} y^a|\nabla U_+|^2\,dX+\int_{\mathbb{R}^{n+1}_+} y^a\langle\nabla U_\varepsilon,\nabla U_+\rangle\,dX \\
& -\frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\,\left((U_\varepsilon+U_+)^{q+1}-U_\varepsilon^{q+1}\right)\,dx \\
& -\frac{1}{p+1}\int_{\mathbb{R}^n}\left((U_\varepsilon+U_+)^{p+1}-U_\varepsilon^{p+1}\right)\,dx,
\end{aligned}$$

where in the last equality we have used the fact that both U_ε and $U_\varepsilon+U_+$ are positive. Hence, the last two formulas give that

$$\begin{aligned}
\mathcal{I}_\varepsilon(U) \; = \; & \frac{1}{2}\int_{\mathbb{R}^{n+1}_+} y^a|\nabla U_-|^2\,dX+\mathcal{F}_\varepsilon(U_\varepsilon+U_+)-\mathcal{F}_\varepsilon(U_\varepsilon) \\
& -\int_{\mathbb{R}^{n+1}_+} y^a\langle\nabla U_\varepsilon,\nabla U_+\rangle\,dX \\
& +\varepsilon\int_{\mathbb{R}^n} h(x)U_\varepsilon^q U_+\,dx+\int_{\mathbb{R}^n} U_\varepsilon^p U_+\,dx.
\end{aligned}$$

Using (5.1.2) with $V:=U_+$, we obtain

$$\mathcal{I}_\varepsilon(U)=\frac{1}{2}\int_{\mathbb{R}^{n+1}_+} y^a|\nabla U_-|^2\,dX+\mathcal{F}_\varepsilon(U_\varepsilon+U_+)-\mathcal{F}_\varepsilon(U_\varepsilon).$$

Moreover, we observe that $[U_+]_a\leqslant\eta$, thanks to (5.1.3). Hence, from (5.1.1) we deduce that

$$\mathcal{I}_\varepsilon(U)\geqslant\frac{1}{2}\int_{\mathbb{R}^{n+1}_+} y^a|\nabla U_-|^2\,dX\geqslant 0=\mathcal{I}_\varepsilon(0).$$

This shows the desired result. □

5.2. Some preliminary lemmata towards the proof of Theorem 1.2.4

In this section we show some preliminary lemmata, that we will use in the sequel to prove that a Palais-Smale sequence is bounded.

We start with a basic inequality.

LEMMA 5.2.1. *For every $\delta > 0$ there exists $M_\delta > 0$ such that the following inequality holds true for every α, $\beta \geqslant 0$ and $m > 0$:*

$$(\alpha+\beta)^{m+1}-\alpha^{m+1}-(m+1)\alpha^m\beta-\beta\big((\alpha+\beta)^m-\alpha^m\big)\leqslant\delta\beta^{m+1}+M_\delta\alpha^{m+1}. \tag{5.2.1}$$

PROOF. First of all, we observe that the left hand side of (5.2.1) vanishes when $\alpha = 0$, therefore we can suppose that

$$\alpha \neq 0. \tag{5.2.2}$$

For any $\tau \geqslant 0$, let

$$f(\tau) := (1+\tau)^{m+1} - 1 - (m+1)\tau - \tau\big((1+\tau)^m - 1\big).$$

We observe that

$$\lim_{\tau\to+\infty}\frac{f(\tau)}{\tau^{m+1}} = 0$$

therefore there exists $\tau_\delta > 0$ such that $\frac{f(\tau)}{\tau^{m+1}} \leqslant \delta$ for any $\tau \geqslant \tau_\delta$. Let also

$$M_\delta := \max_{\tau\in[0,\tau_\delta]} f(\tau).$$

Then, by looking separately at the cases $\tau \in [0, \tau_\delta]$ and $\tau \in [\tau_\delta, +\infty)$, we see that

$$f(\tau) \leqslant \delta\tau^{m+1} + M_\delta.$$

As a consequence, recalling (5.2.2) and taking $\tau := \beta/\alpha$,

$$\begin{aligned}
&(\alpha+\beta)^{m+1}-\alpha^{m+1}-(m+1)\alpha^m\beta-\beta\big((\alpha+\beta)^m-\alpha^m\big)\\
&\quad= \alpha^{m+1}\Big[(1+\tau)^{m+1}-1-(m+1)\tau-\tau\big((1+\tau)^m-1\big)\Big]\\
&\quad= \alpha^{m+1}f(\tau)\\
&\quad\leqslant \alpha^{m+1}\big(\delta\tau^{m+1}+M_\delta\big)\\
&\quad= \delta\beta^{m+1}+M_\delta\alpha^{m+1}.
\end{aligned}$$

□

We recall (1.2.16) and (1.2.17), and we have the following estimates.

COROLLARY 5.2.2. *For any $U \in H^s_a(\mathbb{R}^{n+1}_+)$ and any $\delta \in (0, 1)$, we have that*

$$\int_{\mathbb{R}^n} G(x,U(x,0))-\frac{1}{p+1}g(x,U(x,0))\,U(x,0)\,dx \leqslant C(\varepsilon+\delta)\|U_+(\cdot,0)\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^n)}+C_{\delta,\varepsilon}, \tag{5.2.3}$$

for suitable C, $C_{\delta,\varepsilon} > 0$. Moreover,

$$\int_{\mathbb{R}^n} g(x,U(x,0))\,U(x,0)\,dx \geqslant \frac{\|U_+(\cdot,0)\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^n)}}{8} - C_\varepsilon, \tag{5.2.4}$$

for a suitable $C_\varepsilon > 0$.

PROOF. By (1.2.16) and (1.2.17), we can write $g = g_1 + g_2$ and $G = G_1 + G_2$, where

$$\begin{aligned}
&g_1(x,t) := \varepsilon h(x)\big((U_\varepsilon(x,0)+t_+)^q - U_\varepsilon^q(x,0)\big),\\
&g_2(x,t) := (U_\varepsilon(x,0)+t_+)^p - U_\varepsilon^p(x,0),\\
&G_1(x,t) := \frac{\varepsilon h(x)}{q+1}\big((U_\varepsilon(x,0)+t_+)^{q+1} - U_\varepsilon^{q+1}(x,0)\big) - \varepsilon h(x) U_\varepsilon^q(x,0)\, t_+\\
\text{and}\qquad &G_2(x,t) := \frac{1}{p+1}\big((U_\varepsilon(x,0)+t_+)^{p+1} - U_\varepsilon^{p+1}(x,0)\big) - U_\varepsilon^p(x,0)\, t_+.
\end{aligned}$$

We observe that for any $\tau \geqslant 0$,

$$\begin{aligned}
&(1+\tau)^q - 1 = q\int_0^\tau (1+\theta)^{q-1}\,d\theta \leqslant q\int_0^\tau \theta^{q-1}\,d\theta = \tau^q\\
\text{and}\qquad &(1+\tau)^{q+1} - 1 = (q+1)\int_0^\tau (1+\theta)^q\,d\theta \leqslant (q+1)(1+\tau)^q\tau,
\end{aligned}$$

since $q \in (0,1)$. Therefore, taking $\tau := t_+/U_\varepsilon$,

$$(U_\varepsilon + t_+)^q - U_\varepsilon^q = U_\varepsilon^q\big((1+\tau)^q - 1\big) \leqslant U_\varepsilon^q \tau^q = t_+^q$$

and

$$\begin{aligned}
(U_\varepsilon + t_+)^{q+1} - U_\varepsilon^{q+1} &= U_\varepsilon^{q+1}\big((1+\tau)^{q+1} - 1\big)\\
&\leqslant (q+1)U_\varepsilon^{q+1}(1+\tau)^q\tau = (q+1)(U_\varepsilon + t_+)^q t_+.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
&|g_1| \leqslant \varepsilon|h|t_+^q\\
\text{and}\qquad &|G_1| \leqslant \varepsilon|h|(U_\varepsilon + t_+)^q t_+ + \varepsilon|h|U_\varepsilon^q t_+ \leqslant 2\varepsilon|h|(U_\varepsilon + t_+)^q t_+.
\end{aligned}$$

Thus we obtain

$$|G_1(x,t)| + |g_1(x,t)t| \leqslant 2\varepsilon|h|(U_\varepsilon + t_+)^q t_+ + \varepsilon|h|t_+^{q+1} \leqslant 3\varepsilon|h|(U_\varepsilon + t_+)^q t_+.$$

Since U_ε is bounded (recall Corollary 4.1.2), we obtain that

$$|G_1(x,t)| + |g_1(x,t)t| \leqslant C\varepsilon|h|\,(1 + t_+^{q+1}), \tag{5.2.5}$$

for some $C > 0$. By considering the cases $t_+ \in [0,1]$ and $t_+ \in [1,+\infty)$, we see that

$$t_+^{q+1} \leqslant t_+^{p+1} + 1,$$

since $q < p$. This and (5.2.5) give that

$$|G_1(x,t)| + |g_1(x,t)t| \leqslant C\varepsilon|h|\,(1 + t_+^{p+1}), \tag{5.2.6}$$

up to changing the constants. Now we fix $\delta \in (0,1)$. Using (5.2.1) with $\alpha := U_\varepsilon$, $\beta := t_+$ and $m := p$, we have that

$$\begin{aligned} & G_2(x,t) - \frac{1}{p+1} g_2(x,t)t \\ = \; & \frac{1}{p+1}\big((U_\varepsilon + t_+)^{p+1} - U_\varepsilon^{p+1}\big) - U_\varepsilon^p\, t_+ - (U_\varepsilon + t_+)^p + U_\varepsilon^p \\ = \; & \frac{1}{p+1}\left((\alpha+\beta)^{m+1} - \alpha^{m+1} - (m+1)\alpha^m\beta - \beta\big((\alpha+\beta)^m - \alpha^m\big)\right) \\ \leqslant \; & \frac{1}{p+1}\left(\delta\beta^{m+1} + M_\delta \alpha^{m+1}\right) \\ \leqslant \; & \delta\, t_+^{p+1} + M_\delta U_\varepsilon^{p+1}. \end{aligned}$$

This, together with (5.2.6), implies that

$$\begin{aligned} & G(x,t) - \frac{1}{p+1} g(x,t)t \\ = \; & G_1(x,t) - \frac{1}{p+1} g_1(x,t)t + G_2(x,t) - \frac{1}{p+1} g_2(x,t)t \\ \leqslant \; & C\varepsilon|h|\,(1+t_+^{p+1}) + \delta\, t_+^{p+1} + M_\delta U_\varepsilon^{p+1} \\ \leqslant \; & C(\varepsilon+\delta)t_+^{p+1} + C|h| + M_\delta U_\varepsilon^{p+1}. \end{aligned}$$

As a consequence, and recalling that $p+1 = 2_s^*$, we obtain

$$\int_{\mathbb{R}^n} G(x,U(x,0)) - \frac{1}{p+1} g(x,U(x,0))\,U(x,0)\,dx \leqslant C(\varepsilon+\delta)\|U_+(\cdot,0)\|_{L^{2_s^*}(\mathbb{R}^n)}^{2_s^*} + C_{\delta,\varepsilon}$$

for some C, $C_{\delta,\varepsilon} > 0$. This proves (5.2.3) and we now focus on the proof of (5.2.4). To this goal, for any $\tau \geqslant 0$, we set $\ell(\tau) := \frac{\tau^p}{2} - (1+\tau)^p + 1$. We observe that $\ell(0) = 0$ and

$$\lim_{\tau \to +\infty} \ell(\tau) = -\infty,$$

therefore

$$L := \sup_{\tau \geqslant 0} \ell(\tau) \in [0, +\infty).$$

As a consequence,

$$(1+\tau)^p - 1 = \frac{\tau^p}{2} - \ell(\tau) \geqslant \frac{\tau^p}{2} - L.$$

By taking $\tau := \frac{U_+}{U_\varepsilon}$, this implies that

$$\begin{aligned} g_2(x,U) &= (U_\varepsilon + U_+)^p - U_\varepsilon^p = U_\varepsilon^p\big((1+\tau)^p - 1\big) \\ &\geqslant U_\varepsilon^p \left(\frac{\tau^p}{2} - L\right) = \frac{U_+^p}{2} - LU_\varepsilon^p. \end{aligned}$$

Integrating this formula and using the Young inequality, we obtain

$$\int_{\mathbb{R}^n} g_2(x,U(x,0))\,U(x,0)\,dx \geqslant \frac{\|U_+(\cdot,0)\|_{L^{p+1}(\mathbb{R}^n)}^{p+1}}{4} - C_\varepsilon, \tag{5.2.7}$$

for some $C_\varepsilon > 0$. On the other hand, by (5.2.6), we have that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} g_1(x, U(x,0))\, U(x,0)\, dx\right| &\leqslant C\varepsilon \int_{\mathbb{R}^n} |h(x)|\,(1+U_+^{p+1}(x,0))\, dx \\ &\leqslant C + C\varepsilon \|U_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}.\end{aligned}$$

By combining this and (5.2.7), we get

$$\begin{aligned}&\int_{\mathbb{R}^n} g(x, U(x,0))\, U(x,0)\, dx \\ =\;& \int_{\mathbb{R}^n} g_1(x, U(x,0))\, U(x,0)\, dx + \int_{\mathbb{R}^n} g_2(x, U(x,0))\, U(x,0)\, dx \\ \geqslant\;& \frac{\|U_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}}{8} - C_\varepsilon,\end{aligned}$$

if ε is small enough, up to renaming constants. Recalling that $p+1 = 2_s^*$, the formula above gives the proof of (5.2.4). □

Finally, we recall (1.2.15) and (1.2.18) and we show the following:

Corollary 5.2.3. *Let ε, $\kappa > 0$. There exists $M > 0$, possibly depending on κ, ε, n and s, such that the following statement holds true.*

For any $U \in H^s_a(\mathbb{R}^{n+1}_+)$ such that

$$|\mathcal{I}_\varepsilon(U)| + \sup_{\substack{V\in H^s_a(\mathbb{R}^{n+1}_+)\\ [V]_a=1}} \left|\langle \mathcal{I}'_\varepsilon(U), V\rangle\right| \leqslant \kappa$$

one has that

$$[U]_a \leqslant M.$$

Proof. If $[U]_a = 0$ we are done, so we suppose that $[U]_a \neq 0$ and we obtain that

$$\left|\langle \mathcal{I}'_\varepsilon(U), \frac{U}{[U]_a}\rangle\right| \leqslant \kappa.$$

This and (1.2.18) give that

$$\left|[U]_a^2 - \int_{\mathbb{R}^n} g(x, U(x,0))U(x,0)\, dx\right| \leqslant \kappa\, [U]_a. \tag{5.2.8}$$

Therefore

$$\begin{aligned}\kappa + \frac{\kappa\,[U]_a}{2} &\geqslant \mathcal{I}_\varepsilon(U) - \frac{1}{2}\left([U]_a^2 - \int_{\mathbb{R}^n} g(x, U(x,0))U(x,0)\, dx\right) \\ &= -\int_{\mathbb{R}^n} G(x, U(x,0))\, dx + \frac{1}{2}\int_{\mathbb{R}^n} g(x, U(x,0))U(x,0)\, dx \\ &= \frac{1}{p+1}\int_{\mathbb{R}^n} g(x, U(x,0))U(x,0)\, dx - \int_{\mathbb{R}^n} G(x, U(x,0))\, dx \\ &\qquad + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n} g(x, U(x,0))U(x,0)\, dx.\end{aligned}$$

Consequently, by fixing $\delta \in (0,1)$, to be taken conveniently small in the sequel, and using (5.2.3) and (5.2.4),

$$\begin{aligned}\kappa + \frac{\kappa\,[U]_a}{2} \quad &\geqslant \quad -C(\varepsilon+\delta)\|U_+(\cdot,0)\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^n)} - C_{\delta,\varepsilon} \\ &\qquad + \left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{\|U_+(\cdot,0)\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^n)}}{8} - C_\varepsilon\right),\end{aligned}$$

for suitable C, $C_{\delta,\varepsilon}$ and $C_\varepsilon > 0$. By taking δ and ε appropriately small, we thus obtain that

$$\kappa + \frac{\kappa\,[U]_a}{2} \geqslant \left(\frac{1}{2}-\frac{1}{p+1}\right)\frac{\|U_+(\cdot,0)\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^n)}}{16} - C_\varepsilon,$$

up to renaming the latter constant (this fixes δ once and for all). That is

$$\|U_+(\cdot,0)\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^n)} \leqslant M_1([U]_a+1), \tag{5.2.9}$$

for a suitable M_1, possibly depending on κ, ε, n and s.

Now we recall (5.2.8) and (5.2.3) (used here with $\delta := 1$), and we see that

$$\begin{aligned}&\kappa + \frac{\kappa\,[U]_a}{p+1} \\ \geqslant\;& \mathcal{I}_\varepsilon(U) - \frac{1}{p+1}\left([U]_a^2 - \int_{\mathbb{R}^n} g(x,U(x,0))U(x,0)\,dx\right) \\ =\;& \left(\frac{1}{2}-\frac{1}{p+1}\right)[U]_a^2 + \frac{1}{p+1}\int_{\mathbb{R}^n} g(x,U(x,0))U(x,0)\,dx \\ &\quad - \int_{\mathbb{R}^n} G(x,U(x,0))\,dx \\ \geqslant\;& \left(\frac{1}{2}-\frac{1}{p+1}\right)[U]_a^2 - C\|U_+(\cdot,0)\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^n)} - C_\varepsilon,\end{aligned}$$

for suitable C, $C_\varepsilon > 0$. As a consequence,

$$[U]_a^2 \leqslant M_2\big([U]_a + \|U_+(\cdot,0)\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^n)} + 1\big),$$

for a suitable M_2, possibly depending on κ, ε, n and s. Hence, from (5.2.9),

$$[U]_a^2 \leqslant M_3\big([U]_a + +1\big),$$

for some M_3, possibly depending on κ, ε, n and s. This implies the desired result. □

5.3. Some convergence results in view of Theorem 1.2.4

In this section we collect two convergence results that we will need in the sequel.

The first one shows that weak convergence to 0 in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$ implies a suitable integral convergence.

LEMMA 5.3.1. *Let α, $\beta > 0$ with $\alpha + \beta \leqslant 2_s^*$. Let $V_k \in \dot{H}_a^s(\mathbb{R}_+^{n+1})$ be a sequence such that V_k converges to 0 weakly in $\dot{H}_a^s(\mathbb{R}_+^{n+1})$. Let $U_o \in \dot{H}_a^s(\mathbb{R}_+^{n+1})$, with $U_o(\cdot, 0) \in L^\infty(\mathbb{R}^n)$, and $\psi \in L^{\mathrm{a}}(\mathbb{R}^n) \cap L^{\mathrm{b}}(\mathbb{R}^n)$, where*

$$\mathrm{a} := \begin{cases} \frac{2_s^*}{2_s^* - \alpha - \beta} & \text{if } \alpha + \beta < 2_s^*, \\ +\infty & \text{if } \alpha + \beta = 2_s^*, \end{cases}$$

and

$$\mathrm{b} := \begin{cases} \frac{2_s^* + \alpha}{2_s^* - \alpha - \beta} & \text{if } \alpha + \beta < 2_s^*, \\ +\infty & \text{if } \alpha + \beta = 2_s^*, \end{cases}$$

Then, up to a subsequence,

$$\lim_{k \to +\infty} \int_{\mathbb{R}^n} |\psi(x)| \left|(V_k)_+(x, 0)\right|^\alpha \left|U_o(x, 0)\right|^\beta dx = 0.$$

PROOF. Since weakly convergent sequences are bounded, we have that $[V_k]_a \leqslant C_o$, for every $k \in \mathbb{N}$ and a suitable $C_o > 0$. Accordingly, by (1.2.6), we obtain that $[v_k]_{\dot{H}^s(\mathbb{R}^n)} \leqslant C_o$, where $v_k(x) := V_k(x, 0)$. As a consequence, by Theorem 7.1 of [**21**], we know that, up to a subsequence, v_k converges to some v in $L^\gamma_{\mathrm{loc}}(\mathbb{R}^n)$ for any $\gamma \in [1, 2_s^*)$, and a.e.: we claim that

$$v = 0. \tag{5.3.1}$$

To prove this, let $\eta \in C_0^\infty(\mathbb{R}^n)$ and ψ be the solution of

$$(-\Delta)^s \psi = \eta \text{ in } \mathbb{R}^n. \tag{5.3.2}$$

Also, let Ψ be the extension of ψ according to (1.2.3). In particular, $\mathrm{div}(y^a \nabla \Psi) = 0$ in $\mathbb{R}_+^{n+1}$, therefore

$$\int_{\mathbb{R}_+^{n+1}} \mathrm{div}(y^a V_k \nabla \Psi)\, dX = \int_{\mathbb{R}_+^{n+1}} y^a \langle \nabla V_k, \nabla \Psi \rangle\, dX.$$

The latter term is infinitesimal as $k \to +\infty$, thanks to the weak convergence of V_k in $\dot{H}_a^s(\mathbb{R}_+^{n+1})$. Thus, using the Divergence Theorem in the left hand side of the identity above, we obtain

$$\lim_{k \to +\infty} \int_{\mathbb{R}^n} v_k(x) \partial_y \Phi(x)\, dx = 0.$$

That is, recalling (5.3.2) and the convergence of v_k,

$$\int_{\mathbb{R}^n} v(x) \eta(x)\, dx = \lim_{k \to +\infty} \int_{\mathbb{R}^n} v_k(x) \eta(x)\, dx = 0.$$

Since η is arbitrary, we have established (5.3.1).

Now we set $u_o(x) := U_o(x, 0)$ and we observe that $u_o \in L^{2_s^*}(\mathbb{R}^n)$, thanks to Proposition 1.2.1. Therefore, we can fix $\varepsilon > 0$ and find $R_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_{R_\varepsilon}} |u_o(x)|^{2_s^*}\, dx \leqslant \varepsilon.$$

In virtue of (1.2.7), $\|v_k\|_{L^{2_s^*}(\mathbb{R}^n)} \leqslant S^{-1/2}C_o$. Consequently, using the Hölder inequality with exponents a, $2_s^*/\alpha$ and $2_s^*/\beta$, we deduce that

(5.3.3)

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B_{R_\varepsilon}} \left|\psi(x)\right| \left|(V_k)_+(x,0)\right|^\alpha \left|U_o(x,0)\right|^\beta dx \\
&\qquad \leqslant \|\psi\|_{L^{\mathrm{a}}(\mathbb{R}^n)} \left[\int_{\mathbb{R}^n\setminus B_{R_\varepsilon}} \left|V_k(x,0)\right|^{2_s^*} dx\right]^{\frac{\alpha}{2_s^*}} \left[\int_{\mathbb{R}^n\setminus B_{R_\varepsilon}} \left|U_o(x,0)\right|^{2_s^*} dx\right]^{\frac{\beta}{2_s^*}} \\
&\qquad \leqslant \|\psi\|_{L^{\mathrm{a}}(\mathbb{R}^n)} \left(S^{-1/2}C_o\right)^{\frac{\alpha}{2_s^*}} \varepsilon^{\frac{\beta}{2_s^*}}.
\end{aligned}$$

Now we fix $\gamma := \frac{\alpha+2_s^*}{2}$. Notice that $\gamma \in (1, 2_s^*)$, thus, using the convergence of v_k and (5.3.1), we see that

$$\lim_{k\to+\infty} \|v_k\|_{L^\gamma(B_{R_\varepsilon})} = 0.$$

In addition,

$$\int_{\mathbb{R}^n} |u_o(x)|^{2_s^*+\alpha} dx \leqslant \|u_o\|^\alpha_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |u_o(x)|^{2_s^*} dx \leqslant C_*,$$

for some $C_* > 0$. Therefore we use the Hölder inequality with exponents b, $\frac{\gamma}{\alpha} = \frac{\alpha+2_s^*}{2\alpha}$ and $\frac{2_s^*+\alpha}{\beta}$, and we obtain

$$\begin{aligned}
&\lim_{k\to+\infty} \int_{B_{R_\varepsilon}} \left|\psi(x)\right| \left|(V_k)_+(x,0)\right|^\alpha \left|U_o(x,0)\right|^\beta dx \\
\leqslant\ & \|\psi\|_{L^{\mathrm{b}}(\mathbb{R}^n)} \lim_{k\to+\infty} \left[\int_{B_{R_\varepsilon}} \left|v_k(x)\right|^\gamma\right]^{\frac{\alpha}{\gamma}} \left[\int_{\mathbb{R}^n} \left|u_o(x)\right|^{2_s^*+\alpha} dx\right]^{\frac{\beta}{2_s^*+\alpha}} \\
=\ & 0.
\end{aligned}$$

From this and (5.3.3), we see that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} \left|\psi(x)\right| \left|(V_k)_+(x,0)\right|^\alpha \left|U_o(x,0)\right|^\beta dx \leqslant \|\psi\|_{L^{\mathrm{a}}(\mathbb{R}^n)} \left(S^{-1/2}C_o\right)^{\frac{\alpha}{2_s^*}} \varepsilon^{\frac{\beta}{2_s^*}}.$$

The desired result then follows by taking ε as small as we wish. □

As a corollary we have

Corollary 5.3.2. *Let V_k, U_o and ψ as in Lemma 5.3.1. Then*

(5.3.4)

$$\|(U_o + (V_k)_+)(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} = \|U_o\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} + \|(V_k)_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} + o_k(1),$$

and

$$\left|\int_{\mathbb{R}^n} \psi(x) \left((U_o + (V_k)_+)^{q+1}(x,0) - U_o^{q+1}(x,0)\right) dx\right| \leqslant C + o_k(1), \tag{5.3.5}$$

for some $C > 0$.

Proof. Formula 5.3.4 plainly follows from Lemma 5.3.1, by taking $\psi := 1$ (notice that $p+1 = 2^*_s$).

