Oberwolfach Preprints

OWP 2009 - 16

Bernhard H. Haak and Birgit Jacob

Control of Volterra Systems with Scalar Kernels

Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

Oberwolfach Preprints (OWP)

Starting in 2007, the MFO publishes a preprint series which mainly contains research results related to a longer stay in Oberwolfach. In particular, this concerns the Research in Pairs-Programme (RiP) and the Oberwolfach-Leibniz-Fellows (OWLF), but this can also include an Oberwolfach Lecture, for example.

A preprint can have a size from 1 - 200 pages, and the MFO will publish it on its website as well as by hard copy. Every RiP group or Oberwolfach-Leibniz-Fellow may receive on request 30 free hard copies (DIN A4, black and white copy) by surface mail.

Of course, the full copy right is left to the authors. The MFO only needs the right to publish it on its website *www.mfo.de* as a documentation of the research work done at the MFO, which you are accepting by sending us your file.

In case of interest, please send a **pdf file** of your preprint by email to *rip@mfo.de* or *owlf@mfo.de*, respectively. The file should be sent to the MFO within 12 months after your stay as RiP or OWLF at the MFO.

There are no requirements for the format of the preprint, except that the introduction should contain a short appreciation and that the paper size (respectively format) should be DIN A4, "letter" or "article".

On the front page of the hard copies, which contains the logo of the MFO, title and authors, we shall add a running number (20XX - XX).

We cordially invite the researchers within the RiP or OWLF programme to make use of this offer and would like to thank you in advance for your cooperation.

Imprint:

Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

CONTROL OF VOLTERRA SYSTEMS WITH SCALAR KERNELS

BERNHARD H. HAAK AND BIRGIT JACOB

Abstract. Volterra observations systems with scalar kernels are studied. New sufficient conditions for admissibility of observation operators are developed and some examples are discussed.

1. Introduction

Consider the following scalar abstract Volterra system

$$x(t) = f(t) + \int_0^t a(t-s)\, Ax(s)\, ds. \tag{1}$$

Here, the operator A is supposed to be a closed operator with dense domain on a Banach space X having its spectrum contained in some open sectorial region of the complex plane, symmetric to the real axis and open to the left:

$$\sigma(A) \subseteq -\Sigma_\omega \qquad \text{where} \qquad \Sigma_\omega = \{z \in \mathbb{C} : \; |\arg(z)| < \omega\}$$

for some $\omega \in (0, \pi)$. Moreover, the resolvent of A is supposed to satisfy a growth condition of the type $\|\lambda R(\lambda, A)\| \le M$ uniformly on each sector $\Sigma_{\pi-\omega-\varepsilon}$. Typical examples of such operators are generators of bounded strongly continuous semigroups, where $\omega \le \pi/2$. We call $-A$ a *sectorial operator of type* $\omega \in (0, \pi)$, but we mention that 'sectoriality' may have different meanings for different authors in the literature.

The kernel function $a \in L^1_{\text{loc}}$ is supposed to be of sub-exponential growth so that its Laplace transform $\widehat{a}(\lambda)$ exists for all λ with positive real part. The kernel is called *sectorial of angle* $\theta \in (0, \pi)$ if

$$\widehat{a}(\lambda) \in \Sigma_\theta \qquad \text{for all } \lambda \text{ with positive real part.}$$

We will consider only *parabolic* equations (1) in the sense of Pruess [15]. In the case that $-A$ is sectorial of some angle $\omega \in (0, \pi)$ this is equivalent to require $\widehat{a}(\lambda) \neq 0$ and $\frac{1}{\widehat{a}(\lambda)} \in \varrho(A)$ for all λ with positive real part.

In particular, when $-A$ and a are both sectorial in the respective sense with angles that sum up to a constant strictly inferior to π, the Volterra equation is parabolic.

The kernel function is said *k-regular* if there is a constant $K > 0$ such that

$$|\lambda^n \widehat{a}^{(n)}(\lambda)| \le K |\widehat{a}(\lambda)|$$

for all $n = 0, 1 \ldots k$ and all λ with positive real part. In Pruess [15, Theorem I.3.1] it is shown that parabolic equations with a k-regular kernel for $k \ge 1$ admit a unique *solution family*, i.e. a family of bounded linear operators $(S(t))_{t \ge 0}$ on X, such that

(a) $S(0) = I$ and $S(\cdot)$ is strongly continuous on $\mathbb{R}_+$.
(b) $S(t)$ commutes with A, which means $S(t)(D(A)) \subset D(A)$ for all $t \ge 0$, and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \ge 0$.

