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Type II singular perturbation approximation for linear systems with Lévy noise

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Abstract

When solving linear stochastic partial differential equations numerically, usually a high order spatial discretisation is needed. Model order reduction (MOR) techniques are often used to reduce the order of spatially-discretised systems and hence reduce computational complexity. A particular MOR technique to obtain a reduced order model (ROM) is singular perturbation approximation (SPA), a method which has been extensively studied for deterministic systems. As so-called type I SPA it has already been extended to stochastic equations. We provide an alternative generalisation of the deterministic setting to linear systems with Lévy noise which is called type II SPA. It turns out that the ROM from applying type II SPA has better properties than the one of using type I SPA. In this paper, we provide new energy interpretations for stochastic reachability Gramians, show the preservation of mean square stability in the ROM by type II SPA and prove two different error bounds for type II SPA when applied to Lévy driven systems.

1 Introduction

Many phenomena in real life can be described by partial differential equations (PDEs). Famous examples are the motion of viscous fluids, the description of water or sound waves and the distribution of heat. For an accurate mathematical modeling of these real world applications it is often required to take random effects into account. Uncertainties in a PDE model can for example be represented by an additional noise term. This leads to stochastic PDEs (SPDEs). A possible way is to consider equations driven by Wiener noise. We refer to Da Prato, Zabczyk [10]; Gawarecki, Mandrekar [14] and Prévôt, Röckner [28] who treat infinite dimensional Wiener processes as well as Wiener driven SPDEs. Dealing with Wiener noise yields just continuous systems. This has the disadvantage of not covering models with jumps. Lévy processes, which in general are not continuous, provide a possible solution to this problem. One can find detailed information regarding Lévy processes in infinite dimensional spaces in Peszat, Zabczyk [27], where the work of [10, 14, 28] is extended. They provide a comprehensive book containing the stochastic analysis of infinite dimensional Lévy processes and the theory of Lévy driven SPDEs with various examples.

It is necessary to discretise a time-dependent SPDE in space and time in order to solve it numerically. As a possible strategy discretising in space can be considered as a first step.

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By numerical approximations, an SPDE can be reduced to a finite dimensional equation. A possibility to do that is the spectral Galerkin method which is for example investigated in Grecksch, Kloeden [15]; Hausenblas [17]; Jentzen, Kloeden [19]; Blömker, Jentzen [9] for Wiener driven systems. Alternatively, finite element methods can be applied. Kruse investigates this scheme in [21, 22] for SPDEs with Wiener noise. Barth [3] and Barth, Lang [4] consider finite element approaches for equations with more general noise processes such as Lévy processes.

Semi-discretising an SPDE in space usually leads to a high dimensional SDE. Solving such complex SDE systems causes large computational cost which are aimed to be reduced. In this regard, model order reduction (MOR) becomes a key ingredient. MOR is used to save computational time by replacing large scale systems by systems of low order in which the main information of the original system should be captured. A particular class of MOR schemes is called balancing related MOR. They are based on reachability and observability concepts and corresponding energy functionals. The idea of balancing a system with inputs and outputs is to create a system, where the dominant reachable and observable states are the same. Then, the difficult to observe and difficult to reach states (states producing the least observation energy and causing the most energy to reach, respectively) are neglected. A famous representative of this class is balanced truncation (BT). This was considered first in Moore [25] for linear deterministic system; see Antoulas [1] or Obinata, Anderson [26] for a thorough treatment of the topic. BT was also established for deterministic bilinear systems in Benner, Damm [5] and Zhang et al. [33]. An alternative method to obtain a reduced order model (ROM) is the singular perturbation approximation (SPA), see Liu, Anderson [23] and Fernando, Nicholson [13] for deterministic linear systems and Hartmann et al. [16] for deterministic bilinear systems.

Recently, BT and SPA have been extended to stochastic systems. BT was considered first for SDEs with Wiener noise in Benner, Damm [5] and for systems with Lévy noise by Benner, Redmann in [7]. This is the so-called type I ansatz which relies on a reachability Gramian P_1 that is defined by the fundamental solution of the system. A second way to generalise BT to stochastic systems is discussed in Benner et al. [6]; Benner, Damm [12] and Redmann, Benner [30]. It is based on another reachability Gramian P_2. This new approach, the so-called type II BT, is motivated by the aim of achieving an $\mathcal{H}_\infty$-error bound which cannot be proven in the ansatz used in [7]. Redmann and Benner [29] studied type I SPA for SDEs with Lévy noise but so far no work has been done on type II SPA for stochastic systems. This will be the main focus of this paper.

In Section 2, we will briefly discuss mean square asymptotic stability in a linear system with Lévy noise. This section contains results generalising the Wiener case, see Damm [11] and Khasminskii [20]. Mean square asymptotic stability is a necessary assumption to define reachability and observability Gramians to a stochastic system. In Section 3, we discuss two different reachability Gramians P_1 and P_2 which were e.g. considered in [5, 7] and [6, 12], respectively. First attempts to characterise the meaning of P_1 to the corresponding stochastic system can be found in [5, 7]. The same was done for P_2 in [30]. Unfortunately, all these characterisations are based on a reachability concept involving the mean state of the system. Considering the mean state ignores the information in the diffusion term of a stochastic differential equation. That is why the energy interpretations in [5, 7, 30] might be unsatisfactory. For that reason, we provide new energy interpretations for both P_1 and P_2 which involve the full information of the stochastic system in Section 3. We briefly discuss the meaning of an observability Gramian Q as well. The energy interpretations allow us to characterise the degree of reachability and observability of a state in the system. Hence, unimportant states (difficult to reach and observe states) can be identified from these Gramians. In Section 4, we discuss how to balance a system based on the Gramians P_2 and Q (type II balancing) and show how the difficult to reach and observe states are removed from the resulting balanced system. From this procedure, we obtain the ROM corresponding to type II SPA which is then analysed in detail. It will be shown that type II SPA preserves mean square asymptotic stability which is an extension of the result in [23]. So far, this stability result has not yet been obtained for the type I ansatz, see [29]. In Section 5, we provide both an $\mathcal{H}_2$- and an $\mathcal{H}_\infty$-error bound for type II SPA. The existence of an $\mathcal{H}_\infty$-error bound of the same form is not given for type I SPA which can be seen from examples in [6, 12]. The $\mathcal{H}_2$-bound will be proved for a simplified ROM because the ROM has to have the same structure as the original model. Moreover, the $\mathcal{H}_2$-bound relies on the preservation of mean square asymptotic stability in the ROM which is given here. The $\mathcal{H}_\infty$-error bound in Section 5 is again an extension of the work for deterministic systems, see [23]. There, transfer functions are used that are not available in the more general stochastic case. Therefore, in the stochastic case, the proof has to be conducted in the time domain. In contrast to the deterministic case, there seems to be no link between the case of type II BT (investigated in [5, 12, 30]) and type II SPA in terms of the $\mathcal{H}_\infty$-error bound. This makes the analysis more complicated here. Additionally, we encounter the problem of a change in the structure from the original to the ROM so that different arguments compared to the standard ones have to be used. Both the $\mathcal{H}_2$- and the $\mathcal{H}_\infty$-type error bound of using type II SPA depend on the $n-r$ smallest Hankel singular values of the original system and therefore similar conclusions as in the deterministic case can be made, e.g., type II SPA performs well if these truncated Hankel singular values are small which is the case if only unimportant states are removed from the system.

2 Setting and mean square asymptotic stability

We begin with a stochastic stability concept first, where we consider a linear controlled system driven by Lévy noise. The corresponding Lévy process $M = (M_1, \dots, M_q)^T$ is $\mathbb{R}^q$-valued with mean zero and existing second moments. We investigate

$$dx(t) = [Ax(t) + Bu(t)]dt + N(x(t-))\,dM(t), \quad x(0) = x_0, \quad t \geq 0, \tag{1}$$

where $x(t-) := \lim_{s\uparrow t}, x(s)$ $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $N : \mathbb{R}^n \to \mathbb{R}^{n\times q}$ is a linear mapping defined by $N(x) = \begin{bmatrix} N_1 x & \dots & N_q x \end{bmatrix}$ for $x \in \mathbb{R}^n$ with $N_1, \dots, N_q \in \mathbb{R}^{n\times n}$. Below, $x(t, x_0, u)$, $t \geq 0$, denotes the solution to (1) with initial condition $x_0 \in \mathbb{R}^n$ and control process u. The control u is assumed to be an adapted càdlàg process with

$$\|u\|_{L^2_T}^2 := \mathbb{E}\int_0^T u^T(t)u(t)dt = \mathbb{E}\int_0^T \|u(t)\|_2^2\, dt < \infty$$

for every $T > 0$. Furthermore, by Theorem 4.44 in [27], we know that the covariance function of M is linear in time, i.e., $\mathbb{E}[M(t)M^T(t)] = \mathcal{Q}t$. We call $\mathcal{Q} = (q_{ij})_{i,j=1,\dots,q}$ covariance matrix of M.

Throughout this paper, we assume that (1) is mean square asymptotically stable, that is

$$\mathbb{E}\left\|x(t,x_0,0)\right\|_2^2 \to 0 \tag{2}$$

for $t \to \infty$ and every $x_0 \in \mathbb{R}^n$. Below, we will analyse this property which is vital for the considerations in Section 3. The following Lemma is essential for the stability analysis of the uncontrolled equation (1).

Lemma 2.1. *The matrix-valued function* $\mathbb{E}\left[x(t,x_0,0)x^T(t,x_0,0)\right]$, $t \geq 0$, *solves*

$$\dot{X}(t) = AX(t) + X(t)A^T + \sum_{i,j=1}^q N_i X(t) N_j^T q_{ij}, \quad X(0) = x_0 x_0^T, \tag{3}$$

where $q_{ij} = \mathbb{E}[M_i(1)M_j(1)]$ *is the* ij*th entry of the covariance matrix of* M.

Proof. We replace $x(t,x_0,0)$ by $x(t)$ to shorten the notation in the proof. Using Ito's formula in Corollary A.1, we obtain for $x(t)x^T(t)$, $t \geq 0$:

$$x(t)x^T(t) = x_0x_0^T + \int_0^t x(s-)dx^T(s) + \int_0^t dx(s)x^T(s-) + \left([e_i^Tx, e_j^Tx]_t\right)_{i,j=1,\dots,n},$$

where e_i is the ith unit vector of $\mathbb{R}^n$. Inserting the stochastic differential of $x(t)$ yields

$$\int_0^t x(s-)dx^T(s) = \int_0^t x(s-)x^T(s)A^T ds + \int_0^t x(s-)dM^T(s)N^T\left(x(s-)\right) \quad \text{and}$$
$$\int_0^t dx(s)x^T(s-) = \int_0^t Ax(s)x^T(s-)ds + \int_0^t N\left(x(s-)\right)dM(s)x^T(s-).$$

Since the Ito integrals have mean zero, we get

$$\mathbb{E}\left[x(t)x^T(t)\right] = x_0x_0^T + \int_0^t \mathbb{E}\left[x(s)x^T(s)\right]A^T ds + \int_0^t A\mathbb{E}\left[x(s)x^T(s)\right]ds + \left(\mathbb{E}[e_i^Tx, e_j^Tx]_t\right)_{i,j=1,\dots,n},$$

where we furthermore replaced $x(s-)$ by $x(s)$. This does not impact the integrals since a càdlàg process has at most countably many jumps on a finite time interval (see Theorem

2.7.1 in [2]). Due to (64) it is clear that only the jumps and the continuous martingale part of a semimartingale enter in the compensator process. Since the jumps and the martingale part of $x(t)$, $t \geq 0$, are given by $\mathcal{M}(t) := \int_0^t N\left(x(s-)\right) dM(s)$ (compare (1)), we obtain that

$$[e_i^T \mathcal{M}, e_j^T \mathcal{M}]_t = [e_i^T x, e_j^T x]_t$$

for $i, j = 1, \dots, n$. Now let $(h_k)_{k=1,\dots,q}$ be an orthonormal basis of eigenvectors of $\mathcal{Q}$ with the corresponding eigenvalues $(\lambda_k)_{k=1,\dots,q}$. Then, we can represent the Lévy process as

$$M(t) = \sum_{k=1}^{q} \langle M(t), h_k \rangle_2 h_k, \tag{4}$$

where the scalar Lévy processes $\tilde{M}_k(t) = \langle M(t), h_k \rangle_2$, $t \geq 0$, are uncorrelated since $\mathbb{E}\tilde{M}_i(t)\tilde{M}_j(t) = h_i^T \mathcal{Q} h_j t = \begin{cases} \lambda_i t & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

Inserting representation (4) yields $\mathcal{M}(t) = \sum_{k=1}^q \int_0^t N\left(x(s-)\right) h_k d\tilde{M}_k(s)$. Since all noise processes are uncorrelated, we can apply the result from [7] (Section 2.4) concerning the mean of the compensator process. It is

$$\mathbb{E}[e_i^T \mathcal{M}, e_j^T \mathcal{M}]_t = \sum_{k=1}^{q} \mathbb{E} \int_0^t e_i^T N\left(x(s-)\right) h_k e_j^T N\left(x(s-)\right) h_k ds \; \lambda_k,$$

such that we have

$$\begin{aligned} \left(\mathbb{E}[e_i^T x, e_j^T x]_t\right)_{i,j=1,\dots,n} &= \sum_{k=1}^{q} \mathbb{E} \int_0^t N\left(x(s-)\right) h_k (N\left(x(s-)\right) h_k)^T ds \; \lambda_k \\ &= \sum_{k=1}^{q} \mathbb{E} \int_0^t N\left(x(s)\right) \mathcal{Q}^{\frac{1}{2}} h_k (N\left(x(s)\right) \mathcal{Q}^{\frac{1}{2}} h_k)^T ds \end{aligned}$$

again using that x has at most countably many jumps. With $N\left(x(s)\right) \mathcal{Q}^{\frac{1}{2}} h_k = \sum_{i=1}^q N_i \langle \mathcal{Q}^{\frac{1}{2}} h_k, e_i \rangle_2 x(s) = \sum_{i=1}^q N_i \langle h_k, \mathcal{Q}^{\frac{1}{2}} e_i \rangle_2 x(s)$, we obtain

$$\left(\mathbb{E}[e_i^T x, e_j^T x]_t\right)_{i,j=1,\dots,n} = \sum_{k=1}^{q} \sum_{i,j=1}^{q} \mathbb{E} \int_0^t N_i \langle h_k, \mathcal{Q}^{\frac{1}{2}} e_i \rangle_2 x(s) x^T(s) N_j^T \langle h_k, \mathcal{Q}^{\frac{1}{2}} e_j \rangle_2 ds.$$

Changing the order of the summation and using the following elementary rearrangements $\sum_{k=1}^q \langle h_k, \mathcal{Q}^{\frac{1}{2}} e_i \rangle_2 \langle h_k, \mathcal{Q}^{\frac{1}{2}} e_j \rangle_2 = \langle \mathcal{Q}^{\frac{1}{2}} e_i, \mathcal{Q}^{\frac{1}{2}} e_j \rangle_2 = e_i^T \mathcal{Q} e_j = q_{ij}$ leads to the desired result. □

The above lemma is vital to state equivalent conditions for asymptotic mean square stability for equations with Lévy noise. The arguments of proving the following theorem go back to the ones from the case of Wiener noise, studied intensively in [11, 20]. Asymptotic mean stability is required for the existence of Gramians which we use to characterise reachability in equation (1) and observability in the corresponding output equation that will be introduced in Section 3.

Theorem 2.2. *The following are equivalent:*

(i) The uncontrolled equation (1) is asymptotically mean square stable.

(ii) The uncontrolled equation (1) is exponentially mean square stable, that is, there exist $k, \beta > 0$, such that

$$\mathbb{E}\left\|x(t,x_0,0)\right\|_2^2 \leq \left\|x_0\right\|_2^2 k\,\mathrm{e}^{-\beta t}.$$

(iii) The eigenvalues of $I_n \otimes A + A \otimes I_n + \sum_{i,j=1}^q N_i \otimes N_j q_{ij}$ have negative real parts.

(iv) There exists a matrix $X > 0$, such that

$$A^T X + XA + \sum_{i,j=1}^q N_i^T X N_j q_{ij} < 0.$$

(v) For all $Y > 0$, there exists a matrix $X > 0$, such that

$$A^T X + XA + \sum_{i,j=1}^q N_i^T X N_j q_{ij} = -Y.$$

Proof. With Lemma 2.1 we make use of the techniques applied in the Wiener case to prove the more general case of having Lévy noise [11, 20].