To prove (5.3.5), we use Lemma 5.3.1 to see that

(5.3.6)

$$\begin{aligned}
&\int_{\mathbb{R}^n} \psi(x)\left((U_o + (V_k)_+)^{q+1}(x,0) - U_o^{q+1}(x,0)\right)\,dx\\
&= \int_{\mathbb{R}^n} \psi(x)(U_o + (V_k)_+)^{q+1}(x,0)\,dx - \int_{\mathbb{R}^n} \psi(x)U_o^{q+1}(x,0)\,dx\\
&= \int_{\mathbb{R}^n} \psi(x)U_o^{q+1}(x,0)\,dx + \int_{\mathbb{R}^n} \psi(x)(V_k)_+^{q+1}(x,0)\,dx + o_k(1) - \int_{\mathbb{R}^n} \psi(x)U_o^{q+1}(x,0)\,dx\\
&= \int_{\mathbb{R}^n} \psi(x)(V_k)_+^{q+1}(x,0)\,dx + o_k(1).
\end{aligned}$$

By Hölder inequality with exponents $\frac{2^*_s}{2^*_s - q - 1}$ and $\frac{2^*_s}{q+1}$ and by Proposition 1.2.1 we have that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \psi(x)(V_k)_+^{q+1}(x,0)\,dx\\
&\quad\leqslant \left(\int_{\mathbb{R}^n} |\psi(x)|^{\frac{2^*_s}{2^*_s-q-1}}(x,0)\,dx\right)^{\frac{2^*_s-q-1}{2^*_s}} \left(\int_{\mathbb{R}^n} (V_k)_+^{2^*_s}(x,0)\,dx\right)^{\frac{q+1}{2^*_s}}\\
&\quad\leqslant \|\psi\|_{L^{\frac{2^*_s}{2^*_s-q-1}}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} (V_k)_+^{2^*_s}(x,0)\,dx\right)^{\frac{q+1}{2^*_s}}\\
&\quad\leqslant S^{-\frac{q+1}{2}} \|\psi\|_{L^{\frac{2^*_s}{2^*_s-q-1}}(\mathbb{R}^n)} [V_k]_a.
\end{aligned}$$

Now notice that in this case $\alpha + \beta = q + 1 < 2^*_s$, and so $\psi \in L^{\mathrm{a}}(\mathbb{R}^n)$, with $\mathrm{a} = \frac{2^*_s}{2^*_s - \mathrm{q} - 1}$, by hypothesis. Moreover, since V_k is a weakly convergent sequence in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$, then V_k is uniformly bounded in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$. Hence

$$\int_{\mathbb{R}^n} \psi(x)(V_k)_+^{q+1}(x,0)\,dx \leqslant C,$$

for a suitable $C > 0$. Plugging this information into (5.3.6), we obtain that

$$\left|\int_{\mathbb{R}^n} \psi(x)\left((U_o + (V_k)_+)^{q+1}(x,0) - U_o^{q+1}(x,0)\right)\,dx\right| \leqslant C + o_k(1),$$

as desired. □

Now we show that, under an assumption on the positivity of the limit function, weak convergence in $\dot{H}^s_a(\mathbb{R}^{n+1})$ implies weak convergence of the positive part.

Lemma 5.3.3. *Assume that W_m is a sequence of functions in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$ that converges weakly in $\dot{H}^s(\mathbb{R}^{n+1}_+)$ to $W \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$.*

Suppose also that for any bounded set $K \subset \mathbb{R}^{n+1}_+$, we have that

$$\inf_K W > 0.$$

Then, up to a subsequence, $(W_m)_+ \in H^s_a(\mathbb{R}^{n+1}_+)$ and it also converges weakly in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$ to W.

PROOF. Notice that $|\nabla(W_m)_+| = |\nabla W_m|\chi_{\{W_m>0\}} \leqslant |\nabla W_m|$ a.e., which shows that $(W_m)_+ \in H^s_a(\mathbb{R}^{n+1}_+)$.

We also recall that, since weakly convergent sequences are bounded,

$$\sup_{m\in\mathbb{N}} \int_{\mathbb{R}^{n+1}_+} y^a |\nabla W_m|^2\, dX \leqslant C_o, \tag{5.3.7}$$

for some $C_o > 0$.

Now we claim that

$$\begin{aligned} &\text{for any } \Phi \in C^\infty_0(\overline{\mathbb{R}^{n+1}_+}),\\ &\lim_{m\to+\infty} \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla (W_m)_+, \nabla\Phi\rangle\, dX = \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla W, \nabla\Phi\rangle\, dX. \end{aligned} \tag{5.3.8}$$

For this, we let K be the support of Φ. Up to a subsequence, we know that W_m converges a.e. to W. Therefore, by Egorov Theorem, fixed $\varepsilon > 0$, there exists $K_\varepsilon \subseteq K$ such that W_m converges to W uniformly in K_ε and $|K \setminus K_\varepsilon| \leqslant \varepsilon$. Then, for any $X \in K_\varepsilon$,

$$W_m(X) \geqslant W(X) - \|W_m - W\|_{L^\infty(K_\varepsilon)} \geqslant \inf_K W - \|W_m - W\|_{L^\infty(K_\varepsilon)} \geqslant \frac{1}{2}\inf_K W > 0,$$

as long as m is large enough, say $m \geqslant m_\star(K,\varepsilon)$.

Accordingly, $\nabla(W_m)_+ = \nabla W_m$ a.e. in K_ε if $m \geqslant m_\star(K,\varepsilon)$ and therefore

$$\lim_{m\to+\infty} \int_{K_\varepsilon} y^a \langle \nabla (W_m)_+, \nabla\Phi\rangle\, dX = \int_{K_\varepsilon} y^a \langle \nabla W, \nabla\Phi\rangle\, dX. \tag{5.3.9}$$

Moreover, for any $\eta > 0$, the absolute continuity of the integral gives that

$$\int_{K\setminus K_\varepsilon} y^a |\nabla W|^2\, dX + \int_{K\setminus K_\varepsilon} y^a |\nabla \Phi|^2\, dX \leqslant \eta,$$

provided that ε is small enough, say $\varepsilon \in (0, \varepsilon_\star(\eta))$, for a suitable $\varepsilon_\star(\eta)$. As a consequence, recalling (5.3.7),

$$\begin{aligned} &\left| \int_{K\setminus K_\varepsilon} y^a \langle \nabla (W_m)_+, \nabla\Phi\rangle\, dX \right| + \left| \int_{K\setminus K_\varepsilon} y^a \langle \nabla W, \nabla\Phi\rangle\, dX \right| \\ \leqslant\ & \sqrt{\int_{K\setminus K_\varepsilon} y^a |\nabla (W_m)_+|^2\, dX} \cdot \sqrt{\int_{K\setminus K_\varepsilon} y^a |\nabla\Phi|^2\, dX} \\ &\quad + \sqrt{\int_{K\setminus K_\varepsilon} y^a |\nabla W|^2\, dX} \cdot \sqrt{\int_{K\setminus K_\varepsilon} y^a |\nabla\Phi|^2\, dX} \\ \leqslant\ & \sqrt{C_o \eta} + \eta. \end{aligned}$$

Using this and (5.3.9), we obtain that

$$\begin{aligned}
& \left| \lim_{m\to+\infty} \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla (W_m)_+, \nabla\Phi\rangle \, dX - \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla W, \nabla\Phi\rangle \, dX \right| \\
= \; & \left| \lim_{m\to+\infty} \int_{K} y^a \langle \nabla (W_m)_+, \nabla\Phi\rangle \, dX - \int_{K} y^a \langle \nabla W, \nabla\Phi\rangle \, dX \right| \\
\leqslant \; & \left| \lim_{m\to+\infty} \int_{K\setminus K_\varepsilon} y^a \langle \nabla (W_m)_+, \nabla\Phi\rangle \, dX - \int_{K\setminus K_\varepsilon} y^a \langle \nabla W, \nabla\Phi\rangle \, dX \right| \\
\leqslant \; & \sqrt{C_o\eta} + \eta.
\end{aligned}$$

By taking η as small as we like, we complete the proof of (5.3.8).

Now we finish the proof of Lemma 5.3.3 by a density argument. Let $\Phi \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$ and $\varepsilon > 0$. We take $\Phi_\varepsilon \in C^\infty_0(\mathbb{R}^{n+1}_+)$ such that

$$\int_{\mathbb{R}^{n+1}_+} y^a |\nabla(\Phi - \Phi_\varepsilon)|^2 \, dX \leqslant \varepsilon.$$

The existence of such Φ_ε is guaranteed by (1.2.5). Then, recalling (5.3.7) and (5.3.8) for the function Φ_ε, we obtain that

$$\begin{aligned}
& \left| \lim_{m\to+\infty} \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla (W_m)_+, \nabla\Phi\rangle \, dX - \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla W, \nabla\Phi\rangle \, dX \right| \\
\leqslant \; & \left| \lim_{m\to+\infty} \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla (W_m)_+, \nabla\Phi_\varepsilon\rangle \, dX - \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla W, \nabla\Phi_\varepsilon\rangle \, dX \right| \\
& \quad + \left(\sqrt{C_o} + \sqrt{\int_{\mathbb{R}^{n+1}_+} y^a |\nabla W|^2 \, dX} \right) \sqrt{\int_{\mathbb{R}^{n+1}_+} y^a |\nabla(\Phi - \Phi_\varepsilon)|^2 \, dX} \\
\leqslant \; & 0 + \left(\sqrt{C_o} + \sqrt{\int_{\mathbb{R}^{n+1}_+} y^a |\nabla W|^2 \, dX} \right) \sqrt{\varepsilon}.
\end{aligned}$$

Accordingly, by taking ε as small as we please, we obtain that

$$\lim_{m\to+\infty} \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla (W_m)_+, \nabla\Phi\rangle \, dX = \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla W, \nabla\Phi\rangle \, dX,$$

for any $\Phi \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$, thus completing the proof of Lemma 5.3.3. □

5.4. Palais-Smale condition for $\mathcal{J}_\varepsilon$

Once we have found a minimum of $\mathcal{J}_\varepsilon$, we apply a contradiction procedure to prove the existence of a second critical point.

Roughly speaking, the idea is the following: let us suppose that $U = 0$ is the only critical point; thus, we prove some compactness and geometric properties of the functional (based on the fact that the critical point is unique), and these facts allow us to apply the Mountain Pass Theorem, that provides a second critical point. Hence, we reach a contradiction, so $U = 0$ cannot be the only critical point of $\mathcal{J}_\varepsilon$.

As we did in Proposition 3.2.1 for the minimal solution, also to find the second solution we need to prove that a Palais-Smale condition holds true below a certain threshold, as stated in the following result:

PROPOSITION 5.4.1. *There exists $C > 0$, depending on h, q, n and s, such that the following statement holds true. Let $\{U_k\}_{k\in\mathbb{N}} \subset \dot{H}^s_a(\mathbb{R}^{n+1}_+)$ be a sequence satisfying*

(i) $\lim_{k\to+\infty} \mathcal{J}_\varepsilon(U_k) = c_\varepsilon$, *with*

$$c_\varepsilon + C\varepsilon^{\frac{1}{2\gamma}} < \frac{s}{n} S^{\frac{n}{2s}}, \tag{5.4.1}$$

where $\gamma = 1 + \frac{2}{n-2s}$ and S is the Sobolev constant appearing in Proposition 1.2.1,

(ii) $\lim_{k\to+\infty} \mathcal{J}'_\varepsilon(U_k) = 0$.

Assume also that $U = 0$ is the only critical point of $\mathcal{J}_\varepsilon$.

Then $\{U_k\}_{k\in\mathbb{N}}$ contains a subsequence strongly convergent in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$.

REMARK 5.4.2. The limit in (ii) is intended in the following way

$$\begin{aligned}\lim_{k\to+\infty} &\|\mathcal{J}'_\varepsilon(U_k)\|_{\mathcal{L}(\dot{H}^s_a(\mathbb{R}^{n+1}_+), \dot{H}^s_a(\mathbb{R}^{n+1}_+))} \\ &= \lim_{k\to+\infty} \sup_{V\in\dot{H}^s_a(\mathbb{R}^{n+1}_+), [V]_a=1} \left|\langle \mathcal{J}'_\varepsilon(U_k), V\rangle\right| = 0,\end{aligned}$$

where $\mathcal{L}(\dot{H}^s_a(\mathbb{R}^{n+1}_+), \dot{H}^s_a(\mathbb{R}^{n+1}_+))$ consists of all the linear functionals from $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$ in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$.

We observe that a sequence that satisfies the assumptions of Proposition 5.4.1 is weakly convergent. The precise statement goes as follows:

LEMMA 5.4.3. *Let $\{U_k\}_{k\in\mathbb{N}} \subset \dot{H}^s_a(\mathbb{R}^{n+1}_+)$ be a sequence satisfying the hypotheses of Proposition 5.4.1. Assume also that $U = 0$ is the only critical point of $\mathcal{J}_\varepsilon$.*

Then, up to a subsequence, U_k weakly converges to 0 in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$ as $k \to +\infty$.

PROOF. Notice that assumptions (i) and (ii) imply that there exists $\kappa > 0$ such that

$$|\mathcal{J}_\varepsilon(U_k)| + \sup_{\substack{V\in H^s_a(\mathbb{R}^{n+1}_+) \\ [V]_a=1}} \left|\langle \mathcal{J}'_\varepsilon(U_k), V\rangle\right| \leqslant \kappa.$$

Hence, by Corollary 5.2.3 we have that there exists a positive constant M (independent of k) such that

$$[U_k]_a \leqslant M. \tag{5.4.2}$$

Therefore, there exists a subsequence (that we still denote by U_k) converging weakly to some function $U_\infty \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$, that is,

$$U_k \rightharpoonup U_\infty \quad \text{in } \dot{H}^s_a(\mathbb{R}^{n+1}_+), \tag{5.4.3}$$

as $k \to +\infty$. We now claim that

$$U_\infty = 0. \tag{5.4.4}$$

For this, we first observe that, thanks to (5.4.2) and Theorem 7.1 in [**21**], we have that

$$U_k(\cdot, 0) \to U_\infty(\cdot, 0) \qquad \text{in } L^\alpha_{\text{loc}}(\mathbb{R}^n), \qquad 1 \leqslant \alpha < 2^*_s, \tag{5.4.5}$$

and so

$$U_k(\cdot, 0) \to U_\infty(\cdot, 0) \quad \text{a. e. } \mathbb{R}^n. \tag{5.4.6}$$

Let now $\Psi \in C^\infty_0(\mathbb{R}^{n+1})$, $\psi := \Psi(\cdot, 0)$ and $K := supp(\psi)$. According to (1.2.18),

$$\begin{aligned} \langle \mathcal{I}'_\varepsilon(U_k), \Psi\rangle &= \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_k, \nabla \Psi\rangle \, dX - \int_{\mathbb{R}^n} g(x, U_k(x,0))\psi(x)\, dx \\ &= \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_k, \nabla \Psi\rangle \, dX - \int_K g(x, U_k(x,0))\psi(x)\, dx. \end{aligned} \tag{5.4.7}$$

Thanks to (5.4.3), we have that

$$\int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_k, \nabla \Psi\rangle \, dX \to \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_\infty, \nabla \Psi\rangle \, dX \tag{5.4.8}$$

as $k \to +\infty$. Moreover, (5.4.6) implies that

$$g(x, U_k(\cdot, 0)) \to g(x, U_\infty(\cdot, 0)) \quad \text{a. e. } \mathbb{R}^n,$$

as $k \to +\infty$. Also, notice that

$$(1+r)^p - 1 \leqslant C(1 + r^p),$$

for any $r \geqslant 0$ and for some positive constant $C > 0$. Hence, recalling that $U_\varepsilon > 0$, thanks to Proposition 1.2.3, we can use this with $r := t/U_\varepsilon$ and we obtain that

$$(U_\varepsilon + t)^p - U_\varepsilon^p = U_\varepsilon^p \left[\left(1 + \frac{t}{U_\varepsilon}\right)^p - 1\right] \leqslant C\, U_\varepsilon^p \left(1 + \frac{t^p}{U_\varepsilon^p}\right) = C\,(U_\varepsilon^p + t^p).$$

This, formulas (3.1.3) and (1.2.16) give that

$$|g(x,t)| \leqslant C\,(U_\varepsilon^p + t^p) + \varepsilon|h|. \tag{5.4.9}$$

Hence, for any $k \in \mathbb{N}$,

$$|g(x, U_k(x,0))|\,|\psi| \leqslant C|\psi| \left(U_\varepsilon^p(x,0) + U_k^p(x,0)\right) + \varepsilon|h||\psi|. \tag{5.4.10}$$

This means that the sequence $g(\cdot, U_k(\cdot, 0))$ is bounded by a sequence that is strongly convergent in $L^1_{\text{loc}}(\mathbb{R}^n)$. Moreover, by Theorem 4.9 in [**10**] we have that there exists a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that, up to a subsequence,

$$C|\psi| \left(U_\varepsilon^p(\cdot,0) + U_k^p(\cdot,0)\right) + \varepsilon|\psi||h| \leqslant |f|. \tag{5.4.11}$$

Formulas (5.4.10) and (5.4.11) together with (5.4.6) imply that we can use the Dominated Convergence Theorem (see e.g. Theorem 4.2 in [**10**]) and we obtain that

$$\int_K g(x, U_k(x,0))\psi(x)\,dx \to \int_K g(x, U_\infty(x,0))\psi(x)\,dx,$$

as $k \to +\infty$. Using this and (5.4.8) into (5.4.7) we have

$$\langle \mathcal{J}'_\varepsilon(U_k), \Psi\rangle \to \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_\infty, \nabla \Psi\rangle\,dX - \int_{\mathbb{R}^n} g(x, U_\infty(x,0))\psi(x)\,dx = \langle \mathcal{J}'_\varepsilon(U_\infty), \Psi\rangle,$$

as $k \to +\infty$. On the other hand, assumption (ii) implies that

$$\langle \mathcal{J}'_\varepsilon(U_k), \Psi\rangle \to 0$$

as $k \to +\infty$. The last two formulas imply that

$$\langle \mathcal{J}'_\varepsilon(U_\infty), \Psi\rangle = 0, \quad \text{for any } \Psi \in C_0^\infty(\mathbb{R}^{n+1}_+). \tag{5.4.12}$$

Let now $\Psi \in \dot{H}^s_a(\mathbb{R}^{n+1})$, with $\psi := \Psi(\cdot, 0)$. Then by (1.2.5) there exists a sequence of functions $\Psi_m \in C_0^\infty(\mathbb{R}^{n+1}_+)$, with $\psi_m := \Psi_m(\cdot, 0)$, such that

$$\Psi_m \to \Psi \text{ in } \dot{H}^s_a(\mathbb{R}^{n+1}_+) \text{ as } m \to +\infty. \tag{5.4.13}$$

By Proposition 1.2.1 this implies also that

$$\psi_m \to \psi \text{ in } L^{2^*_s}(\mathbb{R}^n) \text{ as } m \to +\infty. \tag{5.4.14}$$

Therefore, from (5.4.12) we deduce that for any $m \in \mathbb{N}$

$$0 = \langle \mathcal{J}'_\varepsilon(U_\infty), \Psi_m\rangle = \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_\infty, \nabla \Psi_m\rangle\,dX - \int_{\mathbb{R}^n} g(x, U_\infty(x,0))\psi_m(x)\,dx. \tag{5.4.15}$$

Now, (5.4.13) implies that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_\infty, \nabla \Psi_m\rangle\,dX - \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_\infty, \nabla \Psi\rangle\,dX \right| \\
&\leqslant \int_{\mathbb{R}^{n+1}_+} y^a |\nabla U_\infty|\,|\nabla(\Psi_m - \Psi)|\,dX \\
&\leqslant \sqrt{\int_{\mathbb{R}^{n+1}_+} y^a |\nabla U_\infty|^2\,dX} \cdot \sqrt{\int_{\mathbb{R}^{n+1}_+} y^a |\nabla(\Psi_m - \Psi)|^2\,dX} \to 0,
\end{aligned}$$

as $m \to +\infty$. Moreover, by Hölder inequality with exponents 2^*_s and $\frac{2^*_s}{2^*_s-1}$ we get

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} g(x, U_\infty(x,0))\psi_m(x)\,dx - \int_{\mathbb{R}^n} g(x, U_\infty(x,0))\psi(x)\,dx \right| \\
&\leqslant \int_{\mathbb{R}^n} |g(x, U_\infty(x,0))|\,|\psi_m(x) - \psi(x)|\,dx \\
&\leqslant C \int_{\mathbb{R}^n} (U_\varepsilon^p + U_\infty^p)\,|\psi_m(x) - \psi(x)|\,dx + \varepsilon \int_{\mathbb{R}^n} |h|\,|\psi_m(x) - \psi(x)|\,dx \\
&\leqslant C \left(\int_{\mathbb{R}^n} (U_\varepsilon^p + U_\infty^p)^{\frac{2^*_s}{2^*_s-1}}\,dx \right)^{\frac{2^*_s-1}{2^*_s}} \left(\int_{\mathbb{R}^n} |\psi_m(x) - \psi(x)|^{2^*_s}\,dx \right)^{\frac{1}{2^*_s}} \\
&\quad + \varepsilon \left(\int_{\mathbb{R}^n} |h|^{\frac{2^*_s}{2^*_s-1}}\,dx \right)^{\frac{2^*_s-1}{2^*_s}} \left(\int_{\mathbb{R}^n} |\psi_m(x) - \psi(x)|^{2^*_s}\,dx \right)^{\frac{1}{2^*_s}},
\end{aligned}$$

where we have used (5.4.9). Furthermore, noticing that $\frac{2^*_s p}{2^*_s-1} = 2^*_s$, we have that

$$(U_\varepsilon^p + U_\infty^p)^{\frac{2^*_s}{2^*_s-1}} \leqslant (U_\varepsilon + U_\infty)^{\frac{2^*_s p}{2^*_s-1}} \leqslant (U_\varepsilon + U_\infty)^{2^*_s},$$

up to renaming constants. Thus, since U_ε, $U_\infty \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$, and $h \in L^r(\mathbb{R}^n)$ for every $1 \leqslant r \leqslant +\infty$, by Proposition 1.2.1 we deduce that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} g(x, U_\infty(x,0))\psi_m(x)\,dx - \int_{\mathbb{R}^n} g(x, U_\infty(x,0))\psi(x)\,dx \right| \\
&\leqslant C \left(\int_{\mathbb{R}^n} |\psi_m(x) - \psi(x)|^{2^*_s}\,dx \right)^{\frac{1}{2^*_s}} \to 0
\end{aligned}$$

as $m \to +\infty$, thanks to (5.4.14). All in all and going back to (5.4.15) we obtain that

$$0 = \lim_{m \to +\infty} \langle \mathcal{I}'_\varepsilon(U_\infty), \Psi_m \rangle = \langle \mathcal{I}'_\varepsilon(U_\infty), \Psi \rangle,$$

and this shows that (5.4.12) holds true for any $\Psi \in \dot{H}^s_a(\mathbb{R}^{n+1})$. Namely, U_∞ is a critical point for $\mathcal{I}_\varepsilon$. Since $U = 0$ is the only critical point of $\mathcal{I}_\varepsilon$, we obtain the claim in (5.4.4). This concludes the proof of Lemma 5.4.3. □

As we did in the first part to obtain the existence of the minimum, (see in particular Lemma 3.2.5), to prove Proposition 5.4.1 we first need to show that the sequence is tight, according to Definition 2.2.1. Then we can prove the following:

LEMMA 5.4.4. *Let $\{U_k\}_{k\in\mathbb{N}} \subset \dot{H}^s_a(\mathbb{R}^{n+1}_+)$ be a sequence satisfying the hypotheses of Proposition 5.4.1. Assume also that $U = 0$ is the only critical point of $\mathcal{I}_\varepsilon$.*

Then, for all $\eta > 0$ there exists $\rho > 0$ such that for every $k \in \mathbb{N}$ there holds

$$\int_{\mathbb{R}^{n+1}_+ \setminus B^+_\rho} y^a |\nabla U_k|^2\,dX + \int_{\mathbb{R}^n \setminus \{B_\rho \cap \{y=0\}\}} |U_k(x,0)|^{2^*_s}\,dx < \eta.$$

In particular, the sequence $\{U_k\}_{k\in\mathbb{N}}$ is tight.

PROOF. From Lemma 5.4.3 we have that

$$\begin{aligned} U_k \rightharpoonup 0 \quad &\text{in } \dot H^s_a(\mathbb{R}^{n+1}_+) \quad \text{as } k\to+\infty \\ \text{and } U_k(\cdot,0)\to 0 \quad &\text{a.e. in } \mathbb{R}^n \quad \text{as } k\to+\infty. \end{aligned} \tag{5.4.16}$$

Now we proceed by contradiction. That is, we suppose that there exists $\eta_0 > 0$ such that for every $\rho > 0$ there exists $k\in\mathbb{N}$ such that

$$\int_{\mathbb{R}^{n+1}_+\setminus B^+_\rho} y^a|\nabla U_k|^2\,dX + \int_{\mathbb{R}^n\setminus(B_\rho\cap\{y=0\})} (U_k)_+^{2^*_s}(x,0)\,dx \geqslant \eta_0. \tag{5.4.17}$$

Proceeding as in (3.2.7), one can prove that actually $k\to+\infty$ as $\rho\to+\infty$.

Let U_ε be the local minimum of the functional $\mathcal{F}_\varepsilon$ found in Theorem 1.2.2. Since $U_\varepsilon\in\dot H^s_a(\mathbb{R}^{n+1}_+)$, from Propositions 2.1.1 and 1.2.1 we have that for any $\varepsilon>0$ there exists $r:=r_\varepsilon>0$ such that

$$\begin{aligned} &\int_{\mathbb{R}^{n+1}_+\setminus B^+_r} y^a|\nabla U_\varepsilon|^2\,dX + \int_{\mathbb{R}^{n+1}_+\setminus B^+_r} y^a|U_\varepsilon|^{2\gamma}\,dX \\ &\qquad + \int_{\mathbb{R}^n\setminus\left(B^+_r\cap\{y=0\}\right)} |U_\varepsilon(x,0)|^{2^*_s}\,dx < \varepsilon, \end{aligned} \tag{5.4.18}$$

where $\gamma := 1+\frac{2}{n-2s}$. Moreover, by (5.4.2) and again by Propositions 2.1.1 and 1.2.1 we deduce that

$$\begin{aligned} &\int_{\mathbb{R}^{n+1}_+} y^a|\nabla U_k|^2\,dX + \int_{\mathbb{R}^{n+1}_+} y^a|U_k|^{2\gamma}\,dX + \int_{\mathbb{R}^n} |U_k(x,0)|^{2^*_s}\,dx \\ &\quad + \int_{\mathbb{R}^{n+1}_+} y^a|\nabla(U_k+U_\varepsilon)|^2\,dX + \int_{\mathbb{R}^{n+1}_+} y^a(|U_k|+U_\varepsilon)^{2\gamma}\,dX \\ &\quad + \int_{\mathbb{R}^n} (|U_k(x,0)|+U_\varepsilon(x,0))^{2^*_s}\,dx \leqslant \tilde M, \end{aligned} \tag{5.4.19}$$

for some $\tilde M>0$.