Date: April 8, 2009.

(c) For all $x \in D(A)$ and all $t \geq 0$ the resolvent equations hold:

$$S(t)x = x + \int_0^t a(t-s)AS(s)x\,ds. \tag{2}$$

Moreover, $S \in C^{k-1}((0,\infty), B(X))$ and $\|t^n S^{(n)}(t)\| \leq K$ for all $n = 0, \dots, k-1$.

The purpose of this article is to present conditions for the admissibility of observation operators to parabolic Volterra equations, that is, we consider the 'observed' system

$$\begin{cases} x(t) = f(t) + \int_0^t a(t-s)\,Ax(s)\,ds \\ y(t) = Cx(t) \end{cases} \tag{V}$$

Additionally to the sectoriality condition on A and the parabolicity condition on the Volterra equation, the operator C in the second line is supposed to be an operator from X into another Banach space Y that acts as a bounded operator from $X_1 \to Y$ where $X_1 = \mathcal{D}(A)$ endowed by the graph norm of A.

Definition 1.1. A bounded linear operator $C : X_1 \to Y$ is called *finite-time admissible* for the Volterra equation (1) if there are constants $\eta, K > 0$ such that

$$\left(\int_0^t \|CS(r)x\|^2\,dr\right)^{1/2} \leq Ke^{\eta t}\|x\|$$

for all $t \geq 0$ and all $x \in \mathcal{D}(A)$.

The Laplace transform of S, denoted by H, is given by

$$H(\lambda)x = \frac{1}{\lambda}(I - \hat{a}(\lambda)A)^{-1}x, \qquad \operatorname{Re}\lambda > 0.$$

The following necessary condition for admissibility was shown in [11].

Proposition 1.2. *If C is a finite-time admissible observation operator for the Volterra equation (1), then there is a constant $M > 0$ such that*

$$\|\sqrt{\operatorname{Re}\lambda}\,CH(\lambda)\| \leq M, \qquad \operatorname{Re}\lambda > 0. \tag{3}$$

In [11] it is shown that (3) is also sufficient for admissibility if X is a Hilbert space, Y is finite-dimensional and A generates a contraction semigroup. However, in general this condition is not sufficient (see e.g. [10]).

We show that the slightly stronger growth condition on the resolvent

$$\sup_{r>0}\left\| \,(1+\log^+ r)^\alpha r^{1/2} CH(r)\,\right\| < \infty, \tag{4}$$

is sufficient for admissibility if $\alpha > 1/2$ (see Theorem 3.4). This result generalizes the sufficient condition of Zwart [16] for Cauchy systems to general Volterra systems (1).

Our second main result, Theorem 3.1 provides a perturbation argument to obtain admissibility for the controlled Volterra equation from the admissibility of the control operator for the underlying Cauchy equation. In the particular case of diagonal semigroups and one-dimensional output spaces Y this improves a direct Carleson measure criterion from Haak, Jacob, Partington and Pott [5].

We proceed as follows. In Section 2 we obtain an integral representation for the solution family $(S(t))_{t\geq 0}$ and several regularity results of the corresponding kernel. Section 3 is devoted to sufficient condition for admissibility of observation operators. A perturbation result as well as a general sufficient condition is obtain. Several examples are included as well.

To enhance readability of the calculations, for rest of this article, K denotes some positive constant that may change from one line to the other unless explicitly quantified.

2. Regularity transfer

The main result of this section is formulated in the following proposition. Let $s(t,\mu)$ denote the solution of the scalar equation

$$s(t,\mu) + \mu \int_0^t a(t-r)s(r,\mu)\,dr = 1 \qquad t>0, \mu\in\mathbb{C}.$$

Proposition 2.1. *Suppose that the kernel $a \in L^1_{loc}(\mathbb{R}_+)$ is 1-regular, sectorial of angle $\theta < \pi/2$. Then there exists a family of functions v_t such that*

$$\mathcal{L}(v_t)(\mu) = s(t,\mu) \quad \text{and} \quad S(t) = \int_0^\infty v_t(s)T(s)\,ds$$

satisfying

(a) $\sup_{t>0} \|v_t\|_{L^1(\mathbb{R}_+)} < \infty$

(b) $\|v_t\|_{L^2(\mathbb{R}_+)} \le Kt^{-\theta/\pi}$ *where K depends only on θ and $C^{reg}_{a,1}$.*

(c) $\|v_t\|_{W^{1,1}} \le K(1+t^{-\frac{2\theta}{\pi}})$.