Again, we use $x(t)$ instead of $x(t, x_0, 0)$. From Lemma 2.1 it is known that $\mathbb{E}\left[x(t)x^T(t)\right]$ is the solution of the matrix differential equation

$$\dot{X}(t) = X(t)\ A^T + A\ X(t) + \sum_{i,j=1}^q N_i\ X(t)\ N_j^T\ q_{ij}. \tag{5}$$

Vectorising equation (5) leads to the following equivalent ODE

$$\frac{d}{dt}\operatorname{vec}\left(X(t)\right) = \left(I \otimes A + A \otimes I + \sum_{i,j=1}^q N_i \otimes N_j q_{ij}\right)\operatorname{vec}\left(X(t)\right). \tag{6}$$

We first show $(iii) \Rightarrow (ii)$. From (iii) the asymptotic stability of (6) follows. Asymptotic stability of (6) implies exponential stability, such that

$$\left\|\operatorname{vec}\left(X(t)\right)\right\|_2^2 \leq \left\|\operatorname{vec}\left(x_0x_0^T\right)\right\|_2^2 K_1\,\mathrm{e}^{-\beta_1 t} = \left\|x_0x_0^T\right\|_F^2 K_1\,\mathrm{e}^{-\beta_1 t} \leq \left\|x_0x_0^T\right\|_{2,ind}^2 \tilde{c}K_1\,\mathrm{e}^{-\beta_1 t}$$

for $K_1, \beta_1, \tilde{c} > 0$, where $\|\cdot\|_{2,ind}$ is the matrix norm that is induced by $\|\cdot\|_2$. Since

$$\|X(t)\|_{2,ind}^2 \leq \|X(t)\|_F^2 = \|\mathrm{vec}\,(X(t))\|_2^2$$

holds, equation (5) is exponentially stable and hence (ii) follows. It is obvious that (ii) implies (i). We now focus on $(i) \Rightarrow (iii)$. From (i) we conclude that equation (5) is asymptotically stable. The asymptotic stability of (6) follows by

$$\|\mathrm{vec}\,(X(t))\|_2^2 = \|X(t)\|_F^2 \leq \tilde{c}\,\|X(t)\|_{2,ind}^2$$

and asymptotic stability of (6) implies (iii). We continue with the proof of $(iii) \Rightarrow (v)$. Obviously, condition (iii) is equivalent to

$$\sigma\left(I_n \otimes A^T + A^T \otimes I_n + \sum_{i,j=1}^{q} N_i^T \otimes N_j^T q_{ij}\right) \subset \mathbb{C}_-$$

which, by the considerations above, is again equivalent to the exponentially mean square stability of the following equation

$$dx_d(t) = A^T x_d(t)dt + \sum_{i=1}^{q} N_i^T x_d(t-)dM_i(t), \quad t \geq 0. \tag{7}$$

Let Φ_d be the fundamental solution to the dual system (7), i.e., Φ_d satisfies

$$\Phi_d(t) = I_n + \int_0^t A^T \Phi_d(s)ds + \sum_{i=1}^{q} \int_0^t N_i^T \Phi_d(s-)dM_i(s).$$

For an arbitrary matrix $Y > 0$ the integral $\mathbb{E}\int_0^\infty \Phi_d(t) Y \Phi_d^T(t)dt = X > 0$ exists by the exponentially mean square stability of (7). We set $X(t) := \Phi_d(t) Y \Phi_d^T(t)$ and as in Lemma 2.1, we obtain

$$X(t) = Y + \int_0^t X(s)ds\; A + A^T \int_0^t X(s)ds + \sum_{i,j=1}^{q} N_i^T \int_0^t X(s)ds\; N_j q_{ij}$$

for $t \geq 0$. Letting $t \to \infty$ and using the exponentially mean square stability of the dual system, we find

$$-Y = X\; A + A^T\; X + \sum_{i,j=1}^{q} N_i^T X\; N_j q_{ij}$$

which is the desired result. Since (v) implies (iv), it remains to show that $(iv) \Rightarrow (ii)$. Let $X > 0$ such that

$$A^T X + XA + \sum_{i,j=1}^{q} N_i^T X N_j q_{ij} = -Y < 0. \tag{8}$$

So, due to Lemma 2.1, we have

$$\begin{aligned}
\mathbb{E}\left[x^T(t)Xx(t)\right] &= \mathbb{E}\left[\operatorname{tr}\left(Xx(t)x^T(t)\right)\right] = \operatorname{tr}\left(X\mathbb{E}\left[x(t)x^T(t)\right]\right) \\
&= \operatorname{tr}\left(X\left(x_0x_0^T + \int_0^t \mathbb{E}\left[x(s)x^T(s)\right] ds\ A^T + A\int_0^t \mathbb{E}\left[x(s)x^T(s)\right] ds \right.\right. \\
&\qquad \left.\left. + \sum_{i,j=1}^q N_i \int_0^t \mathbb{E}\left[x(s)x^T(s)\right] ds\ N_j^T q_{ij}\right)\right) \\
&= x_0^T X x_0 + \mathbb{E}\left[\int_0^t x^T(s)A^T X x(s) ds + \int_0^t x^T(s) X A x(s) ds\right] \\
&\qquad + \mathbb{E}\left[\int_0^t \sum_{i,j=1}^q x^T(s) N_i^T X N_j q_{ij} x(s) ds\right].
\end{aligned}$$

Inserting equation (8) yields

$$\mathbb{E}\left[x^T(t)Xx(t)\right] = x_0^T X x_0 - \mathbb{E}\left[\int_0^t x^T(s) Y x(s) ds\right].$$

and hence

$$\dot{g}(t) = -\mathbb{E}\left[x^T(t)Yx(t)\right],$$

where $g(t) := \mathbb{E}\left[x^T(t)Xx(t)\right]$. Now, let k_1 be the smallest and k_2 be the largest eigenvalue of X such that $k_1 v^T v \leq v^T X v \leq k_2 v^T v$. Furthermore, we assume k_3 to be the smallest eigenvalue of Y, then we obtain

$$\dot{g}(t) \leq -k_3 \mathbb{E}\left[x^T(t)x(t)\right] \leq -\frac{k_3}{k_2}\mathbb{E}\left[x^T(t)Xx(t)\right] = -\frac{k_3}{k_2} g(t).$$

By Gronwall's inequality, we have

$$\mathbb{E}\left[x^T(t)x(t)\right] \leq \frac{1}{k_1}\mathbb{E}\left[x^T(t)Xx(t)\right] \leq \frac{1}{k_1} x_0^T X x_0 \,\mathrm{e}^{-\frac{k_3}{k_2}t} \leq \frac{k_2}{k_1} x_0^T x_0 \,\mathrm{e}^{-\frac{k_3}{k_2}t}$$

which yields the required result and concludes the proof. □

Having discussed mean square asymptotic stability we will introduce reachability and observability Gramians and corresponding energy interpretations in the next section.

3 Characterising reachability and observability using Gramians

Starting from zero ($x_0 = 0$) in (1) we investigate how much the noise and the control u can steer the state away from zero. To do so we introduce two different reachability Gramians

below and we will see that they provide certain information about the degree of reachability of a state. In the context of model order reduction, it is of particular interest to identify the difficult to reach states (states, where a large control u has to be used to steer the system to these states). Those states are seen to be unimportant in the system dynamics.
Moreover, we briefly discuss an observability Gramian which allows us to identify difficult to observe states in a system. These states are unimportant for the system dynamics too.

Reachability Gramian type I ansatz In the following, we introduce an infinite Gramian P_1 corresponding to the type I ansatz, compare [5, 7]. It provides necessary conditions for reachability as we will see later. We define $P_1 := \mathbb{E}\int_0^\infty \Phi(s)BB^T\Phi^T(s)ds$, where Φ is the fundamental solution of (1), i.e., it satisfies

$$\Phi(t) = I_n + \int_0^t A\Phi(s)ds + \sum_{i=1}^q \int_0^t N_i\Phi(s-)dM_i(s), \quad t \geq 0.$$

The infinite integral P_1 is well-defined due to the asymptotic mean square stability of system (1), that is condition (2).

The solution $x(t)$, $t \geq 0$, to (1) can be expressed using the fundamental matrix Φ:

$$x(t, x_0, u) = \Phi(t)x_0 + \int_0^t \Phi(t)\Phi^{-1}(s)Bu(s)ds. \tag{9}$$

The above representation is a consequence of the classical product rule applied to the product $\Phi(t)f(t)$, where $f(t) := x_0 + \int_0^t \Phi^{-1}(s)Bu(s)ds$. Since f is continuous with a zero martingale part, the compensator processes are zero (see (64)). The Gramian P_1 has already been used in [5] (M is a Wiener process) and [7] (M is a vector of uncorrelated Lévy processes) in a different context. In both references the reachability of the mean state to the stochastic process

$$\hat{x}(t, 0, \hat{u}) = \int_0^t \Phi(t)\Phi^{-1}(s)B\hat{u}(s)ds$$

was analysed using P_1, where $\hat{u}$ is a square integrable stochastic process which is not necessarily adapted. The processes $\hat{x}(t, 0, \hat{u})$ and $x(t, 0, \hat{u})$ coincide if $\hat{u}$ is an adapted control. If $x \equiv \hat{x}$ it does not make too much sense to investigate the reachability of the mean state because the diffusion term in (1) chancels out when applying the mean. Here we use this Gramian P_1 to analyse reachability in (1) including the entire information in this equation. Furthermore, equation (1) is even more general than the ones considered in [5, 7]. For that reason, P_1 is the unique solution to a more general matrix equation

$$AP_1 + P_1A^T + \sum_{i,j=1}^q N_iP_1N_j^Tq_{ij} = -BB^T. \tag{10}$$

In [5, 7], the mixed terms are $q_{ij} = 0$ ($i \neq j$). To see the relation in (10), we make use of the partition $B = \left[\begin{array}{ccc} b_1 & \dots & b_m \end{array}\right]$ which yields $\Phi(t)B = \left[\begin{array}{ccc} x(t, b_1, 0) & \dots & x(t, b_m, 0) \end{array}\right]$ and hence the following identity $\mathbb{E}\left[\Phi(t)BB^T\Phi^T(t)\right] = \sum_{k=1}^m \mathbb{E}\left[x(t, b_k, 0)x^T(t, b_k, 0)\right]$. Applying Lemma 2.1 to every summand leads to

$$\begin{aligned}\mathbb{E}\left[\Phi(t)BB^T\Phi^T(t)\right] = BB^T &+ A\int_0^t \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right] ds \\ &+ \int_0^t \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right] ds A^T \\ &+ \sum_{i,j=1}^q N_i \int_0^t \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right] N_j^T q_{ij}.\end{aligned} \tag{11}$$

Taking the limit $t \to \infty$ in (11) and due to the asymptotic mean square stability the left side tends to zero. This provides equation (10). Below, we will make use of a solution representation to (1). We now analyse the process $\langle x(t, 0, u), \tilde{x}\rangle_2$, where $\tilde{x} \in \mathbb{R}^n$. We set $\Phi(t, s) = \Phi(t)\Phi^{-1}(s)$, $t \geq s \geq 0$. Inserting (9) yields a first bound

$$\begin{aligned}\mathbb{E}\left|\langle x(t, 0, u), \tilde{x}\rangle_2\right| &= \mathbb{E}\left|\int_0^t \langle \tilde{x}, \Phi(t, s)Bu(s)\rangle_2 ds\right| = \mathbb{E}\left|\int_0^t \langle B^T\Phi^T(t, s)\tilde{x}, u(s)\rangle_2 ds\right| \\ &\leq \mathbb{E}\int_0^t \left\|B^T\Phi^T(t, s)\tilde{x}\right\|_2 \left\|u(s)\right\|_2 ds.\end{aligned}$$

By Cauchy's inequality it follows that

$$\mathbb{E}\left|\langle x(t, 0, u), \tilde{x}\rangle_2\right| \leq \left(\mathbb{E}\int_0^t \left\|B^T\Phi^T(t, s)\tilde{x}\right\|_2^2 ds\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^t \left\|u(s)\right\|_2^2 ds\right)^{\frac{1}{2}}.$$

Following the arguments in Section 4 of [7], we know that $\mathbb{E}\left[\Phi(t, \tau)BB^T\Phi^T(t, \tau)\right] = \mathbb{E}\left[\Phi(t - \tau)BB^T\Phi^T(t - \tau)\right]$, since both functions satisfy the integral equation (11) with initial time $\tau \leq t \leq T$ which is uniquely solvable. Hence, we obtain

$$\begin{aligned}&\mathbb{E}\int_0^t \left\|B^T\Phi^T(t, \tau)\tilde{x}\right\|_2^2 ds = \tilde{x}^T\mathbb{E}\int_0^t \Phi(t, \tau)BB^T\Phi^T(t, \tau) ds\ \tilde{x} \\ &= \tilde{x}^T\mathbb{E}\int_0^t \Phi(t - \tau)BB^T\Phi^T(t - \tau) ds\ \tilde{x} = \tilde{x}^T\mathbb{E}\int_0^t \Phi(s)BB^T\Phi^T(s) ds\ \tilde{x} \leq \tilde{x}^T P_1 \tilde{x}\end{aligned}$$

and consequently

$$\sup_{t\in[0,T]} \mathbb{E}\left|\langle x(t, 0, u), \tilde{x}\rangle_2\right| \leq \left(\tilde{x}^T P_1 \tilde{x}\right)^{\frac{1}{2}} \left\|u\right\|_{L^2_T}. \tag{12}$$

If $\tilde{x} \in \ker P_1$, then the left side of (12) is zero which implies that $\langle x(t, 0, u), \tilde{x}\rangle_2 = 0$, $t \in [0, T]$, $\mathbb{P}$-a.s. regardless of the control that is chosen. That means that the trajectories of x are orthogonal to $\ker P_1$ and thus

$$\mathbb{P}\left\{x(t, 0, u) \in \operatorname{im} P_1, \quad t \in [0, T]\right\} = 1$$

for every $u \in L^2_T$, so that no state outside $\operatorname{im} P_1$ is reachable (from zero).

Let $(p_{1,k})_{k=1,\dots,n}$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of P_1. Then the following representation

$$x(t, 0, u) = \sum_{k=1}^{n} \langle x(t, 0, u), p_{1,k} \rangle_2 \, p_{1,k}$$

holds. With (12) we can answer the question how difficult it is to reach a state in the direction of $p_{1,k}$ by analysing the corresponding Fourier coefficient:

$$\sup_{t \in [0,T]} \mathbb{E} \left| \langle x(t, 0, u), p_{1,k} \rangle_2 \right| \leq \lambda_{1,k}^{\frac{1}{2}} \left\| u \right\|_{L^2_T}, \tag{13}$$

where $\lambda_{1,k}$ is the corresponding eigenvalue. Consequently, the Fourier coefficient in (13) is small (in the L^1-sense) if $\lambda_{1,k}$ is small assuming that the chosen control u is not too large. This implies that difficult to reach states have a “large” component in the eigenspaces of P_1 belonging to the small eigenvalues.

Reachability Gramian type II ansatz We now going to study an alternative Gramian P_2 which was initially introduced in [12] in order to guarantee the existence of an $\mathcal{H}_\infty$-error bound for balanced truncation model order reduction based on this Gramian. The type II Gramian was furthermore analysed in [6]. In both references, linear systems with Wiener noise were investigated. Using P_2, a first result on characterising the energy, when controlling the average state of a stochastic linear system, was given in [30], where the driver was a Lévy process with uncorrelated components. Since considering the average state neglects the diffusion term of the stochastic differential equation completely, we give another energy interpretation for P_2 now.

First of all, we introduce P_2 in a more general setting compared to [6, 12, 30]. We define it to be a positive definite solution to

$$A^T P_2^{-1} + P_2^{-1} A + \sum_{i,j=1}^{q} N_i^T P_2^{-1} N_j q_{ij} \leq -P_2^{-1} B B^T P_2^{-1}. \tag{14}$$

An inequality is considered in (14) since the existence of a positive definite solution is not ensured when having an equality. The existence of such kind of solution in the case of an inequality goes back to [12]. We state this result again due to the more general situation.

Proposition 3.1. *There exists a positive definite solution $P_2 > 0$ to inequality (14).*

Proof. Since system (1) is assumed to be mean square asymptotically stable, by Theorem 2.2 part (v), we have

$$A^T (\epsilon P) + (\epsilon P) A + \sum_{i,j=1}^{q} N_i^T (\epsilon P) N_j q_{ij} = -\epsilon Y \tag{15}$$

for an arbitrary matrix $Y > 0$ and $\epsilon > 0$, where the matrix $P > 0$ is positive definite. This we obtain by multiplying the equation in Theorem 2.2 (v) with ϵ. For a sufficiently small ϵ we can guarantee that $-\epsilon Y \leq -(\epsilon P)BB^T(\epsilon P)$ such that $P_2 = (\epsilon P)^{-1}$ solves (14). □

Remark. *We can fix the right side of (15) to $Y = I$ in order to practically compute a solution to (14). From the proof of Proposition 3.1 we know that $I = Y \geq \epsilon PBB^TP$, where P is the solution to (15) with $Y = I$. Hence, we set $\epsilon = \left(\lambda_{max}(PBB^TP)\right)^{-1}$ to ensure this property and then obtain a solution given by $P_2 = \lambda_{max}(PBB^TP)P^{-1}$. Of course, the maximal eigenvalue of PBB^TP can be fairly large and this solution candidate can be far from optimal. So, as mentioned in [6], an optimisation procedure for the solution to (14) is required. So far it is not clear how to do this.*

Let us now turn our attention to an energy interpretation of type (13) for the alternative Gramian P_2. Let $(p_{2,k})_{k=1,\dots,n}$ be eigenvectors of P_2 such that they represent an orthonormal basis of $\mathbb{R}^n$. The corresponding eigenvalues are denoted by $(\lambda_{2,k})_{k=1,\dots,n}$. Then,

$$\begin{aligned}
\mathbb{E}\langle x(t,0,u), p_{2,k}\rangle_2^2 &\leq \lambda_{2,k}\ \mathbb{E}\sum_{i=1}^n \lambda_{2,i}^{-1}\langle x(t,0,u), p_{2,i}\rangle_2^2 \\
&= \lambda_{2,k}\ \mathbb{E}\left\|\sum_{i=1}^n \lambda_{2,i}^{-\frac{1}{2}}\langle x(t,0,u), p_{2,i}\rangle_2\ p_{2,i}\right\|_2^2 = \lambda_{2,k}\ \mathbb{E}\left\|P_2^{-\frac{1}{2}}x(t,0,u)\right\|_2^2 \\
&= \lambda_{2,k}\ \mathbb{E}\left[x(t,0,u)^T P_2^{-1} x(t,0,u)\right].
\end{aligned}$$

We can argue like in the proof of Lemma 2.1 to find an equation for $\mathbb{E}\left[x(t,0,u)x(t,0,u)^T\right]$ since $x(t,0,u)$ and $x(t,x_0,0)$ have the same compensator process. This is because the additional control term only effects the drift and hence there is no change in the martingale part or in the jumps, compare (64). To shorten the notation we write $x(t)$ instead of $x(t,0,u)$ from time to time below. So, by the Ito product formula, we have

$$\begin{aligned}
\mathbb{E}\left[x(t)x^T(t)\right] = &\int_0^t \mathbb{E}\left[x(s)x^T(s)\right] A^T ds + \int_0^t \mathbb{E}\left[x(s)u^T(s)\right] B^T ds \\
&+ \int_0^t A\mathbb{E}\left[x(s)x^T(s)\right] ds + \int_0^t B\mathbb{E}\left[u(s)x^T(s)\right] ds \\
&+ \int_0^t \sum_{i,j=1}^q N_i \mathbb{E}\left[x(s)x^T(s)\right] N_j^T q_{ij} ds,
\end{aligned}$$

where the control terms that do not occur in Lemma 2.1 come from additional terms in $\mathbb{E}\int_0^t x(s-)dx^T(s)$ and $\mathbb{E}\int_0^t dx(s)x^T(s-)$. Using the trace operator and inserting the above equation, we find

$$\begin{aligned}\mathbb{E}\langle x(t,0,u),p_{2,k}\rangle_2^2 &\leq \lambda_{2,k}\ \operatorname{tr}\left(P_2^{-1}\mathbb{E}\left[x(t,0,u)x(t,0,u)^T\right]\right)\\ &= \lambda_{2,k}\ \operatorname{tr}\left(P_2^{-1}\left[\int_0^t \mathbb{E}\left[x(s)x^T(s)\right]A^T ds + \int_0^t \mathbb{E}\left[x(s)u^T(s)\right]B^T ds\right.\right.\\ &\quad + \int_0^t A\mathbb{E}\left[x(s)x^T(s)\right]ds + \int_0^t B\mathbb{E}\left[u(s)x^T(s)\right]ds\\ &\quad \left.\left.+ \int_0^t \sum_{i,j=1}^q N_i\mathbb{E}\left[x(s)x^T(s)\right]N_j^T q_{ij} ds\right]\right).\end{aligned}$$

Using again properties of the trace operator, we obtain

$$\begin{aligned}\mathbb{E}\langle x(t,0,u),p_{2,k}\rangle_2^2 \leq \lambda_{2,k}&\left[\mathbb{E}\int_0^t x^T(s)(A^TP_2^{-1}+P_2^{-1}A+\sum_{i,j=1}^q N_i^TP_2^{-1}N_jq_{ij})x(s)\ ds\right.\\ &\left.+2\mathbb{E}\int_0^t x^T(s)P_2^{-1}Bu(s)ds\right].\end{aligned}$$

We insert inequality (14) which gives

$$\begin{aligned}\mathbb{E}\langle x(t,0,u),p_{2,k}\rangle_2^2 &\leq \lambda_{2,k}\mathbb{E}\int_0^t 2x^T(s)P_2^{-1}Bu(s) - x^T(s)P_2^{-1}BB^TP_2^{-1}x(s)ds\\ &= \lambda_{2,k}\mathbb{E}\int_0^t \left\|u(s)\right\|_2^2 - \left\|B^TP_2^{-1}x(s)-u(s)\right\|_2^2 ds.\end{aligned}$$

Consequently, we have

$$\sup_{t\in[0,T]}\sqrt{\mathbb{E}\langle x(t,0,u),p_{2,k}\rangle_2^2} \leq \lambda_{2,k}^{\frac{1}{2}}\left\|u\right\|_{L^2_T}. \tag{16}$$

So, by (16), $x(t,0,u)$ is small in the direction of $p_{2,k}$ (in the L^2-sense) if $\lambda_{2,k}$ is small whenever the control u is not too large. This implies that difficult to reach states have a “large” component in the eigenspaces of P_2 belonging to the small eigenvalues. Hence, we have a similar interpretation as in the type I ansatz (compare with (13)) but in a different norm.

