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Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP)   ISSN 1864-7596

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The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

The Weyl Group of the Cuntz Algebra

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16 October 2011

Abstract

The Weyl group of the Cuntz algebra $\mathcal{O}_n$ is investigated. This is (isomorphic to) the group of polynomial automorphisms λ_u of $\mathcal{O}_n$, namely those induced by unitaries u that can be written as finite sums of words in the canonical generating isometries S_i and their adjoints. A necessary and sufficient algorithmic combinatorial condition is found for deciding when a polynomial endomorphism λ_u restricts to an automorphism of the canonical diagonal MASA. Some steps towards a general criterion for invertibility of λ_u on the whole of $\mathcal{O}_n$ are also taken. A condition for verifying invertibility of a certain subclass of polynomial endomorphisms is given. First examples of polynomial automorphisms of $\mathcal{O}_n$ not inner related to permutative ones are exhibited, for every $n \geq 2$. In particular, the image of the Weyl group in the outer automorphism group of $\mathcal{O}_n$ is strictly larger than the image of the reduced Weyl group analyzed in previous papers. Results about the action of the Weyl group on the spectrum of the diagonal are also included.

MSC 2010: Primary 46L40, 46L55, Secondary 37B10

Keywords: Cuntz algebra, MASA, automorphism, endomorphism, Cantor set

[*]This research was supported through the programme "Research in Pairs" by the Mathematisches Forschungsinstitut Oberwolfach in 2011.

[†]This work was supported by National Research Foundation of Korea Grant funded by the Korean Government (KRF–2008–313-C00039).

[‡]Partially supported by the FNU Rammebevilling 'Operator algebras and applications', the NordForsk Research Network 'Operator algebra and dynamics' (grant #11580), and the FNU Forskningsprojekt 'Structure and Symmetry'.

1 Introduction

Consider a finite alphabet $\{1, 2, \ldots, n\}$ with $n \geq 2$ letters, and let $\mathcal{W}$ be the set of finite words on this alphabet. We say that two words are orthogonal if one is not the initial subword of the other. Let Σ be the collection of finite subsets of $\mathcal{W}$ consisting of mutually orthogonal words. We consider the set of n words $\{\alpha 1, \alpha 2, \ldots, \alpha n\}$ (beginning with the same subword α and ending with all the distinct letters of the alphabet) equivalent to the single word α, and this extends to an equivalence relation on Σ. The set of equivalence classes is denoted $\widetilde{\Sigma}$. We fix $U \in \Sigma$, comprised of two ordered subsets: $\{\alpha_1, \ldots, \alpha_r\}$ and $\{\beta_1, \ldots, \beta_r\}$, with the property that both $\{\alpha_1, \ldots, \alpha_r\}$ and $\{\beta_1, \ldots, \beta_r\}$ are equivalent to the empty word. Such a U determines recursively a sequence of transformations $T_k : \widetilde{\Sigma} \to \widetilde{\Sigma}$ such that:

if $\gamma = \alpha_j \mu$ for some j then $T_1(\gamma) = \beta_j \mu$, and

if $T_{k-1}(\gamma) = \nu \alpha_j \mu$ for some j and a word ν of length $k-1$ then $T_k(\gamma) = \nu \beta_j \mu$.

Thus each transformation T_k is determined by a certain Turing machine, [10], and hence it is computable for any finite set of inputs. We are interested in the following **stabilization problem** of the recursive process $T_k \circ T_{k-1} \circ \ldots \circ T_1 : \widetilde{\Sigma} \to \widetilde{\Sigma}$. For what U it holds that for each $A \in \widetilde{\Sigma}$ there exists an m such that for all $k \geq m$ we have $T_k \circ \ldots \circ T_m \circ \ldots \circ T_1 = T_m \circ \ldots \circ T_1$? We provide a surprisingly simple complete solution to this (suitably reformulated in more algebraic terms) stabilization problem in Theorem 3.7, below.

We can reformulate the above described combinatorial setup in topological terms, as follows. Let X_n be the space of all (one-sided) infinite words. Then X_n is a Cantor set with the product topology and elements of $\widetilde{\Sigma}$ are in bijective correspondence with its clopen subsets. Our stabilization problem is then equivalent to injectivity of a certain continuous map $\psi_U : X_n \to X_n$ determined naturally by U. Then, by the Gelfand duality, this problem is equivalent to surjectivity of a unital, injective $*$-homomorphism $\tilde{\psi}_U : C(X_n) \to C(X_n)$, dual to ψ_U. That is, we ask if ψ_U is a homeomorphism of X_n or, equivalently, if $\tilde{\psi}_U$ is an automorphism of $C(X_n)$.

Somewhat paradoxically, it is most natural to view this problem in the context of much larger and noncommutative Cuntz algebras $\mathcal{O}_n$, [8]. These are C^*-algebras generated by n isometries $S_1, \ldots, S_n$ of a Hilbert space with ranges adding up to the identity. In our setting, the element U gives rise to a unitary $u = \sum_{j=1}^{r} S_{\alpha_j} S_{\beta_j}^*$ in $\mathcal{O}_n$, which in turn leads to a necessarily injective, unital $*$-endomorphism λ_u of $\mathcal{O}_n$ such that $\lambda_u(S_j) = uS_j$ for all $j = 1, \ldots, n$. The C^*-subalgebra $\mathcal{D}_n$ of $\mathcal{O}_n$ generated by ranges of all finite products of $S_1, \ldots, S_n$ is maximal abelian in $\mathcal{O}_n$ and naturally isomorphic to $C(X_n)$. The restriction of endomorphism λ_u to $\mathcal{D}_n$ coincides with $\tilde{\psi}_U$. Thus, our combinatorial stabilization problem is equivalent to the **problem of surjectivity** of $\lambda_u|_{\mathcal{D}_n}$. The question of surjectivity of λ_u itself is very interesting as well and closely related to the so called Weyl group of the Cuntz algebra. This last problem appears very difficult and an algorithm for deciding surjectivity of an arbitrary λ_u has not been found yet, although we make some headway towards its solution, below.

The present paper is a continuation of our investigations of the subgroup $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$ of automorphisms of $\mathcal{O}_n$ which globally preserve the canonical diagonal MASA $\mathcal{D}_n$, and of related endomorphisms of $\mathcal{O}_n$, [6, 4, 5, 11, 1, 2]. As shown in [9], the quotient of $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$ by its normal subgroup $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$, consisting of those automorphisms which fix $\mathcal{D}_n$ point-wise, is discrete. Since $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$ is a maximal abelian subgroup of $\mathrm{Aut}(\mathcal{O}_n)$, [9], it is natural to call this quotient the *Weyl group* of $\mathcal{O}_n$. The Weyl group contains a natural interesting subgroup corresponding to those automorphisms which also globally preserve the core UHF-subalgebra $\mathcal{F}_n$ of $\mathcal{O}_n$, called the *restricted Weyl group* of $\mathcal{O}_n$. It was shown in [2] that the image of the restricted Weyl group in the outer automorphism group of $\mathcal{O}_n$ can be embedded into the quotient of the automorphism group of the full two-sided n-shift by its center, and this embedding is surjective whenever n is prime. In the present article, we focus our attention on the (full) Weyl group. It was shown in [6] that the Weyl group is isomorphic with the group of those automorphisms $\lambda_u \in \mathrm{Aut}(\mathcal{O}_n)$ whose corresponding unitaries u may be written as a sum of words in $\{S_i, S_j^*\}$. (The collection of all such unitaries in $\mathcal{O}_n$ is denoted $\mathcal{S}_n$.) The structure of the Weyl group is highly complicated. For example, it contains the Thompson F group in its intersection with $\mathrm{Inn}(\mathcal{O}_n)$, [13]. Our main objective here is investigation of the structure of the Weyl group of $\mathcal{O}_n$, its action on the diagonal MASA, and determining which unitaries $u \in \mathcal{S}_n$ give rise to automorphisms.

The present paper is organized as follows. In section 2, we set up notation and review some basic facts on Cuntz algebras and their endomorphisms. In section 3, we study the restriction of an endomorphism λ_u, $u \in \mathcal{S}_n$, to the diagonal $\mathcal{D}_n$. We give an algorithmic criterion for $\lambda_u|_{\mathcal{D}_n}$ to be an automorphism of $\mathcal{D}_n$, Theorem 3.7. Its proof is combinatorial and involves equivalence of surjectivity of $\lambda_u|_{\mathcal{D}_n}$ with the stabilization problem mentioned above. In section 4, we investigate the problem when λ_u is an automorphism of the entire $\mathcal{O}_n$. In Proposition 4.3, we present a combinatorial procedure for deciding this question for a certain large class of unitaries $u \in \mathcal{S}_n$. In section 5, we exhibit endomorphisms λ_u, $u \in \mathcal{S}_n$, which are not inner related to the ones of the form λ_w with w a unitary in the core UHF-subalgebra $\mathcal{F}_n$. In particular, we show with concrete examples that the image in $\mathrm{Out}(\mathcal{O}_n)$ of the Weyl group is strictly larger then the image of the restricted Weyl group, Theorem 5.2 and Corollary 5.3. Finally, in section 6, we look at the action induced by λ_u on the space X_n, the spectrum of the diagonal $\mathcal{D}_n$. We characterize homeomorphisms of X_n corresponding to automorphisms $\mathrm{Ad}(u)$, $u \in \mathcal{S}_n$, and describe the fixed points in X_n for some exotic automorphisms λ_u.

