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Digital DC blocker filters

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Abstract: This paper mathematically investigates a special kind of digital infinite-impulse response (IIR) filters, suitable for filtering out very low frequencies near zero from digital signals. We investigate the transfer functions of such filters from 1st to 3rd order and provide formulas to calculate the filter coefficients from the desired cutoff frequency.

Keywords: DC blocker; digital filter; IIR filter.

1 Introduction

Digital filtering plays a very important role in many data processing systems. Sometimes it is necessary to remove a time-invariant additive offset from a signal [1, 2]. Such signal components are often named “DC offset”, the respective filter “DC blocker” (DC = “Directed Current” in contrast to “Alternating Current” = AC). For example, Analog-to-Digital converters (ADCs) often possess such a small constant offset that needs compensation or removal [2, 3]. The required filter is a high-pass filter that lets uniformly pass all signal frequencies except extremely low ones, say few percent of the sampling rate or even less.

This paper investigates a special kind of digital DC blocker filters. We provide formulas to calculate the filter coefficients when the desired corner frequency is known. The filter structure can be implemented in software or digital hardware.

2 Digital filter basics

A digital filter is a hardware or software unit that processes an (infinite) series of digital input samples $\{x_k\}$, producing a series of output samples $\{y_k\}$. Each output sample y_k is a linear combination of a number M of preceding input samples x_{k-m} ($m = 0$–M) and (optionally) a number N of previous output samples y_{k-n} ($n = 1$–N). If $N = 0$, we call it a “Finite Impulse Response” (FIR) filter, since a singular non-zero pulse at the input will lead to a finite number of output pulses (at most M of them). If $N > 0$, the number of emerging output pulses may theoretically be infinite. This is the “Infinite-Impulse Response” (IIR) filter. It may possess some complications such as insufficient stability, but normally the desired spectral shaping of the output signal, i.e., the filtering, can be achieved with less computational effort than in FIR filters.

Mathematically, the filter is described by its equation

$$y_k = \sum_{m=0}^{M} b_m x_{k-m} + \sum_{n=1}^{N} a_n y_{k-n} \tag{1}$$

where the x_{k-m} are the input samples, y_{k-n} the output samples, and the b_m and a_n are constant real coefficients. Their values determine the characteristics of the filter, i.e., the spectral shaping of the signal when it passes the filter.

The samples are almost always taken at constant intervals in time Δt, or, in other words, at a constant sampling frequency $fs = 1/\Delta t$. The input and output samples can then be written as $x_k = x(k \cdot \Delta t)$ and $y_\mathrm{k} = y(k \cdot \Delta t)$, respectively. Equation (1) then transforms to

$$y(\mathrm{k}\cdot\Delta t) = \sum_{m=0}^{M} b_m x((k-m)\cdot\Delta t) + \sum_{n=1}^{N} a_n y((k-n)\cdot\Delta t) \tag{2}$$

For investigating the spectral properties, we perform a Fourier transform (in the following we only show important final results, for more details consult a textbook like [4–6]).

$$Y(\Omega) = X(\Omega)\sum_{m=0}^{M} b_m e^{-i\Omega m} + Y(\Omega)\sum_{n=1}^{N} a_n e^{-i\Omega n} \tag{3}$$

Here, $X(\Omega)$ and $Y(\Omega)$ are the Fourier-transformed input and output signals. Ω is the normalized angular frequency. We normalize all frequencies to the sampling frequency f_S.

$$\Omega = 2\pi \cdot f \cdot \Delta t = 2\pi \cdot f / f_S \tag{4}$$

According to the Nyquist theorem, the highest frequency component in the signal, which makes sense, is the Nyquist frequency $f_\mathrm{N} = ½ f_\mathrm{S}$. Thus, the sensible range for Ω is from 0 to $\pi = 180° =$ Nyquist frequency. The sampling frequency f_S would correspond to $\Omega = 2\pi$.

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The ratio of the output signal to the input signal in frequency domain is denoted "transfer function" $H(\Omega)$.

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{m=0}^{M} b_m e^{-i\Omega \cdot m}}{1 - \sum_{n=1}^{N} a_n e^{-i\Omega \cdot n}} \tag{5}$$

The transfer function is a complex function of the (normalized) frequency, i.e., it has a real and an imaginary part, or an absolute value and a phase. The squared absolute value $|H(\Omega)|^2$ is the most important feature of the filter since it describes its spectral properties, i.e., which frequency components are attenuated or amplified. Also, the phase $\varphi(\Omega)$ is sometimes considered, but we will not discuss this property in this paper. The absolute value computes as:

$$|H(\Omega)|^2 = \frac{\sum_{m=0}^{M} b_m^2 + 2 \cdot \sum_{m=1}^{M} \left(\cos(\Omega \cdot m) \cdot \left(\sum_{k=m}^{M} b_k b_{k-m} \right) \right)}{1 + \sum_{n=1}^{N} a_n^2 - 2 \cdot \sum_{n=1}^{N} \left(\cos(\Omega \cdot n) \cdot \left(a_n - \sum_{k=n+1}^{N} a_k a_{k-n} \right) \right)} \tag{6}$$

Finally, we investigate the stability of the filter. This means, that the filter output samples $\{y_k\}$ are and remain finite (limited) for any finite input sequence $\{x_k\}$. In particular, they must remain finite after a single input pulse. Mathematically, instability occurs at poles of the transfer function $H(\Omega)$. For investigating this property, we transform (5) into the z domain by substituting $e^{i\Omega} = z$:

$$H(\Omega) = \frac{\sum_{m=0}^{M} b_m e^{-i\Omega \cdot m}}{1 - \sum_{n=1}^{N} a_n e^{-i\Omega \cdot n}} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{1 - \sum_{n=1}^{N} a_n z^{-n}} \tag{7}$$

The poles occur when the denominator is zero, i.e.,

$$1 - \sum_{n=1}^{N} a_n z^{-n} = 0 \quad \text{or} \quad z^N - \sum_{n=1}^{N} a_n z^{N-n} = 0 \tag{8}$$

The filter is stable when all complex poles z_k of (7), equivalent to the polynomial roots of (8), are inside the unit circle of the z domain or, in other words, all $|z_k| < 1$ [6].

