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GREGORY A. CHECHKIN, TARAS A. MEL'NYK

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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
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Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

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Enhanced Spatial Skin–Effect for Free Vibrations of a Thick Cascade Junction with "Super Heavy" Concentrated Masses

Gregory A. Chechkin[♮], Taras A. Mel'nyk[♯]

[♮] Department of Differential Equations,
Faculty of Mechanics and Mathematics,
M.V.Lomonosov Moscow State University,
Moscow 119991, Russia,
chechkin@mech.math.msu.su

[♯] Department of Mathematical Physics,
Faculty of Mechanics and Mathematics,
Taras Shevchenko National University of Kyiv ,
Volodymyrska st., 64/13
Kyiv 01601, Ukraine,
melnyk@imath.kiev.ua

Abstract

The asymptotic behavior (as $\varepsilon \to 0$) of eigenvalues and eigenfunctions of a boundary-value problem for the Laplace operator in a thick cascade junction with concentrated masses is studied. This cascade junction consists of the junction's body and a great number $5N = \mathcal{O}(\varepsilon^{-1})$ of ε-alternating thin rods belonging to two classes. One class consists of rods of finite length and the second one consists of rods of small length of order $\mathcal{O}(\varepsilon)$. The mass density is of order $\mathcal{O}(\varepsilon^{-\alpha})$ on the rods from the second class and $\mathcal{O}(1)$ outside of them. There exist five qualitatively different cases in the asymptotic behavior of eigen-magnitudes as $\varepsilon \to 0$, namely the case of "light" concentrated masses ($\alpha \in (0,1)$), "intermediate" concentrated masses ($\alpha = 1$) and "heavy" concentrated masses ($\alpha \in (1,+\infty)$) that we divide into "slightly heavy" concentrated masses ($\alpha \in (1,2)$), "moderate heavy" concentrated masses ($\alpha = 2$), and "super heavy" concentrated masses ($\alpha > 2$).

In the paper we study the influence of the concentrated masses on the asymptotic behavior of the eigen-magnitudes in the cases $\alpha = 2$ and $\alpha > 2$. The leading terms of asymptotic expansions both for the eigenvalues and eigenfunctions are constructed and the corresponding asymptotic estimates are proved. In addition, a new kind of high-frequency vibrations is found.

Contents

1 Introduction

In present paper we continue our investigation of a spectral problem with concentrated masses in a new kind of thick junctions, namely *thick cascade junctions*, which we have begun in [1, 2, 3, 4, 5]. Thick cascade junctions are prototypes of widely used nanotechnological, microtechnical, modern engineering constructions (microstrip radiator, ferrite-filled rod radiator), as well as many physical and biological systems (see examples in the papers mentioned above).

Vibrating systems with a concentration of masses on a small set of diameter $\mathcal{O}(\varepsilon)$ have been studied for a long time. It was experimentally established that such concentration leads to the big reduction of the main frequencies and to the large localization of vibrations near concentrated masses. The new impulse in this research was given by E. Sánchez-Palencia in the paper [6], in which the effect of local vibrations was mathematically described. After this paper, many articles appeared. The reader can find widely presented bibliography on spectral problems with concentrated masses and problems in thick junctions in [1, 2, 3, 4, 5].

In the papers [1, 2] we have studied the cases $\alpha \in (0,1)$ and $\alpha = 1$. The cases of "heavy" concentrated masses $\alpha \in (1, +\infty)$ we have begun to study in [3, 4, 5], where the case of "slightly heavy" ($\alpha \in (1,2)$) was considered and a new spatial skin-effect for eigenvibrations was found out. As far as we know, for the first time the skin-effect for systems with many concentrated masses near the boundary was discovered in [10].

In the present paper we continue to study the spatial skin-effect in the case of "moderate havy" ($\alpha = 2$) and "super heavy" ($\alpha > 2$) concentrated masses.

It is known that for spectral problems with concentrated masses there exist other converging sequences of eigenvalues $\lambda_{n(\varepsilon)}(\varepsilon)$ ($n(\varepsilon) \to +\infty$ as $\varepsilon \to 0$); the corresponding vibrations are usually called high frequency vibrations (see for instance [6, 7, 8, 9, 10, 11, 12]). Convergence of eigenvalues $\lambda_n(\varepsilon)$ as $\varepsilon \to 0$ at each fixed index n are called the *low-frequency convergence* of the spectrum.

Also in [2] we proved the low- and high-frequency convergences of the spectrum of problem (1.1) as $\varepsilon \to 0$, constructed and justified the leading terms of the asymptotics both for the eigenfunctions and eigenvalues in both cases $\alpha \in (0,1)$ and $\alpha = 1$. In addition, as in the paper [11], we found *pseudovibrations* in problem (1.1), having rapidly oscillating character, and in which different rods of the junction vibrate individually, i.e., each rod has its own frequency.

Here we will show that there is a new kind of high-frequency vibrations in problem (1.1), so called *high-frequency cell-vibrations*, which appear at each case of the concentrated masses mentioned above.

The paper is organized as follows. After the statement of the problem we describe and compare the main results. In Section 2 we construct the leading terms of the asymptotics both for eigenfunctions and eigenvalues in the case $\alpha \in (m, m+1)$, $m \in \mathbb{N}$, $m \geq 2$. And then in Section 3 we justify the constructed asymptotics and prove the corresponding asymptotic estimates. Similar investigation is done in Section 4 for the case $\alpha = m \in \mathbb{N}$, $m \geq 2$. In Section 5 we study high-frequency cell-vibrations of problem (1.1).

1.1 Statement of the problem

Let $a,\ b_1,\ b_2,\ h_1,\ h_2$ be positive numbers such that

$$0 < b_1 < b_2 < \frac{1}{2}, \quad 0 < b_1 - \frac{h_1}{2}, \quad b_1 + \frac{h_1}{2} < b_2 - \frac{h_1}{2}, \quad b_2 + \frac{h_1}{2} < \frac{1}{2} - \frac{h_2}{2}.$$

These inequalities mean that the intervals

$$\Big(b_1 - \frac{h_1}{2}\,, \, b_1 + \frac{h_1}{2}\Big), \quad \Big(b_2 - \frac{h_1}{2}\,, \, b_2 + \frac{h_1}{2}\Big), \quad \Big(\frac{1-h_2}{2}\,, \, \frac{1+h_2}{2}\Big),$$

$$\Big(1 - b_2 - \frac{h_1}{2}\,, \, 1 - b_2 + \frac{h_1}{2}\Big), \quad \Big(1 - b_1 - \frac{h_1}{2}\,, \, 1 - b_1 + \frac{h_1}{2}\Big)$$

are disjoint and they are subintervals of $(0,1)$.