2 Notation and preliminaries

If n is an integer greater than 1, then the Cuntz algebra $\mathcal{O}_n$ is a unital, simple, purely infinite C^*-algebra generated by n isometries $S_1, \ldots, S_n$ satisfying $\sum_{i=1}^n S_i S_i^* = 1$, [8]. We denote by W_n^k the set of k-tuples $\mu = (\mu_1, \ldots, \mu_k)$ with $\mu_m \in \{1, \ldots, n\}$, and by W_n the union $\cup_{k=0}^{\infty} W_n^k$, where $W_n^0 = \{0\}$. We call elements of W_n multi-indices. If $\mu \in W_n^k$ then $|\mu| = k$ is the length of μ. For $\mu, \nu \in W_n$ we write $\mu \prec \nu$ if μ is an initial subword of ν. If $\mu \in W_n^k$, $\nu \in W_n^m$ and $\mu \prec \nu$, then we denote by $\nu - \mu$ the word in W_n^{m-k}

obtained from ν by removing its initial segment μ. Also, if $\mu \in W_n^k$ then we denote by $s(\mu)$ its first letter, and by $\tilde{\mu}$ the word in W_n^{k-1} obtained from μ by removing $s(\mu)$. We denote by $\mu \wedge \nu$ the collection of all non-empty words η such that both $\eta \prec \mu$ and $\eta \prec \nu$. If $\mu = (\mu_1, \dots, \mu_k) \in W_n$ then $S_\mu = S_{\mu_1} \dots S_{\mu_k}$ ($S_0 = 1$ by convention) is an isometry with range projection $P_\mu = S_\mu S_\mu^*$. Every word in $\{S_i, S_i^* \mid i = 1, \dots, n\}$ can be uniquely expressed as $S_\mu S_\nu^*$, for $\mu, \nu \in W_n$ [8, Lemma 1.3].

We denote by $\mathcal{F}_n^k$ the C^*-subalgebra of $\mathcal{O}_n$ spanned by all words of the form $S_\mu S_\nu^*$, $\mu, \nu \in W_n^k$, which is isomorphic to the matrix algebra $M_{n^k}(\mathbb{C})$. The norm closure $\mathcal{F}_n$ of $\cup_{k=0}^\infty \mathcal{F}_n^k$ is the UHF-algebra of type n^∞, called the core UHF-subalgebra of $\mathcal{O}_n$, [8]. We denote by τ the unique normalized trace on $\mathcal{F}_n$. The core UHF-subalgebra $\mathcal{F}_n$ is the fixed-point algebra for the gauge action $\gamma : U(1) \to \mathrm{Aut}(\mathcal{O}_n)$, such that $\gamma_z(S_j) = zS_j$ for $z \in U(1)$ and $j = 1, \dots, n$. We denote by E the faithful conditional expectation from $\mathcal{O}_n$ onto $\mathcal{F}_n$ given by averaging with respect to the normalized Haar measure:

$$E(x) = \int_{z \in U(1)} \gamma_z(x) dz.$$

For an integer $m \in \mathbb{Z}$ we denote $\mathcal{O}_n^{(m)} := \{x \in \mathcal{O}_n : \gamma_z(x) = z^m x, \forall z \in U(1)\}$, a spectral subspace for γ. Then $\mathcal{O}_n^{(0)} = \mathcal{F}_n$ and for each positive integer m and each $\alpha \in W_n^m$ we have $\mathcal{O}_n^{(m)} = \mathcal{F}_n S_\alpha$ and $\mathcal{O}_n^{(-m)} = S_\alpha^* \mathcal{F}_n$.

The C^*-subalgebra of $\mathcal{O}_n$ generated by projections P_μ, $\mu \in W_n$, is a MASA (maximal abelian subalgebra) in $\mathcal{O}_n$. We call it the *diagonal* and denote $\mathcal{D}_n$. Every projection in $\mathcal{D}_n$ of the form P_α for some $\alpha \in W_n$ will be called *standard*. The spectrum of $\mathcal{D}_n$ is naturally identified with X_n — the full one-sided n-shift space. For $d \in \mathcal{D}_n$ we denote by M_d a map $M_d : \mathcal{D}_n \to \mathcal{D}_n$ such that $M_d(x) = dx$.

As shown by Cuntz in [9], there exists the following bijective correspondence between unitaries in $\mathcal{O}_n$ (whose collection is denoted $\mathcal{U}(\mathcal{O}_n)$) and unital $*$-endomorphisms of $\mathcal{O}_n$ (whose collection we denote $\mathrm{End}(\mathcal{O}_n)$). A unitary $u \in \mathcal{U}(\mathcal{O}_n)$ determines an endomorphism λ_u by

$$\lambda_u(S_i) = uS_i, \quad i = 1, \dots, n.$$

Conversely, if $\rho : \mathcal{O}_n \to \mathcal{O}_n$ is an endomorphism, then $\sum_{i=1}^n \rho(S_i)S_i^* = u$ gives a unitary $u \in \mathcal{O}_n$ such that $\rho = \lambda_u$. Composition of endomorphisms corresponds to a 'convolution' multiplication of unitaries as follows:

$$\lambda_u \circ \lambda_w = \lambda_{\lambda_u(w)u}. \tag{1}$$

If A is either a unital C^*-subalgebra of $\mathcal{O}_n$ or a subset of $\mathcal{U}(\mathcal{O}_n)$, then we denote $\lambda(A) = \{\lambda_u \in \mathrm{End}(\mathcal{O}_n) : u \text{ unitary in } A\}$ and $\lambda(A)^{-1} = \{\lambda_u \in \mathrm{Aut}(\mathcal{O}_n) : u \text{ unitary in } A\}$.

We denote by φ the canonical shift:

$$\varphi(x) = \sum_{i=1}^n S_i x S_i^*, \quad x \in \mathcal{O}_n.$$

If we take $u = \sum_{i,j=1}^n S_i S_j S_i^* S_j^*$ then $\varphi = \lambda_u$. For all $u \in \mathcal{U}(\mathcal{O}_n)$ we have $\mathrm{Ad}(u) = \lambda_{u\varphi(u^*)}$. It is well-known that φ leaves $\mathcal{D}_n$ globally invariant. We denote by ϕ the standard left inverse of φ, defined as

$$\phi(x) = \frac{1}{n}\sum_{i=1}^{n} S_i^* x S_i, \quad x \in \mathcal{O}_n.$$

If $u \in \mathcal{U}(\mathcal{O}_n)$ then for each positive integer k we denote

$$u_k = u\varphi(u)\cdots\varphi^{k-1}(u). \tag{2}$$

Here $\varphi^0 = \mathrm{id}$, and we agree that u_k^* stands for $(u_k)^*$. If α and β are multi-indices of length k and m, respectively, then $\lambda_u(S_\alpha S_\beta^*) = u_k S_\alpha S_\beta^* u_m^*$. This is established through a repeated application of the identity $S_i x = \varphi(x) S_i$, valid for all $i = 1, \ldots, n$ and $x \in \mathcal{O}_n$.

We often consider elements of $\mathcal{O}_n$ of the form $w = \sum_{(\alpha,\beta)\in\mathcal{J}} c_{\alpha,\beta} S_\alpha S_\beta^*$, where $\mathcal{J}$ is a finite collection of pairs (α, β) of words $\alpha, \beta \in W_n$ and $c_{\alpha,\beta} \in \mathbb{C}$. We denote $\mathcal{J}_1 = \{\alpha : \exists(\alpha, \beta) \in \mathcal{J}\}$ and $\mathcal{J}_2 = \{\beta : \exists(\alpha, \beta) \in \mathcal{J}\}$. Of course, such a presentation (if it exists) is not unique, but once it is chosen then we associate with it two integers: $\ell = \ell(\mathcal{J}) = \max\{|\alpha| : (\alpha, \beta) \in \mathcal{J}\}$ and $\ell' = \ell'(\mathcal{J}) = \max\{|\alpha|, |\beta| : (\alpha, \beta) \in \mathcal{J}\}$. Note that if $w \in \mathcal{F}_n$ then $w \in \mathcal{F}_n^\ell$. We have $\varphi(w) = \sum_{(\mu,\nu)\in\varphi(\mathcal{J})} c_{\mu,\nu} S_\mu S_\nu^*$, where $\varphi(\mathcal{J}) = \{((i, \alpha), (\beta, i)) : i \in W_n^1,\ (\alpha, \beta) \in \mathcal{J}\}$ and $c_{(i,\alpha),(\beta,i)} = c_{\alpha,\beta}$. Then $\ell(\varphi(\mathcal{J})) = \ell(\mathcal{J}) + 1$ and $\ell'(\varphi(\mathcal{J})) = \ell'(\mathcal{J}) + 1$. In particular, we consider the group $\mathcal{S}_n$ of those unitaries in $\mathcal{O}_n$ which can be written as finite sums of words, i.e. in the form $u = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^*$. Note that such a sum is a unitary if and only if $\sum_{\alpha\in\mathcal{J}_1} P_\alpha = 1 = \sum_{\beta\in\mathcal{J}_2} P_\beta$. We also write $\mathcal{P}_n = \mathcal{S}_n \cap \mathcal{F}_n$ and $\mathcal{P}_n^k = \mathcal{S}_n \cap \mathcal{F}_n^k$ for the subgroups of $\mathcal{S}_n$ consisting of permutative unitaries.

For algebras $A \subseteq B$ we denote by $\mathcal{N}_B(A) = \{u \in \mathcal{U}(B) : uAu^* = A\}$ the normalizer of A in B and by $A' \cap B = \{b \in B : (\forall a \in A)\ ab = ba\}$ the relative commutant of A in B. We also denote by $\mathrm{Aut}(B, A)$ the collection of all those automorphisms α of B such that $\alpha(A) = A$, and by $\mathrm{Aut}_A(B)$ those automorphisms of B which fix A point-wise.