The stability of a filter depends only on the recursive (feedback) coefficients a_n. FIR filters, where all $a_n = 0$, are always stable.

3 DC blocker filters

The DC blocker filter shall block very low frequencies near zero and let all higher signal components pass unchanged. This is the high-pass behavior. The transfer function shall be $H(\Omega) = 0$ for $\Omega = 0$ and $H(\Omega) = 1$ for $\Omega = \pi$ (Nyquist frequency).

3.1 First order

The textbook DC blocker filter ([4] chapter 13.23, [2]) is a 1st order IIR filter having the transfer function

$$H(\Omega) = \frac{1 - e^{-i\Omega}}{1 - \alpha \cdot e^{-i\Omega \cdot}} \tag{1.1}$$

This corresponds to the filter equation

$$y_k = (x_k - x_{k-1}) + \alpha \cdot y_{k-1} \tag{1.2}$$

Parameter α determines the corner frequency. For the DC blocker it is slightly less than one, say in the range 0.95–0.99.

We will now derive these equations from our general formulas in Section 2. Naming and normalization will slightly differ from the conventions in (1.1) and (1.2). For example: from (1.1) we deduce a pass-band gain of $H(\Omega = \pi) = 2/(1+\alpha)$; in our normalization we get $H(\Omega = \pi) = 1$ for any high-pass filter.

Inserting $M = N = 1$ into (5) leads to

$$H(\Omega) = \frac{b_0 + b_1 \cdot e^{-i\Omega}}{1 - a_1 \cdot e^{-i\Omega}} \tag{1.3}$$

For a high-pass filter we set the boundary conditions $H(\Omega = 0) = 0$ (frequency zero shall be blocked) and $H(\Omega = \pi) = 1$ (the Nyquist frequency component shall pass unchanged). This allows elimination of two of the unknown coefficients b_0, b_1 and a_1.

From the first condition we get with $e^{-i0} = 1$:

$$H(\Omega = 0) = \frac{b_0 + b_1}{1 - a_1} = 0 \tag{1.4}$$

or, in other words, $b_1 = -b_0$ (provided that $a_1 \neq 1$).

From the second condition we get with $e^{-i\pi} = -1$:

$$H(\Omega = \pi) = \frac{b_0 - b_1}{1 + a_1} = 1 \tag{1.5}$$

or, in other words, $b_0 - b_1 = 1 + a_1$. We redefine the one remaining free parameter as b in the following way:

$$b_0 = b \quad b_1 = -b \quad a_1 = b_0 - b_1 - 1 = 2b - 1 \tag{1.6}$$

We insert this into (6):

$$\begin{aligned} |H(\Omega)|^2 &= \frac{b_0^2 + b_1^2 + 2 \cdot b_0 b_1 \cdot \cos(\Omega)}{1 + a_1^2 - 2 \cdot a_1 \cdot \cos(\Omega)} \\ &= \frac{b^2 + b^2 - 2b^2 \cdot \cos(\Omega)}{1 + (2b-1)^2 - 2 \cdot (2b-1) \cdot \cos(\Omega)} \\ &= \frac{2b^2 \cdot (1 - \cos(\Omega))}{4b^2 - 4b \cdot (1 + \cos(\Omega)) + 2 \cdot (1 + \cos(\Omega))} \end{aligned} \tag{1.7}$$

We transform this equation using the trigonometric relations $(1 - \cos(\Omega)) = 2{\cdot}\sin^2(^1/_2 \cdot \Omega)$, $(1 + \cos(\Omega)) = 2{\cdot}\cos^2(^1/_2 \cdot \Omega)$, and $\sin^2(^1/_2 \cdot \Omega) + \cos^2(^1/_2 \cdot \Omega) = 1$.

$$|H(\Omega)|^2 = \frac{4b^2 \cdot \sin^2\left(\tfrac{1}{2}\Omega\right)}{4b^2 + 4(1-2b)\cdot\cos^2\left(\tfrac{1}{2}\Omega\right)}$$

$$= \frac{b^2 \cdot \sin^2\left(\tfrac{1}{2}\Omega\right)}{b^2 \cdot \left(\sin^2\left(\tfrac{1}{2}\Omega\right) + \cos^2\left(\tfrac{1}{2}\Omega\right)\right) + (1-2b)\cdot\cos^2\left(\tfrac{1}{2}\Omega\right)}$$

$$= \frac{b^2 \cdot \sin^2\left(\tfrac{1}{2}\Omega\right)}{b^2 \cdot \sin^2\left(\tfrac{1}{2}\Omega\right) + \left(1-2b+b^2\right)\cdot\cos^2(\tfrac{1}{2}\Omega)} \tag{1.8}$$

$$|H(\Omega)|^2 = \frac{1}{1 + \dfrac{(1-b)^2 \cdot \cos^2\left(\tfrac{1}{2}\Omega\right)}{b^2 \cdot \sin^2\left(\tfrac{1}{2}\Omega\right)}}$$

We re-formulate the free parameter again as $b = 1 - {}^1/_2\,\omega$ or $\omega = 2(1 - b)$ (we will see the reason later):

$$|H(\Omega)|^2 = \frac{1}{1 + \left(\dfrac{\omega}{2-\omega} \cdot \cot\left(\dfrac{\Omega}{2}\right)\right)^2} \tag{1.9}$$

For Ω in the range $0 \le \Omega \le \pi$, $|H(\Omega)|^2$ is a monotonic function. Figure 1 shows some graphs. The larger ω, the higher the corner frequency of the filter, i.e., the transition point from blocking to passing behavior. In case $\omega = 0$ Equation (1.9) has a singularity at $\Omega = 0$.