Let Ω_0 be a bounded domain in $\mathbb{R}^2$ with the Lipschitz boundary $\partial\Omega_0$ and $\Omega_0 \subset \{x := (x_1, x_2) \in \mathbb{R}^2 : \ x_2 > 0\}$. Let $\partial\Omega_0$ contains the segment $I_0 = \{x : x_1 \in [0,a], \ \ x_2 = 0\}$. We also assume that there exists a positive number δ_0 such that $\Omega_0 \cap \{x : \ 0 < x_2 < \delta_0\} = \{x : \ x_1 \in (0,a), \ x_2 \in (0, \delta_0)\}$.

Let us divide the segment $[0,a]$ into N equal segments $[\varepsilon j, \varepsilon(j+1)], \ j = 0, \ldots, N-1$. Here N is a big positive integer, hence the value $\varepsilon = a/N$ is a small discrete parameter.

A model *thick cascade junction* Ω_ε (see Fig. 1) consists of the junction's body Ω_0 and a large number of thin rods

$$G_j^{(1)}(d_k, \varepsilon) = \Big\{x \in \mathbb{R}^2 : \ |x_1 - \varepsilon\,(j + d_k)| < \frac{\varepsilon h_1}{2}, \quad x_2 \in (-\varepsilon\, l_1, 0]\Big\}, \quad k = 1, \ldots, 4,$$

$$G_j^{(2)}(\varepsilon) = \Big\{x \in \mathbb{R}^2 : \ |x_1 - \varepsilon\,(j + \frac{1}{2})| < \frac{\varepsilon h_2}{2}, \quad x_2 \in (-l_2, 0]\Big\}, \quad j = 0, 1, \ldots, N-1,$$

where $d_1 = b_1, \ d_2 = b_2, \ d_3 = 1 - b_2, \ d_4 = 1 - b_1,$ that is $\Omega_\varepsilon = \Omega_0 \cup G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}$, where

$$G_\varepsilon^{(1)} = \bigcup_{j=0}^{N-1} \Big(\bigcup_{k=1}^{4} G_j^{(1)}(d_k, \varepsilon)\Big), \qquad G_\varepsilon^{(2)} = \bigcup_{j=0}^{N-1} G_j^{(2)}(\varepsilon).$$

Thus the number of the thin rods is equal to $5N$; the thin rods are divided into two classes $G_\varepsilon^{(1)}$ and $G_\varepsilon^{(2)}$ subject to their length and thickness. The length and thickness of the rods from the first class are equal to εl_1 and εh_1 respectively, and these magnitudes are equal to l_2 and εh_2 for the rods from the second class. In addition, the thin rods from each classes are ε-periodically alternated along the segment I_0.

In Ω_ε we consider the following spectral problem

$$\begin{cases} -\Delta_x\, u(\varepsilon, x) &= \ \lambda(\varepsilon)\ \rho_\varepsilon(x) u(\varepsilon, x), & x \in \Omega_\varepsilon; \\ u(\varepsilon, x) &= \ 0, & x \in \Gamma_1; \\ -\partial_\nu u(\varepsilon, x) &= \ 0, & x \in \partial\Omega_\varepsilon \setminus \Gamma_1; \\ [u]_{|x_2=0} &= \ [\partial_{x_2} u]_{|x_2=0} = 0, & x_1 \in Q_\varepsilon. \end{cases} \tag{1.1}$$

Here $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative; the brackets denote the jump of the enclosed quantities; Γ_1 is a curve on $\partial\Omega_0$, located in $\{x : \ x_2 > \delta_0\}$; the density

$$\rho_\varepsilon(x) = \begin{cases} 1, & x \in \Omega_0 \cup G_\varepsilon^{(2)}, \\ \varepsilon^{-\alpha}, & x \in G_\varepsilon^{(1)}; \end{cases}$$

the parameter $\alpha \in \mathbb{R}$ (if $\alpha > 0$, then concentrated masses are presented on the thin rods from the first class $G_\varepsilon^{(1)}$); $Q_\varepsilon = Q_\varepsilon^{(1)} \cup Q_\varepsilon^{(2)}$, $Q_\varepsilon^{(i)} = G_\varepsilon^{(i)} \cap \{x : x_2 = 0\}, \ i = 1, 2.$

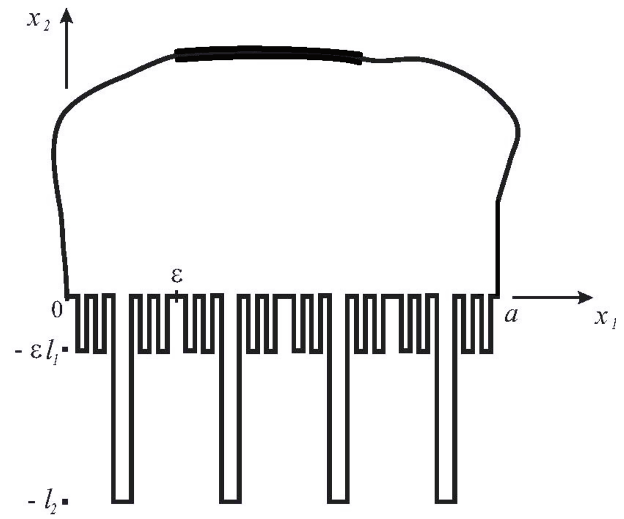

Figure 1: The thick cascade junction Ω_ε.