$\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$ is a normal subgroup of $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$, and the corresponding quotient is called the *Weyl group* of $\mathcal{O}_n$. It was shown in [9] that the Weyl group is discrete, and more recently in [6] that it is isomorphic to $\lambda(\mathcal{S}_n)^{-1}$. The quotient of $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \cap \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$ by $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$ is called the *restricted Weyl group* of $\mathcal{O}_n$. It is isomorphic to $\lambda(\mathcal{P}_n)^{-1}$, [6]. The image of $\lambda(\mathcal{S}_n)^{-1}$ in $\mathrm{Out}(\mathcal{O}_n)$ is called the *outer Weyl group* of $\mathcal{O}_n$ and such image of $\lambda(\mathcal{P}_n)^{-1}$ is called the *restricted outer Weyl group* of $\mathcal{O}_n$. As shown in [3, Theorem 3.7], the outer Weyl group is just the quotient of $\lambda(\mathcal{S}_n)^{-1}$ by $\{\mathrm{Ad}(u) : u \in \mathcal{S}_n\}$. Likewise, the restricted outer Weyl group is the quotient of $\lambda(\mathcal{P}_n)^{-1}$ by $\{\mathrm{Ad}(w) : w \in \mathcal{P}_n\}$.

3 The automorphisms of the diagonal

In this section, we give an algorithmic crierion for deciding if the restriction to $\mathcal{D}_n$ of an endomorphisms λ_u, $u \in \mathcal{S}_n$, gives rise to an automorphism of the diagonal $\mathcal{D}_n$.

Lemma 3.1 *Let $u \in \mathcal{S}_n$ be such that $u = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^*$, and let $\ell = \ell(\mathcal{J})$. Then $\lambda_u(\mathcal{D}_n^k) \subseteq \mathcal{D}_n^{k\ell}$ for all $k \in \mathbb{N}$.*

Proof. We proceed by induction on k. For $k = 1$ and $i \in W_n^1$ we have

$$\lambda_u(P_i) = uP_iu^* = \sum_{(\alpha,\beta),\,(\alpha',\beta')\in\mathcal{J}} S_\alpha S_\beta^* P_i S_{\beta'} S_{\alpha'}^* = \sum_{(\alpha,\beta)\in\mathcal{J},\, s(\beta)=i} S_\alpha S_\alpha^*$$

and thus $\lambda_u(\mathcal{D}_n^1) \subseteq \mathcal{D}_n^\ell$. For the inductive step, suppose that $\lambda_u(\mathcal{D}_n^k) \subseteq \mathcal{D}_n^{k\ell}$. Then

$$\lambda_u(\mathcal{D}_n^{k+1}) = \lambda_u(\mathcal{D}_n^1\varphi(\mathcal{D}_n^k)) = \lambda_u(\mathcal{D}_n^1)(\mathrm{Ad}(u)\varphi\lambda_u)(\mathcal{D}_n^k) \subseteq \mathcal{D}_n^\ell(\mathrm{Ad}(u)\varphi)(\mathcal{D}_n^{k\ell}) \subseteq \mathcal{D}_n^\ell\mathcal{D}_n^{(k+1)\ell}$$

and thus $\lambda_u(\mathcal{D}_n^{k+1}) \subseteq \mathcal{D}_n^{(k+1)\ell}$. □

Proposition 3.2 *Let $u \in \mathcal{S}_n$. Then the following hold.*

1. *$\lambda_u|_{\mathcal{D}_n}$ is an automorphism of $\mathcal{D}_n$ if and only if for each $\alpha \in W_n$ the sequence $\{u_k^* P_\alpha u_k\}$ eventually stabilizes.*
2. *λ_u is an automorphism of $\mathcal{O}_n$ if and only if:*
   - *(a) $\lambda_u|_{\mathcal{D}_n}$ is an automorphism of $\mathcal{D}_n$, and*
   - *(b) there exists a $w \in \mathcal{S}_n$ such that $\lambda_w|_{\mathcal{D}_n} = (\lambda_u|_{\mathcal{D}_n})^{-1}$.*

Proof. Ad 1. This is well-known, [9]. Indeed, the sequence $\{u_k^* P_\alpha u_k\}$ eventually stabilizes if and only if P_α belongs to the range of λ_u (and then $\lambda_u(\lim u_k^* P_\alpha u_k) = P_\alpha$). Thus, condition 1. is equivalent to $\lambda_u(\mathcal{D}_n) = \mathcal{D}_n$, i.e. to $\lambda_u|_{\mathcal{D}_n}$ being an automorphism of $\mathcal{D}_n$.

Ad 2. If λ_u is automorphism of $\mathcal{O}_n$, then $\lambda_u(\mathcal{D}_n) \subseteq \mathcal{D}_n$ since $u \in \mathcal{N}_{\mathcal{O}_n}(\mathcal{D}_n)$. Thus $\lambda_u(\mathcal{D}_n) = \mathcal{D}_n$, since $\mathcal{D}_n$ is a MASA in $\mathcal{O}_n$. Also, there exists $w \in \mathcal{S}_n$ such that $\lambda_u^{-1} = \lambda_w$, [14, 6, 12]. This gives one implication of part 2. For the reversed implication, suppose that (a) and (b) hold. Then $\lambda_u\lambda_w|_{\mathcal{D}_n} = \mathrm{id}$. Thus $\lambda_u\lambda_w$ is an automorphism of $\mathcal{O}_n$ by [1, Proposition 3.2]. Consequently, λ_u being surjective is automorphism of $\mathcal{O}_n$. □

Example 3.3 **(a)** If $u = S_1S_1S_1^* + S_1S_2S_1^*S_2^* + S_2S_2^*S_2^* \in \mathcal{S}_2$ then $\lambda_u|_{\mathcal{D}_2}$ is not surjective. Indeed, a straightforward calculation shows that $\lambda_u(\mathcal{D}_2)P_2 = \mathbb{C}P_2$.

(b) If $u = S_2S_1S_1^* + S_2S_2S_1^*S_2^* + S_1S_2^*S_2^* \in \mathcal{S}_2$ then $\lambda_u|_{\mathcal{D}_2}$ is not surjective. Indeed, projection P_{12} does not satisfy (Condition 1) of Proposition 3.2.

(c) If $u = u^* = S_1S_2^*S_2^* + P_{21} + S_2S_2S_1^* \in \mathcal{S}_2$ then $\lambda_u|_{\mathcal{D}_2}$ is not surjective. Indeed, projection P_{11} does not satisfy (Condition 1) of Proposition 3.2.

Our next result shows that in order to verify (Condition 1) in Proposition 3.2 it is enough to check it only for finitely many projections. Before that, we note the following. Let $u \in \mathcal{S}_n$ be such that $u = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^*$. Then for each word $\mu \in W_n$ and for each $(\alpha,\beta) \in \mathcal{J}$ we have

$$\mathrm{Ad}(u)(P_{\beta\mu}) = P_{\alpha\mu}. \tag{3}$$

In particular, $\mathrm{Ad}(P_\beta) = P_\alpha$.

Lemma 3.4 *Let $u \in \mathcal{S}_n$ be such that $u = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^*$, and let $\ell' = \ell'(\mathcal{J})$. Then $\lambda_u|_{\mathcal{D}_n}$ is an automorphism of $\mathcal{D}_n$ if and only if for each $\gamma \in W_n^{\ell'}$ the sequence $\{u_k^* P_\gamma u_k\}$ eventually stabilizes.*

Proof. For short, say a projection $Q \in \mathcal{D}_n$ is "bad" (relative to u) if the sequence $\{u_k^* Q u_k\}$ does not stabilize, and "good" otherwise. Also, let r be the non-negative integer uniquely defined by requiring that all projections in $\mathcal{D}_n^r$ are good, but there is a bad projection in $\mathcal{D}_n^{r+1}$. Then at least one of the minimal projections in $\mathcal{D}_n^{r+1}$ is bad as well. We claim that $r+1 \leq \ell'$.

Reasoning by way of contradiction, suppose that $\ell' < r+1$ and let $p = P_\gamma$, $\gamma \in W_n^{r+1}$, be such a bad minimal projection in $\mathcal{D}_n^{r+1}$. Now, u^*pu can be computed using equation (3), with u replaced by u^*, and hence it is still of the form P_{γ_1} for some $\gamma_1 \in W_n$. In this process, by replacing the initial α-segment of γ with the corresponding β, the last $r+1-\ell'$ digits will remain unaltered. Now, the assumption that p is bad easily implies that the projection $P_\delta := n\phi(u^*pu)$, obtained from u^*pu by deleting the first digit of γ_1, is still bad. By assumption, one must have $|\delta| \geq r+1$, and hence $u^*pu \notin \mathcal{D}_n^{r+1}$. In other words, when computing u^*pu we have replaced a word α in γ with a longer word β. This implies that when in the next step we consider $P_{\gamma_2} := \varphi(u)^* u^* pu \varphi(u)$, the last $r+1-\ell'$ digits of γ_2 will coincide again with those of γ. Also, $(n\phi)^2(P_{\gamma_2})$ must be bad, i.e. $u_2^* p u_2 = P_{\gamma_2} \notin \mathcal{D}_n^{r+2}$. Repeating this argument, one can indeed show that $u_k^* p u_k = P_{\gamma_k} \notin \mathcal{D}_n^{r+k}$ for all $k = 1, 2, \ldots$, and moreover the last $r+1-\ell'$ digits of γ_k coincide with those of γ for any k. All in all, this means that these last digits of γ indeed play no role in the whole process and defining γ' simply to be the multi-index obtained from γ by deleting its last digit, the very same argument would readily show that $P_{\gamma'}$ is still bad. But then $P_{\gamma'} \in \mathcal{D}_n^r$, contradicting our assumption.