A common criterion for the corner frequency is the point, where the filter's attenuation reaches 3 dB, i.e., $|H(\Omega)|^2 = ½$. We calculate this point from (1.9):

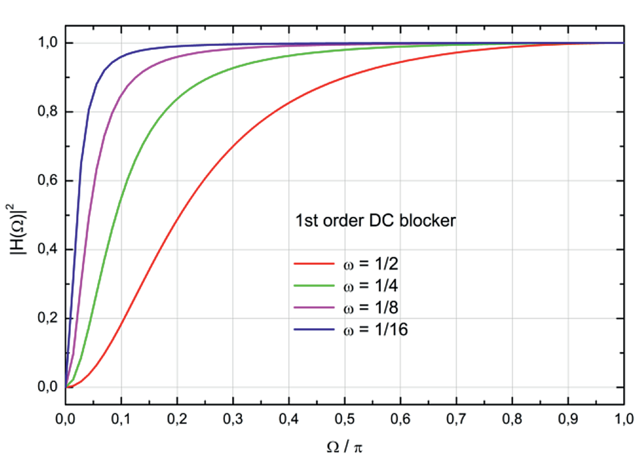

Figure 1: Transfer function of 1st order DC blocker filters (Equation [1.9]).

$$|H(\Omega_{3dB})|^2 = \frac{1}{2} = \frac{1}{1 + \left(\dfrac{\omega}{2-\omega}\cdot\cot\left(\dfrac{\Omega_{3dB}}{2}\right)\right)^2}$$

$$\left(\frac{\omega}{2-\omega}\cdot\cot\left(\frac{\Omega_{3dB}}{2}\right)\right)^2 = 1$$

$$\tan\left(\frac{\Omega_{3dB}}{2}\right) = \pm\frac{\omega}{2-\omega} \tag{1.10}$$

For a DC blocker, the corner frequency shall be low: $\Omega_{3dB} << 1$. In this case, we apply the series expansion for $\tan(x) = x - 1/3\,x^3 + \ldots$ Also ω will be small, so that $\omega << 2$.

$$\Omega_{3dB} \approx \frac{2\cdot\omega}{2-\omega} \approx \omega \tag{1.11}$$

Our parameter ω equals the (normalized, angular) 3 dB corner frequency of the filter. If the latter is given, we can calculate the filter coefficients b_0, b_1 and a_1 (see (1.6) and text above (1.9)):

$$a_1 = 1 - \Omega_{3dB}$$

$$b_0 = +\left(1 - \tfrac{1}{2}\,\Omega_{3dB}\right)$$

$$b_1 = -\left(1 - \tfrac{1}{2}\,\Omega_{3dB}\right) \tag{1.12}$$

These equations are valid only for small corner frequencies. Equation (1.10), however, applies to any frequency.

Finally, we verify the stability of the filter, which is not always certain for IIR filters. For doing this, we have to find the poles of (1.3). Inserting (1.12), the equation reads as:

$$H(\Omega) = \frac{b_0 + b_1 e^{-i\Omega}}{1 - a_1 e^{-i\Omega\cdot}} = \frac{(1-\tfrac{1}{2}\omega)\cdot(1-e^{-i\Omega})}{1-(1-\omega)\cdot e^{-i\Omega\cdot}}$$

We substitute $e^{i\Omega} = z$ for transforming into the z domain:

$$H(\Omega) = \frac{(1-\tfrac{1}{2}\omega)\cdot(1-z^{-1})}{1-(1-\omega)\cdot z^{-1}} = \frac{(1-\tfrac{1}{2}\omega)\cdot(z-1)}{z-(1-\omega)}$$

The pole occurs when the denominator is zero, i.e.,

$$z - (1-\omega) = 0 \quad \text{or} \quad z = 1 - \omega$$

In the z domain, the filter is stable when $|z| < 1$ [6]. This is the case when $0 < \omega < 2$, which is fulfilled for a DC blocker filter. However, the nearer ω to 0 (i.e., the smaller the corner frequency), the more the filter approaches the stability limit.

3.2 Second order

Figure 1 depicts that the first order filters are not very steep. For many applications, a smaller transition range between stop and pass behavior is highly desired. This can be achieved by increasing the order of the filter, i.e., the number of coefficients.

For a 2nd order filter we have to set $M = N = 2$ in (5)

$$H(\Omega) = \frac{b_0 + b_1 \cdot e^{-i\Omega} + b_2 \cdot e^{-i2\Omega}}{1 - a_1 \cdot e^{-i\Omega} - a_2 \cdot e^{-i2\Omega}} \tag{2.1}$$

The boundary conditions for high-pass filter do not change: $H(\Omega = 0) = 0$ (frequency zero shall be blocked) and $H(\Omega = \pi) = 1$ (Nyquist frequency shall pass unchanged). This again allows elimination of two of the unknown coefficients, but we have five of them now: b_0, b_1, b_2 and a_1, a_2.

From the first condition we get with $e^{-i0} = 1$:

$$H(\Omega = 0) = \frac{b_0 + b_1 + b_2}{1 - a_1 - a_2} = 0 \tag{2.2}$$

or, in other words, $b_1 = -(b_0 + b_2)$ (provided that $a_1 + a_2 \neq 1$).