Now let $j_\varepsilon\in\mathbb{N}$ be integer part of $\frac{\tilde M}{\varepsilon}$, and set, for any $l\in\{0,1,\dots,j_\varepsilon\}$

$$I_l := \{(x,y)\in\mathbb{R}^{n+1}_+ : r+l\leqslant|(x,y)|\leqslant r+l+1\}.$$

Notice that $j_\varepsilon\to+\infty$ as $\varepsilon\to0$. Therefore, by (5.4.19) we have that

$$\begin{aligned} (j_\varepsilon+1)\varepsilon &\geqslant \frac{\tilde M}{\varepsilon}\varepsilon \\ &\geqslant \sum_{l=0}^{j_\varepsilon}\Big(\int_{I_l} y^a|\nabla U_k|^2\,dX + \int_{I_l} y^a|U_k|^{2\gamma}\,dX + \int_{I_l\cap\{y=0\}} |U_k(x,0)|^{2^*_s}\,dx \\ &\qquad + \int_{I_l} y^a|\nabla(U_k+U_\varepsilon)|^2\,dX + \int_{I_l} y^a(|U_k|+U_\varepsilon)^{2\gamma}\,dX \\ &\qquad + \int_{I_l\cap\{y=0\}} (|U_k(x,0)|+U_\varepsilon(x,0))^{2^*_s}\,dx\Big). \end{aligned}$$

This implies that there exists $\bar{l} \in \{0, 1, \ldots, j_\varepsilon\}$ such that, up to a subsequence,

$$
\begin{aligned}
&\int_{I_{\bar{l}}} y^a |\nabla U_k|^2 \, dX + \int_{I_{\bar{l}}} y^a |U_k|^{2\gamma} \, dX + \int_{I_{\bar{l}} \cap \{y=0\}} |U_k(x,0)|^{2^*_s} \, dx \\
&\quad + \int_{I_{\bar{l}}} y^a |\nabla (U_k + U_\varepsilon)|^2 \, dX + \int_{I_{\bar{l}}} y^a (|U_k| + U_\varepsilon)^{2\gamma} \, dX \\
&\quad + \int_{I_{\bar{l}} \cap \{y=0\}} (|U_k(x,0)| + U_\varepsilon(x,0))^{2^*_s} \, dx \leqslant \varepsilon.
\end{aligned} \tag{5.4.20}
$$

Let now $\chi \in C_0^\infty(\mathbb{R}^{n+1}_+, [0,1])$ be a cut-off function such that

$$
\chi(x,y) = \begin{cases} 1, & |(x,y)| \leqslant r + \bar{l}, \\ 0, & |(x,y)| \geqslant r + \bar{l} + 1, \end{cases} \qquad \text{and} \qquad |\nabla \chi| \leqslant 2. \tag{5.4.21}
$$

Define, for any $k \in \mathbb{N}$,

$$
W_{1,k} := \chi U_k \quad \text{and} \quad W_{2,k} := (1 - \chi) U_k. \tag{5.4.22}
$$

Hence $W_{1,k} + W_{2,k} = U_k$ for any $k \in \mathbb{N}$. Moreover,

$$
W_{1,k}, \, W_{2,k} \rightharpoonup 0 \quad \text{in } \dot{H}^s_a(\mathbb{R}^{n+1}_+), \tag{5.4.23}
$$

as $k \to +\infty$. Indeed, for any $\Psi \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$ with $[\Psi]_a = 1$ and $\delta > 0$, we have that

$$
\left| \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_k, \nabla \Psi \rangle \, dX \right| \leqslant \frac{\delta}{2},
$$

for any k sufficiently large, say $k \geqslant \bar{k}(\delta)$, thanks to (5.4.16). Moreover, the compactness result in Lemma 2.1.2 implies that

$$
\int_{I_{\bar{l}}} y^a |U_k|^2 \, dX \leqslant \frac{\delta^2}{16},
$$

for k large enough (say $k \geqslant \bar{k}(\delta)$, up to renaming $\bar{k}(\delta)$). Therefore, recalling (5.4.22) and (5.4.21) and using Hölder inequality, we obtain that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla W_{1,k}, \nabla\Psi\rangle\, dX\right| \\
= &\left|\int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla (\chi\, U_k), \nabla\Psi\rangle\, dX\right| \\
\leqslant &\left|\int_{\mathbb{R}^{n+1}_+} y^a \chi \langle \nabla U_k, \nabla\Psi\rangle\, dX\right| + \left|\int_{\mathbb{R}^{n+1}_+} y^a U_k \langle \nabla \chi, \nabla\Psi\rangle\, dX\right| \\
\leqslant &\left|\int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_k, \nabla\Psi\rangle\, dX\right| + \int_{\mathbb{R}^{n+1}_+} y^a |U_k|\, |\nabla\chi|\, |\nabla\Psi|\, dX \\
\leqslant &\ \frac{\delta}{2} + 2\sqrt{\int_{\mathbb{R}^{n+1}_+} y^a |U_k|^2\, dX} \cdot \sqrt{\int_{\mathbb{R}^{n+1}_+} y^a |\nabla\Psi|^2\, dX} \\
\leqslant &\ \frac{\delta}{2} + 2\frac{\delta}{4} \\
= &\ \delta,
\end{aligned}$$

which proves (5.4.23) for $W_{1,k}$. The proof for $W_{2,k}$ is similar, and so we omit it.

Furthermore, from (5.4.23) and Theorem 7.1 in [**21**] we have that

$$\begin{aligned}
&W_{i,k}(\cdot,0) \to 0 \quad \text{a.e. } \mathbb{R}^n, \\
&\text{and } (U_\varepsilon + W_{i,k})(\cdot,0) \to U_\varepsilon(\cdot,0) \ \text{ in } L^\alpha_{\text{loc}}(\mathbb{R}^n), \ \ \forall\, 1 \leqslant \alpha < 2^*_s,\ i=1,2,
\end{aligned} \tag{5.4.24}$$

as $k \to +\infty$. Notice also that there exists a positive constant C (independent of k) such that

$$[U_\varepsilon + W_{i,k}]_a \leqslant C, \tag{5.4.25}$$

for $i \in \{1,2\}$. Let us show (5.4.25) only for $W_{1,k}$, being the proof for $W_{2,k}$ similar. From (5.4.22) we obtain that

$$\begin{aligned}
[W_{i,k}]_a^2 &= \int_{\mathbb{R}^{n+1}_+} y^a |\nabla W_{1,k}|^2\, dX = \int_{\mathbb{R}^{n+1}_+} y^a |\nabla(\chi U_k)|^2\, dX \\
&\leqslant 2\int_{\mathbb{R}^{n+1}_+} y^a \chi^2 |\nabla U_k|^2\, dX + 2\int_{\mathbb{R}^{n+1}_+} y^a |U_k|^2 |\nabla\chi|^2\, dX \\
&\leqslant 2M + 8\int_{I_{\bar{l}}} y^a |U_k|^2\, dX \leqslant 2M + 8C,
\end{aligned}$$

for some $C > 0$ independent of k, thanks to (5.4.2), (5.4.21) and Lemma 2.1.2. This, together with the fact that $U_\varepsilon \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$, gives (5.4.25).

Therefore, using hypothesis (ii),

$$\lim_{k\to+\infty}\langle \mathcal{I}'_\varepsilon(U_k), U_\varepsilon+W_{i,k}\rangle = 0, \qquad i=1,2. \tag{5.4.26}$$

On the other hand, by (1.2.18),

$$\begin{aligned}
&\left|\langle \mathcal{I}'_\varepsilon(U_k)-\mathcal{I}'_\varepsilon(W_{1,k}), U_\varepsilon+W_{1,k}\rangle\right| \\
\leqslant& \left|\int_{\mathbb{R}^{n+1}_+} y^a\langle\nabla(1-\chi)U_k, \nabla(U_\varepsilon+W_{1,k})\rangle\,dX\right| \\
&+\left|\varepsilon\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon+(W_{1,k})_+)^q-(U_\varepsilon+(U_k)_+)^q\right)(x,0)(U_\varepsilon+W_{1,k})(x,0)\,dx\right| \\
&+\left|\int_{\mathbb{R}^n} \left((U_\varepsilon+(W_{1,k})_+)^p-(U_\varepsilon+(U_k)_+)^p\right)(x,0)(U_\varepsilon+W_{1,k})(x,0)\,dx\right| \\
=:& \, I_1+I_2+I_3.
\end{aligned} \tag{5.4.27}$$

To estimate I_1, notice that $I_1\leqslant I_{1,1}+I_{1,2}$, where

$$\begin{aligned}
I_{1,1} &:= \left|\int_{\mathbb{R}^{n+1}_+} y^a\langle\nabla(1-\chi)U_k, \nabla U_\varepsilon\rangle\,dX\right| \\
\text{and}\quad I_{1,2} &:= \left|\int_{\mathbb{R}^{n+1}_+} y^a\langle\nabla(1-\chi)U_k, \nabla(\chi U_k)\rangle\,dX\right|.
\end{aligned}$$

We split further $I_{1,1}$ as

$$I_{1,1}\leqslant \left|\int_{\mathbb{R}^{n+1}_+\setminus B^+_{r+\bar l}} y^a(1-\chi)\langle\nabla U_k, \nabla U_\varepsilon\rangle\,dX\right| + \left|\int_{I_{\bar l}} y^a U_k\langle\nabla(1-\chi), \nabla U_\varepsilon\rangle\,dX\right|$$

Since $B^+_r\subset B^+_{r+\bar l}$, by Hölder inequality, (5.4.18) and (5.4.2) we have that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{n+1}_+\setminus B^+_{r+\bar l}} y^a(1-\chi)\langle\nabla U_k, \nabla U_\varepsilon\rangle\,dX\right| \\
&\qquad\leqslant \sqrt{\int_{\mathbb{R}^{n+1}_+} y^a|\nabla U_k|^2\,dX}\cdot\sqrt{\int_{\mathbb{R}^{n+1}_+\setminus B^+_r} y^a|\nabla U_\varepsilon|^2\,dX}\leqslant M\varepsilon^{1/2}.
\end{aligned}$$

Moreover, by (5.4.21) and applying twice the Hölder inequality (first with exponent $1/2$ and then with exponents γ and $\frac{\gamma}{\gamma-1}$) we obtain that

$$\begin{aligned}
&\left|\int_{I_{\bar l}} y^a U_k \langle \nabla(1-\chi), \nabla U_\varepsilon\rangle \, dX\right| \\
&\quad \leqslant 2\left(\int_{I_{\bar l}} y^a |U_k|^2\, dX\right)^{1/2} \left(\int_{I_{\bar l}} y^a |\nabla U_\varepsilon|^2\, dX\right)^{1/2} \\
&\quad \leqslant 2\left(\int_{I_{\bar l}} y^a |U_k|^2\, dX\right)^{1/2} \left(\int_{\mathbb{R}^{n+1}_+} y^a |\nabla U_\varepsilon|^2\, dX\right)^{1/2} \\
&\quad \leqslant C\left(\int_{I_{\bar l}} y^a \, dX\right)^{\frac{\gamma-1}{2\gamma}} \left(\int_{I_{\bar l}} y^a |U_k|^{2\gamma}\, dX\right)^{\frac{1}{2\gamma}} \leqslant C\varepsilon^{1/2\gamma},
\end{aligned}$$

up to renaming constants, where (5.4.20) was used in the last line. Hence,

$$I_{1,1} \leqslant C\varepsilon^{1/2\gamma},$$

for a suitable constant $C > 0$. Let us estimate $I_{1,2}$:

$$\begin{aligned}
I_{1,2} \;\leqslant\; & \left|\int_{I_{\bar l}} y^a |U_k|^2 \langle \nabla(1-\chi), \nabla\chi\rangle \, dX\right| + \left|\int_{I_{\bar l}} y^a U_k \chi \langle \nabla(1-\chi), \nabla U_k\rangle \, dX\right| \\
& + \left|\int_{I_{\bar l}} y^a \chi(1-\chi) |\nabla U_k|^2 \, dX\right| + \left|\int_{I_{\bar l}} y^a (1-\chi) U_k \langle \nabla U_k, \nabla\chi\rangle \, dX\right|.
\end{aligned}$$

Thus, in the same way as before, and using (5.4.20) once more, we obtain that $I_{1,2} \leqslant C\varepsilon^{1/2\gamma}$ for some $C > 0$. Therefore

$$I_1 \leqslant C\varepsilon^{1/2\gamma}, \tag{5.4.28}$$

for some positive constant C.

We estimate now I_2. For this, we first observe that formulas (3.1.3) and (5.4.22) give that

$$|(U_\varepsilon + (W_{1,k})_+)^q - (U_\varepsilon + (U_k)_+)^q| \leqslant L|(W_{1,k})_+ - (U_k)_+|^q = L(U_k)_+^q |1-\chi|^q,$$

for a suitable constant $L > 0$. Consequently, applying Hölder inequality with exponents $\frac{2_s^*}{2_s^*-1-q}$, $\frac{2_s^*}{q}$ and 2_s^* we obtain that

$$\begin{aligned}
I_2 \;&\leqslant\; \varepsilon \int_{\mathbb{R}^n} |h|\, |(U_\varepsilon + (W_{1,k})_+)^q - (U_\varepsilon + (U_k)_+)^q|\, |U_\varepsilon + W_{1,k}| \, dx \\
&\leqslant\; \varepsilon L \int_{\mathbb{R}^n} |h| (U_k)_+^q |U_\varepsilon + W_{1,k}| \, dx \\
&\leqslant\; \varepsilon L \|h\|_{L^{\frac{2_s^*}{2_s^*-1-q}}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} (U_k)_+^{2_s^*} \, dx\right)^{\frac{q}{2_s^*}} \left(\int_{\mathbb{R}^n} |U_\varepsilon + W_{1,k}|^{2_s^*} \, dx\right)^{\frac{1}{2_s^*}} \\
&\leqslant\; C\varepsilon,
\end{aligned}$$

for some $C > 0$, thanks to (1.1.5), (5.4.2) and Proposition 1.2.1.

To estimate I_3, let us define the auxiliary function

$$f(t) := (U_\varepsilon + t\chi(U_k)_+ + (1-t)(U_k)_+)^p, \qquad t \in [0,1].$$

Thus, recalling (5.4.22), we have that

$$\begin{aligned}
&|(U_\varepsilon + (W_{1,k})_+)^p - (U_\varepsilon + (U_k)_+)^p| = |(U_\varepsilon + \chi(U_k)_+)^p - (U_\varepsilon + (U_k)_+)^p| \\
&\quad = |f(1) - f(0)| = \left| \int_0^1 f'(t)\, dt \right| \\
&\quad \leqslant p(1-\chi)(U_k)_+ \int_0^1 |U_\varepsilon + t\chi(U_k)_+ + (1-t)(U_k)_+|^{p-1}\, dt \\
&\quad \leqslant p(1-\chi)(U_k)_+ (U_\varepsilon + (U_k)_+)^{p-1} \leqslant C(1-\chi)(U_k)_+ U_\varepsilon^{p-1} + C(1-\chi)(U_k)_+^p,
\end{aligned}$$

for a suitable positive constant C. Therefore,

$$\begin{aligned}
I_3 \leqslant C\Big(&\int_{\mathbb{R}^n} (1-\chi(x,0))(U_k)_+(x,0) U_\varepsilon^p(x,0)\, dx \\
&+ \int_{\mathbb{R}^n} (1-\chi(x,0))(U_k)_+^p(x,0) U_\varepsilon(x,0)\, dx \\
&+ \int_{I_{\bar l}} \chi(x,0)(1-\chi(x,0)) U_\varepsilon^{p-1}(x,0)(U_k)_+^2(x,0)\, dx \\
&+ \int_{I_{\bar l}} \chi(x,0)(1-\chi(x,0))(U_k)_+^{p+1}(x,0)\, dx \Big) \\
=:\ & I_{3,1} + I_{3,2} + I_{3,3} + I_{3,4}.
\end{aligned}$$

Concerning $I_{3,1}$ and $I_{3,2}$ we are in the position to apply Lemma 5.3.1 with $V_k := U_k$, $U_o := U_\varepsilon$ and $\psi := 1 - \chi(\cdot, 0)$ (notice that $\alpha := 1$ and $\beta := p$ and $\alpha := p$ and $\beta := 1$, respectively). So we obtain that both $I_{3,1} = o_k(1)$ and $I_{3,2} = o_k(1)$.

Moreover, using Hölder inequality with exponent $\frac{p+1}{p-1}$ and $\frac{p+1}{2}$, Proposition 1.2.1 and (5.4.20), we have

$$I_{3,3} \leqslant \left(\int_{I_{\bar l}} U_\varepsilon^{p+1}(x,0)\, dx \right)^{\frac{p-1}{p+1}} \left(\int_{I_{\bar l}} (U_k)_+^{p+1}(x,0)\, dx \right)^{\frac{2}{p+1}} \leqslant C\varepsilon^{\frac{2}{p+1}},$$

for a suitable $C > 0$. Finally, making use of (5.4.20) once again we obtain that $I_{3,4} \leqslant C\varepsilon$, for some $C > 0$. Consequently, putting all these informations together we get

$$I_3 \leqslant C\varepsilon^{\frac{2}{p+1}} + o_k(1).$$

All in all, from (5.4.27) we obtain that

$$|\langle \mathcal{I}'_\varepsilon(U_k) - \mathcal{I}'_\varepsilon(W_{1,k}), U_\varepsilon + W_{1,k} \rangle| \leqslant C\varepsilon^{\frac{1}{2\gamma}} + o_k(1). \tag{5.4.29}$$

Likewise, it can be checked that

$$|\langle \mathcal{I}'_\varepsilon(U_k) - \mathcal{I}'_\varepsilon(W_{2,k}), U_\varepsilon + W_{2,k} \rangle| \leqslant C\varepsilon^{\frac{1}{2\gamma}} + o_k(1). \tag{5.4.30}$$

Therefore, using this and (5.4.26),

$$|\langle \mathcal{J}'_\varepsilon(W_{i,k}), U_\varepsilon + W_{i,k}\rangle| \leqslant C\varepsilon^{\frac{1}{2\gamma}} + o_k(1), \qquad i = 1, 2. \tag{5.4.31}$$

From now on we organize the proof in three steps as follows: in the forthcoming Step 1 and 2 we show lower bounds for $\mathcal{J}_\varepsilon(W_{1,k})$ and $\mathcal{J}_\varepsilon(W_{2,k})$, respectively. Then, in Step 3 we use these estimates to obtain a lower bound for $\mathcal{J}_\varepsilon(U_k)$ that will give a contradiction with the assumptions on $\mathcal{J}_\varepsilon$, and so the desired claim in Lemma 5.4.4 follows.

Step 1: Lower bound for $\mathcal{J}_\varepsilon(W_{1,k})$. From (1.2.15) and (1.2.18) we have

(5.4.32)

$$\begin{aligned}
&\mathcal{J}_\varepsilon(W_{1,k}) - \frac{1}{2}\langle \mathcal{J}'_\varepsilon(W_{1,k}), U_\varepsilon + W_{1,k}\rangle \\
= &-\frac{1}{2}\int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_\varepsilon, \nabla W_{1,k}\rangle \, dX \\
&- \frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (W_{1,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\right) dx \\
&+ \varepsilon \int_{\mathbb{R}^n} h(x) U_\varepsilon^q(x,0)(W_{1,k})_+(x,0)\, dx \\
&- \frac{1}{p+1}\int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{1,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\right) dx \\
&+ \int_{\mathbb{R}^n} U_\varepsilon^p(x,0)(W_{1,k})_+(x,0)\, dx \\
&+ \frac{\varepsilon}{2}\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (W_{1,k})_+)^q(x,0) - U_\varepsilon^q(x,0)\right)(U_\varepsilon + W_{1,k})(x,0)\, dx \\
&+ \frac{1}{2}\int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{1,k})_+)^p(x,0) - U_\varepsilon(x,0)\right)(U_\varepsilon + W_{1,k})(x,0)\, dx.
\end{aligned}$$

Thanks to (5.4.23), we have that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_\varepsilon, \nabla W_{1,k}\rangle \, dX = 0.$$

Moreover, from Lemma 5.3.1 applied here with $V_k := W_{1,k}$, $U_o := U_\varepsilon$, $\psi := h$, $\alpha := 1$ and $\beta := q$ we have that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} h(x) U_\varepsilon^q(x,0)(W_{1,k})_+(x,0)\, dx = 0. \tag{5.4.33}$$

Analogously, by taking $V_k := W_{1,k}$, $U_o := U_\varepsilon$, $\psi := 1$, $\alpha := 1$ and $\beta := p$ in Lemma 5.3.1 (notice that in this case $\alpha + \beta = p + 1 = 2^*_s$ and $\psi \in L^\infty(\mathbb{R}^n)$) we obtain that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} U_\varepsilon^p(x,0)(W_{1,k})_+(x,0)\, dx = 0. \tag{5.4.34}$$

Taking the limit as $k \to +\infty$ in (5.4.32) and using the last three formulas, we obtain that

(5.4.35)

$$
\begin{aligned}
&\lim_{k\to+\infty} \mathcal{I}_\varepsilon(W_{1,k}) - \frac{1}{2}\langle \mathcal{I}'_\varepsilon(W_{1,k}), U_\varepsilon + W_{1,k}\rangle \\
&= \lim_{k\to+\infty} \Big(-\frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (W_{1,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\right)\,dx \\
&\qquad - \frac{1}{p+1}\int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{1,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\right)\,dx \\
&\qquad + \frac{\varepsilon}{2}\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (W_{1,k})_+)^{q}(x,0) - U_\varepsilon^{q}(x,0)\right)(U_\varepsilon + W_{1,k})(x,0)\,dx \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{1,k})_+)^{p}(x,0) - U_\varepsilon^{p}(x,0)\right)(U_\varepsilon + W_{1,k})(x,0)\,dx \Big).
\end{aligned}
$$

Now we observe that if $x \in \mathbb{R}^n$ is such that $W_{1,k}(x,0) \leqslant 0$, then

$$
(U_\varepsilon + (W_{1,k})_+)^q(x,0) - U_\varepsilon^q(x,0) = U_\varepsilon^q(x,0) - U_\varepsilon^q(x,0) = 0,
$$

and so

$$
\begin{aligned}
&\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (W_{1,k})_+)^{q}(x,0) - U_\varepsilon^{q}(x,0)\right)(U_\varepsilon + W_{1,k})(x,0)\,dx \\
=\;& \int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (W_{1,k})_+)^{q}(x,0) - U_\varepsilon^{q}(x,0)\right)(U_\varepsilon + (W_{1,k})_+)(x,0)\,dx \\
=\;& \int_{\mathbb{R}^n} h(x)(U_\varepsilon + (W_{1,k})_+)^{q+1}(x,0)\,dx - \int_{\mathbb{R}^n} h(x)U_\varepsilon^{q+1}(x,0)\,dx \\
&\quad - \int_{\mathbb{R}^n} h(x)U_\varepsilon^{q}(x,0)(W_{1,k})_+(x,0)\,dx \\
=\;& \int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (W_{1,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\right)\,dx \\
&\quad - \int_{\mathbb{R}^n} h(x)U_\varepsilon^{q}(x,0)(W_{1,k})_+(x,0)\,dx.
\end{aligned}
$$

Analogously

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{1,k})_+)^{p}(x,0) - U_\varepsilon^{p}(x,0)\right)(U_\varepsilon + W_{1,k})(x,0)\,dx \\
=\;& \int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{1,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\right)\,dx - \int_{\mathbb{R}^n} U_\varepsilon^{p}(x,0)(W_{1,k})_+(x,0)\,dx.
\end{aligned}
$$

Therefore, using once more (5.4.33) and (5.4.34), from (5.4.35) we obtain that

(5.4.36)

$$\begin{aligned}
&\lim_{k\to+\infty} \mathcal{J}_\varepsilon(W_{1,k}) - \frac{1}{2}\langle \mathcal{J}'_\varepsilon(W_{1,k}), U_\varepsilon + W_{1,k}\rangle \\
&= \lim_{k\to+\infty} \Big(-\frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\big((U_\varepsilon+(W_{1,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\big)\,dx \\
&\qquad - \frac{1}{p+1}\int_{\mathbb{R}^n} \big((U_\varepsilon+(W_{1,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\big)\,dx \\
&\qquad + \frac{\varepsilon}{2}\int_{\mathbb{R}^n} h(x)\big((U_\varepsilon+(W_{1,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\big)\,dx \\
&\qquad - \frac{\varepsilon}{2}\int_{\mathbb{R}^n} h(x)U_\varepsilon^{q}(x,0)(W_{1,k})_+(x,0)\,dx \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}^n} \big((U_\varepsilon+(W_{1,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\big)\,dx \\
&\qquad - \frac{1}{2}\int_{\mathbb{R}^n} U_\varepsilon^{p}(x,0)(W_{1,k})_+(x,0)\,dx\Big) \\
&= \lim_{k\to+\infty} \Big(-\frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\big((U_\varepsilon+(W_{1,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\big)\,dx \\
&\qquad - \frac{1}{p+1}\int_{\mathbb{R}^n} \big((U_\varepsilon+(W_{1,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\big)\,dx \\
&\qquad + \frac{\varepsilon}{2}\int_{\mathbb{R}^n} h(x)\big((U_\varepsilon+(W_{1,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\big)\,dx \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}^n} \big((U_\varepsilon+(W_{1,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\big)\,dx \\
&= \lim_{k\to+\infty} \Big(-\varepsilon\Big(\frac{1}{q+1}-\frac{1}{2}\Big)\int_{\mathbb{R}^n} h(x)\big((U_\varepsilon+(W_{1,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\big)\,dx \\
&\qquad + \Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^n} \big((U_\varepsilon+(W_{1,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\big)\,dx\Big).
\end{aligned}$$

Now we claim that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} h(x)\big((U_\varepsilon+(W_{1,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\big)\,dx = 0. \tag{5.4.37}$$

For this, notice that if $x \in \mathbb{R}^n \setminus B_{r+\bar{l}+1}$, then $W_{1,k}(x,0) = 0$, thanks to (5.4.22) and (5.4.21). Therefore, for any $x \in \mathbb{R}^n \setminus B_{r+\bar{l}+1}$ we have that

$$(U_\varepsilon+(W_{1,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0) = U_\varepsilon^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0) = 0.$$

Thus

$$\begin{aligned}
&\int_{\mathbb{R}^n} h(x)\big((U_\varepsilon+(W_{1,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\big)\,dx \\
&= \int_{B_{r+\bar{l}+1}} h(x)\big((U_\varepsilon+(W_{1,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\big)\,dX.
\end{aligned} \tag{5.4.38}$$

Thanks to (5.4.24), we have that $W_{1,k}(\cdot,0)$ converges to zero a.e. in $\mathbb{R}^n$, and so $(W_{1,k})_+(\cdot,0)$ converges to zero a.e. in $\mathbb{R}^n$, as $k\to+\infty$. Therefore

$$U_\varepsilon+(W_{1,k})_+^{q+1}(x,0)\to U_\varepsilon^{q+1}(x,0)\qquad\text{for a.e. } x\in\mathbb{R}^n,$$

as $k\to+\infty$. Moreover the strong convergence of $W_{1,k}(\cdot,0)$ in $L^{q+1}_{\text{loc}}(\mathbb{R}^n)$ (due again to (5.4.24)) and Theorem 4.9 in [**10**] imply that there exists a function $F\in L^{q+1}_{\text{loc}}(\mathbb{R}^n)$ such that $|W_{1,k}(x,0)|\leqslant|F(x)|$ for a.e. $x\in\mathbb{R}^n$. This and the boundedness of U_ε (see Corollary 4.1.2) give that

$$h(U_\varepsilon+(W_{1,k})_+)^{q+1}\leqslant|h|(|U_\varepsilon|+|W_{1,k}|)^{q+1}\leqslant C|h|(1+|F|^{q+1})\in L^1(B_{r+\bar{l}+1}),$$

for a suitable $C>0$. Thus, the Dominated Convergence Theorem applies, and together with (5.4.38) give the convergence in (5.4.37).