Moreover, for $\gamma\in[0,1]$, $|\mu^\gamma s(t,\mu)| \le Kt^{-\frac{2\gamma\theta}{\pi}}$

For the proof of this proposition the following two lemmas are needed.

Lemma 2.2. *Suppose $a \in L^1_{loc}(\mathbb{R}_+)$ is 1-regular and sectorial of angle $\theta \le \pi$. Let $\rho_0 := 2\theta/\pi$. Then there exists a constant $c>0$ such that*

$$|\widehat{a}(\lambda)| \ge \begin{cases} c|\lambda|^{-\rho_0} & |\lambda|\ge 1 \\ c|\lambda|^{\rho_0} & |\lambda| \le 1 \end{cases}$$

for all $\lambda \in \mathbb{C}_+$.

Proof. We borrow the argument from the proof of [14, Proposition 1]: we start with the analytic completion of the Poisson formula for the harmonic function $H(\lambda) = \arg\widehat{a}(\lambda)$, that is,

$$\log\widehat{a}(\lambda) = \kappa_0 + \frac{i}{\pi}\int_{-\infty}^{\infty} \left[\frac{1-i\rho\lambda}{\lambda - i\rho}\right] h(i\rho)\frac{d\rho}{1+\rho^2},$$

where $\kappa_0 \in \mathbb{R}$ is a constant. An easy calculation shows

$$|\mathrm{Re}\,\log\widehat{a}(\lambda)| \le \kappa_0 + \rho_0|\log\lambda|$$

for real $\lambda>0$, and thus

$$|\widehat{a}(\lambda)| = e^{\log(|\widehat{a}(\lambda)|)} = e^{\mathrm{Re}\,\log\widehat{a}(\lambda)} \ge \begin{cases} c\lambda^{-\rho_0} & \lambda \ge 1 \\ c\lambda^{\rho_0} & 0\le\lambda\le 1 \end{cases},$$

where $c := e^{-\kappa_0} > 0$. This estimate, together with [15, Lemma 8.1] stating the existence of a constant $c>0$ such that $c^{-1} \le \left|\widehat{a}(|\lambda|)/\widehat{a}(\lambda)\right| \le c$ for all $\lambda\in\mathbb{C}_+$ completes the proof. □

Lemma 2.3. *Let $\theta\in(0,\pi)$. Then there exists $c_\theta>0$ such that*

$$1+|\lambda| \le c_\theta\,|1+\lambda| \tag{5}$$

for all $\lambda\in\Sigma_{\pi-\theta}$.

Proof. Clearly, $\alpha > \widetilde{\alpha}$, see Figure 1. Since $\widetilde{\alpha} = \frac{\theta}{2}$, the assertion follows then from the fact that $\frac{|1+\lambda|}{1+|\lambda|} = \frac{\sin(\alpha)}{\sin(\theta-\alpha)} \ge \sin(\alpha) \ge \sin(\theta/2)$. □

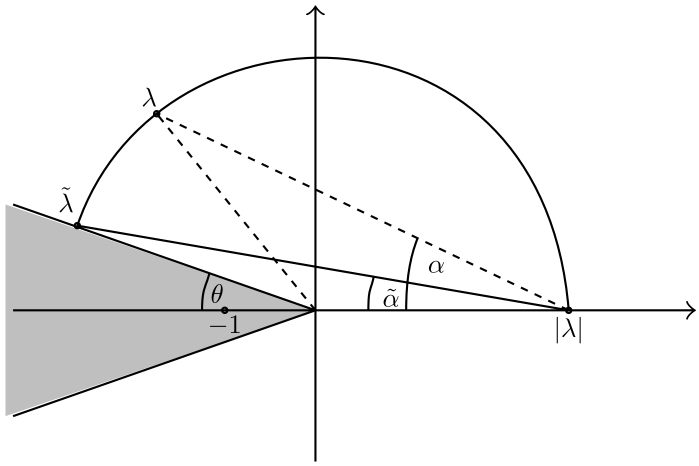

FIGURE 1. Illustration of (5)