Observability Gramian We conclude this section by introducing an output equation

$$y(t,x_0,u) = Cx(t,x_0,u), \quad t\geq 0. \tag{17}$$

corresponding to (1). We recall arguments from [5, 7] below. We aim to characterise the importance of certain initial states in the system dynamics in the uncontrolled situation where $u\equiv 0$. In an observation problem an unknown initial state x_0 is supposed to be reconstructed from the observation $y(t,x_0,0)$ on the entire time line $[0,\infty)$.

In order to describe the energy caused by the observations of x_0, we introduce the observability Gramian Q as the unique solution to the following matrix equation

$$A^TQ + QA + \sum_{i,j=1}^{q} N_i^T Q N_j q_{ij} = -C^TC. \tag{18}$$

The definition of Q makes sense due to condition (2). With the following relation

$$\mathbb{E}\left[x(t,x_0,0)^T Q x(t,x_0,0)\right] = \operatorname{tr}(Q\mathbb{E}\left[x(t,x_0,0)x(t,x_0,0)^T\right]) \tag{19}$$

we can insert the result of Lemma 2.1. Since we can change the order in a matrix product within the trace, we then have

$$\begin{aligned}&\mathbb{E}\left[x(t,x_0,0)^T Q x(t,x_0,0)\right]\\ &= x_0^T Q x_0 + \mathbb{E}\int_0^t x(s,x_0,0)^T \left(A^TQ + QA + \sum_{i,j=1}^{q} N_i^T Q N_j q_{ij}\right) x(s,x_0,0)ds.\end{aligned}$$

We plug in equation (18) and obtain

$$\mathbb{E}\left[x(t,x_0,0)^T Q x(t,x_0,0)\right] = x_0^T Q x_0 - \mathbb{E}\int_0^t x(s,x_0,0)^T C^T C x(s,x_0,0)ds. \tag{20}$$

Because system (1) is mean square asymptotically stable, the left side of (20) tends to zero if $t \to \infty$. Hence, the observation energy is given by

$$\mathbb{E}\int_0^\infty \left\|y(s,x_0,0)\right\|_2^2 ds = x_0^T Q x_0. \tag{21}$$

The difficult to observe, and hence unimportant, initial states are those producing only little observation energy. From (21) we see that the difficult to observe states are contained in the eigenspaces spanned by the eigenvectors of Q corresponding to the small eigenvalues.

Moreover, it is easy to find a representation for Q. Inserting the solution representation $Cx(t,x_0,0) = C\Phi(t)x_0$ to (21), we then also have

$$x_0^T\, \mathbb{E}\int_0^\infty \Phi^T(t)C^TC\Phi(t)dt\; x_0 = x_0^T Q x_0.$$

Since this is true for every $x_0 \in \mathbb{R}^n$, this yields

$$Q = \mathbb{E}\int_0^\infty \Phi^T(t)C^TC\Phi(t)dt. \tag{22}$$

The infinite integral in (22) indeed exists due to the mean square asymptotic stability of the system.

4 Type II singular perturbation approximation and stability preservation

4.1 Balancing related MOR

Before considering singular perturbation approximation (SPA) based on the Gramians P_2 and Q (type II ansatz), we summarise the general theory of balancing and discuss how to find a balancing transformation.

States that are difficult to reach can be characterised by P_1, cf. (13). These states have large components in the span of the eigenvectors corresponding to small eigenvalues of the reachability Gramian P_1. Similarly, states that are difficult to observe are the ones that have large components in the span of eigenvectors corresponding to small eigenvalues of the observability Gramian Q, see (21). Now, balancing a system relies on the idea to create a system, where dominant reachable and observable states are the same, i.e., reachability and observability Gramians are simultaneously transformed such that they are equal and diagonal. Balancing related MOR based on the Gramians P_1 and Q (type I ansatz) was already studied intensively. Type I balanced truncation (BT) for systems with Wiener noise are investigated in [5] and systems with Lévy noise are studied in [7]. An alternative balancing method is type I SPA which can be found in [29].

In this paper, we consider the so called type II ansatz. This approach is based on the Gramians P_2 and Q. P_2 characterises difficult to reach states in a similar fashion as P_1, see (16). So, balancing with using P_2 instead of P_1 definitely makes sense too. For BT this is done in [6, 12, 30]. However, the type II ansatz has not yet been applied to SPA. For that reason, we will mainly discuss this approach in the following.

We consider a control system consisting of state equation (1) and output equation (17)

$$\begin{aligned} dx(t) &= [Ax(t) + Bu(t)]dt + \sum_{i=1}^{q} N_i x(t-) dM_i(t), \\ y(t) &= Cx(t), \quad t \geq 0, \end{aligned} \tag{23}$$

Recall that the state equation in (23) is mean square asymptotically stable, i.e., property (2) is satisfied. Introduce a transformation matrix $T \in \mathbb{R}^{n\times n}$ which is assumed to be non-singular, the states are transformed as follows:

$$\hat{x}(t) = Tx(t),$$

such that system (23) becomes

$$\begin{aligned} d\hat{x}(t) &= [\hat{A}\hat{x}(t) + \hat{B}u(t)]dt + \sum_{i=1}^{q} \hat{N}_i x(t-) dM_i(t), \\ y(t) &= \hat{C}\hat{x}(t), \quad t \geq 0, \end{aligned} \tag{24}$$

where $\hat{A} = TAT^{-1}$, $\hat{B} = TB$, $\hat{C} = CT^{-1}$ and $\hat{N}_i = TN_iT^{-1}$. The input-output map remains the same, only the state and the systems matrices are transformed.

P_2 and Q, the reachability and observability Gramians of system (23), which satisfy (14) and (18) can be transformed into reachability and observability Gramians of the transformed system (24):

$$\hat{P}_2 = TP_2T^T \quad \text{and} \quad \hat{Q} = T^{-T}QT^{-1}.$$

The above relation is obtained by multiplying (14) and (18) with T^{-T} from the left and T^{-1} from the right. The Hankel singular values (HSVs) $\sigma_1 \geq \ldots \geq \sigma_n$, where $\sigma_i = \sqrt{\lambda_i(P_2Q)}$ ($i = 1, \ldots, n$), of the original and transformed system are the same. The above transformation is a balancing transformation if the transformed Gramians are equal and diagonal. Such a transformation always exists if $Q > 0$ (observation energy is always non zero for every $x_0 \neq 0$). We also need that $P_2 > 0$ but this is automatically given by Proposition 3.1. A balanced system is obtained by choosing

$$T = \Sigma^{-\frac{1}{2}}U^TL^T \quad \text{and} \quad T^{-1} = KV\Sigma^{-\frac{1}{2}},$$

where $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n) > 0$ is the diagonal matrix of HSVs. Y, Z, L and K are computed as follows. Let $P_2 = KK^T$, $Q = LL^T$ be square root factorisations of P_2 and Q, then an SVD of $K^TL = V\Sigma U^T$ gives the required matrices. With this transformation $\hat{P}_2 = \hat{Q} = \Sigma$. This implies that Σ characterises both the reachability and observability in system (24). The smaller the diagonal entry of Σ, the less important the corresponding state component in the system dynamics of (24).

Below, let T be the balancing transformation as stated above, then we partition the coefficients of the balanced realisation as follows:

$$TAT^{-1} = \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right], \quad TB = \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right], \quad CT^{-1} = \left[\begin{smallmatrix} C_1 & C_2 \end{smallmatrix}\right], \quad TN_iT^{-1} = \left[\begin{smallmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{smallmatrix}\right],$$

where $A_{11} \in \mathbb{R}^{r \times r}$ etc. Furthermore, by setting $\hat{x} = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$, where $x_1(t) \in \mathbb{R}^r$, we obtain the transformed partitioned system

$$\begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \left[\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}u(t)\right]dt + \sum_{i=1}^{q}\begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{bmatrix}\begin{bmatrix} x_1(t-) \\ x_2(t-) \end{bmatrix}dM_i(t), \tag{25}$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad t \geq 0. \tag{26}$$

From this system we aim to obtain a approximating system with reduced dimension $r \ll n$. The ROM is of the form

$$dx_r(t) = [A_rx_r(t) + B_ru(t)dt] + \sum_{i=1}^{q}[N_{i,r}x_r(t-) + E_{i,r}u(t-)]dM_i(t), \tag{27}$$
$$y_r(t) = C_rx_r(t) + D_ru(t), \quad t \geq 0,$$

where $A_r, N_{i,r} \in \mathbb{R}^{r\times r}$, $B_r, E_{i,r} \in \mathbb{R}^{r\times m}$, $D_r \in \mathbb{R}^{p\times m}$ and $C_r \in \mathbb{R}^{p\times r}$. In equations (25) and (26), the difficult to reach and observe states are represented by x_2, which correspond to the smallest HSVs $\sigma_{r+1}, \dots, \sigma_n$, but of course r has to be chosen such that the neglected HSVs are small ($\sigma_{r+1} \ll \sigma_r$).

For BT the second row in (25) is truncated and the remaining x_2 components in the first row of (25) and in (26) are set to zero. This leads to a ROM having the same structure as (23). For BT the corresponding matrices are

$$(A_r, B_r, C_r, D_r, E_{i,r}, N_{i,r}) = (A_{11}, B_1, C_1, 0, 0, N_{i,11}).$$

We refer to [5, 6, 7, 12, 30] for more details on BT for stochastic systems.

An alternative method is SPA which has been studied already in [29] using the reachability Gramian P_1. From [29] it is known that the reduced coefficients are given by

$$(A_r, B_r, C_r, D_r, E_{i,r}, N_{i,r}) = (\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{E}_i, \bar{N}_i), \tag{28}$$

where we set

$$\begin{aligned} &\bar{A} := A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad \bar{B} := B_1 - A_{12}A_{22}^{-1}B_2, \quad \bar{C} := C_1 - C_2A_{22}^{-1}A_{21}, \\ &\bar{D} := -C_2A_{22}^{-1}B_2, \quad \bar{E}_i := -N_{i,12}A_{22}^{-1}B_2, \quad \bar{N}_i := N_{i,11} - N_{i,12}A_{22}^{-1}A_{21}. \end{aligned}$$

Hence, we see that this ROM has a different structure than (23) since D_r and $E_{i,r}$ are non-zero. The matrices (28) of the ROM using SPA are obtained by setting $dx_2(t) = 0$ in (25). One then solves for x_2 in the resulting algebraic constraint and inserts the result in (25) and (26). This straight forward ansatz is based on observations from the deterministic case ($N_i = 0$). There, x_2 represents the fast variables, i.e., $\dot{x}_2(t) \approx 0$ after a short time. Consequently, assuming $\dot{x}_2(t) = 0$ can lead to a good approximation, see [23].

This ansatz of setting the increments dx_2 equal to zero for the stochastic system is rather unsatisfactory, since this might be false, no matter how small the HSVs corresponding to x_2 are. Despite the fact that for the motivation, a maybe less convincing argument is used, this leads to a viable model reduction method with reasonable properties as we will see later.

An averaging principle would be a mathematically well-founded alternative to this naive approach. Averaging principles for stochastic systems have for example been investigated in [31, 32]. A further strategy to derive a reduced model in this context can be found in [8].

We conclude this subsection by introducing a simplified ROM based on SPA. It relies on the idea that the structure of the original model (23) should be preserved. It has already been discussed in [29] and is obtained by setting $B_2 = 0$ in (28):

$$(A_r, B_r, C_r, D_r, E_{i,r}, N_{i,r}) = (\bar{A}, B_1, \bar{C}, 0, 0, \bar{N}_i). \tag{29}$$

In the rest of this paper, properties of ROMs with matrices (28) or (29) are analysed. This means that we investigate type II SPA, a balancing related model order reduction technique based on the Gramians P_2 and Q. Advantages of using P_2 instead of P_1 can be seen in Subsections 4.2 and 5.2. So, the choice of P_2 guarantees the existence of an $\mathcal{H}_\infty$-type error bound (Subsection 5.2) which is not true for the type I ansatz. Mean square asymptotic stability is also preserved for type II SPA (Subsection 4.2) which is still an open problem when the system is balanced based on P_1 and Q, see [29].

4.2 Preservation of mean square asymptotic stability for type II singular perturbation approximation

In this subsection, we discuss mean square asymptotic stability in the ROMs (27) with coefficients (28) or (29). In the stability analysis it does not matter whether the reduced order matrices (28) or the simplified version (29) is considered. This is because the uncontrolled case is considered and only the matrices $\bar{A}$ and $\bar{N}_i$ $(i = 1, \dots, q)$ characterise the stability, compare Theorem 2.2. We will see that some ideas of proving asymptotic mean square stability can be adopted from the deterministic case, compare [23]. This is not true for type I SPA (reachability Gramian P_1 is used) which is explained in [29].

For simplicity we assume that the original model (23) is already balanced, i.e., the following relations hold true:

$$A^T\Sigma^{-1} + \Sigma^{-1}A + \sum_{i,j=1}^{q} N_i^T\Sigma^{-1}N_j q_{ij} \leq -\Sigma^{-1}BB^T\Sigma^{-1}, \tag{30}$$

$$A^T\Sigma + \Sigma A + \sum_{i,j=1}^{q} N_i^T\Sigma N_j q_{ij} = -C^T C, \tag{31}$$

where the Gramians coincide with the diagonal matrix $\Sigma > 0$ of HSVs. We multiply A^{-T} from the left and A^{-1} from the right in equations (30) and (31). Hence, we get

$$\tilde{A}^T\Sigma^{-1} + \Sigma^{-1}\tilde{A} + \sum_{i,j=1}^{q} \tilde{N}_i^T\Sigma^{-1}\tilde{N}_j q_{ij} \leq -A^{-T}\Sigma^{-1}BB^T\Sigma^{-1}A^{-1} \leq 0, \tag{32}$$

$$\tilde{A}^T\Sigma + \Sigma\tilde{A} + \sum_{i,j=1}^{q} \tilde{N}_i^T\Sigma\tilde{N}_j q_{ij} = -\tilde{C}^T\tilde{C} \leq 0, \tag{33}$$

where $\tilde{A} = A^{-1}$, $\tilde{N}_i = N_i A^{-1}$ and $\tilde{C} = CA^{-1}$. From Theorem 2.2 part (iv) it can be easily seen that the stability of the system with the transformed coefficients $\tilde{A}$ and $\tilde{N}_i$ is equivalent to the stability of the system with matrices A and N_i. In the following theorem it is proven that mean square asymptotic stability is preserved when considering the ROM with the left upper blocks $\tilde{A}_{11}$ and $\tilde{N}_{i,11}$ of the transformed matrices $\tilde{A}$ and $\tilde{N}_i$. This is the case of type II BT, where the stability preservation is investigated in [6]. The problem that is considered here can be reduced to the situation in [6] as we will see below.

Theorem 4.1. *Let $\tilde{A}_{11}$, $\tilde{N}_{i,11} \in \mathbb{R}^{r\times r}$ be the left upper blocks of $\tilde{A}$ and $\tilde{N}_i$, respectively. If $\sigma_r \neq \sigma_{r+1}$, then equation*

$$dx_r(t) = \tilde{A}_{11}x_r(t)dt + \sum_{i=1}^{q} \tilde{N}_{i,11}x_r(t-)dM_i(t), \quad t \geq 0,$$

is mean square asymptotically stable.