Consequently, from (5.4.36) and (5.4.37) we obtain that

(5.4.39)
$$\begin{aligned}&\lim_{k\to\infty}\left(\mathcal{I}_\varepsilon(W_{1,k})-\frac{1}{2}\langle\mathcal{I}'_\varepsilon(W_{1,k}),U_\varepsilon+W_{1,k}\rangle\right)\\ &=\left(\frac12-\frac{1}{p+1}\right)\lim_{k\to\infty}\int_{\mathbb{R}^n}\big((U_\varepsilon+(W_{1,k})_+)^{p+1}(x,0)-U_\varepsilon^{p+1}(x,0)\big)\,dx\geqslant 0\end{aligned}$$

(recall that $p+1=2^*_s>2$). In particular, by (5.4.31) and (5.4.39), there holds

(5.4.40)
$$\begin{aligned}\mathcal{I}_\varepsilon(W_{1,k})&=\mathcal{I}_\varepsilon(W_{1,k})-\frac12\langle\mathcal{I}'_\varepsilon(W_{1,k}),U_\varepsilon+W_{1,k}\rangle+\frac12\langle\mathcal{I}'_\varepsilon(W_{1,k}),U_\varepsilon+W_{1,k}\rangle\\ &\geqslant\mathcal{I}_\varepsilon(W_{1,k})-\frac12\langle\mathcal{I}'_\varepsilon(W_{1,k}),U_\varepsilon+W_{1,k}\rangle-C\varepsilon^{\frac{1}{2\gamma}}+o_k(1)\\ &\geqslant -C\varepsilon^{\frac{1}{2\gamma}}+o_k(1),\end{aligned}$$

where C is a positive constant that may change from line to line.

Formula (5.4.40) provides the desired estimate from below for $\mathcal{I}_\varepsilon(W_{1,k})$. Next step is to obtain an estimate from below for $\mathcal{I}_\varepsilon(W_{2,k})$ as well.

Step 2: Lower bound for $\mathcal{I}_\varepsilon(W_{2,k})$. We first observe that formula (3.1.3) implies that there exists a constant $L>0$ such that

$$|(U_\varepsilon+(W_{2,k})_+)^q(x,0)-U_\varepsilon^q(x,0)|\leqslant L(W_{2,k})_+^q(x,0).$$

Hence

$$\begin{aligned}&\left|\varepsilon\int_{\mathbb{R}^n}h(x)\left((U_\varepsilon+(W_{2,k})_+)^q(x,0)-U_\varepsilon^q(x,0)\right)(U_\varepsilon+W_{2,k})(x,0)\,dx\right|\\ &\quad\leqslant\varepsilon\int_{\mathbb{R}^n}|h(x)|\,|(U_\varepsilon+(W_{2,k})_+)^q(x,0)-U_\varepsilon^q(x,0)|\,|(U_\varepsilon+W_{2,k})(x,0)|\,dx\\ &\quad\leqslant\varepsilon L\int_{\mathbb{R}^n}|h(x)|(W_{2,k})_+^q(x,0)|(U_\varepsilon+W_{2,k})(x,0)|\,dx\\ &\quad\leqslant\varepsilon L\left(\int_{\mathbb{R}^n}|h(x)|(W_{2,k})_+^q(x,0)U_\varepsilon(x,0)\,dx+\int_{\mathbb{R}^n}|h(x)|(W_{2,k})_+^{q+1}(x,0)\,dx\right).\end{aligned}$$

Thanks to Lemma 5.3.1 (applied here with $V_k := W_{2,k}$, $U_o := U_\varepsilon$, $\psi := h$, $\alpha := q$ and $\beta := 1$) we have that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}|h(x)|(W_{2,k})_+^q(x,0)U_\varepsilon(x,0)\,dx=0.$$

Moreover, by Hölder inequality with exponents $\frac{2_s^*}{2_s^*-1-q}$ and $\frac{2_s^*}{q+1}$ we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^n}|h(x)|(W_{2,k})_+^{q+1}(x,0)\,dx\\&\quad\leqslant\left(\int_{\mathbb{R}^n}|h(x)|^{\frac{2_s^*}{2_s^*-1-q}}\,dx\right)^{\frac{2_s^*-1-q}{2_s^*}}\left(\int_{\mathbb{R}^n}(W_{2,k})_+^{2_s^*}(x,0)\,dx\right)^{\frac{q+1}{2_s^*}}\\&\quad\leqslant C\left(\int_{\mathbb{R}^n}(U_k)_+^{2_s^*}(x,0)\,dx\right)^{\frac{q+1}{2_s^*}}\leqslant C[U_k]_a^{q+1}\leqslant C,\end{aligned}$$

for some constant $C>0$, where we have also used (1.1.5), Proposition 1.2.1 and Corollary 4.1.2. The last three formulas imply that

$$\left|\varepsilon\int_{\mathbb{R}^n}h(x)\left((U_\varepsilon+(W_{2,k})_+)^q(x,0)-U_\varepsilon^q(x,0)\right)(U_\varepsilon+W_{2,k})(x,0)\,dx\right|\leqslant C\varepsilon+o_k(1),$$

for a suitable $C>0$. This, together with (1.2.18), (5.4.23) and (5.4.31) (with $i=2$) gives

(5.4.41)

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+}y^a|\nabla W_{2,k}|^2\,dX\\&=\langle\mathcal{J}_\varepsilon'(W_{2,k},U_\varepsilon+W_{2,k}\rangle-\int_{\mathbb{R}^{n+1}_+}y^a\langle\nabla W_{2,k},\nabla U_\varepsilon\rangle,dX\\&\qquad+\varepsilon\int_{\mathbb{R}^n}h(x)\left((U_\varepsilon+(W_{2,k})_+)^q(x,0)-U_\varepsilon^q(x,0)\right)(U_\varepsilon+W_{2,k})(x,0)\,dx\\&\qquad+\int_{\mathbb{R}^n}\left((U_\varepsilon+(W_{2,k})_+)^p(x,0)-U_\varepsilon^p(x,0)\right)(U_\varepsilon+W_{2,k})(x,0)\,dx\\&\leqslant C\varepsilon^{\frac{1}{2\gamma}}+o_k(1)+\int_{\mathbb{R}^n}\left((U_\varepsilon+(W_{2,k})_+)^p(x,0)-U_\varepsilon^p(x,0)\right)(U_\varepsilon+W_{2,k})(x,0)\,dx.\end{aligned}$$

Now notice that if $x\in\mathbb{R}^n$ is such that $W_{2,k}\leqslant 0$ then

$$(U_\varepsilon+(W_{2,k})_+)^p(x,0)-U_\varepsilon^p(x,0)=0,$$

and so

$$\begin{aligned}&\int_{\mathbb{R}^n}\left((U_\varepsilon+(W_{2,k})_+)^p(x,0)-U_\varepsilon^p(x,0)\right)(U_\varepsilon+W_{2,k})(x,0)\,dx\\&=\int_{\mathbb{R}^n}\left((U_\varepsilon+(W_{2,k})_+)^p(x,0)-U_\varepsilon^p(x,0)\right)(U_\varepsilon+(W_{2,k})_+)(x,0)\,dx\\&=\int_{\mathbb{R}^n}\left((U_\varepsilon+(W_{2,k})_+)^{p+1}(x,0)-U_\varepsilon^{p+1}(x,0)\right)\,dx-\int_{\mathbb{R}^n}U_\varepsilon^p(x,0)(W_{2,k})_+(x,0)\,dx.\end{aligned}$$

According to Lemma 5.3.1 (applied here with $V_k := W_{2,k}$, $U_o := U_\varepsilon$, $\psi := 1$, $\alpha := 1$ and $\beta := p$) we have that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} U_\varepsilon^p(x,0)(W_{2,k})_+(x,0)\,dx = 0.$$

Therefore, (5.4.41) becomes

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+} y^a |\nabla W_{2,k}|^2\,dX \\ &\leqslant \int_{\mathbb{R}^n} \big((U_\varepsilon + (W_{2,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\big)\,dx + C\varepsilon^{\frac{1}{2\gamma}} + o_k(1).\end{aligned} \tag{5.4.42}$$

Furthermore, it is not difficult to see that that there exist two constants $0 < c_1 < c_2$ such that

$$c_1 \leqslant \frac{(1+t)^{p+1} - 1 - t^{p+1}}{t^p + t} \leqslant c_2, \qquad t > 0$$

Thus, setting $t := \frac{(W_{2,k})_+}{U_\varepsilon}$, one has

$$\begin{aligned}(U_\varepsilon + (W_{2,k})_+)^{p+1} - U_\varepsilon^{p+1} &= U_\varepsilon^{p+1}\left[\left(1 + \frac{(W_{2,k})_+}{U_\varepsilon}\right)^{p+1} - 1\right] \\ &\leqslant U_\varepsilon^{p+1}\left[c_2\left(\frac{(W_{2,k})_+^p}{U_\varepsilon^p} + \frac{(W_{2,k})_+}{U_\varepsilon}\right) + \frac{(W_{2,k})_+^{p+1}}{U_\varepsilon^{p+1}}\right] \\ &= c_2 U_\varepsilon (W_{2,k})_+^p + c_2 U_\varepsilon^p (W_{2,k})_+ + (W_{2,k})_+^{p+1}.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_{\mathbb{R}^n} ((U_\varepsilon + (W_{2,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0))\,dx \\ &\leqslant c_2 \int_{\mathbb{R}^n} U_\varepsilon^p(x,0)(W_{2,k})_+(x,0)\,dx + c_2 \int_{\mathbb{R}^n} (W_{2,k})_+^p(x,0)U_\varepsilon(x,0)\,dx \\ &\quad + \int_{\mathbb{R}^n} (W_{2,k})_+^{p+1}(x,0)\,dx.\end{aligned}$$

Applying Lemma 5.3.1 once more, we obtain that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} U_\varepsilon^p(x,0)(W_{2,k})_+(x,0)\,dx = 0$$

$$\text{and} \quad \lim_{k\to+\infty} \int_{\mathbb{R}^n} (W_{2,k})_+^p(x,0)U_\varepsilon(x,0)\,dx = 0.$$

Hence, going back to (5.4.42), we get

$$\int_{\mathbb{R}^{n+1}_+} y^a |\nabla W_{2,k}|^2\,dX \leqslant \int_{\mathbb{R}^n} (W_{2,k})_+^{p+1}(x,0)\,dx + C\varepsilon^{\frac{1}{2\gamma}} + o_k(1). \tag{5.4.43}$$

Now we observe that, thanks to (5.4.22), $W_{2,k} = U_k$ outside $B_{r+\bar{l}+1}$. So, using (5.4.17) with $\rho := r + \bar{l} + 1$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+\setminus B^+_{r+\bar{l}+1}} y^a |\nabla W_{2,k}|^2\,dX + \int_{\mathbb{R}^n\setminus\left(B_{r+\bar{l}+1}\cap\{y=0\}\right)} (W_{2,k})_+^{2^*_s}(x,0)\,dx \\
&\quad = \int_{\mathbb{R}^{n+1}_+\setminus B^+_{r+\bar{l}+1}} y^a |\nabla U_k|^2\,dX + \int_{\mathbb{R}^n\setminus\left(B_{r+\bar{l}+1}\cap\{y=0\}\right)} (U_k)_+^{2^*_s}(x,0)\,dx \geqslant \eta_0,
\end{aligned}$$

for some k that depends on ρ. This implies that either

$$\int_{\mathbb{R}^{n+1}_+\setminus B^+_{r+\bar{l}+1}} y^a |\nabla W_{2,k}|^2\,dX \geqslant \frac{\eta_0}{2}$$

or

$$\int_{\mathbb{R}^n\setminus\{B_{r+\bar{l}+1}\cap\{y=0\}\}} (W_{2,k})_+^{p+1}(x,0)\,dx \geqslant \frac{\eta_0}{2}.$$

In the first case we have

$$\int_{\mathbb{R}^{n+1}_+} y^a |\nabla W_{2,k}|^2\,dX \geqslant \int_{\mathbb{R}^{n+1}_+\setminus B^+_{r+\bar{l}+1}} y^a |\nabla W_{2,k}|^2\,dX \geqslant \frac{\eta_0}{2}.$$

From this and (5.4.43) it follows

$$\int_{\mathbb{R}^n} (W_{2,k})_+^{p+1}(x,0)\,dx > \frac{\eta_0}{4}. \tag{5.4.44}$$

In the second case, this inequality holds trivially.

Accordingly, we can define $\psi_k := \alpha_k W_{2,k}$, where

$$\alpha_k^{p-1} := \frac{[W_{2,k}]_a^2}{\|(W_{2,k})_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}}. \tag{5.4.45}$$

We claim that

$$[W_{2,k}]_a^2 \leqslant \|(W_{2,k})_+(\cdot,0)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} + C\varepsilon^{\frac{2}{p+1}} + o_k(1), \tag{5.4.46}$$

for a suitable positive constant C. For this, notice that (1.2.18), (5.4.23) and (5.4.31) give

(5.4.47)

$$
\begin{aligned}
[W_{2,k}]_a^2 = & \langle \mathcal{J}'_\varepsilon(W_{2,k}), U_\varepsilon + W_{2,k}\rangle - \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_\varepsilon, \nabla W_{2,k}\rangle \, dX \\
& + \varepsilon \int_{\mathbb{R}^n} h(x) \left((U_\varepsilon + (W_{2,k})_+)^q(x,0) - U_\varepsilon^q(x,0)\right) (U_\varepsilon + W_{2,k})(x,0)\, dx \\
& + \int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{2,k})_+)^p(x,0) - U_\varepsilon^p(x,0)\right) (U_\varepsilon + W_{2,k})(x,0)\, dx \\
\leqslant & \, C\varepsilon^{\frac{1}{2\gamma}} + o_k(1) \\
& + \varepsilon \int_{\mathbb{R}^n} h(x) \left((U_\varepsilon + (W_{2,k})_+)^q(x,0) - U_\varepsilon^q(x,0)\right) (U_\varepsilon + W_{2,k})(x,0)\, dx \\
& + \int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{2,k})_+)^p(x,0) - U_\varepsilon^p(x,0)\right) (U_\varepsilon + W_{2,k})(x,0)\, dx.
\end{aligned}
$$

We can rewrite

$$
\begin{aligned}
& \int_{\mathbb{R}^n} h(x) \left((U_\varepsilon + (W_{2,k})_+)^q(x,0) - U_\varepsilon^q(x,0)\right) (U_\varepsilon + W_{2,k})(x,0)\, dx \\
= & \int_{\mathbb{R}^n} h(x) \left((U_\varepsilon + (W_{2,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\right) \, dx \\
& \quad - \int_{\mathbb{R}^n} h(x) U_\varepsilon^q(x,0) (W_{2,k})_+(x,0)\, dx \\
= & \int_{\mathbb{R}^n} h(x) \left((U_\varepsilon + (W_{2,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\right) \, dx + o_k(1),
\end{aligned}
$$

where we have applied once again Lemma 5.3.1. Analogously,

$$
\begin{aligned}
& \int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{2,k})_+)^p(x,0) - U_\varepsilon^p(x,0)\right) (U_\varepsilon + W_{2,k})(x,0)\, dx \\
= & \int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{2,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\right) \, dx + o_k(1).
\end{aligned}
$$

Plugging these informations into (5.4.47), we obtain that

$$
\begin{aligned}
[W_{2,k}]_a^2 \leqslant & \varepsilon \int_{\mathbb{R}^n} h(x) \left((U_\varepsilon + (W_{2,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\right) \, dx \\
& + \int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{2,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\right) \, dx \\
& + C\varepsilon^{\frac{1}{2\gamma}} + o_k(1).
\end{aligned} \tag{5.4.48}
$$

So using (5.3.5) into (5.4.48) we obtain

$$
\begin{aligned}
[W_{2,k}]_a^2 \leqslant & \int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{2,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\right) \, dx \\
& + C\varepsilon^{\frac{1}{2\gamma}} + o_k(1),
\end{aligned} \tag{5.4.49}
$$

up to renaming constants. Now we use (5.3.4) with $V_k := W_{2,k}$ and $U_o := U_\varepsilon$ and we get

$$\begin{aligned}
[W_{2,k}]_a^2 &\leqslant \int_{\mathbb{R}^n} (U_\varepsilon + (W_{2,k})_+)^{p+1}(x,0)\,dx - \int_{\mathbb{R}^n} U_\varepsilon^{p+1}(x,0)\,dx \\
&\qquad + C\varepsilon^{\frac{1}{2\gamma}} + o_k(1) \\
&= \|(U_\varepsilon + (W_{2,k})_+)(\cdot,0)\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} - \|U_\varepsilon\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} + C\varepsilon^{\frac{1}{2\gamma}} + o_k(1) \\
&= \|(W_{2,k})_+(\cdot,0)\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} + C\varepsilon^{\frac{1}{2\gamma}} + o_k(1),
\end{aligned}$$

and this shows (5.4.46).

From (5.4.45), (5.4.46) and (5.4.44) we have that

$$\alpha_k^{p-1} = \frac{[W_{2,k}]_a^2}{\|(W_{2,k})_+(\cdot,0)\|_{L^{p+1}(\mathbb{R}^n)}^{p+1}} \leqslant 1 + C\varepsilon^{\frac{1}{2\gamma}} + o_k(1). \tag{5.4.50}$$

Notice also that, with the choice of α_k in (5.4.45), it holds

$$[\psi_k]_a^2 = \alpha_k^2 [W_{2,k}]_a^2 = \alpha_k^{p+1} \|(W_{2,k})_+(\cdot,0)\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} = \|(\psi_k)_+(\cdot,0)\|_{L^{p+1}(\mathbb{R}^n)}^{p+1}.$$

Hence, by (1.2.6) and Proposition 1.2.1, we have that

$$\begin{aligned}
S &\leqslant \frac{[\psi_k(\cdot,0)]_{\dot H^s(\mathbb{R}^n)}^2}{\|(\psi_k)_+(\cdot,0)\|_{L^{p+1}(\mathbb{R}^n)}^2} = \frac{[\psi_k]_a^2}{\|(\psi_k)_+(\cdot,0)\|_{L^{p+1}(\mathbb{R}^n)}^2} \\
&= \frac{\|(\psi_k)_+(\cdot,0)\|_{L^{p+1}(\mathbb{R}^n)}^{p+1}}{\|(\psi_k)_+(\cdot,0)\|_{L^{p+1}(\mathbb{R}^n)}^2} = \|(\psi_k)_+(\cdot,0)\|_{L^{p+1}(\mathbb{R}^n)}^{p-1}.
\end{aligned}$$

Accordingly,

$$\|(W_{2,k})_+(\cdot,0)\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} = \frac{\|(\psi_k)_+(\cdot,0)\|_{L^{p+1}(\mathbb{R}^n)}^{p+1}}{\alpha_k^{p+1}} \geqslant S^{n/2s} \frac{1}{\alpha_k^{p+1}},$$

where we have used the fact that $p - 1 = 2_s^* - 2 = \frac{4s}{n-2s}$. This, together with (5.4.50), gives that

$$\begin{aligned}
S^{n/2s} &\leqslant (1 + C\varepsilon^{\frac{2}{p+1}} + o_k(1))^{\frac{p+1}{p-1}} \|(W_{2,k})_+(\cdot,0)\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \\
&\leqslant \|(W_{2,k})_+(\cdot,0)\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} + C\varepsilon^{\frac{1}{2\gamma}} + o_k(1).
\end{aligned} \tag{5.4.51}$$

Moreover, by (5.4.23) and Lemma 5.3.1 we have that

$$\begin{aligned}
&\mathcal{I}_\varepsilon(W_{2,k}) - \frac{1}{2}\langle \mathcal{I}'_\varepsilon(W_{2,k}), U_\varepsilon + W_{2,k}\rangle \\
= & -\int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla W_{2,k}, \nabla U_\varepsilon\rangle \, dX \\
& -\frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (W_{2,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\right)\, dx \\
& +\varepsilon \int_{\mathbb{R}^n} h(x) U_\varepsilon^q(x,0)(W_{2,k})_+(x,0)\, dx \\
& -\frac{1}{p+1}\int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{2,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\right)\, dx \\
& +\int_{\mathbb{R}^n} U_\varepsilon^p(x,0)(W_{2,k})_+(x,0)\, dx \\
& +\frac{\varepsilon}{2}\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (W_{2,k})_+)^q(x,0) - U_\varepsilon^q(x,0)\right)(U_\varepsilon + W_{2,k})(x,0))\, dx \\
& +\frac{1}{2}\int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{2,k})_+)^p(x,0) - U_\varepsilon^p(x,0)\right)(U_\varepsilon + W_{2,k})(x,0))\, dx \\
= & -\varepsilon\left(\frac{1}{q+1} - \frac{1}{2}\right)\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (W_{2,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\right)\, dx \\
& +\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{2,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\right)\, dx + o_k(1).
\end{aligned}$$

We observe that $\frac{1}{2} - \frac{1}{p+1} = \frac{s}{n}$. Thus, using (5.3.5) we have that

$$\begin{aligned}
&\mathcal{I}_\varepsilon(W_{2,k}) - \frac{1}{2}\langle \mathcal{I}'_\varepsilon(W_{2,k}), U_\varepsilon + W_{2,k}\rangle \\
\geqslant\ & \frac{s}{n}\int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{2,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\right)\, dx - C\varepsilon + o_k(1),
\end{aligned}$$

for some $C > 0$. Therefore, using (5.3.4) with $V_k := W_{2,k}$ and $U_o := U_\varepsilon$, we have that

$$\mathcal{I}_\varepsilon(W_{2,k}) - \frac{1}{2}\langle \mathcal{I}'_\varepsilon(W_{2,k}), U_\varepsilon + W_{2,k}\rangle \geqslant \frac{s}{n}\|(W_{2,k})_+\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} - C\varepsilon + o_k(1).$$

Furthermore, by (5.4.51),

$$\mathcal{I}_\varepsilon(W_{2,k}) - \frac{1}{2}\langle \mathcal{I}'_\varepsilon(W_{2,k}), U_\varepsilon + W_{2,k}\rangle \geqslant \frac{s}{n}S^{n/2s} - C\varepsilon^{\frac{1}{2\gamma}} + o_k(1).$$

This and (5.4.31) give the desired estimate for $\mathcal{I}_\varepsilon(W_{2,k})$, namely

$$\mathcal{I}_\varepsilon(W_{2,k}) \geqslant \frac{s}{n}S^{n/2s} - C\varepsilon^{\frac{1}{2\gamma}} + o_k(1). \tag{5.4.52}$$

Step 3: Lower bound for $\mathcal{I}_\varepsilon(U_k)$. Now, keeping in mind the estimates obtained in (5.4.40) and (5.4.52) for $\mathcal{I}_\varepsilon(W_{1,k})$ and $\mathcal{I}_\varepsilon(W_{2,k})$ respectively, we will produce an estimate for $\mathcal{I}_\varepsilon(U_k)$. Indeed, notice first that $U_k =$

$\chi U_k+(1-\chi)U_k = W_{1,k}+W_{2,k}$, thanks to (5.4.22). Hence, recalling (1.2.15), we have

$$\begin{aligned}
&\mathcal{J}_\varepsilon(U_k)\\
=\;& \mathcal{J}_\varepsilon(W_{1,k})+\mathcal{J}_\varepsilon(W_{2,k})+\int_{\mathbb{R}^{n+1}_+} y^a\langle\nabla W_{1,k},\nabla W_{2,k}\rangle\,dX\\
&-\frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon+(U_k)_+)^{q+1}(x,0)-U_\varepsilon^{q+1}(x,0)\right)\,dx\\
&+\varepsilon\int_{\mathbb{R}^n} h(x)U_\varepsilon^q(x,0)(U_k)_+(x,0)\,dx\\
&-\frac{1}{p+1}\int_{\mathbb{R}^n}\left((U_\varepsilon+(U_k)_+)^{p+1}(x,0)-U_\varepsilon^{p+1}(x,0)\right)\,dx\\
&+\int_{\mathbb{R}^n} U_\varepsilon^p(x,0)(U_k)_+(x,0)\,dx\\
&+\frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon+(W_{1,k})_+)^{q+1}(x,0)-U_\varepsilon^{q+1}(x,0)\right)\,dx\\
&-\varepsilon\int_{\mathbb{R}^n} h(x)U_\varepsilon^q(x,0)(W_{1,k})_+(x,0)\,dx\\
&+\frac{1}{p+1}\int_{\mathbb{R}^n}\left((U_\varepsilon+(W_{1,k})_+)^{p+1}(x,0)-U_\varepsilon^{p+1}(x,0)\right)\,dx\\
&-\int_{\mathbb{R}^n} U_\varepsilon^p(x,0)(W_{1,k})_+(x,0)\,dx\\
&+\frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon+(W_{2,k})_+)^{q+1}(x,0)-U_\varepsilon^{q+1}(x,0)\right)\,dx\\
&-\varepsilon\int_{\mathbb{R}^n} h(x)U_\varepsilon^q(x,0)(W_{2,k})_+(x,0)\,dx\\
&+\frac{1}{p+1}\int_{\mathbb{R}^n}\left((U_\varepsilon+(W_{2,k})_+)^{p+1}(x,0)-U_\varepsilon^{p+1}(x,0)\right)\,dx\\
&-\int_{\mathbb{R}^n} U_\varepsilon^p(x,0)(W_{2,k})_+(x,0)\,dx.
\end{aligned}$$