Obviously, that for each fixed value of ε there is a sequence of eigenvalues

$$0 < \lambda_1(\varepsilon) < \lambda_2(\varepsilon) \leq \ldots \leq \lambda_n(\varepsilon) \leq \cdots \to +\infty \quad \text{as} \quad n \to \infty, \tag{1.2}$$

of problem (1.1). The corresponding eigenfunctions $\{u_n(\varepsilon, \cdot)\}_{n\in\mathbb{N}}$, which belong to $\mathcal{H}_\varepsilon$, can be orthonormalized as follows

$$(u_n, u_k)_{L_2(\Omega_0 \cup G_\varepsilon^{(2)})} + \varepsilon^{-\alpha}(u_n, u_k)_{L_2(G_\varepsilon^{(1)})} = \delta_{n,k}, \quad \{n, k\} \in \mathbb{N}. \tag{1.3}$$

Here and below $\delta_{n,k}$ is the Kronecker delta, $\mathcal{H}_\varepsilon$ is the Sobolev space $\{u \in H^1(\Omega_\varepsilon) : u|_{\Gamma_1} = 0$ *in sense of the trace*$\}$ with the scalar product

$$(u, v)_{\mathcal{H}_\varepsilon} := \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, dx \quad \forall\, u, v \in \mathcal{H}_\varepsilon.$$

Our aim is to study the asymptotic behavior of the eigenvalues $\{\lambda_n(\varepsilon)\}_{n\in\mathbb{N}}$ and the eigenfunctions $\{u_n(\varepsilon, \cdot)\}_{n\in\mathbb{N}}$ as $\varepsilon \to 0$, i.e., when the number of the attached thin rods from each class infinitely increases and their thickness decreases to zero.

It should be noted that the limit process is accompanied by the influence of the concentrated masses on the rods from the first class. In fact, we have two kinds of perturbations for problem (1.1): the domain perturbation and the density perturbation. We are going to study the influence of both these factors on the asymptotic behavior of the eigenvalues and eigenfunctions as well.

1.2 The outline of results

We establish five qualitatively different cases in the asymptotic behavior of eigenvalues and eigenfunctions of problem (1.1) as $\varepsilon \to 0$, namely the case of "light" concentrated masses ($\alpha \in (0, 1)$), "intermediate" concentrated masses ($\alpha = 1$), and "heavy" concentrated masses

($\alpha \in (1, +\infty)$) that we divide into "slightly heavy" concentrated masses ($\alpha \in (1, 2)$), "moderate heavy" concentrated masses ($\alpha = 2$), and "super heavy" concentrated masses ($\alpha > 2$).

In the cases of "light" and "intermediate" concentrated masses (see [1, 2]) the perturbation of domain plays the leading role in the asymptotic behavior.

If $\boldsymbol{\alpha \in (0, 1)}$, then the spectrum of the homogenized problem coincides with the spectrum of the problem in domain without concentrated masses (see for instance the papers [8, 13, 14, 15, 16, 17], where it was discovered a remarkable peculiarity in the geometric structure of the spectrum (the presence of *lacunas*)). The concentrated masses influence only to the second term of the asymptotic expansion, in particular the asymptotic expansion for an eigenvalue $\lambda_n(\varepsilon)$ of problem (1.1) is as follows

$$\lambda(\varepsilon) \approx \lambda_0 + \varepsilon^{1-\alpha}\lambda_{1-\alpha} + \varepsilon\lambda_1 + \varepsilon^{2-\alpha}\lambda_{2-\alpha} + \dots \tag{1.4}$$

Here we omit the index n.

The concentrated masses are revealed in the corresponding homogenized spectral problem in the case $\boldsymbol{\alpha = 1}$. This influence appears through the following additional term $4h_1 l_1 \lambda_0 \, v_0^+(x_1, 0)$ with the spectral parameter λ_0 in the jump of the derivatives in the joint, i.e.

$$\partial_{x_2} v_0^+(x_1, 0) - h_2 \partial_{x_2} v_0^-(x_1, 0) = -4h_1 l_1 \lambda_0 \, v_0^+(x_1, 0), \quad x_1 \in (0, a). \tag{1.5}$$

This term shows also the influence of the geometrical structure of thin rectangles from the first class on the asymptotics. In this case the asymptotic expansion for an eigenvalue has the form

$$\lambda(\varepsilon) \approx \lambda_0 + \varepsilon\lambda_1 + \dots \tag{1.6}$$

It turned out that we cannot directly substitute $\alpha = 1$ in (1.4) to obtain (1.6).

If $\boldsymbol{\alpha > 1}$, then the concentrated masses begin to play the leading role in the asymptotic behavior of the eigenvalues and the eigenfunctions. The principal differences between this and previous cases are the following: all eigenvalues $\{\lambda_n(\varepsilon)\}$ converge to zero with the rate $\varepsilon^{\alpha-1}$, i.e., for any $n \in \mathbb{N}$

$$\lambda_n(\varepsilon) \sim \varepsilon^{\alpha-1}\lambda_0^{(n)} \quad \text{as} \quad \varepsilon \to 0.$$

This fact was proved in the following lemma.

Lemma 1.1 (see [5]). *If $\alpha > 1$, then for any fixed $n \in \mathbb{N}$ there exist constants C_0, C_1 and ε_0 such that for all value of ε from the interval $(0, \varepsilon_0)$ the following estimates hold*

$$0 < \lambda_n(\varepsilon) \le C_0 \, \varepsilon^{\alpha-1}, \qquad \|u_n\|_{\mathcal{H}_\varepsilon} \le C_1 \, \varepsilon^{\frac{\alpha-1}{2}}.$$

In addition, there is a positive constant c_0 (depending neither on ε nor on n) such that

$$0 < c_0 \, \varepsilon^{\alpha-1} \le \lambda_n(\varepsilon) \tag{1.7}$$

for all $n \in \mathbb{N}$ and $\varepsilon \in (0, \varepsilon_0)$.