Proof. We transform a term that appears in (30) and (31) such that our problem is reduced to the case of Wiener noise. Let e_i be the ith unit vector of $\mathbb{R}^q$. We then have

$$\begin{aligned}\sum_{i,j=1}^{q} \tilde{N}_i^T\Sigma^{-1}\tilde{N}_j q_{ij} &= \sum_{i,j=1}^{q} \tilde{N}_i^T\Sigma^{-1}\tilde{N}_j e_i^T \mathcal{Q}^{\frac{1}{2}}\mathcal{Q}^{\frac{1}{2}} e_j \\ &= \sum_{i,j=1}^{q} \tilde{N}_i^T\Sigma^{-1}\tilde{N}_j \sum_{k=1}^{q}\langle \mathcal{Q}^{\frac{1}{2}}e_i, e_k\rangle_2 \langle \mathcal{Q}^{\frac{1}{2}}e_j, e_k\rangle_2 \\ &= \sum_{k=1}^{q}(\sum_{i=1}^{q} \tilde{N}_i\langle \mathcal{Q}^{\frac{1}{2}}e_i, e_k\rangle_2)^T\Sigma^{-1}(\sum_{j=1}^{q} \tilde{N}_j\langle \mathcal{Q}^{\frac{1}{2}}e_j, e_k\rangle_2).\end{aligned}$$

We define $\Psi_k := \sum_{i=1}^{q} \tilde{N}_i\langle \mathcal{Q}^{\frac{1}{2}}e_i, e_k\rangle_2$ and insert the above rearrangement. With (30) and (31) we apply the stability result in [6] and thus

$$I_r \otimes \tilde{A}_{11} + \tilde{A}_{11} \otimes I_r + \sum_{k=1}^{q} \Psi_{k,11} \otimes \Psi_{k,11}$$

has only eigenvalues with negative real parts, where $\Psi_{k,11}$ is the $r \times r$ left upper block of Ψ_k. With Theorem 2.2 this is equivalent to

$$\tilde{A}_{11}^T X + X\tilde{A}_{11} + \sum_{k=1}^{q} \Psi_{k,11}^T X \Psi_{k,11} < 0$$

for a positive definite matrix $X > 0$. Since

$$\begin{aligned}\sum_{k=1}^{q} \Psi_{k,11}^T X\Psi_{k,11} &= \sum_{k=1}^{q}(\sum_{i=1}^{q} \tilde{N}_{i,11}\langle \mathcal{Q}^{\frac{1}{2}}e_i, e_k\rangle_2)^T X(\sum_{j=1}^{q} \tilde{N}_{j,11}\langle \mathcal{Q}^{\frac{1}{2}}e_j, e_k\rangle_2) \\ &= \sum_{i,j=1}^{q} \tilde{N}_{i,11}^T X\tilde{N}_{j,11} q_{ij},\end{aligned}$$

the claim of this theorem follows by Theorem 2.2 (iv). □

The next Corollary states that type II SPA preserves mean square asymptotic stability.

Corollary 4.2. *If $\sigma_r \neq \sigma_{r+1}$, then for the following ROM of order r*

$$dx_r(t) = \bar{A}x_r(t)dt + \sum_{i=1}^{q} \bar{N}_i x_r(t-)dM_i(t), \quad t \geq 0,$$

is mean square asymptotically stable, where $\bar{A}$ and $\bar{N}_i$ are defined below (28).

Proof. Since one can show that

$$\tilde{A} = \begin{bmatrix} \bar{A}^{-1} & -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\ -A_{22}^{-1}A_{21}\bar{A}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix}, \tag{34}$$

we have $\tilde{A}_{11} = \bar{A}^{-1}$ and $\tilde{N}_{i,11} = \bar{N}_i\bar{A}^{-1}$. Using the result in Theorem 4.1 and using the equivalent condition in Theorem 2.2 (iv) yields

$$\bar{A}^{-T}X + X\bar{A}^{-1} + \sum_{i,j=1}^{q} \bar{A}^{-T}\bar{N}_i^T X \bar{N}_j \bar{A}^{-1} q_{ij} < 0$$

for a matrix $X > 0$. Multiplying $\bar{A}^T$ from the left and $\bar{A}$ from the right provides

$$\bar{A}^T X + X\bar{A} + \sum_{i,j=1}^{q} \bar{N}_i^T X \bar{N}_j q_{ij} < 0$$

and hence the result of this Corollary follows. □

5 Error bounds for type II SPA

In this section, we establish two error bounds namely an $\mathcal{H}_2$-type and an $\mathcal{H}_\infty$-type error bound. The $\mathcal{H}_2$-error bound is proven in Subsection 5.1 for the simplified type II SPA with matrices defined in (29). For the existence of this bound the stability result in Subsection 4.2 is vital. Moreover, the reason to analyse the $\mathcal{H}_2$-error bound for the simplified scheme is that we need to have the same structure in the ROM as in the original one. Then the solution representation in (9) can be applied.
An explicit solution representation is not needed when proving the $\mathcal{H}_\infty$-error bound in Subsection 5.2. There, the error between the full model and the type II ROM with coefficients (28) is investigated. The bound is shown by removing the HSVs step by step. Since the structure of the ROM with the matrices (28) differs from the structure of the original system, two cases have to be studied. So, we prove the case of removing the smallest HSV first and then we investigate the error of two ROMs of different dimensions.

5.1 $\mathcal{H}_2$-error bound

In this subsection, we provide an error bound between the original system (23) ($x_0 = 0$) and the output of the simplified ROM from using type II SPA, that is

$$dx_r(t) = [\bar{A}x_r(t) + B_1u(t)]dt + \sum_{i=1}^{q} \bar{N}_i x_r(t-)dM_i(t), \quad x_r(0) = 0, \tag{35}$$
$$y_r(t) = \bar{C}x_r(t), \quad t \geq 0,$$

where the above matrices are defined below (28). Let us now exploit the explicit solution representations for the full state variable x, see (9). Since ROM (35) has the same structure as the original model, we have the same representation for the ROM, too. We consequently obtain for the corresponding outputs that

$$y(t) = Cx(t) = C\int_0^t \Phi(t,s)Bu(s)ds,$$
$$y_r(t) = \bar{C}x_r(t) = \bar{C}\int_0^t \Phi_r(t,s)B_1u(s)ds,$$

where $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$ and $\Phi_r(t,s) = \Phi_r(t)\Phi_r^{-1}(s)$, $t \geq s \geq 0$. Here, Φ and Φ_r are the fundamental solutions to the full system and the ROM, respectively. Simple calculations give

$$\begin{aligned}
\mathbb{E}\left\|y(t) - y_r(t)\right\|_2 &= \mathbb{E}\left\|C\int_0^t \Phi(t,s)Bu(s)ds - \bar{C}\int_0^t \Phi_r(t,s)B_1u(s)ds\right\|_2 \\
&\leq \mathbb{E}\int_0^t \left\|\left(C\Phi(t,s)B - \bar{C}\Phi_r(t,s)B_1\right)u(s)\right\|_2 ds \\
&\leq \mathbb{E}\int_0^t \left\|C\Phi(t,s)B - \bar{C}\Phi_r(t,s)B_1\right\|_F \left\|u(s)\right\|_2 ds,
\end{aligned}$$

where $\left\|\cdot\right\|_F$ denotes the Frobenius norm. Using Cauchy's inequality, it holds that

$$\mathbb{E}\left\|y(t) - y_r(t)\right\|_2 \leq \left(\mathbb{E}\int_0^t \left\|C\Phi(t,s)B - \bar{C}\Phi_r(t,s)B_1\right\|_F^2 ds\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^t \left\|u(s)\right\|_2^2 ds\right)^{\frac{1}{2}}.$$

Applying the arguments that are used in Section 4 of [7], we know that

$$\begin{aligned}
\mathbb{E}\left[\Phi(t,s)BB^T\Phi^T(t,s)\right] &= \mathbb{E}\left[\Phi(t-s)BB^T\Phi^T(t-s)\right], \\
\mathbb{E}\left[\Phi_r(t,s)B_1B_1^T{\Phi_r}^T(t,s)\right] &= \mathbb{E}\left[\Phi_r(t-s)B_1B_1^T{\Phi_r}^T(t-s)\right], \\
\mathbb{E}\left[\Phi(t,s)BB_1^T{\Phi_r}^T(t,s)\right] &= \mathbb{E}\left[\Phi(t-s)BB_1^T{\Phi_r}^T(t-s)\right].
\end{aligned}$$

The above identities yield

$$\mathbb{E}\int_0^t \left\|C\Phi(t,s)B-\bar{C}\Phi_r(t,s)B_1\right\|_F^2 ds = \mathbb{E}\int_0^t \left\|C\Phi(t-s)B-\bar{C}\Phi_r(t-s)B_1\right\|_F^2 ds$$
$$= \mathbb{E}\int_0^t \left\|C\Phi(s)B-\bar{C}\Phi_r(s)B_1\right\|_F^2 ds \leq \mathbb{E}\int_0^\infty \left\|C\Phi(s)B-\bar{C}\Phi_r(s)B_1\right\|_F^2 ds$$
$$= \operatorname{tr}\left(CP_1C^T\right)+\operatorname{tr}\left(\bar{C}P_{r,1}\bar{C}^T\right)-2\ \operatorname{tr}\left(CP_g\bar{C}^T\right),$$

where

$$P_1 = \mathbb{E}\int_0^\infty \Phi(t)BB^T\Phi^T(t)dt$$

is the type I reachability Gramians of the full model solving equation (10). We further set

$$P_g = \mathbb{E}\int_0^\infty \Phi(t)BB_1^T\Phi_r^T(t)dt, \quad P_{r,1} = \mathbb{E}\int_0^\infty \bar{\Phi}(t)B_1B_1^T\bar{\Phi}^T(t)dt.$$

P_g and the type I reachability Gramians $P_{r,1}$ of the ROM exist since mean square asymptotic stability is preserved under the assumptions of Corollary 4.2 ($\sigma_r \neq \sigma_{r+1}$ and $\Sigma > 0$). Practically, $P_{r,1}$ is computed by solving

$$\bar{A}P_{r,1}+P_{r,1}\bar{A}^T+\sum_{i,j=1}^q \bar{N}_iP_{r,1}\bar{N}_j^Tq_{ij} = -B_1B_1^T \tag{36}$$

and the matrix P_g is derived from solving the following equation:

$$AP_g+P_g\bar{A}^T+\sum_{i,j=1}^q N_iP_g\bar{N}_j^Tq_{ij} = -BB_1^T. \tag{37}$$

The identity in (37) can be shown as the relations for P_1 and $P_{r,1}$ (see first paragraph of Section 3). For more details we refer to [7], where this identity is proven for a similar case. In summary, we have

$$\sup_{t\in[0,T]}\mathbb{E}\left\|y(t)-y_r(t)\right\|_2 \leq \left(\operatorname{tr}\left(CP_1C^T\right)+\operatorname{tr}\left(\bar{C}P_{r,1}\bar{C}^T\right)-2\ \operatorname{tr}\left(CP_g\bar{C}^T\right)\right)^{\frac{1}{2}}\left\|u\right\|_{L_T^2}. \tag{38}$$

It is an obvious observation that the type II reachability Gramians P_2 and $P_{r,2}$ do not enter the first bound (38) directly, even though we balance based on P_2. However, P_2 enters indirectly in the consideration since balancing based on P_2 ensures that asymptotic mean square stability is preserved, see Subsection 4.2. As argued above this is vital for the existence of P_g and $P_{r,1}$ and hence the existence of the bound in (38). For type I SPA the stability preservation has not been shown yet in general, see [29]. Another representation for (38) will be proven below. The $n-r$ smallest HSVs of the system will enter there. Consequently, the dependence of the error bound on P_2 can then be seen better.

From now on we assume that system (23) is balanced. Hence, for the type II Gramians it holds that $P_2 = Q = \Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2)$, where $\Sigma_1 = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$ contains the large and $\Sigma_2 = \operatorname{diag}(\sigma_{r+1}, \dots, \sigma_n)$ the neglected small HSVs. We partition the balanced realisation as follows

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \; N_i = \begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{bmatrix}, \; C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \tag{39}$$

where all matrices are of suitable size, i.e., $A_{11} \in \mathbb{R}^{r\times r}$ etc. The next theorem contains the main result of this subsection. We specify the bound in (38), where the resulting representation can be used to emphasise the cases in which type II SPA performs well. In particular, we obtain a bound that depends on the matrix Σ_2 of neglected HSVs.

Theorem 5.1. *Let system (23) be balanced. Under the assumption of Corollary 4.2 the error bound in (38) exists and can be represented as follows:*

$$\begin{aligned}
&\operatorname{tr}\left(CP_1C^T\right) + \operatorname{tr}\left(\bar{C}P_{r,1}\bar{C}^T\right) - 2\ \operatorname{tr}\left(CP_g\bar{C}^T\right) \\
&= \operatorname{tr}\left(\Sigma_2(B_2B_2^T - 2(A_{22}P_g^2 + A_{21}P_g^1)(A_{22}^{-1}A_{21})^T)\right) \\
&\quad + \operatorname{tr}\left(\Sigma_2\ 2\sum_{i,j=1}^{q}(N_{i,22}P_g^2 + N_{i,21}P_g^1)(N_{j,21} - N_{j,22}A_{22}^{-1}A_{21})^T q_{ij}\right) \\
&\quad - \operatorname{tr}\left(\Sigma_2\sum_{i,j=1}^{q}(N_{i,21} - N_{i,22}A_{22}^{-1}A_{21})P_{r,1}(N_{j,21} - N_{j,22}A_{22}^{-1}A_{21})^T q_{ij}\right),
\end{aligned}$$

where P_g^1 is the matrix of the first r and P_g^2 the matrix of the last $n-r$ rows of P_g. Moreover, q_{ij} represents the ijth entry of the covariance matrix $\mathcal{Q}$ of the Lévy process M.

Proof. Below, we make use of Einstein's summation convention which we indicate by writing q^{ij} instead of q_{ij}. We define $\mathcal{E} := \left(\operatorname{tr}\left(CP_1C^T\right) + \operatorname{tr}\left(\bar{C}P_{r,1}\bar{C}^T\right) - 2\ \operatorname{tr}\left(CP_g\bar{C}^T\right)\right)^{\frac{1}{2}}$. We easily see that

$$\begin{aligned}
\operatorname{tr}\left(CP_1C^T\right) &= \operatorname{tr}\left(P_1C^TC\right) = -\operatorname{tr}\left(P_1(A^T\Sigma + \Sigma A + N_i^T\Sigma N_j q^{ij})\right) \\
&= -\operatorname{tr}\left(\Sigma(AP_1 + P_1A^T + N_iP_1N_j^T q^{ij})\right) = \operatorname{tr}\left(B^T\Sigma B\right)
\end{aligned} \tag{40}$$

using the properties of the trace operator and inserting equations (31) and (10). From the partitioned error expression, we obtain

$$\begin{aligned}
\mathcal{E}^2 &= \operatorname{tr}\left(\begin{bmatrix} B_1^T & B_2^T \end{bmatrix}\begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\right) + \operatorname{tr}\left(\bar{C}P_{r,1}\bar{C}^T\right) - 2\ \operatorname{tr}\left(\begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} P_g^1 \\ P_g^2 \end{bmatrix}\bar{C}^T\right) \\
&= \operatorname{tr}(B_2^T\Sigma_2B_2 + B_1^T\Sigma_1B_1 + \bar{C}P_{r,1}\bar{C}^T - 2C_1P_g^1\bar{C}^T - 2C_2P_g^2\bar{C}^T).
\end{aligned}$$

We now use the partitions in (39) and the representation (34) for the inverse of A. In order to find equations for the matrices $\bar{C}^TC_1$ and $\bar{C}^TC_2$, we multiply (31) with A^{-T} from the left.

The left and right upper block of this equation are then

$$\begin{aligned}-\bar{A}^{-T}\bar{C}^T C_1 =&\Sigma_1 + \bar{A}^{-T}[\Sigma_1 A_{11} - A_{21}^T A_{22}^{-T}\Sigma_2 A_{21}\\ &+ \bar{N}_i^T \Sigma_1 N_{j,11} q^{ij} + (N_{i,21} - N_{i,22}A_{22}^{-1}A_{21})^T \Sigma_2 N_{j,21} q^{ij}],\\ -\bar{A}^{-T}\bar{C}^T C_2 =&\bar{A}^{-T}[\Sigma_1 A_{12} - A_{21}^T A_{22}^{-T}\Sigma_2 A_{22}\\ &+ \bar{N}_i^T \Sigma_1 N_{j,12} q^{ij} + (N_{i,21} - N_{i,22}A_{22}^{-1}A_{21})^T \Sigma_2 N_{j,22} q^{ij}].\end{aligned}$$

We multiply $\bar{A}^T$ from the left and thus

$$\begin{aligned}-\bar{C}^T C_1 =&\bar{A}^T\Sigma_1 + \Sigma_1 A_{11} - A_{21}^T A_{22}^{-T}\Sigma_2 A_{21} \quad (41)\\ &+ \bar{N}_i^T \Sigma_1 N_{j,11} q^{ij} + (N_{i,21} - N_{i,22}A_{22}^{-1}A_{21})^T \Sigma_2 N_{j,21} q^{ij},\\ -\bar{C}^T C_2 =&\Sigma_1 A_{12} - A_{21}^T A_{22}^{-T}\Sigma_2 A_{22} \quad (42)\\ &+ \bar{N}_i^T \Sigma_1 N_{j,12} q^{ij} + (N_{i,21} - N_{i,22}A_{22}^{-1}A_{21})^T \Sigma_2 N_{j,22} q^{ij}.\end{aligned}$$

From the partitioned equation (37)

$$\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}P_g^1\\ P_g^2\end{bmatrix} + \begin{bmatrix}P_g^1\\ P_g^2\end{bmatrix}\bar{A}^T + \begin{bmatrix}N_{i,11} & N_{i,12}\\ N_{i,21} & N_{i,22}\end{bmatrix}\begin{bmatrix}P_g^1\\ P_g^2\end{bmatrix}\bar{N}_j^T q^{ij} = -\begin{bmatrix}B_1 B_1^T\\ B_2 B_1^T\end{bmatrix},$$

we obtain

$$A_{11}P_g^1 + A_{12}P_g^2 + P_g^1\bar{A}^T + N_{i,11}P_g^1\bar{N}_j^T q^{ij} + N_{i,12}P_g^2\bar{N}_j^T q^{ij} = -B_1 B_1^T \quad (43)$$

by evaluating the first line. Now, plugging in (42) yields

$$\begin{aligned}&\operatorname{tr}(-C_2 P_g^2\bar{C}^T) = \operatorname{tr}(-\bar{C}^T C_2 P_g^2)\\ &= \operatorname{tr}([\Sigma_1 A_{12} - A_{21}^T A_{22}^{-T}\Sigma_2 A_{22} + \bar{N}_i^T\Sigma_1 N_{j,12}q^{ij} + \bar{N}_{i,21}^T\Sigma_2 N_{j,22}q^{ij}]P_g^2)\\ &= \operatorname{tr}(A_{12}P_g^2\Sigma_1 - A_{21}^T A_{22}^{-T}\Sigma_2 A_{22}P_g^2 + N_{i,12}P_g^2\bar{N}_j^T\Sigma_1 q^{ij} + \bar{N}_{i,21}^T\Sigma_2 N_{j,22}P_g^2 q^{ij}),\end{aligned}$$

where we set $\bar{N}_{i,21} = N_{i,21} - N_{i,22}A_{22}^{-1}A_{21}$. With equation (43), we obtain

$$\begin{aligned}\operatorname{tr}(-C_2 P_g^2\bar{C}^T) =&\operatorname{tr}(-A_{21}^T A_{22}^{-T}\Sigma_2 A_{22}P_g^2 + \bar{N}_{i,21}^T\Sigma_2 N_{j,22}P_g^2 q^{ij})\\ &- \operatorname{tr}([B_1 B_1^T + P_g^1\bar{A}^T + A_{11}P_g^1 + N_{i,11}P_g^1\bar{N}_j^T q^{ij}]\Sigma_1).\end{aligned}$$

