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# Nonlocal minimal surfaces: Interior regularity, quantitative estimates and boundary stickiness

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ABSTRACT. We consider surfaces which minimize a nonlocal perimeter functional and we discuss their interior regularity and rigidity properties, in a quantitative and qualitative way, and their (perhaps rather surprising) boundary behavior. We present at least a sketch of the proofs of these results, in a way that aims to be as elementary and self contained as possible, referring to the papers [CRS10, SV13, CV13, BFV14, FV, DSV15, CSV16] for full details.

*...taurino quantum possent circumdare tergo...*

## 1. INTRODUCTION

The study of surfaces which minimize the perimeter is a classical topic in analysis and geometry and probably one of the oldest problems in the mathematical literature: according to the first book of Virgil's Aeneid, Dido, the legendary queen of Carthage, needed to study these questions in order to found her reign in 814 B.C. (in spite of the great mathematical talent of Dido and of her vivid geometric intuition, Aeneas broke his betrothal with her after a short time to seal the Mediterranean towards the coasts of Italy, but this is another story).

The first problem in the study of these surfaces of minimal perimeter (minimal surfaces, for short) lies in proving that minimizers do exist. Indeed "nice" sets, for which one can compute the perimeter using an intuitive notion known from elementary school, turn out to be a "non compact" family (roughly speaking, for instance, an "ugly" set can be approximated by a sequence of "nice" sets, thus the limit point of the sequence may end up outside the family). To overcome this difficulty, a classical tool of the calculus of variation is to look for minimizers in a wider family of candidates: this larger family will then possess the desired compactness properties to ensure the existence of a minimum, and then the regularity of the minimal candidate can be (hopefully) proved a posteriori.

To this end, one needs to set up an appropriate notion of perimeter for the sets in the enlarged family of candidates, since no intuitive notion of perimeter is available, in principle, in this generality. The classical approach of Caccioppoli (see e.g. [Cac27]) to this question lies in the observation that if $\Omega$ and $E$ are[1] smooth sets and $\nu$ is the external normal of $E$, then, for any vector field $T \in C^1_0(\Omega, \mathbb{R}^n)$ with $|T(x)| \leqslant 1$ for any $x \in \Omega$, we have that

$$T \cdot \nu \leqslant |T|\,|\nu| \leqslant 1.$$

Consequently, the perimeter of $E$ in $\Omega$, i.e. the measure of the boundary of $E$ inside $\Omega$ (that is, the $(n-1)$-dimensional Hausdorff measure of $\partial E$ in $\Omega$), satisfies the inequality

$$\operatorname{Per}(E,\Omega) = \mathcal{H}^{n-1}\big((\partial E)\cap\Omega\big) \geqslant \int_{\partial E} T\cdot\nu \, d\mathcal{H}^{n-1} = \int_E \operatorname{div} T(x)\,dx, \tag{1.1}$$

for every vector field $T \in C^1_0(\Omega, \mathbb{R}^n)$ with $\|T\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \leqslant 1$, where the Divergence Theorem has been used in the last identity.

Viceversa, if $E$ is a smooth set, its normal vector can be extended near $\partial E$, and then to the whole of $\mathbb{R}^n$, to a vector field $\nu_* \in C^1_0(\Omega, \mathbb{R}^n)$, with $|\nu_*(x)| \leqslant 1$ for any $x \in \mathbb{R}^n$. Then, if $\eta \in C^\infty_0(\Omega, [0,1])$,

[1] From now on, we reserve the name of $\Omega$ to an open set, possibly with smooth boundary, which can be seen as the "ambient space" for our problem.

with $\eta = 1$ in an $\varepsilon$-neighborhood of $\Omega$, one can take $T := \eta\nu_*$ and find that $T \in C^1_0(\Omega, \mathbb{R}^n)$, $|T(x)| \leqslant 1$ for any $x \in \mathbb{R}^n$ and

$$\begin{aligned}\int_E \operatorname{div} T(x)\, dx &= \int_{\partial E} T \cdot \nu \, d\mathcal{H}^{n-1} \\ &= \int_{\partial E} \eta\nu_* \cdot \nu \, d\mathcal{H}^{n-1} = \int_{\partial E} \eta \, d\mathcal{H}^{n-1} \\ &\geqslant \mathcal{H}^{n-1}\big((\partial E) \cap \Omega\big) - O(\varepsilon) = \operatorname{Per}(E, \Omega) - O(\varepsilon).\end{aligned}$$

By taking $\varepsilon$ as small as we wish and recalling (1.1), we obtain that

$$\operatorname{Per}(E, \Omega) = \sup_{\substack{T \in C^1_0(\Omega,\mathbb{R}^n) \\ \|T\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)} \leqslant 1}} \int_E \operatorname{div} T(x)\, dx. \tag{1.2}$$

While (1.2) was obtained for smooth sets $E$, the classical approach for minimal surfaces is in fact to take (1.2) as *definition* of perimeter of a (not necessarily smooth) set $E$ in $\Omega$. The class of sets obtained in this way indeed has the necessary compactness properties (and the associated functional has the desired lower semicontinuity properties) to give the existence of minimizers: that is, one finds (at least) one set $E \subseteq \mathbb{R}^n$ satisfying

$$\operatorname{Per}(E, \Omega) \leqslant \operatorname{Per}(F, \Omega) \tag{1.3}$$

for any $F \subseteq \mathbb{R}^n$ such that $F$ coincides with $E$ in a neighborhood of $\Omega^c$.

The boundary of this minimal set $E$ satisfies, a posteriori, a bunch of additional regularity properties – just to recall the principal ones:

(1.4) If $n \leqslant 7$ then $(\partial E) \cap \Omega$ is smooth;

(1.5) If $n \geqslant 8$ then $\big((\partial E) \cap \Omega\big) \setminus \Sigma$ is smooth,
being $\Sigma$ a closed set of Hausdorff dimension at most $n - 8$;

(1.6) The statement in (1.5) is sharp, since there exist
examples in which the singular set $\Sigma$
has Hausdorff dimension $n - 8$.

We refer to [Giu77] for complete statements and proofs (in particular, the claim in (1.4) here corresponds to Theorem 10.11 in [Giu77], the claim in (1.5) here to Theorem 11.8 there, and the claim in (1.6) here to Theorem 16.4 there).

A natural problem that is closely related to these regularity results is the complete description of classical minimal surfaces in the whole of the space which are also graphs in some direction (the so-called minimal graphs). These questions, that go under the name of Bernstein's problem, have, in the classical case, the following positive answer:

(1.7) If $n \leqslant 8$ and $E$ is a minimal graph, then $E$ is a halfspace;

(1.8) The statement in (1.7) is sharp, since there exist
examples of minimal graphs in dimension $9$ and higher
that are not halfspaces.

We refer to Theorems 17.8 and 17.10 in [Giu77] for further details on the claims in (1.7) and (1.8), respectively.

It is also worth recalling that

(1.9) surfaces minimizing perimeters have zero mean curvature, see e.g. Chapter 10 in [Giu77].

Recently, and especially in light of the seminal paper [CRS10], some attention has been devoted to a variation of the classical notion of perimeters which takes into account also long-range interactions between sets, as well as the corresponding minimization problem. This type of *nonlocal minimal surfaces* arises naturally, for instance, in the study of fractals [Vis91], cellular automata [Imb09, CS10] and phase transitions [SV12] (see also [BV16] for a detailed introduction to the topic).

A simple idea for defining a notion of nonlocal perimeter may be described as follows. First of all, such nonlocal perimeter should compute the interaction $I$ of all the points of $E$ against all the points of the complement of $E$, which we denote by $E^c$.

On the other hand, if we want to localize these contributions inside the domain $\Omega$, it is convenient to split $E$ into $E \cap \Omega$ and $E \setminus \Omega$, as well as the set $E^c$ into $E^c \cap \Omega$ and $E^c \setminus \Omega$, and so consider the four possibilities of interaction between $E$ and $E^c$ given by

$$\begin{aligned} &I(E \cap \Omega, E^c \cap \Omega), \qquad I(E \cap \Omega, E^c \setminus \Omega), \\ &I(E \setminus \Omega, E^c \cap \Omega), \qquad \text{and} \qquad I(E \setminus \Omega, E^c \setminus \Omega). \end{aligned} \tag{1.10}$$

Among these interactions, we observe that the latter one only depends on the configuration of the set outside $\Omega$, and so

$$I(E \setminus \Omega, E^c \setminus \Omega) = I(F \setminus \Omega, F^c \setminus \Omega)$$

for any $F \subseteq \mathbb{R}^n$ such that $F \setminus \Omega = E \setminus \Omega$. Therefore, in a minimization process with fixed data outside $\Omega$, the term $I(E \setminus \Omega, E^c \setminus \Omega)$ does not change the minimizers. It is therefore natural to omit this term in the energy functional (and, as a matter of fact, omitting this term may turn out to be important from the mathematical point of view, since this term may provide an infinite contribution to the energy): for this reason, the nonlocal perimeter considered in [CRS10] is given by the sum of the first three terms in (1.10), namely one defines

$$\mathrm{Per}_s\,(E, \Omega) := I(E \cap \Omega, E^c \cap \Omega) + I(E \cap \Omega, E^c \setminus \Omega) + I(E \setminus \Omega, E^c \cap \Omega).$$

As for the interaction $I(\cdot, \cdot)$, of course some freedom is possible, and basically any interaction for which $\mathrm{Per}_s\,(E, \Omega)$ is finite, say, for smooth sets $E$ makes perfect sense. A natural choice performed in [CRS10] is to take the interaction as a weighted Lebesgue measure, where the weight is translation invariant, isotropic and homogeneous: more precisely, for any disjoint sets $S_1$ and $S_2$, one defines

$$I(S_1, S_2) := \iint_{S_1 \times S_2} \frac{dx\, dy}{|x-y|^{n+2s}}, \tag{1.11}$$

with $s \in \left(0, \frac{1}{2}\right)$. With this choice of the fractional parameter $s$, one sees that

$$\begin{aligned} [\chi_E]_{W^{\sigma,p}(\mathbb{R}^n)} &:= \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\chi_E(x) - \chi_E(y)|^p}{|x-y|^{n+p\sigma}}\, dx\, dy \\ &= 2 \iint_{E \times E^c} \frac{dx\, dy}{|x-y|^{n+p\sigma}} = 2I(E, E^c) = 2\,\mathrm{Per}_s\,(E, \mathbb{R}^n) \end{aligned}$$

as long as $p\sigma = 2s$, that is the fractional perimeter of a set coincides (up to normalization constants) to a fractional Sobolev norm of the corresponding characteristic function (see e.g. [DNPV12] for a simple introduction to fractional Sobolev spaces).

Moreover, for any fixed $y \in \mathbb{R}^n$,

$$\mathrm{div}_x \frac{x-y}{|x-y|^{n+2s}} = -\frac{2s}{|x-y|^{n+2s}}. \tag{1.12}$$

Also, for any fixed $x \in \mathbb{R}^n$,

$$\mathrm{div}_y \, \frac{\nu(x)}{|x-y|^{n+2s-2}} = (n+2s-2) \frac{\nu(x) \cdot (x-y)}{|x-y|^{n+2s}}.$$

Accordingly, by the Divergence Theorem[2]

$$\begin{aligned}
& \mathrm{Per}_s\,(E, \mathbb{R}^n) \\
=& \; - \frac{1}{2s} \int_{E^c} dy \left[ \int_E \mathrm{div}_x \, \frac{x-y}{|x-y|^{n+2s}} \, dx \right] \\
=& \; - \frac{1}{2s} \int_{E^c} dy \left[ \int_{\partial E} \frac{\nu(x) \cdot (x-y)}{|x-y|^{n+2s}} \, d\mathcal{H}^{n-1}(x) \right] \\
=& \; - \frac{1}{2s\,(n+2s-2)} \int_{\partial E} d\mathcal{H}^{n-1}(x) \, \left[ \int_{E^c} \mathrm{div}_y \, \frac{\nu(x)}{|x-y|^{n+2s-2}} \, dy \right] \\
=& \; \frac{1}{2s\,(n+2s-2)} \iint_{(\partial E) \times (\partial E)} \frac{\nu(x) \cdot \nu(y)}{|x-y|^{n+2s-2}} \, d\mathcal{H}^{n-1}(x) \, d\mathcal{H}^{n-1}(y).
\end{aligned} \tag{1.13}$$

That is,

$$\mathrm{Per}_s\,(E, \mathbb{R}^n) = \frac{1}{4s\,(n+2s-2)} \iint_{(\partial E) \times (\partial E)} \frac{2 - |\nu(x) - \nu(y)|^2}{|x-y|^{n+2s-2}} \, d\mathcal{H}^{n-1}(x) \, d\mathcal{H}^{n-1}(y),$$

which suggests that the fractional perimeter is a weighted measure of the variation of the normal vector around the boundary of a set. As a matter of fact, as $s \nearrow 1/2$, the $s$-perimeter recovers the classical perimeter from many point of views (a sketchy discussion about this will be given in Appendix A).

Also, in Appendix C, we briefly discuss the second variation of the $s$-perimeter on surfaces of vanishing nonlocal mean curvature and we show that graphs with vanishing nonlocal mean curvature cannot have horizontal normals.

Let us now recall (among the others) an elementary, but useful, application of this notion of fractional perimeter in the framework of digital image reconstruction. Suppose that we have a black and white digitalized image, say a bitmap, in which each pixel is either colored in black or in white. We call $E$ the “black set” and we are interested in measuring its perimeter (the reason for that may be, for instance, that noises or impurities could be distinguished by having “more perimeter” than the “real” picture, since they may present irregular or fractal boundaries). In doing that, we need to be able to compute such perimeter with a very good precision. Of course, numerical errors could affect the computation, since the digital process replaced the real picture by a pixel representation of it, but we would like that our computation becomes more and more reliable if the resolution of the image is sufficiently high, i.e. if the size of the pixels is sufficiently small.

Unfortunately, we see that, in general, an accurate computation of the perimeter is not possible, not even for simple sets, since the numerical error produced by the pixel may not become negligible, even when the pixels are small. To observe this phenomenon (see e.g. [CSV16]) we can consider a grid of square pixels of small side $\varepsilon$ and a black square $E$ of side $1$, with the black square rotated by $45$ degrees with respect to the orientation of the pixels. Now, the digitalization of the square will produce a numerical error, since, say, the pixels that intersect the square are taken as black, and so each side of the square is replaced by a “sawtooth” curve (see Figure 1).

[2]We will often use the Divergence Theorem here in a rather formal way, by neglecting the possible singularity of the kernel – for a rigorous formulation one has to check that the possible singular contributions average out, at least for smooth sets.

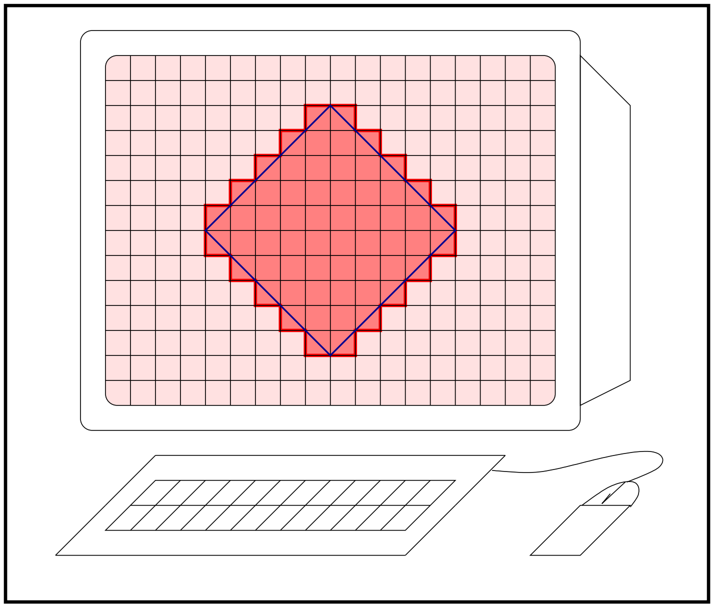

FIGURE 1. *Numerical error in computing the perimeter.*

Notice that the length of each of these sawtooth curves is $\sqrt{2}$ (independently on how small each teeth is, that is independently on the size of $\varepsilon$). As a consequence, the perimeter of the digitalized image is $4\sqrt{2}$, instead of $4$, which was the original perimeter of the square.

This shows the rather unpleasant fact that the perimeter may be poorly approximated numerically, even in case of high precision digitalization processes. It is a rather remarkable fact that fractional perimeters do not present the same inconvenience and indeed the numerical error in computing the fractional perimeter becomes small when the pixels are small enough. Indeed, the number of pixels which intersect the sides of the original square is $O(\varepsilon^{-1})$ (recall that the side of the square is $1$ and the side of each pixel is of size $\varepsilon$). Also, the $s$-perimeter of each pixel is $O(\varepsilon^{2-2s})$ (since this is the natural scale factor of the interaction in (1.11), with $n = 2$). Then, the numerical error in the fractional perimeter comes from the contributions of all these pixels[3] and it is therefore $O(\varepsilon^{-1}) \cdot O(\varepsilon^{2-2s}) = O(\varepsilon^{1-2s})$, which tends to zero for small $\varepsilon$, thus showing that the nonlocal perimeters are more efficient than classical ones in this type of digitalization process.