From the second condition we get:

$$H(\Omega = \pi) = \frac{b_0 - b_1 + b_2}{1 + a_1 - a_2} = 1 \tag{2.3}$$

or, in other words, $b_0 - b_1 + b_2 = 1 + a_1 - a_2$. Combining this with the result from (2.2) leads to $2(b_0 + b_2) = -2b_1 = 1 + a_1 - a_2$ or $a_1 = 2(b_0 + b_2) + a_2 - 1$. Inserting into (6) yields

$$|H(\Omega)|^2 = \frac{b_0^2 + (b_0 + b_2)^2 + b_2^2 - 2(b_0 + b_2)^2 \cdot \cos(\Omega) + 2b_0 b_2 \cdot \cos(2 \cdot \Omega)}{1 + (2(b_0 + b_2) + a_2 - 1)^2 + a_2^2 - 2(2(b_0 + b_2) + a_2 - 1) \cdot (1 - a_2) \cdot \cos(\Omega) - 2a_2 \cdot \cos(2 \cdot \Omega)} \tag{2.4}$$

We transform this equation in a way similar to (1.6). Only main intermediate results are given.

$$\begin{aligned}
|H(\Omega)|^2 &= \frac{2(b_0+b_2)^2 \cdot (1-\cos(\Omega)) - 2b_0 b_2 \cdot (1-\cos(2\Omega))}{1+a_2^2+(a_2-1)^2+4(b_0+b_2)^2+4(b_0+b_2)\cdot(a_2-1)+2(2(b_0+b_2)+a_2-1)\cdot(a_2-1)\cos(\Omega)-2a_2\cos(2\Omega)} \\
&= \frac{4(b_0+b_2)^2 \cdot \sin^2\left(\Omega/2\right) - 4b_0 b_2 \cdot (1-\cos^2(\Omega))}{2(a_2-1)^2+4(b_0+b_2)^2+8(b_0+b_2)\cdot(a_2-1)\cdot\cos^2\left(\Omega/2\right)+2(a_2-1)^2\cdot\cos(\Omega)+2a_2\cdot(1-\cos(2\cdot\Omega))} \\
&= \frac{4(b_0+b_2)^2 \cdot \sin^2\left(\Omega/2\right) - 4b_0 b_2 \cdot (1-\cos(\Omega))\cdot(1+\cos(\Omega))}{4(b_0+b_2)^2+8(b_0+b_2)\cdot(a_2-1)\cdot\cos^2\left(\Omega/2\right)+2(a_2-1)^2\cdot(1+\cos(\Omega))+4a_2(1-\cos(\Omega))\cdot(1+\cos(\Omega))} \\
&= \frac{4(b_0+b_2)^2 \cdot \sin^2\left(\Omega/2\right) - 16\cdot b_0 b_2 \cdot \sin^2\left(\Omega/2\right)\cdot\left(1-\sin^2\left(\Omega/2\right)\right)}{4\left((b_0+b_2)^2+4a_2\right)\cdot\sin^2\left(\Omega/2\right)+4\left((b_0+b_2)^2+(2(b_0+b_2)+(a_2-1))\cdot(a_2-1)\right)\cdot\cos^2\left(\Omega/2\right)-16a_2\sin^4\left(\Omega/2\right)} \\
&= \frac{4(b_0-b_2)^2 \cdot \sin^2\left(\Omega/2\right) + 16\cdot b_0 b_2 \cdot \sin^4\left(\Omega/2\right)}{4\left((b_0+b_2)^2+4a_2\right)\cdot\sin^2\left(\Omega/2\right)+4((b_0+b_2)+(a_2-1))^2\cdot\cos^2\left(\Omega/2\right)-16a_2\sin^4\left(\Omega/2\right)} \\
&= \frac{(b_0-b_2)^2 \cdot \sin^2\left(\Omega/2\right) + 4\cdot b_0 b_2 \cdot \sin^4\left(\Omega/2\right)}{(b_0-b_2)^2\cdot\sin^2\left(\Omega/2\right)+4(b_0 b_2+a_2)\cdot\sin^2\left(\Omega/2\right)-4a_2\cdot\sin^4\left(\Omega/2\right)+((b_0+b_2)+(a_2-1))^2\cdot\cos^2\left(\Omega/2\right)}
\end{aligned} \tag{2.5}$$

This formula already looks pretty similar to the 1st order transfer function (1.8), provided that $b_0 = b_2$ and $a_2 = -b_0 \cdot b_2$. It will turn out that this setting is indeed a good solution. Thus, we may set our remaining free parameters as $b_0 = b_2 = b$ and $a_2 = -b_0 \cdot b_2 = -b^2$.

$$|H(\Omega)|^2 = \frac{4b^2 \cdot \sin^4\left(\Omega/2\right)}{4b^2 \cdot \sin^4\left(\Omega/2\right) + \left(2b - b^2 - 1\right)^2 \cdot \cos^2\left(\Omega/2\right)} = \frac{1}{1 + \left(\frac{(1-b)^2}{2b \cdot \sin(\Omega/2)} \cdot \cot\left(\frac{\Omega}{2}\right)\right)^2} \tag{2.6}$$

More generally, we may set $b_0 = b + \beta$, $b_2 = b - \beta$ and $a_2 = \alpha - b_0 \cdot b_2 = \alpha - (b^2 - \beta^2)$. Inserting into (2.5) leads to:

$$|H(\Omega)|^2 = \frac{4\beta^2 \cdot \sin^2\left(\Omega/2\right) + 4 \cdot \left(b^2 - \beta^2\right) \cdot \sin^4\left(\Omega/2\right)}{4\beta^2 \cdot \sin^2\left(\Omega/2\right) + 4\alpha \cdot \sin^2\left(\Omega/2\right) - 4\alpha \cdot \sin^4\left(\Omega/2\right) + 4\left(b^2 - \beta^2\right) \cdot \sin^4\left(\Omega/2\right) + \left(\alpha + \beta^2 - \left(1 - 2b + b^2\right)\right)^2 \cdot \cos^2\left(\Omega/2\right)}$$

$$= \frac{4\beta^2 \cdot \sin^2\left(\Omega/2\right) \cdot \cos^2\left(\Omega/2\right) + 4b^2 \cdot \sin^4\left(\Omega/2\right)}{4\beta^2 \cdot \sin^2\left(\Omega/2\right) \cdot \cos^2\left(\Omega/2\right) + 4b^2 \cdot \sin^4\left(\Omega/2\right) + 4\alpha \cdot \sin^2\left(\Omega/2\right) \cdot \cos^2\left(\Omega/2\right) + \left(\alpha + \beta^2 - (1-b)^2\right)^2 \cdot \cos^2\left(\Omega/2\right)}$$

$$= \frac{1}{1 + \frac{4\alpha \cdot \sin^2\left(\Omega/2\right) + \left((1-b)^2 - \alpha - \beta^2\right)^2}{4\beta^2 \cdot \cos^2\left(\Omega/2\right) + 4b^2 \cdot \sin^2\left(\Omega/2\right)} \cdot \cot^2\left(\frac{\Omega}{2}\right)} \tag{2.7}$$