In [3, 4, 5] the problem (1.1) was completely studied for $\boldsymbol{\alpha \in (1, 2)}$. There we have proved that the eigenvibrations $\{u_n(\varepsilon, \cdot)\}_{n \in \mathbb{N}}$ have a new type of the skin effect which we call *spatial skin-effect*. It means that vibrations of the thin rods from the second class repeat the shape of vibrations of the joint zone in the first term of the asymptotics. This first term is equal to

$$v_0(x) = \begin{cases} v_0^+(x), & x \in \ \Omega_0, \\ v_0^-(x_1) \equiv v_0^+(x_1, 0), & x \in \ D_2 = (0, a) \times (-l_2, 0), \end{cases} \tag{1.8}$$

and it and the corresponding number λ_0 are solutions of the following Steklov problem:

$$\begin{cases} \Delta_x\, v_0^+(x) &= 0, & x\in\Omega_0 \\ \partial_\nu\, v_0^+(x) &= 0, & x\in\Gamma_2, \\ v_0^+(x) &= 0, & x\in\Gamma_1, \\ \partial_{x_2} v_0^+(x_1,0) &= -4h_1l_1\lambda_0\, v_0^+(x_1,0), & x_1\in(0,a). \end{cases} \tag{1.9}$$

The number λ_0 is the first term in the asymptotic expansion

$$\lambda(\varepsilon)\approx\varepsilon^{\alpha-1}\Big(\lambda_0+\varepsilon^{\alpha-1}\lambda_{\alpha-1}+\ldots\Big) \tag{1.10}$$

for eigenvalues of problem (1.1). The second term in (1.10) depends on the geometrical characteristics both of the thin rods from the first class $G_\varepsilon^{(1)}$ and the thin rods from the second class $G_\varepsilon^{(2)}$ and the domain Ω_0. It is equal to

$$\lambda_{\alpha-1}=-\frac{\lambda_0}{4h_1l_1}\Big(h_2l_2+\int\limits_{\Omega_0}(v_0^+)^2\,dx\Big). \tag{1.11}$$

The corresponding second term $v_{\alpha-1}^-$ in the asymptotics for eigenfunctions in D_2 depends also on the geometrical parameters h_2 and l_2 and in addition on the variable x_2, which does not take place for the first term v_0^- (see (1.8)).

If $\boldsymbol{\alpha\in(m,m+1)}$, $m\in\mathbb{N}$, $m\ge 2$, then we will show that the asymptotic expansion for an eigenvalue of problem (1.1) is as follows:

$$\lambda(\varepsilon)\approx\varepsilon^{\alpha-1}\left(\lambda_0+\varepsilon^{\alpha-m}\lambda_{\alpha-m}+\ldots+\varepsilon\lambda_1+\varepsilon^{\alpha-m+1}\lambda_{\alpha-m+1}+\ldots\ldots\right), \tag{1.12}$$

where λ_0 is an eigenvalue of problem (1.9) and the second term

$$\lambda_{\alpha-m}=-\frac{1}{4h_1l_1}\int_{I_0}v_0^+(x_1,0)\,\partial_{x_2}v_0^+(x_1,0)\,dx. \tag{1.13}$$

The second term in (1.12) depends only on the geometrical characteristics of the thin rods from the first class $G_\varepsilon^{(1)}$ where the concentrated masses are presented. This means the growing influence of concentrated masses on the asymptotics of the eigenvalues of problem (1.1).

As concerns the corresponding eigenfunctions, we observe the enhancement of the spatial skin-effect. This means that both the first and second terms of the asymptotics are independent of x_2 in D_2, namely the first term is the same as v_0 for $\alpha\in(1,2)$ (see (1.8)) and the second one has the similar form

$$v_{\alpha-m}(x)=\begin{cases} v_{\alpha-m}^+(x), & x\in\ \Omega_0, \\ v_{\alpha-m}^-(x_1)=v_{\alpha-m}^+(x_1,0), & x\in\ D_2. \end{cases} \tag{1.14}$$

Moreover, the α is the nearest to m, the more terms are between $\varepsilon^{\alpha-m}\lambda_{\alpha-m}$ and $\varepsilon\lambda_1$ in (1.12). Therefore, hereinafter we have written down "…" between $\varepsilon^{\alpha-m}\lambda_{\alpha-m}$ and $\varepsilon\lambda_1$. This means that for integer α we cannot use (1.12) at $\alpha=m$ and it is necessary to reapply a formal procedure for this case.

Thus, for $\boldsymbol{\alpha=m}$, $m\in\mathbb{N}$, $m\ge 2$, we propose the following asymptotic ansatz for an eigenvalue: $\lambda_n(\varepsilon)$ (next we omit the index n)

$$\lambda(\varepsilon)\approx\varepsilon^{m-1}\Big(\lambda_0+\varepsilon\lambda_1(m)+\varepsilon^2\lambda_2(m)+\ldots\Big), \tag{1.15}$$

where λ_0 is an eigenvalue of problem (1.9); and if $m = 2$, then

$$\begin{aligned}\lambda_1 = &- \frac{\lambda_0}{4h_1 l_1}\left(h_2 l_2 + \int\limits_{\Omega_0} (v_0^+)^2\, dx\right) - \frac{\varsigma_{(0,0)}}{4h_1 l_1} - \varsigma_{(2,0)} \int_{I_0} \partial^2_{x_1 x_1} v_0^+(x_1,0)\, v_0^+(x_1,0)\, dx_1 \\ &- \lambda_0 \int_{\Pi_{l_1}} Z_1^{(2)}(\eta)\, d\eta \int_{I_0} v_0^+(x_1,0)\, \partial_{x_2} v_0^+(x_1,0)\, dx_1;\end{aligned} \tag{1.16}$$

and if $m \geq 3$, then

$$\begin{aligned}\lambda_1 = &- \frac{\varsigma_{(0,0)}}{4h_1 l_1} - \varsigma_{(2,0)} \int_{I_0} \partial^2_{x_1 x_1} v_0^+(x_1,0)\, v_0^+(x_1,0)\, dx_1 \\ &- \lambda_0 \int_{\Pi_{l_1}} Z_1^{(2)}(\eta)\, d\eta \int_{I_0} v_0^+(x_1,0)\, \partial_{x_2} v_0^+(x_1,0)\, dx_1.\end{aligned} \tag{1.17}$$

Comparing formulas for the second terms in the asymptotics for eigenvalues of problem (1.1) (see (1.11) for $\alpha \in (1,2)$, (1.16) for $\alpha = 2$, and (1.13) and (1.17) for $\alpha > 2$), we see the reducing of the influence of geometry of the domain Ω_0 and the thin rods from the second class $G_\varepsilon^{(2)}$, on the asymptotic behaviour of the eigenvalues.