Thanks to Lemma 5.3.1 we have that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} h(x)U_\varepsilon^q(x,0)(U_k)_+(x,0)\,dx = 0,$$
$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} U_\varepsilon^p(x,0)(U_k)_+(x,0)\,dx,$$
$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} h(x)U_\varepsilon^q(x,0)(W_{1,k})_+(x,0)\,dx = 0,$$
$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} U_\varepsilon^p(x,0)(W_{1,k})_+(x,0)\,dx = 0,$$
$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} h(x)U_\varepsilon^q(x,0)(W_{2,k})_+(x,0)\,dx$$
$$\text{and}\qquad \lim_{k\to+\infty}\int_{\mathbb{R}^n} U_\varepsilon^p(x,0)(W_{2,k})_+(x,0)\,dx = 0.$$

Therefore,

$$\begin{aligned}
\mathcal{I}_\varepsilon(U_k) &= \mathcal{I}_\varepsilon(W_{1,k}) + \mathcal{I}_\varepsilon(W_{2,k}) + \int_{\mathbb{R}^{n+1}_+} y^a\langle\nabla W_{1,k},\nabla W_{2,k}\rangle\,dX\\
&\quad -\frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon+(U_k)_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\right)\,dx\\
&\quad -\frac{1}{p+1}\int_{\mathbb{R}^n} \left((U_\varepsilon+(U_k)_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\right)\,dx\\
&\quad +\frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon+(W_{1,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\right)\,dx\\
&\quad +\frac{1}{p+1}\int_{\mathbb{R}^n} \left((U_\varepsilon+(W_{1,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\right)\,dx\\
&\quad +\frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon+(W_{2,k})_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\right)\,dx\\
&\quad +\frac{1}{p+1}\int_{\mathbb{R}^n} \left((U_\varepsilon+(W_{2,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\right)\,dx + o_k(1).
\end{aligned}$$

Since the terms with ε in front are bounded (see (5.3.5) and notice that it holds also for U_k and $W_{1,k}$), we have that

(5.4.53)
$$\begin{aligned}\mathcal{J}_\varepsilon(U_k) \geqslant\, &\mathcal{J}_\varepsilon(W_{1,k}) + \mathcal{J}_\varepsilon(W_{2,k}) + \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla W_{1,k}, \nabla W_{2,k}\rangle \, dX \\ &- \frac{1}{p+1}\int_{\mathbb{R}^n} \big((U_\varepsilon + (U_k)_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\big)\, dx \\ &+ \frac{1}{p+1}\int_{\mathbb{R}^n} \big((U_\varepsilon + (W_{1,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\big)\, dx \\ &+ \frac{1}{p+1}\int_{\mathbb{R}^n} \big((U_\varepsilon + (W_{2,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\big)\, dx \\ &- C\varepsilon + o_k(1).\end{aligned}$$

Now notice that

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla W_{1,k}, \nabla W_{2,k}\rangle \, dX \\ &\quad = \frac{1}{2}\int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla (U_k - W_{1,k}), \nabla W_{1,k}\rangle \, dX \\ &\qquad + \frac{1}{2}\int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla (U_k - W_{2,k}), \nabla W_{2,k}\rangle \, dX.\end{aligned} \tag{5.4.54}$$

Moreover, from (1.2.18) we have that for any $i \in \{1,2\}$

$$\begin{aligned}&\langle \mathcal{J}'_\varepsilon(U_k) - \mathcal{J}'_\varepsilon(W_{i,k}), U_\varepsilon + W_{i,k}\rangle \\ &= \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla (U_k - W_{i,k}), \nabla (W_{i,k} + U_\varepsilon)\rangle \, dX \\ &\quad - \varepsilon \int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (U_k)_+)^q - U_\varepsilon^q\right)(U_\varepsilon + W_{i,k})\, dx \\ &\quad - \int_{\mathbb{R}^n} \left((U_\varepsilon + (U_k)_+)^p - U_\varepsilon^p\right)(U_\varepsilon + W_{i,k})\, dx \\ &\quad + \varepsilon \int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (W_{i,k})_+)^q - U_\varepsilon^q\right)(U_\varepsilon + W_{i,k})\, dx \\ &\quad + \int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{i,k})_+)^p - U_\varepsilon^p\right)(U_\varepsilon + W_{i,k})\, dx.\end{aligned} \tag{5.4.55}$$

We claim that

(5.4.56)
$$\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (U_k)_+)^q(x,0) - U_\varepsilon^q(x,0)\right)(U_\varepsilon + W_{i,k})(x,0)\, dx \leqslant C + o_k(1)$$

and $\displaystyle\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (W_{i,k})_+)^q(x,0) - U_\varepsilon^q(x,0)\right)(U_\varepsilon + W_{i,k})(x,0)\, dx \leqslant C + o_k(1),$

for some $C > 0$. Let us prove the first estimate in (5.4.56). For this, we notice that if $x \in \mathbb{R}^n$ is such that $U_k(x,0) \geqslant 0$ then also $W_{i,k}(x,0) \geqslant 0$, thanks to the definition of $W_{i,k}$ given in (5.4.22). Hence

$$\begin{aligned}
&\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (U_k)_+)^q(x,0) - U_\varepsilon^q(x,0)\right)(U_\varepsilon + W_{i,k})(x,0)\,dx \\
&\quad= \int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (U_k)_+)^q(x,0) - U_\varepsilon^q(x,0)\right)(U_\varepsilon + (W_{i,k})_+)(x,0)\,dx \\
&\quad\leqslant \int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (U_k)_+)^q(x,0) - U_\varepsilon^q(x,0)\right)(U_\varepsilon + (U_k)_+)(x,0)\,dx \\
&\quad= \int_{\mathbb{R}^n} h(x)\left((U_\varepsilon + (U_k)_+)^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0)\right)\,dx \\
&\qquad - \int_{\mathbb{R}^n} h(x)U_\varepsilon^q(x,0)(U_k)_+)(x,0)\,dx \\
&\quad\leqslant C + o_k(1),
\end{aligned}$$

for a suitable $C > 0$, thanks to (5.3.5) (that holds true also for U_k) and Lemma 5.3.1. Analogously one can prove also the second estimate in (5.4.56), and this finishes the proof of (5.4.56).

Hence, from (5.4.53), (5.4.54), (5.4.55) and (5.4.56) we get

$$\begin{aligned}
&\mathcal{I}_\varepsilon(U_k) \\
\geqslant\ &\mathcal{I}_\varepsilon(W_{1,k}) + \mathcal{I}_\varepsilon(W_{2,k}) + \frac{1}{2}\langle \mathcal{I}'_\varepsilon(U_k) - \mathcal{I}'_\varepsilon(W_{1,k}), U_\varepsilon + W_{1,k}\rangle \\
&+\frac{1}{2}\langle \mathcal{I}'_\varepsilon(U_k) - \mathcal{I}'_\varepsilon(W_{2,k}), U_\varepsilon + W_{2,k}\rangle \\
&-\frac{1}{p+1}\int_{\mathbb{R}^n}\left((U_\varepsilon + (U_k)_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\right)\,dx \\
&+\frac{1}{p+1}\int_{\mathbb{R}^n}\left((U_\varepsilon + (W_{1,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\right)\,dx \\
&+\frac{1}{p+1}\int_{\mathbb{R}^n}\left((U_\varepsilon + (W_{2,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\right)\,dx \\
&+\frac{1}{2}\int_{\mathbb{R}^n}\left((U_\varepsilon + (U_k)_+)^p(x,0) - U_\varepsilon^p(x,0)\right)(U_\varepsilon + W_{1,k})(x,0)\,dx \\
&-\frac{1}{2}\int_{\mathbb{R}^n}\left((U_\varepsilon + (W_{1,k})_+)^p(x,0) - U_\varepsilon^p(x,0)\right)(U_\varepsilon + W_{1,k})(x,0)\,dx \\
&+\frac{1}{2}\int_{\mathbb{R}^n}\left((U_\varepsilon + (U_k)_+)^p(x,0) - U_\varepsilon^p(x,0)\right)(U_\varepsilon + W_{2,k})(x,0)\,dx \\
&-\frac{1}{2}\int_{\mathbb{R}^n}\left((U_\varepsilon + (W_{2,k})_+)^p(x,0) - U_\varepsilon^p(x,0)\right)(U_\varepsilon + W_{1,k})(x,0)\,dx \\
&-C\varepsilon + o_k(1).
\end{aligned}$$

Moreover, the estimates in (5.4.29) and (5.4.30) give

$$\begin{aligned}
\mathcal{J}_\varepsilon(U_k) \;\geqslant\;& \mathcal{J}_\varepsilon(W_{1,k}) + \mathcal{J}_\varepsilon(W_{2,k}) \\
& -\frac{1}{p+1}\int_{\mathbb{R}^n} \big((U_\varepsilon + (U_k)_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\big)\, dx \\
& +\frac{1}{p+1}\int_{\mathbb{R}^n} \big((U_\varepsilon + (W_{1,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\big)\, dx \\
& +\frac{1}{p+1}\int_{\mathbb{R}^n} \big((U_\varepsilon + (W_{2,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\big)\, dx \\
& +\frac{1}{2}\int_{\mathbb{R}^n} \left((U_\varepsilon + (U_k)_+)^{p}(x,0) - U_\varepsilon^{p}(x,0)\right)(U_\varepsilon + W_{1,k})(x,0)\, dx \\
& -\frac{1}{2}\int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{1,k})_+)^{p}(x,0) - U_\varepsilon^{p}(x,0)\right)(U_\varepsilon + W_{1,k})(x,0)\, dx \\
& +\frac{1}{2}\int_{\mathbb{R}^n} \left((U_\varepsilon + (U_k)_+)^{p}(x,0) - U_\varepsilon^{p}(x,0)\right)(U_\varepsilon + W_{2,k})(x,0)\, dx \\
& -\frac{1}{2}\int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{2,k})_+)^{p}(x,0) - U_\varepsilon^{p}(x,0)\right)(U_\varepsilon + W_{1,k})(x,0)\, dx \\
& -C\varepsilon^{\frac{1}{2\gamma}} + o_k(1).
\end{aligned}$$

Now we use Lemma 5.3.1 once more to see that, for $i \in \{1,2\}$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left((U_\varepsilon + (W_{i,k})_+)^{p}(x,0) - U_\varepsilon^{p}(x,0)\right)(U_\varepsilon + W_{i,k})(x,0)\, dx \\
&\qquad = \int_{\mathbb{R}^n} (U_\varepsilon + (W_{i,k})_+)^{p+1}(x,0)\, dx - \int_{\mathbb{R}^n} U_\varepsilon^{p+1}(x,0)\, dx + o_k(1).
\end{aligned}$$

Hence, using this and collecting some terms, we have

(5.4.57)

$$\begin{aligned}
&\mathcal{J}_\varepsilon(U_k) \\
&\geqslant \mathcal{J}_\varepsilon(W_{1,k}) + \mathcal{J}_\varepsilon(W_{2,k}) \\
&\quad - \frac{1}{p+1}\int_{\mathbb{R}^n} \big((U_\varepsilon + (U_k)_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\big)\, dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n} \left((U_\varepsilon + (U_k)_+)^{p}(x,0) - U_\varepsilon^{p}(x,0)\right)(U_\varepsilon + W_{1,k})(x,0)\, dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n} \left((U_\varepsilon + (U_k)_+)^{p}(x,0) - U_\varepsilon^{p}(x,0)\right)(U_\varepsilon + W_{2,k})(x,0)\, dx \\
&\quad - \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n} \big((U_\varepsilon + (W_{1,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\big)\, dx \\
&\quad - \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^n} \big((U_\varepsilon + (W_{2,k})_+)^{p+1}(x,0) - U_\varepsilon^{p+1}(x,0)\big)\, dx \\
&\quad - C\varepsilon^{\frac{1}{2\gamma}} + o_k(1).
\end{aligned}$$

Now we claim that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}\left((U_\varepsilon+(U_k)_+)^p(x,0)-U_\varepsilon^p(x,0)\right)U_\varepsilon(x,0)\,dx=0. \tag{5.4.58}$$

Indeed, we first observe that for any $a\geqslant b\geqslant 0$

$$a^p-b^p=p\int_b^a t^{p-1}\,dt\leqslant pa^{p-1}(a-b).$$

Hence, taking $a:=U_\varepsilon+(U_k)_+$ and $b:=U_\varepsilon$, we have that

$$|(U_\varepsilon+(U_k)_+)^p-U_\varepsilon^p|\leqslant p(U_\varepsilon+(U_k)_+)^{p-1}(U_k)_+.$$

Accordingly,

$$\begin{aligned}&\int_{\mathbb{R}^n}\left((U_\varepsilon+(U_k)_+)^p(x,0)-U_\varepsilon^p(x,0)\right)U_\varepsilon(x,0)\,dx\\&\qquad\leqslant p\int_{\mathbb{R}^n}(U_\varepsilon+(U_k)_+)^{p-1}(x,0)(U_k)_+(x,0)U_\varepsilon(x,0)\,dx.\end{aligned}$$

We now use Hölder inequality with exponents $\frac{2^*_s}{p-1}=\frac{n}{2s}$ and $\frac{n}{n-2s}$ and obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}\left((U_\varepsilon+(U_k)_+)^p(x,0)-U_\varepsilon^p(x,0)\right)U_\varepsilon(x,0)\,dx\\&\quad\leqslant p\left(\int_{\mathbb{R}^n}(U_\varepsilon+(U_k)_+)^{2^*_s}(x,0)\,dx\right)^{\frac{2s}{n}}\left(\int_{\mathbb{R}^n}(U_k)_+^{\frac{n}{n-2s}}(x,0)U_\varepsilon^{\frac{n}{n-2s}}(x,0)\,dx\right)^{\frac{n-2s}{n}}\\&\quad\leqslant C[U_\varepsilon+(U_k)_+]_a^{p-1}\left(\int_{\mathbb{R}^n}(U_k)_+^{\frac{n}{n-2s}}(x,0)U_\varepsilon^{\frac{n}{n-2s}}(x,0)\,dx\right)^{\frac{n-2s}{n}}\\&\quad\leqslant C\left(\int_{\mathbb{R}^n}(U_k)_+^{\frac{n}{n-2s}}(x,0)U_\varepsilon^{\frac{n}{n-2s}}(x,0)\,dx\right)^{\frac{n-2s}{n}},\end{aligned}$$

for some positive C that may change from line to line, thanks to Proposition 1.2.1 and (5.4.2). Now the desired claim in (5.4.58) simply follows by using Lemma 5.3.1 with $V_k:=U_k$, $U_o:=U_\varepsilon$, $\psi:=1$, $\alpha:=\frac{n}{n-2s}$ and $\beta:=\frac{n}{n-2s}$ (notice that $\alpha+\beta=2^*_s$).

From (5.4.58) we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^n}\left((U_\varepsilon+(U_k)_+)^p(x,0)-U_\varepsilon^p(x,0)\right)(U_\varepsilon+W_{1,k})(x,0)\,dx\\&\qquad+\int_{\mathbb{R}^n}\left((U_\varepsilon+(U_k)_+)^p(x,0)-U_\varepsilon^p(x,0)\right)(U_\varepsilon+W_{2,k})(x,0)\,dx\\&\qquad=\int_{\mathbb{R}^n}\left((U_\varepsilon+(U_k)_+)^p(x,0)-U_\varepsilon^p(x,0)\right)(U_\varepsilon+W_{1,k}+W_{2,k})(x,0)\,dx+o_k(1)\\&\qquad=\int_{\mathbb{R}^n}\left((U_\varepsilon+(U_k)_+)^p(x,0)-U_\varepsilon^p(x,0)\right)(U_\varepsilon+U_k)(x,0)\,dx+o_k(1)\\&\qquad=\int_{\mathbb{R}^n}\left((U_\varepsilon+(U_k)_+)^{p+1}(x,0)-U_\varepsilon^{p+1}(x,0)\right)(x,0)\,dx+o_k(1),\end{aligned}$$

where Lemma 5.3.1 was used once again in the last line. Plugging this information into (5.4.57) we obtain

$$\begin{aligned}
&\mathcal{J}_\varepsilon(U_k)\\
\geqslant\ &\mathcal{J}_\varepsilon(W_{1,k})+\mathcal{J}_\varepsilon(W_{2,k})\\
&\quad+\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n}\left((U_\varepsilon+(U_k)_+)^{p+1}(x,0)-U_\varepsilon^{p+1}(x,0)\right)\,dx\\
&\quad-\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n}\left((U_\varepsilon+(W_{1,k})_+)^{p+1}(x,0)-U_\varepsilon^{p+1}(x,0)\right)\,dx\\
&\quad-\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n}\left((U_\varepsilon+(W_{2,k})_+)^{p+1}(x,0)-U_\varepsilon^{p+1}(x,0)\right)\,dx-C\varepsilon^{\frac{1}{2\gamma}}+o_k(1).
\end{aligned}$$

Now we use (5.3.4) with $U_o:=U_\varepsilon$ and $V_k:=U_k$, $V_k:=W_{1,k}$ and $V_k:=W_{2,k}$ respectively, and so

$$\begin{aligned}
\mathcal{J}_\varepsilon(U_k)\ \geqslant\ &\mathcal{J}_\varepsilon(W_{1,k})+\mathcal{J}_\varepsilon(W_{2,k})+\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n}(U_k)_+^{p+1}(x,0)\,dx\\
&-\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n}(W_{1,k})_+^{p+1}(x,0)\,dx\\
&-\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n}(W_{2,k})_+^{p+1}(x,0)\,dx-C\varepsilon^{\frac{1}{2\gamma}}+o_k(1).
\end{aligned}$$

Notice now that for any $x\in\mathbb{R}^n$

$$\begin{aligned}
&(U_k)_+^{p+1}(x,0)-(W_{1,k})_+^{p+1}(x,0)-(W_{2,k})_+^{p+1}(x,0)\\
=\ &(U_k)_+^{p+1}(x,0)-\chi^{p+1}(x,0)(U_k)_+^{p+1}(x,0)-(1-\chi(x,0))^{p+1}(U_k)_+^{p+1}(x,0)\\
=\ &(U_k)_+^{p+1}(x,0)\left(1-\chi^{p+1}(x,0)-(1-\chi)^{p+1}(x,0)\right)\geqslant 0.
\end{aligned}$$

This and the fact that $p+1>2$ give

$$\mathcal{J}_\varepsilon(U_k)\geqslant\mathcal{J}_\varepsilon(W_{1,k})+\mathcal{J}_\varepsilon(W_{2,k})-C\varepsilon^{\frac{1}{2\gamma}}+o_k(1).$$

Finally, this, together with (5.4.40) and (5.4.52), implies that

$$\mathcal{J}_\varepsilon(U_k)\geqslant\frac{s}{n}S^{n/2s}-C\varepsilon^{\frac{1}{2\gamma}}+o_k(1),$$

up to renaming constants. Therefore, taking the limit as $k\to+\infty$ we have

$$c_\varepsilon=\lim_{k\to+\infty}\mathcal{J}_\varepsilon(U_k)\geqslant\frac{s}{n}S^{n/2s}-C\varepsilon^{\frac{1}{2\gamma}}.$$

This gives a contradiction with (5.4.1) and finishes the proof of Lemma 5.4.4. □

We are now in the position to show that the functional $\mathcal{J}_\varepsilon$ introduced in (1.2.15) satisfies a Palais-Smale condition.

Proof of Proposition 5.4.1. Thanks to Lemma 5.4.3 we know that the sequence U_k weakly converges to 0 in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$ as $k\to+\infty$.

For any $k \in \mathbb{N}$, we set $V_k := U_\varepsilon + U_k$, where U_ε is the local minimum of $\mathcal{F}_\varepsilon$ found Theorem 1.2.2. Since U_ε is a critical point of $\mathcal{F}_\varepsilon$, from (1.2.11) we deduce that

$$\int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_\varepsilon, \nabla U_k \rangle \, dX = \varepsilon \int_{\mathbb{R}^n} h(x) U_\varepsilon^q(x,0) U_k(x,0)\, dx + \int_{\mathbb{R}^n} U_\varepsilon^p(x,0) U_k(x,0)\, dx.$$

Therefore, recalling (1.2.10) and (1.2.15) we have

(5.4.59)

$$\begin{aligned}
\mathcal{F}_\varepsilon(V_k) =& \frac{1}{2}\int_{\mathbb{R}^{n+1}_+} y^a |\nabla (U_\varepsilon + U_k)|^2 \, dX \\
&- \frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)(U_\varepsilon + U_k)_+^{q+1}(x,0)\, dx - \frac{1}{p+1}\int_{\mathbb{R}^n} (U_\varepsilon + U_k)_+^{p+1}(x,0)\, dx \\
=& \mathcal{I}_\varepsilon(U_k) + \mathcal{F}_\varepsilon(U_\varepsilon) + \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla U_\varepsilon, \nabla U_k\rangle \, dX \\
&+ \frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\Big((U_\varepsilon + (U_k)_+)^{q+1}(x,0) - (U_\varepsilon + U_k)_+^{q+1}(x,0)\Big)\, dx \\
&+ \frac{1}{p+1}\int_{\mathbb{R}^n} \Big((U_\varepsilon + (U_k)_+)^{p+1}(x,0) - (U_\varepsilon + U_k)_+^{p+1}(x,0)\Big)\, dx \\
&- \varepsilon \int_{\mathbb{R}^n} h(x) U_\varepsilon^q(x,0)(U_k)_+(x,0)\, dx - \int_{\mathbb{R}^n} U_\varepsilon^p(x,0)(U_k)_+(x,0) \\
=& \mathcal{I}_\varepsilon(U_k) + \mathcal{F}_\varepsilon(U_\varepsilon) \\
&+ \frac{\varepsilon}{q+1}\int_{\mathbb{R}^n} h(x)\Big((U_\varepsilon + (U_k)_+)^{q+1}(x,0) - (U_\varepsilon + U_k)_+^{q+1}(x,0) \\
&\qquad + (q+1)U_\varepsilon^q(x,0)(U_k - (U_k)_+)(x,0)\Big)\, dx \\
&+ \frac{1}{p+1}\int_{\mathbb{R}^n} \Big((U_\varepsilon + (U_k)_+)^{p+1}(x,0) - (U_\varepsilon + U_k)_+^{p+1}(x,0) \\
&\qquad + (p+1)U_\varepsilon^p(x,0)(U_k - (U_k)_+)(x,0)\Big)\, dx.
\end{aligned}$$

We now claim that

(5.4.60)

$$(U_\varepsilon + (U_k)_+)^{r+1}(x,0) - (U_\varepsilon + U_k)_+^{r+1}(x,0) + (r+1)U_\varepsilon^r(x,0)(U_k - (U_k)_+)(x,0) \leqslant 0,$$

for any $x \in \mathbb{R}^n$ and $r \in \{p,q\}$. Indeed, the claim is trivially true if $U_k(x,0) \geqslant 0$. Hence we suppose that $U_k(x,0) < 0$, and so (5.4.60) becomes

(5.4.61)

$$U_\varepsilon^{r+1}(x,0) - (U_\varepsilon + U_k)_+^{r+1}(x,0) + (r+1)U_\varepsilon^r(x,0)(U_k - (U_k)_+)(x,0) \leqslant 0.$$

Given $a > 0$, the function $f(t) := (a+t)_+^{r+1}$, for $t \in \mathbb{R}$, is convex, and therefore it satisfies for any $b < 0$

$$f(b) \geqslant f(0) + f'(0)b,$$

that is

$$(a+b)_+^{r+1} \geqslant a^{r+1} + (r+1)a^r b.$$

Thus, taking $a := U_\varepsilon(x,0)$ and $b := U_k(x,0)$ we have

$$(U_\varepsilon + U_k)_+^{r+1}(x,0) \geqslant U_\varepsilon^{r+1}(x,0) + (r+1)U_\varepsilon^r(x,0)U_k(x,0),$$

which shows (5.4.61), and in turn (5.4.60).

Accordingly, using (5.4.60) into (5.4.59) we get

$$\mathcal{F}_\varepsilon(V_k) \leqslant \mathcal{I}_\varepsilon(U_k) + \mathcal{F}_\varepsilon(U_\varepsilon). \tag{5.4.62}$$

This and assumption (i) in Proposition 5.4.1 imply that

$$|\mathcal{F}_\varepsilon(V_k)| \leqslant C, \tag{5.4.63}$$

for a suitable $C > 0$ independent of k.