By the above, if there are bad projections at all, we can find at least one of them in $\mathcal{D}_n^{\ell'}$. As a sum of good projections is clearly good, it is also clear that in that case there is always such a bad projection of the form P_γ, where $|\gamma| = \ell'$. ☐

All in all, for $u \in \mathcal{S}_n$ one has

$$\lambda_u(\mathcal{D}_n) = \mathcal{D}_n \;\Leftrightarrow\; \mathcal{D}_n^{\ell'} \subseteq \lambda_u(\mathcal{D}_n), \tag{4}$$

where ℓ' is as in the statement of Lemma 3.4.

In view of Lemma 3.4, the process of determination if an endomorphism $\lambda_u|_{\mathcal{D}_n}$, $u \in \mathcal{S}_n$, is an automorphism of the diagonal can be reduced to verification if a certain finite collection of projections is contained in its range. This is a very significant reduction but still it is not clear a priori if this process can be carried out in finately many steps even for a single projection! This question has a positive answer in the case of a permutative unitary $u \in \mathcal{P}_n$, as shown in [14, 6], but the present case is much more complicated. Now, we will describe a key construction of the present paper, producing a certain finite directed graph corresponding to a unitary $u \in \mathcal{S}_n$. Non occurence of closed paths on the graph will turn out to be equivalent to $\lambda_u|_{\mathcal{D}_n}$ being automorphism of $\mathcal{D}_n$.

Given $u = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^*$ in $\mathcal{S}_n$, we define a **finite directed graph** Γ_u, whose vertices Γ_u^0 will be identified with certain subsets of $\mathcal{J}_1$. In order to construct the graph Γ_u, we proceed by induction.

The initial step. To begin with, we include in Γ_u^0 each singleton subset $\{\alpha\}$ of $\mathcal{J}_1$ and the empty set $\emptyset$. Now, given $(\alpha, \beta) \in \mathcal{J}$, one of the following three cases takes place:

(i) $\beta = (i)$ for some $i \in W_n^1$,

(ii) $\beta = (i, \alpha', \mu)$ for some $i \in W_n^1$, $\alpha' \in \mathcal{J}_1$, and a word μ (possibly empty),

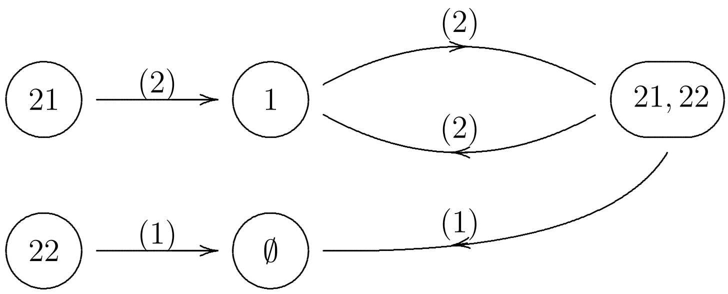
(iii) $\beta = (i, \mu)$ for some $i \in W_n^1$ and a word μ which is an initial segment of at least two elements of $\mathcal{J}_1$, namely $\alpha'_1, \ldots, \alpha'_r$.

Depending on the case, we enlarge the graph Γ_u as follows. In case (i), we add an edge from vertex $\{\alpha\}$ to vertex $\emptyset$ with label i. In case (ii), we add an edge from vertex $\{\alpha\}$ to vertex $\{\alpha'\}$ with label i. In case (iii), we add a vertex $A = \{\alpha'_1, \ldots, \alpha'_r\}$ and an edge from $\{\alpha\}$ to A with label i.

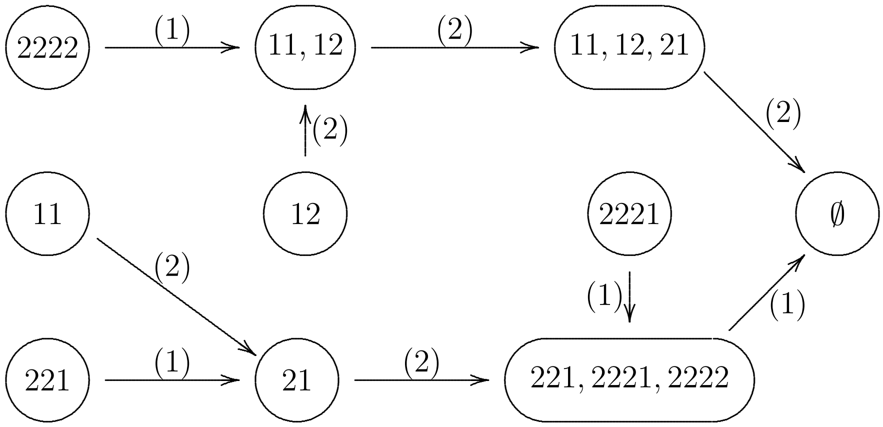
The inductive step. Let $A \subseteq \mathcal{J}_1$ be a vertex added to Γ_u^0 in the preceding step, but $A \neq \emptyset$ and A not a singleton set. For each $j \in W_n^1$ we proceed as follows. Let B_k, $k = 1, \ldots, m$, be the collection of all those already constructed vertices of Γ_u that there exists an $\alpha \in A$ and an edge from $\{\alpha\}$ to B_k with label j. If $\bigcup_{k=1}^m B_k = \mathcal{J}_1$ then we add an edge from A to $\emptyset$ with label j. If $\bigcup_{k=1}^m B_k \neq \mathcal{J}_1$ then we add a vertex $B = \bigcup_{k=1}^m B_k$ (if such a vertex does not exist already), and we add an edge from A to B with label j.

Continuing inductively in the above described manner, we produce the desired graph Γ_u. This is a finite, directed, and labeled graph. Each vertex emits at most n edges, carrying distinct labels from the set W_n^1. Any finite path on the graph Γ_u may be uniquely identified with a pair (A, ν), where $A \in \Gamma_u^0$ is the initial vertex of the path and $\nu = (\nu_1, \nu_2, \ldots, \nu_k)$ is the word such that ν_j is the label of the j^{th} edge entering this path. For such a path (A, ν), we denote its terminal vertex by $\nu(A)$. We will denote by Γ_u^1 the set of edges of the graph, by Γ_u^k the set of paths of length k, and by Γ_u^* the set of finite paths. $\Gamma_u^*(A)$ and $\Gamma_u^k(A)$, respectively, are the sets of finite paths and paths of length k which begin at the vertex A.

Example 3.5 Let $u = S_1 S_2^* S_2^* + S_2 S_1 S_1^* S_2^* + S_2 S_2 S_1^*$. Then the corresponding graph Γ_u has five vertices and five edges, and looks as follows. In particular, there is a closed (directed) path on the graph.



Example 3.6 Let $u = S_{12}S_{21}^* + S_{11}S_{221}^* + S_{21}S_{222}^* + S_{2222}S_{11}^* + S_{2221}S_{122}^* + S_{221}S_{121}^*$. Then the corresponding graph Γ_u looks as follows:



For $\alpha \in \mathcal{J}_1$, we say that $\{\alpha\}$ is a *splitting vertex* if it emits an edge to a vertex $A \subseteq \mathcal{J}_1$ such that A contains at least two elements. This happens when for $(\alpha, \beta) \in \mathcal{J}$ we have that β is an initial subword of more than one $\alpha \in \mathcal{J}_1$. For example, $\alpha = 1$ in Example 3.5 and $\alpha_1 = 12$, $\alpha_2 = 21$, $\alpha_3 = 2221$ and $\alpha_4 = 2222$ in Example 3.6 are all splitting vertices.

The point of introducing graph Γ_u is that it conveniently captures the essential features of the process of calculating $u_k^* P_\alpha u_k$, appearing in part 1 of Proposition 3.2. Indeed, for $A \in \Gamma_u^0$ denote $P_A := \sum_{\alpha \in A} P_\alpha$. Then we have

$$\mathrm{Ad}(u^*)(P_A) = \sum_{(A,j)\in\Gamma_u^1(A)} \sum_{\alpha\in j(A)} P_{j\alpha}. \tag{5}$$

Clearly, if $\mathrm{Ad}(u^*)(P_\mu) = \sum_k P_{\nu_k}$, then for each $i \in W_n^1$ we have

$$\mathrm{Ad}(\varphi(u^*))(P_{i\mu}) = \sum_k P_{i\nu_k}. \tag{6}$$

Combining (5) with (6) and proceeding by induction on k, we see that for any $A \in \Gamma_u^0$ and a non-negative integer k we have

$$\mathrm{Ad}(u_k^*)(P_A) = \sum_{(A,\nu)\in\Gamma_u^k(A)} \sum_{\alpha\in\nu(A)} P_{\nu\alpha} + \sum_{m=1}^{k-1} \sum_{\substack{(A,\mu)\in\Gamma_u^m(A)\\ \mu(A)=\emptyset}} P_\mu. \tag{7}$$

Now, we are ready to prove a theorem which gives an algorithmic (finite) procedure for determining if an endomorphism λ_u, $u \in \mathcal{S}_n$, restricts to an automorphism of the diagonal $\mathcal{D}_n$.