This and the facts mentioned above justify the separation of the "heavy" concentrated masses into "slightly heavy" ($\alpha \in (1,2)$), "moderate heavy" ($\alpha = 2$), and "super heavy" concentrated masses ($\alpha > 2$).

We recall that the cases $\alpha = 2$ and $\alpha > 2$ are of our interest in the present paper.

High-frequency cell-vibrations. As for vibrations of fastened membranes with concentrated masses on a small set of diameter $\mathcal{O}(\varepsilon)$ (see for instance [18, 19, 20, 21, 22, 23] and reference therein) there exist three qualitatively different cases for such spectral problems: $\alpha < 2$, $\alpha = 2$, $\alpha > 2$. It was proved in these papers that there are two kinds of eigenvibrations: the local vibrations, for which the corresponding eigenfunctions are of order $O(1)$ only in a region near the concentrated masses; and the global vibrations, for which the corresponding eigenfunctions are located on the whole membrane. The local and global vibrations can exist only for $\alpha \geq 2$, and the local vibrations are low-frequency vibrations. Local vibrations are not found in the case $\alpha < 2$. The associated eigenvalues for the local vibrations have the asymptotics

$$\lambda_n(\varepsilon) = \varepsilon^{\alpha-2}\lambda_n + o(\varepsilon^{\alpha-2}), \tag{1.18}$$

where λ_n is an eigenvalue of the corresponding spectral local problem. The formula (1.18) shows the structure of the low-frequency convergence of the spectrum.

In contrast to results of papers [18, 19, 20, 21], we show that there are free-vibrations in problem (1.1), which correspond to local vibrations of the concentrated masses; they present at each value of the parameter $\alpha \in (0, +\infty)$; and they are always high-frequency vibrations (see Sec. 5). The associated eigenvalues for these vibrations have the asymptotics

$$\lambda(\varepsilon) = \varepsilon^{\alpha-2}\Lambda + o(\varepsilon^{\alpha-2}), \tag{1.19}$$

where Λ is an eigenvalue of the corresponding spectral cell-problem (see (5.1)). We see from this formula that these eigenvalues are of order $\mathcal{O}(1)$ at $\alpha = 2$. This is another reason to distinguish the case $\alpha = 2$ among "heavy" masses.

1.3 Rescaling of problem (1.1)

Keeping in mind the bounds from Lemma 1.1 for "heavy" concentrated masses we rescale the eigenvalues and eigenfunctions as follows:

$$\lambda(\varepsilon) = \varepsilon^{\alpha-1}\Lambda(\varepsilon), \qquad u(\varepsilon,x) = \varepsilon^{\frac{\alpha-1}{2}} v(\varepsilon,x). \tag{1.20}$$

Under this rescaling the problem (1.1) becomes

$$\begin{cases} -\Delta_x\, v(\varepsilon,x) &= \varepsilon^{\alpha-1}\Lambda(\varepsilon)\, v(\varepsilon,x), & x\in\Omega_0\cup G_\varepsilon^{(2)}; \\ -\Delta_x\, v(\varepsilon,x) &= \varepsilon^{-1}\Lambda(\varepsilon)\, v(\varepsilon,x), & x\in G_\varepsilon^{(1)}; \\ -\partial_\nu v(\varepsilon,x) &= 0, & x\in\partial\Omega_\varepsilon\setminus\Gamma_1; \\ v(\varepsilon,x) &= 0, & x\in\Gamma_1; \\ [v]_{|x_2=0} &= [\partial_{x_2}v]_{|x_2=0} = 0, & x_1\in Q_\varepsilon. \end{cases} \tag{1.21}$$

Let us define an operator $A_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_\varepsilon$ that corresponds to problem (1.21) by the following equality:

$$(A_\varepsilon u, v)_{\mathcal{H}_\varepsilon} = (u,v)_{\mathcal{V}_\varepsilon} \quad \forall\, u,v\in\mathcal{H}_\varepsilon, \tag{1.22}$$

where $\mathcal{V}_\varepsilon$ is the weighted space $L^2(\Omega_\varepsilon)$ with the scalar product

$$(u,v)_{\mathcal{V}_\varepsilon} := \varepsilon^{\alpha-1}\int\limits_{\Omega_0\cup G_\varepsilon^{(2)}} u\,v\,dx + \varepsilon^{-1}\int\limits_{G_\varepsilon^{(1)}} u\,v\,dx.$$

It is easy to see that the operator A_ε is self-adjoint, positive, and compact. In addition, problem (1.21) is equivalent to the spectral problem $A_\varepsilon u = \lambda^{-1}(\varepsilon)\,u$ in $\mathcal{H}_\varepsilon$.

Therefore, for each fixed value of ε there is a sequence of eigenvalues of problem (1.21)

$$0 < \Lambda_1(\varepsilon) < \Lambda_2(\varepsilon) \le \ldots \le \Lambda_n(\varepsilon) \le \cdots \to +\infty \quad \text{as} \quad n\to\infty. \tag{1.23}$$

The corresponding eigenfunctions $\{v_n(\varepsilon,\cdot)\}_{n\in\mathbb{N}}$ can be orthonormalized in the following way:

$$(v_n, v_k)_{\mathcal{V}_\varepsilon} = \delta_{n,k}, \quad \{n,\, k\}\in\mathbb{N}. \tag{1.24}$$