Moreover, using equation (41), we have

$$\begin{aligned}&\operatorname{tr}([P_g^1\bar{A}^T + A_{11}P_g^1 + N_{i,11}P_g^1\bar{N}_j^T q^{ij}]\Sigma_1) = \operatorname{tr}([\bar{A}^T\Sigma_1 + \Sigma_1 A_{11} + \bar{N}_i^T\Sigma_1 N_{j,11}q^{ij}]P_g^1)\\ &= -\operatorname{tr}(\bar{C}^T C_1 P_g^1 + \bar{N}_{i,21}^T\Sigma_2 N_{j,21}P_g^1 q^{ij} - (A_{22}^{-1}A_{21})^T\Sigma_2 A_{21}P_g^1)\end{aligned}$$

and hence, inserting all derived identities yields

$$\begin{aligned}\mathcal{E}^2 =&\operatorname{tr}(B_2^T\Sigma_2 B_2 - B_1^T\Sigma_1 B_1 + \bar{C}P_{r,1}\bar{C}^T)\\ &+ 2\operatorname{tr}(\bar{N}_{i,21}^T\Sigma_2 N_{j,22}P_g^2 q^{ij} - (A_{22}^{-1}A_{21})^T\Sigma_2 A_{22}P_g^2)\\ &+ 2\operatorname{tr}(\bar{N}_{i,21}^T\Sigma_2 N_{j,21}P_g^1 q^{ij} - (A_{22}^{-1}A_{21})^T\Sigma_2 A_{21}P_g^1).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{E}^2 =& \operatorname{tr}(\Sigma_2(B_2B_2^T - 2(A_{22}P_g^2 + A_{21}P_g^1)(A_{22}^{-1}A_{21})^T)) \\ &+ \operatorname{tr}(2\Sigma_2(N_{i,22}P_g^2 + N_{i,21}P_g^1)\bar{N}_{j,21}^T q^{ij}) \\ &+ \operatorname{tr}(\bar{C}P_{r,1}\bar{C}^T - B_1^T\Sigma_1 B_1).\end{aligned} \tag{44}$$

Analogous to equation (40), we get $\operatorname{tr}(\bar{C}P_{r,1}\bar{C}^T) = \operatorname{tr}(B_1^T Q_r B_1)$, where the ROM observability Gramian satisfies

$$\bar{A}^T Q_r + Q_r\bar{A} + \bar{N}_i^T Q_r \bar{N}_j q^{ij} = -\bar{C}^T\bar{C}. \tag{45}$$

When inserting this into (44), we see that it remains to analyse the term $\operatorname{tr}(B_1^T(Q_R - \Sigma_1)B_1)$. We use the partition in (39) for equation (33). We evaluate the left upper block of (33), make use of the representation in (34) and then obtain

$$\bar{A}^T\Sigma_1 + \Sigma_1\bar{A} + \bar{N}_i^T\Sigma_1\bar{N}_j q^{ij} = -(\bar{C}^T\bar{C} + \bar{N}_{i,21}^T\Sigma_2\bar{N}_{j,21}q^{ij}).$$

With (45) we thus know that

$$\bar{A}^T(Q_r - \Sigma_1) + (Q_r - \Sigma_1)\bar{A} + \bar{N}_i^T(Q_r - \Sigma_1)\bar{N}_j q^{ij} = \bar{N}_{i,21}^T\Sigma_2\bar{N}_{j,21}q^{ij}. \tag{46}$$

Applying equations (36) and (46) yields

$$\begin{aligned}\operatorname{tr}(B_1^T(Q_r - \Sigma_1)B_1) =& -\operatorname{tr}([\bar{A}P_{r,1} + P_{r,1}\bar{A}^T + \bar{N}_iP_{r,1}\bar{N}_j^T q^{ij}](Q_R - \Sigma_1)) \\ =& -\operatorname{tr}(P_{r,1}[(Q_r - \Sigma_1)\bar{A} + \bar{A}^T(Q_r - \Sigma_1) + \bar{N}_i^T(Q_r - \Sigma_1)\bar{N}_j q^{ij}]) \\ =& -\operatorname{tr}(P_{r,1}\bar{N}_{i,21}^T\Sigma_2\bar{N}_{j,21}q^{ij}).\end{aligned}$$

We apply these results to (44) and obtain

$$\begin{aligned}\mathcal{E}^2 =& \operatorname{tr}(\Sigma_2(B_2B_2^T - 2(A_{22}P_g^2 + A_{21}P_g^1)(A_{22}^{-1}A_{21})^T)) \\ &+ \operatorname{tr}(2\Sigma_2(N_{i,22}P_g^2 + N_{i,21}P_g^1)\bar{N}_{j,21}^T q^{ij}) - \operatorname{tr}(\Sigma_2\bar{N}_{i,21}P_{r,1}\bar{N}_{j,21}^T q^{ij}),\end{aligned}$$

which gives the required result. □

From Theorem 5.1 it can be seen that the $\mathcal{H}_2$-type error bound can be written as an expression depending on Σ_2, the matrix of the $n - r$ smallest HSVs $\sigma_{r+1}, \dots, \sigma_n$ of the original system. These values correspond to the truncated state components. If these components are unimportant, i.e., they are difficult to reach and difficult to observe, then the values $\sigma_{r+1}, \dots, \sigma_n$ are small. Consequently, the error bound would be small which indicates that the ROM from applying type II SPA has a good quality.

5.2 $\mathcal{H}_\infty$-error bound

An $\mathcal{H}_\infty$-error bound for deterministic systems ($N_i = 0$, $i = 1, \dots, q$) can be found in [23], which uses tools that are not available in the more general stochastic case such as transfer functions. Using transfer functions the link between SPA and BT is shown, so that the $\mathcal{H}_\infty$-bound for SPA can be directly concluded from the $\mathcal{H}_\infty$-bound of BT. In the stochastic case, the proof has to be conducted in the time domain. Moreover, in terms of the $\mathcal{H}_\infty$-error bound, there seems to be no link between the case of type II BT (investigated in [5, 12, 30]) and type II SPA. This makes the analysis more complicated here. Additionally, we encounter the problem of a change in the structure from the original to the ROM such that the arguments in the first paragraph below can not just simply be repeated when removing the HSVs step by step. Hence, the consideration of a second case, where the error between two different ROMs is studied, is needed. When comparing these two ROMs, we can not rely on having matrix inequality (30) for the ROM too. This can be seen by evaluating the left upper block of (32). For that reason, we will link to the full matrix inequality (30) in our proof, although we compare two systems in the second paragraph that are reduced already.

Before we start with the actual proof of the $\mathcal{H}_\infty$-error bound, we introduce two straight forward results which are frequently needed below.

Lemma 5.2. *Let $a, b_1, \dots, b_q$ be $\mathbb{R}^d$-valued processes, where a is adapted and almost surely Lebesgue integrable and the functions b_i are integrable with respect to the mean zero square integrable Lévy process $M = (M_1, \dots, M_q)^T$. If the process x is given by*

$$dx(t) = a(t)dt + \sum_{i=1}^{q} b_i(t)dM_i,$$

then, we have

$$\frac{d}{dt}\mathbb{E}\left[x^T(t)x(t)\right] = 2\mathbb{E}\left[x^T(t)a(t)\right] + \sum_{i,j=1}^{q} \mathbb{E}\left[b_i^T(t)b_j(t)\right] q_{ij}.$$

Proof. We define the matrix-valued process $b := [b_1, \dots, b_q]$ and apply Corollary A.1 to get

$$x^T(t)x(t) = x^T(0)x(0) + 2\int_0^t x^T(s-)dx(s) + \sum_{k=1}^{d} [e_k^T x, e_k^T x]_t.$$

Inserting the differential of x and taking the expectation yields

$$\mathbb{E}\left[x^T(t)x(t)\right] = \mathbb{E}\left[x^T(0)x(0)\right] + 2\int_0^t \mathbb{E}\left[x^T(s)a(s)\right] ds + \mathbb{E}\left[\sum_{k=1}^{d} [e_k^T x, e_k^T x]_t\right].$$

With the arguments in the proof of Lemma 2.1 and Ito's isometry it can be shown that

$$\mathbb{E}\left[\sum_{k=1}^{d}[e_k^T x, e_k^T x]_t\right] = \mathbb{E}\left\|\int_0^t b(s)dM(s)\right\|_2^2 = \mathbb{E}\int_0^t \left\|b(s)\mathcal{Q}^{\frac{1}{2}}\right\|_F^2 ds$$
$$= \mathbb{E}\int_0^t \operatorname{tr}\left(b^T(s)b(s)\mathcal{Q}\right)ds = \mathbb{E}\int_0^t \sum_{i,j=1}^{q} b_i^T(s)b_j(s)q_{ij}ds,$$

where $\mathcal{Q}$ denotes the covariance matrix of M. This concludes the proof. □

Proposition 5.3. *Let $A_1, \dots, A_q$ be $d_1 \times d_2$ matrices and $K = (k_{ij})_{i,j=1,\dots,q}$ be a positive semidefinite matrix, then*

$$\tilde{K} := \sum_{i,j=1}^{q} A_i^T A_j k_{ij}$$

is also positive semidefinite.

Proof. Let x be an arbitrary vector in $\mathbb{R}^{d_2}$, then

$$x^T\tilde{K}x = \sum_{i,j=1}^{q}(A_i x)^T A_j x k_{ij} = \sum_{i,j=1}^{q}(A_i x)^T A_j x e_i^T K^{\frac{1}{2}} K^{\frac{1}{2}} e_j$$
$$= \sum_{i,j=1}^{q}(A_i x)^T A_j x \sum_{k=1}^{q}\langle K^{\frac{1}{2}}e_i, e_k\rangle_2 \langle K^{\frac{1}{2}}e_j, e_k\rangle_2$$
$$= \sum_{k=1}^{q}\Big(\sum_{i=1}^{q} A_i x\langle K^{\frac{1}{2}}e_i, e_k\rangle_2\Big)^T\Big(\sum_{j=1}^{q} A_j x\langle K^{\frac{1}{2}}e_j, e_k\rangle_2\Big) \geq 0.$$

□

Error bound of removing the smallest Hankel singular value In this paragraph, we determine a bound for the error between the full model and the ROM of only removing one HSV. This represents the first step of proving the general $\mathcal{H}_\infty$-error bound for type II SPA in Theorem 5.6.

We recall the original model that we aim to reduce:

$$dx(t) = [Ax(t) + Bu(t)]dt + \sum_{i=1}^{q} N_i x(t-)dM_i(t), \quad x(0) = 0, \tag{47}$$
$$y(t) = Cx(t), \quad t \geq 0,$$

where the matrices and vectors above are partitioned as follows

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \ N_i = \begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{bmatrix}, \ C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$

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To simplify the notation, we assume that system (47) is balanced already, i.e., we applied the balancing transformation from Subsection 4.1 already. Hence, the Gramians P_2 and Q are equal and coincide with the diagonal matrix $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2)$, where $\Sigma_1 = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$ is the matrix of large and $\Sigma_2 = \operatorname{diag}(\sigma_{r+1}, \dots, \sigma_n)$ the matrix of neglected small HSVs. Consequently, the matrix (in)equalities (30) and (31) hold. To show the error bound below, we do not need the equality in (31). It can be replaced by a more general inequality.
The following ROM is supposed to be compare with the original model (47):

$$
\begin{aligned}
dx_r(t) &= [\bar{A}x_r(t) + \bar{B}u(t)]dt + \sum_{i=1}^{q} \left[\bar{N}_i x_r(t-) + \bar{E}_i u(t-)\right] dM_i(t), \quad x_r(0) = 0, \\
y_r(t) &= \bar{C}x_r(t) + \bar{D}u(t), \quad t \geq 0,
\end{aligned} \tag{48}
$$

where the matrices are defined below (28). The next theorem deals with the error of removing the smallest HSV, i.e., we consider the case of Σ_2 being a multiple of the identity matrix.

Theorem 5.4. *If $\Sigma_2 = \sigma I$, then*

$$
\left\|y - y_r\right\|_{L^2_T} \leq 2\sigma \left\|u\right\|_{L^2_T}.
$$

Proof. We sometimes omit the time dependence of the functions in this proof to keep the notation as easy as possible. For the same reason, we make use of Einstein's summation convention which we indicate by writing q^{ij} instead of q_{ij}. Inserting for y and y_r yields

$$
\begin{aligned}
-\mathbb{E}\left\|y - y_r\right\|_2^2 &= -\mathbb{E}\left\|C_1[x_1 - x_r] + C_2[x_2 + A_{22}^{-1}A_{21}x_r + A_{22}^{-1}B_2 u]\right\|_2^2 \\
&= -\mathbb{E}\left(\begin{bmatrix} x_1 - x_r \\ x_2 + A_{22}^{-1}(A_{21}x_r + B_2 u)\end{bmatrix}^T C^T C \begin{bmatrix} x_1 - x_r \\ x_2 + A_{22}^{-1}(A_{21}x_r + B_2 u)\end{bmatrix}\right).
\end{aligned}
$$

The partitioned matrix (in)equality (31)

$$
\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}^T \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2\end{bmatrix} + \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2\end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix} + \begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22}\end{bmatrix}^T \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2\end{bmatrix}\begin{bmatrix} N_{j,11} & N_{j,12} \\ N_{j,21} & N_{j,22}\end{bmatrix} q^{ij} \leq -C^T C \tag{49}
$$

leads to

$$
\begin{aligned}
&-\mathbb{E}\left\|y - y_r\right\|_2^2 \geq \\
&\mathbb{E}\Bigg(2\begin{bmatrix} x_1 - x_r \end{bmatrix}^T \Sigma_1 \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}\begin{bmatrix} x_1 - x_r \\ x_2 + A_{22}^{-1}(A_{21}x_r + B_2 u)\end{bmatrix} \\
&+ \left(\begin{bmatrix} N_{i,11} & N_{i,12} \end{bmatrix}\begin{bmatrix} x_1 - x_r \\ x_2 + A_{22}^{-1}(A_{21}x_r + B_2 u)\end{bmatrix}\right)^T \Sigma_1 \begin{bmatrix} N_{j,11} & N_{j,12} \end{bmatrix}\begin{bmatrix} x_1 - x_r \\ x_2 + A_{22}^{-1}(A_{21}x_r + B_2 u)\end{bmatrix} q^{ij} \\
&+ 2\begin{bmatrix} x_2 + A_{22}^{-1}(A_{21}x_r + B_2 u) \end{bmatrix}^T \Sigma_2 \begin{bmatrix} A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1 - x_r \\ x_2 + A_{22}^{-1}(A_{21}x_r + B_2 u)\end{bmatrix} \\
&+ \left(\begin{bmatrix} N_{i,21} & N_{i,22} \end{bmatrix}\begin{bmatrix} x_1 - x_r \\ x_2 + A_{22}^{-1}(A_{21}x_r + B_2 u)\end{bmatrix}\right)^T \Sigma_2 \begin{bmatrix} N_{j,21} & N_{j,22} \end{bmatrix}\begin{bmatrix} x_1 - x_r \\ x_2 + A_{22}^{-1}(A_{21}x_r + B_2 u)\end{bmatrix} q^{ij}\Bigg).
\end{aligned}
$$