Proof of Proposition 2.1. (a) is [15, Proposition I.3.5]

(b) Let $\sigma(\lambda,\mu) = \big(\mathcal{L}s(\cdot,\mu)\big)(\lambda)$, i.e. $\sigma(\lambda,\mu) = \frac{1}{\lambda(1+\mu\widehat{a}(\lambda))}$. Fix $t>0$ and $\varepsilon>0$. Then

$$s(t,\mu) = \tfrac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{\lambda t}\sigma(\lambda,\mu)\,d\lambda.$$

The existence of the integral as an improper Riemann integral is a standard argument using partial integration. Then, by partial integration

$$\begin{aligned} s(t,\mu) &= \lim_{R\to\infty} \tfrac{1}{2\pi i}\Big[\tfrac{1}{t}e^{\lambda t}\sigma(\lambda,\mu)\Big]_{\lambda=\varepsilon-iR}^{\lambda=\varepsilon+iR} - \tfrac{1}{2\pi i}\int_{\varepsilon-iR}^{\varepsilon+iR}\tfrac{1}{t}e^{\lambda t}\frac{d}{d\lambda}\sigma(\lambda,\mu)\,d\lambda \\ &= -\tfrac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\tfrac{1}{t}e^{\lambda t}\frac{d}{d\lambda}\sigma(\lambda,\mu)\,d\lambda \end{aligned}$$

An elementary calculation gives

$$\frac{d}{d\lambda}\frac{1}{\lambda(1+\mu\widehat{a}(\lambda))} = -\frac{1+\mu\widehat{a}(\lambda)\Big(1+\Big(\frac{\lambda\widehat{a}'(\lambda)}{\widehat{a}(\lambda)}\Big)\Big)}{\lambda^2(1+\mu\widehat{a}(\lambda))^2}$$

By 1-sectoriality of the kernel, $\Big|\frac{\lambda\widehat{a}'(\lambda)}{\widehat{a}(\lambda)}\Big| \le C = C_{a,1}^{\mathrm{reg}}$ and so the Lemma yields for any $\delta>0$,

$$\begin{aligned} &\Big(\int_{-\infty}^{\infty}\Big|s(t,\delta+iy)\Big|^2\,dy\Big)^{1/2} \\ \le\;& C_\theta(1+C)\frac{e^{\varepsilon t}}{2\pi t}\Big(\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2+x^2)(1+|\delta+iy||\widehat{a}(\varepsilon+ix)|)}\,dx\Big)^2 dy\Big)^{1/2} \\ \le\;& \sqrt{2}C_\theta(1+C)\frac{e^{\varepsilon t}}{2\pi t}\Big(\int_{0}^{\infty}\Big(\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2+x^2)(1+|y||\widehat{a}(\varepsilon+ix)|)}\,dx\Big)^2 dy\Big)^{1/2} \\ \le\;& \sqrt{2}C_\theta(1+C)\frac{e^{\varepsilon t}}{2\pi t}\int_{-\infty}^{\infty}\Big(\int_{0}^{\infty}\frac{1}{(\varepsilon^2+x^2)^2(1+|y||\widehat{a}(\varepsilon+ix)|)^2}\,dy\Big)^{1/2} dx \end{aligned}$$

$$= \sqrt{2}C_\theta(1+C)\frac{e^{\varepsilon t}}{2\pi t}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2+x^2)|\widehat{a}(\varepsilon+ix)|^{1/2}}\left(\int_0^\infty \frac{1}{(1+u)^2}\,du\right)^{1/2} dx$$
$$= \sqrt{2}C_\theta(1+C)\frac{e^{\varepsilon t}}{2\pi t}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2+x^2)|\widehat{a}(\varepsilon+ix)|^{1/2}}\,dx$$

Now, if $|\widehat{a}(\lambda)| \geq m|\lambda|^{-2\theta/\pi}$, a substitution gives

$$\|s(t,\cdot)\|_{H^2} \leq \widetilde{C}_\theta\frac{e^{\varepsilon t}}{t\varepsilon^{1-\theta/\pi}}$$

Letting $\varepsilon = 1/t$ gives the assertion.