Now we recall that U_ε is a critical point of $\mathcal{F}_\varepsilon$. Hence, from (1.2.11) we deduce that for any $\Psi \in \dot{H}_a^s(\mathbb{R}_+^{n+1})$ with $\psi := \Psi(\cdot, 0)$

$$\int_{\mathbb{R}_+^{n+1}} y^a \langle \nabla U_\varepsilon, \nabla \Psi \rangle \, dX = \varepsilon \int_{\mathbb{R}^n} h(x) U_\varepsilon^q(x,0)\psi(x)\, dx + \int_{\mathbb{R}^n} U_\varepsilon^p(x,0)\psi(x)\, dx. \tag{5.4.64}$$

Moreover, from (1.2.11) and (1.2.18) we have that

$$\begin{aligned}
&\langle \mathcal{F}_\varepsilon'(V_k), \Psi \rangle \\
= &\int_{\mathbb{R}_+^{n+1}} y^a \langle \nabla V_k, \nabla \Psi \rangle \, dX - \varepsilon \int_{\mathbb{R}^n} h(x)(V_k)_+^q(x,0)\psi(x)\, dx - \int_{\mathbb{R}^n} (V_k)_+^p(x,0)\psi(x)\, dx \\
= &\langle \mathcal{I}_\varepsilon'(U_k), \Psi \rangle + \int_{\mathbb{R}_+^{n+1}} y^a \langle \nabla U_\varepsilon, \nabla \Psi \rangle \, dX \\
&\quad + \varepsilon \int_{\mathbb{R}^n} h(x) \left((U_\varepsilon + (U_k)_+)^q(x,0) - U_\varepsilon^q(x,0) \right) \psi(x)\, dx \\
&\quad + \int_{\mathbb{R}^n} \left((U_\varepsilon + (U_k)_+)^p(x,0) - U_\varepsilon^p(x,0) \right) \psi(x)\, dx \\
&\quad - \varepsilon \int_{\mathbb{R}^n} h(x)(V_k)_+^q(x,0)\psi(x)\, dx - \int_{\mathbb{R}^n} (V_k)_+^p(x,0)\psi(x)\, dx.
\end{aligned}$$

Using (5.4.64) in the formula above and recalling that $V_k = U_\varepsilon + U_k$, we obtain

$$\begin{aligned}
\langle \mathcal{F}_\varepsilon'(V_k), \Psi \rangle &= \langle \mathcal{I}_\varepsilon'(U_k), \Psi \rangle \\
&\quad + \varepsilon \int_{\mathbb{R}^n} h(x) \big((U_\varepsilon + (U_k)_+)^q(x,0) - (U_\varepsilon + U_k)_+^q(x,0) \big) \psi(x)\, dx \\
&\quad + \int_{\mathbb{R}^n} \big((U_\varepsilon + (U_k)_+)^p(x,0) - (U_\varepsilon + U_k)_+^p(x,0) \big) \psi(x)\, dx.
\end{aligned} \tag{5.4.65}$$

We claim that

(5.4.66)
$$\begin{aligned}&\lim_{k\to+\infty}\int_{\mathbb{R}^n} h(x)\left((U_\varepsilon+(U_k)_+)^q(x,0)-(U_\varepsilon+U_k)_+^q(x,0)\right)\psi(x)\,dx=0\\ \text{and }&\lim_{k\to+\infty}\int_{\mathbb{R}^n}\left((U_\varepsilon+(U_k)_+)^p(x,0)-(U_\varepsilon+U_k)_+^p(x,0)\right)\psi(x)\,dx=0.\end{aligned}$$

Notice that if $U_k(x,0)\geqslant 0$ then

$$\begin{aligned}&(U_\varepsilon+(U_k)_+)^q(x,0)-(U_\varepsilon+U_k)_+^q(x,0)\\ &\qquad=(U_\varepsilon+U_k)^q(x,0)-(U_\varepsilon+U_k)^q(x,0)=0\\ \text{and}\qquad&(U_\varepsilon+(U_k)_+)^p(x,0)-(U_\varepsilon+U_k)_+^p(x,0)\\ &\qquad=(U_\varepsilon+U_k)^p(x,0)-(U_\varepsilon+U_k)^p(x,0)=0.\end{aligned}$$

Therefore the claim becomes

(5.4.67)
$$\begin{aligned}&\lim_{k\to+\infty}\int_{\mathbb{R}^n\cap\{U_k(\cdot,0)<0\}} h(x)\left(U_\varepsilon^q(x,0)-(U_\varepsilon+U_k)_+^q(x,0)\right)\psi(x)\,dx=0\\ \text{and }&\lim_{k\to+\infty}\int_{\mathbb{R}^n\cap\{U_k(\cdot,0)<0\}}\left(U_\varepsilon^p(x,0)-(U_\varepsilon+U_k)_+^p(x,0)\right)\psi(x)\,dx=0.\end{aligned}$$

Now, we recall that Lemma 5.4.3 here and the compact embedding in Theorem 7.1 in [**21**] imply that $U_k(\cdot,0)\to 0$ a.e. in $\mathbb{R}^n$ as $k\to+\infty$. Moreover, we notice that, by the Hölder inequality with exponents $\frac{2_s^*}{2_s^*-1-q}$, $\frac{2_s^*}{q}$ and 2_s^*,

$$\begin{aligned}&\left|\int_{\mathbb{R}^n\cap\{U_k(\cdot,0)<0\}} h(x)U_\varepsilon^q(x,0)\psi(x)\,dx\right|\\ &\quad\leqslant\left(\int_{\mathbb{R}^n\cap\{U_k(\cdot,0)<0\}}|h(x)|^{\frac{2_s^*}{2_s^*-1-q}}\,dx\right)^{\frac{2_s^*-1-q}{2_s^*}}\left(\int_{\mathbb{R}^n\cap\{U_k(\cdot,0)<0\}}U_\varepsilon^{2_s^*}(x,0)\,dx\right)^{\frac{q}{2_s^*}}\\ &\qquad\cdot\left(\int_{\mathbb{R}^n\cap\{U_k(\cdot,0)<0\}}|\psi(x)|^{2_s^*}\,dx\right)^{\frac{1}{2_s^*}}\\ &\quad\leqslant\|h\|_{L^{\frac{2_s^*}{2_s^*-1-q}}(\mathbb{R}^n)}S^{-q/2}[U_\varepsilon]_a^qS^{-1/2}[\Psi]_a\leqslant C,\end{aligned}$$

for some $C>0$, thanks to (1.1.5) and Proposition 1.2.1. Consequently

$$h\left(U_\varepsilon^q(\cdot,0)-(U_\varepsilon+U_k)_+^q(\cdot,0)\right)\psi\leqslant|h|U_\varepsilon^q(\cdot,0)|\psi|\in L^1(\mathbb{R}^n\cap\{U_k(\cdot,0)<0\}).$$

Hence, by the Dominated Convergence Theorem we get the first limit in (5.4.67).

To prove the second limit in (5.4.67), we use the Hölder inequality with exponents $\frac{2_s^*}{p} = \frac{2n}{n+2s}$ and $2_s^* = \frac{2n}{n-2s}$ to see that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n\cap\{U_k(\cdot,0)<0\}} U_\varepsilon^p(x,0)\psi(x)\,dx\right| \\
\leqslant\ & \left(\int_{\mathbb{R}^n\cap\{U_k(\cdot,0)<0\}} U_\varepsilon^{2_s^*}(x,0)\,dx\right)^{\frac{n+2s}{2n}} \left(\int_{\mathbb{R}^n\cap\{U_k(\cdot,0)<0\}} |\psi(x)|^{2_s^*}\,dx\right)^{\frac{1}{2_s^*}} \\
\leqslant\ & S^{-p/2}[U_\varepsilon]_a^p S^{-1/2}[\Psi]_a \leqslant C,
\end{aligned}$$

for a suitable $C > 0$, where (1.2.7) was also used. Therefore,

$$\left(U_\varepsilon^p(\cdot,0) - (U_\varepsilon + U_k)_+^p(\cdot,0)\right)\psi \leqslant U_\varepsilon^p(\cdot,0)|\psi| \in L^1(\mathbb{R}^n \cap \{U_k(\cdot,0) < 0\}).$$

So the second limit in (5.4.67) follows from the Dominated Convergence Theorem. This shows (5.4.67) and so the proof of (5.4.66) is finished.

As a consequence of (5.4.65), (5.4.66) and assumption (ii) in Proposition 5.4.1 we have that

$$\mathcal{F}'_\varepsilon(V_k) \to 0 \text{ as } k \to +\infty \tag{5.4.68}$$

in the sense of Remark 5.4.2. This, together with (5.4.63) and Lemma 3.2.3, implies that the sequence V_k is uniformly bounded in $\dot{H}_a^s(\mathbb{R}_+^{n+1})$, namely there exists a constant $M > 0$ such that

$$[V_k]_a \leqslant M \text{ for all } k \in \mathbb{N}. \tag{5.4.69}$$

Hence, V_k is weakly convergent in $\dot{H}_a^s(\mathbb{R}_+^{n+1})$ to some function V_0. Since $V_k = U_\varepsilon + U_k$ and U_k weakly converges to 0 in $\dot{H}_a^s(\mathbb{R}_+^{n+1})$ as $k \to +\infty$ (see Lemma 5.4.3), it turns out that $V_0 = U_\varepsilon$. Also, we recall that U_ε is positive, thanks to Proposition 1.2.3. Therefore, we are in the position to apply Lemma 5.3.3 with $W_m := V_k$ and $W := U_\varepsilon$, and we obtain that

$$(V_k)_+ \text{ weakly converges to } U_\varepsilon \text{ in } \dot{H}_a^s(\mathbb{R}_+^{n+1}) \text{ as } k \to +\infty. \tag{5.4.70}$$

We also show that

$$\text{the sequence } \{V_k\}_k \text{ is tight, according to Definition 2.2.1.} \tag{5.4.71}$$

For this, we fix $\eta > 0$. Thanks to Lemma 5.4.4, we have that there exists $\rho_1 > 0$ such that

$$\int_{\mathbb{R}_+^{n+1}\setminus B_{\rho_1}^+} y^a |\nabla U_k|^2\,dX < \frac{\eta}{4},$$

for any $k \in \mathbb{N}$. Moreover, since $U_\varepsilon \in \dot{H}_a^s(\mathbb{R}_+^{n+1})$, there exists $\rho_2 > 0$ such that

$$\int_{\mathbb{R}_+^{n+1}\setminus B_{\rho_2}^+} y^a |\nabla U_\varepsilon|^2\,dX < \frac{\eta}{4}.$$

We take $\rho := \max\{\rho_1, \rho_2\}$, and so the two formulas above give that

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+\setminus B^+_\rho} y^a|\nabla V_k|^2\,dX \\
= \;& \int_{\mathbb{R}^{n+1}_+\setminus B^+_\rho} y^a|\nabla U_k|^2\,dX + \int_{\mathbb{R}^{n+1}_+\setminus B^+_\rho} y^a|\nabla U_\varepsilon|^2\,dX \\
&\quad +2\int_{\mathbb{R}^{n+1}_+\setminus B^+_\rho} y^a\langle\nabla U_k, \nabla U_\varepsilon\rangle\,dX \\
\leqslant\;& \int_{\mathbb{R}^{n+1}_+\setminus B^+_\rho} y^a|\nabla U_k|^2\,dX + \int_{\mathbb{R}^{n+1}_+\setminus B^+_\rho} y^a|\nabla U_\varepsilon|^2\,dX \\
&\quad +2\sqrt{\int_{\mathbb{R}^{n+1}_+\setminus B^+_\rho} y^a|\nabla U_k|^2\,dX}\cdot\sqrt{\int_{\mathbb{R}^{n+1}_+\setminus B^+_\rho} y^a|\nabla U_\varepsilon|^2\,dX} \\
\leqslant\;& \frac{\eta}{4}+\frac{\eta}{4}+2\frac{\sqrt{\eta}}{2}\frac{\sqrt{\eta}}{2} = \eta.
\end{aligned}$$

This shows (5.4.71).

Also, Theorem 1.1.4 in [**24**] gives the existence of two measures on $\mathbb{R}^n$ and $\mathbb{R}^{n+1}_+$, ν and μ respectively, such that $(V_k)_+^{2^*_s}(\cdot,0)$ converges to ν and $y^a|\nabla(V_k)_+|^2$ converges to μ as $k\to+\infty$, according to Definition 1.1.2 in [**24**] (see also Definition 2.2.2). This, (5.4.70) and (5.4.71) imply that the hypotheses of Proposition 2.2.3 are satisfied, and so there exist an at most countable set J and three families $\{x_j\}_{j\in J}\in\mathbb{R}^n$, $\{\nu_j\}_{j\in J}$ and $\{\mu_j\}_{j\in J}$, with $\nu_j,\mu_j\geqslant 0$ such that

$$(V_k)_+^{2^*_s} \text{ converges to } \nu = U_\varepsilon^{2^*_s} + \textstyle\sum_{j\in J}\nu_j\delta_{x_j} \text{ as } k\to+\infty, \tag{5.4.72}$$

(5.4.73)
$$y^a|\nabla(V_k)_+|^2 \text{ converges to } \mu \geqslant y^a|\nabla U_\varepsilon|^2 + \textstyle\sum_{j\in J}\mu_j\delta_{(x_j,0)} \text{ as } k\to+\infty$$

and

$$\mu_j \geqslant S\nu_j^{2/2^*_s} \quad \text{for all } j\in J. \tag{5.4.74}$$

We claim now that $\nu_j=\mu_j=0$ for every $j\in J$. To prove this, we argue by contradiction and we suppose that there exists $j\in J$ such that $\mu_j=0$. We denote $X_j:=(x_j,0)$, we fix $\delta>0$ and we take a cut-off function $\phi_\delta\in C^\infty(\mathbb{R}^{n+1}_+,[0,1])$ such that

$$\phi_\delta(X)=\begin{cases}1, & \text{if } X\in B^+_{\delta/2}(X_j),\\ 0, & \text{if } X\in (B^+_\delta(X_j))^c,\end{cases} \qquad \text{and} \qquad |\nabla\phi_\delta|\leqslant\frac{C}{\delta},$$

for some $C>0$.

Now, it is not difficult to show that the sequence $\phi_\delta(V_k)_+$ is uniformly bounded in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$. Therefore, from (5.4.68) and (1.2.11) we have that

(5.4.75)
$$\begin{aligned}
0 &= \lim_{k\to+\infty} \langle \mathcal{F}'_\varepsilon(V_k), \phi_\delta(V_k)_+\rangle \\
&= \lim_{k\to+\infty} \Big(\int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla V_k, \nabla(\phi_\delta(V_k)_+)\rangle \, dX \\
&\qquad - \varepsilon \int_{\mathbb{R}^n} h(x)(V_k)_+^{q+1}(x,0)\phi_\delta(x,0)\, dx - \int_{\mathbb{R}^n} (V_k)_+^{p+1}(x,0)\phi_\delta(x,0)\, dx \Big) \\
&= \lim_{k\to+\infty} \Big(\int_{\mathbb{R}^{n+1}_+} y^a |\nabla (V_k)_+|^2 \phi_\delta \, dX + \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla (V_k)_+, \nabla \phi_\delta\rangle (V_k)_+ \, dX \\
&\qquad - \varepsilon \int_{\mathbb{R}^n} h(x)(V_k)_+^{q+1}(x,0)\phi_\delta(x,0)\, dx - \int_{\mathbb{R}^n} (V_k)_+^{p+1}(x,0)\phi_\delta(x,0)\, dx \Big).
\end{aligned}$$

We recall that $p+1 = 2^*_s$ and we use (5.4.72) and (5.4.73) to see that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} (V_k)_+^{p+1}(x,0)\phi_\delta(x,0)\, dx = \int_{\mathbb{R}^n} \phi_\delta(x,0)\, d\nu \tag{5.4.76}$$

$$\text{and} \quad \lim_{k\to+\infty} \int_{\mathbb{R}^{n+1}_+} y^a |\nabla (V_k)_+|^2 \phi_\delta \, dX = \int_{\mathbb{R}^{n+1}_+} \phi_\delta \, d\mu. \tag{5.4.77}$$

Also, the weak convergence in (5.4.70), (1.2.6) and Theorem 7.1 in [**21**] imply that $(V_k)_+(\cdot,0)$ strongly converges to $U_\varepsilon(\cdot,0)$ in $L^r_{\mathrm{loc}}(\mathbb{R}^n)$ as $k\to+\infty$, for any $r\in[1,2^*_s)$. Accordingly,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} h(x)(V_k)_+^{q+1}(x,0)\phi_\delta(x,0)\, dx - \int_{\mathbb{R}^n} h(x) U_\varepsilon^{q+1}(x,0)\phi_\delta(x,0)\, dx \right| \\
&\qquad \leqslant \|h\|_{L^\infty(\mathbb{R}^n)} \left| \int_{B^+_\delta(X_j)\cap\{y=0\}} \Big((V_k)_+^{q+1}(x,0) - U_\varepsilon^{q+1}(x,0) \Big)\, dx \right| \to 0,
\end{aligned}$$

as $k\to+\infty$, since $1 < q+1 < 2^*_s$. This implies that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} h(x)(V_k)_+^{q+1}(x,0)\phi_\delta(x,0)\, dx = \int_{\mathbb{R}^n} h(x) U_\varepsilon^{q+1}(x,0)\phi_\delta(x,0)\, dx.$$

Taking the limit as $\delta\to 0$ we have

$$\begin{aligned}
&\lim_{\delta\to 0}\lim_{k\to+\infty} \int_{\mathbb{R}^n} h(x)(V_k)_+^{q+1}(x,0)\phi_\delta(x,0)\, dx \\
&\qquad = \lim_{\delta\to 0} \int_{B^+_\delta(X_j)\cap\{y=0\}} h(x) U_\varepsilon^{q+1}(x,0)\phi_\delta(x,0)\, dx = 0.
\end{aligned} \tag{5.4.78}$$

Finally, we claim that

$$\lim_{\delta\to 0}\lim_{k\to+\infty} \int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla (V_k)_+, \nabla\phi_\delta\rangle (V_k)_+ \, dX = 0. \tag{5.4.79}$$

For this, we apply the Hölder inequality and we use (5.4.69) to obtain that

(5.4.80)
$$\begin{aligned}&\left|\int_{\mathbb{R}^{n+1}_+} y^a\langle\nabla(V_k)_+,\nabla\phi_\delta\rangle(V_k)_+\,dX\right|\\&\quad=\left|\int_{B^+_\delta(X_j)} y^a\langle\nabla(V_k)_+,\nabla\phi_\delta\rangle(V_k)_+\,dX\right|\\&\quad\leqslant\left(\int_{B^+_\delta(X_j)} y^a|\nabla(V_k)_+|^2\,dX\right)^{1/2}\left(\int_{B^+_\delta(X_j)} y^a(V_k)^2_+|\nabla\phi_\delta|^2\,dX\right)^{1/2}\\&\quad\leqslant M\left(\int_{B^+_\delta(X_j)} y^a(V_k)^2_+|\nabla\phi_\delta|^2\,dX\right)^{1/2}.\end{aligned}$$

Again by (5.4.69) and Lemma 2.1.2, we deduce that

$$\begin{aligned}&\left|\int_{B^+_\delta(X_j)} y^a(V_k)^2_+|\nabla\phi_\delta|^2\,dX-\int_{B^+_\delta(X_j)} y^aU^2_\varepsilon|\nabla\phi_\delta|^2\,dX\right|\\&\quad\leqslant\frac{C^2}{\delta^2}\left|\int_{B^+_\delta(X_j)} y^a(V_k)^2_+\,dX-\int_{B^+_\delta(X_j)} y^aU^2_\varepsilon\,dX\right|\to 0,\end{aligned}$$

as $k\to+\infty$. Hence

$$\lim_{k\to+\infty}\int_{B^+_\delta(X_j)} y^a(V_k)^2_+|\nabla\phi_\delta|^2\,dX=\int_{B^+_\delta(X_j)} y^aU^2_\varepsilon|\nabla\phi_\delta|^2\,dX. \tag{5.4.81}$$

Now by the Hölder inequality with exponents γ and $\frac{\gamma}{\gamma-1}$ we have that

$$\begin{aligned}&\int_{B^+_\delta(X_j)} y^aU^2_\varepsilon|\nabla\phi_\delta|^2\,dX\\&\leqslant\left(\int_{B^+_\delta(X_j)} y^aU^{2\gamma}_\varepsilon\,dX\right)^{\frac{1}{\gamma}}\left(\int_{B^+_\delta(X_j)} y^a|\nabla\phi_\delta|^{\frac{2\gamma}{\gamma-1}}\,dX\right)^{\frac{\gamma-1}{\gamma}}\\&\leqslant\frac{C^2}{\delta^2}\left(\int_{B^+_\delta(X_j)} y^aU^{2\gamma}_\varepsilon\,dX\right)^{\frac{1}{\gamma}}\left(\int_{B^+_\delta(X_j)} y^a\,dX\right)^{\frac{\gamma-1}{\gamma}}\\&\leqslant C\delta^{\frac{(n+a+1)(\gamma-1)}{\gamma}-2}\left(\int_{B^+_\delta(X_j)} y^aU^{2\gamma}_\varepsilon\,dX\right)^{\frac{1}{\gamma}},\end{aligned}$$

up to renaming constants. Since $\frac{(n+a+1)(\gamma-1)}{\gamma}-2=0$, this implies that

$$\int_{B^+_\delta(X_j)} y^aU^2_\varepsilon|\nabla\phi_\delta|^2\,dX\leqslant C\left(\int_{B^+_\delta(X_j)} y^aU^{2\gamma}_\varepsilon\,dX\right)^{\frac{1}{\gamma}},$$

for a suitable positive constant C. Hence,

$$\lim_{\delta\to 0}\int_{B_\delta^+(X_j)} y^a U_\varepsilon^2 |\nabla \phi_\delta|^2\, dX = 0.$$

This, together with (5.4.80) and (5.4.81), proves (5.4.79).

From (5.4.75), (5.4.76), (5.4.77), (5.4.78) and (5.4.79) we obtain that

$$\begin{aligned} 0 &= \lim_{\delta\to 0}\lim_{k\to+\infty}\langle \mathcal{F}'_\varepsilon(V_k), \phi_\delta (V_k)_+\rangle \\ &= \lim_{\delta\to 0}\Big(\int_{\mathbb{R}^{n+1}_+}\phi_\delta\, d\mu - \int_{\mathbb{R}^n}\phi_\delta(x,0)\, d\nu\Big) \geqslant \mu_j - \nu_j. \end{aligned}$$

Therefore, this and (5.4.74) give that $\nu_j \geqslant \mu_j \geqslant S\nu_j^{2/2_s^*}$. Hence, either $\nu_j = \mu_j = 0$ or $\nu_j^{1-2/2_s^*} \geqslant S$. Since we are in the case $\mu_j \neq 0$, the first possibility cannot occur. As a consequence,

$$\nu_j \geqslant S^{n/2s}. \tag{5.4.82}$$

Now, taking the limit as $k \to +\infty$ in (5.4.62) and recalling assumption (i) of Proposition 5.4.1, (5.4.68) and (5.4.69), we have that

$$\begin{aligned} &c_\varepsilon + \mathcal{F}_\varepsilon(U_\varepsilon) \\ &\geqslant \lim_{k\to+\infty}\left(\mathcal{F}_\varepsilon(V_k) - \frac{1}{2}\langle \mathcal{F}'_\varepsilon(V_k), V_k\rangle\right) \\ &= \lim_{k\to+\infty}\left[\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n}(V_k)_+^{p+1}(x,0)\,dx - \varepsilon\left(\frac{1}{q+1}-\frac{1}{2}\right)\int_{\mathbb{R}^n} h(x)(V_k)_+^{q+1}(x,0)\,dx\right]. \end{aligned} \tag{5.4.83}$$

We claim that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}(V_k)_+^{p+1}(x,0)\,dx \geqslant S^{n/2s} + \int_{\mathbb{R}^n} U_\varepsilon^{p+1}(x,0)\,dx. \tag{5.4.84}$$

For this, we take a sequence $\{\varphi_m\}_{m\in\mathbb{N}} \in C_0^\infty(\mathbb{R}^n,[0,1])$ such that $\lim_{m\to+\infty}\varphi_m(x) = 1$ for any $x\in\mathbb{R}^n$. By (5.4.72) we have that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}(V_k)_+^{p+1}(x,0)\,dx \geqslant \lim_{k\to+\infty}\int_{\mathbb{R}^n}(V_k)_+^{p+1}(x,0)\varphi_m(x)\,dx = \int_{\mathbb{R}^n}\varphi_m(x)\,d\nu.$$

Moreover, thanks to Fatou's lemma and (5.4.82),

$$\lim_{m\to+\infty}\int_{\mathbb{R}^n}\varphi_m(x)\,d\nu \geqslant \int_{\mathbb{R}^n} d\mu \geqslant S^{n/2s} + \int_{\mathbb{R}^n} U_\varepsilon^{p+1}(x,0)\,dx.$$

The last two formulas imply that

$$\begin{aligned} \lim_{k\to+\infty}\int_{\mathbb{R}^n}(V_k)_+^{p+1}(x,0)\,dx &= \lim_{m\to+\infty}\lim_{k\to+\infty}\int_{\mathbb{R}^n}(V_k)_+^{p+1}(x,0)\,dx \\ &\geqslant \lim_{m\to+\infty}\int_{\mathbb{R}^n}\varphi_m(x)\,d\nu \geqslant S^{n/2s} + \int_{\mathbb{R}^n} U_\varepsilon^{p+1}(x,0)\,dx, \end{aligned}$$

which gives the desired result in (5.4.84). We now show that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} h(x)(V_k)_+^{q+1}(x,0)\,dx = \int_{\mathbb{R}^n} h(x)U_\varepsilon^{q+1}(x,0)\,dx. \tag{5.4.85}$$

Indeed, thanks to (5.4.69) we know that $[(V_k)_+]_a \leqslant M$. Therefore, Proposition 1.2.1 and Theorem 7.1 in [**21**] imply that

$$\begin{aligned} &\|(V_k)_+(\cdot,0)-U_\varepsilon(\cdot,0)\|_{L^{2_s^*}(\mathbb{R}^n)} \leqslant 2M \\ \text{and}\qquad &(V_k)_+(\cdot,0)\to U_\varepsilon(\cdot,0) \ \text{ in } L^{q+1}_{\rm loc}(\mathbb{R}^n) \ \text{ as } k\to+\infty. \end{aligned}$$

Thus, we fix $R>0$ and we use (1.1.3), (1.1.5) and the Hölder inequality to obtain that

$$\begin{aligned} &\left|\int_{\mathbb{R}^n} h(x)\left((V_k)_+(x,0)-U_\varepsilon(x,0)\right)^{q+1}\,dx\right| \\ &\quad\leqslant \int_{B_R} |h(x)|\ |(V_k)_+(x,0)-U_\varepsilon(x,0)|^{q+1}\,dx \\ &\qquad + \int_{\mathbb{R}^n\setminus B_R} |h(x)|\ |(V_k)_+(x,0)-U_\varepsilon(x,0)|^{q+1}\,dx \\ &\quad\leqslant \|h\|_{L^\infty(\mathbb{R}^n)}\|(V_k)_+(\cdot,0)-U_\varepsilon(\cdot,0)\|_{L^{q+1}(B_R)} \\ &\qquad + \|h\|_{L^{\frac{2_s^*}{2_s^*-q-1}}(\mathbb{R}^n\setminus B_R)}\|(V_k)_+(\cdot,0)-U_\varepsilon(\cdot,0)\|_{L^{2_s^*}(\mathbb{R}^n)} \\ &\quad\leqslant C\|(V_k)_+(\cdot,0)-U_\varepsilon(\cdot,0)\|_{L^{q+1}(B_R)}\,dx + (2M)^{q+1}\|h\|_{L^{\frac{2_s^*}{2_s^*-q-1}}(\mathbb{R}^n\setminus B_R)}. \end{aligned}$$

Hence, letting first $k\to+\infty$ and then $R\to+\infty$, we obtain (5.4.85).