Comparing (2.6) and (2.7) one finds that a small non-zero β is almost equivalent to a change in b, i.e., to a shift of the corner frequency. Moreover, only β^2 is present in (2.7), i.e., the sign of β is irrelevant. In contrast, a negative α can lead to $|H(\Omega)|^2 > 1$ when the term $4\alpha \cdot \sin^2(^1/_2 \cdot \Omega) + ((1-b)^2 - \alpha - \beta^2)^2$ becomes negative.

Figure 2 shows some numerical simulations. It turns out that $\alpha = 0$ is in fact a good choice. Positive α leads to flatter curve shape of the transfer function, whereas a negative α may result in a peak of $|H(\Omega)|^2$. Both is not desired. A slightly improved curve shape may be obtained for small negative $\alpha \approx -(1-b)^2/8 = -\omega^2/16$.

Coming back to our simple case (2.6), we re-define the parameter b that determines the corner frequency as $\omega = \sqrt{2}\,(1 - b)$. The 2nd order Formula (2.8) is now very similar to the 1st order Formula (1.9), but it has an additional factor $\omega/2 \sin(^1/_2 \cdot \Omega)$ before the cot() term. This term is responsible for the steeper transfer curve of the 2nd order filter, compared to the 1st order one.

$$|H(\Omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{2 - \sqrt{2} \cdot \omega} \cdot \frac{\omega}{2 \cdot \sin(\Omega/2)} \cdot \cot\left(\frac{\Omega}{2}\right)\right)^2} \tag{2.8}$$

For Ω in the range $0 \le \Omega \le \pi$, $|H(\Omega)|^2$ is a monotonic function. Figure 3 shows some graphs. The larger ω, the higher the corner frequency of the filter, i.e., the transition from blocking to passing behavior. In the case $\omega = 0$ Equation (2.8) has a singularity at $\Omega = 0$.

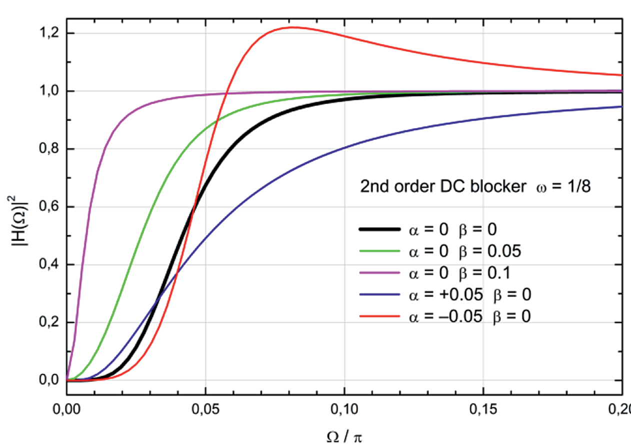

Figure 2: Influence of non-zero α and β in Equation (2.7).

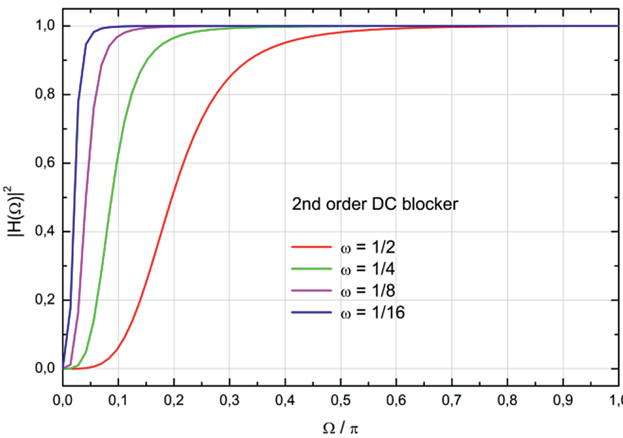

Figure 3: Transfer function of 2nd order DC blocker filters (Equation [2.8]).

We calculate the 3 dB corner frequency from (2.8):

$$|H(\Omega_{3\mathrm{dB}})|^2 = \frac{1}{2} = \frac{1}{1+\left(\dfrac{\omega^2}{4-\sqrt{8}\cdot\omega}\cdot\dfrac{\cot(\frac{1}{2}\Omega_{3\mathrm{dB}})}{\sin(\frac{1}{2}\Omega_{3\mathrm{dB}})}\right)^2}$$

$$\left(\frac{\omega^2}{4-\sqrt{8}\cdot\omega}\cdot\frac{\cot(\frac{1}{2}\Omega_{3\mathrm{dB}})}{\sin(\frac{1}{2}\Omega_{3\mathrm{dB}})}\right)^2 = 1$$

$$\tan(\tfrac{1}{2}\Omega_{3\mathrm{dB}})\cdot\sin(\tfrac{1}{2}\Omega_{3\mathrm{dB}}) = \pm\left(\frac{\omega^2}{4-\sqrt{8}\cdot\omega}\right) \quad (2.9)$$

For a DC blocker, the corner frequency shall be low: $\Omega_{3\mathrm{dB}} << 1$. In this case, we apply the series expansion for $\tan(x) = x - 1/3\,x^3 + \ldots$ and $\sin(x) = x - 1/6\,x^3 + \ldots$ Also ω will be small, so that $\sqrt{8}\omega << 4$.