Thus, given its mathematical interest and its importance in concrete applications, it is desirable to reach a better understanding of the surfaces which minimize the $s$-perimeter (that one can call $s$-minimal surfaces). To start with, let us remark that an analogue of (1.9) holds true, in the sense that $s$-minimal surfaces have vanishing $s$-mean curvature in a sense that we now briefly describe. Given a set $E$ with smooth boundary and $p \in \partial E$, we define

$$H^s_E(p) := \int_{\mathbb{R}^n} \frac{\chi_{E^c}(x) - \chi_E(x)}{|x-p|^{n+2s}}\, dx. \tag{1.14}$$

[3]More precisely, when the computer changes the "real" square with the discretized one and produces a staircase border, the only interactions changed are the ones affecting the union of the triangles (that are "half pixels") that are added to the square in this procedure. In the "real" picture, these triangles interact with the square, while in the digitalized picture they interact with the exterior. To compute the error obtained one takes the signed superposition of these effects, therefore, to estimate the error in absolute value, one can just sum up these contributions, which in turn are bounded by the sum of the interactions of each triangle with its complement, see Figure 2.

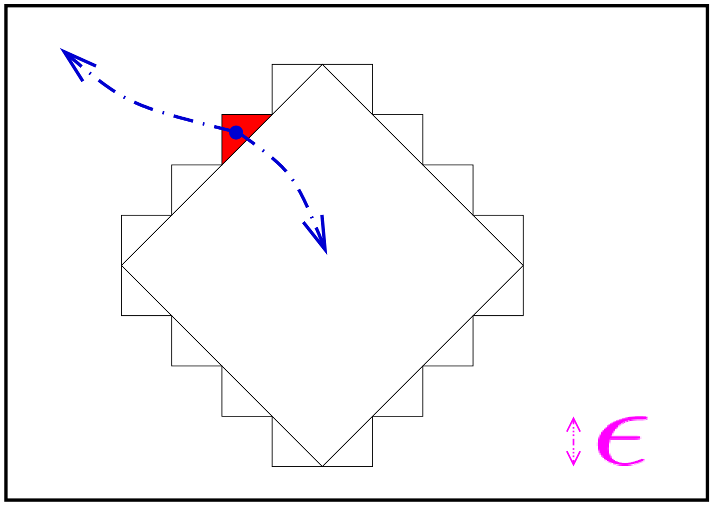

FIGURE 2. *Pixel interactions and numerical errors.*

The expression in (1.14) is intended in the principal value sense, namely the singularity is taken in an averaged limit, such as

$$H_E^s(p) = \lim_{\rho \searrow 0} \int_{\mathbb{R}^n \setminus B_\rho(p)} \frac{\chi_{E^c}(x) - \chi_E(x)}{|x-p|^{n+2s}}\, dx.$$

For simplicity, we omit the principal value from the notation. It is also useful to recall (1.12) and to remark that $H_E^s$ can be computed as a weighted boundary integral of the normal, namely

$$\begin{aligned} H_E^s(p) &= -\frac{1}{2s}\int_{\mathbb{R}^n} \big(\chi_{E^c}(x) - \chi_E(x)\big)\, \mathrm{div}\, \frac{x-p}{|x-p|^{n+2s}}\, dx \\ &= -\frac{1}{s}\int_{\partial E} \frac{\nu(x)\cdot(p-x)}{|p-x|^{n+2s}}\, d\mathcal{H}^{n-1}(x). \end{aligned} \tag{1.15}$$

This quantity $H_E^s$ is what we call the nonlocal mean curvature of $E$ at the point $p$, and the name is justified by the following observation:

**Lemma 1.1.** *If $E$ is a set with smooth boundary that minimizes the $s$-perimeter in $\Omega$, then $H_E^s(p) = 0$ for any $p \in (\partial E) \cap \Omega$.*

The proof of Lemma 1.1 will be given in Section 2. We refer to [CRS10] for a version of Lemma 1.1 that holds true (in the viscosity sense) without assuming that the set has smooth boundary. See also [AV14] for further comments on this notion of nonlocal mean curvature.

Let us now briefly discuss the fractional analogue of the regularity results in (1.4) and (1.5). At the moment, a complete regularity theory in the fractional case is still not available. At best, one can obtain regularity results either in low dimension or when $s$ is sufficiently close to $\frac{1}{2}$ (see [SV13, CV13] and also [BFV14] for higher regularity results): namely, the analogue of (1.4) is:

**Theorem 1.2** (Interior regularity results for $s$-minimal surfaces - I)**.** *Let $E \subset \mathbb{R}^n$ be a minimizer for the $s$-perimeter in $\Omega$. Assume that*

- *either $n = 2$,*
- *or $n \leqslant 7$ and $\frac{1}{2} - s \leqslant \varepsilon_*$, for some $\varepsilon_* > 0$ sufficiently small.*

*Then, $(\partial E) \cap \Omega$ is smooth.*

Similarly, a fractional analogue of (1.5) is known, by now, only when $s$ is sufficiently close to $\frac{1}{2}$:

**Theorem 1.3** (Interior regularity results for $s$-minimal surfaces - II)**.** *Let $E \subset \mathbb{R}^n$ be a minimizer for the $s$-perimeter in $\Omega$. Assume that $n \geqslant 8$ and $\frac{1}{2} - s \leqslant \varepsilon_n$, for some $\varepsilon_n > 0$ sufficiently small. Then, $\big((\partial E) \cap \Omega\big) \setminus \Sigma$ is smooth, being $\Sigma$ a closed set of Hausdorff dimension at most $n-8$.*

Differently from the statement in (1.6), it is not known if Theorems 1.2 and 1.3 are sharp, and in fact there are no known examples of $s$-minimal surfaces with singular sets: and, as a matter of fact, in dimension $n \leqslant 6$, these pathological examples – if they exist – cannot be built by symmetric cones (which means that they either do not exist or are pretty hard to find!), see [DdW].

In [CSV16], several quantitative regularity estimates for local minimizers are given (as a matter of fact, these estimates are valid in a much more general setting, but, for simplicity, we focus here on the most basic statements and proofs). For instance, minimizers of the $s$-perimeter have locally finite perimeter (that is, classical perimeter, not only fractional perimeter), as stated in the next result:

**Theorem 1.4.** *Let $E \subset \mathbb{R}^n$ be a minimizer for the $s$-perimeter in $B_R$. Then*

$$\mathrm{Per}\,(E, B_{1/2}) \leqslant C R^{n-1},$$

*for a suitable constant $C > 0$.*

We stress that Theorem 1.4 presents several novelties with respect to the existing literature. First of all, it provides a scaling invariant regularity estimate that goes beyond the natural scaling of the $s$-perimeter, that is valid in any dimension and without any topological restriction on the $s$-minimal surface (analogous results for the classical perimeter are not known in this generality). Also, in spite of the fact that, for the sake of simplicity, we state and prove Theorem 1.4 only in the case of minimizers of the $s$-perimeter, more general versions of this result hold true for stable solutions and for more general interaction kernels (even for kernels without any regularizing effect) and this type of results also leads to new compactness and existence theorems, see [CSV16] for full details on this topic.

As a matter of fact, we stress that the analogue of Theorem 1.4 for stable surfaces which are critical points of the classical perimeter is only known, up to now, for two-dimensional surfaces that are simply connected and immersed in $\mathbb{R}^3$ (hence, this is a case in which the nonlocal theory can go beyond the local one).

Now, we briefly discuss the fractional analogue of the Bernstein's problem. Let us start by pointing out that, by combining (1.4) and (1.7), we have an "abstract" version of the Bernstein's problem, which states that *if $E$ is a minimal graph in $\mathbb{R}^{n+1}$ and the minimal surfaces in $\mathbb{R}^n$ are smooth, then $E$ is a halfspace.*

Of course, for the way we have written (1.4) and (1.7), this abstract statement seems only to say that $8 = 7 + 1$: nevertheless this abstract version of the Bernstein's problem is very useful in the classical case, since it admits a nice fractional counterpart, which is:

**Theorem 1.5** (Bernstein result for $s$-minimal surfaces - I)**.** *If $E$ is an $s$-minimal graph in $\mathbb{R}^{n+1}$ and the $s$-minimal surfaces in $\mathbb{R}^n$ are smooth, then $E$ is a halfspace.*

This result was proved in [FV]. By combining it with Theorem 1.2 (using the notation $N := n+1$), we obtain:

**Theorem 1.6** (Bernstein result for $s$-minimal surfaces - II)**.** *Let $E \subset \mathbb{R}^N$ be an $s$-minimal graph. Assume that*

- *either $N = 3$,*
- *or $N \leqslant 8$ and $\frac{1}{2} - s \leqslant \varepsilon_*$, for some $\varepsilon_* > 0$ sufficiently small.*

*Then, $E$ is a halfspace.*

This is, at the moment, the fractional counterpart of (1.7) (we stress, however, that any improvement in the fractional regularity theory would give for free an improvement in the fractional Bernstein's problem, via Theorem 1.5).

We remark again that, differently from the claim in (1.8), it is not known if the statement in Theorem 1.6 is sharp, since there are no known examples of $s$-minimal graphs other than the hyperplanes.

It is worth recalling that, by a blow-down procedure, one can deduce from Theorem 1.2 that global $s$-minimal surfaces are hyperplanes, as stated in the following result:

**Theorem 1.7** (Flatness of $s$-minimal surfaces)**.** *Let $E \subset \mathbb{R}^n$ be a minimizer for the $s$-perimeter in any domain of $\mathbb{R}^n$. Assume that*

- *either $n = 2$,*
- *or $n \leqslant 7$ and $\frac{1}{2} - s \leqslant \varepsilon_*$, for some $\varepsilon_* > 0$ sufficiently small.*

*Then, $E$ is a halfspace.*

Of course, a very interesting spin-off of the regularity theory in Theorem 1.7 lies in finding quantitative flatness estimates: namely, if we know that a set $E$ is an $s$-minimizer in a large domain, can we say that it is sufficiently close to be a halfspace, and if so, how close, and in which sense?

This question has been recently addressed in [CSV16]. As a matter of fact, the results in [CSV16] are richer than the ones we present here, and they are valid for a very general class of interaction kernels and of perimeters of nonlocal type – nevertheless we think it is interesting to give a flavor of them even in their simpler form, to underline their connection with the regularity theory that we discussed till now.

In this setting, we present here the following result when $n = 2$ (see indeed [CSV16] for more general statements):

**Theorem 1.8.** *Let $R \geqslant 2$. Let $E \subset \mathbb{R}^2$ be a minimizer for the $s$-perimeter in $B_R$. Then there exists a halfplane $h$ such that*

$$\big|(E\Delta h) \cap B_1\big| \leqslant \frac{C}{R^s}, \tag{1.16}$$

*where $\Delta$ is here the symmetric difference of the two sets (i.e. $E\Delta h := (E \setminus h) \cup (h \setminus E)$) and $C > 0$ is a constant.*

We stress that Theorem 1.8 may be seen as a quantitative version of Theorem 1.7 when $n = 2$: indeed if $E \subset \mathbb{R}^n$ is a minimizer for the $s$-perimeter in any domain of $\mathbb{R}^n$ we can send $R \nearrow +\infty$ in (1.16) and obtain that $E$ is a halfplane.

We observe that, till now, we have presented and discussed a series of results which are somehow in accordance, as much as possible, with the classical case. Now we present something with striking difference from the classical case. The minimizers of the classical perimeter in a convex domain reach continuously the boundary data (see e.g. Theorem 15.9 in [Giu77]). Quite surprisingly, the minimizers of the fractional perimeter have the tendency to stick at the boundary. This phenomenon has been discovered in [DSV15], where several explicit stickiness examples have been given.

Roughly speaking, the stickiness phenomenon may be described as follows. We know from Lemma 1.1 that nonlocal minimal surfaces in a domain $\Omega$ need to adjust their shape in order to make the nonlocal minimal curvature vanish inside $\Omega$. This is a rather strong condition, since the nonlocal minimal curvature "sees" the set all over the space. As a consequence, in many cases in which the boundary data are "not favorable" for this condition to hold, the nonlocal minimal surfaces may prefer to modify their shape by sticking at the boundary, where the condition is not prescribed, in order to compensate the values of the nonlocal mean curvature inside $\Omega$.

In many cases, for instance, the nonlocal minimal set may even prefer to "disappear", i.e. its contribution inside $\Omega$ becomes empty and its boundary sticks completely to the boundary of $\Omega$. In concrete cases, the fact that the nonlocal minimal set disappears may be induced by a suitable choice of the data outside $\Omega$ or by an appropriate choice of the fractional parameter. As a prototype example of these two phenomena, we recall here the following results given in [DSV15]:

**Theorem 1.9** (Stickiness for small data)**.** *For any $\delta > 0$, let*

$$K_\delta := \big(B_{1+\delta} \setminus B_1\big) \cap \{x_n < 0\}.$$

*Let $E_\delta$ be the $s$-minimal set among all the sets $E$ such that $E \setminus B_1 = K_\delta$.*

*Then, there exists $\delta_o > 0$, depending on $s$ and $n$, such that for any $\delta \in (0, \delta_o]$ we have that*

$$E_\delta = K_\delta.$$

**Theorem 1.10** (Stickiness for small $s$)**.** *As $s \to 0^+$, the $s$-minimal set in $B_1 \subset \mathbb{R}^2$ that agrees with a sector outside $B_1$ sticks to the sector.*

*More precisely: let $E_s$ be the $s$-minimizer among all the sets $E$ such that*

$$E \setminus B_1 = \Sigma := \{(x, y) \in \mathbb{R}^2 \setminus B_1 \text{ s.t. } x > 0 \text{ and } y > 0\}.$$

*Then, there exists $s_o > 0$ such that for any $s \in (0, s_o]$ we have that $E_s = \Sigma$.*

We stress the sharp difference between the local and the nonlocal cases exposed in Theorems 1.9 and 1.10: indeed, in the local framework, in both cases the minimal surface is a segment inside the ball $B_1$, while in the nonlocal case it coincides with a piece of the circumference $\partial B_1$.

The stickiness phenomenon of nonlocal minimal surfaces may also be caused by a sufficiently high oscillation of the data outside $\Omega$. This concept is exposed in the following result:

**Theorem 1.11** (Stickiness coming from large oscillations of the data)**.** *Let $M > 1$ and let $E_M \subset \mathbb{R}^2$ be $s$-minimal in $(-1, 1) \times \mathbb{R}$ with datum outside $(-1, 1) \times \mathbb{R}$ given by $J_M := J_M^- \cup J_M^+$, where*

$$J_M^- := (-\infty, -1] \times (-\infty, -M) \quad \text{and} \quad J_M^+ := [1, +\infty) \times (-\infty, M).$$

*Then, if $M$ is large enough, $E_M$ sticks at the boundary. Moreover, the stickiness region gets close to the origin, up to a power of $M$.*

*More precisely: there exist $M_o > 0$ and $C_o \geqslant C_o' > 0$, depending on $s$, such that if $M \geqslant M_o$ then*

$$\begin{aligned} &[-1, 1) \times [C_o M^{\frac{1+2s}{2+2s}},\, M] \subseteq E_M^c \\ \text{and} \quad &(-1, 1] \times [-M,\, -C_o M^{\frac{1+2s}{2+2s}}] \subseteq E_M. \end{aligned} \tag{1.17}$$

It is worth to remark that the stickiness phenomenon in Theorem 1.11 becomes "more and more visible" as the oscillation of the data increase, since, referring to (1.17), we have that

$$\lim_{M \to +\infty} \frac{M^{\frac{1+2s}{2+2s}}}{M} = 0,$$

hence the sticked portion of $E_M$ on $\partial\Omega$ becomes, proportionally to $M$, larger and larger when $M \to +\infty$.

Also, the exponent $\frac{1+2s}{2+2s}$ in (1.17) is optimal, see again [DSV15]. The stickiness phenomenon detected in Theorem 1.11 is described in Figure 3.

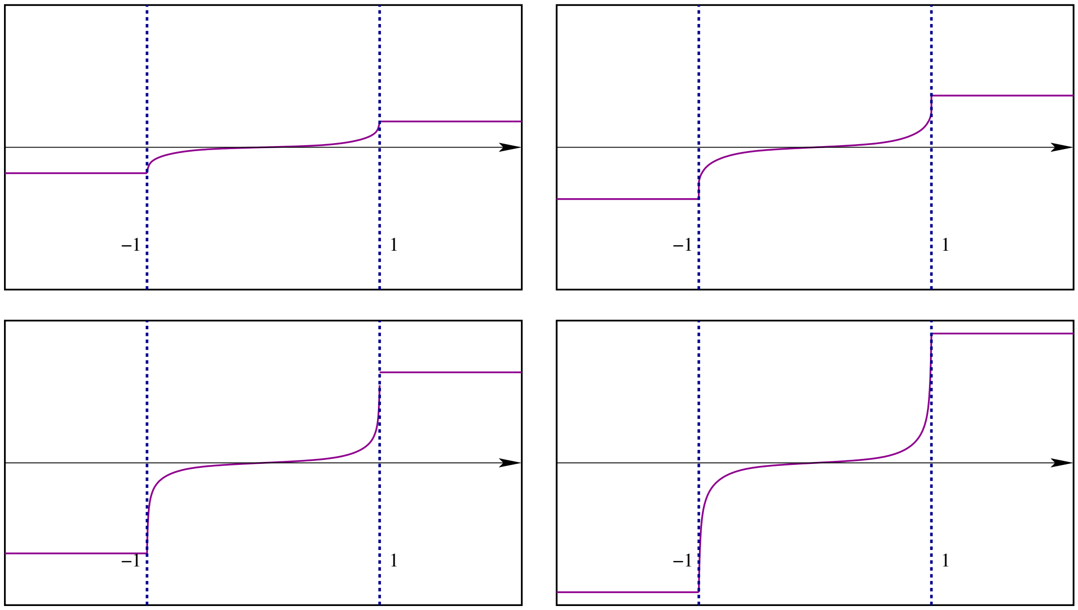


FIGURE 3. *Stickiness coming from large oscillations of the data with the oscillation progressively larger.*

We believe that the stickiness phenomenon is indeed rather common among nonlocal minimal surfaces, and indeed it may occur even under small modifications of boundary data for which the nonlocal minimal surfaces cut the boundary in a transversal way.