(c) We argue in the same spirit as above: by partial integration

$$\tfrac{d}{d\mu}\Big(\mu s(t,\mu)\Big) = \tfrac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\tfrac{1}{t}e^{\lambda t}\frac{d^2}{d\mu\,d\lambda}\Big(\mu\sigma(\lambda,\mu)\Big)\,d\lambda$$

An elementary calculation gives

$$\frac{d^2}{d\lambda\,d\mu}\frac{\mu\widehat{a}(\lambda)}{(\lambda(1+\mu\widehat{a}(\lambda)))^2} = \frac{1+\mu\widehat{a}(\lambda)\Big(1+2\Big(\frac{\lambda\widehat{a}'(\lambda)}{\widehat{a}(\lambda)}\Big)\Big)}{\lambda^2(1+\mu\widehat{a}(\lambda))^3}$$

By 1-sectoriality of the kernel, $\left|\frac{\lambda\widehat{a}'(\lambda)}{\widehat{a}(\lambda)}\right| \leq C$ and so the Lemma yields any $\delta > 0$,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\Big|\frac{d}{d\mu}\Big(\mu s(t,\delta+iy)\Big)\Big|\,dy\\
\leq\;& C_\theta(1+2C)\frac{e^{\varepsilon t}}{2\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2+x^2)(1+|\delta+iy||\widehat{a}(\varepsilon+ix)|)^2}\,dx\,dy\\
\leq\;& C_\theta(1+2C)\frac{e^{\varepsilon t}}{2\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2+x^2)(1+|y||\widehat{a}(\varepsilon+ix)|)^2}\,dx\,dy\\
=\;& 2C_\theta(1+2C)\frac{e^{\varepsilon t}}{2\pi t}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2+x^2)}\frac{1}{|\widehat{a}(\varepsilon+ix)|}\int_0^\infty\frac{1}{(1+u)^2}\,du\,dx\\
=\;& C_\theta(1+2C)\frac{e^{\varepsilon t}}{2\pi t}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2+x^2)}\frac{1}{|\widehat{a}(\varepsilon+ix)|}\,dx\\
\leq\;& Kt^{-\frac{2\theta}{\pi}}
\end{aligned}$$

by choosing $\varepsilon = 1/t$. This shows that $f_t(\mu) = \frac{d}{d\mu}\Big(\mu s(t,\mu)\Big) \in H^1(\mathbb{C}_+)$. We may apply Hardy's inequality (see e.g. [1, p.198], [7, Theorem 4.2]),

$$\int_0^\infty\frac{|\check{f}_t(r)|}{r}\,dr \leq \tfrac{1}{2}\int_{-\infty}^{\infty}|f_t(i\omega)|\,d\omega$$

so that $\frac{\check{f}_t(r)}{r} \in L^1(\mathbb{R}_+)$ is Laplace transformable for every $t > 0$. Since

$$\mathcal{L}\Big(\frac{\check{f}_t(r)}{r}\Big)(\sigma) = \int_\sigma^\infty f_t(\mu)\,d\mu = \sigma s(t,\sigma),$$

we find that $\mu \mapsto \mu s(t,\mu) \in H^\infty(\mathbb{C}_+)$ with a norm controlled by $t^{-\frac{2\theta}{\pi}}$. This implies that $v_t' \in L^1(\mathbb{R}_+)$. together with (a) the claim follows

Finally, the same technique gives an estimate for the growth of $s(t,\mu)$:

$$\begin{aligned}
\mu^\gamma s(t,\mu) \;&\leq\; K\frac{|\mu|^\gamma e^{\varepsilon t}}{t}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2+r^2)(1+|\mu||\widehat{a}(\varepsilon+ir)|)}\,dr\\
&\leq\; K\frac{e^{\varepsilon t}}{t}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2+r^2)|\widehat{a}(\varepsilon+ir)|^\gamma}\,\frac{|\mu|^\gamma|\widehat{a}(\varepsilon+ir)|^\gamma}{(1+|\mu||\widehat{a}(\varepsilon+ir)|)}\,dr
\end{aligned}$$

$$\begin{aligned} &\leq \quad K\frac{e^{\varepsilon t}}{t}\int_{-\infty}^{\infty}\frac{1}{(\varepsilon^2+r^2)^{1-\frac{\gamma\theta}{\pi}}}\,dr \\ &\stackrel{\varepsilon=\frac{1}{t}}{\leq} \quad Kt^{-\frac{2\gamma\theta}{\pi}}. \end{aligned}$$

□

3. Sufficient conditions for finite-time admissibility

In this section we present the two main results of this paper.