2 Formal asymptotics for $\alpha\in(m,m+1)$, $m\in\mathbb{N}$, $m\ge 2$

2.1 Construction of asymptotics

Combining the algorithm of constructing asymptotics in thin domains with the methods of homogenization theory, we seek the main terms of the asymptotics for the eigenvalue $\Lambda_n(\varepsilon)$ and the eigenfunction $v_n(\varepsilon,\cdot)$ in the form (index n is omitted):

$$\Lambda(\varepsilon) \approx \lambda_0 + \varepsilon^{\alpha-m}\lambda_{\alpha-m} + \ldots \tag{2.1}$$

$$v(\varepsilon,x) \approx v_0^+(x) + \varepsilon^{\alpha-m}v_{\alpha-m}^+(x) + \ldots \quad \text{in domain } \Omega_0; \tag{2.2}$$

$$v(\varepsilon,x) \approx v_0^-(x_1,x_2,\frac{x_1}{\varepsilon}-j) + \varepsilon^{\alpha-m}v_{\alpha-m}^-(x_1,x_2,\frac{x_1}{\varepsilon}-j) + \ldots \tag{2.3}$$

in the thin rectangles $G_j^{(2)}, \varepsilon)$ $(j = 0, \dots, N-1;$ and in the junction zone of the body and thin rectangles of both classes (which we call internal expansion) the series of the following type:

$$\begin{aligned} v(\varepsilon, x) \approx\ & v_0^+(x_1, 0) + \varepsilon^{\alpha-m} v_{\alpha-m}^+(x_1, 0) + \dots + \varepsilon\Big(Z_1^{(0)}(\tfrac{x}{\varepsilon}) v_0^+(x_1, 0) + \sum_{i=1}^{2} Z_1^{(i)}(\tfrac{x}{\varepsilon}) \partial_{x_i} v_0^+(x_1, 0)\Big) + \\ & + \varepsilon^{\alpha-m+1}\Big(Z_{\alpha-m+1}^{(0)}(\tfrac{x}{\varepsilon}) v_0^+(x_1, 0) + Z_{\alpha-m+1}^{(2)}(\tfrac{x}{\varepsilon}) \partial_{x_2} v_0^+(x_1, 0) + \\ & + X_{\alpha-m+1}^{(0)}(\tfrac{x}{\varepsilon}) v_{\alpha-m}^+(x_1, 0) + \sum_{i=1}^{2} X_{\alpha-m+1}^{(i)}(\tfrac{x}{\varepsilon}) \partial_{x_i} v_{\alpha-m}^+(x_1, 0)\Big) + \dots + \\ & + \varepsilon^2 \sum_{|\beta| \le 2} Z_2^{(\beta)}(\eta) D^\beta v_0^+(x_1, 0) + \dots \end{aligned} \tag{2.4}$$

We used the following standard notation: $\partial_{x_i} = \dfrac{\partial}{\partial x_i}$, $D^\beta = \dfrac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}$, where $\beta = (\beta_1, \beta_2)$, $|\beta| = \beta_1 + \beta_2$, $\beta_i \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.

Denote $\Gamma_2 := \partial\Omega_0 \setminus (\Gamma_1 \cup I_0)$. Substituting (2.1) and (2.2) in problem (1.21) and collecting terms with equal order of ε, we get:

$$\begin{cases} -\Delta_x\, v_0^+(x) = 0, \quad x \in \Omega_0, \\ \partial_\nu v_0^+(x)|_{x\in\Gamma_2} = 0, \qquad v_0^+(x)|_{x\in\Gamma_1} = 0. \end{cases} \tag{2.5}$$

Collecting terms of order $\varepsilon^{\alpha-m}$, we have

$$\begin{cases} -\Delta_x\, v_{\alpha-m}^+(x) = 0, \quad x \in \Omega_0, \\ \partial_\nu v_{\alpha-m}^+(x)|_{x\in\Gamma_2} = 0, \qquad v_{\alpha-m}^+(x)|_{x\in\Gamma_1} = 0. \end{cases} \tag{2.6}$$

To complete these problems we have to find conditions on I_0; this is done in Subsection 2.2.

2.1.1 Formal asymptotics in each thin rectangle $G_j^{(2)}(\varepsilon)$

Using Taylor series for the functions $\{v_\gamma^-\}$ in (2.3) in a neighborhood of the point $x_1 = \varepsilon(j + \frac{1}{2})$, we get

$$\begin{aligned} v(\varepsilon, x) \approx W_0^{(j)}(x_2, \eta_1) + \varepsilon^{\alpha-m} W_{\alpha-m}^{(j)}(x_2, \eta_1) + \dots + \varepsilon W_1^{(j)}(x_2, \eta_1) + \\ + \varepsilon^{\alpha-m+1} W_{\alpha-m+1}^{(j)}(x_2, \eta_1) + \dots + \varepsilon^2 W_2^{(j)}(x_2, \eta_1) + \dots, \end{aligned} \tag{2.7}$$

where

$$W_\gamma^{(j)}(x_2, \eta_1) = v_\gamma^-\big(\varepsilon(j + \tfrac{1}{2}), x_2, \eta_1 - j\big) \quad \text{for} \quad \gamma \in \{0, \alpha - m\}, \tag{2.8}$$

$$W_\gamma^{(j)}(x_2, \eta_1) = v_\gamma^-\big(\varepsilon(j + \tfrac{1}{2}), x_2, \eta_1 - j\big) + \big(\eta_1 - j - \tfrac{1}{2}\big) \frac{\partial v_{\gamma-1}^-}{\partial x_1} \big(\varepsilon(j + \tfrac{1}{2}), x_2, \eta_1 - j\big) \tag{2.9}$$

for $\gamma \in \{1, \alpha - m + 1\}$ and

$$\begin{aligned} W_\gamma^{(j)}(x_2, \eta_1) = v_\gamma^-\big(\varepsilon(j + \tfrac{1}{2}), x_2, \eta_1 - j\big) + \big(\eta_1 - j - \tfrac{1}{2}\big) \frac{\partial v_{\gamma-1}^-}{\partial x_1} \big(\varepsilon(j + \tfrac{1}{2}), x_2, \eta_1 - j\big) \\ + \frac{1}{2} \big(\eta_1 - j - \tfrac{1}{2}\big)^2 \frac{\partial^2 v_{\gamma-2}^-}{\partial x_1^2} \big(\varepsilon(j + \tfrac{1}{2}), x_2, \eta_1 - j\big) \end{aligned} \tag{2.10}$$

for $\gamma \in \{2, \alpha - m + 2\}$; here $\eta_1 = \frac{x_1}{\varepsilon}$.