A typical, and rather striking, example of this situation happens for perturbation of halfplanes in $\mathbb{R}^2$. That is, an arbitrarily small perturbation of the data corresponding to halfplanes is sufficient for the stickiness phenomenon to occur. Of course, the smaller the perturbation, the smaller the stickiness: nevertheless, small perturbations are enough to cause the fact that the boundary data of nonlocal minimal surfaces are not attained in a continuous way, and indeed they may exhibit jumps (notice that this lack of boundary regularity for $s$-minimal surfaces is rather surprising, especially after the interior regularity results discussed in Theorem 1.2 and 1.3 and it shows that the boundary behavior of the halfplanes is rather unstable).

A detailed result goes as follows:

**Theorem 1.12** (Stickiness arising from perturbation of halfplanes)**.** *There exists $\delta_0 > 0$ such that for any $\delta\ in(0, \delta_0]$ the following statement holds true.*

*Let $\Omega := (-1, 1) \times \mathbb{R}$. Let also*

$$F_- := [-3, -2] \times [0, \delta], \qquad F_+ := [2, 3] \times [0, \delta], \qquad H := \mathbb{R} \times (-\infty, 0).$$

*Assume that $F \subseteq \mathbb{R}^2$, with*

$$F \supseteq H \cup F_- \cup F_+.$$

*Let $E$ be an $s$-minimal set in $\Omega$ among all the sets which coincide with $F$ outside $\Omega$.*

*Then,*

$$E \supseteq (-1, 1) \times [0, \delta^\gamma],$$

*for a suitable $\gamma > 1$.*

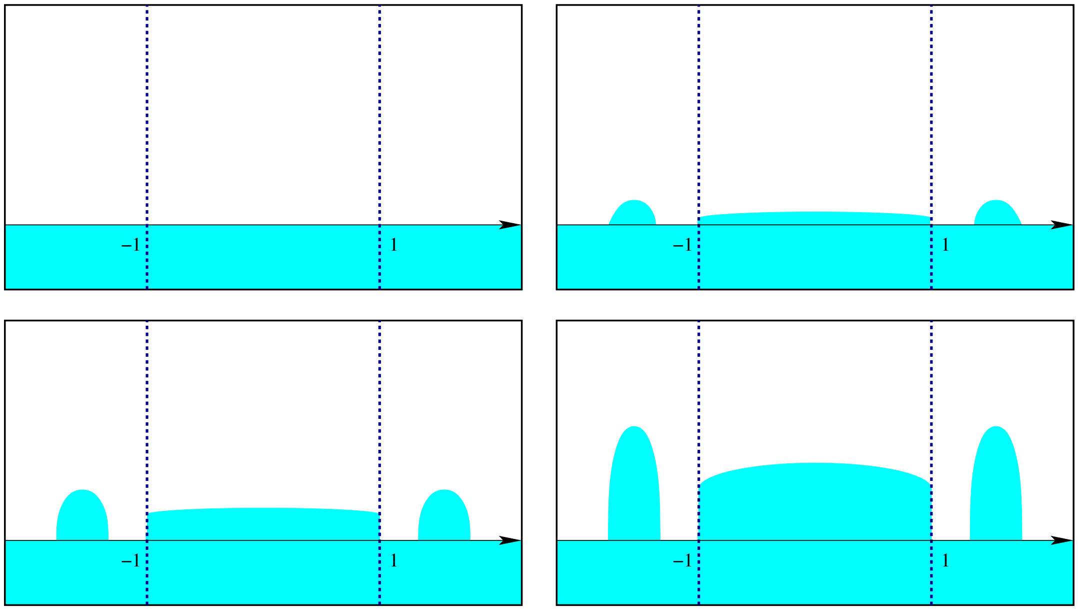


FIGURE 4. *Stickiness arising from perturbation of halfplanes, with the perturbation progressively larger.*

The result of Theorem 1.12 is depicted in Figure 4.

Let us briefly give some further comments on the stickiness phenomena discussed above. First of all, we would like to convince the reader (as well as ourselves) that this type of behaviors indeed occurs in the nonlocal case.

To this end, let us make an investigation to find how the $s$-minimal set $E_\alpha$ in $\Omega := (-1,1)\times\mathbb{R}\subset\mathbb{R}^2$ with datum

$$C_\alpha := \{(x,y)\in\mathbb{R}^2 \text{ s.t. } y<\alpha|x|\}$$

looks like.

When $\alpha = 0$, then $E_\alpha = C_\alpha$ is the halfplane, so the interesting case is when $\alpha \neq 0$; say, up to symmetries, $\alpha > 0$. Now, we know how an investigation works: we need to place all the usual suspects in a row and try to find the culprit.

The line of suspect is on Figure 5 (remember that we have to find the $s$-minimal set among them). Some of the suspects resemble our prejudices on how the culprit should look like. For instance, for what we saw on TV, we have the prejudice that serial killers always wear black gloves and raincoats. Similarly, for what we learnt from the hyperplanes, we may have the prejudice that $s$-minimal surfaces meet the boundary data in a smooth fashion (this prejudice will turn out to be wrong, as we will see). In this sense, the usual suspects number 1 and 2 in Figure 5 are the ones who look like the serial killers.

Then, we have the regular guys with some strange hobbies, we know from TV that they are also quite plausible candidates for being guilty; in our analogy, these are the usual suspects number 3 and 4, which meet the boundary data in a Lipschitz or Hölder fashion (and one may also observe that number 3 is the minimal set in the local case).

Then, we have the candidates which look above suspicion, the ones to which nobody ever consider to be guilty, usually the postman or the butler. In our analogy, these are the suspects number 5 and 6, which are discontinuous at the boundary.

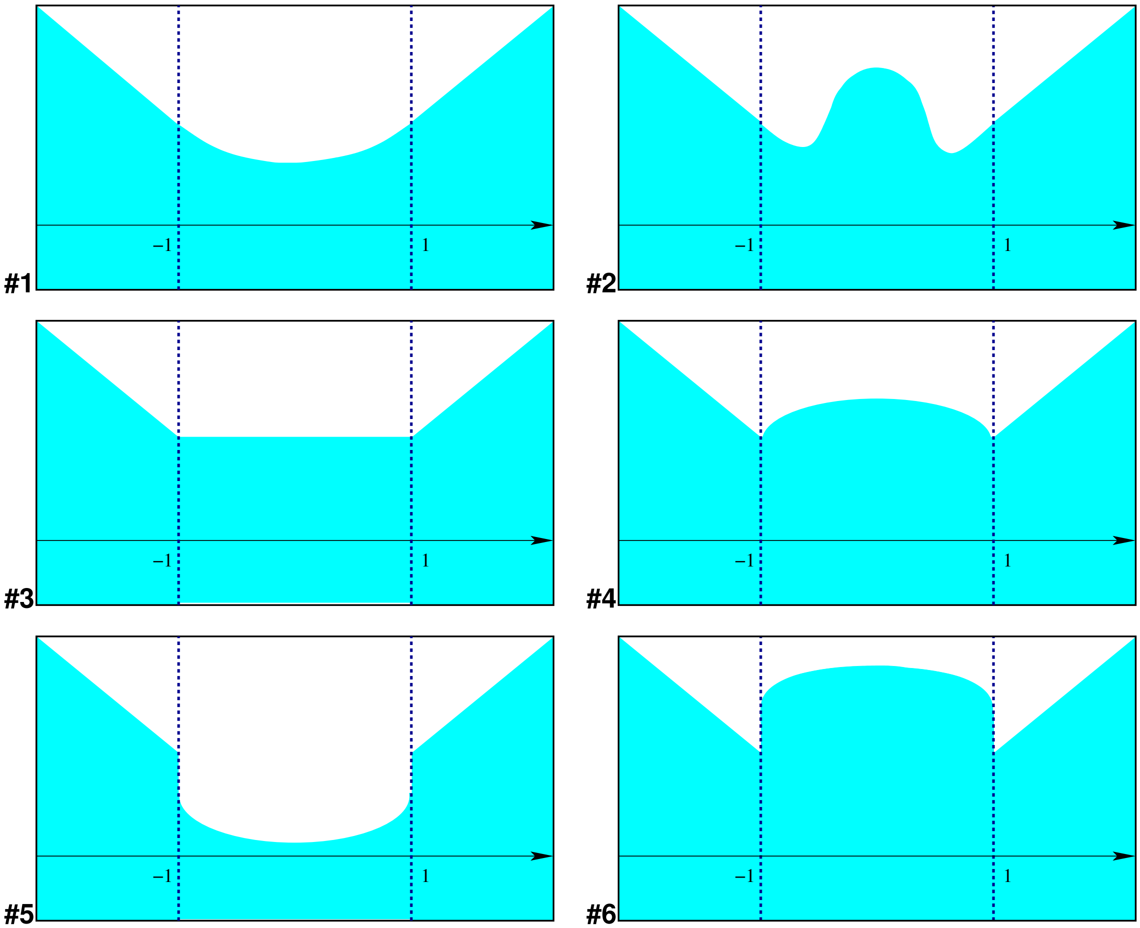


FIGURE 5. *Confrontation between the suspects.*

Now, we know from TV how we should proceed: if a suspect has a strong and verified alibi, we can rule him or her out of the list. In our case, an alibi can be offered by the necessary condition for $s$-minimality given in Lemma 1.1. Indeed, if one of our suspects $E$ does not satisfy that $H^s_E = 0$ along $(\partial E) \cap \Omega$, then $E$ cannot be $s$-minimal and we can cross out $E$ from our list of suspects ($E$ has an alibi!).

Now, it is easily seen that all the suspects number 1, 2, 3, 4 and 5 have an alibi: indeed, from Figure 6 we see that $H^s_E(p) \neq 0$, since the set $E$ occupies (in measure, weighted by the kernel in (1.14)) more then a halfplane passing through $p$: in Figure 6 the point $p$ is the big dot and the halfplane is marked by the line passing through it, so a quick inspection confirms that the alibis of number 1, 2, 3, 4 and 5 check out, hence their nonlocal mean curvature does not vanish at $p$ and consequently they are not $s$-minimal sets.

On the other hand, the alibi of number 6 doesn't hold water. Indeed, near $p$, the set $E$ is confined below the horizontal line, but at infinity the set $E$ go well beyond such line: these effects might compensate each other and produce a vanishing mean curvature.

So, having ruled out all the suspects but number 6, we have only to remember what the old investigators have taught us (e.g., "When you have eliminated the impossible, whatever remains, however improbable, must be the truth"), to find that the only possible (though, in principle, rather improbable) culprit is number 6.

Of course, once that we know that the butler did it, i.e. that number 6 is $s$-minimal, it is our duty to prove it beyond any reasonable doubt. Many pieces of evidence, and a complete proof, is given in [DSV15]

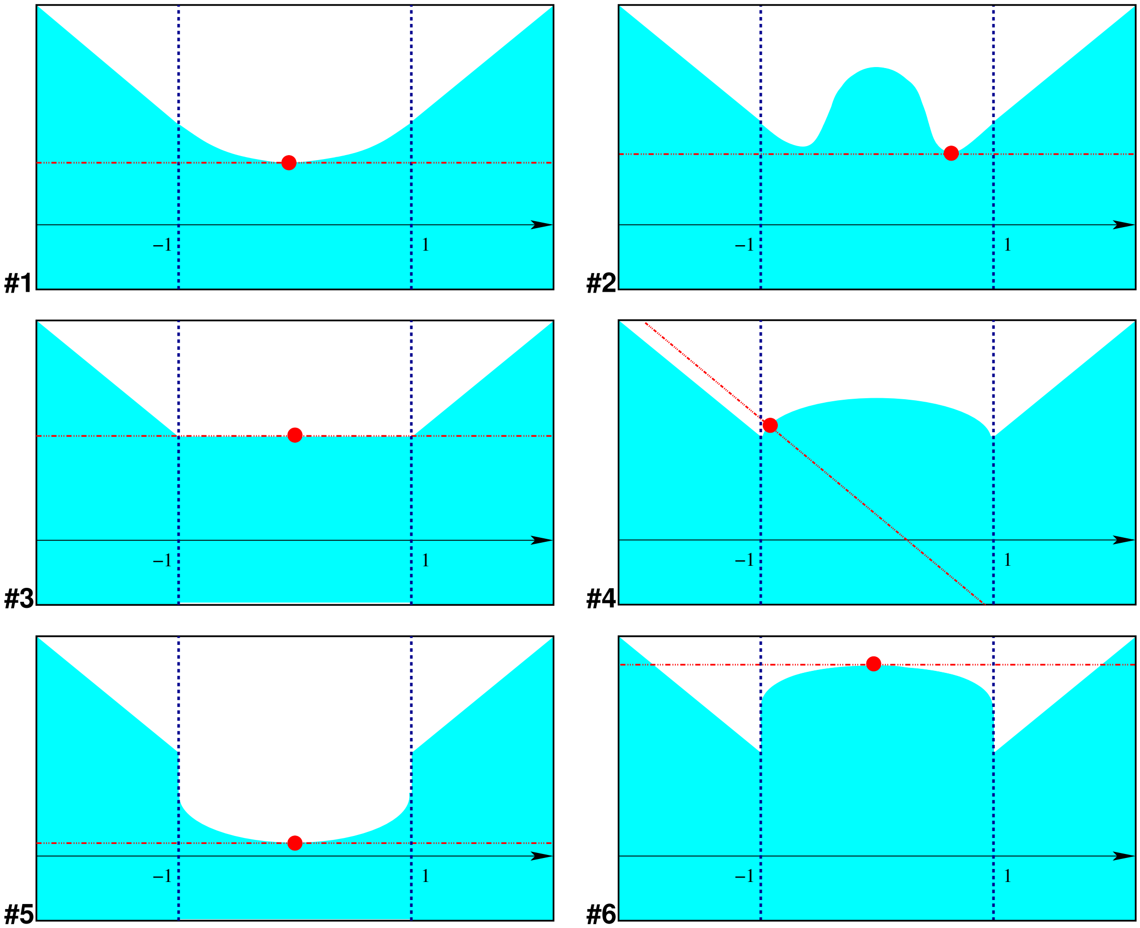


FIGURE 6. *The alibis of the suspects.*

(where indeed the more general version given in Theorem 1.12 is established). Here, we provide some ideas towards the proof of Theorem 1.12 in Section 5.

This set of notes is organized as follows. In Section 2 we present the proof of Lemma 1.1. Sections 3 and 4 are devoted to the proofs of the quantitative estimates in Theorems 1.4 and 1.8, respectively. Then, Section 5 is dedicated to a sketch of the proof of Theorem 1.12. We also provide Appendix A to discuss briefly the asymptotics of the $s$-perimeter as $s \nearrow 1/2$ and as $s \searrow 0$ and Appendix B to discuss the asymptotic expansion of the nonlocal mean curvature as $s \searrow 0$.