Theorem 3.7 *Let $u \in \mathcal{S}_n$ and let Γ_u be the directed graph corresponding to u. Then $\lambda_u|_{\mathcal{D}_n}$ is an automorphism of $\mathcal{D}_n$ if and only if graph Γ_u does not contain any closed (directed) paths.*

Proof. Firstly, suppose that there is a closed path

$$A_1 \xrightarrow{(i_1)} A_2 \xrightarrow{(i_2)} \ldots \xrightarrow{(i_{r-1})} A_r \xrightarrow{(i_r)} A_1$$

in the graph Γ_u. We denote $\nu = (i_1, i_2, \ldots, i_r)$ and $\nu^k = \nu\nu\cdots\nu$ (k-fold composition). With help of formula (7) we see that

$$\mathrm{Ad}(u_{kr}^*)(P_{A_1}) = P_{\nu^k}\varphi^{kr}(P_{A_1}) + \sum_{\mu\wedge\nu^k=\emptyset} P_\mu. \tag{8}$$

Given any $k < k'$ there exists a non-zero projection $q \in \mathcal{D}_n$ such that $q \le P_{\nu^k}$ and $qP_{\nu^{k'}} = 0$. But then formula (8) implies that $q\,\mathrm{Ad}(u_{kr}^*)(P_{A_1}) \neq 0$ while $q\,\mathrm{Ad}(u_{k'r}^*)(P_{A_1}) = 0$. Thus the sequence $\{\mathrm{Ad}(u_m^*)(P_{A_1})\}$ never stabilizes and, consequently, projection P_{A_1} does not belong to $\lambda_u(\mathcal{D}_n)$, Proposition 3.2.

Conversely, suppose that graph Γ_u does not contain any closed paths. By virtue of Lemma 3.4, it suffices to show that the sequence $\{\mathrm{Ad}(u_k^*)(P_\mu)\}$ eventually stabilizes for each $\mu \in W_n^{\ell'}$ with $\ell' = \ell'(\mathcal{J})$. To this end, consider the following three cases.

Firstly, we consider the case of P_α, $\alpha \in \mathcal{J}_1$. Since Γ_u is a finite graph without closed paths, there are only finitely many paths and each of them terminates at a sink. By construction, graph Γ_u contains exactly one sink, namely vertex $\emptyset$. Thus formula (7) applied to $A = \{\alpha\}$ shows that for sufficiently large k we have

$$\mathrm{Ad}(u_k^*)(P_\alpha) = \sum_{\substack{(A,\mu)\in\Gamma_u^*(\{\alpha\})\\ \mu(\{\alpha\})=\emptyset}} P_\mu,$$

and thus the sequence $\{\mathrm{Ad}(u_k^*)(P_\alpha)\}$ eventually stabilizes.

Secondly, we consider a word μ such that there exists an $\alpha \in \mathcal{J}_1$ with $\mu \prec \alpha$. Then $P_\mu = \sum P_{\alpha'}$, where the sum is over all such $\alpha' \in \mathcal{J}_1$ that $\mu \prec \alpha'$. In this case, the sequence $\{\mathrm{Ad}(u_k^*)(P_\mu)\}$ stabilizes by the preceding argument.

Thirdly, we must consider the case with μ a word of length at most ℓ' for which there exists an $\alpha \in \mathcal{J}_1$ such that $\alpha \prec \mu$. Then write $\mu = \alpha\nu$. Let $(\{\alpha\}, \eta)$ be the maximal path beginning at $\{\alpha\}$ and such that each vertex on the path is a singleton subset of $\mathcal{J}_1$. Let k be the length of this path. Using formula (7), we see that

$$\mathrm{Ad}(u_k^*)(P_\mu) = P_{\eta\alpha'\nu} \tag{9}$$

for some $\alpha' \in \mathcal{J}_1$. Now, one of the following two cases happens: either $\{\alpha'\}$ emits an edge (with label j) to the sink $\emptyset$, or $\{\alpha'\}$ is a splitting vertex. In the former case, we have $\mathrm{Ad}(u_{k+1}^*)(P_\mu) = P_{\eta j\nu}$, and the question of stabilization of the sequence corresponding to the word μ reduces to the same question for the sequence corresponding to the word ν, which is strictly shorter then μ. In the latter case, let $\{\alpha'\}$ emit an edge (with label i) to a vertex A. Then we have $\mathrm{Ad}(u^*)(\mathcal{P}_{\alpha'}) = P_{\beta'} = \sum_{j=1}^m P_{i\alpha_j\nu_j}$, for some $\alpha_j \in \mathcal{J}_1$ and words ν_j such that each ν_j is strictly shorter then ν. Taking into account formula (9), we obtain $\mathrm{Ad}(u_{k+1}^*)(P_\mu) = \sum_{j=1}^m P_{\eta i\alpha_j\nu_j}$. Thus, the question if the sequence $\{\mathrm{Ad}(u_k^*)(P_\mu)\}$ stabilizes (with $\mu = \alpha\nu$) reduces to the same question for all $\mu_j = \alpha_j\nu_j$, where $|\nu_j| < |\nu|$. Consequently, the claim follows for all words $\mu = \alpha\nu$, $\alpha \in \mathcal{J}_1$, by induction on $|\nu|$. □

Remark 3.8 We note that for certain special classes of unitaries $u \in \mathcal{S}_n$, different criteria for $\lambda_u|_{\mathcal{D}_n} \in \mathrm{Aut}(\mathcal{D}_n)$ were given earlier in [7].

4 The invertibility

In this section, we consider the problem when λ_u, $u \in \mathcal{S}_n$, is an automorphism of $\mathcal{O}_n$. Recall that $E : \mathcal{O}_n \to \mathcal{F}_n$ is the gauge invariant conditional expectation, and for a $\beta \in W_n^k$ the symbol $\tilde{\beta}$ denotes the word in W_n^{k-1} obtained from β by removing its first letter.

Lemma 4.1 *If $u \in \mathcal{S}_n$ is arbitrary then there exists a $v \in \mathcal{S}_n$ such that $E(w) \neq 0$ for $w = vu\varphi(v^*)$.*

Proof. Let $u = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^*$ and suppose that $E(u) = 0$. If $v \in \mathcal{S}_n$ then $vu\varphi(v^*) = \sum_{(\alpha,\beta)\in\mathcal{J}} vS_\alpha S_{\tilde{\beta}}^* v^* S_{\beta_1}^*$. Thus, it suffices to find a $v \in \mathcal{S}_n$ such that for certain $(\alpha, \beta) \in \mathcal{J}$ we have $vS_\alpha S_{\tilde{\beta}}^* v^* \in \mathcal{O}_n^{(1)}$. Since $E(u) = 0$, there exists $(\alpha, \beta) \in \mathcal{J}$ with $|\alpha| > |\beta|$. Now, one of the following two cases takes place: either P_α is orthogonal to $P_{\tilde{\beta}}$ or $\tilde{\beta} \prec \alpha$ and $\tilde{\beta} \neq \alpha$. In the former case, put $v = S_1^2 S_\alpha^* + S_2 S_{\tilde{\beta}}^*$ +(other terms). In the latter, we have $\alpha = \tilde{\beta}\mu$. Take any $\nu \neq \mu$ with $|\nu| = |\mu|$ and put $v = S_1 S_{\tilde{\beta}\mu}^* + S_2^{|\mu|} S_{\tilde{\beta}\nu}^*$ + (other terms). Then $w = vu\varphi(v^*)$ has the required form. □

Let $u = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^*$ and $\ell = \ell(\mathcal{J})$. Assume $\mathcal{D}_n^\ell \subseteq \lambda_u(\mathcal{O}_n)$. Then for each $(\alpha, \beta) \in \mathcal{J}$ and $j = 1, \ldots, n$ the element $S_\alpha S_\beta^* S_j = P_\alpha \lambda_u(S_j)$ belongs to $\lambda_u(\mathcal{O}_n)$. Denote by $\mathcal{Z}_u$ the collection of all finite products of these elements $S_\alpha S_{\tilde{\beta}}^*$ and their adjoints. The linear span of $\mathcal{Z}_u$ is dense in $\lambda_u(\mathcal{O}_n)$. Also, we denote by $\langle \mathcal{Z}_u \rangle$ the collection of all sums of elements from $\mathcal{Z}_u$.

Lemma 4.2 *Let $u = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^*$. Denote $\ell = \ell(\mathcal{J})$ and let $k \geq \ell$ be any integer such that there exists a $z \in \mathcal{Z}_u$, a word of length $2k - 1$, with $z \in \mathcal{O}_n^{(1)}$. Assume that $\lambda_u(\mathcal{D}_n) = \mathcal{D}_n$ and $E(u) \neq 0$. Then for λ_u to be an automorphism of $\mathcal{O}_n$ it suffices that $\mathcal{F}_n^k \subseteq \lambda_u(\mathcal{O}_n)$. If this is the case then each $S_\mu S_\nu^*$ with $\mu, \nu \in W_n^k$ belongs to $\langle \mathcal{Z}_u \rangle$.*

Proof. At first we note that a word z, as in the statement of this lemma, exists since $E(u) \neq 0$ by assumption. Then observe that $\varphi^k(S_i)$ belongs to $\lambda_u(\mathcal{O}_n)$ for all $i = 1, \ldots, n$. Hence $\varphi^k(\mathcal{F}_n) \subseteq \lambda_u(\mathcal{O}_n)$, and consequently $\lambda_u(\mathcal{O}_n)$ contains the entire $\mathcal{F}_n$. Thus $\mathcal{F}_n$ and $\varphi^k(S_i)$ are contained in $\lambda_u(\mathcal{O}_n)$ and we conclude that $\lambda_u(\mathcal{O}_n) = \mathcal{O}_n$.