Theorem 3.1. *Let A generate an exponentially stable strongly continuous semigroup $(T(t))_{t\geq 0}$ and let $C : X_1 \to Y$ be bounded. Further we assume that the kernel $a \in L^1_{loc}(\mathbb{R}_+)$ is 1-regular and sectorial of angle $\theta < \pi/2$. Then finite-time admissibility of C for the semigroup $(T(t))_{t\geq 0}$ implies that of C for the solution family $(S(t))_{t\geq 0}$.*

Proof. By Proposition 2.1 there exists a family of functions v_t such that $\|v_t\|_{L^2(\mathbb{R}_+)} \leq Kt^{-\theta/\pi}$ for some constant $K > 0$ independent of $t \geq 0$ and

$$S(t) = \int_0^\infty v_t(r)\, T(r)\, dr, \qquad t \geq 0.$$

For $x \in \mathcal{D}(A)$ we have thus

$$CS(t)x = \int_0^\infty v_t(r)\, CT(r)x\, dr.$$

Note that finite-time admissibility of C for $(T(t))_{t\geq 0}$ implies the existence of a constant $M > 0$ such that

$$\|CT(t)x\|_{L^2(\mathbb{R}_+)} \leq M\|x\|, \qquad x \in \mathcal{D}(A),$$

thanks to the exponential stability of $(T(t))_{t\geq 0}$. Thus the result follows from Cauchy-Schwarz inequality. □

By replacing the Cauchy-Schwarz inequality by Hölder's inequality, similar arguments can be used to obtain sufficient conditions for L^p-admissibility.

Corollary 3.2. *Assume in addition to the hypotheses of the theorem that one of the following conditions is satisfied:*

(a) *Y is finite-dimensional, X is a Hilbert space and A generates a contraction semigroup;*
(b) *X is a Hilbert space and A generates a normal, analytic semigroup;*
(c) *A generates an analytic semigroup and $(-A)^{1/2}$ is an finite-time admissible observation operator for $(T(t))_{t\geq 0}$.*

If there exists a constant $M > 0$ such that

$$\|C(\lambda - A)^{-1}\| \leq \frac{M}{\sqrt{\operatorname{Re}\lambda}}, \qquad \operatorname{Re}\lambda > 0, \tag{6}$$

then C is a finite-time admissible observation operator for $(S(t))_{t\geq 0}$.

Proof. Under the assumption of the corollary, the inequality (6) implies that C is a finite-time admissible observation operator for $(S(t))_{t\geq 0}$, see [9], [6], [13]. Thus the result follows from Theorem 3.1. □

The following corollary is an immediate consequence of the Carleson measure criterion of Ho and Russell [8].

Corollary 3.3. *Assume in addition to the hypotheses of the theorem that A admits a Riesz basis of eigenfunctions (e_n) on a Hilbert space X with corresponding eigenvalues λ_n. If $Y = \mathbb{C}$ and if*

$$\mu = \sum_n |Ce_n|^2 \delta_{-\lambda_n}$$

is a Carleson measure on $\mathbb{C}_+$, then C is finite-time admissible for the solution family $(S(t))_{t\geq 0}$.

Next we generalize a sufficient condition for admissibility for Cauchy problem given by Zwart [16] to the context of Volterra equations.

Theorem 3.4. *Let $S(\cdot)$ be a bounded solution family to (V) with a 1–regular kernel function a. Let $C : \mathcal{D}(A) \to Y$ be bounded and assume that for some $\alpha > 1/2$,*

$$\sup_{r>0} \| \ (1 + \log^+ r)^\alpha r^{1/2} CH(r) \, \| < \infty. \tag{7}$$

Then C is finite-time admissible for $(S(t))_{t\geq 0}$.

Note that the exponent $\alpha > 1/2$ is optimal in the sense that for $\alpha < 1/2$ it is even wrong in the case $a \equiv 1$, see [12]. About the case $\alpha = 1/2$ nothing is known at the moment.