## 2. PROOF OF LEMMA 1.1

*Proof of Lemma 1.1.* We consider a diffeomorphism $T_\varepsilon(x) := x + \varepsilon v(x)$, with $v \in C_0^\infty(\Omega, \mathbb{R}^n)$ and we take $E_\varepsilon := T_\varepsilon(E)$. By minimality, we know that $\operatorname{Per}_s(E_\varepsilon, \Omega) \geqslant \operatorname{Per}_s(E, \Omega)$ for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, with $\varepsilon_0 > 0$ sufficiently small, hence

$$\operatorname{Per}_s(E_\varepsilon, \Omega) - \operatorname{Per}_s(E, \Omega) = o(\varepsilon). \tag{2.1}$$

Suppose, for simplicity, that $I(E \setminus \Omega, E^c \setminus \Omega) < +\infty$, so that we can write

$$\operatorname{Per}_s(E_\varepsilon, \Omega) - \operatorname{Per}_s(E, \Omega) = I(E_\varepsilon, E_\varepsilon^c) - I(E, E^c).$$

Moreover, if we use the notation $X := T_\varepsilon^{-1}(x)$, we have that

$$dx = |\det DT_\varepsilon(X)|\, dX = \big(1 + \varepsilon \operatorname{div} v(X) + o(\varepsilon)\big)\, dX.$$

Similarly, if $Y := T_\varepsilon^{-1}(y)$, we find that

$$\begin{aligned}
& |x-y|^{-n-2s} \\
= \;& |T_\varepsilon(X)-T_\varepsilon(Y)|^{-n-2s} \\
= \;& \big|X-Y+\varepsilon\big(v(X)-v(Y)\big)\big|^{-n-2s} \\
= \;& \big|X-Y\big|^{-n-2s} - (n+2s)\,\varepsilon\,|X-Y|^{-n-2s-2}(X-Y)\cdot\big(v(X)-v(Y)\big)+o(\varepsilon).
\end{aligned}$$

As a consequence,

$$\begin{aligned}
& \mathrm{Per}_s\,(E_\varepsilon,\Omega)-\mathrm{Per}_s\,(E,\Omega) \\
= \;& \iint_{E_\varepsilon\times E_\varepsilon^c}\frac{dx\,dy}{|x-y|^{n+2s}}-\iint_{E\times E^c}\frac{dx\,dy}{|x-y|^{n+2s}} \\
= \;& \iint_{E\times E^c}\Big[\big|X-Y\big|^{-n-2s}-(n+2s)\,\varepsilon\,|X-Y|^{-n-2s-2}(X-Y)\cdot\big(v(X)-v(Y)\big)\Big] \\
& \quad \cdot\big(1+\varepsilon\,\mathrm{div}\,v(X)\big)\big(1+\varepsilon\,\mathrm{div}\,v(Y)\big)\,dX\,dY \\
& \quad -\iint_{E\times E^c}\frac{dx\,dy}{|x-y|^{n+2s}}+o(\varepsilon) \\
= \;& -(n+2s)\,\varepsilon\iint_{E\times E^c}\frac{(x-y)\cdot\big(v(x)-v(y)\big)}{|x-y|^{n+2s+2}}\,dx\,dy \\
& \quad +\varepsilon\iint_{E\times E^c}\frac{\mathrm{div}\,v(x)+\mathrm{div}\,v(y)}{|x-y|^{n+2s}}\,dx\,dy+o(\varepsilon).
\end{aligned}$$

Now we point out that

$$\mathrm{div}_x\,\frac{v(x)}{|x-y|^{n+2s}}=-(n+2s)\frac{v(x)\cdot(x-y)}{|x-y|^{n+2s+2}}+\frac{\mathrm{div}_x\,v(x)}{|x-y|^{n+2s}}$$

and so, interchanging the names of the variables,

$$\mathrm{div}_y\,\frac{v(y)}{|x-y|^{n+2s}}=(n+2s)\frac{v(y)\cdot(x-y)}{|x-y|^{n+2s+2}}+\frac{\mathrm{div}_y\,v(y)}{|x-y|^{n+2s}}.$$

Consequently,

$$\begin{aligned}
& \mathrm{Per}_s\,(E_\varepsilon,\Omega)-\mathrm{Per}_s\,(E,\Omega) \\
= \;& \varepsilon\iint_{E\times E^c}\left[\mathrm{div}_x\,\frac{v(x)}{|x-y|^{n+2s}}+\mathrm{div}_y\,\frac{v(y)}{|x-y|^{n+2s}}\right]\,dx\,dy+o(\varepsilon).
\end{aligned}$$

Now, using the Divergence Theorem and changing the names of the variables we have that

$$\begin{aligned}
\iint_{E\times E^c}\mathrm{div}_x\,\frac{v(x)}{|x-y|^{n+2s}}\,dx\,dy &= \int_{E^c}dy\left[\int_{\partial E}\frac{v(x)\cdot\nu(x)}{|x-y|^{n+2s}}\,d\mathcal{H}^{n-1}(x)\right] \\
&= \int_{E^c}dx\left[\int_{\partial E}\frac{v(y)\cdot\nu(y)}{|x-y|^{n+2s}}\,d\mathcal{H}^{n-1}(y)\right]
\end{aligned}$$

and

$$\iint_{E\times E^c}\mathrm{div}_y\,\frac{v(y)}{|x-y|^{n+2s}}\,dx\,dy=-\int_E dx\left[\int_{\partial E}\frac{v(y)\cdot\nu(y)}{|x-y|^{n+2s}}\,d\mathcal{H}^{n-1}(y)\right].$$

Accordingly, we find that

$$\begin{aligned}
&\mathrm{Per}_s\,(E_\varepsilon,\Omega)-\mathrm{Per}_s\,(E,\Omega)\\
=\;&\varepsilon\int_{\partial E} d\mathcal{H}^{n-1}(y)v(y)\cdot\nu(y)\left[\int_{E^c}\frac{dx}{|x-y|^{n+2s}}-\int_E\frac{dx}{|x-y|^{n+2s}}\right]+o(\varepsilon)\\
=\;&\varepsilon\int_{\partial E} v(y)\cdot\nu(y)\,H^s_E(y)\,d\mathcal{H}^{n-1}(y)+o(\varepsilon).
\end{aligned}$$

Comparing with (2.1), we see that

$$\int_{\partial E} v(y)\cdot\nu(y)\,H^s_E(y)\,d\mathcal{H}^{n-1}(y)=0$$

and so, since $v$ is an arbitrary vector field supported in $\Omega$, the desired result follows. $\square$

## 3. Proof of Theorem 1.4

The basic idea goes as follows. One uses the appropriate combination of two general facts: on the one hand, one can perturb a given set by a smooth flow and compare the energy at time $t$ with the one at time $-t$, thus obtaining a second order estimate; on the other hand, the nonlocal interaction always charges a mass on points that are sufficiently close, thus providing a natural measure for the discrepancy between the original set and its flow. One can appropriately combining these two facts with the minimality (or more generally, the stability) property of a set. Indeed, by choosing as smooth flow a translation near the origin, the above arguments lead to an integral estimate of the discrepancy between the set and its translations, which in turn implies a perimeter estimate.

We now give the details of the proof of Theorem 1.4. To do this, we fix $R\geqslant 1$, a direction $v\in S^{n-1}$, a function $\varphi\in C^\infty_0(B_{9/10})$ with $\varphi=1$ in $B_{3/4}$, and a small scalar quantity $t\in\left(-\frac{1}{100},\frac{1}{100}\right)$, and we consider the diffeomorphism $\Phi^t\in C^\infty_0(B_{9/10})$ given by $\Phi^t(x):=x+t\varphi(x/R)\,v$. Notice that

$$\Phi^t(x)=x+tv \text{ for any } x\in B_{3R/4}. \tag{3.1}$$

We also define $E_t:=\Phi^t(E)$. We have the following useful auxiliary estimates (that will be used in the proofs of both Theorem 1.4 and Theorem 1.8):

**Lemma 3.1.** *Let $E$ be a minimizer for the $s$-perimeter in $B_R$. Then*

$$\mathrm{Per}_s\,(E_t,B_R)+\mathrm{Per}_s\,(E_{-t},B_R)-2\mathrm{Per}_s\,(E,B_R)\leqslant CR^{n-2s-2}\,t^2, \tag{3.2}$$

$$2I(E_t\setminus E,E\setminus E_t)\leqslant CR^{n-2s-2}\,t^2, \tag{3.3}$$

$$\min\Big\{\big|((E+tv)\setminus E)\cap B_{R/2}\big|,\ \big|(E\setminus(E+tv))\cap B_{R/2}\big|\Big\}\leqslant C\,R^{\frac{n-2s-2}{2}}\,|t|, \tag{3.4}$$

*and*

$$\min\left\{\int_{B_{R/2}}\big(\chi_E(x+tv)-\chi_E(x)\big)_+\,dx,\ \int_{B_{R/2}}\big(\chi_E(x+tv)-\chi_E(x)\big)_-\,dx\right\}\leqslant C\,R^{\frac{n-2s-2}{2}}\,|t|, \tag{3.5}$$

*for some $C>0$.*

*Proof.* First we observe that

$$\mathrm{Per}_s\,(E_t,B_R)+\mathrm{Per}_s\,(E_{-t},B_R)-2\mathrm{Per}_s\,(E,B_R)\leqslant\frac{Ct^2}{R^2}\mathrm{Per}_s\,(E,B_R), \tag{3.6}$$

for some $C>0$. This is indeed a general estimate, which does not use minimality, and which follows by changing variable in the integrals of the fractional perimeter (and noticing that the linear term in $t$

simplifies). We provide some details of the proof of (3.6) for the facility of the reader. To this aim, we observe that

$$|\det D\Phi^t(X)| = |\det(\mathbf{1} + tR^{-1}\nabla\varphi(X/R)\otimes v)| = 1 + tR^{-1}\nabla\varphi(X/R)\cdot v + O(t^2R^{-2}).$$

Moreover, if, for any $\xi$, $\eta \in \mathbb{R}^n$, we set

$$g(\xi,\eta) := \frac{\big(\varphi(\xi)-\varphi(\eta)\big)\,v}{|\xi-\eta|},$$

we have that $g$ is bounded and

$$\begin{aligned}
|\Phi^t(X)-\Phi^t(Y)| &= \big|X-Y+t\big(\varphi(X/R)-\varphi(Y/R)\big)\,v\big| \\
&= |X-Y|\left|\frac{X-Y}{|X-Y|}+tR^{-1}\frac{\big(\varphi(X/R)-\varphi(Y/R)\big)\,v}{|(X/R)-(Y/R)|}\right| \\
&= |X-Y|\left|\frac{X-Y}{|X-Y|}+tR^{-1}g(X/R,Y/R)\right|.
\end{aligned}$$

Therefore

$$\begin{aligned}
|\Phi^t(X)-\Phi^t(Y)|^{-n-2s} &= |X-Y|^{-n-2s}\left|\frac{X-Y}{|X-Y|}+tR^{-1}g(X/R,Y/R)\right|^{-n-2s} \\
&= |X-Y|^{-n-2s}\left(1-(n+2s)tR^{-1}\frac{X-Y}{|X-Y|}\cdot g(X/R,Y/R)+O(t^2R^{-2})\right).
\end{aligned}$$

Now we observe that $\Phi^t$ is the identity outside $B_R$ and therefore if $A \in \{B_R, B_R^c, \mathbb{R}^n\}$ then $E_t \cap A = \Phi^t(E\cap A)$. Accordingly, for any $A$, $B \in \{B_R, B_R^c, \mathbb{R}^n\}$, a change of variables $x := \Phi^t(X)$ and $y := \Phi^t(Y)$ gives that

$$\begin{aligned}
& I(E_t\cap A, E_t^c\cap B) \\
=\; & \int_{\Phi^t(E\cap A)}\int_{\Phi^t(E\cap B)} |x-y|^{-n-2s}\,dx\,dy \\
=\; & \int_{E\cap A}\int_{E\cap B} |\Phi^t(X)-\Phi^t(Y)|^{-n-2s}\,|\det D\Phi^t(X)|\,|\det D\Phi^t(Y)|\,dX\,dY \\
=\; & \int_{E\cap A}\int_{E\cap B} |X-Y|^{-n-2s}\left(1-(n+2s)tR^{-1}\frac{X-Y}{|X-Y|}\cdot g(X/R,Y/R)+O(t^2R^{-2})\right) \\
& \qquad \cdot\big(1+tR^{-1}\nabla\varphi(X/R)\cdot v+O(t^2R^{-2})\big)\big(1+tR^{-1}\nabla\varphi(Y/R)\cdot v+O(t^2R^{-2})\big)\,dX\,dY \\
=\; & \int_{E\cap A}\int_{E\cap B} |X-Y|^{-n-2s}\big(1-(n+2s)tR^{-1}\tilde g(X/R,Y/R)+O(t^2R^{-2})\big)\,dX\,dY,
\end{aligned}$$

for a suitable scalar function $\tilde g$.

Then, replacing $t$ with $-t$ and summing up, the linear term in $t$ simplifies and we obtain

$$I(E_t\cap A, E_t^c\cap B)+I(E_{-t}\cap A, E_{-t}^c\cap B) = \big(2+O(t^2R^{-2})\big)\int_{E\cap A}\int_{E\cap B}\frac{dX\,dY}{|X-Y|^{n+2s}}.$$

This, choosing $A$ and $B$ appropriately, establishes (3.6).

On the other hand, the $s$-minimality of $E$ gives that $\mathrm{Per}_s\,(E,B_R) \leqslant \mathrm{Per}_s\,(E\cup B_R, B_R)$, which, in turn, is majorized by the interaction between $B_R$ and $B_R^c$, namely $I(B_R,B_R^c)$, which is a constant (only depending on $n$ and $s$) times $R^{n-2s}$, due to scale invariance of the fractional perimeter. That is, we have that $\mathrm{Per}_s\,(E,B_R) \leqslant CR^{n-2s}$, for some $C>0$, and then we can make the right hand side of (3.6) uniform in $E$ and obtain (3.2), up to renaming $C>0$.

The next step is to charge mass in a ball. Namely, one defines $E_t^{\cup} := E \cup E_t$ and $E_t^{\cap} := E \cap E_t$. By counting the interactions of the different sets, one sees that

$$\text{Per}_s\,(E, B_R) + \text{Per}_s\,(E_t, B_R) - \text{Per}_s\,(E_t^{\cup}, B_R) - \text{Per}_s\,(E_t^{\cap}, B_R) = 2I(E_t \setminus E, E \setminus E_t). \tag{3.7}$$

To check this, one observes indeed that the set $E_t \setminus E$ interacts with $E \setminus E_t$ in the computations of $\text{Per}_s\,(E, B_R)$ and $\text{Per}_s\,(E_t, B_R)$, while these two sets do not interact in the computations of $\text{Per}_s\,(E_t^{\cup}, B_R)$ and $\text{Per}_s\,(E_t^{\cap}, B_R)$ (the interactions of the other sets simplify). This proves (3.7). We remark that, again, formula (3.7) is a general fact and is not based on minimality. Changing $t$ with $-t$, we also obtain from (3.7) that

$$\text{Per}_s\,(E, B_R) + \text{Per}_s\,(E_{-t}, B_R) - \text{Per}_s\,(E_{-t}^{\cup}, B_R) - \text{Per}_s\,(E_{-t}^{\cap}, B_R) = 2I(E_{-t} \setminus E, E \setminus E_{-t}).$$

This and (3.7) give that

$$\begin{aligned}
& \text{Per}_s\,(E_t, B_R) + \text{Per}_s\,(E_{-t}, B_R) - 2\text{Per}_s\,(E, B_R) \\
= \; & \text{Per}_s\,(E_t^{\cup}, B_R) + \text{Per}_s\,(E_t^{\cap}, B_R) + \text{Per}_s\,(E_{-t}^{\cup}, B_R) + \text{Per}_s\,(E_{-t}^{\cap}, B_R) \\
& \quad -4\text{Per}_s\,(E, B_R) + 2I(E_t \setminus E, E \setminus E_t) + 2I(E_{-t} \setminus E, E \setminus E_{-t}) \\
\geqslant \; & 2I(E_t \setminus E, E \setminus E_t) + 2I(E_{-t} \setminus E, E \setminus E_{-t}),
\end{aligned}$$

thanks to the $s$-minimality of $E$. In particular,

$$\text{Per}_s\,(E_t, B_R) + \text{Per}_s\,(E_{-t}, B_R) - 2\text{Per}_s\,(E, B_R) \geqslant 2I(E_t \setminus E, E \setminus E_t).$$

This and (3.2) imply (3.3).

Now, the interaction kernel is bounded away from zero in $B_{R/2}$, and so

$$I(E_t \setminus E, E \setminus E_t) \geqslant \big|(E_t \setminus E) \cap B_{R/2}\big| \cdot \big|(E \setminus E_t) \cap B_{R/2}\big|.$$

This is again a general fact, not using minimality. By plugging this into (3.3), we conclude that

$$\begin{aligned}
CR^{n-2s-2}\,t^2 \; & \geqslant \; \big|(E_t \setminus E) \cap B_{R/2}\big| \cdot \big|(E \setminus E_t) \cap B_{R/2}\big| \\
& \geqslant \; \min\Big\{\big|(E_t \setminus E) \cap B_{R/2}\big|^2,\; \big|(E \setminus E_t) \cap B_{R/2}\big|^2\Big\}
\end{aligned}$$

and so, again up to renaming $C$,

$$\min\Big\{\big|(E_t \setminus E) \cap B_{R/2}\big|,\; \big|(E \setminus E_t) \cap B_{R/2}\big|\Big\} \leqslant CR^{\frac{n-2s-2}{2}}\,t. \tag{3.8}$$

Now, we recall (3.1) and we observe that $E_t \cap B_{R/2} = (E + tv) \cap B_{R/2}$. Hence, the estimate in (3.8) becomes

$$\min\Big\{\big|((E+tv) \setminus E) \cap B_{R/2}\big|,\; \big|(E \setminus (E+tv)) \cap B_{R/2}\big|\Big\} \leqslant CR^{\frac{n-2s-2}{2}}\,t. \tag{3.9}$$

Since this is valid for any $v \in S^{n-1}$, we may also switch the sign of $v$ and obtain that

$$\min\Big\{\big|((E-tv) \setminus E) \cap B_{R/2}\big|,\; \big|(E \setminus (E-tv)) \cap B_{R/2}\big|\Big\} \leqslant CR^{\frac{n-2s-2}{2}}\,t. \tag{3.10}$$

From (3.9) and (3.10) we obtain (3.4).

Now we observe that, for any sets $A$ and $B$,

$$\chi_{A \setminus B}(x) \geqslant \chi_A(x) - \chi_B(x). \tag{3.11}$$

Indeed, this formula is clearly true if $x \in B$, since in this case the right hand side is nonpositive. The formula is also true if $x \in A \setminus B$, since in this case the left hand side is $1$ and the right hand side is less or equal than $1$. It remains to consider the case in which $x \notin A \cup B$. In this case, $\chi_A(x) = 0$, hence the right hand side is nonpositive, which gives that (3.11) holds true.