$$\Omega_{3\mathrm{dB}}\cdot\Omega_{3\mathrm{dB}} \approx \frac{4\cdot\omega^2}{4-\sqrt{8}\cdot\omega} \approx \omega^2 \quad (2.10)$$

Our parameter ω again equals the (normalized, angular) 3 dB corner frequency of the filter. If the latter is given, we can calculate the filter coefficients:

$$a_1 = 3-\left(1+\sqrt{\tfrac{1}{2}}\cdot\Omega_{3\mathrm{dB}}\right)^2$$

$$= 2\cdot\left(1-\sqrt{\tfrac{1}{2}}\cdot\Omega_{3\mathrm{dB}}\right) - \tfrac{1}{2}\cdot\Omega_{3\mathrm{dB}}^2$$

$$a_2 = -\left(1-\sqrt{\tfrac{1}{2}}\cdot\Omega_{3\mathrm{dB}}\right)^2$$

$$b_0 = 1-\sqrt{\tfrac{1}{2}}\cdot\Omega_{3\mathrm{dB}}$$

$$b_1 = -2\cdot\left(1-\sqrt{\tfrac{1}{2}}\cdot\Omega_{3\mathrm{dB}}\right)$$

$$b_2 = 1-\sqrt{\tfrac{1}{2}}\cdot\Omega_{3\mathrm{dB}} \quad (2.11)$$

These equations are valid only for small corner frequencies. Equation (2.8), however, applies to any frequency.

Finally, we verify the stability of the filter. We have to find the poles of (2.1). Inserting (2.10), the z-transformed equation reads as:

$$H(\Omega) = \frac{\left(1-\sqrt{\tfrac{1}{2}}\cdot\omega\right)\cdot(z^2-2z+1)}{z^2-\left(3-\left(1+\sqrt{\tfrac{1}{2}}\cdot\omega\right)^2\right)z+\left(1-\sqrt{\tfrac{1}{2}}\cdot\omega\right)^2}$$

The poles occur when the denominator is zero, i.e.,

$$z^2-2\cdot\left(1-\sqrt{\tfrac{1}{2}}\cdot\omega-\tfrac{1}{4}\cdot\omega^2\right)z+\left(1-\sqrt{\tfrac{1}{2}}\cdot\omega\right)^2 = 0$$

This is a quadratic polynomial. Its roots are given by Vieta's formula:

$$z = \left(1-\sqrt{\tfrac{1}{2}}\cdot\omega-\tfrac{1}{4}\cdot\omega^2\right)$$

$$\pm\sqrt{\left(1-\sqrt{\tfrac{1}{2}}\cdot\omega-\tfrac{1}{4}\cdot\omega^2\right)^2-\left(1-\sqrt{\tfrac{1}{2}}\cdot\omega\right)^2}$$

$$= \left(1-\sqrt{\tfrac{1}{2}}\cdot\omega-\tfrac{1}{4}\cdot\omega^2\right)\pm\sqrt{\tfrac{1}{16}\cdot\omega^4+\sqrt{\tfrac{1}{8}}\cdot\omega^3-\tfrac{1}{2}\cdot\omega^2}$$

For small $\omega << 1$ the term under the square root is negative, i.e., the two poles are conjugate complex. Their absolute value is given by the sum of the squares of real and imaginary parts:

$$|z|^2 = \left(1-\sqrt{\tfrac{1}{2}}\cdot\omega-\tfrac{1}{4}\omega^2\right)^2+\left(\tfrac{1}{2}\cdot\omega^2-\sqrt{\tfrac{1}{8}}\cdot\omega^3-\tfrac{1}{16}\omega^4\right)$$

$$= 1+\tfrac{1}{2}\omega^2+\tfrac{1}{16}\omega^4-\sqrt{2}\cdot\omega-\tfrac{1}{2}\omega^2+\sqrt{\tfrac{1}{8}}\cdot\omega^3$$

$$+\tfrac{1}{2}\cdot\omega^2-\sqrt{\tfrac{1}{8}}\cdot\omega^3-\tfrac{1}{16}\omega^4$$

$$= 1-\sqrt{2}\cdot\omega+\tfrac{1}{2}\cdot\omega^2$$

$$|z|^2 = \left(1-\sqrt{\tfrac{1}{2}}\cdot\omega\right)^2$$

The filter is stable when $|z| < 1$. This is the case when $0 < \omega < 2\sqrt{2}$, which is fulfilled for a DC blocker filter. However, the nearer ω to 0 (i.e., the smaller the corner frequency), the more the filter approaches the stability limit.

3.3 Third order

In order to further increase the steepness of the transfer function, we may further increase the order of the filter. We will provide here the results for the 3rd order filter, without giving all intermediate steps and results. The way of calculations is similar to the 2nd order filter.

For a 3rd order filter we have to set $M = N = 3$ in (5)

$$H(\Omega) = \frac{b_0 + b_1 \cdot e^{-i\Omega} + b_2 \cdot e^{-i2\Omega} + a_3 \cdot e^{-i3\Omega}}{1 - a_1 \cdot e^{-i\Omega} - a_2 \cdot e^{-i2\Omega} - a_3 \cdot e^{-i3\Omega}} \tag{3.1}$$

The boundary conditions for high-pass filter do not change: $H(\Omega = 0) = 0$ (frequency zero shall be blocked) and $H(\Omega = \pi) = 1$ (Nyquist frequency shall pass unchanged). This again allows elimination of two of the unknown coefficients, but we now have seven of them: b_0, b_1, b_2, b_3 and a_1, a_2, a_3.