We define with the above summands:

$$\begin{aligned}
\mathcal{T}_1 :&= \mathbb{E}\left(2\left[\,x_1-x_r\,\right]^T\,\Sigma_1\,\left[\,A_{11}\;A_{12}\,\right]\left[\begin{smallmatrix} x_1-x_r \\ x_2+A_{22}^{-1}(A_{21}x_r+B_2u)\end{smallmatrix}\right]\right)\\
&= \mathbb{E}\left(2\left[\,x_1-x_r\,\right]^T\,\Sigma_1\,\left[\,A_{11}x_1+A_{12}x_2-\bar{A}x_r+(B_1-\bar{B})u\,\right]\right),\\
\mathcal{T}_2 :&= \mathbb{E}\left(\left[\,N_{i,11}\;N_{i,12}\,\right]\left[\begin{smallmatrix} x_1-x_r \\ x_2+A_{22}^{-1}(A_{21}x_r+B_2u)\end{smallmatrix}\right]\right)^T\,\Sigma_1\,\left[\,N_{j,11}\;N_{j,12}\,\right]\left[\begin{smallmatrix} x_1-x_r \\ x_2+A_{22}^{-1}(A_{21}x_r+B_2u)\end{smallmatrix}\right]\,q^{ij}\\
&= \mathbb{E}\left(\left[\,N_{i,11}x_1+N_{i,12}x_2-\bar{N}_ix_r-\bar{E}_iu\,\right]^T\,\Sigma_1\,\left[\,N_{j,11}x_1+N_{j,12}x_2-\bar{N}_jx_r-\bar{E}_ju\,\right]\,q^{ij}\right),\\
\mathcal{T}_3 :&= \mathbb{E}\left(2\left[\,x_2+A_{22}^{-1}(A_{21}x_r+B_2u)\,\right]^T\,\Sigma_2\,\left[\,A_{21}\;A_{22}\,\right]\left[\begin{smallmatrix} x_1-x_r \\ x_2+A_{22}^{-1}(A_{21}x_r+B_2u)\end{smallmatrix}\right]\right)\\
&= \mathbb{E}\left(2\left[\,x_2+A_{22}^{-1}(A_{21}x_r+B_2u)\,\right]^T\,\Sigma_2\,\left[\,A_{21}x_1+A_{22}x_2+B_2u\,\right]\right),\\
\mathcal{T}_4 :&= \mathbb{E}\left(\left[\,N_{i,21}\;N_{i,22}\,\right]\left[\begin{smallmatrix} x_1-x_r \\ x_2+A_{22}^{-1}(A_{21}x_r+B_2u)\end{smallmatrix}\right]\right)^T\,\Sigma_2\,\left[\,N_{j,21}\;N_{j,22}\,\right]\left[\begin{smallmatrix} x_1-x_r \\ x_2+A_{22}^{-1}(A_{21}x_r+B_2u)\end{smallmatrix}\right]\,q^{ij}\\
&= \mathbb{E}\left(\left[\,N_{i,21}x_1+N_{i,22}x_2-\bar{N}_{i,21}x_r+N_{i,22}A_{22}^{-1}B_2u\,\right]^T\,\Sigma_2\,\left[\,N_{j,21}x_1+N_{j,22}x_2-\bar{N}_{j,21}x_r+N_{j,22}A_{22}^{-1}B_2u\,\right]\,q^{ij}\right),
\end{aligned}$$

where $\bar{N}_{i,21} = N_{i,21} - N_{i,22}A_{22}^{-1}A_{21}$. Since, we have

$$\begin{aligned}
d(x_1(t) - x_r(t)) =& [A_{11}x_1(t) + A_{12}x_2(t) + (B_1 - \bar{B})u(t) - \bar{A}x_r(t)]dt\\
&+ \sum_{i=1}^{q}[N_{i,11}x_1(t) + N_{i,12}x_2(t) + -\bar{E}_iu(t) - \bar{N}_ix_r(t)]dM_i(t)
\end{aligned}$$

by Lemma 5.2, we obtain

$$\frac{d}{dt}\mathbb{E}\left((x_1(t) - x_r(t))^T\Sigma_1(x_1(t) - x_r(t))\right) = \mathcal{T}_1 + \mathcal{T}_2.$$

The variable x_2 obeys

$$dx_2(t) = [A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t)]dt + \sum_{i=1}^{q}[N_{i,21}x_1(t) + N_{i,22}x_2(t)]dM_i(t). \tag{50}$$

Again with Lemma 5.2, we have

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left(x_2(t)^T\Sigma_2x_2(t)\right) = 2\mathbb{E}\left(x_2^T(t)\Sigma_2(A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t))\right)\\
&+ \mathbb{E}\left((N_{i,21}x_1(t) + N_{i,22}x_2(t))^T\Sigma_2(N_{j,21}x_1(t) + N_{j,22}x_2(t))q^{ij}\right).
\end{aligned}$$

This yields

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}\left(x_2(t)^T\Sigma_2x_2(t)\right) =& \left[\mathcal{T}_3 - 2\mathbb{E}(\left[\,A_{22}^{-1}(A_{21}x_r+B_2u)\,\right]^T\,\Sigma_2\,\left[\,A_{21}x_1+A_{22}x_2+B_2u\,\right])\right]\\
&+ \left[\mathcal{T}_4 - 2\mathbb{E}\left(\left[\,N_{i,22}A_{22}^{-1}B_2u-\bar{N}_{i,21}x_r\,\right]^T\,\Sigma_2\,\left[\,N_{j,21}x_1+N_{j,22}x_2\,\right]\,q^{ij}\right)\right.\\
&\left.-\mathbb{E}\left(\left[\,N_{i,22}A_{22}^{-1}B_2u-\bar{N}_{i,21}x_r\,\right]^T\,\Sigma_2\,\left[\,N_{j,22}A_{22}^{-1}B_2u-\bar{N}_{j,21}x_r\,\right]\,q^{ij}\right)\right].
\end{aligned}$$

Summarising the above computations, we obtain

$$
\begin{aligned}
-\mathbb{E}\left\|y-y_r\right\|_2^2 \geq & \frac{d}{dt}\mathbb{E}\left((x_1(t)-x_r(t))^T\Sigma_1(x_1(t)-x_r(t))\right)+\frac{d}{dt}\mathbb{E}\left(x_2(t)^T\Sigma_2 x_2(t)\right) \\
& + 2\mathbb{E}\left(\left[A_{22}^{-1}(A_{21}x_r+B_2u)\right]^T \Sigma_2 \left[A_{21}x_1+A_{22}x_2+B_2u\right]\right) \\
& + 2\mathbb{E}\left(\left[N_{i,22}A_{22}^{-1}B_2u-\bar{N}_{i,21}x_r\right]^T \Sigma_2 \left[N_{j,21}x_1+N_{j,22}x_2\right] q^{ij}\right) \\
& + \mathbb{E}\left(\left[N_{i,22}A_{22}^{-1}B_2u-\bar{N}_{i,21}x_r\right]^T \Sigma_2 \left[N_{j,22}A_{22}^{-1}B_2u-\bar{N}_{j,21}x_r\right] q^{ij}\right).
\end{aligned}
$$

Using Proposition 5.3 and the assumption that $\Sigma_2=\sigma I$ provides

$$
\begin{aligned}
\mathbb{E}\int_0^T \left\|y(t)-y_r(t)\right\|_2^2 dt \leq & -2\sigma^2\Bigg[\mathbb{E}\int_0^T \left[A_{22}^{-1}(A_{21}x_r+B_2u)\right]^T \Sigma_2^{-1} \left[A_{21}x_1+A_{22}x_2+B_2u\right] dt \\
& +\mathbb{E}\int_0^T \left[N_{i,22}A_{22}^{-1}B_2u-\bar{N}_{i,21}x_r\right]^T \Sigma_2^{-1} \left[N_{j,21}x_1+N_{j,22}x_2\right] q^{ij}\, dt\Bigg].
\end{aligned} \tag{51}
$$

Inequality (30) and the Schur complement condition on definiteness implies

$$
\begin{bmatrix} A^T\Sigma^{-1}+\Sigma^{-1}A+N_i^T\Sigma^{-1}N_jq^{ij} & \Sigma^{-1}B \\ B^T\Sigma^{-1} & -I \end{bmatrix} \leq 0. \tag{52}
$$

If we multiply $\begin{bmatrix} x_1+x_r \\ x_2-A_{22}^{-1}(A_{21}x_r+B_2u) \\ 2u \end{bmatrix}^T$ from the left and $\begin{bmatrix} x_1+x_r \\ x_2-A_{22}^{-1}(A_{21}x_r+B_2u) \\ 2u \end{bmatrix}$ from the right to matrix inequality (52) and furthermore take the expected value, then we get

$$
\begin{aligned}
4\mathbb{E}\left\|u\right\|_2^2 \geq & \mathbb{E}\Bigg(2\left[x_1+x_r\right]^T \Sigma_1^{-1}\left(\begin{bmatrix} A_{11} & A_{12}\end{bmatrix}\begin{bmatrix} x_1+x_r \\ x_2-A_{22}^{-1}(A_{21}x_r+B_2u)\end{bmatrix}+2B_1u\right) \\
& + \left(\begin{bmatrix} N_{i,11} & N_{i,12}\end{bmatrix}\begin{bmatrix} x_1+x_r \\ x_2-A_{22}^{-1}(A_{21}x_r+B_2u)\end{bmatrix}\right)^T \Sigma_1^{-1}\begin{bmatrix} N_{j,11} & N_{j,12}\end{bmatrix}\begin{bmatrix} x_1+x_r \\ x_2-A_{22}^{-1}(A_{21}x_r+B_2u)\end{bmatrix} q^{ij} \\
& + 2\left[x_2-A_{22}^{-1}(A_{21}x_r+B_2u)\right]^T \Sigma_2^{-1}\left(\begin{bmatrix} A_{21} & A_{22}\end{bmatrix}\begin{bmatrix} x_1+x_r \\ x_2-A_{22}^{-1}(A_{21}x_r+B_2u)\end{bmatrix}+2B_2u\right) \\
& + \left(\begin{bmatrix} N_{i,21} & N_{i,22}\end{bmatrix}\begin{bmatrix} x_1+x_r \\ x_2-A_{22}^{-1}(A_{21}x_r+B_2u)\end{bmatrix}\right)^T \Sigma_2^{-1}\begin{bmatrix} N_{j,21} & N_{j,22}\end{bmatrix}\begin{bmatrix} x_1+x_r \\ x_2-A_{22}^{-1}(A_{21}x_r+B_2u)\end{bmatrix} q^{ij}\Bigg).
\end{aligned}
$$

The above terms are used to define

$$
\begin{aligned}
\mathcal{T}_5 :&= \mathbb{E}\left(2\left[x_1+x_r\right]^T \Sigma_1^{-1}\left(\begin{bmatrix} A_{11} & A_{12}\end{bmatrix}\begin{bmatrix} x_1+x_r \\ x_2-A_{22}^{-1}(A_{21}x_r+B_2u)\end{bmatrix}+2B_1u\right)\right) \\
&= \mathbb{E}\left(2\left[x_1+x_r\right]^T \Sigma_1^{-1}\left[A_{11}x_1+A_{12}x_2+\bar{A}x_r+(B_1+\bar{B})u\right]\right), \\
\mathcal{T}_6 :&= \mathbb{E}\left(\left(\begin{bmatrix} N_{i,11} & N_{i,12}\end{bmatrix}\begin{bmatrix} x_1+x_r \\ x_2-A_{22}^{-1}(A_{21}x_r+B_2u)\end{bmatrix}\right)^T \Sigma_1^{-1}\begin{bmatrix} N_{j,11} & N_{j,12}\end{bmatrix}\begin{bmatrix} x_1+x_r \\ x_2-A_{22}^{-1}(A_{21}x_r+B_2u)\end{bmatrix} q^{ij}\right) \\
&= \mathbb{E}\left(\left[N_{i,11}x_1+N_{i,12}x_2+\bar{N}_ix_r+\bar{E}_iu\right]^T \Sigma_1^{-1}\left[N_{j,11}x_1+N_{j,12}x_2+\bar{N}_jx_r+\bar{E}_ju\right] q^{ij}\right), \\
\mathcal{T}_7 :&= \mathbb{E}\left(2\left[x_2-A_{22}^{-1}(A_{21}x_r+B_2u)\right]^T \Sigma_2^{-1}\left(\begin{bmatrix} A_{21} & A_{22}\end{bmatrix}\begin{bmatrix} x_1+x_r \\ x_2-A_{22}^{-1}(A_{21}x_r+B_2u)\end{bmatrix}+2B_2u\right)\right)
\end{aligned}
$$

$$= \mathbb{E}\left(2\left[\, x_2 - A_{22}^{-1}(A_{21}x_r + B_2u)\,\right]^T\ \Sigma_2^{-1}\ \left[\, A_{21}x_1 + A_{22}x_2 + B_2u\,\right]\right),$$

$$\mathcal{T}_8 := \mathbb{E}\left(\left[\, N_{i,21}\ N_{i,22}\,\right]\begin{bmatrix} x_1 + x_r \\ x_2 - A_{22}^{-1}(A_{21}x_r + B_2u)\end{bmatrix}\right)^T\ \Sigma_2^{-1}\ \left[\, N_{j,21}\ N_{j,22}\,\right]\begin{bmatrix} x_1 + x_r \\ x_2 - A_{22}^{-1}(A_{21}x_r + B_2u)\end{bmatrix}\ q^{ij}$$

$$= \mathbb{E}\left(\left[\, N_{i,21}x_1 + N_{i,22}x_2 + \bar{N}_{i,21}x_r - N_{i,22}A_{22}^{-1}B_2u\,\right]^T\ \Sigma_2^{-1}\ \left[\, N_{j,21}x_1 + N_{j,22}x_2 + \bar{N}_{j,21}x_r - N_{j,22}A_{22}^{-1}B_2u\,\right]\ q^{ij}\right).$$

Using Lemma 5.2, exploiting the following equation

$$\begin{aligned} d(x_1(t) + x_r(t)) =& [A_{11}x_1(t) + A_{12}x_2(t) + (B_1 + \bar{B})u(t) + \bar{A}x_r(t)]dt \\ &+ \sum_{i=1}^{q} [N_{i,11}x_1(t) + N_{i,12}x_2(t) + \bar{E}_i u(t) + \bar{N}_i x_r(t)]dM_i(t) \end{aligned}$$

and with (50), we the first of all find that

$$\frac{d}{dt}\mathbb{E}\left((x_1(t) + x_r(t))^T\Sigma_1^{-1}(x_1(t) + x_r(t))\right) = \mathcal{T}_5 + \mathcal{T}_6$$

and secondly obtain the following identity

$$\begin{aligned} \frac{d}{dt}\mathbb{E}\left(x_2(t)^T\Sigma_2^{-1}x_2(t)\right) =& \left[\mathcal{T}_7 + 2\mathbb{E}(\left[\, A_{22}^{-1}(A_{21}x_r + B_2u)\,\right]^T\ \Sigma_2^{-1}\ \left[\, A_{21}x_1 + A_{22}x_2 + B_2u\,\right])\right] \\ &+ \left[\mathcal{T}_8 + 2\mathbb{E}\left(\left[\, N_{i,22}A_{22}^{-1}B_2u - \bar{N}_{i,21}x_r\,\right]^T\ \Sigma_2^{-1}\ \left[\, N_{j,21}x_1 + N_{j,22}x_2\,\right]\ q^{ij}\right)\right. \\ &\left. - \mathbb{E}\left(\left[\, N_{i,22}A_{22}^{-1}B_2u - \bar{N}_{i,21}x_r\,\right]^T\ \Sigma_2^{-1}\ \left[\, N_{j,22}A_{22}^{-1}B_2u - \bar{N}_{j,21}x_r\,\right]\ q^{ij}\right)\right]. \end{aligned}$$

This all then provides

$$\begin{aligned} 4\mathbb{E}\left\|u(t)\right\|_2^2 \geq& \frac{d}{dt}\mathbb{E}\left((x_1(t) + x_r(t))^T\Sigma_1^{-1}(x_1(t) + x_r(t))\right) + \frac{d}{dt}\mathbb{E}\left(x_2(t)^T\Sigma_2^{-1}x_2(t)\right) \\ &- 2\mathbb{E}\left(\left[\, A_{22}^{-1}(A_{21}x_r + B_2u)\,\right]^T\ \Sigma_2^{-1}\ \left[\, A_{21}x_1 + A_{22}x_2 + B_2u\,\right]\right) \\ &- 2\mathbb{E}\left(\left[\, N_{i,22}A_{22}^{-1}B_2u - \bar{N}_{i,21}x_r\,\right]^T\ \Sigma_2^{-1}\ \left[\, N_{j,21}x_1 + N_{j,22}x_2\,\right]\ q^{ij}\right) \\ &+ \mathbb{E}\left(\left[\, N_{i,22}A_{22}^{-1}B_2u - \bar{N}_{i,21}x_r\,\right]^T\ \Sigma_2^{-1}\ \left[\, N_{j,22}A_{22}^{-1}B_2u - \bar{N}_{j,21}x_r\,\right]\ q^{ij}\right). \end{aligned}$$

Since the last summand is non-negative because of Proposition 5.3, we find that

$$\begin{aligned} 4\mathbb{E}\int_0^T \left\|u(t)\right\|_2^2 dt \geq& -2\left[\mathbb{E}\int_0^T \left[\, A_{22}^{-1}(A_{21}x_r + B_2u)\,\right]^T\ \Sigma_2^{-1}\ \left[\, A_{21}x_1 + A_{22}x_2 + B_2u\,\right] dt\right. \\ &\left. + \mathbb{E}\int_0^T \left[\, N_{i,22}A_{22}^{-1}B_2u - \bar{N}_{i,21}x_r\,\right]^T\ \Sigma_2^{-1}\ \left[\, N_{j,21}x_1 + N_{j,22}x_2\,\right]\ q^{ij}\ dt\right]. \end{aligned}$$

Combining this inequality with (51) leads to the claim. □

Error bound for neighbouring reduced order models In this paragraph, an error bound between two particular ROM is analysed since this is the second ingredient of proving an $\mathcal{H}_\infty$-error bound for type II SPA in Theorem 5.6 later.