By (3.11),

$$\chi_{A\setminus B}(x) \geqslant \big(\chi_A(x) - \chi_B(x)\big)_+.$$

As a consequence,

$$\begin{aligned}
\big|((E-tv)\setminus E)\cap B_{R/2}\big| &= \int_{B_{R/2}} \chi_{(E-tv)\setminus E}(x)\,dx \\
&\geqslant \int_{B_{R/2}} \big(\chi_{E-tv}(x) - \chi_E(x)\big)_+\,dx = \int_{B_{R/2}} \big(\chi_E(x+tv) - \chi_E(x)\big)_+\,dx \\
\text{and} \qquad \big|(E\setminus (E-tv))\cap B_{R/2}\big| &= \int_{B_{R/2}} \chi_{E\setminus(E-tv)}(x)\,dx \\
&\geqslant \int_{B_{R/2}} \big(\chi_E(x) - \chi_{E-tv}(x)\big)_+\,dx = \int_{B_{R/2}} \big(\chi_E(x) - \chi_E(x+tv)\big)_+\,dx \\
&= \int_{B_{R/2}} \big(\chi_E(x+tv) - \chi_E(x)\big)_-\,dx.
\end{aligned}$$

This and (3.10) give that

$$CR^{\frac{n-2s-2}{2}}\,t \geqslant \min\left\{ \int_{B_{R/2}} \big(\chi_E(x+tv) - \chi_E(x)\big)_+\,dx,\ \int_{B_{R/2}} \big(\chi_E(x+tv) - \chi_E(x)\big)_-\,dx \right\},$$

which is (3.5). This ends the proof of Lemma 3.1. □

With the preliminary work done in Lemma 3.1 (to be used here with $R = 1$), we can now complete the proof of Theorem 1.4. To this end, we observe that

$$\begin{aligned}
&\left| \int_{B_{1/2}} \big(\chi_E(x+tv) - \chi_E(x)\big)_+\,dx - \int_{B_{1/2}} \big(\chi_E(x+tv) - \chi_E(x)\big)_-\,dx \right| \\
&= \left| \int_{B_{1/2}} \big(\chi_E(x+tv) - \chi_E(x)\big)\,dx \right| \\
&= \left| \int_{B_{1/2}-tv} \chi_E(x)\,dx - \int_{B_{1/2}} \chi_E(x)\,dx \right| \\
&\leqslant \big|(B_{1/2} - tv)\Delta B_{1/2}\big| \\
&\leqslant C\,t,
\end{aligned} \tag{3.12}$$

for some $C > 0$.

Also, we observe that, for any $a, b \in \mathbb{R}$,

$$a + b \leqslant |a-b| + 2\min\{a,b\}. \tag{3.13}$$

Indeed, up to exchanging $a$ and $b$, we may suppose that $a \geqslant b$; thus

$$a + b = a - b + 2b = |a-b| + 2\min\{a,b\},$$

which proves (3.13).

Using (3.5), (3.12) and (3.13), we obtain that

$$\begin{aligned}
&\int_{B_{1/2}} \big|\chi_E(x+tv)-\chi_E(x)\big|\,dx\\
=\;&\int_{B_{1/2}} \big(\chi_E(x+tv)-\chi_E(x)\big)_+\,dx+\int_{B_{1/2}} \big(\chi_E(x+tv)-\chi_E(x)\big)_-\,dx\\
\leqslant\;&\left|\int_{B_{1/2}} \big(\chi_E(x+tv)-\chi_E(x)\big)_+\,dx-\int_{B_{1/2}} \big(\chi_E(x+tv)-\chi_E(x)\big)_-\,dx\right|\\
&\quad+2\min\left\{\int_{B_{1/2}} \big(\chi_E(x+tv)-\chi_E(x)\big)_+\,dx,\ \int_{B_{1/2}} \big(\chi_E(x+tv)-\chi_E(x)\big)_-\,dx\right\}\\
\leqslant\;& Ct,
\end{aligned}$$

up to renaming $C$. Dividing by $t$ and sending $t \searrow 0$ (up to subsequences), one finds that

$$\int_{B_{1/2}} |\partial_v \chi_E(x)|\,dx \leqslant C,$$

for any $v \in S^{n-1}$, in the bounded variation sense. Since the direction $v$ is arbitrary, this proves that

$$\operatorname{Per}(E, B_{1/2}) = \int_{B_{1/2}} |\nabla \chi_E(x)|\,dx \leqslant C.$$

This proves Theorem 1.4 with $R=1$, and the general case follows from scaling.

## 4. Proof of Theorem 1.8

In this part, we will make use of some integral geometric formulas which compute the perimeter of a set by averaging the number of intersections of straight lines with the boundary of a set.

For this, we recall the notation of the positive and negative part of a function $u$, namely

$$u_+(x) := \max\{u(x),\ 0\} \qquad \text{and} \qquad u_-(x) := \max\{-u(x),\ 0\}.$$

Notice that $u_\pm \geqslant 0$, that $|u| = u_+ + u_-$ and that $u = u_+ - u_-$.

Also, if $v \in \partial B_1$ and $p \in \mathbb{R}^n$, we define

$$\begin{aligned}
& v^\perp := \{y \in \mathbb{R}^n \text{ s.t. } y\cdot v = 0\}\\
\text{and}\qquad & p+\mathbb{R}v := \{p+tv \text{ s.t. } t \in \mathbb{R}\}.
\end{aligned}$$

That is, $v^\perp$ is the orthogonal linear space to $v$ and $p+\mathbb{R}v$ is the line passing through $p$ with direction $v$.

Now, given a Caccioppoli set $E \subseteq \mathbb{R}^n$ with exterior normal $\nu$ (and reduced boundary denoted by $\partial^* E$), and $v \in \partial B_1$, we set

$$I_{v,\pm}(y) := \sup \mp \int_{y+\mathbb{R}v} \chi_E(x)\,\phi'(x)\,d\mathcal{H}^1(x), \tag{4.1}$$

with the $\sup$ taken over all smooth $\phi$ supported in the segment $B_1 \cap (y+\mathbb{R}v)$ with image in $[0,1]$. We have (see e.g. Proposition 4.4 in [CSV16]) that one can compute the directional derivative in the sense of bounded variation by the formula

$$\int_{B_1} (\partial_v \chi_E)_\pm(x)\,dx = \int_{y\in v^\perp} I_{v,\pm}(y)\,d\mathcal{H}^{n-1}(y) \tag{4.2}$$

and we also have that $I_{v,\pm}(y)$ is the number of points $x$ that lie in $B_1 \cap (\partial^* E) \cap (y + \mathbb{R}v)$ and such that $\mp v \cdot \nu(x) > 0$. That is, the quantity $I_{v,+}(y)$ (resp., $I_{v,-}(y)$) counts the number of intersections in the ball $B_1$ between the line $y + \mathbb{R}v$ and the (reduced) boundary of $E$ that occur at points $x$ in which $v \cdot \nu(x)$ is negative (resp., positive). In particular,

$$I_{v,\pm}(y) \in \mathbb{Z} \cap [0, +\infty) = \{0, 1, 2, 3, \dots\}. \tag{4.3}$$

Furthermore, the vanishing of $I_{v,+}(y)$ (resp., $I_{v,-}(y)$) is related to the fact that, moving along the segment $B_1 \cap (y + \mathbb{R}v)$, one can only exit (resp., enter) the set $E$, according to the following result:

**Lemma 4.1.** *If $I_{v,+}(y) = 0$, then the map $B_1 \cap (y + \mathbb{R}v) \ni x \mapsto \chi_E(x)$ is nonincreasing.*

*Proof.* For any smooth $\phi$ supported in the segment $B_1 \cap (y + \mathbb{R}v)$ with image in $[0, 1]$,

$$0 = I_{v,+}(y) \geqslant -\int_{y+\mathbb{R}v} \chi_E(x)\, \phi'(x)\, d\mathcal{H}^1(x),$$

that is

$$\int_{y+\mathbb{R}v} \chi_E(x)\, \phi'(x)\, d\mathcal{H}^1(x) \geqslant 0,$$

which gives the desired result. □

Now we define

$$\Phi_\pm(v) := \int_{y \in v^\perp} I_{v,\pm}(y)\, d\mathcal{H}^{n-1}(y). \tag{4.4}$$

By (4.2),

$$\Phi_\pm(v) = \int_{B_1} (\partial_v \chi_E)_\pm(x)\, dx. \tag{4.5}$$

We observe that

**Lemma 4.2.** *Let $\mathrm{Per}\,(E, B_1) < +\infty$ and $n \geqslant 2$. Then the functions $\Phi_\pm$ are continuous on $S^{n-1}$. Moreover, there exists $v_\star$ such that*

$$\Phi_+(v_\star) = \Phi_-(v_\star). \tag{4.6}$$

*Proof.* Let $v, w \in S^{n-1}$. By (4.5),

$$\begin{aligned} \big|\Phi_+(v) - \Phi_+(w)\big| &\leqslant \int_{B_1} \big|(\partial_v \chi_E)_+(x) - (\partial_w \chi_E)_+(x)\big|\, dx \\ &\leqslant \int_{B_1} \big|\partial_v \chi_E(x) - \partial_w \chi_E(x)\big|\, dx \leqslant |v - w| \int_{B_1} |\nabla \chi_E(x)|\, dx = |v - w|\, \mathrm{Per}\,(E, B_1). \end{aligned}$$

This shows that $\Phi_+$ is continuous. Similarly, one sees that $\Phi_-$ is continuous.

Now we prove (4.6). For this, let $\Psi(v) := \Phi_+(v) - \Phi_-(v)$. By (4.5),

$$\Phi_\pm(-v) = \Phi_\mp(v).$$

Therefore

$$\Psi(-v) = \Phi_+(-v) - \Phi_-(-v) = \Phi_-(v) - \Phi_+(v) = -\Psi(v). \tag{4.7}$$

Now, if $\Psi(e_1) = 0$, we can take $v_\star := e_1$ and (4.6) is proved. So we can assume that $\Psi(e_1) > 0$ (the case $\Psi(e_1) < 0$ is analogous). By (4.7), we obtain that $\Psi(-e_1) < 0$. Hence, since $\Psi$ is continuous, it must have a zero on any path joining $e_1$ to $-e_1$, and this proves (4.6). □

A control on the function $\Phi_\pm$ implies a quantitative flatness bound on the set $E$, as stated here below:

**Lemma 4.3.** *Let $n = 2$. There exists $\mu_o > 0$ such that for any $\mu \in (0, \mu_o]$ the following statement holds. Assume that*

$$\Phi_-(e_2) \leqslant \mu \tag{4.8}$$

*and that*

$$\max\{\Phi_+(e_1),\ \Phi_-(e_1)\} \leqslant \mu. \tag{4.9}$$

*Then, there exists a horizontal halfplane $h \subset \mathbb{R}^2$ such that*

$$\big|(E \setminus h) \cap B_1\big| + \big|(h \setminus E) \cap B_1\big| \leqslant C\mu, \tag{4.10}$$

*for some $C > 0$.*

*Proof.* Given $v \in \partial B_1$, we take into account the sets of $y \in v^\perp$ which give a positive contribution to $I_{v,\pm}(y)$. For this, we define

$$\mathcal{B}_\pm(v) := \{y \in v^\perp \text{ s.t. } I_{v,\pm}(y) \neq 0\}.$$

From (4.3), we know that if $y \in \mathcal{B}_\pm(v)$, then $I_{v,\pm}(y) \geqslant 1$. As a consequence of this and of (4.4), we have that

$$\Phi_\pm(v) \geqslant \int_{\mathcal{B}_\pm(v)} I_{v,\pm}(y)\, d\mathcal{H}^1(y) \geqslant \mathcal{H}^1(\mathcal{B}_\pm(v)).$$

Accordingly, by (4.8) and (4.9), we see that

$$\mathcal{H}^1(\mathcal{B}_-(e_2)) \leqslant \mu \tag{4.11}$$

and

$$\mathcal{H}^1(\mathcal{B}_\pm(e_1)) \leqslant \mu. \tag{4.12}$$

Furthermore, for any $y \in v^\perp \setminus \mathcal{B}_+(v)$ (resp. $y \in v^\perp \setminus \mathcal{B}_-(v)$), we have that $I_{v,+}(y) = 0$ (resp., $I_{v,-}(y) = 0$) and thus, by Lemma 4.1, the map $B_1 \cap (y + \mathbb{R}v) \ni x \mapsto \chi_E(x)$ is nonincreasing (resp., nondecreasing).

Therefore, by (4.12), we have that for any vertical coordinate $y \in e_1^\perp$ outside the small set $\mathcal{B}_-(e_1) \cup \mathcal{B}_+(e_1)$ (which has total length of size $2\mu$), the vertical line $y + \mathbb{R}e_1$ is either all contained in $E$ or in its complement (see Figure 7).

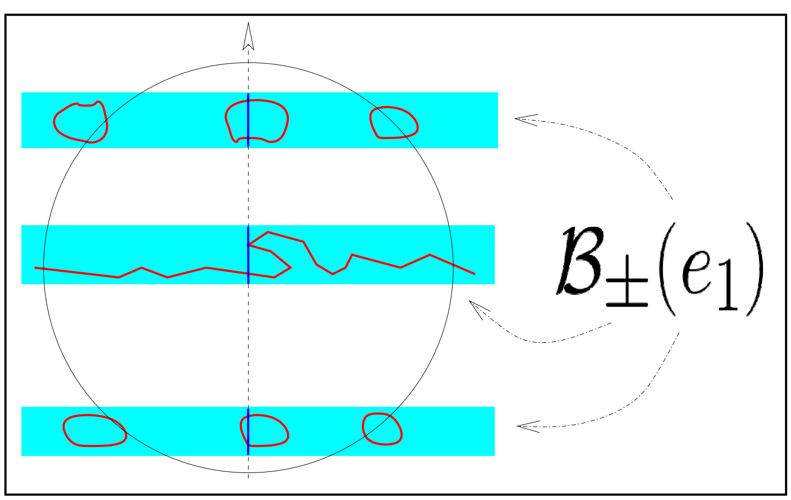


FIGURE 7. *Horizontal lines do not meet the boundary of $E$, with the exception of a small set $\mathcal{B}_\pm(e_1)$.*

That is, we can denote by $\mathcal{G}_E$ the set of vertical coordinates $y$ for which the portion in $B_1$ of the horizontal line passing through $y$ lies in $E$ and, similarly, by $\mathcal{G}_{E^c}$ the set of vertical coordinates $y$ for which the portion in $B_1$ of the horizontal line passing through $y$ lies in $E^c$ and we obtain that $\mathcal{G}_E \cup \mathcal{G}_{E^c}$ exhaust the whole of $(-1,1)$, up to a set of size at most $2\mu$.

We also remark that $\mathcal{G}_E$ lies below $\mathcal{G}_{E^c}$: indeed, by (4.11), we have that vertical lines can only exit the set $E$ (possibly with the exception of a small set of size $\mu$). The situation is depicted in Figure 8.

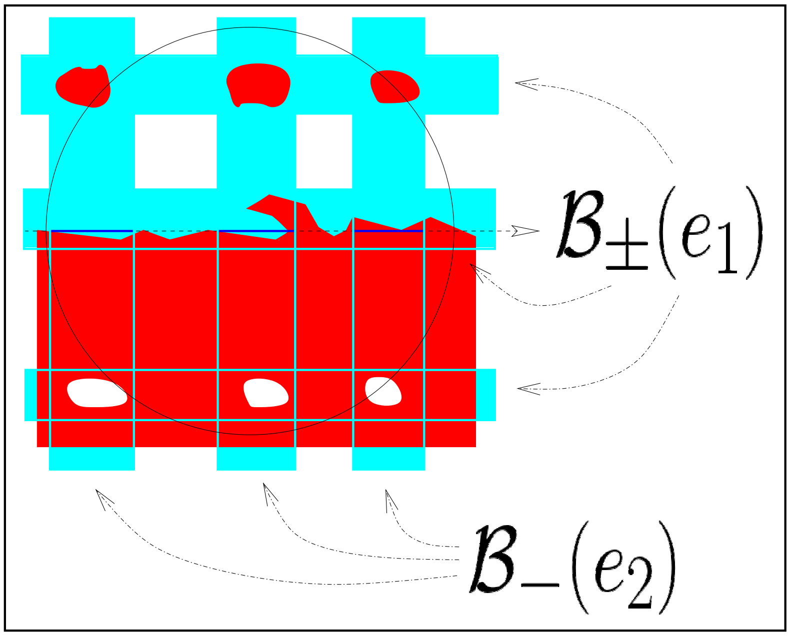


FIGURE 8. *Vertical lines do not meet the boundary of $E$, with the exception of a small set $\mathcal{B}_-(e_2)$.*

Hence, if we take $h$ to be a horizontal halfplane which separates $\mathcal{G}_E$ and $\mathcal{G}_{E^c}$, we obtain (4.10). $\square$

With this, we can now complete the proof of Theorem 1.8. The main tool for this goal is Lemma 4.3. In order to apply it, we need to check that (4.8) and (4.9) are satisfied. To this end, we argue as follows. First of all, fixed a large $R > 2$, we consider, as in Section 3, a diffeomorphism $\Phi^t$ such that $\Phi^t(x) = x$ for any $x \in \mathbb{R}^n \setminus B_{9R/10}$, and $\Phi^t(x) = x + tv$ for any $x \in B_{3R/4}$, and we set $E_t := \Phi^t(E)$. From (3.5) (recall that here $n = 2$), we have that

$$\min\left\{\int_{B_{R/2}} \big(\chi_E(x+tv) - \chi_E(x)\big)_+\,dx,\ \int_{B_{R/2}} \big(\chi_E(x+tv) - \chi_E(x)\big)_-\,dx\right\} \leqslant \frac{Ct}{R^s},$$

for some $C > 0$. Thus, dividing by $t$ and sending $t \searrow 0$,

$$\min\left\{\int_{B_{R/2}} \big(\partial_v \chi_E(x)\big)_+\,dx,\ \int_{B_{R/2}} \big(\partial_v \chi_E(x)\big)_-\,dx\right\} \leqslant \frac{C}{R^s}.$$

That is, recalling (4.5),

$$\min\left\{\Phi_+(v),\ \Phi_-(v)\right\} \leqslant \frac{C}{R^s}. \tag{4.13}$$

We also observe that $E$ has finite perimeter in $B_1$, thanks to Theorem 1.4, and so we can make use of Lemma 4.2. In particular, by (4.6), after a rotation of coordinates, we may assume that $\Phi_+(e_1) = \Phi_-(e_1)$.