From these boundary conditions we get:

$$H(\Omega = 0) = \frac{b_0 + b_1 + b_2 + b_3}{1 - a_1 - a_2 - a_3} = 0 \tag{3.2}$$

$$H(\Omega = \pi) = \frac{b_0 - b_1 + b_2 - b_3}{1 + a_1 - a_2 + a_3} = 1 \tag{3.3}$$

The calculations proceed in the same way as for 2nd order. Since we have two more free coefficients, we need more criteria to pre-select some of them. On one-hand side, we have chosen the b_k coefficients to behave like binomial coefficients (i.e., to have a relation $b_0 = b$, $b_1 = -3b$, $b_2 = 3b$, $b_3 = -b$) in concordance with the lower order filter's b_k coefficients. On the other hand, the target $|H(\Omega)|^2$ transfer function should have only one $\cot^2(\Omega/2)$ term. The calculations are quite lengthy, so we report only the final results. The transfer function reads as:

$$|H(\Omega)|^2 = \frac{1}{1 + \left(\frac{\omega^3}{4 \cdot (1-\omega) \cdot (2-\omega) \cdot \sin^2\left(\frac{\Omega}{2}\right)} \cdot \cot\left(\frac{\Omega}{2}\right) \right)^2} \tag{3.4}$$

Compared with (2.8), the power of the sin() term is increased to 2nd power, which results in a higher steepness at the corner frequency (see Figure 4). The corner frequency is calculated from

$$\left(\frac{\omega^3}{4 \cdot (1-\omega)(2-\omega)} \cdot \frac{\cot(\frac{1}{2}\Omega_{3dB})}{\sin^2(\frac{1}{2}\Omega_{3dB})} \right)^2 = 1 \tag{3.5}$$

For small corner frequencies the series expansion again finally results in

$$\omega \approx \Omega_{3dB} \tag{3.6}$$

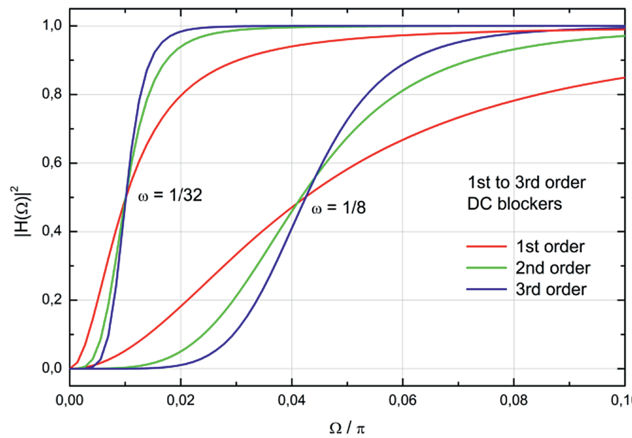

Figure 4: Transfer function of 1st–3rd order DC blocker filters.

The filter coefficients compute as:

$$a_1 = +\frac{6 - 7 \cdot \Omega_{3dB}}{2 - \Omega_{3dB}}$$

$$a_2 = -\frac{6 + \Omega_{3dB}}{2 - \Omega_{3dB}} \cdot (1 - \Omega_{3dB})^2$$

$$a_3 = +(1 - \Omega_{3dB})^2$$

$$b_0 = +(1 - \Omega_{3dB})$$

$$b_1 = -3 \cdot (1 - \Omega_{3dB})$$

$$b_2 = +3 \cdot (1 - \Omega_{3dB})$$

$$b_3 = -(1 - \Omega_{3dB}) \tag{3.7}$$

Finally, we report without proof that for small Ω_{3dB} the filter is stable. The pole's absolute values are approximately at $|z| \approx (1 - \omega)$.

Figure 4 shows the transfer functions 1st–3rd order for filters with corner frequencies $\Omega_{3dB} = 1/32$ and $\Omega_{3dB} = 1/8$. It is clearly visible that the steepness of the transition region between pass and stop regions increases with increasing filter order. Please notice the reduced x axis range when comparing with Figures 1 and 3.

4 Low-pass filters

The same principle can also be applied to low-pass filters with a corner frequency near to the Nyquist frequency. Also such filters are needed in communication systems, for example in those based on OFDM modulation [7, 8].

We sketch the calculations for the 1st order filter here. The starting point is again (1.3), but for a low-pass filter we have to swap the boundary conditions: $H(\Omega = 0) = 1$ (frequency zero shall pass) and $H(\Omega = \pi) = 0$ (Nyquist frequency shall be blocked). Equations (1.4) and (1.5) are the replaced by

$$H(\Omega = 0) = \frac{b_0 + b_1}{1 - a_1} = 1 \tag{4.1}$$

$$H(\Omega = \pi) = \frac{b_0 - b_1}{1 + a_1} = 0 \tag{4.2}$$

or, in other words, $b_0 = b_1$ and $b_0 + b_1 = 1 - a_1$. Combining leads to $2b_0 = 2b_1 = 1 - a_1$. We set $\omega = 1 + a_1$ and insert these relations into (6):

$$\begin{aligned}|H(\Omega)|^2 &= \frac{½(2-\omega)^2\cdot(1+\cos(\Omega))}{1+(\omega-1)^2-2\cdot(\omega-1)\cdot\cos(\Omega)} \\ &= \frac{1}{1+\left(\frac{\omega}{2-\omega}\tan\left(\frac{\Omega}{2}\right)\right)^2}\end{aligned} \tag{4.3}$$

Compared with (1.9), the cot() function is replaced with tan(). The corner frequency, where the filter's attenuation reaches 3 dB, computes as:

$$|H(\Omega_{3dB})|^2 = \frac{1}{2} = \frac{1}{1+\left(\frac{\omega}{2-\omega}\cdot\tan\left(\frac{\Omega_{3dB}}{2}\right)\right)^2}$$

$$\left(\frac{\omega}{2-\omega}\cdot\tan\left(\frac{\Omega_{3dB}}{2}\right)\right)^2 = 1$$

$$\cot\left(\frac{\Omega_{3dB}}{2}\right) = \pm\frac{\omega}{2-\omega} \tag{4.4}$$

The corner frequency is now near to the Nyquist frequency, which corresponds to $\Omega = \pi$. This means $(\pi - \Omega_{3dB}) << 1$. Under this condition, we transform (4.4) using the relation $\cot(x) = \tan(\pi/2 - x)$. After that, the normal series expansion for $\tan(x)$ is possible.