We now use a even finer partition than above, where all matrices are of suitable size:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \ B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}, \ N_i = \begin{bmatrix} N_{i,11} & N_{i,12} & N_{i,13} \\ N_{i,21} & N_{i,22} & N_{i,23} \\ N_{i,31} & N_{i,32} & N_{i,33} \end{bmatrix}, \ C = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}. \tag{53}$$

The diagonal and equal Gramians are then of the form

$$\Sigma = \begin{bmatrix} \Sigma_1 & & \\ & \Sigma_2 & \\ & & \Sigma_3 \end{bmatrix}. \tag{54}$$

We want to compare two ROMs of different dimensions. The ROM of removing Σ_3 only is given by

$$\begin{aligned} d\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \left[\bar{A}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \bar{B}u(t)\right]dt + \sum_{i=1}^{q}\left[\bar{N}_i\begin{bmatrix} x_1(t-) \\ x_2(t-) \end{bmatrix} + \bar{E}_i u(t-)\right]dM_i(t), \\ \bar{y}(t) &= \bar{C}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \bar{D}u(t), \quad t \geq 0, \end{aligned}$$

where $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and the matrices are defined in the sense of (28):

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} A_{13} \\ A_{23} \end{bmatrix} A_{33}^{-1} \begin{bmatrix} A_{31} & A_{32} \end{bmatrix}, \ \bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - \begin{bmatrix} A_{13} \\ A_{23} \end{bmatrix} A_{33}^{-1} B_3, \\ \bar{N}_i &= \begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{bmatrix} - \begin{bmatrix} N_{i,13} \\ N_{i,23} \end{bmatrix} A_{33}^{-1} \begin{bmatrix} A_{31} & A_{32} \end{bmatrix}, \ \bar{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} - C_3 A_{33}^{-1} \begin{bmatrix} A_{31} & A_{32} \end{bmatrix}, \\ \bar{D} &= -C_3 A_{33}^{-1} B_3, \ \bar{E}_i = -\begin{bmatrix} N_{i,13} \\ N_{i,23} \end{bmatrix} A_{33}^{-1} B_3. \end{aligned}$$

The above ROM is going to be compared with a smaller ROM, where Σ_2 and Σ_3 are removed. The corresponding state x_r has the same dimension as x_1 in the larger ROM above. We consider

$$\begin{aligned} dx_r(t) &= [\bar{A}_r x_r(t) + \bar{B}_r u(t)]dt + \sum_{i=1}^{q}\left[\bar{N}_{r,i} x_r(t-) + \bar{E}_{r,i} u(t-)\right]dM_i(t), \quad x_r(0) = 0, \\ \bar{y}_r(t) &= \bar{C}_r x_r(t) + \bar{D}_r u(t), \quad t \geq 0. \end{aligned}$$

For the definition of the above matrices, we set

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} := \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{-1}.$$

It can be shown that

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & -A_{22}^{-1} A_{23} \tilde{A}_{22} \\ -A_{33}^{-1} A_{32} \tilde{A}_{11} & \tilde{A}_{22} \end{bmatrix}, \tag{55}$$

where $\tilde{A}_{11} = (A_{22} - A_{23}A_{33}^{-1}A_{32})^{-1}$ and $\tilde{A}_{22} = (A_{33} - A_{32}A_{22}^{-1}A_{23})^{-1}$. Now, the matrices are of the form described in (28):

$$\begin{aligned}
\bar{A}_r &= A_{11} - \left(A_{12}\left[\begin{smallmatrix}\tilde{A}_{11} & \tilde{A}_{12}\end{smallmatrix}\right]\left[\begin{smallmatrix}A_{21}\\ A_{31}\end{smallmatrix}\right] + A_{13}\left[\begin{smallmatrix}\tilde{A}_{21} & \tilde{A}_{22}\end{smallmatrix}\right]\left[\begin{smallmatrix}A_{21}\\ A_{31}\end{smallmatrix}\right]\right),\\
\bar{B}_r &= B_1 - \left(A_{12}\left[\begin{smallmatrix}\tilde{A}_{11} & \tilde{A}_{12}\end{smallmatrix}\right]\left[\begin{smallmatrix}B_2\\ B_3\end{smallmatrix}\right] + A_{13}\left[\begin{smallmatrix}\tilde{A}_{21} & \tilde{A}_{22}\end{smallmatrix}\right]\left[\begin{smallmatrix}B_2\\ B_3\end{smallmatrix}\right]\right)\\
\bar{N}_{r,i} &= N_{i,11} - \left(N_{i,12}\left[\begin{smallmatrix}\tilde{A}_{11} & \tilde{A}_{12}\end{smallmatrix}\right]\left[\begin{smallmatrix}A_{21}\\ A_{31}\end{smallmatrix}\right] + N_{i,13}\left[\begin{smallmatrix}\tilde{A}_{21} & \tilde{A}_{22}\end{smallmatrix}\right]\left[\begin{smallmatrix}A_{21}\\ A_{31}\end{smallmatrix}\right]\right),\\
\bar{C}_r &= C_1 - \left(C_2\left[\begin{smallmatrix}\tilde{A}_{11} & \tilde{A}_{12}\end{smallmatrix}\right]\left[\begin{smallmatrix}A_{21}\\ A_{31}\end{smallmatrix}\right] + C_3\left[\begin{smallmatrix}\tilde{A}_{21} & \tilde{A}_{22}\end{smallmatrix}\right]\left[\begin{smallmatrix}A_{21}\\ A_{31}\end{smallmatrix}\right]\right),\\
\bar{D}_r &= -\left(C_2\left[\begin{smallmatrix}\tilde{A}_{11} & \tilde{A}_{12}\end{smallmatrix}\right]\left[\begin{smallmatrix}B_2\\ B_3\end{smallmatrix}\right] + C_3\left[\begin{smallmatrix}\tilde{A}_{21} & \tilde{A}_{22}\end{smallmatrix}\right]\left[\begin{smallmatrix}B_2\\ B_3\end{smallmatrix}\right]\right),\\
\bar{E}_{r,i} &= -\left(N_{i,12}\left[\begin{smallmatrix}\tilde{A}_{11} & \tilde{A}_{12}\end{smallmatrix}\right]\left[\begin{smallmatrix}B_2\\ B_3\end{smallmatrix}\right] + N_{i,13}\left[\begin{smallmatrix}\tilde{A}_{21} & \tilde{A}_{22}\end{smallmatrix}\right]\left[\begin{smallmatrix}B_2\\ B_3\end{smallmatrix}\right]\right).
\end{aligned}$$

Below, we investigate the error between $\bar{y}$ and $\bar{y}_r$ when the corresponding ROMs are neighbouring, i.e., they are chosen such that $\Sigma_2 = \sigma I$ in (54).

Theorem 5.5. *If $\Sigma_2 = \sigma I$ in (54), then*

$$\left\|\bar{y} - \bar{y}_r\right\|_{L^2_T} \leq 2\sigma \left\|u\right\|_{L^2_T}.$$

Proof. We mostly omit the time dependence of the functions below to keep the notation as easy as possible. We also make use of Einstein's summation convention which we indicate by replacing q_{ij} by the notation q^{ij} with upper indices. We insert for y and y_r and obtain

$$\begin{aligned}
&-\mathbb{E}\left\|\bar{y} - y_r\right\|_2^2\\
&= -\mathbb{E}\Big\| C_1x_1 + C_2x_2 - C_3A_{33}^{-1}(A_{31}x_1 + A_{32}x_2 + B_3u)\\
&\quad -C_1x_r + C_2\left[\begin{smallmatrix}\tilde{A}_{11} & \tilde{A}_{12}\end{smallmatrix}\right]\left(\left[\begin{smallmatrix}A_{21}\\ A_{31}\end{smallmatrix}\right]x_r + \left[\begin{smallmatrix}B_2\\ B_3\end{smallmatrix}\right]u\right) + C_3\left[\begin{smallmatrix}\tilde{A}_{21} & \tilde{A}_{22}\end{smallmatrix}\right]\left(\left[\begin{smallmatrix}A_{21}\\ A_{31}\end{smallmatrix}\right]x_r + \left[\begin{smallmatrix}B_2\\ B_3\end{smallmatrix}\right]u\right)\Big\|_2^2\\
&= -\mathbb{E}\left(\begin{bmatrix}x_1-x_r\\ x_2+k_1\\ k_2+k_3\end{bmatrix}^T C^TC\begin{bmatrix}x_1-x_r\\ x_2+k_1\\ k_2+k_3\end{bmatrix}\right),
\end{aligned}$$

where $k_1 = \left[\begin{smallmatrix}\tilde{A}_{11} & \tilde{A}_{12}\end{smallmatrix}\right]\left(\left[\begin{smallmatrix}A_{21}\\ A_{31}\end{smallmatrix}\right]x_r + \left[\begin{smallmatrix}B_2\\ B_3\end{smallmatrix}\right]u\right)$, $k_2 = -A_{33}^{-1}(A_{31}x_1 + A_{32}x_2 + B_3u)$ and $k_3 = \left[\begin{smallmatrix}\tilde{A}_{21} & \tilde{A}_{22}\end{smallmatrix}\right]\left(\left[\begin{smallmatrix}A_{21}\\ A_{31}\end{smallmatrix}\right]x_r + \left[\begin{smallmatrix}B_2\\ B_3\end{smallmatrix}\right]u\right)$. Since $-C^TC$ is bounded from below as follows

$$\begin{aligned}
&\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}^T\begin{bmatrix}\Sigma_1 & & \\ & \Sigma_2 & \\ & & \Sigma_3\end{bmatrix} + \begin{bmatrix}\Sigma_1 & & \\ & \Sigma_2 & \\ & & \Sigma_3\end{bmatrix}\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}\\
&+\begin{bmatrix}N_{i,11} & N_{i,12} & N_{i,13}\\ N_{i,21} & N_{i,22} & N_{i,23}\\ N_{i,31} & N_{i,32} & N_{i,33}\end{bmatrix}^T\begin{bmatrix}\Sigma_1 & & \\ & \Sigma_2 & \\ & & \Sigma_3\end{bmatrix}\begin{bmatrix}N_{j,11} & N_{j,12} & N_{j,13}\\ N_{j,21} & N_{j,22} & N_{j,23}\\ N_{j,31} & N_{j,32} & N_{j,33}\end{bmatrix}q^{ij} \leq -C^TC,
\end{aligned} \tag{56}$$

we consequently have

$$\begin{aligned}
&-\mathbb{E}\left\|\bar{y} - \bar{y}_r\right\|_2^2 \geq\\
&\mathbb{E}\Bigg(2\left[\begin{smallmatrix}x_1-x_r\end{smallmatrix}\right]^T\Sigma_1\left[\begin{smallmatrix}A_{11} & A_{12} & A_{13}\end{smallmatrix}\right]\begin{bmatrix}x_1-x_r\\ x_2+k_1\\ k_2+k_3\end{bmatrix}\\
&+\left(\left[\begin{smallmatrix}N_{i,11} & N_{i,12} & N_{i,13}\end{smallmatrix}\right]\begin{bmatrix}x_1-x_r\\ x_2+k_1\\ k_2+k_3\end{bmatrix}\right)^T\Sigma_1\left[\begin{smallmatrix}N_{j,11} & N_{j,12} & N_{j,13}\end{smallmatrix}\right]\begin{bmatrix}x_1-x_r\\ x_2+k_1\\ k_2+k_3\end{bmatrix}q^{ij}
\end{aligned}$$

$$
\begin{aligned}
&+ 2\left[\begin{smallmatrix} x_2+k_1 \end{smallmatrix}\right]^T \Sigma_2 \left[\begin{smallmatrix} A_{21} & A_{22} & A_{23} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{smallmatrix}\right] \\
&+ \left(\left[\begin{smallmatrix} N_{i,21} & N_{i,22} & N_{i,23} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{smallmatrix}\right]\right)^T \Sigma_2 \left[\begin{smallmatrix} N_{j,21} & N_{j,22} & N_{j,23} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{smallmatrix}\right] q^{ij} \\
&+ 2\left[\begin{smallmatrix} k_2+k_3 \end{smallmatrix}\right]^T \Sigma_3 \left[\begin{smallmatrix} A_{31} & A_{32} & A_{33} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{smallmatrix}\right] \\
&+ \left(\left[\begin{smallmatrix} N_{i,31} & N_{i,32} & N_{i,33} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{smallmatrix}\right]\right)^T \Sigma_3 \left[\begin{smallmatrix} N_{j,31} & N_{j,32} & N_{j,33} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{smallmatrix}\right] q^{ij} \Bigg).
\end{aligned}
$$

In the following, the terms depending on Σ_3 can be neglected. For the last summand this is because it is non-negative by Proposition 5.3 and the penultimate term vanishes since

$$
\left[\begin{smallmatrix} A_{31} & A_{32} & A_{33} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{smallmatrix}\right] = A_{31}(x_1 - x_r) + A_{32}x_2 + A_{33}k_2 + A_{32}k_1 + A_{33}k_3.
$$

Inserting for k_2 yields

$$
\left[\begin{smallmatrix} A_{31} & A_{32} & A_{33} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{smallmatrix}\right] = -(A_{31}x_r + B_3u) + A_{32}k_1 + A_{33}k_3.
$$

We plug in k_1 and k_3 and use (55):

$$
\begin{aligned}
A_{32}k_1 + A_{33}k_3 &= \left(A_{32}\left[\begin{smallmatrix} \tilde{A}_{11} & \tilde{A}_{12} \end{smallmatrix}\right] + A_{33}\left[\begin{smallmatrix} \tilde{A}_{21} & \tilde{A}_{22} \end{smallmatrix}\right]\right)\left(\left[\begin{smallmatrix} A_{21} \\ A_{31} \end{smallmatrix}\right] x_r + \left[\begin{smallmatrix} B_2 \\ B_3 \end{smallmatrix}\right] u\right) \qquad (57) \\
&= \left(\left[\begin{smallmatrix} A_{32}\tilde{A}_{11} & -A_{32}A_{22}^{-1}A_{23}\tilde{A}_{22} \end{smallmatrix}\right] + \left[\begin{smallmatrix} -A_{32}\tilde{A}_{11} & A_{33}\tilde{A}_{22} \end{smallmatrix}\right]\right)\left(\left[\begin{smallmatrix} A_{21} \\ A_{31} \end{smallmatrix}\right] x_r + \left[\begin{smallmatrix} B_2 \\ B_3 \end{smallmatrix}\right] u\right) \\
&= \left[\begin{smallmatrix} 0 & I \end{smallmatrix}\right]\left(\left[\begin{smallmatrix} A_{21} \\ A_{31} \end{smallmatrix}\right] x_r + \left[\begin{smallmatrix} B_2 \\ B_3 \end{smallmatrix}\right] u\right) = A_{31}x_r + B_3u.
\end{aligned}
$$

Hence, we have $\left[\begin{smallmatrix} A_{31} & A_{32} & A_{33} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{smallmatrix}\right] = 0$. Furthermore, we know that

$$
\begin{aligned}
dx_1 &= \left[\left(\left[\begin{smallmatrix} A_{11} & A_{12} \end{smallmatrix}\right] - A_{13}A_{33}^{-1}\left[\begin{smallmatrix} A_{31} & A_{32} \end{smallmatrix}\right]\right)\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] + \left(B_1 - A_{13}A_{33}^{-1}B_3\right)u\right] dt \\
&\quad + \sum_{i=1}^q \left[\left(\left[\begin{smallmatrix} N_{i,11} & N_{i,12} \end{smallmatrix}\right] - N_{i,13}A_{33}^{-1}\left[\begin{smallmatrix} A_{31} & A_{32} \end{smallmatrix}\right]\right)\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] - N_{i,13}A_{33}^{-1}B_3u\right] dM_i(t) \\
&= \left[\left[\begin{smallmatrix} A_{11} & A_{12} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] + A_{13}k_2 + B_1u\right] dt + \sum_{i=1}^q \left[\left[\begin{smallmatrix} N_{i,11} & N_{i,12} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] + N_{i,13}k_2\right] dM_i(t).
\end{aligned} \tag{58}
$$

By the definition of k_1 and k_3, we have

$$
dx_r = [A_{11}x_r - A_{12}k_1 - A_{13}k_3 + B_1u]\, dt + \sum_{i=1}^q [N_{i,11}x_r - N_{i,12}k_1 - N_{i,13}k_3]\, dM_i(t) \tag{59}
$$

and hence by Lemma 5.2, we obtain

$$
\begin{aligned}
\frac{d}{dt}\mathbb{E}\left((x_1 - x_r))^T\Sigma_1(x_1 - x_r)\right) &= \mathbb{E}\Bigg(2\left[\begin{smallmatrix} x_1-x_r \end{smallmatrix}\right]^T \Sigma_1 \left[\begin{smallmatrix} A_{11} & A_{12} & A_{13} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{smallmatrix}\right] \\
&\quad + \left(\left[\begin{smallmatrix} N_{i,11} & N_{i,12} & N_{i,13} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{smallmatrix}\right]\right)^T \Sigma_1 \left[\begin{smallmatrix} N_{j,11} & N_{j,12} & N_{j,13} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{smallmatrix}\right] q^{ij} \Bigg),
\end{aligned}
$$

such that overall, it holds

$$
\begin{aligned}
-\mathbb{E}\left\|\bar{y}-\bar{y}_r\right\|_2^2 \geq & \frac{d}{dt}\mathbb{E}\left((x_1-x_r))^T\Sigma_1(x_1-x_r)\right) \\
& \mathbb{E}\left(2\begin{bmatrix} x_2+k_1 \end{bmatrix}^T \Sigma_2 \begin{bmatrix} A_{21} & A_{22} & A_{23} \end{bmatrix}\begin{bmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{bmatrix}\right. \\
& \left.+\left(\begin{bmatrix} N_{i,21} & N_{i,22} & N_{i,23} \end{bmatrix}\begin{bmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{bmatrix}\right)^T \Sigma_2 \begin{bmatrix} N_{j,21} & N_{j,22} & N_{j,23} \end{bmatrix}\begin{bmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{bmatrix} q^{ij}\right).
\end{aligned}
$$

So, it remains to analyse the terms depending on Σ_2. First of all, it holds that

$$
\begin{aligned}
dx_2 = & \left[\left(\begin{bmatrix} A_{21} & A_{22} \end{bmatrix} - A_{23}A_{33}^{-1}\begin{bmatrix} A_{31} & A_{32} \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left(B_2 - A_{23}A_{33}^{-1}B_3\right)u\right]dt \\
& + \sum_{i=1}^{q}\left[\left(\begin{bmatrix} N_{i,21} & N_{i,22} \end{bmatrix} - N_{i,23}A_{33}^{-1}\begin{bmatrix} A_{31} & A_{32} \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - N_{i,23}A_{33}^{-1}B_3u\right]dM_i(t) \\
& = \left[\begin{bmatrix} A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + A_{23}k_2 + B_2u\right]dt + \sum_{i=1}^{q}\left[\begin{bmatrix} N_{i,21} & N_{i,22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + N_{i,23}k_2\right]dM_i(t),
\end{aligned}
$$

so that by Lemma 5.2, we have

$$
\begin{aligned}
\frac{d}{dt}\mathbb{E}\left(x_2^T\Sigma_2x_2\right) = & \mathbb{E}\left(2\, x_2{}^T \Sigma_2 \left(\begin{bmatrix} A_{21} & A_{22} & A_{23} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ k_2 \end{bmatrix} + B_2u\right)\right. \\
& \left.+\left(\begin{bmatrix} N_{i,21} & N_{i,22} & N_{i,23} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ k_2 \end{bmatrix}\right)^T \Sigma_2 \begin{bmatrix} N_{j,21} & N_{j,22} & N_{j,23} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ k_2 \end{bmatrix} q^{ij}\right).
\end{aligned} \tag{60}
$$