Hence (4.13) says that

$$\max\{\Phi_+(e_1),\ \Phi_-(e_1)\} = \min\{\Phi_+(e_1),\ \Phi_-(e_1)\} \leqslant \frac{C}{R^s}. \tag{4.14}$$

Also, up to a change of orientation, we may suppose that $\Phi_-(e_2) \leqslant \Phi_+(e_2)$, hence in this case (4.13) says that

$$\Phi_-(e_2) \leqslant \frac{C}{R^s}.$$

From this and (4.14), we see that (4.8) and (4.9) are satisfied (with $\mu = C/R^s$) and so by Lemma 4.3 we conclude that

$$\big|(E\setminus h)\cap B_1\big| + \big|(h\setminus E)\cap B_1\big| \leqslant \frac{C}{R^s},$$

for some halfplane $h$. This completes the proof of Theorem 1.8: as a matter of fact, the result proven is even stronger, since it says that, after removing horizontal and vertical slabs of size $C/R^s$, we have that $\partial E$ in $B_1$ is a graph of oscillation bounded by $C/R^s$, see Figure 8 (in fact, more general statements and proofs can be find in [CSV16]).

## 5. Sketch of the proof of Theorem 1.12

The core of the proof of Theorem 1.12 consists in constructing a suitable barrier that can be slided "from below" and which exhibits the desired stickiness phenomenon: if this is possible, since the $s$-minimal surface cannot touch the barrier, it has to stay above the barrier and stick at the boundary as well.

So, the barrier we are looking for should have negative fractional mean curvature, coincide with $F$ outside $(-1,1)\times\mathbb{R}$ and contain $(-1,1)\times(-\infty,\delta^\gamma)$.

Such barrier is constructed in [DSV15] in an iterative way, that we now try to describe.

*Step 1.* Let us start by looking at the subgraph of the function $y = \frac{x_+}{\ell}$, given $\ell \geqslant 0$. Then, at all the boundary points $X=(x,y)$ with positive abscissa $x>0$, the fractional mean curvature is at most

$$-\frac{c}{\max\{1,\ell\}\,|X|^{2s}}, \tag{5.1}$$

for some $c>0$. The full computation is given in Lemma 5.1 of [DSV15], but we can give a heuristic justification of it, by saying that for small $X$ the boundary point gets close to the origin, where there is a corner and the curvature blows up (with a negative sign, since there is "more than a hyperplane" contained in the set), see Figure 9. Also, the power $2s$ in (5.1) follows by scaling.

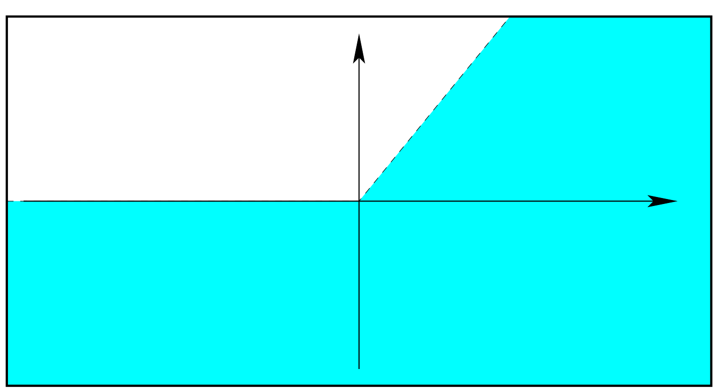

FIGURE 9. *Description of Step 1.*

In addition, if $\ell$ is close to $0$, this first barrier is close to a ninety degree angle, while if $\ell$ is large it is close to a flat line, and these considerations are also in agreement with (5.1).

*Step 2.* Having understood in Step 1 what happens for the "angles", now we would like to "shift iteratively in a smooth way from one slope to another", see Figure 10.

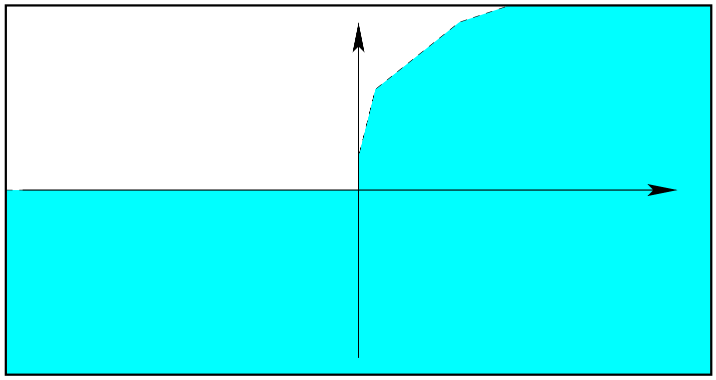

FIGURE 10. *Description of Step 2.*

The detailed statement is given in Proposition 5.3 in [DSV15], but the idea is as follows. For any $K \in \mathbb{N}$, $K \geqslant 1$, one looks at the subgraph of a nonnegative function $v_K$ such that

- $v_K(x) = 0$ if $x < 0$,
- $v_K(x) \geqslant a_K$ if $x > 0$, for some $a_K > 0$,
- $v_K(x) = \frac{x+q_K}{\ell_K}$ for any $x \geqslant \ell_K - q_K$, for some $\ell_K \geqslant K$ and $q_K \in \left[0, \frac{1}{K}\right]$,
- at all the boundary points $X = (x, y)$ with positive abscissa $x > 0$, the fractional mean curvature is at most $-\frac{c}{\ell_K |X|^{2s}}$, for some $c > 0$.

*Step 3.* If $K$ is sufficiently large in Step 2, the final slope is almost horizontal. In this case, one can smoothly glue such barrier with a power like function like $x^{\frac{1}{2}+s+\varepsilon_0}$. Here, $\varepsilon_0$ is any fixed positive exponent (the power $\gamma$ in the statement of Theorem 1.12 is related to $\varepsilon_0$, since $\gamma := \frac{2+\varepsilon_0}{1-2s}$). The details of the barrier constructed in this way are given in Proposition 6.3 of [DSV15]. In this case, one can still control the fractional mean curvature at all the boundary points $X = (x, y)$ with positive abscissa $x > 0$, but the estimate is of the type either $|X|^{-2s}$, for small $|X|$, or $|X|^{-\frac{1}{2}-s+\varepsilon_0}$, for large $|X|$. A sketch of such barrier is given in Figure 11.

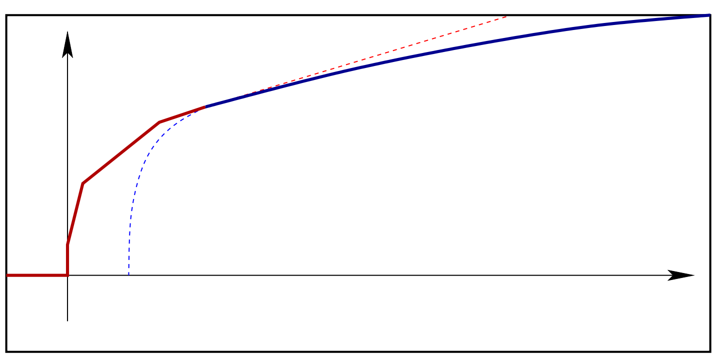

FIGURE 11. *Description of Step 3.*

*Step 4.* Now we use the barrier of Step 3 to construct a compactly supported object. The idea is to take such barrier, to reflect it and to glue it at a "horizontal level", see Figure 12.

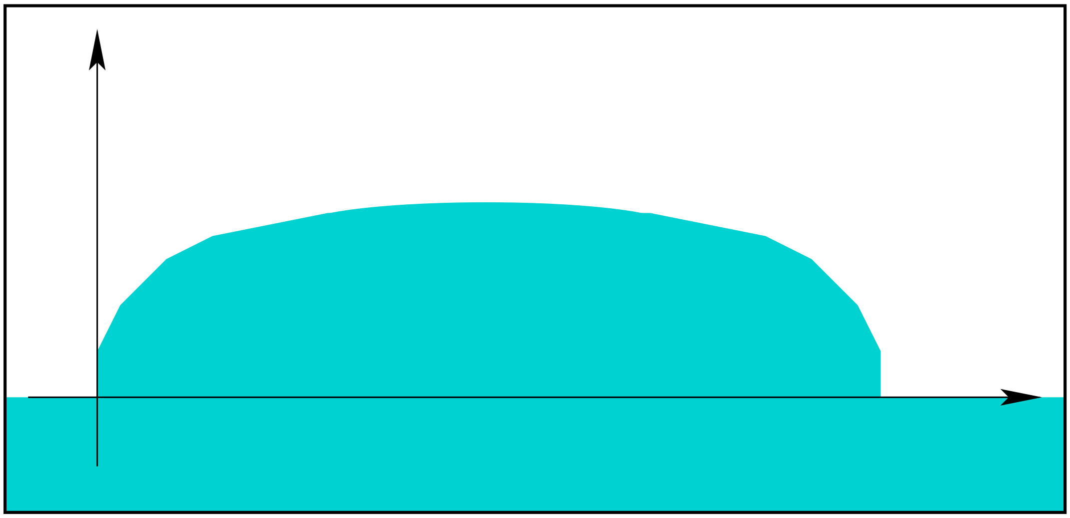

FIGURE 12. *Description of Step 4.*

We remark that such barrier has a vertical portion at the origin and one can control its fractional mean curvature from above with a negative quantity for the boundary points $X = (x, y)$ with positive, but not too large, abscissa.

Of course, this type of estimate cannot hold at the maximal point of the barrier, where "more than a hyperplane" is contained in the complement of the set, and therefore the fractional mean curvature is positive (the precise quantitative estimate is given in Proposition 7.1. of [DSV15]).

*Step 5.* Nevertheless, we can now compensate this error in the fractional mean curvature near the maximal point of the barrier by adding two suitably large domains on the sides of the barriers, see Figure 13.

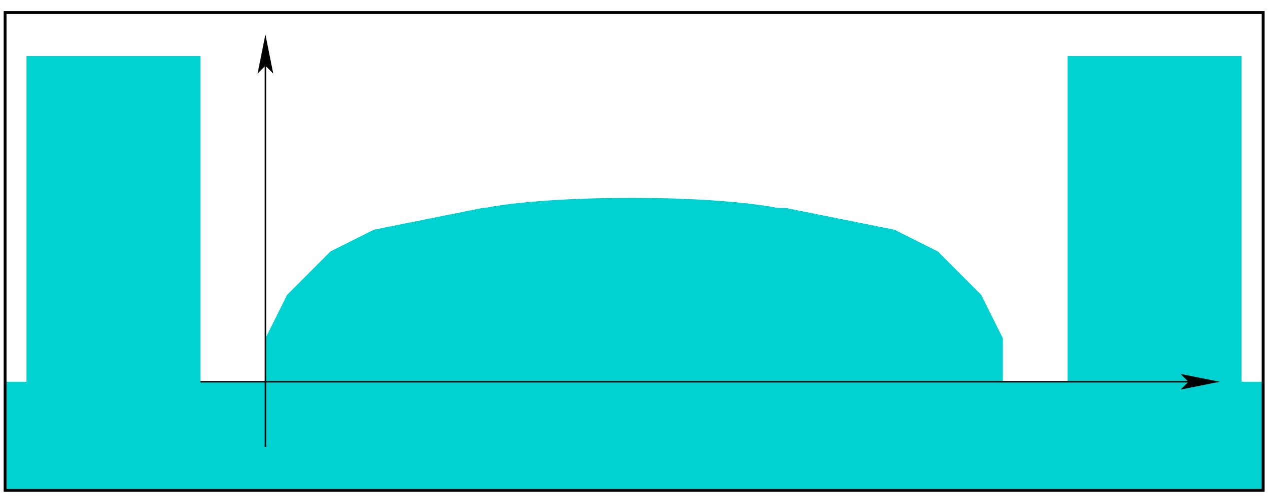

FIGURE 13. *Description of Step 5.*

The barrier constructed in this way is described in details in Proposition 7.3 of [DSV15] and its basic feature is to possess a vertical portion near the origin and to possess negative fractional mean curvature.

By keeping good track of the quantitative estimates on the bumps of the barriers and on their fractional mean curvatures, one can now scale the latter barrier and slide it from below, in order to prove Theorem 1.12. The full details are given in Section 8 of [DSV15].

## APPENDIX A. A SKETCHY DISCUSSION ON THE ASYMPTOTICS OF THE $s$-PERIMETER

In this appendix, we would like to emphasize the fact that, as $s \nearrow 1/2$, the $s$-perimeter recovers (under different perspectives) the classical perimeter, while, as $s \searrow 0$, the nonlocal features become predominant and the problem produces the Lebesgue measure âĂŤ or, better to say, convex combinations of Lebesgue measures by interpolation parameters of nonlocal type.

First of all, we show that if $E$ is a bounded set with smooth boundary, then

$$\lim_{s \nearrow 1/2} (1-2s)\,\mathrm{Per}_s\,(E, \mathbb{R}^n) = \kappa_{n-1}\,\mathrm{Per}\,(E, \mathbb{R}^n), \tag{A.1}$$

where we denoted by $\kappa_n$ the $n$-dimensional volume of the $n$-dimensional unit ball.

For further convenience, we also use the notation

$$\varpi_n := \mathcal{H}^{n-1}(S^{n-1}).$$

Notice that, by polar coordinates,

$$\kappa_n = \int_{S^{n-1}} \left[ \int_0^1 \rho^{n-1}\, d\rho \right] d\mathcal{H}^{n-1}(x) = \frac{\varpi_n}{n}. \tag{A.2}$$

We point out that formula (A.1) is indeed a simple version of more general approximation results, for which we refer to [BBM02, Dáv02, ADPM11, Pon04, CV11] and to [CV13] for the regularity results that can be achieved by approximation methods.

The proof of (A.1) can be performed by different methods; here we give a simple argument which uses formula (1.13). To this aim, we fix $x \in \partial E$ and $\delta > 0$. If $y \in (\partial E) \cap B_\delta(x)$ and $\delta$ is sufficiently small, then $\nu(y) = \nu(x) + O(\delta)$. Moreover, for any $\varrho \in (0, \delta]$, the $(n-2)$-dimensional contribution of $\partial E$ in $\partial B_\varrho(x)$ coincides, up to higher orders in $\delta$, with the one of the $(n-2)$-dimensional sphere, that is $\varpi_{n-1}\,\varrho^{n-2}$, see Figure 14.