We have $\langle \mathcal{Z}_u \rangle = \lambda_u(\langle\{S_\mu S_\nu^* : \mu, \nu \in W_n\}\rangle) \subseteq \langle\{S_\mu S_\nu^* : \mu, \nu \in W_n\}\rangle$. Now, if λ_u is invertible then there exists a unitary $w \in \mathcal{S}_n$ such that $\lambda_u^{-1} = \lambda_w$. Thus we have

$$\lambda_u^{-1}(\langle\{S_\mu S_\nu^* : \mu, \nu \in W_n\}\rangle) \subseteq \langle\{S_\mu S_\nu^* : \mu, \nu \in W_n\}\rangle$$

and, consequently, $\langle \mathcal{Z}_u \rangle = \langle\{S_\mu S_\nu^* : \mu, \nu \in W_n\}\rangle$. □

For a while, we restrict our attention to unitaries $u = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^*$ such that $|\alpha| - |\beta| \in \{0, \pm 1\}$ for all $(\alpha, \beta) \in \mathcal{J}$. (We note that endomorphisms corresponding to such unitaries were studied earlier in [5].)

Given a unitary u as above, we may always find its presentation such that the lengths of all α coincide. Let k be this common length. Then the collection of all α entering the presentation of u is equal to W_n^k. Now, we define a **new directed graph** Δ_u, as follows. The set of vertices of Δ_u is just W_n^k. We put an edge from α_1 to α_2 whenever $P_{\alpha_2} \leq P_{\tilde{\beta}_1}$. In view of our assumptions on u, the difference $|\alpha| - |\tilde{\beta}|$ is 0, 1 or 2. We call this difference the degree of vertex α and of each edge emitted by α. If d is the degree of α then the vertex α emits exactly n^d edges, which end at distinct vertices. With each edge of degree $d > 0$ from α_1 to α_2, we associate a *label*, which is the terminal subword of length d of α_2. Edges of degree 0 carry empty labels. We extend so defined labels from edges to finite directed paths on Δ_u by concatenation. Also, we define the *degree* of a path on Δ_u as the sum of the degrees of its edges. We denote the label of a path x by $L(x)$ and its degree by $\deg(x)$.

Now, let Δ_u^* be the set of all finite directed paths. In what follows, we consider pairs (x, y) in $\Delta_u^* \times \Delta_u^*$ such that x and y end at the same vertex. Let $x = x'e$ and $y = y'f$, where e from α_1 to α and f from α_2 to α are the last edges of x and y, respectively. Since e and f end at the same vertex, $P_{\tilde{\beta}_1} P_{\tilde{\beta}_2} \neq 0$ and thus either $\tilde{\beta}_1 \prec \tilde{\beta}_2$ or $\tilde{\beta}_2 \prec \tilde{\beta}_1$. Let μ be the word of length $||\tilde{\beta}_1| - |\tilde{\beta}_2||$ such that $\tilde{\beta}_1 = \tilde{\beta}_2\mu$ or $\tilde{\beta}_2 = \tilde{\beta}_1\mu$. We say that the pair (x, y) is *balanced* if the following condition holds: $L(x') = L(y')\mu$ if $\tilde{\beta}_1 = \tilde{\beta}_2\mu$, and $L(x')\mu = L(y')$ if $\tilde{\beta}_2 = \tilde{\beta}_1\mu$. Then we define the *total label* of (x, y) as $L(x')$ in the former case, and $L(y')$ in the latter. Now, we define a subset Ω_u of the cartesian product $\Delta_u^* \times \Delta_u^*$, as follows. A pair (x, y) belongs to Ω_u if and only if:

(i) The paths x and y end at the same vertex, but they begin at distinct vertices.

(ii) The paths x and y have identical degrees.

(iii) The pair (x, y) is balanced.

The importance of the set Ω_u for our purposes comes from the following Proposition 4.3. Unfortunately, it is not clear to us at the moment if its hypothesis may be algorithmically verified in all cases (i.e. for all applicable unitaries $u \in \mathcal{S}_n$). However, in many concrete situations this can be done fairly easily, but preferably with help of a computer. Thus, combined with Theorem 3.7, Lemma 4.1 and Lemma 4.2, Proposition 4.3 gives a criterion for deciding invertibility of endomorphism λ_u.

Proposition 4.3 *Let $u = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^*$ be such that $|\alpha| - |\beta| \in \{0, \pm 1\}$ and let Δ_u be the corresponding graph. We assume that $\lambda_u(\mathcal{D}_n) = \mathcal{D}_n$. Then the following hold.*

1. *Let $(x, y) \in \Omega_u$ have the total label γ and let the paths x, y begin at α and α', respectively. Then $\mathcal{Z}_u$ contains $S_\alpha P_\gamma S_{\alpha'}^*$.*

2. *If $\alpha, \alpha' \in W_n^k$ then $S_\alpha S_{\alpha'}^*$ belongs to $\langle \mathcal{Z}_u \rangle$ if and only if there exists a finite collection $(x_1, y_1), \ldots, (x_m, y_m)$ in Δ_u with the total labels $\gamma_1, \ldots, \gamma_m$, respectively, and with*

all x_j beginning at α and all y_j beginning at α', such that

$$1 = \sum_{j=1}^{m} P_{\gamma_j}.$$

Proof. Ad 1. Let $(\alpha, \alpha_1, \ldots, \alpha_m)$ be the consecutive vertices through which the path x passes, and likewise let $(\alpha', \alpha'_1, \ldots, \alpha'_r)$ be such vertices for y. Then our definition of Ω_u ensures that

$$S_\alpha P_\gamma S^*_{\alpha'} = S_\alpha S^*_{\tilde\beta} S_{\alpha_1} S^*_{\tilde\beta_1} \cdots S_{\alpha_m} S^*_{\tilde\beta_m} (S_{\alpha'} S^*_{\tilde\beta'} S_{\alpha'_1} S^*_{\tilde\beta'_1} \cdots S_{\alpha'_r} S^*_{\tilde\beta'_r})^*,$$

and thus $S_\alpha P_\gamma S^*_{\alpha'} \in \mathcal{Z}_u$.

Ad 2. Suppose that $S_\alpha S^*_{\alpha'} \in \langle \mathcal{Z}_u \rangle$, and let $S_\alpha S^*_{\alpha'} = \sum_{j=1}^m S_{\mu_j} S^*_{\nu_j}$, with each $S_{\mu_j} S^*_{\nu_j}$ in $\mathcal{Z}_u$. Since there are no cancellations among words, each $S_{\mu_j} S^*_{\nu_j}$ must be of the form $S_\alpha P_{\gamma_j} S^*_{\alpha'}$ for some $\gamma_j \in W_n$. Now, it is not difficult to verify that an element of $\mathcal{Z}_u$ has this form if and only if there exists a pair (x_j, y_j) in Ω_u with the total label γ_j and such that x_j and y_j begin at α and α', respectively.

The reverse implication is an immediate consequence of part 1 of this proposition. □

We end this section with some examples of invertible endomorphisms λ_u, $u \in \mathcal{S}_n \setminus \mathcal{P}_n$.

Example 4.4 Let μ, ν be two words such that $\tilde\nu = j_1 \cdots j_r \tilde\mu$ with $j_k \in W_n^1$ and $j_k \notin \{\mu_1, \nu_1\}$ for all $k = 1, \ldots, r$. Let

$$u = S_\nu S^*_\mu + S_\mu S^*_\nu + 1 - P_\nu - P_\mu.$$

Suppose that $\lambda_u(\mathcal{D}_n) = \mathcal{D}_n$. We claim that then λ_u is automatically invertible. Indeed, it suffices to check that $S_\nu S^*_\mu \in \lambda_u(\mathcal{O}_n)$. But we have $S_\nu S^*_\mu = S_\nu S^*_{\tilde\mu} S_{\tilde\mu} S^*_{\tilde\nu} S_{\tilde\nu} S^*_\mu$. Now, $S_\nu S^*_{\tilde\mu} = P_\nu \lambda_u(S_{\mu_1})$ and $S_{\tilde\nu} S^*_\mu = \lambda_u(S^*_{\nu_1}) P_\mu$ are both in $\lambda_u(\mathcal{O}_n)$. Also, $S_{\tilde\nu} S^*_{\tilde\mu} = S_{\tilde\nu} S^*_{\tilde\nu} S_{j_1} \cdots S_{j_r} = P_{\tilde\nu} \lambda_u(S_{j_1}) \cdots \lambda_u(S_{j_r})$ and hence $S_{\tilde\mu} S^*_{\tilde\nu} \in \lambda_u(\mathcal{O}_n)$. Consequently, λ_u is invertible, as required.

Example 4.5 Let $\alpha_1, \alpha_2, \alpha_3$ be such that $\{P_{\alpha_j}\}$ are mutually orthogonal and each α_j begins with the same letter i. Furthermore, suppose that $\tilde\alpha_j = \gamma_j \mu$ for some γ_j which do not contain the letter i. Let

$$u = S_{\alpha_1} S^*_{\alpha_2} + S_{\alpha_2} S^*_{\alpha_3} + S_{\alpha_3} S^*_{\alpha_1} + 1 - P_{\alpha_1} - P_{\alpha_2} - P_{\alpha_3}.$$

If $\lambda_u(\mathcal{D}_n) = \mathcal{D}_n$ then automatically $\lambda_u \in \mathrm{Aut}(\mathcal{O}_n)$. Indeed, we have $S_k = \lambda_u(S_k)$ for all $k \neq i$ and thus $S_{\alpha_1} S^*_{\alpha_2} = P_{\alpha_1} \lambda(S_i) S_{\gamma_2} S^*_{\gamma_3} \lambda(S^*_i) P_{\alpha_2}$ belongs to $\lambda_u(\mathcal{O}_n)$. Similarly, $S_{\alpha_2} S^*_{\alpha_3}$ and $S_{\alpha_3} S^*_{\alpha_1}$ are in $\lambda_u(\mathcal{O}_n)$ as well. Thus $u \in \lambda_u(\mathcal{O}_n)$ and λ_u is invertible. A concrete example in $\mathcal{O}_2$ is obtained by putting

$$w = S_{11} S^*_{121} + S_{121} S^*_{1221} + S_{1221} S^*_{11} + P_{1222} + P_2,$$

and then indeed λ_w is an automorphism of $\mathcal{O}_2$.