Proof. Let $\lambda \in \mathbb{C}_+$ and let φ such that $\lambda = |\lambda| e^{2i\varphi}$. Then, by resolvent identity,

$$\begin{aligned}
&(1 + (\log^+ (\operatorname{Re} \lambda))^\alpha \lambda^{1/2} CH(\lambda) \\
= \ &(1 + (\log^+ (\operatorname{Re} \lambda))^\alpha \lambda^{-1/2} C \frac{1}{\widehat{a}(\lambda)} R(\frac{1}{\widehat{a}(\lambda)}, A) \\
= \ &(1 + \log^+ |\lambda|)^\alpha |\lambda|^{1/2} CH(|\lambda|) \, e^{-i\varphi} \frac{\widehat{a}(|\lambda|)}{\widehat{a}(\lambda)} \Big[I + (\frac{1}{\widehat{a}(|\lambda|)} - \frac{1}{\widehat{a}(\lambda)}) R(\frac{1}{\widehat{a}(\lambda)}, A) \Big] \\
= \ &(1 + \log^+ |\lambda|)^\alpha |\lambda|^{1/2} CH(|\lambda|) \, e^{-i\varphi} \Big[I + (1 - \frac{\widehat{a}(|\lambda|)}{\widehat{a}(\lambda)}) AR(\frac{1}{\widehat{a}(\lambda)}, A) \Big].
\end{aligned}$$

By [15, Lemma 8.1], $c^{-1} \leq \big| \widehat{a}(|\lambda|)/\widehat{a}(\lambda) \big| \leq c$ for some $c > 0$. This yields uniform boundedness of expression in brackets and so the assumed estimate (7) gives

$$\| \lambda \mapsto CH(r+\lambda) \|_{H^\infty(\mathbb{C}_+)} \leq K (1 + \log^+ r)^{-\alpha} r^{-1/2}. \tag{8}$$

Since $(S(t))_{t\geq 0}$ is bounded,

$$\| \lambda \mapsto H(r+\lambda)x \|_{H^2(\mathbb{C}_+)} = \| e^{-rt} S(t) x \|_{H^2(\mathbb{C}_+)} \leq K r^{-1/2} \, \|x\| \qquad \forall r > 0$$

and together with (8), we infer

$$\| \lambda \mapsto CH(r+\lambda)^2 x \|_{H^2(\mathbb{C}_+)} \leq \frac{K}{(1 + \log^+ r)^\alpha r} \, \|x\| \qquad \forall r > 0. \tag{9}$$

Moreover, the estimate

$$\big\| \lambda \mapsto \frac{1}{r+\lambda} CH(r+\lambda) x \big\|_{H^2(\mathbb{C}_+)} \leq \big\| \lambda \mapsto CH(r+\lambda) x \big\|_{H^\infty(\mathbb{C}_+)} \big\| \lambda \mapsto \frac{1}{r+\lambda} \big\|_{H^2(\mathbb{C}_+)}$$

implies

$$\big\| \lambda \mapsto \frac{1}{r+\lambda} CH(r+\lambda) x \big\|_{H^2(\mathbb{C}_+)} \leq \frac{K}{(1 + \log^+ r)^\alpha r} \|x\| \qquad \forall r > 0. \tag{10}$$

Since $\frac{d}{d\lambda} H(\lambda) = \Big(\frac{\lambda \widehat{a}'(\lambda)}{\widehat{a}(\lambda)} \Big) H(\lambda)^2 - \frac{1}{\lambda} \big(1 + \frac{\lambda \widehat{a}'(\lambda)}{\widehat{a}(\lambda)} \big) H(\lambda)$ we infer from (9) and (10) that

$$\Big\| \mu \mapsto \frac{d}{d\mu} CH(r + \mu) x \Big\|_{H^2(\mathbb{C}_+)} \leq \frac{K}{(1 + \log^+ r)^\alpha r} \|x\| \qquad \forall r > 0.$$