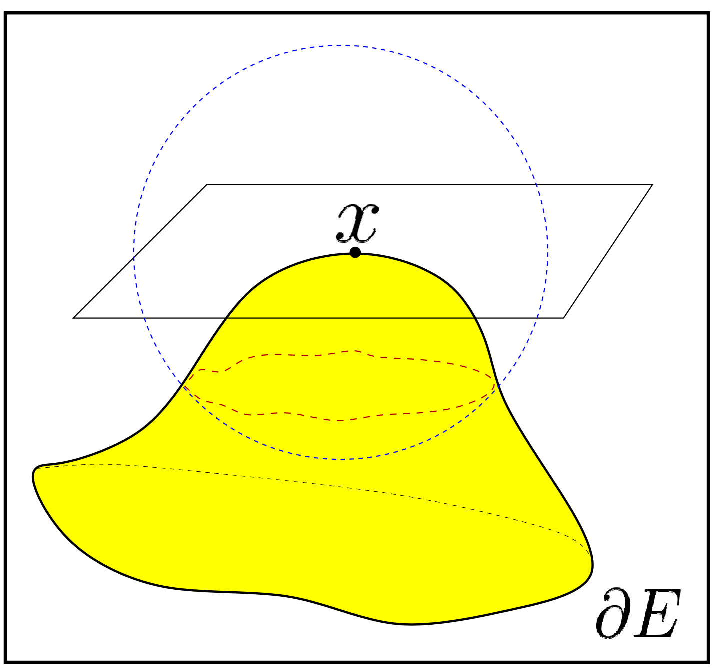


FIGURE 14. $\mathcal{H}^{n-2}\big((\partial E) \cap \partial B_\varrho(x)\big)$ *(in the picture, $n = 3$).*

As a consequence of these observations, we have that

$$\begin{aligned}
&\int_{(\partial E)\cap B_\delta(x)} \frac{\nu(x)\cdot\nu(y)}{|x-y|^{n+2s-2}}\,d\mathcal{H}^{n-1}(y) = \int_{(\partial E)\cap B_\delta(x)} \frac{1+O(\delta)}{|x-y|^{n+2s-2}}\,d\mathcal{H}^{n-1}(y)\\
&\quad= \big(1+O(\delta)\big) \int_0^\delta \frac{\mathcal{H}^{n-2}\big((\partial E)\cap(\partial B_\varrho)\big)}{\varrho^{n+2s-2}}\,d\varrho\\
&\quad= \big(1+O(\delta)\big)\,\varpi_{n-1} \int_0^\delta \frac{\varrho^{n-2}}{\varrho^{n+2s-2}}\,d\varrho\\
&\quad= \frac{\big(1+O(\delta)\big)\,\varpi_{n-1}\,\delta^{1-2s}}{1-2s}.
\end{aligned}$$

On the other hand,

$$\int_{(\partial E)\setminus B_\delta(x)} \frac{\nu(x)\cdot\nu(y)}{|x-y|^{n+2s-2}}\,d\mathcal{H}^{n-1}(y) \leqslant \frac{\mathcal{H}^{n-1}(\partial E)}{\delta^{n+2s-2}}.$$

Therefore

$$\int_{\partial E} \frac{\nu(x)\cdot\nu(y)}{|x-y|^{n+2s-2}}\,d\mathcal{H}^{n-1}(y) = \frac{\big(1+O(\delta)\big)\,\varpi_{n-1}\,\delta^{1-2s}}{1-2s} + O(\delta^{-n-2s+2}).$$

Accordingly, recalling (1.13),

$$\begin{aligned}
&\lim_{s\nearrow 1/2} (1-2s)\,\mathrm{Per}_s\,(E,\mathbb{R}^n)\\
&\quad= \lim_{s\nearrow 1/2} \frac{1-2s}{2s\,(n+2s-2)} \int_{\partial E} \left[\int_{\partial E} \frac{\nu(x)\cdot\nu(y)}{|x-y|^{n+2s-2}}\,d\mathcal{H}^{n-1}(y)\right] d\mathcal{H}^{n-1}(x)\\
&\quad= \lim_{s\nearrow 1/2} \frac{1-2s}{n-1} \int_{\partial E} \left[\frac{\big(1+O(\delta)\big)\,\varpi_{n-1}\,\delta^{1-2s}}{1-2s} + O(\delta^{-n-2s+2})\right] d\mathcal{H}^{n-1}(x)\\
&\quad= \lim_{s\nearrow 1/2} \frac{\big(1+O(\delta)\big)\,\varpi_{n-1}\,\delta^{1-2s} + (1-2s)\,O(\delta^{-n-2s+2})}{n-1}\,\mathcal{H}^{n-1}(\partial E)\\
&\quad= \frac{\big(1+O(\delta)\big)\,\varpi_{n-1}}{n-1}\,\mathcal{H}^{n-1}(\partial E).
\end{aligned}$$

Hence, by taking $\delta$ arbitrarily small,

$$\lim_{s\nearrow 1/2} (1-2s)\,\mathrm{Per}_s\,(E,\mathbb{R}^n) = \frac{\varpi_{n-1}}{n-1}\,\mathcal{H}^{n-1}(\partial E),$$

which gives (A.1), in view of (A.2).

Now we show that, if $n\geqslant 3$ and $E$ is a bounded set with smooth boundary,

$$\lim_{s\searrow 0} s\,\mathrm{Per}_s\,(E,\mathbb{R}^n) = \frac{\varpi_n}{2}\,|E|. \tag{A.3}$$

Once again, more general (and subtle) statements hold true, see [MS02, DFPV13] for details.

To prove (A.3), we denote by

$$\Gamma(x) := \frac{1}{(n-2)\,\varpi_n\,|x|^{n-2}}$$

the fundamental solution[4] of the Laplace operator when $n \geqslant 3$, that is

$$-\Delta\Gamma(x) = \delta_0(x),$$

where $\delta_0$ is the Dirac's Delta centered at the origin. Then, from (1.13),

$$\begin{aligned}
\lim_{s\searrow 0} s\,\mathrm{Per}_s\,(E,\mathbb{R}^n) &= \frac{1}{2\,(n-2)}\int_{\partial E}\left[\int_{\partial E}\frac{\nu(x)\cdot\nu(y)}{|x-y|^{n-2}}\,d\mathcal{H}^{n-1}(y)\right]\,d\mathcal{H}^{n-1}(x)\\
&= \frac{\varpi_n}{2}\int_{\partial E}\left[\int_{\partial E}\nu(y)\cdot\big(\nu(x)\Gamma(x-y)\big)\,d\mathcal{H}^{n-1}(y)\right]\,d\mathcal{H}^{n-1}(x)\\
&= \frac{\varpi_n}{2}\int_{\partial E}\left[\int_{E}\mathrm{div}_y\big(\nu(x)\Gamma(x-y)\big)\,dy\right]\,d\mathcal{H}^{n-1}(x)\\
&= \frac{\varpi_n}{2}\int_{\partial E}\left[\int_{E}\nu(x)\cdot\nabla_y\Gamma(x-y)\,dy\right]\,d\mathcal{H}^{n-1}(x)\\
&= \frac{\varpi_n}{2}\int_{E}\left[\int_{\partial E}\nu(x)\cdot\nabla_y\Gamma(x-y)\,d\mathcal{H}^{n-1}(x)\right]\,dy\\
&= \frac{\varpi_n}{2}\int_{E}\left[\int_{E}\mathrm{div}_x\,\big(\nabla_y\Gamma(x-y)\big)\,dx\right]\,dy\\
&= -\frac{\varpi_n}{2}\iint_{E\times E}\Delta\Gamma(x-y)\,dx\,dy\\
&= \frac{\varpi_n}{2}\iint_{E\times E}\delta_0(x-y)\,dx\,dy\\
&= \frac{\varpi_n}{2}\int_{E}1\,dy\\
&= \frac{\varpi_n}{2}\,|E|,
\end{aligned}$$

that is (A.3).

We remark that formula (A.3) is actually a particular case of a more general phenomenon, described in [DFPV13]. For instance, if the following limit exists

$$a(E) := \lim_{s\searrow 0}\frac{2s}{\varpi_n}\int_{E\setminus B_1}\frac{dx}{|x|^{n+2s}},$$

then

$$\lim_{s\searrow 0}\frac{2s}{\varpi_n}\,\mathrm{Per}_s\,(E,\Omega) = (1-a(E))\,|E\cap\Omega| + a(E)\,|\Omega\setminus E|. \tag{A.4}$$

[4]It is interesting to understand how the fundamental solution of the Laplacian also occurs when $n=2$. In this case, we observe that if $c_E := \int_{\partial E}\nu(y)\,d\mathcal{H}^{n-1}(y)$, then of course

$$\iint_{(\partial E)\times(\partial E)}\nu(x)\cdot\nu(y)\,d\mathcal{H}^{n-1}(x)\,d\mathcal{H}^{n-1}(y) = \int_{\partial E}\nu(x)\cdot c_E\,d\mathcal{H}^{n-1}(x) = \int_E \mathrm{div}_x c_E\,dx = \int_E 0\,dx = 0.$$

Hence, we write

$$\frac{1}{|x-y|^{2s}} = \exp\left(-2s\log|x-y|\right) = 1 - 2s\log|x-y| + O(s^2),$$

thus

$$\frac{1}{2s}\iint_{(\partial E)\times(\partial E)}\frac{\nu(x)\cdot\nu(y)}{|x-y|^{2s}}\,d\mathcal{H}^{n-1}(x)\,d\mathcal{H}^{n-1}(y) = -\iint_{(\partial E)\times(\partial E)}\nu(x)\cdot\nu(y)\,\log|x-y|\,d\mathcal{H}^{n-1}(x)\,d\mathcal{H}^{n-1}(y) + O(s)$$

and one can use the same fundamental solution trick as in the case $n \geqslant 3$.

Notice indeed that (A.3) is a particular case of (A.4), since when $E$ is bounded, then $a(E)=0$. Equation (A.4) has also a suggestive interpretation, since it says that, in a sense, as $s\searrow 0$, the fractional perimeter is a convex interpolation of measure contributions inside the reference set $\Omega$: namely it weights the measures of two contributions of $E$ and the complement of $E$ inside $\Omega$ by a convex parameter $a(E)\in[0,1]$ which in turn takes into account the behavior of $E$ at infinity.

## APPENDIX B. A SKETCHY DISCUSSION ON THE ASYMPTOTICS OF THE $s$-MEAN CURVATURE

As $s\nearrow 1/2$, the $s$-mean curvature recovers the classical mean curvature (see [AV14] for details).

A very natural question raised to us by Jun-Cheng Wei dealt with the asymptotics as $s\searrow 0$ of the $s$-mean curvature. Notice that, by (A.3), we know that $2s$ times the $s$-perimeter approaches $\varpi_n$ times the volume. Since the variation of the volume along normal deformations is $1$, if one is allowed to "exchange the limits" (i.e. to identify the limit of the variation with the variation of the limit), then she or he may guess that $2s$ times the $s$-mean curvature should approach $\varpi_n$.

This is indeed the case, and higher orders can be computed as well, according to the following observation: if $E$ has smooth boundary, $p\in\partial E$ and $E\subseteq B_R(p)$ for some $R>0$, then

$$2s\,H^s_E(p)=\varpi_n+2s\left(\int_{B_R(p)}\frac{\chi_{E^c}(x)-\chi_E(x)}{|x-p|^n}\,dx-\varpi_n\,\log R\right)+o(s), \tag{B.1}$$

as $s\searrow 0$. To prove this, we first observe that, up to a translation, we can take $p=0$. Moreover, since $E$ lies inside $B_R$,

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_R}\frac{\chi_{E^c}(x)-\chi_E(x)}{|x|^{n+2s}}\,dx&=\int_{\mathbb{R}^n\setminus B_R}\frac{dx}{|x|^{n+2s}}\\&=\frac{\varpi_n}{2s\,R^{2s}}=\frac{\varpi_n}{2s}\,\exp(-2s\,\log R)\\&=\frac{\varpi_n}{2s}\big(1-2s\,\log R+o(s)\big).\end{aligned} \tag{B.2}$$

In addition, since $\partial E$ is smooth, we have that (possibly after a rotation) there exists $\delta_o\in\big(0,\,\min\{1,R\}\big)$ such that, for any $\delta\in(0,\delta_o]$, $E\cap B_\delta$ contains $\{x_n\leqslant -M|x'|^2\}$ and is contained in $\{x_n\leqslant M|x'|^2\}$ (here, $M>0$ only depends on the curvatures of $E$). Therefore, we have that $\chi_{E^c}(x)-\chi_E(x)=-1$ for any $x\in B_\delta\cap\{x_n\leqslant -M|x'|^2\}$ and $\chi_{E^c}(x)-\chi_E(x)=1$ for any $x\in B_\delta\cap\{x_n\geqslant M|x'|^2\}$. In this way, a cancellation gives that

$$\int_{B_\delta\cap\{|x_n|\geqslant M|x'|^2\}}\frac{\chi_{E^c}(x)-\chi_E(x)}{|x|^{n+2s}}\,dx=0.$$

As a consequence, for any $\sigma\in[0,s]$, if $s\in(0,\,1/4)$,

$$\begin{aligned}\left|\int_{B_\delta}\frac{\chi_{E^c}(x)-\chi_E(x)}{|x|^{n+2\sigma}}\,dx\right|&\leqslant\int_{\{|x'|\leqslant\delta\}}dx'\int_{\{|x_n|\leqslant M\,|x'|^2\}}dx_n\,\frac{1}{|x|^{n+2\sigma}}\\&\leqslant 2M\int_{\{|x'|\leqslant\delta\}}\frac{|x'|^2}{|x'|^{n+2\sigma}}\,dx'\leqslant\frac{2M\,\varpi_n\,\delta^{1-2\sigma}}{1-2\sigma}\leqslant 4M\,\varpi_n\,\delta^{1/2}.\end{aligned}$$

Therefore, we use this inequality with $\sigma := 0$ and $\sigma := s$ and the Dominated Convergence Theorem, to find that

$$\begin{aligned}
& \lim_{s\searrow 0}\left|\int_{B_R} \frac{\chi_{E^c}(x)-\chi_E(x)}{|x|^n}\,dx - \int_{B_R} \frac{\chi_{E^c}(x)-\chi_E(x)}{|x|^{n+2s}}\,dx\right| \\
\leqslant\; & \lim_{s\searrow 0}\left|\int_{B_R\setminus B_\delta} \frac{\chi_{E^c}(x)-\chi_E(x)}{|x|^n}\,dx - \int_{B_R\setminus B_\delta} \frac{\chi_{E^c}(x)-\chi_E(x)}{|x|^{n+2s}}\,dx\right| + 8M\,\varpi_n\,\delta^{1/2} \\
=\; & 8M\,\varpi_n\,\delta^{1/2}.
\end{aligned}$$

Hence, since we can now take $\delta$ arbitrarily small, we conclude that

$$\lim_{s\searrow 0}\left|\int_{B_R} \frac{\chi_{E^c}(x)-\chi_E(x)}{|x|^n}\,dx - \int_{B_R} \frac{\chi_{E^c}(x)-\chi_E(x)}{|x|^{n+2s}}\,dx\right| = 0.$$

In view of this, and recalling (1.14) and (B.2), we find that

$$\begin{aligned}
& \lim_{s\searrow 0}\frac{1}{s}\left|2s\,H^s_E(0)-\varpi_n-2s\left(\int_{B_R} \frac{\chi_{E^c}(x)-\chi_E(x)}{|x|^n}\,dx-\varpi_n\,\log R\right)\right| \\
\leqslant\; & \lim_{s\searrow 0}\frac{1}{s}\left|2s\int_{\mathbb{R}^n} \frac{\chi_{E^c}(x)-\chi_E(x)}{|x|^{n+2s}}\,dx-\varpi_n-2s\left(\int_{B_R} \frac{\chi_{E^c}(x)-\chi_E(x)}{|x|^{n+2s}}\,dx-\varpi_n\,\log R\right)\right| \\
& \qquad +2\left|\int_{B_R} \frac{\chi_{E^c}(x)-\chi_E(x)}{|x|^n}\,dx - \int_{B_R} \frac{\chi_{E^c}(x)-\chi_E(x)}{|x|^{n+2s}}\,dx\right| \\
=\; & \lim_{s\searrow 0}\frac{1}{s}\left|\varpi_n\big(1-2s\,\log R+o(s)\big)-\varpi_n+2s\varpi_n\,\log R\right| \\
=\; & 0.
\end{aligned}$$

This proves (B.1).

## Appendix C. Second variation formulas and graphs of zero nonlocal mean curvature

In this appendix, we show that the second variation of the fractional perimeter of surfaces with vanishing mean curvature is given by

$$-2\int_{\partial E} \frac{\eta(y)-\eta(x)}{|x-y|^{n+2s}}\,d\mathcal{H}^{n-1}(y) + \int_{\partial E} \frac{\eta(x)\,\big[1-\nu(x)\cdot\nu(y)\big]}{|x-y|^{n+2s}}\,d\mathcal{H}^{n-1}(y).$$

This expression is related with the Jacobi field along surfaces of vanishing nonlocal mean curvature. We refer to [DdW] for full details about this type of formulas (see in particular formula (1.6) there, which gives the details of this formula, Lemma A.2 there, which shows that, as $s \nearrow 1/2$, the first integral approaches the Laplace-Beltrami operator and Lemma A.4 there, which shows that the latter integral produces, as $s \nearrow 1/2$, the norm squared of the second fundamental form, in agreement with the classical case).