Substituting (2.1) and (2.7) in the problem (1.21) instead of $\Lambda_n(\varepsilon)$ and $v_n(\varepsilon, \cdot)$ respectively, collecting terms with equal powers of ε, we obtain the following boundary-value problems:

$$\left\{ \begin{array}{rcll} -\partial^2_{\eta_1\eta_1} W^{(j)}_\gamma(x_2, \eta_1) & = & 0, & \eta_1 \in \left(\frac{1-h_2}{2}, \frac{1+h_2}{2}\right), \\ \partial_{\eta_1} W^{(j)}_\gamma(x_2, \frac{1\pm h_2}{2}) & = & 0, & \end{array} \right. \tag{2.11}$$

for $\gamma \in \{0, \alpha - m, 1, \alpha - m + 1\}$. Here the variable x_2 is regarded as a parameter, $\partial_{\eta_1} = \frac{\partial}{\partial \eta_1}$. From (2.11) we deduce that the solutions $W^{(j)}_\gamma$, $\gamma \in \{0, \alpha - m, 1, \alpha - m + 1\}$, are independent of η_1.

Then, for $\gamma \in \{2, 3, \alpha - m + 2, \alpha - m + 3\}$ we get the following problems:

$$\left\{ \begin{array}{rcll} -\partial^2_{\eta_1\eta_1} W^{(j)}_\gamma(x_2, \eta_1) & = & \partial^2_{x_2x_2} W^{(j)}_{\gamma-2}(x_2), & \eta_1 \in \left(\frac{1-h_2}{2}, \frac{1+h_2}{2}\right), \\ \partial_{\eta_1} W^{(j)}_2(x_2, \frac{1\pm h_2}{2}) & = & 0. & \end{array} \right. \tag{2.12}$$

The solvability condition for (2.12) gives us the relations

$$\partial^2_{x_2x_2} W^{(j)}_{\gamma-2}(x_2) = 0, \quad x_2 \in (-l_2, 0), \quad \gamma \in \{2, 3, \alpha - m + 2, \alpha - m + 3\}.$$

If $\gamma = 2$ and $\alpha - m + 2$ it is the same that

$$\partial^2_{x_2x_2} v^-_{\gamma-2}\big(\varepsilon(j + \tfrac{1}{2}), x_2\big) = 0, \quad x_2 \in (-l_2, 0), \tag{2.13}$$

because of (2.8). Bearing in mind the boundary conditions of the original problem at $x_2 = -l_2$, we should add the following condition $\partial_{x_2} v^-_{\gamma-2}\big(\varepsilon(j+\frac{1}{2}), -l_2\big) = 0$ to (2.13). These two relations mean that v^-_0 and $v^-_{\alpha-m}$ are independent of x_2.

Similarly, but now with regard to (2.9) we get

$$v^-_1\big(\varepsilon(j + \tfrac{1}{2}), \eta_1 - j\big) + \big(\eta_1 - j - \tfrac{1}{2}\big) \frac{\partial v^-_0}{\partial x_1} \big(\varepsilon(j + \tfrac{1}{2})\big) = \Phi_1\big(\varepsilon(j + \tfrac{1}{2})\big) \tag{2.14}$$

if $\gamma = 3$, and

$$v^-_{\alpha-m+1}\big(\varepsilon(j + \tfrac{1}{2}), \eta_1 - j\big) + \big(\eta_1 - j - \tfrac{1}{2}\big) \frac{\partial v^-_{\alpha-m}}{\partial x_1} \big(\varepsilon(j + \tfrac{1}{2})\big) = \Phi_{\alpha-m+1}\big(\varepsilon(j + \tfrac{1}{2})\big) \tag{2.15}$$

if $\gamma = \alpha - m + 3$; here the values Φ_1 and $\Phi_{\alpha-m+1}$ will be defined in subsection 2.2.

Since the points $\{x_1 = \varepsilon(j + \frac{1}{2}) : j = 0, \ldots, N - 1\}$ form the ε-net in the interval $(0, a)$, then we extend relations (2.14) and (2.15) to the whole of $(0, a)$.

2.1.2 Junction-layer solutions

Let us pass to the "fast" variables $\eta = \dfrac{x}{\varepsilon}$ in (1.21). Under this transformation as $\varepsilon \to 0$ the domain Ω_0 transforms to $\{\eta : \eta_i > 0, \, i = 1, 2\}$, the thin rectangle $G^{(2)}_0(\varepsilon)$ to the semistrip

$$\Pi^- = \left(\frac{1}{2} - \frac{h_2}{2}, \frac{1}{2} + \frac{h_2}{2}\right) \times (-\infty, 0]$$

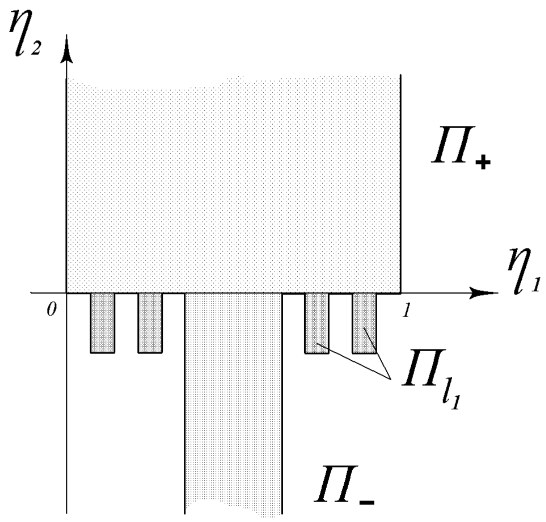

Figure 2: The cell of periodicity.

and rectangle $G_0^{(1)}(d_k,\varepsilon)$ to the fixed rectangle

$$\Pi_k = \left(d_k - \frac{h_1}{2}, d_k + \frac{h_1}{2}\right) \times (-l_1, 0].$$