Also, we observe that $\frac{1}{2}-\frac{1}{p+1}=\frac{s}{n}$. Using this and plugging (5.4.84) and (5.4.85) into (5.4.83) we obtain that

$$\begin{aligned} & c_\varepsilon + \mathcal{F}_\varepsilon(U_\varepsilon) \\ \geqslant\ & \frac{s}{n}S^{n/2s} + \left(\frac{1}{2}-\frac{1}{p+1}\right)\int_{\mathbb{R}^n} U_\varepsilon^{p+1}(x,0)\,dx \\ & \qquad -\varepsilon\left(\frac{1}{q+1}-\frac{1}{2}\right)\int_{\mathbb{R}^n} h(x)U_\varepsilon^{q+1}(x,0)\,dx \\ =\ & \frac{s}{n}S^{n/2s} + \mathcal{F}_\varepsilon(U_\varepsilon). \end{aligned}$$

Hence

$$c_\varepsilon \geqslant \frac{s}{n}S^{n/2s},$$

and this is a contradiction with (5.4.1).

As a consequence, necessarily $\mu_j=\nu_j=0$ for any $j\in J$. Hence, by (5.4.72)

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n} (V_k)_+^{2_s^*}(x,0)\varphi(x)\,dx = \int_{\mathbb{R}^n} U_\varepsilon^{2_s^*}(x,0)\varphi(x)\,dx, \tag{5.4.86}$$

for any $\varphi \in C_0(\mathbb{R}^n)$. Furthermore, by Lemma 5.4.4 and the fact that $U_\varepsilon(\cdot,0) \in L^{2^*_s}(\mathbb{R}^n)$ (thanks to Proposition 1.2.1), we have that for any $\eta > 0$ there exists $\rho > 0$ such that

$$\int_{\mathbb{R}^n\setminus B_\rho} (V_k)_+^{2^*_s}(x,0)\,dx < \eta.$$

Thus we are in the position to apply Lemma 3.1.2 with $v_k := (V_k)_+(\cdot,0)$ and $v := U_\varepsilon(\cdot,0)$, and we obtain that $(V_k)_+(\cdot,0) \to U_\varepsilon(\cdot,0)$ in $L^{2^*_s}(\mathbb{R}^n)$ as $k \to +\infty$. Then, by Lemma 3.1.1 (again applied with $v_k := (V_k)_+(\cdot,0)$ and $v := U_\varepsilon(\cdot,0)$) we have that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} \left|(V_k)_+^q(x,0) - U_\varepsilon^q(x,0)\right|^{\frac{2^*_s}{q}}\,dx = 0$$

$$\text{and} \qquad \lim_{k\to+\infty} \int_{\mathbb{R}^n} \left|(V_k)_+^p(x,0) - U_\varepsilon^p(x,0)\right|^{\frac{2n}{n+2s}}\,dx = 0.$$

Therefore, we can fix $\delta \in (0,1)$ (that we will take arbitrarily small in the sequel), and say that

$$\begin{aligned}\int_{\mathbb{R}^n} &\left|(V_k)_+^q(x,0) - (V_m)_+^q(x,0)\right|^{\frac{2^*_s}{q}}\,dx \\ &+ \int_{\mathbb{R}^n} \left|(V_k)_+^p(x,0) - (V_m)_+^p(x,0)\right|^{\frac{2n}{n+2s}}\,dx \leqslant \delta\end{aligned} \tag{5.4.87}$$

for k and m sufficiently large (say bigger that some $k_\star(\delta)$).

We now take $\Psi \in \dot{H}^s_a(\mathbb{R}^{n+1}_+)$ with $\psi := \Psi(\cdot,0)$ and such that

$$[\Psi]_a = 1. \tag{5.4.88}$$

From (5.4.68) we have that, for large k (say $k \geqslant k_\star(\delta)$, up to renaming $k_\star(\delta)$), we deduce that

$$\left|\langle \mathcal{F}'_\varepsilon(V_k), \Psi\rangle\right| \leqslant \delta.$$

As a consequence of this and (1.2.11),

$$\begin{aligned}\Bigg|\int_{\mathbb{R}^{n+1}_+} & y^a \langle \nabla V_k, \nabla \Psi\rangle\,dX \\ &-\varepsilon \int_{\mathbb{R}^n} h(x)(V_k)_+^q(x,0)\psi(x)\,dx - \int_{\mathbb{R}^n} (V_k)_+^p(x,0)\psi(x)\,dx\Bigg| \leqslant \delta.\end{aligned}$$

In particular, for $k, m \geqslant k_\star(\delta)$,

$$\begin{aligned}\Bigg|\int_{\mathbb{R}^{n+1}_+} & y^a \langle \nabla(V_k - V_m), \nabla\Psi\rangle\,dX \\ &-\varepsilon \int_{\mathbb{R}^n} h(x)\left((V_k)_+^q(x,0) - (V_m)_+^q(x,0)\right)\psi(x)\,dx \\ &- \int_{\mathbb{R}^n} \left((V_k)_+^p(x,0) - (V_m)_+^p(x,0)\right)\psi(x)\,dx\Bigg| \leqslant 2\delta.\end{aligned}$$

Now we use the Hölder inequality with exponents $\frac{2n}{n+2s-q(n-2s)}$, $\frac{2^*_s}{q} = \frac{2n}{q(n-2s)}$ and $2^*_s = \frac{2n}{n-2s}$, and with exponents $\frac{2^*_s}{p} = \frac{2n}{n+2s}$ and 2^*_s, and we obtain that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla(V_k - V_m), \nabla\Psi\rangle \, dX\right| \\
\leqslant\ & \varepsilon \left|\int_{\mathbb{R}^n} h(x) \left((V_k)^q_+(x,0) - (V_m)^q_+(x,0)\right) \psi(x)\, dx\right| \\
&\quad + \left|\int_{\mathbb{R}^n} \left((V_k)^p_+(x,0) - (V_m)^p_+(x,0)\right) \psi(x)\, dx\right| + 2\delta \\
\leqslant\ & \varepsilon \left[\int_{\mathbb{R}^n} |h(x)|^{\frac{2n}{n+2s-q(n-2s)}} \, dx\right]^{\frac{n+2s-q(n-2s)}{2n}} \\
&\quad \cdot \left[\int_{\mathbb{R}^n} \left|(V_k)^q_+(x,0) - (V_m)^q_+(x,0)\right|^{\frac{2^*_s}{q}} \, dx\right]^{\frac{q(n-2s)}{2n}} \left[\int_{\mathbb{R}^n} |\psi(x)|^{2^*_s}\, dx\right]^{\frac{1}{2^*_s}} \\
&\quad + \left[\int_{\mathbb{R}^n} \left|(V_k)^p_+(x,0) - (V_m)^p_+(x,0)\right|^{\frac{2n}{n+2s}} \, dx\right]^{\frac{n+2s}{2n}} \left[\int_{\mathbb{R}^n} |\psi(x)|^{2^*_s}\, dx\right]^{\frac{1}{2^*_s}} + 2\delta.
\end{aligned}$$

Hence, from (1.1.5) and (5.4.87) we have that

$$\left|\int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla(V_k - V_m), \nabla\Psi\rangle \, dX\right| \leqslant C\delta^{\frac{q(n-2s)}{2n}} \|\psi\|_{L^{2^*_s}(\mathbb{R}^n)} + C\delta^{\frac{n+2s}{2n}} \|\psi\|_{L^{2^*_s}(\mathbb{R}^n)} + 2\delta,$$

for a suitable positive constant C. Now notice that (1.2.7) and (5.4.88) imply that $\|\psi\|_{L^{2^*_s}(\mathbb{R}^n)} \leqslant S^{-1/2}[\Psi]_a = S^{-1/2}$, and so

$$\left|\int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla(V_k - V_m), \nabla\Psi\rangle \, dX\right| \leqslant C\delta^a, \tag{5.4.89}$$

for some $C, a > 0$, as long as $k, m \geqslant k_\star(\delta)$. Also,

$$\nabla(V_k - V_m) = \nabla V_k - \nabla V_m = \nabla(U_k + U_\varepsilon) - \nabla(U_m + U_\varepsilon) = \nabla U_k - \nabla U_m.$$

Hence, plugging this into (5.4.89), we have

$$\left|\int_{\mathbb{R}^{n+1}_+} y^a \langle \nabla(U_k - U_m), \nabla\Psi\rangle \, dX\right| \leqslant C\delta^a.$$

Since this inequality is valid for any Ψ satisfying (5.4.88), we deduce that

$$[U_k - U_m]_a \leqslant C\delta^a,$$

namely U_k is a Cauchy sequence in $\dot{H}^s_a(\mathbb{R}^{n+1}_+)$. Then, the desired result plainly follows. ☐

5.5. Bound on the minimax value

The goal of this section is to show that the minimax value (computed along a suitable path) lies below the critical threshold given by Proposition 5.4.1. The chosen path will be a suitably cut-off rescaling of the fractional Sobolev minimizers introduced in (1.1.6).

To start with, we set

$$\tilde{G}(x,U) := \int_0^U \tilde{g}(x,t)\,dt \tag{5.5.1}$$

and

$$\tilde{g}(x,t) := \begin{cases} (U_\varepsilon + t)^q - U_\varepsilon^q, & \text{if } t \geqslant 0,\\ 0 \text{ if } t < 0. \end{cases}$$

We observe that

$$\tilde{G}(x,0) = 0. \tag{5.5.2}$$

Also, we see that $\tilde{g}(x,t) \geqslant 0$ for any $t \in \mathbb{R}$, and so

$$\tilde{G}(x,U) \geqslant 0 \text{ for any } U \geqslant 0. \tag{5.5.3}$$

Moreover, recalling (1.1.4), we write the ball B in (1.1.4) as $B_{\mu_0}(\xi)$ for some $\xi \in \mathbb{R}^n$ and $\mu_0 > 0$. We fix a cut-off function $\bar{\phi} \in C_0^\infty(B_{\mu_0}(\xi), [0,1])$ with

$$\bar{\phi}(x) = 1 \text{ for any } x \in B_{\mu_0/2}(\xi). \tag{5.5.4}$$

The quantities $\xi \in \mathbb{R}^n$ and $\mu_0 > 0$, as well as the cut-off $\bar{\phi}$, are fixed from now on. Also, if z is as in (1.1.6), given $\mu > 0$, we let

$$z_{\mu,\xi}(x) := \mu^{-\frac{n-2s}{2}} z\left(\frac{x-\xi}{\mu}\right). \tag{5.5.5}$$

Let also $\bar{Z}_{\mu,\xi}$ be the extension of $\bar{\phi} z_{\mu,\xi}$, according to (1.2.3).

From (1.1.6), we know that

$$S = \frac{[z]^2_{\dot{H}^s(\mathbb{R}^n)}}{\|z\|^2_{L^{2^*_s}(\mathbb{R}^n)}} \tag{5.5.6}$$

and $(-\Delta)^s z = z^p$. Thus, by testing this equation against z itself, we obtain that

$$[z]^2_{\dot{H}^s(\mathbb{R}^n)} = \|z\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^n)},$$

which, together with (5.5.6), gives that

$$\|z\|_{L^{2^*_s}(\mathbb{R}^n)} = S^{\frac{n-2s}{4s}}$$

and so

$$[z]^2_{\dot{H}^s(\mathbb{R}^n)} = S^{\frac{n}{2s}}.$$

Moreover, by scaling, we have that

$$\|z_{\mu,\xi}\|_{L^{2^*_s}(\mathbb{R}^n)} = \|z\|_{L^{2^*_s}(\mathbb{R}^n)} = S^{\frac{n-2s}{4s}} \tag{5.5.7}$$

and

$$[z_{\mu,\xi}]^2_{\dot H^s(\mathbb{R}^n)} = [z]^2_{\dot H^s(\mathbb{R}^n)} = S^{\frac{n}{2s}}.$$

From the equivalence of norms in (1.2.6) and Proposition 21 in [**36**], we have that

$$[\bar Z_{\mu,\xi}]^2_a = [\bar\phi z_{\mu,\xi}]^2_{\dot H^s(\mathbb{R}^n)} \leqslant S^{\frac{n}{2s}} + C\mu^{n-2s}, \tag{5.5.8}$$

for some $C > 0$.

This setting is fixed from now on, together with the minimum $u_\varepsilon(x) = U_\varepsilon(x, 0)$ given in Theorem 1.2.2. Now we show that the effect of the cut-off on the Lebesgue norm of the rescaled Sobolev minimizers is negligible when μ is small. The quantitative statement is the following:

Lemma 5.5.1. *We have that*

$$\int_{\mathbb{R}^n} |\bar\phi^{2^*_s} - 1|\, z^{2^*_s}_{\mu,\xi}\, dx \leqslant C\mu^n,$$

for some $C > 0$.

Proof. We observe that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_{\frac{\mu_0}{2}}(\xi)} z^{2^*_s}_{\mu,\xi}(x)\, dx &= \mu^{-n}\int_{\mathbb{R}^n\setminus B_{\frac{\mu_0}{2}}(\xi)} z^{2^*_s}\left(\frac{x-\xi}{\mu}\right) dx \\ &= \int_{\mathbb{R}^n\setminus B_{\frac{\mu_0}{2\mu}}} z^{2^*_s}(y)\, dy \leqslant C\int_{\mathbb{R}^n\setminus B_{\frac{\mu_0}{2\mu}}} |y|^{-2n}\, dy \leqslant C\mu^n,\end{aligned}$$

for some $C > 0$ (that may vary from line to line and may also depend on μ_0). As a consequence, recalling (5.5.4), we have that

$$\int_{\mathbb{R}^n} |\bar\phi^{2^*_s} - 1|\, z^{2^*_s}_{\mu,\xi}\, dx = \int_{\mathbb{R}^n\setminus B_{\frac{\mu_0}{2}}(\xi)} |\bar\phi^{2^*_s} - 1|\, z^{2^*_s}_{\mu,\xi}\, dx \leqslant C\mu^n. \qquad \square$$

The next result states that we can always "concentrate the mass near the positivity set of h", in order to detect a positive integral out of it.

Lemma 5.5.2. *We have that*

$$\int_{\mathbb{R}^n} h(x)\, \tilde G\big(x, t\bar\phi(x)\, z_{\mu,\xi}(x)\big)\, dx \geqslant 0, \tag{5.5.9}$$

for any $\mu > 0$ *and any* $t \geqslant 0$.

Proof. We have that $\bar\phi(x) = 0$ if $x \in \mathbb{R}^n \setminus B_{\mu_0}(\xi)$. Thus, using (5.5.2), we have that

$$\tilde G\big(x, t\bar\phi(x)\, z_{\mu,\xi}(x)\big) = 0$$

for any $x \in \mathbb{R}^n \setminus B_{\mu_0}(\xi)$. Therefore

$$\int_{\mathbb{R}^n} h(x)\, \tilde G\big(x, t\bar\phi(x)\, z_{\mu,\xi}(x)\big)\, dx = \int_{B_{\mu_0}(\xi)} h(x)\, \tilde G\big(x, t\bar\phi(x)\, z_{\mu,\xi}(x)\big)\, dx.$$

Then, the desired result follows from (1.1.4) and (5.5.3). $\square$

Now we check that the geometry of the mountain pass is satisfied by the functional $\mathcal{J}_\varepsilon$. Indeed, we first observe that Proposition 5.1.1 gives that 0 is a local minimum for the functional $\mathcal{J}_\varepsilon$. The next result shows that the path induced by the function $\bar{Z}_{\mu,\xi}$ attains negative values, in a somehow uniform way (the uniform estimates in μ in Lemma 5.5.3 will be needed in the subsequent Corollary 5.5.8 and, from these facts, we will be able to deduce the mountain pass geometry, check that the minimax values stays below the critical threshold and complete the proof of Theorem 1.2.4 in the forthcoming Section 5.6). To this goal, it is useful to introduce the auxiliary functional

$$\mathcal{J}^\star_\varepsilon(U) := \frac{1}{2}\int_{\mathbb{R}^{n+1}_+} y^a |\nabla U|^2\, dX - \int_{\mathbb{R}^n} G^\star(x, U(x,0))\, dx, \tag{5.5.10}$$

where

$$G^\star(x,U) := \int_0^U g^\star(x,t)\, dt$$

and

$$g^\star(x,t) := \begin{cases} (U_\varepsilon + t)^p - U_\varepsilon^p, & \text{if } t \geqslant 0,\\ 0 \ \text{ if } t < 0. \end{cases}$$

By (1.2.15) and (5.5.1), we see that $G = G^\star + \varepsilon h \tilde{G}$. Thus, as a consequence of Lemma 5.5.2, we have that

$$\mathcal{J}_\varepsilon(t\bar{Z}_{\mu,\xi}) = \mathcal{J}^\star_\varepsilon(t\bar{Z}_{\mu,\xi}) - \varepsilon\int_{\mathbb{R}^n} h(x)\, \tilde{G}(x, t\bar{Z}_{\mu,\xi}(x))\, dx \leqslant \mathcal{J}^\star_\varepsilon(t\bar{Z}_{\mu,\xi}). \tag{5.5.11}$$

Then we have:

Lemma 5.5.3. *There exists $\mu_1 \in (0,\mu_0)$ such that*

$$\lim_{t\to+\infty} \sup_{\mu\in(0,\mu_1)} \mathcal{J}^\star_\varepsilon(t\bar{Z}_{\mu,\xi}) = -\infty.$$

In particular, there exists $T_1 > 0$ such that

$$\sup_{\mu\in(0,\mu_1)} \mathcal{J}^\star_\varepsilon(t\bar{Z}_{\mu,\xi}) \leqslant 0 \tag{5.5.12}$$

for any $t \geqslant T_1$.

Proof. We observe that, if $U \geqslant 0$,

$$G^\star(x,U) = \int_0^U (U_\varepsilon + t)^p - U_\varepsilon^p\, dt = \frac{(U_\varepsilon + U)^{p+1} - U_\varepsilon^{p+1}}{p+1} - U_\varepsilon^p\, U. \tag{5.5.13}$$

Moreover,

$$U_\varepsilon^p(x,0)\, \bar{Z}_{\mu,\xi}(x,0) \leqslant u_\varepsilon^{p+1}(x) + z_{\mu,\xi}^{p+1}(x).$$

Using this and (5.5.13), we obtain that

$$\begin{aligned} & G^\star(x, t\bar{Z}_{\mu,\xi}(x,0)) \\ = \; & \frac{1}{p+1}\big((U_\varepsilon(x,0)+t\bar{Z}_{\mu,\xi}(x,0))^{p+1} - U_\varepsilon^{p+1}(x,0)\big) - tU_\varepsilon^p(x,0)\,\bar{Z}_{\mu,\xi}(x,0) \\ \geqslant \; & \frac{1}{p+1}\Big((t\bar{\phi}(x)z_{\mu,\xi}(x))^{p+1} - u_\varepsilon^{p+1}(x)\Big) - tu_\varepsilon^{p+1}(x) - tz_{\mu,\xi}^{p+1}(x). \end{aligned}$$

Thus, integrating over $\mathbb{R}^n$ and recalling (1.1.5), (5.5.7) and the fact that $2_s^* = p+1$, we get

$$\begin{aligned} & \int_{\mathbb{R}^n} G^\star(x, t\bar{Z}_{\mu,\xi})\,dx \\ & \qquad \geqslant \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^n} \bar{\phi}^{p+1}(x) z_{\mu,\xi}^{p+1}(x)\,dx - C - Ct, \end{aligned} \tag{5.5.14}$$

for some $C > 0$ (up to renaming constants).

Now we deduce from Lemma 5.5.1 that there exists $\mu_1 \in (0,1)$ such that if $\mu \in (0,\mu_1)$ then

$$\int_{\mathbb{R}^n} \bar{\phi}^{p+1}(x) z_{\mu,\xi}^{p+1}(x)\,dx \geqslant \frac{S^{\frac{n}{2s}}}{2}.$$

Now, by inserting this into (5.5.14), we obtain that, if $\mu \in (0,\mu_1)$, then

$$\int_{\mathbb{R}^n} G^\star(x, t\bar{Z}_{\mu,\xi})\,dx \geqslant \frac{S^{\frac{n}{2s}}\, t^{p+1}}{2(p+1)} - C - Ct.$$

This and (5.5.10) give that

$$\mathcal{I}_\varepsilon^\star(t\bar{Z}_{\mu,\xi}) \leqslant \frac{t^2[\bar{Z}_{\mu,\xi}]_a^2}{2} + C(1+t) - \frac{S^{\frac{n}{2s}}\, t^{p+1}}{2(p+1)}.$$

Hence, recalling (5.5.8),

$$\mathcal{I}_\varepsilon^\star(t\bar{Z}_{\mu,\xi}) \leqslant C(1+t+t^2) - \frac{S^{\frac{n}{2s}}\, t^{p+1}}{2(p+1)},$$

up to renaming constants, for any $\mu \in (0,\mu_1)$. Since $p+1 > 2$, the desired claim easily follows. □

Now we introduce a series of purely elementary, but useful, estimates.

LEMMA 5.5.4. *For any a, $b \geqslant 0$ and any $p > 1$, we have that*

$$(a+b)^p \geqslant a^p + b^p. \tag{5.5.15}$$

Also, if a, $b > 0$, we have that

$$(a+b)^p > a^p + b^p. \tag{5.5.16}$$

PROOF. If either $a = 0$ or $b = 0$, then (5.5.15) is obvious. So we can suppose that $a \neq 0$ and $b \neq 0$. We let $f(b) := (a+b)^p - a^p - b^p$. Notice that $f'(b) = p\big((a+b)^{p-1} - b^{p-1}\big) > 0$, since $a > 0$. Hence

$$(a+b)^p - a^p - b^p = f(b) > f(0) = 0,$$

since $b > 0$, as desired. $\square$

The result in Lemma 5.5.4 can be made more precise when $p \geqslant 2$, as follows:

LEMMA 5.5.5. *Let $p \geqslant 2$. Then, there exists $c_p > 0$ such that, for any a and $b \geqslant 0$,*

$$(a+b)^p \geqslant a^p + b^p + c_p\, a^{p-1} b.$$

PROOF. If $a = 0$, then we are done, so we suppose $a \neq 0$ and we set $t_o := b/a$. For any $t > 0$, we let

$$f(t) := \frac{(1+t)^p - 1 - t^p}{t}.$$

From (5.5.16) (used here with $a := 1$ and $b := t$), we know that $f(t) > 0$ for any $t > 0$. Moreover

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} \frac{1 + pt + o(t) - 1 - t^p}{t} = p,$$

hence f can be continuously extended over $[0, +\infty)$ by setting $f(0) := p$. Furthermore,

$$\begin{aligned}\lim_{t\to+\infty} f(t) &= \lim_{t\to+\infty} t^{p-1}\left(\left(\frac{1}{t}+1\right)^p - \frac{1}{t^p} - 1\right)\\ &= \lim_{t\to+\infty} t^{p-1}\left(1 + \frac{p}{t} + o\left(\frac{1}{t}\right) - \frac{1}{t^p} - 1\right)\\ &= \lim_{t\to+\infty} t^{p-2}\left(p + \frac{o\left(\frac{1}{t}\right)}{\frac{1}{t}}\right) - \frac{1}{t} = \begin{cases} p & \text{if } p = 2,\\ +\infty & \text{if } p > 2.\end{cases}\end{aligned}$$

In any case,

$$\lim_{t\to+\infty} f(t) \geqslant f(0) = p,$$

hence

$$c_p := \inf_{[0,+\infty)} f = \min_{[0,+\infty)} f > 0.$$

As a consequence,

$$\begin{aligned}&(a+b)^p - a^p - b^p - c_p\, a^{p-1} b\\ =\;& a^p\big((1+t_o)^p - 1 - t_o^p - c_p\, t_o\big)\\ =\;& a^p t_o\big(f(t_o) - c_p\big)\\ \geqslant\;& 0,\end{aligned}$$

as desired. $\square$

It is worth to stress that the result in Lemma 5.5.5 does not hold when $p \in (1,2)$, differently than what is often stated in the literature: as a counterexample, one can take $b=1$ and observe that

$$\begin{aligned}\lim_{a\to 0}\frac{(a+b)^p-a^p-b^p}{a^{p-1}b}&=\lim_{a\to 0}\frac{(a+1)^p-a^p-1}{a^{p-1}}\\&=\lim_{a\to 0}\frac{1+pa+o(a)-a^p-1}{a^{p-1}}=\lim_{a\to 0}pa^{2-p}+a^{1-p}o(a)-a=0\end{aligned}$$

when $p\in(1,2)$. In spite of this additional difficulty, when $p\in(1,2)$ one can obtain a variant of Lemma 5.5.5 under an additional assumption on the size of b. The precise statement goes as follows:

LEMMA 5.5.6. *Let $p\in(1,2)$ and $\kappa>0$. Then, there exists $c_{p,\kappa}>0$ such that, for any $a>0$, $b\geqslant 0$, with $\frac{b}{a}\in[0,\kappa]$, we have*

$$(a+b)^p\geqslant a^p+b^p+c_{p,\kappa}\,a^{p-1}b.$$

PROOF. The proof is a variation of the one of Lemma 5.5.5. Full details are provided for the facility of the reader. We set $t_o:=\frac{b}{a}\in[0,\kappa]$. For any $t>0$, we let

$$f(t):=\frac{(1+t)^p-1-t^p}{t}.$$

From (5.5.16) (used here with $a:=1$ and $b:=t$), we know that $f(t)>0$ for any $t>0$. Moreover, f can be continuously extended over $[0,+\infty)$ by setting $f(0):=p$. Therefore

$$c_{p,\kappa}:=\min_{[0,\kappa]}f>0.$$

As a consequence,

$$\begin{aligned}&(a+b)^p-a^p-b^p-c_{p,\kappa}\,a^{p-1}b\\=\;&a^p\big((1+t_o)^p-1-t_o^p-c_{p,\kappa}\,t_o\big)\\=\;&a^pt_o\big(f(t_o)-c_{p,\kappa}\big)\\\geqslant\;&0,\end{aligned}$$

as desired. □

Now we consider the functional introduced in (5.5.10), deal with the path induced by the function z in (1.1.6) (suitably scaled and cut-off) and show that the associated mountain pass level for $\mathcal{J}^\star_\varepsilon$ lies below the critical threshold $\frac{s}{n}S^{\frac{n}{2s}}$ (see Proposition 5.4.1). The precise result goes as follows:

LEMMA 5.5.7. *There exists $\mu_\star\in(0,\mu_0)$ such that if $\mu\in(0,\mu_\star)$ then we have*

$$\sup_{t\geqslant 0}\mathcal{J}^\star_\varepsilon(t\bar Z_{\mu,\xi})<\frac{s}{n}S^{\frac{n}{2s}}. \tag{5.5.17}$$

PROOF. We will take $\mu_\star\leqslant\mu_1$, where $\mu_1>0$ was introduced in Lemma 5.5.3. We also take T_1 as in Lemma 5.5.3. Then, by (5.5.12),

$$\sup_{t\geqslant T_1}\sup_{\mu\in(0,\mu_\star)}\mathcal{J}^\star_\varepsilon(t\bar Z_{\mu,\xi})\leqslant\sup_{t\geqslant T_1}\sup_{\mu\in(0,\mu_1)}\mathcal{J}^\star_\varepsilon(t\bar Z_{\mu,\xi})\leqslant 0<\frac{s}{n}S^{\frac{n}{2s}}. \tag{5.5.18}$$

Consequently, we have that the claim in (5.5.17) holds true if we prove that, for any $\mu \in (0, \mu_\star)$,

$$\sup_{t\in[0,T_1]} \mathcal{J}^\star_\varepsilon(t\bar{Z}_{\mu,\xi}) < \frac{s}{n} S^{\frac{n}{2s}}. \tag{5.5.19}$$

To this goal, we set

$$m := \begin{cases} 2 & \text{if } n > 4s, \\ 2^*_s - 1 & \text{if } n \in (2s, 4s], \end{cases} \tag{5.5.20}$$

and

$$\Omega := \begin{cases} B_{2\sqrt{\mu}}(\xi) \setminus B_{\sqrt{\mu}}(\xi) & \text{if } n > 4s, \\ \mathbb{R}^n & \text{if } n \in (2s, 4s]. \end{cases} \tag{5.5.21}$$

For further reference, we point out that, if $n \in (2s, 4s]$, then $m-2 = \frac{6s-n}{n-2s} > 0$, and so

$$m - 2 \geqslant 0 \text{ for every } n > 2s. \tag{5.5.22}$$

We claim that, for any $t \in [0, T_1]$, any $\mu \in (0, \mu_\star)$ and any $x \in \Omega$, we have

$$G^\star\big(x, t\bar\phi(x) z_{\mu,\xi}(x)\big) \geqslant \frac{t^{2^*_s}\bar\phi^{2^*_s}(x)\, z^{2^*_s}_{\mu,\xi}(x)}{2^*_s} + \frac{c\, u_\varepsilon^{2^*_s-m}(x)\, t^m \bar\phi^m(x)\, z^m_{\mu,\xi}(x)}{m}, \tag{5.5.23}$$

for some $c > 0$.