5 The outer Weyl group

In this section, we consider the question if an endomorphism corresponding to a unitary in $\mathcal{S}_n$ may be equivalent (via an inner automorphism) to one corresponding to a unitary in the core UHF-subalgebra $\mathcal{F}_n$.

Proposition 5.1 *There exist unitaries $u \in \mathcal{S}_n$ such that $\lambda_u \notin \mathrm{Aut}(\mathcal{O}_n)\lambda(\mathcal{F}_n)$.*

Proof. At first we observe that if $w \in \mathcal{U}(\mathcal{F}_n)$ and $Q \neq 0$ is a projection in $\mathcal{O}_n$ then the space $\lambda_w(\mathcal{D}_n)Q$ is infinite dimensional. Indeed, since $E(Q)$ is a non-zero, positive element of $\mathcal{F}_n$, there is a non-zero projection $q \in \mathcal{F}_n$ and a scalar $t > 0$ such that $tq \leq E(Q)$. There exists a sequence of indices $j_k \in W_n^1$ such that if $\alpha_k \in W_n^k$ are defined recursively as $\alpha_1 = j_1$, $\alpha_{k+1} = (\alpha_k, j_{k+1})$ then $\lambda_w(P_{\alpha_k})q \neq 0$ for all k. The sequence $\{q\lambda_w(P_{\alpha_k})q\}$ never stabilizes. Indeed, if $q\lambda_w(P_{\alpha_{k+m}})q = q\lambda_w(P_{\alpha_k})q$ for all m then

$$0 \neq \tau(q\lambda_w(P_{\alpha_k})q) = \tau(q\lambda_w(P_{\alpha_{k+m}})q) = \tau(\lambda_w(P_{\alpha_{k+m}})q\lambda_w(P_{\alpha_{k+m}})) \leq \tau(P_{\alpha_{k+m}}) \underset{m\to\infty}{\longrightarrow} 0,$$

a contradiction. The inequality above holds since w being in $\mathcal{U}(\mathcal{F}_n)$ the corresponding endomorphism λ_w is τ-preserving. Thus, there is a strictly decreasing, infinite sequence of projections $f_1 > f_2 > \ldots$ in $\lambda_w(\mathcal{D}_n)$ such that $qf_kq > qf_{k+1}q$ for all k. Thus $(f_k - f_{k+1})q \neq 0$ for all k, and hence

$$0 \neq (f_k - f_{k+1})tq(f_k - f_{k+1}) \leq (f_k - f_{k+1})E(Q)(f_k - f_{k+1}).$$

Thus $(f_k - f_{k+1})Q \neq 0$ and, consequently, $\{(f_k - f_{k+1})Q\}$ is an infinite sequence of linearly independent elements of $\lambda_w(\mathcal{D}_n)Q$, since these are non-zero operators with mutually orthogonal ranges.

Now, the same conclusion as above holds if λ_w is replaced by $\psi\lambda_w$ for some automorphism $\psi \in \mathrm{Aut}(\mathcal{O}_n)$, since the dimension of $(\psi\lambda_w)(\mathcal{D}_n)Q$ is the same as that of $\lambda_w(\mathcal{D}_n)\psi^{-1}(Q)$. Thus, the conclusion of the proposition follows from Example 3.3 (a), where a unitary $u \in \mathcal{S}_2$ is exhibited such that $\lambda_u(\mathcal{D}_2)P_2$ is one-dimensional. □

Of course, the method of Proposition 5.1 cannot give any information about automorphisms. We treat the automorphism case in the Theorem 5.2, below. To the best of our knowledge, the automorphism entering its proof is the first known example of an automorphism of $\mathcal{O}_n$ in $\lambda(\mathcal{S}_n)^{-1}$ not inner related to a permutative automorphism.

Theorem 5.2 *There exist automorphisms λ_u, $u \in \mathcal{S}_n$, of $\mathcal{O}_n$ such that for all $w \in \mathcal{U}(\mathcal{O}_n)$ and $v \in \mathcal{P}_n$ we have $\lambda_u \neq \mathrm{Ad}(w)\lambda_v$.*

Proof. Suppose $\lambda_u = \mathrm{Ad}(\tilde{w})\lambda_v$ for some $u \in \mathcal{S}_n$, $v \in \mathcal{P}_n$, $\tilde{w} \in \mathcal{U}(\mathcal{O}_n)$, where λ_u and λ_v are automorphisms of $\mathcal{O}_n$. Then $\tilde{w}$ belongs to the normalizer of $\mathcal{D}_n$. Thus $\tilde{w} = dw$ for some $d \in \mathcal{U}(\mathcal{D}_n)$ and $w \in \mathcal{S}_n$. Then $\lambda_d = \lambda_u \,\mathrm{Ad}(w^*)\lambda_v^{-1}$, and thus $d = 1$. Consequently, we may suppose from the start that $w \in \mathcal{S}_n$.

We claim that there exists a constant $C > 0$ such that

$$\frac{1}{C}\tau(q) \leq \tau(\mathrm{Ad}(w)\lambda_v(q)) \leq C\tau(q) \tag{10}$$

for all projections $q \in \mathcal{D}_n$. Indeed, since $\lambda_v(\mathcal{D}_n) = \mathcal{D}_n$ and λ_v preserves the trace, it suffices to show that $1/C \le \tau(\mathrm{Ad}(w)(q)) \le C$ for any such q. Let $w = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^*$, and let $\kappa = \max\{||\alpha| - |\beta|| : (\alpha, \beta) \in \mathcal{J}\}$. If f is a subprojection of P_β then

$$\frac{1}{n^\kappa}\tau(f) \le \frac{1}{n^{-||\alpha|-|\beta||}}\tau(f) \le \tau(\mathrm{Ad}(w)(f)) \le n^{||\alpha|-|\beta||}\tau(f) \le n^\kappa \tau(f).$$

Thus, writing $q = \sum_{\beta\in\mathcal{J}_2} qP_\beta$, we see that the claim holds with $C = n^\kappa$.

Now, we consider the following self-adjoint element of $\mathcal{S}_2$ (c.f. Example 4.4):

$$u = S_{11}S_{121}^* + S_{121}S_{11}^* + P_{122} + P_2. \tag{11}$$

One easily checks that

$$\lambda_u(S_1) = S_1(S_1S_{21}^* + S_{21}S_1^* + P_{22}) \quad \text{and} \quad \lambda_u(S_2) = S_2.$$

This yields $\lambda_u^2 = \mathrm{id}$. Now, let $\mu_k = (11\ldots1)$, a word of length k consisting of 1's only, and let $\nu_k = (1212\ldots121)$, a word of length $2k-1$ with alternating 1's and 2's and beginning with 1. One easily checks that $\lambda_u(P_{\mu_k}) = P_{\nu_k}$. Thus

$$\frac{\tau(\lambda_u(P_{\mu_k}))}{\tau(P_{\mu_k})} = \frac{1/n^{2k-1}}{1/n^k} = \frac{1}{n^{k-1}},$$

contradicting the double inequality (10). Thus, there is no $w \in \mathcal{U}(\mathcal{O}_2)$, $v \in \mathcal{P}_2$ such that $\lambda_u = \mathrm{Ad}(w)\lambda_v$.

Now, if $n \ge 2$ is arbitrary, then we consider $\tilde{u} = S_{11}S_{121}^* + S_{121}S_{11}^* + P_{122} + 1 - P_1$, and the same argument as above applies. □

As immediate consequences of Theorem 5.2, we obtain the following two corollaries.

Corollary 5.3 *The restricted outer Weyl group of $\mathcal{O}_n$ is a proper subgroup of the outer Weyl group of $\mathcal{O}_n$.*

As shown in [2], the restricted outer Weyl group of $\mathcal{O}_n$ is residually finite and nonamenable. Thus the outer Weyl group is nonamenable as well, but we do not know if it is residually finite.

Corollary 5.4 *There exist unital subalgebras $\mathcal{A}$ of $\mathcal{O}_n$ isomorphic to the UHF algebra of type $\{n^\infty\}$ and containing the diagonal $\mathcal{D}_n$ such that the pairs $\mathcal{D}_n \subseteq \mathcal{F}_n$ and $\mathcal{D}_n \subseteq \mathcal{A}$ are conjugated inside $\mathcal{O}_n$ (by an automorphism of $\mathcal{O}_n$) but not inner conjugated.*

6 The action on the shift space

Equality (3) easily implies that for all $d \in \mathcal{D}_n$ and all $k > \ell'(\mathcal{J})$ we have

$$\mathrm{Ad}(u)(\varphi^k(d)) = \sum_{(\alpha,\beta)\in\mathcal{J}} \varphi^{k+|\alpha|-|\beta|}(d)P_\alpha. \tag{12}$$

Consider a map $f : \mathcal{D}_n \to \mathcal{D}_n$. We say that f eventually preseves standard projections if there exists an integer $m \in \mathbb{N}$ such that for each $\alpha \in W_n$, $|\alpha| \geq m$, the image $f(P_\alpha)$ is a standard projection. If $u \in \mathcal{S}_n$ then $\mathrm{Ad}(u)$ eventually preserves standard projections.