Here, for simplicity, we reduce to the case in which $E$ is a graph and we consider a small normal deformation of its boundary, plus an additional small translation, and we write the resulting manifold as an appropriate normal deformation. The details go as follows:

**Lemma C.1.** *Let $\Sigma \subset \mathbb{R}^n$ be a graph of class $C^2$, and let $E$ be the corresponding epigraph. Let $\nu = (\nu_1, \dots, \nu_n)$ be the exterior normal of $\Sigma = \partial E$.*

*Given $\varepsilon > 0$ and $\bar{x} \in \Sigma$, we set*

(C.1) $$\Sigma^*_\varepsilon := \{x + \varepsilon\eta(x)\,\nu(x) - \varepsilon\eta(\bar{x})\,\nu(\bar{x}),\ x \in \Sigma\}.$$

*Then, if $\varepsilon$ is sufficiently small, $\Sigma^*_\varepsilon$ is a graph, with epigraph a suitable $E^*_\varepsilon$, with $\bar x \in \partial E^*_\varepsilon$, and*

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\big(H^s_E(\bar x)-H^s_{E^*_\varepsilon}(\bar x)\big) &= \int_\Sigma \frac{\eta(y)-\eta(\bar x)\,\nu(\bar x)\cdot\nu(y)}{|\bar x-y|^{n+2s}}\,d\mathcal{H}^{n-1}(y)\\ &= \int_\Sigma \frac{\eta(y)-\eta(\bar x)}{|\bar x-y|^{n+2s}}\,d\mathcal{H}^{n-1}(y)+\int_\Sigma \frac{\eta(\bar x)\,\big[1-\nu(\bar x)\cdot\nu(y)\big]}{|\bar x-y|^{n+2s}}\,d\mathcal{H}^{n-1}(y).\end{aligned}$$

*Proof.* We denote by $\gamma:\mathbb{R}^{n-1}\to\mathbb{R}$ the graph of class $C^2$ that describes $\Sigma$. In this way, we can write $E=\{x_n<\gamma(x')\}$ and

$$\nu(x)=\nu\big(x',\gamma(x')\big)=\frac{\big(-\nabla\gamma(x'),1\big)}{\sqrt{1+|\nabla\gamma(x')|^2}}.$$

We also write $\kappa=(\kappa',\kappa_n):=\eta(\bar x)\,\nu(\bar x)$. Then

$$\begin{aligned}\Sigma^*_\varepsilon &= \left\{\big(x',\gamma(x')\big)+\varepsilon\eta\big(x',\gamma(x')\big)\,\frac{\big(-\nabla\gamma(x'),1\big)}{\sqrt{1+|\nabla\gamma(x')|^2}}-\varepsilon\kappa,\ x'\in\mathbb{R}^{n-1}\right\}\\ &= \left\{\left(x'-\varepsilon\kappa'-\frac{\varepsilon\eta\big(x',\gamma(x')\big)\nabla\gamma(x')}{\sqrt{1+|\nabla\gamma(x')|^2}},\ \gamma(x')-\varepsilon\kappa_n+\frac{\varepsilon\eta\big(x',\gamma(x')\big)}{\sqrt{1+|\nabla\gamma(x')|^2}}\right),\ x'\in\mathbb{R}^{n-1}\right\}.\end{aligned}$$

So we define

$$y'=y'(x'):=x'-\varepsilon\kappa'-\frac{\varepsilon\eta\big(x',\gamma(x')\big)\nabla\gamma(x')}{\sqrt{1+|\nabla\gamma(x')|^2}}. \tag{C.2}$$

Notice that, if $\varepsilon$ is sufficiently small

$$\det\frac{\partial y'(x')}{\partial x'}\neq 0.$$

Moreover, $|\nabla\gamma(x')|\leqslant 1+|\nabla\gamma(x')|^2$ and therefore

$$|y'(x)|\geqslant |x'|-\varepsilon\kappa'-\varepsilon\to+\infty \text{ as } |x|\to+\infty.$$

Hence, by the Global Inverse Function Theorem (see e.g. Corollary 4.3 in [Pal59]), we have that $y'$ is a global diffeomorphism of class $C^2$ of $\mathbb{R}^{n-1}$, with inverse diffeomorphism $x'=x'(y')$. Thus, we obtain

$$\Sigma^*_\varepsilon=\left\{\left(y',\ \gamma\big(x'(y')\big)-\varepsilon\kappa_n+\frac{\varepsilon\eta\big(x'(y'),\gamma\big(x'(y')\big)\big)}{\sqrt{1+\big|\nabla\gamma\big(x'(y')\big)\big|^2}}\right),\ y'\in\mathbb{R}^{n-1}\right\}.$$

This is clearly a graph, whose corresponding epigraph can be written as $E^*_\varepsilon=\{y_n<\gamma^*_\varepsilon(y')\}$, with

$$\gamma^*_\varepsilon(y'):=\gamma\big(x'(y')\big)-\varepsilon\kappa_n+\frac{\varepsilon\eta\big(x'(y'),\gamma\big(x'(y')\big)\big)}{\sqrt{1+\big|\nabla\gamma\big(x'(y')\big)\big|^2}}.$$

By (C.2), we have that $y'(x'_0)=x'_0$, therefore $\gamma^*_\varepsilon(\bar x')=\gamma(\bar x')$ and so $\bar x\in\partial E^*_\varepsilon$. We also notice that

$$\begin{aligned}\gamma^*_\varepsilon(y') &= \gamma(y')+\nabla\gamma(y')\cdot\big(x'(y)-y'\big)-\varepsilon\kappa_n+\frac{\varepsilon\eta(y',\gamma(y'))}{\sqrt{1+|\nabla\gamma(y')|^2}}+\varepsilon^2R(y')\\ &= \gamma(y')+\nabla\gamma(y')\cdot\left(\varepsilon\kappa'+\frac{\varepsilon\eta(y',\gamma(y'))\nabla\gamma(y')}{\sqrt{1+|\nabla\gamma(y')|^2}}\right)-\varepsilon\kappa_n+\frac{\varepsilon\eta(y',\gamma(y'))}{\sqrt{1+|\nabla\gamma(y')|^2}}+\varepsilon^2R(y')\\ &= \gamma(y')+\varepsilon\sqrt{1+|\nabla\gamma(y')|^2}\left(\eta(y',\gamma(y'))-\kappa\cdot\frac{(\nabla\gamma(y'),-1)}{\sqrt{1+|\nabla\gamma(y')|^2}}\right)+\varepsilon^2R(y')\end{aligned}$$

for a suitable remainder functions $R$ (possibly varying from line to line), that are bounded if so is $|D^2\gamma|$. Accordingly,

$$\begin{aligned} E^*_\varepsilon \setminus E &= \{\gamma(y') \leqslant y_n < \gamma^*_\varepsilon(y')\} \\ &= \left\{\gamma(y') \leqslant y_n < \gamma(y') + \varepsilon\big(\Xi(y') + \varepsilon^2 R(y')\big)^+\right\}, \end{aligned}$$

where

$$\Xi(y') := \sqrt{1+|\nabla\gamma(y')|^2}\;\big(\eta(y',\gamma(y')) - \kappa\cdot\tilde\nu(y')\big)$$

and $$\tilde\nu(y') := \frac{\big(\nabla\gamma(y'),-1\big)}{\sqrt{1+|\nabla\gamma(y')|^2}}.$$

Notice that $\tilde\nu(y') = \nu\big(y',\gamma(y')\big)$. Similarly,

$$E\setminus E^*_\varepsilon \subseteq \left\{\gamma(y') - \varepsilon\big(\Xi(y') + \varepsilon^2 R(y')\big)^- \leqslant y_n < \gamma(y')\right\}.$$

Therefore

$$\begin{aligned} \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{E^*_\varepsilon\setminus E}\frac{dy}{|\bar x - y|^{n+2s}} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\mathbb{R}^{n-1}}\left[\int_{\gamma(y')}^{\gamma(y')+\varepsilon\,(\Xi(y')+\varepsilon R(y'))^+}\frac{dy_n}{|\bar x - y|^{n+2s}}\right]dy' \\ &= \int_{\mathbb{R}^{n-1}}\frac{\Xi^+(y')}{\big(|\bar x' - y'|^2 + |\bar x_n - \gamma(y')|^2\big)^{\frac{n+2s}{2}}}\,dy' \end{aligned}$$

and, similarly

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{E\setminus E^*_\varepsilon}\frac{dy}{|\bar x - y|^{n+2s}} = \int_{\mathbb{R}^{n-1}}\frac{\Xi^-(y')}{\big(|\bar x' - y'|^2 + |\bar x_n - \gamma(y')|^2\big)^{\frac{n+2s}{2}}}\,dy'.$$

As a consequence,

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\big(H^s_E(\bar x) - H^s_{E^*_\varepsilon}(\bar x)\big) \\ =\;& \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_{E^*_\varepsilon\setminus E}\frac{dy}{|\bar x - y|^{n+2s}} - \int_{E\setminus E^*_\varepsilon}\frac{dy}{|\bar x - y|^{n+2s}}\right] \\ =\;& \int_{\mathbb{R}^{n-1}}\frac{\Xi(y')}{\big(|\bar x' - y'|^2 + |\bar x_n - \gamma(y')|^2\big)^{\frac{n+2s}{2}}}\,dy' \\ =\;& \int_{\mathbb{R}^{n-1}}\sqrt{1+|\nabla\gamma(y')|^2}\,\frac{\eta(y',\gamma(y')) - \kappa\cdot\tilde\nu(y')}{\big(|\bar x' - y'|^2 + |\bar x_n - \gamma(y')|^2\big)^{\frac{n+2s}{2}}}\,dy' \\ =\;& \int_\Sigma\frac{\eta(y) - \kappa\cdot\nu(y)}{|\bar x - y|^{n+2s}}\,d\mathcal{H}^{n-1}(y), \end{aligned}$$

that is the desired result. □

An interesting consequence of Lemma C.1 is that graphs with vanishing nonlocal mean curvature cannot have horizontal normals, as given by the following result:

**Theorem C.2.** *Let $E \subset \mathbb{R}^n$. Suppose that $\partial E$ is globally of class $C^2$ and that $H^s_E(x) = 0$ for any $x \in \partial E$.*

*Let $\nu = (\nu_1(x),\dots,\nu_n(x))$ be the exterior normal of $E$ at $x \in \partial E$.*

*Then $\nu_n(x) \neq 0$, for any $x \in \partial E$.*

To prove Theorem C.2, we first compare deformations and translations of a graph. Namely, we show that a normal deformation of size $\varepsilon\nu_n$ of a graph with normal $\nu = (\nu_1, \dots, \nu_n)$ coincides with a vertical translation of the graph itself, up to order of $\varepsilon^2$. The precise result goes as follows:

**Lemma C.3.** *Let $\Sigma \subset \mathbb{R}^n$ be a graph of class $C^2$ globally, and let $E$ be the corresponding epigraph. Let $\nu = (\nu_1, \dots, \nu_n)$ be the exterior normal of $\Sigma = \partial E$.*

*Given $\varepsilon > 0$, let*

$$\Sigma_\varepsilon := \{x + \varepsilon\nu_n(x)\,\nu(x),\ x \in \Sigma\}. \tag{C.3}$$

*Then, if $\varepsilon$ is sufficiently small, $\Sigma_\varepsilon$ is a graph, for some epigraph $E_\varepsilon$, and there exists a $C^2$-diffeomorphism $\Psi$ of $\mathbb{R}^n$ that is $C\varepsilon^2$-close to the identity in $C^2(\mathbb{R}^n)$, for some $C > 0$, such that*

$$\Psi(E_\varepsilon) = E + \varepsilon e_n.$$

*Proof.* We denote by $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ the graph that describes $\Sigma$. In this way, we can write $E = \{x_n < \gamma(x')\}$ and

$$\nu(x) = \nu\big(x', \gamma(x')\big) = \frac{\big(-\nabla\gamma(x'), 1\big)}{\sqrt{1 + |\nabla\gamma(x')|^2}}.$$

Accordingly,

$$\begin{aligned}\Sigma_\varepsilon &= \left\{\big(x', \gamma(x')\big) + \varepsilon\frac{\big(-\nabla\gamma(x'), 1\big)}{1 + |\nabla\gamma(x')|^2},\ x' \in \mathbb{R}^{n-1}\right\}\\ &= \left\{\left(x' - \varepsilon\frac{\nabla\gamma(x')}{1 + |\nabla\gamma(x')|^2},\ \gamma(x') + \frac{\varepsilon}{1 + |\nabla\gamma(x')|^2}\right),\ x' \in \mathbb{R}^{n-1}\right\}.\end{aligned}$$

To write $\Sigma_\varepsilon$ as a graph, we take as new coordinate

$$y' = y'(x') := x' - \varepsilon\frac{\nabla\gamma(x')}{1 + |\nabla\gamma(x')|^2}. \tag{C.4}$$

Notice that, if $\varepsilon$ is sufficiently small

$$\det\frac{\partial y'(x')}{\partial x'} \neq 0.$$

Moreover, $|\nabla\gamma(x')| \leqslant 1 + |\nabla\gamma(x')|^2$ and therefore

$$|y'(x)| \geqslant |x'| - \varepsilon \to +\infty \text{ as } |x'| \to +\infty.$$

As a consequence, by the Global Inverse Function Theorem (see e.g. Corollary 4.3 in [Pal59]), we have that $y'$ is a global diffeomorphism of class $C^2$ of $\mathbb{R}^{n-1}$, we write $x' = x'(y')$ the inverse diffeomorphism and we have that

$$\Sigma_\varepsilon = \left\{\left(y',\ \gamma\big(x'(y')\big) + \frac{\varepsilon}{1 + \big|\nabla\gamma\big(x'(y')\big)\big|^2}\right),\ y' \in \mathbb{R}^{n-1}\right\}.$$

So we can write the epigraph of $\Sigma_\varepsilon$ as

$$E_\varepsilon = \left\{y_n < \gamma\big(x'(y')\big) + \frac{\varepsilon}{1 + \big|\nabla\gamma\big(x'(y')\big)\big|^2}\right\}.$$

Now we define

$$\Phi(y') := \gamma(y') - \gamma\big(x'(y')\big) + \varepsilon - \frac{\varepsilon}{1 + \big|\nabla\gamma\big(x'(y')\big)\big|^2} \tag{C.5}$$

and $z = \Psi(y) = \Psi(y', y_n) := y + \Phi(y')e_n$. By construction, we have that

$$\Psi(E_\varepsilon) = \big\{z_n < \gamma(z') + \varepsilon\big\} = E + \varepsilon e_n.$$

To complete the proof of Lemma C.3, we need to show that

$$\|\Phi\|_{C^2(\mathbb{R}^n)} \leqslant C\varepsilon^2, \tag{C.6}$$

for some $C > 0$. To this aim, we use (C.4) to see that

$$x' = y' + \varepsilon \frac{\nabla\gamma(y')}{1 + |\nabla\gamma(y')|^2} + \phi_1(y'),$$

with $\|\phi_1\|_{C^2(\mathbb{R}^n)} \leqslant C\varepsilon^2$. Accordingly, by (C.5), we have that

$$\begin{aligned}
\Phi(y') &= \gamma(y') - \gamma\left(y' + \varepsilon \frac{\nabla\gamma(y')}{1 + |\nabla\gamma(y')|^2} + \phi_1(y')\right) + \varepsilon - \frac{\varepsilon}{1 + \left|\nabla\gamma\left(y' + \varepsilon\frac{\nabla\gamma(y')}{1+|\nabla\gamma(y')|^2} + \phi_1(y')\right)\right|^2} \\
&= \gamma(y') - \gamma(y') - \varepsilon\frac{|\nabla\gamma(y')|^2}{1 + |\nabla\gamma(y')|^2} + \varepsilon - \frac{\varepsilon}{1 + |\nabla\gamma(y')|^2} + \phi_2(y') \\
&= \phi_2(y'),
\end{aligned}$$

with $\|\phi_2\|_{C^2(\mathbb{R}^n)} \leqslant C\varepsilon^2$. This proves (C.6), as desired. □

From Lemma C.3 here and Theorem 1.1 in [Coz15], we obtain:

**Corollary C.4.** *In the setting of Lemma C.3, for any $p \in \Sigma_\varepsilon = \partial E_\varepsilon$ we have that*

$$\left|H^s_{E_\varepsilon}(p) - H^s_{E+\varepsilon e_n}\big(\Psi(p)\big)\right| \leqslant C\varepsilon^2,$$

*for some $C > 0$.*

Now we complete the proof of Theorem C.2. To this aim, we observe that

$$\nu_n(x) \geqslant 0 \text{ for any } x \in \partial E, \tag{C.7}$$

since $E$ is a graph. Suppose that, by contradiction,

$$\nu_n(\bar{x}) = 0 \text{ for some } \bar{x} \in \partial E. \tag{C.8}$$

We use this and Lemma C.1 with $\eta := \nu_n$ and we find that

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\big(H^s_E(\bar{x}) - H^s_{E^*_\varepsilon}(\bar{x})\big) = \int_\Sigma \frac{\nu_n(y)}{|\bar{x} - y|^{n+2s}}\, d\mathcal{H}^{n-1}(y). \tag{C.9}$$

Also, comparing (C.1) (with $\eta := \nu_n$) and (C.3), and using again (C.8), we see that $E^*_\varepsilon = E_\varepsilon$ and so Corollary C.4 gives that

$$H^s_{E^*_\varepsilon}(\bar{x}) = H^s_{E+\varepsilon e_n}(\bar{y}) + O(\varepsilon^2),$$

for some $\bar{y} \in \partial E + \varepsilon e_n$. Since $H^s_E$ vanishes, we can use the translation invariance to see that also $H^s_{E+\varepsilon e_n}$ vanishes. So we conclude that

$$H^s_{E^*_\varepsilon}(\bar{x}) = O(\varepsilon^2).$$

These observations and (C.9) imply that

$$\int_\Sigma \frac{\nu_n(y)}{|\bar{x} - y|^{n+2s}}\, d\mathcal{H}^{n-1}(y) = 0.$$

Hence, in view of (C.7), we see that $\nu_n$ must vanish identically along $\Sigma$. This says that $\Sigma$ is a vertical hyperplane, in contradiction with the graph assumption. This ends the proof of Theorem C.2.

## ONLINE LECTURES

There are a few videotaped lectures online which collect some of the material presented in this set of notes. The interest reader may look at

http://www.birs.ca/events/2014/5-day-workshops/14w5017/videos/watch/201405271048-Valdinoci.html

https://www.youtube.com/watch?v=2j2r1ykoyuE

https://www.youtube.com/watch?v=EDJ8uBpYpB4

https://www.youtube.com/watch?v=s_RRzgZ7VcM&list=PLj6jTBBj-5B_Vx5qA-HelhGUnGrCu7SdW&index=7

https://www.youtube.com/watch?v=okXncmRbCZc&index=14&list=PLj6jTBBj-5B_Vx5qA-HelhGUnGrCu7SdW

http://www.fields.utoronto.ca/video-archive/2016/06/2022-15336

http://www.mathtube.org/lecture/video/nonlocal-equations-various-perspectives-lecture-1

http://www.mathtube.org/lecture/video/nonlocal-equations-various-perspectives-lecture-2

http://www.mathtube.org/lecture/video/nonlocal-equations-various-perspectives-lecture-3

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