Taking into account that

$$
\begin{aligned}
\begin{bmatrix} A_{21} & A_{22} & A_{23} \end{bmatrix}\begin{bmatrix} -x_r \\ k_1 \\ k_3 \end{bmatrix} & = \left(A_{22}\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \end{bmatrix} + A_{23}\begin{bmatrix} \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}\right)\left(\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}x_r + \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}u\right) - A_{21}x_r \\
& = \left(\begin{bmatrix} A_{22}\tilde{A}_{11} & -A_{23}\tilde{A}_{22} \end{bmatrix} + \begin{bmatrix} -A_{23}A_{33}^{-1}A_{32}\tilde{A}_{11} & A_{23}\tilde{A}_{22} \end{bmatrix}\right)\left(\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}x_r + \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}u\right) \\
& \quad - A_{21}x_r \\
& = \begin{bmatrix} I & 0 \end{bmatrix}\left(\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}x_r + \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}u\right) - A_{21}x_r = B_2u,
\end{aligned} \tag{61}
$$

we see that

$$
\begin{aligned}
-\mathbb{E}\left\|\bar{y}-\bar{y}_r\right\|_2^2 \geq & \frac{d}{dt}\mathbb{E}\left((x_1-x_r))^T\Sigma_1(x_1-x_r)\right) + \frac{d}{dt}\mathbb{E}\left(x_2^T\Sigma_2x_2\right) \\
& \mathbb{E}\left(2\, k_1{}^T \Sigma_2 \begin{bmatrix} A_{21} & A_{22} & A_{23} \end{bmatrix}\begin{bmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{bmatrix}\right. \\
& + 2\left(\begin{bmatrix} N_{i,21} & N_{i,22} & N_{i,23} \end{bmatrix}\begin{bmatrix} -x_r \\ k_1 \\ k_3 \end{bmatrix}\right)^T \Sigma_2 \begin{bmatrix} N_{j,21} & N_{j,22} & N_{j,23} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ k_2 \end{bmatrix} q^{ij} \\
& \left.+\left(\begin{bmatrix} N_{i,21} & N_{i,22} & N_{i,23} \end{bmatrix}\begin{bmatrix} -x_r \\ k_1 \\ k_3 \end{bmatrix}\right)^T \Sigma_2 \begin{bmatrix} N_{j,21} & N_{j,22} & N_{j,23} \end{bmatrix}\begin{bmatrix} -x_r \\ k_1 \\ k_3 \end{bmatrix} q^{ij}\right)
\end{aligned}
$$

and thus with Proposition 5.3 and $\Sigma_2 = \sigma I$, we obtain

$$
\begin{aligned}
\mathbb{E}\int_0^T \|y(t)-y_r(t)\|_2^2\,dt \leq -2\sigma^2 & \left[\mathbb{E}\int_0^T k_1{}^T \Sigma_2^{-1} \begin{bmatrix} A_{21} & A_{22} & A_{23} \end{bmatrix}\begin{bmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3 \end{bmatrix} dt\right. \\
& \left.+\mathbb{E}\int_0^T \left(\begin{bmatrix} N_{i,21} & N_{i,22} & N_{i,23} \end{bmatrix}\begin{bmatrix} -x_r \\ k_1 \\ k_3 \end{bmatrix}\right)^T \Sigma_2^{-1} \begin{bmatrix} N_{j,21} & N_{j,22} & N_{j,23} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ k_2 \end{bmatrix} q^{ij}\,dt\right].
\end{aligned} \tag{62}
$$

Below, we make use of inequality (52) and insert the partitions in (53) and (54). Multiplying $\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \\ 2u \end{smallmatrix}\right]^T$ from the left and $\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \\ 2u \end{smallmatrix}\right]$ from the right to matrix inequality (52) and taking the expectation leads to

$$\begin{aligned}
4\mathbb{E}\left\|u\right\|_2^2 \geq & \mathbb{E}\Bigg(2\left[\begin{smallmatrix} x_1+x_r \end{smallmatrix}\right]^T \Sigma_1^{-1} \left(\left[\begin{smallmatrix} A_{11} & A_{12} & A_{13} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \end{smallmatrix}\right] + 2B_1 u\right) \\
& + \left(\left[\begin{smallmatrix} N_{i,11} & N_{i,12} & N_{i,13} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \end{smallmatrix}\right]\right)^T \Sigma_1^{-1} \left[\begin{smallmatrix} N_{j,11} & N_{j,12} & N_{j,13} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \end{smallmatrix}\right] q^{ij} \\
& + 2\left[\begin{smallmatrix} x_2-k_1 \end{smallmatrix}\right]^T \Sigma_2^{-1} \left(\left[\begin{smallmatrix} A_{21} & A_{22} & A_{23} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \end{smallmatrix}\right] + 2B_2 u\right) \\
& + \left(\left[\begin{smallmatrix} N_{i,21} & N_{i,22} & N_{i,23} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \end{smallmatrix}\right]\right)^T \Sigma_2^{-1} \left[\begin{smallmatrix} N_{j,21} & N_{j,22} & N_{j,23} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \end{smallmatrix}\right] q^{ij} \\
& + 2\left[\begin{smallmatrix} k_2-k_3 \end{smallmatrix}\right]^T \Sigma_3^{-1} \left(\left[\begin{smallmatrix} A_{31} & A_{32} & A_{33} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \end{smallmatrix}\right] + 2B_3 u\right) \\
& + \left(\left[\begin{smallmatrix} N_{i,31} & N_{i,32} & N_{i,33} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \end{smallmatrix}\right]\right)^T \Sigma_3^{-1} \left[\begin{smallmatrix} N_{j,31} & N_{j,32} & N_{j,33} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \end{smallmatrix}\right] q^{ij}\Bigg).
\end{aligned}$$

Again, we can neglect the terms depending on Σ_3, since the last summand is non-negative by Proposition 5.3 and the penultimate term vanishes, because

$$\left[\begin{smallmatrix} A_{31} & A_{32} & A_{33} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \end{smallmatrix}\right] = A_{31}(x_1+x_r) + A_{32}x_2 + A_{33}k_2 - A_{32}k_1 - A_{33}k_3.$$

Inserting for k_2 yields

$$\left[\begin{smallmatrix} A_{31} & A_{32} & A_{33} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \end{smallmatrix}\right] = A_{31}x_r - B_3 u - (A_{32}k_1 + A_{33}k_3).$$

Using (57), we have $\left[\begin{smallmatrix} A_{31} & A_{32} & A_{33} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \end{smallmatrix}\right] + 2B_3 u = 0$. Combining (58) and (59) and applying Lemma 5.2 yields

$$\begin{aligned}
& \frac{d}{dt}\mathbb{E}\left((x_1+x_r))^T\Sigma_1^{-1}(x_1+x_r)\right) \\
& = \mathbb{E}\Bigg(2\left[\begin{smallmatrix} x_1+x_r \end{smallmatrix}\right]^T \Sigma_1^{-1}\left(\left[\begin{smallmatrix} A_{11} & A_{12} & A_{13} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \end{smallmatrix}\right] + 2B_1 u\right) \\
& \quad + \left(\left[\begin{smallmatrix} N_{i,11} & N_{i,12} & N_{i,13} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \end{smallmatrix}\right]\right)^T \Sigma_1^{-1} \left[\begin{smallmatrix} N_{j,11} & N_{j,12} & N_{j,13} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3 \end{smallmatrix}\right] q^{ij}\Bigg).
\end{aligned}$$

Analogous to (60), we get

$$\begin{aligned}
& \frac{d}{dt}\mathbb{E}\left(x_2^T\Sigma_2^{-1}x_2\right) \\
& = \mathbb{E}\Bigg(2\, x_2{}^T \Sigma_2^{-1}\left(\left[\begin{smallmatrix} A_{21} & A_{22} & A_{23} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1 \\ x_2 \\ k_2 \end{smallmatrix}\right] + B_2 u\right) \\
& \quad + \left(\left[\begin{smallmatrix} N_{i,21} & N_{i,22} & N_{i,23} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1 \\ x_2 \\ k_2 \end{smallmatrix}\right]\right)^T \Sigma_2^{-1} \left[\begin{smallmatrix} N_{j,21} & N_{j,22} & N_{j,23} \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1 \\ x_2 \\ k_2 \end{smallmatrix}\right] q^{ij}\Bigg)
\end{aligned}$$

$$= \mathbb{E}\left(2\, x_2{}^T\, \Sigma_2^{-1}\left(\begin{bmatrix} A_{21} & A_{22} & A_{23}\end{bmatrix}\begin{bmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3\end{bmatrix} + 2B_2u\right)\right.$$
$$\left. + \left(\begin{bmatrix} N_{i,21} & N_{i,22} & N_{i,23}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ k_2\end{bmatrix}\right)^T \Sigma_2^{-1}\begin{bmatrix} N_{j,21} & N_{j,22} & N_{j,23}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ k_2\end{bmatrix} q^{ij}\right).$$

The last equality above is obtained by (61). Hence,

$$\begin{aligned}4\mathbb{E}\left\|u\right\|_2^2 \geq & \frac{d}{dt}\mathbb{E}\left((x_1+x_r))^T\Sigma_1^{-1}(x_1+x_r)\right) + \frac{d}{dt}\mathbb{E}\left(x_2^T\Sigma_2^{-1}x_2\right) \\ & - 2\mathbb{E}\left(k_1{}^T\,\Sigma_2^{-1}\left(\begin{bmatrix} A_{21} & A_{22} & A_{23}\end{bmatrix}\begin{bmatrix} x_1+x_r \\ x_2-k_1 \\ k_2-k_3\end{bmatrix} + 2B_2u\right)\right) \\ & - 2\mathbb{E}\left(\left(\begin{bmatrix} N_{i,21} & N_{i,22} & N_{i,23}\end{bmatrix}\begin{bmatrix} -x_r \\ k_1 \\ k_3\end{bmatrix}\right)^T \Sigma_2^{-1}\begin{bmatrix} N_{j,21} & N_{j,22} & N_{j,23}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ k_2\end{bmatrix} q^{ij}\right) \\ & + \mathbb{E}\left(\left(\begin{bmatrix} N_{i,21} & N_{i,22} & N_{i,23}\end{bmatrix}\begin{bmatrix} x_r \\ -k_1 \\ -k_3\end{bmatrix}\right)^T \Sigma_2^{-1}\begin{bmatrix} N_{j,21} & N_{j,22} & N_{j,23}\end{bmatrix}\begin{bmatrix} x_r \\ -k_1 \\ -k_3\end{bmatrix} q^{ij}\right).\end{aligned}$$

Due to Proposition 5.3 the last term can be omitted. Moreover, we insert (61) for B_2u and then obtain

$$\begin{aligned}4\int_0^T \mathbb{E}\left\|u(t)\right\|_2^2 dt \geq & -2\mathbb{E}\int_0^T \left[k_1{}^T\,\Sigma_2^{-1}\begin{bmatrix} A_{21} & A_{22} & A_{23}\end{bmatrix}\begin{bmatrix} x_1-x_r \\ x_2+k_1 \\ k_2+k_3\end{bmatrix}\right. \\ & \left.\left(\begin{bmatrix} N_{i,21} & N_{i,22} & N_{i,23}\end{bmatrix}\begin{bmatrix} -x_r \\ k_1 \\ k_3\end{bmatrix}\right)^T \Sigma_2^{-1}\begin{bmatrix} N_{j,21} & N_{j,22} & N_{j,23}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ k_2\end{bmatrix} q^{ij}\right] dt.\end{aligned}$$

We plug in this result to (62) and obtain the result of this theorem. □

Main results

Theorem 5.6. *If $x(0) = 0$ and $x_r(0) = 0$, then for all $T > 0$, we have*

$$\left\|y - y_r\right\|_{L_T^2} \leq 2(\tilde{\sigma}_1 + \tilde{\sigma}_2 + \ldots + \tilde{\sigma}_\nu)\left\|u\right\|_{L_T^2},$$

where y is the output of the original system (47), y_r is the output of the type II SPA approach ROM (coefficients as in (28)) and $\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_\nu$ are the distinct diagonal entries of $\Sigma_2 = \operatorname{diag}(\sigma_{r+1}, \ldots, \sigma_n) = \operatorname{diag}(\tilde{\sigma}_1 I, \tilde{\sigma}_2 I, \ldots, \tilde{\sigma}_\nu I)$.

Proof. This proof relies on Theorems 5.4 and 5.5. We will use the common idea of removing the Hankel singular values step by step. The error between the outputs y and y_r can be bounded as follows:

$$\left\|y - y_r\right\|_{L_T^2} \leq \left\|y - y_{r_\nu}\right\|_{L_T^2} + \left\|y_{r_\nu} - y_{r_{\nu-1}}\right\|_{L_T^2} + \ldots + \left\|y_{r_2} - y_r\right\|_{L_T^2},$$

where the dimensions r_i of the corresponding states are defined by $r_{i+1} = r_i + m(\tilde{\sigma}_i)$ for $i = 1, 2\ldots, \nu - 1$. Here, $m(\tilde{\sigma}_i)$ denotes the multiplicity of $\tilde{\sigma}_i$ and $r_1 = r$. In the reduction step from y to y_{r_ν} only the smallest Hankel singular value $\tilde{\sigma}_\nu$ is removed from the system. Hence, by Theorem 5.4, we have

$$\left\|y - y_{r_\nu}\right\|_{L^2_T} \leq 2\tilde{\sigma}_\nu \left\|u\right\|_{L^2_T}.$$

The reduced order outputs y_{r_j} and $y_{r_{j-1}}$ are neighbouring, i.e., only the Hankel singular value $\tilde{\sigma}_{r_{j-1}}$ is removed. Thus, by Theorem 5.5, we obtain

$$\left\|y_{r_j} - y_{r_{j-1}}\right\|_{L^2_T} \leq 2\tilde{\sigma}_{r_{j-1}} \left\|u\right\|_{L^2_T}$$

for $j = 2, \dots, \nu$. This provides the claimed result. □

Since the bound in Theorem 5.6 involves only the sum of distinct diagonal entries of Σ_2, the result is of course also true when using the sum of all diagonal entries instead.

Corollary 5.7. *If $x(0) = 0$ and $x_r(0) = 0$, then for all $T > 0$, we have*

$$\left\|y - y_r\right\|_{L^2_T} \leq 2(\sigma_{r+1} + \sigma_{r+2} + \ldots + \sigma_n) \left\|u\right\|_{L^2_T},$$

where y is the output of the original system (47), y_r is the output of the type II SPA approach ROM (coefficients as in (28)) and $\sigma_{r+1}, \dots, \sigma_n$ are the diagonal entries of Σ_2.

Since the $\mathcal{H}_\infty$-type error of using type II SPA depends on the $n - r$ smallest HSVs of the original system, the same conclusion as from Theorem 5.1 can be made. So, when neglecting the difficult to reach and observe states only, the values $\sigma_{r+1}, \dots, \sigma_n$ are supposed to be small which leads to a good approximation by Corollary 5.7.

6 Conclusions

We have analysed the concept of mean square asymptotic stability for Lévy driven systems based on the stability analysis for the Wiener case. This concept was needed to introduce reachability and observability Gramians which are used to identify difficult to reach and difficult to observe states in a Lévy driven system. We provided new energy interpretations for two different reachability Gramians, P_1 and P_2, that allow a better characterisation of reachability of a state in a stochastic system. So far, only energy interpretations have been available that neglected the diffusion term of stochastic differential equations. Based on the reachability Gramian P_2 and the observability Gramian Q balancing of stochastic systems was explained in this paper (type II balancing). In the resulting balanced system, the unimportant states can be easily identified, because dominant reachable and observable states are the same. We explained in which sense the state components, that contribute only little to the system dynamics, are neglected. This particular approach is called type II singular perturbation approximation (SPA). It generalises the deterministic setting which was studied first. Furthermore, type II SPA provides an alternative to type I SPA. This type I ansatz is based on the reachability Gramian P_1 and was developed for stochastic systems as well but it has worse properties than the type II approach. This is what we pointed out throughout this paper. So, we showed the preservation of mean square asymptotic stability in the reduced order model by type II SPA which has not been shown for type I SPA yet. Moreover, we proved an $\mathcal{H}_2$-type and an $\mathcal{H}_\infty$-type error bound for type II SPA which allow to find the cases in which the approximation performs well. In particular, the $\mathcal{H}_\infty$-type error bound represents an extension of the error bound in the deterministic case. To prove this generalised bound completely different techniques were needed since some tools are not available anymore in the stochastic setting. An $\mathcal{H}_2$-bound for the type I ansatz could already be shown too but an $\mathcal{H}_\infty$-bound does not exist. The existence of an $\mathcal{H}_\infty$-error bound is the main advantage of the approach considered here.

A Ito calculus

Let all stochastic processes appearing in this section be defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$[1]. We denote the set of all càdlàg square integrable $\mathbb{R}$-valued martingales with respect to $(\mathcal{F}_t)_{t\geq 0}$ by $\mathcal{M}^2(\mathbb{R})$.

Let Z_1, Z_2 be scalar semimartingales. We set $\Delta Z_i(s) := Z_i(s) - Z_i(s-)$ with $Z_i(s-) := \lim_{t\uparrow s} Z_i(t)$ for $i = 1, 2$. Then the Ito product formula

$$Z_1(t)Z_2(t) = Z_1(0)Z_2(0) + \int_0^t Z_1(s-)dZ_2(s) + \int_0^t Z_2(s-)dZ_1(s) + [Z_1, Z_2]_t \tag{63}$$

for $t \geq 0$ holds, see [24] or [2] for the special case of Lévy-type integrals. By [18, Theorem 4.52], the compensator process $[Z_1, Z_2]$ is given by

$$[Z_1, Z_2]_t = \langle M_1^c, M_2^c \rangle_t + \sum_{0\leq s\leq t} \Delta Z_1(s)\Delta Z_2(s) \tag{64}$$

for $t \geq 0$, where M_1^c, $M_2^c \in \mathcal{M}^2(\mathbb{R})$ are the continuous martingale parts of Z_1 and Z_2 (cf. [18, Theorem 4.18]). The process $\langle M_1^c, M_2^c \rangle$ is a uniquely defined angle bracket process that ensures that $M_1^c M_2^c - \langle M_1^c, M_2^c \rangle$ is an $(\mathcal{F}_t)_{t\geq 0}$- martingale, see [24, Proposition 17.2]. As a simple consequence of (63), we have:

Corollary A.1. *Let Y be an $\mathbb{R}^d$-valued and Z be an $\mathbb{R}^n$-valued semimartingale, then we have*

$$Y(t)Z^T(t) = Y(0)Z^T(0) + \int_0^t dY(s)Z^T(s-) + \int_0^t Y(s-)dZ^T(s) + \left([Y_i, Z_j]_t\right)_{\substack{i=1,\dots,d\\ j=1,\dots,n}}$$

for all $t \geq 0$.

[1] $(\mathcal{F}_t)_{t\geq 0}$ shall be right continuous and complete.

Proof. Considering the stochastic differential of the ijth component of the matrix-valued process $Y(t)Z^T(t)$, $t \geq 0$, and using (63) gives the result, see also [7]. $\square$

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