Proposition 6.1 *If $f \in \mathrm{Aut}(\mathcal{D}_n)$ then there exists a unitary $u \in \mathcal{S}_n$ such that $f = \mathrm{Ad}(u)|_{\mathcal{D}_n}$ if and only if;*

(i) f eventually preserves standard projections, and

(ii) there exist projections P_i, Q_i, $i = 1, \ldots, r$, in $\mathcal{D}_n$ and non-negative integers k_i, m_i, $i = 1, \ldots, r$, such that $\sum_{i=1}^r P_i = 1 = \sum_{i=1}^r Q_i$ and

$$f \circ M_{P_i} \circ \varphi^{k_i} = M_{Q_i} \circ \varphi^{m_i}, \qquad i = 1, \ldots, r.$$

Proof. Let $f \in \mathrm{Aut}(\mathcal{D}_n)$ satisfy conditions (i) and (ii) of the proposition. For a given $i \in \{1, \ldots, r\}$, we note that for any subprojection p of P_i we have $f \circ M_p \circ \varphi^{k_i} = M_{f(p)} \circ \varphi^{m_i}$. Subdividing P_i into a sum of standard projections and using condition (i), we can assume in condition (ii) that all projections P_i, Q_i are standard, say $P_i = P_{\beta_i}$ and $Q_i = P_{\alpha_i}$. Define $u = \sum_{i=1}^r S_{\alpha_i} S_{\beta_i}^*$, a unitary element of $\mathcal{S}_n$. Then we have

$$(\mathrm{Ad}(u^*) \circ f) \circ M_{P_{\beta_i}} \circ \varphi^{k_i + |\alpha_i| + h} = M_{P_{\beta_i}} \circ \varphi^{m_i + |\beta| + h}, \qquad i = 1, \ldots, r,$$

for all sufficiently large $h \in \mathbb{N}$. We claim that $k_i + |\alpha_i| = m_i + |\beta_i|$ for each i. Indeed, fix an i and suppose that $k_i + |\alpha_i| \geq m_i + |\beta_i|$ (otherwise consider $(\mathrm{Ad}(u^*) \circ f)^{-1}$ instead). Then we have

$$(\mathrm{Ad}(u^*) \circ f)(\varphi^{k_i + |\alpha_i| - m_i - |\beta_i|}(y) P_{\beta_i}) = y P_{\beta_i}, \quad \forall y \in \varphi^{m_i + |\beta_i| + h}(\mathcal{D}_n),$$

for all sufficiently large $h \in \mathbb{N}$. Fix such an h and let $r \geq m_i + |\beta_i| + h$ be such that

$$(\mathrm{Ad}(u^*) \circ f)(\varphi^{k_i + |\alpha_i| - m_i - |\beta_i|}(\mathcal{D}_n^{m_i + |\beta_i| + h}) P_{\beta_i}) \subseteq \mathcal{D}_n^r P_{\beta_i}$$

and $P_{\beta_i} \in \mathcal{D}_n^r$. Then we have

$$\begin{aligned}
&(\mathrm{Ad}(u^*) \circ f)(\varphi^{k_i + |\alpha_i| - m_i - |\beta_i|}(\mathcal{D}_n^r) P_{\beta_i}) \\
&\quad = (\mathrm{Ad}(u^*) \circ f)(\varphi^{k_i + |\alpha_i| - m_i - |\beta_i|}(\mathcal{D}_n^{m_i + |\beta_i| + h}) \varphi^{m_i + |\beta_i| + h}(\mathcal{D}_n^{r - m_i - |\beta_i| - h}) P_{\beta_i}) \\
&\quad \subseteq \mathcal{D}_n^r P_{\beta_i} \varphi^{m_i + |\beta_i| + h}(\mathcal{D}_n^{r - m_i - |\beta_i| - h}) P_{\beta_i} \\
&\quad \subseteq \mathcal{D}_n^r P_{\beta_i}.
\end{aligned}$$

Since $\mathrm{Ad}(u^*) \circ f$ is injective and the dimension of $\varphi^{k_i + |\alpha_i| - m_i - |\beta_i|}(\mathcal{D}_n^r) P_{\beta_i}$ is not smaller than the dimension of $\mathcal{D}_n^r P_{\beta_i}$, it follows that these two dimensions are identical, and this can only happen when $k_i + |\alpha_i| - m_i - |\beta_i| = 0$. Consequently,

$$(\mathrm{Ad}(u^*) \circ f) \circ M_{P_{\beta_i}} \circ \varphi^h = M_{P_{\beta_i}} \circ \varphi^h, \qquad i = 1, \ldots, r,$$

for all sufficiently large $h \in \mathbb{N}$. Summing over i we get

$$(\mathrm{Ad}(u^*) \circ f) \circ \varphi^h = \varphi^h$$

for all sufficiently large $h \in \mathbb{N}$. Therefore $\mathrm{Ad}(u^*) \circ f = \mathrm{Ad}(w)|_{\mathcal{D}_n}$ for some $w \in \mathcal{P}_n$, by [2, Lemma 3.2]. Hence $f = \mathrm{Ad}(uw)|_{\mathcal{D}_n}$ and $uw \in \mathcal{S}_n$. This proves one direction. The opposite direction is clear. Indeed, let $u \in \mathcal{S}_n$ be such that $u = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^*$. Then condition (i) holds, as noted just above this proposition. One easily checks that condition (ii) holds with projections P_β and P_α instead of P_i and Q_i, respectively, and with $|\beta|$ and $|\alpha|$ instead of k_i and m_i, respectively. □

Given $u \in \mathcal{S}_n$ and considering the homeomorphism $\mathrm{Ad}(u)_*$ of the spectrum X_n of $\mathcal{D}_n$, we see that the set of fixed points has a very simple structure, as the following Proposition 6.2 shows.

Proposition 6.2 *For $u \in \mathcal{S}_n$, the set of fixed points in X_n for the homeomorphism $\mathrm{Ad}(u)_*$ consists of the union of a clopen set and a finite set. Furthermore, each of the isolated fixed points is either a local attractor or a local repeller.*

Proof. Let $u = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^*$. It is clear that $\mathrm{Ad}(u)_*$ admits fixed points in X_n if and only if there exists $(\alpha,\beta) \in \mathcal{J}$ such that either $\alpha \prec \beta$ or $\beta \prec \alpha$. Thus we arrive at one of the following three cases. (1) If $\alpha = \beta$ then the clopen set $\{x \in X_n : \beta \prec x\}$ is fixed by $\mathrm{Ad}(u)_*$. (2) If $\alpha = \beta\mu$, $\mu \neq \emptyset$, then $x = \beta\mu\mu\ldots$ is a fixed point and a local attractor. (3) If $\beta = \alpha\mu$, $\mu \neq \emptyset$, then $x = \alpha\mu\mu\ldots$ is a fixed point and a local repeller. □

In contrast to Proposition 6.2 above, the set of fixed points in X_n corresponding to an outer automorphism λ_u, $u \in \mathcal{S}_n$, may have a much more complicated structure, as the following example demonstrates.

Example 6.3 Let u be the unitary in $\mathcal{S}_2$ defined by formula (11). It is not difficult to verify that the corresponding homeomorphism $(\lambda_u)_*$ of X_2 fixes an $x \in X_2$ if and only if x does not contain substrings (11) and (121). These fixed points form a compact, nowhere dense subset K of X_2, in which there are no isolated points. Thus K itself is homeomorphic to the Cantor set and closed under the action of the one-sided shift φ_*.

References

[1] R. Conti, *Automorphisms of the UHF algebra that do not extend to the Cuntz algebra*, J. Austral. Math. Soc. **89** (2010), 309–315.

[2] R. Conti, J. H. Hong and W. Szymański, *The restricted Weyl group of the Cuntz algebra and shift endomorphisms,* arXiv:1006.4791, to appear in J. reine angew. Math.

[3] R. Conti, J. H. Hong and W. Szymański, *Endomorphisms of graph algebras*, Oberwolfach Preprint 2011–04, arXiv:1101.4210.

[4] R. Conti, J. Kimberley and W. Szymański, *More localized automorphisms of the Cuntz algebras*, Proc. Edinburgh Math. Soc. **53** (2010), 619–631.

[5] R. Conti, M. Rørdam and W. Szymański, *Endomorphisms of $\mathcal{O}_n$ which preserve the canonical UHF-subalgebra*, J. Funct. Anal. **259** (2010), 602–617.

[6] R. Conti and W. Szymański, *Labeled trees and localized automorphisms of the Cuntz algebras*, Trans. Amer. Math. Soc. **363** (2011), 5847–5870.

[7] R. Conti and W. Szymański, *Automorphisms of the Cuntz algebras*, arXiv:1108.0860.

[8] J. Cuntz, *Simple C^*-algebras generated by isometries*, Commun. Math. Phys. **57** (1977), 173–185.

[9] J. Cuntz, *Automorphisms of certain simple C^*-algebras*, in *Quantum fields-algebras-processes*, ed. L. Streit, 187–196, Springer, 1980.

[10] S. Eilenberg, *Automata, languages and machines*, Academic Press, New York, 1974.

[11] J. H. Hong, A. Skalski and W. Szymański, *On invariant MASAs for endomorphisms of the Cuntz algebras*, arXiv:1001.1899, to appear in Indiana Univ. Math. J. **59** (2010).

[12] K. Matsumoto, *Orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras*, Pacific J. Math. **246** (2010), 199–225.

[13] V. Nekrashevych, *Cuntz-Pimsner algebras of group actions*, J. Operator Theory **52** (2004), 223–249.

[14] W. Szymański, *On localized automorphisms of the Cuntz algebras which preserve the diagonal subalgebra*, in 'New Development of Operator Algebras', R.I.M.S. Kôkyûroku **1587** (2008), 109–115.

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