$$\cot\left(\frac{\Omega_{3dB}}{2}\right) = \tan\left(\frac{\pi - \Omega_{3dB}}{2}\right) = \pm\frac{\omega}{2-\omega}$$

$$\pi - \Omega_{3dB} \approx \pm\frac{2\cdot\omega}{2-\omega} \approx \omega \tag{4.5}$$

Our parameter ω equals the distance of the (normalized, angular) 3 dB corner frequency of the filter to the Nyquist frequency (half sampling rate). If the corner frequency is given, we can calculate the filter coefficients b_0, b_1 and a_1 (see text above (4.3)):

$$a_1 = (\pi - \Omega_{3dB}) - 1$$

$$b_0 = 1 - ½(\pi - \Omega_{3dB})$$

$$b_1 = 1 - ½(\pi - \Omega_{3dB}) \tag{4.6}$$

The low-pass filter coefficients (4.6) are equivalent to their high-pass counterparts (1.12) except for the sign of a_1 and b_1. Inserting these coefficients into (1) results in the instruction to compute the output samples y_k of the filter from the input x_k.

$$y_k = \left(1 - \frac{\omega}{2}\right)\cdot(x_k + x_{k-1}) - (1-\omega)\cdot y_{k-1} \tag{4.7}$$

The filter equations and coefficients for higher-order low-pass filters can be derived in the same way as for the DC blocker filters. This is left to another publication.

5 Implementation aspects

5.1 Calculating filter coefficients

When practically using the above results for designing one of the investigated IIR high-pass filters, one first has to calculate the normalized angular corner frequency (filter attenuation 3 dB) that equals our filter parameter ω. From (4) we get:

$$\omega = \Omega_{3dB} = 2\pi\cdot\frac{f_{3dB}}{f_S} \tag{9}$$

where $f_{3\,dB}$ is the corner frequency in user units (e.g., MHz) and f_S is the sampling rate of the filter in the same units.

The filter equation for the first order DC blocker (or high-pass) filter is then given by:

$$y_k = \left(1 - \frac{\omega}{2}\right)\cdot(x_k - x_{k-1}) + (1-\omega)\cdot y_{k-1} \tag{10}$$

For the second order filter it is given by:

$$\begin{aligned}y_k = &\left(1 - \sqrt{½}\,\omega\right)\cdot(x_k - 2x_{k-1} + x_{k-2}) \\ &+ \left(2 - \sqrt{2}\cdot\omega - ½\,\omega^2\right)\cdot y_{k-1} - \left(1 - \sqrt{½}\,\omega\right)^2\cdot y_{k-2}\end{aligned} \tag{11}$$

For the third order filter this equation is given by:

$$\begin{aligned}y_k = &(1-\omega)\cdot(x_k - 3x_{k-1} + 3x_{k-2} - x_{k-3}) \\ &+ \frac{6-7\omega}{2-\omega}\cdot y_{k-1} - \frac{6+\omega}{2-\omega}\cdot(1-\omega)^2\cdot y_{k-2} + (1-\omega)^2\cdot y_{k-3}\end{aligned} \tag{12}$$

In many cases, the exact value of the corner frequency is not so critical. Then, one can attempt to choose the parameter ω (or $\sqrt{1/2}\cdot\omega$ in case of the 2nd order filter) to be equal to a negative power of 2, like 1/8, 1/32, or so. For 1st

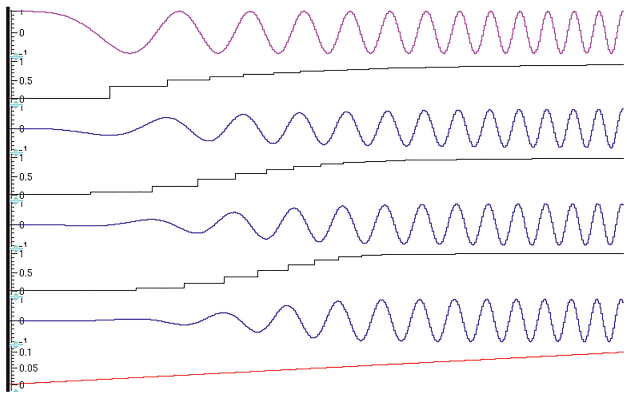

Figure 5: Simulated signal waveforms of 1st–3rd order DC blocker filters.

and 2nd order filters, all filter coefficients can be well expressed as simple fractions in this case. Thus, all mathematical operations can be well executed with integer arithmetics in hardware. This reduces the detrimental effects of rounding errors, which might even lead to instabilities.

5.2 VHDL simulation

To test the designed filter architectures, we have undertaken a simulation of example filters in time domain in the hardware description language VHDL, which would be a preferred candidate for a filter implementation in digital hardware. The simulation presented here uses real arithmetics for all data, but also fixed point simulations were carried out.

Figure 5 shows simulation waveforms of 1st–3rd order filters with $\omega = 1/8$ for the frequency range $0 < \Omega/\pi < 0.1$ (see also Figure 4). The pink wave at the top is the input chirp signal (frequency varies from 0 to 0.1 f_N = 0.05 f_S). The three blue waveforms are the outputs of the 1st–3rd order DC blocker filters. The gray line above each approximates the un-squared transfer function $|H(\Omega)|$ (not $|H(\Omega)|^2$) as determined from the signal maxima. The red curve at the bottom gives the actual signal frequency as fraction of the Nyquist frequency.

6 Conclusions

We have investigated a class of IIR filters, namely high-pass filters with very low corner frequency, also named as DC blockers. The proposed 2nd and 3rd order filters show a significantly smaller and steeper transition range between stop and pass-band than the commonly used 1st order filters. We have proven that the filters are intrinsically stable (i.e., when not considering rounding errors) and provide analytical formulas to calculate the filter coefficients from the desired cutoff frequency.

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