To prove it, we distinguish two cases, according to whether $n > 4s$ or $n \in (2s, 4s]$. If $n > 4s$, we take $a := u_\varepsilon(x)$ and $b \geqslant 0$, with $b \leqslant t\bar\phi(x) z_{\mu,\xi}(x)$, and $x \in \Omega = B_{2\sqrt{\mu}}(\xi) \setminus B_{\sqrt{\mu}}(\xi)$. Notice that, in this case,

$$a \geqslant \inf_{B_{2\sqrt{\mu}}(\xi)\setminus B_{\sqrt{\mu}}(\xi)} u_\varepsilon \geqslant \inf_{B_2(\xi)} u_\varepsilon \geqslant a_0, \tag{5.5.24}$$

for some $a_0 > 0$. Moreover, from (1.1.6),

$$b \leqslant t z_{\mu,\xi}(x) = t\mu^{-\frac{n-2s}{2}} z\left(\frac{x-\xi}{\mu}\right) = \frac{c_\star t \mu^{-\frac{n-2s}{2}}}{\left(1 + \left|\frac{x-\xi}{\mu}\right|^2\right)^{\frac{n-2s}{2}}} = \frac{c_\star t \mu^{\frac{n-2s}{2}}}{(\mu^2 + |x-\xi|^2)^{\frac{n-2s}{2}}}.$$

Since $x \in B_{2\sqrt{\mu}}(\xi) \setminus B_{\sqrt{\mu}}(\xi)$, we obtain that $|x - \xi| \geqslant \sqrt{\mu}$ and so

$$b \leqslant \frac{c_\star t \mu^{\frac{n-2s}{2}}}{(\mu^2 + \mu)^{\frac{n-2s}{2}}} \leqslant \frac{c_\star t \mu^{\frac{n-2s}{2}}}{\mu^{\frac{n-2s}{2}}} \leqslant c_\star T_1.$$

From this and (5.5.24) we obtain that $b/a \leqslant \kappa$, for some $\kappa > 0$, hence we can apply Lemma 5.5.6 and obtain that

$$\begin{aligned} G^\star\big(x, t\bar\phi(x) z_{\mu,\xi}(x)\big) &= \int_0^{t\bar\phi(x) z_{\mu,\xi}(x)} \big[(u_\varepsilon(x)+b)^p - u_\varepsilon^p(x)\big]\,db \\ &= \int_0^{t\bar\phi(x) z_{\mu,\xi}(x)} \big[(a+b)^p - a^p\big]\,db \\ &\geqslant \int_0^{t\bar\phi(x) z_{\mu,\xi}(x)} \big[b^p + c_{p,\kappa} a^{p-1} b\big]\,db \\ &= \frac{\big(t\bar\phi(x) z_{\mu,\xi}(x)\big)^{p+1}}{p+1} + c_{p,\kappa} u_\varepsilon^{p-1}(x)\,\frac{\big(t\bar\phi(x) z_{\mu,\xi}(x)\big)^2}{2}. \end{aligned}$$

This and (5.5.20) complete the proof of (5.5.23) when $n > 4s$ (recall that $p+1 = 2_s^*$).

Now we prove (5.5.23) when $n \in (2s, 4s]$. In this case, we observe that

$$p = \frac{n+2s}{n-2s} \geqslant 2.$$

So we choose $a \geqslant 0$, with $a \leqslant t\bar\phi(x) z_{\mu,\xi}(x)$, and $b := u_\varepsilon(x)$, and we can use Lemma 5.5.5 to obtain that

$$\begin{aligned} G^\star\big(x, t\bar\phi(x) z_{\mu,\xi}(x)\big) &= \int_0^{t\bar\phi(x) z_{\mu,\xi}(x)} \big[(u_\varepsilon(x)+a)^p - u_\varepsilon^p(x)\big]\,da \\ &= \int_0^{t\bar\phi(x) z_{\mu,\xi}(x)} \big[(a+b)^p - b^p\big]\,da \\ &\geqslant \int_0^{t\bar\phi(x) z_{\mu,\xi}(x)} \big[a^p + c_p a^{p-1} b\big]\,da \\ &= \frac{\big(t\bar\phi(x) z_{\mu,\xi}(x)\big)^{p+1}}{p+1} + c_p \frac{\big(t\bar\phi(x) z_{\mu,\xi}(x)\big)^{p}}{p}\, u_\varepsilon(x). \end{aligned}$$

This and (5.5.20) imply (5.5.23) when $n \in (2s, 4s]$. With this, we have completed the proof of (5.5.23).

Now we claim that, for any $t \in [0, T_1]$, any $\mu \in (0, \mu_\star)$ and any $x \in \mathbb{R}^n$,

$$G^\star\big(x, t\bar\phi(x) z_{\mu,\xi}(x)\big) \geqslant \frac{t^{2_s^*} \bar\phi^{2_s^*}(x)\, z_{\mu,\xi}^{2_s^*}(x)}{2_s^*}. \tag{5.5.25}$$

We remark that (5.5.23) is a stronger inequality than (5.5.25), but (5.5.23) only holds in Ω, while (5.5.25) holds in the whole of $\mathbb{R}^n$ (this is an advantadge in the case $n > 4s$, according to (5.5.21)). To prove (5.5.25), we use

Lemma 5.5.4, with $a := u_\varepsilon(x)$ and $b \geqslant 0$, to see that

$$\begin{aligned} G^\star\big(x, t\bar\phi(x) z_{\mu,\xi}(x)\big) &= \int_0^{t\bar\phi(x)z_{\mu,\xi}(x)} \big[(u_\varepsilon(x)+b)^p - u_\varepsilon^p(x)\big]\,db \\ &= \int_0^{t\bar\phi(x)z_{\mu,\xi}(x)} \big[(a+b)^p - a^p\big]\,db \\ &\geqslant \int_0^{t\bar\phi(x)z_{\mu,\xi}(x)} b^p\,db \\ &= \frac{\big(t\bar\phi(x)\,z_{\mu,\xi}(x)\big)^{p+1}}{p+1}, \end{aligned}$$

and this establishes (5.5.25).

By combining (5.5.23) and (5.5.25), we obtain that

(5.5.26)

$$\begin{aligned} &\int_{\mathbb{R}^n} G^\star\big(x, t\bar\phi(x) z_{\mu,\xi}(x)\big)\,dx \\ &\quad= \int_{\mathbb{R}^n\setminus\Omega} G^\star\big(x, t\bar\phi(x) z_{\mu,\xi}(x)\big)\,dx + \int_\Omega G^\star\big(x, t\bar\phi(x) z_{\mu,\xi}(x)\big)\,dx \\ &\quad\geqslant \int_{\mathbb{R}^n\setminus\Omega} \frac{t^{2_s^*}\bar\phi^{2_s^*}(x)\,z_{\mu,\xi}^{2_s^*}(x)}{2_s^*}\,dx \\ &\qquad+ \int_\Omega \frac{t^{2_s^*}\bar\phi^{2_s^*}(x)\,z_{\mu,\xi}^{2_s^*}(x)}{2_s^*} + \frac{c\,u_\varepsilon^{2_s^*-m}(x)\,t^m\bar\phi^m(x)\,z_{\mu,\xi}^m(x)}{m}\,dx \\ &\quad\geqslant \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^n} \bar\phi^{2_s^*}(x)\,z_{\mu,\xi}^{2_s^*}(x)\,dx + \frac{c\,t^m}{m}\int_\Omega u_\varepsilon^{2_s^*-m}(x)\,\bar\phi^m(x)\,z_{\mu,\xi}^m(x)\,dx. \end{aligned}$$

Now we claim that

$$\int_\Omega u_\varepsilon^{2_s^*-m}(x)\,\bar\phi^m(x)\,z_{\mu,\xi}^m(x)\,dx \geqslant c'\mu^\beta, \tag{5.5.27}$$

for some $c' > 0$, where

$$\beta := \begin{cases} \dfrac{n}{2} & \text{if } n > 4s, \\[2ex] \dfrac{n-2s}{2} & \text{if } n \in (2s, 4s], \end{cases} \tag{5.5.28}$$

To prove this, when $n > 4s$ we remark that, for small μ, we have $B_{2\sqrt{\mu}}(\xi) \subseteq B_{\mu_0/2}(\xi)$, and $\bar\phi = 1$ in this set, due to (5.5.4). So, we use (5.5.20) and (5.5.21) and we have that

$$\begin{aligned}\int_\Omega u_\varepsilon^{2^*_s-m}(x)\,\bar\phi^m(x)\,z^m_{\mu,\xi}(x)\,dx &= \int_{B_{2\sqrt{\mu}}(\xi)\setminus B_{\sqrt{\mu}}(\xi)} u_\varepsilon^{2^*_s-2}(x)\,z^2_{\mu,\xi}(x)\,dx\\ &\geqslant \inf_{B_2(\xi)} u_\varepsilon^{2^*_s-2}\int_{B_{2\sqrt{\mu}}(\xi)\setminus B_{\sqrt{\mu}}(\xi)} z^2_{\mu,\xi}(x)\,dx\\ &= \inf_{B_2(\xi)} u_\varepsilon^{2^*_s-2}\,\mu^{-(n-2s)}\int_{B_{2\sqrt{\mu}}(\xi)\setminus B_{\sqrt{\mu}}(\xi)} z^2\left(\frac{x-\xi}{\mu}\right)dx\\ &= \inf_{B_2(\xi)} u_\varepsilon^{2^*_s-2}\,\mu^{2s}\int_{B_{\frac{2}{\sqrt{\mu}}}\setminus B_{\frac{1}{\sqrt{\mu}}}} z^2(y)\,dy.\end{aligned}$$

Thus, recalling (1.1.6) and taking μ suitably small, we have that

$$\begin{aligned}\int_\Omega u_\varepsilon^{2^*_s-m}(x)\,\bar\phi^m(x)\,z^m_{\mu,\xi}(x)\,dx &\geqslant c_1\mu^{2s}\int_{1/\sqrt{\mu}}^{2/\sqrt{\mu}}\frac{\rho^{n-1}d\rho}{(1+\rho^2)^{n-2s}}\\ &\geqslant c_1\mu^{2s}\int_{1/\sqrt{\mu}}^{2/\sqrt{\mu}}\frac{\rho^{n-1}d\rho}{(2\rho^2)^{n-2s}} = c_2\mu^{\frac{n}{2}},\end{aligned}$$

for some c_1, $c_2>0$. This proves (5.5.27) when $n>4s$.

Now we prove (5.5.27) when $n\in(2s,4s]$. For this, we exploit (5.5.20) and (5.5.21) and we observe that

$$\begin{aligned}\int_\Omega u_\varepsilon^{2^*_s-m}(x)\,\bar\phi^m(x)\,z^m_{\mu,\xi}(x)\,dx &= \int_{\mathbb{R}^n} u_\varepsilon(x)\,\bar\phi^{2^*_s-1}(x)\,z^{2^*_s-1}_{\mu,\xi}(x)\,dx\\ &\geqslant \mu^{-\frac{n+2s}{2}}\int_{B_{2\sqrt{\mu}}(\xi)} u_\varepsilon(x)\,z^p\left(\frac{x-\xi}{\mu}\right)dx\\ &\geqslant \mu^{-\frac{n+2s}{2}}\inf_{B_1(\xi)} u_\varepsilon\int_{B_{2\sqrt{\mu}}(\xi)} z^p\left(\frac{x-\xi}{\mu}\right)dx\\ &= \mu^{\frac{n-2s}{2}}\inf_{B_1(\xi)} u_\varepsilon\int_{B_1} z^p(y)\,dy\\ &\geqslant c'\mu^{\frac{n-2s}{2}},\end{aligned}$$

for some $c'>0$, which establishes (5.5.27) when $n\in(2s,4s]$. The proof of (5.5.27) is thus complete.

Now, by inserting (5.5.27) into (5.5.26), we obtain that

$$\int_{\mathbb{R}^n} G^\star\big(x,t\bar\phi(x)z_{\mu,\xi}(x)\big)\,dx \geqslant \frac{t^{2^*_s}}{2^*_s}\int_{\mathbb{R}^n}\bar\phi^{2^*_s}(x)\,z^{2^*_s}_{\mu,\xi}(x)\,dx + \frac{c\,t^m\,\mu^\beta}{m}, \tag{5.5.29}$$

for some $c>0$, up to renaming constants.

By Lemma 5.5.1 and (5.5.29), we conclude that

$$\int_{\mathbb{R}^n} G^\star\big(x,t\bar\phi(x)z_{\mu,\xi}(x)\big)\,dx \geqslant \frac{t^{2^*_s}}{2^*_s}\int_{\mathbb{R}^n} z^{2^*_s}_{\mu,\xi}(x)\,dx + \frac{c\,\mu^\beta\,t^m}{m} - \frac{C\mu^n\,t^{2^*_s}}{2^*_s}.$$

This and (5.5.7) give that

$$\int_{\mathbb{R}^n} G^\star(x, t\bar\phi z_{\mu,\xi})\,dx \geqslant \frac{t^{2_s^*}\,S^{\frac{n}{2s}}}{2_s^*} + \frac{c\,\mu^\beta\,t^m}{m} - \frac{C\mu^n\,t^{2_s^*}}{2_s^*},$$

for some c, $C>0$. As a consequence, recalling (5.5.8), we obtain that

$$\begin{aligned}\mathcal{J}^\star_\varepsilon(t\bar Z_{\mu,\xi}) &\leqslant \frac{t^2\,[\bar Z_{\mu,\xi}]_a^2}{2} - \frac{t^{2_s^*}\,S^{\frac{n}{2s}}}{2_s^*} - \frac{c\,\mu^\beta\,t^m}{m} + \frac{C\mu^n\,t^{2_s^*}}{2_s^*}\\ &\leqslant \frac{t^2\,S^{\frac{n}{2s}}}{2} - \frac{t^{2_s^*}\,S^{\frac{n}{2s}}}{2_s^*} - \frac{c\,\mu^\beta\,t^m}{m} + \frac{C\mu^n\,t^{2_s^*}}{2_s^*} + \frac{C\mu^{n-2s}\,t^2}{2},\end{aligned}$$

and so, up to renaming constants,

$$\mathcal{J}^\star_\varepsilon(t\bar Z_{\mu,\xi}) \leqslant S^{\frac{n}{2s}}\,\Psi(t), \tag{5.5.30}$$

with

$$\Psi(t) := \frac{t^2}{2} - \frac{t^{2_s^*}}{2_s^*} - \frac{c\,\mu^\beta\,t^m}{m} + \frac{C\mu^n\,t^{2_s^*}}{2_s^*} + \frac{C\mu^{n-2s}\,t^2}{2},$$

for some C, $c>0$.

Now we claim that

$$\sup_{t\geqslant 0}\Psi(t) < \frac{s}{n}, \tag{5.5.31}$$

provided that $\mu>0$ is suitably small. To check this, we notice that $\Psi(0)=0$ and

$$\lim_{t\to+\infty}\Psi(t) = -\infty,$$

since $2_s^* > \max\{2,m\}$ (recall (5.5.20)). As a consequence, Ψ attains its maximum at some point $T\in[0,+\infty)$. If $T=0$, then $\Psi(T)=0$ and (5.5.31) is obvious, so we can assume that $T\in(0,+\infty)$. Accordingly, we have that $\Psi'(T)=0$. Therefore

$$0 = \frac{\Psi'(T)}{T} = 1 - T^{2_s^*-2} - c\,\mu^\beta\,T^{m-2} + C\mu^n\,T^{2_s^*-2} + C\mu^{n-2s}.$$

So we set

$$\Phi_\mu(t) := 1 - t^{2_s^*-2} - c\,\mu^\beta\,t^{m-2} + C\mu^n\,t^{2_s^*-2} + C\mu^{n-2s}$$

and we have that $T=T(\mu)$ is a solution of $\Phi_\mu(T)=0$. We remark that

$$\Phi'_\mu(t) = -(2_s^*-2)(1-C\mu^n)t^{2_s^*-3} - c\,\mu^\beta\,(m-2)t^{m-3} < 0,$$

since $m-2\geqslant 0$ and $(2_s^*-2)(1-C\mu^n)\geqslant 0$ for small μ (recall (5.5.22)). This says that Φ_μ is strictly decreasing, hence $T=T(\mu)$ is the unique solution of $\Phi_\mu(T(\mu))=0$. It is now convenient to write $\tau(\mu):=T(\mu^{\frac{1}{\beta}})$ and $\eta:=\mu^\beta$, so that our equation becomes

$$\begin{aligned}0 &= \Phi_\mu(T(\mu)) = \Phi_\mu(\tau(\mu^\beta)) = \Phi_\mu(\tau(\eta))\\ &= 1-(1-C\eta^{\frac{n}{\beta}})(\tau(\eta))^{2_s^*-2} - c\,\eta\,(\tau(\eta))^{m-2} + C\eta^{\frac{n-2s}{\beta}}.\end{aligned}$$

Accordingly, if we differentiate in η, we have that

(5.5.32)
$$\begin{aligned} 0 &= \frac{\partial}{\partial\eta}\Big(1-(1-C\eta^{\frac{n}{\beta}})(\tau(\eta))^{2_s^*-2} - c\,\eta\,(\tau(\eta))^{m-2} + C\eta^{\frac{n-2s}{\beta}}\Big) \\ &= -(2_s^*-2)(1-C\eta^{\frac{n}{\beta}})(\tau(\eta))^{2_s^*-3}\tau'(\eta) + C\,\frac{n}{\beta}\eta^{\frac{n}{\beta}-1}(\tau(\eta))^{2_s^*-2} \\ &\qquad - c\,(\tau(\eta))^{m-2} - c(m-2)\,\eta\,(\tau(\eta))^{m-3}\tau'(\eta) + \frac{C(n-2s)}{\beta}\eta^{\frac{n-2s}{\beta}-1}. \end{aligned}$$

Now we claim that

$$\frac{n-2s}{\beta} - 1 > 0. \tag{5.5.33}$$

Indeed, using (5.5.28), we see that

$$\begin{aligned} \frac{n-2s}{\beta} - 1 &= \begin{cases} \dfrac{2(n-2s)}{n} - 1 & \text{if } n > 4s, \\ 2-1 & \text{if } n \in (2s, 4s], \end{cases} \\ &= \begin{cases} \dfrac{n-4s}{n} & \text{if } n > 4s, \\ 1 & \text{if } n \in (2s, 4s], \end{cases} \end{aligned}$$

which proves (5.5.33).

Now we observe that when $\mu = 0$, we have that $T = 1$ is a solution of $\Phi_0(t) = 0$, i.e. $T(0) = 1$ and so $\tau(0) = 1$. Hence, we evaluate (5.5.32) at $\eta = 0$ and we conclude that

$$0 = -(2_s^*-2)\tau'(0) - c.$$

We remark that (5.5.33) was used here. Then, we obtain

$$\tau'(0) = -\frac{c}{2_s^*-2},$$

which gives that

$$\tau(\eta) = 1 - \frac{c\eta}{2_s^*-2} + o(\eta)$$

and so

$$T(\mu) = \tau(\mu^\beta) = 1 - \frac{c\mu^\beta}{2_s^*-2} + o(\mu^\beta) = 1 - c_o\mu^\beta + o(\mu^\beta),$$

for some $c_o > 0$. Therefore

$$
\begin{aligned}
\sup_{t\geqslant 0}\Psi(t) &= \Psi(T(\mu))\\
&= (1+C\mu^{n-2s})\frac{(T(\mu))^2}{2} - (1-C\mu^n)\frac{(T(\mu))^{2^*_s}}{2^*_s} - \frac{c\,\mu^\beta\,(T(\mu))^m}{m}\\
&= (1+C\mu^{n-2s})\frac{(1-c_o\mu^\beta+o(\mu^\beta))^2}{2} - (1-C\mu^n)\frac{(1-c_o\mu^\beta+o(\mu^\beta))^{2^*_s}}{2^*_s}\\
&\qquad - \frac{c\,\mu^\beta\,(1-c_o\mu^\beta+o(\mu^\beta))^m}{m}\\
&= (1+C\mu^{n-2s})\frac{1-2c_o\mu^\beta}{2} - (1-C\mu^n)\frac{1-2^*_s c_o\mu^\beta}{2^*_s} - \frac{c\,\mu^\beta}{m} + o(\mu^\beta)\\
&= \frac{1-2c_o\mu^\beta}{2} - \frac{1-2^*_s c_o\mu^\beta}{2^*_s} - \frac{c\,\mu^\beta}{m} + o(\mu^\beta)\\
&= \frac{1}{2} - \frac{1}{2^*_s} - \frac{c\,\mu^\beta}{m} + o(\mu^\beta)\\
&< \frac{1}{2} - \frac{1}{2^*_s}\\
&= \frac{s}{n},
\end{aligned}
$$

and this proves (5.5.31).

Using (5.5.30) and (5.5.31), we obtain that

$$\sup_{t\in[0,T_1]} \mathcal{I}^\star_\varepsilon(t\bar{Z}_{\mu,\xi}) \leqslant S^{\frac{n}{2s}} \sup_{t\geqslant 0}\Psi(t) < S^{\frac{n}{2s}}\cdot\frac{s}{n},$$

which proves (5.5.19) and so it completes the proof of Lemma 5.5.7. □

By combining (5.5.11) with Lemmata 5.5.3 and 5.5.7, we obtain:

Corollary 5.5.8. *There exists $\mu_\star > 0$ such that if $\mu\in(0,\mu_\star)$ we have that*

$$
\begin{aligned}
&\lim_{t\to+\infty}\mathcal{I}_\varepsilon(t\bar{Z}_{\mu,\xi}) = -\infty\\
\textit{and}\qquad &\sup_{t\geqslant 0}\mathcal{I}_\varepsilon(t\bar{Z}_{\mu,\xi}) < \frac{s}{n}S^{\frac{n}{2s}}.
\end{aligned}
$$

The result in Corollary 5.5.8 says that the path induced by the function $\bar{Z}_{\mu,\xi}$ is a mountain pass path which lies below the critical threshold given in Proposition 5.4.1 (so, from now on, the value of $\mu\in(0,\mu_\star)$ will be fixed so that Corollary 5.5.8 holds true).

5.6. Proof of Theorem 1.2.4

In this section we establish Theorem 1.2.4. For this, we argue by contradiction and we suppose that $U = 0$ is the only critical point of $\mathcal{I}_\varepsilon$. As a consequence, the functional $\mathcal{J}_\varepsilon$ verifies the Palais-Smale condition below the critical level given in (5.4.1), according to Proposition 5.4.1.

In addition, $\mathcal{J}_\varepsilon$ fulfills the mountain pass geometry, and the minimax level c_ε stays strictly below the level $\frac{s}{n}S^{\frac{n}{2s}}$, as shown in Proposition 5.1.1 and Corollary 5.5.8.

Hence, for small ε, we have that $c_\varepsilon + C\varepsilon^{\frac{1}{2\gamma}}$ remains strictly below $\frac{s}{n}S^{\frac{n}{2s}}$, thus satisfying (5.4.1).

Then, exploiting Proposition 5.4.1 and the Mountain Pass Theorem in [**7, 27**], we obtain the existence of another critical point, in contradiction with the assumption. This ends the proof of Theorem 1.2.4.

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