Finally, (inverse) Laplace transform yields

$$\left\| t \mapsto rte^{-rt} CS(t)x \right\|_{L^2(\mathbb{R}_+)} \leq \frac{K}{(1+\log^+ r)^\alpha} \|x\| \qquad \forall r > 0$$

and that is the estimate we need in the following dyadic decomposition argument: notice that $xe^{-x} \geq 2e^{-2}$ for $x \in [1,2]$. Fix some $t_0 > 0$. Then,

$$\begin{aligned} \int_0^{t_0} \left\| CS(t)x \right\|^2 dt &= \sum_{n=1}^\infty \int_{t_0 2^{-n}}^{t_0 2^{-n+1}} \left\| CS(t)x \right\|^2 dt \\ &\leq \tfrac{e^4}{4} \sum_{n=1}^\infty \int_{t_0 2^{-n}}^{t_0 2^{-n+1}} \left\| t2^n t_0^{-1} e^{t2^n t_0^{-1}} CS(t)x \right\|^2 dt \\ &\leq K \sum_{n=1}^\infty \frac{1}{(1+\log^+(2^n t_0^{-1}))^{2\alpha}} \|x\|^2 \leq K \|x\|^2. \end{aligned}$$

□

Notice that the condition (7) can be reformulated by saying that in the sense of Evans, Opic and Pick (see [3, 2, 4])

$$\forall x \in X : \qquad \|CH(\cdot)x\|_{1/2,\infty.\mathbb{A}} < \infty$$

where $\mathbb{A} = (0, \alpha)$.

Acknowledgement

This research was done at the Mathematisches Forschungsinstitut Oberwolfach during a stay within the Research in Pairs Programme from June 15 to June 28, 2008. We would like to thank the MFO for excellent working conditions.

References

[1] P. L. Duren, *Theory of H^p spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York, 1970.
[2] W. D. Evans and B. Opic, *Real interpolation with logarithmic functors and reiteration*, Canad. J. Math., 52 (2000), pp. 920–960.
[3] W. D. Evans, B. Opic, and L. Pick, *Interpolation of operators on scales of generalized Lorentz-Zygmund spaces*, Math. Nachr., 182 (1996), pp. 127–181.
[4] W. D. Evans, B. Opic, and L. Pick, *Real interpolation with logarithmic functors*, J. Inequal. Appl., 7 (2002), pp. 187–269.
[5] B. Haak, B. Jacob, J. Partington, and S. Pott, *Admissibility and controllability of diagonal volterra equations with scalar inputs*, J. Differential Equations, (2009). To appear.
[6] S. Hansen and G. Weiss, *The operator Carleson measure criterion for admissibility of control operators for diagonal semigroups on l^2*, Systems Control Lett., 16 (1991), pp. 219–227.
[7] E. Hille and J. Tamarkin, *On the absolute integrability of Fourier transforms.*, Fundam. Math., 25 (1935), pp. 329–352.
[8] L. F. Ho and D. L. Russell, *Admissible input elements for systems in Hilbert space and a Carleson measure criterion*, SIAM J. Control Optim., 21 (1983), pp. 614–640. Erratum in the same journal, Vol. 21,No. 6, p. 985–986.
[9] B. Jacob and J. R. Partington, *The Weiss conjecture on admissibility of observation operators for contraction semigroups*, Integral Equations Operator Theory, 40 (2001), pp. 231–243.
[10] ———, *Admissibility of control and observation operators for semigroups: a survey*, in Current trends in operator theory and its applications, vol. 149 of Oper. Theory Adv. Appl., Birkhäuser, Basel, 2004, pp. 199–221.
[11] ———, *A resolvent test for admissibility of Volterra observation operators*, J. Math. Anal. Appl., 332 (2007), pp. 346–355.
[12] B. Jacob, J. R. Partington, and S. Pott, *Zero-class admissibility of observation operators*, Systems Control Lett., (2009). To appear.
[13] C. Le Merdy, *The Weiss conjecture for bounded analytic semigroups*, J. London Math. Soc. (2), 67 (2003), pp. 715–738.

[14] S. Monniaux and J. Prüss, *A theorem of the Dore-Venni type for noncommuting operators*, Trans. Amer. Math. Soc., 349 (1997), pp. 4787–4814.
[15] J. Prüss, *Evolutionary integral equations and applications*, vol. 87 of Monographs in Mathematics, Birkhäuser Verlag, Basel, 1993.
[16] H. Zwart, *Sufficient conditions for admissibility*, Systems Control Lett., 54 (2005), pp. 973–979.

Institut de Mathématiques de Bordeaux, 351 cours de la Libération, 33405 Talence CEDEX, France
E-mail address: Bernhard.Haak@math.u-bordeaux1.fr

Institut für Mathematik, Universität Paderborn, Warburger Strasse 100, D-33098 Paderborn, Germany
E-mail address: jacob@math.uni-paderborn.de