Taking into account the periodic structure of Ω_ε in a neighborhood of I_0, we take the following cell of periodicity

$$\Pi = \Pi^- \cup \Pi^+ \cup \Pi_{l_1},$$

in which we will consider boundary-value problems for coefficients Z, X from (2.4). Here $\Pi^+ = (0,1) \times (0,+\infty)$, $\Pi_{l_1} := \bigcup_{k=1}^4 \overline{\Pi}_k$ (see Fig.2). To find problems for these coefficients we should calculate

$$\begin{aligned}
\partial_{x_1} v(\varepsilon, x) \approx \varepsilon^0 \Bigg(& \partial_{x_1} v_0^+(x_1,0)\Big[1 + \partial_{\eta_1} Z_1^{(1)}\Big] + \partial_{\eta_1} Z_1^{(0)} v_0^+(x_1,0) + \partial_{\eta_1} Z_1^{(2)} \partial_{x_2} v_0^+(x_1,0)\Bigg) + \\
& + \varepsilon^{\alpha-m}\Bigg(\partial_{\eta_1} Z_{\alpha-m+1}^{(0)} v_0^+(x_1,0) + \partial_{\eta_1} X_{\alpha-m+1}^{(0)} v_{\alpha-m}^+(x_1,0) + \\
& + \partial_{x_1} v_{\alpha-m}^+(x_1,0)\Big[1 + \partial_{\eta_1} X_{\alpha-m+1}^{(1)}\Big] + \partial_{\eta_1} X_{\alpha-m+1}^{(2)} \partial_{x_2} v_{\alpha-m}^+(x_1,0) + \\
& + \partial_{\eta_1} Z_{\alpha-m+1}^{(2)}(\eta) \partial_{x_2} v_0^+(x_1,0)\Bigg) + \varepsilon\Bigg(Z_1^{(0)}(\eta) \partial_{x_1} v_0^+(x_1,0) + \\
& + \sum_{i=1}^{2} Z_1^{(i)}(\eta) \partial^2_{x_1 x_i} v_0^+(x_1,0) + \sum_{|\beta|\le 2} \partial_{\eta_1} Z_2^{(\beta)}(\eta) D^\beta v_0^+(x_1,0)\Bigg) + \mathcal{O}(\varepsilon^{\alpha-m+1})
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
\Delta_x v(\varepsilon,x) \approx \varepsilon^{-1}\Big(\Delta_\eta Z_1^{(0)}(\eta) v_0^+(x_1,0) + \sum_{i=1}^{2} \Delta_\eta Z_1^{(i)}(\eta)\partial_{x_i} v_0^+(x_1,0)\Big) + \\
+ \varepsilon^{\alpha-m-1}\Big(\Delta_\eta Z_{\alpha-m+1}^{(0)}(\eta) v_0^+(x_1,0) + \Delta_\eta Z_{\alpha-m+1}^{(2)}(\eta)\partial_{x_2} v_0^+(x_1,0) + \\
+ \Delta_\eta X_{\alpha-m+1}^{(0)}(\eta) v_{\alpha-m}^+(x_1,0) + \sum_{i=1}^{2} \Delta_\eta X_{\alpha-m+1}^{(i)}(\eta)\partial_{x_i} v_{\alpha-m}^+(x_1,0)\Big) + \\
+ \varepsilon^0\Big(\partial^2_{x_1x_1} v_0^+(x_1,0)\Big[1 + 2\partial_{\eta_1} Z_1^{(1)}(\eta)\Big] + 2\partial_{\eta_1} Z_1^{(0)}(\eta)\partial_{x_1} v_0^+(x_1,0) + \\
+ 2\partial_{\eta_1} Z_1^{(2)}(\eta)\partial^2_{x_1x_2} v_0^+(x_1,0) + \sum_{|\beta|\le 2} \Delta_\eta Z_2^{(\beta)}(\eta) D^\beta v_0^+(x_1,0)\Big) + \ldots
\end{aligned} \tag{2.17}$$

Keeping in mind (2.16) and (2.17), substituting the series (2.4) and (2.1) in problem (1.21) and collecting terms with equal powers of ε, we get problems for $Z_1^{(i)}$, $i = 0,1,2$, $X_{\alpha-m+1}^{(i)}$, $i = 0,1,2$, $Z_{\alpha-m+1}^{(i)}$, $i = 0,2$, and $Z_2^{(\beta)}$, $|\beta| \le 2$. Obviously, these solutions have to be 1-periodic in η_1. Therefore, we demand the following periodicity conditions:

$$\begin{aligned}
\partial^s_{\eta_1} Z(0,\eta_2) = \partial^s_{\eta_1} Z(1,\eta_2), \quad \eta_2 > 0, \quad s = 0,1, \\
\partial^s_{\eta_1} X(0,\eta_2) = \partial^s_{\eta_1} X(1,\eta_2), \quad \eta_2 > 0, \quad s = 0,1,
\end{aligned} \tag{2.18}$$

on the vertical sides of semistrip Π^+. In addition, it is easy to see that all these solutions must satisfy the Neumann conditions

$$\begin{aligned}
\partial_{\eta_2} Z(\eta_1,0) = 0, \quad (\eta_1,0) \in \partial\Pi, \quad \partial_{\eta_2} Z(\eta_1,-l_1) = 0, \quad (\eta_1,-l_1) \in \partial\Pi, \\
\partial_{\eta_2} X(\eta_1,0) = 0, \quad (\eta_1,0) \in \partial\Pi, \quad \partial_{\eta_2} X(\eta_1,-l_1) = 0, \quad (\eta_1,-l_1) \in \partial\Pi,
\end{aligned} \tag{2.19}$$

on the horizontal parts of the boundary of Π.

Denote by $\partial\Pi_{\|}$ the vertical part of $\partial\Pi$ laying in $\{\eta :\ \eta_2 < 0\}$.

Thus we get the following problems (to all those problems we must add the respective conditions (2.18) and (2.19)):

$$\begin{cases} -\Delta_\eta\, Z_1^{(0)}(\eta) &= \begin{cases} 0, & \eta \in \Pi^+ \cup \Pi^-, \\ \lambda_0, & \eta \in \Pi_{l_1}, \end{cases} \\ \partial_{\eta_1} Z_1^{(0)}(\eta) &= 0, \qquad \eta \in \partial\Pi_{\|}; \end{cases} \tag{2.20}$$

$$\begin{cases} -\Delta_\eta\, Z_1^{(i)}(\eta) &= 0, \qquad \eta \in \Pi, \\ \partial_{\eta_1} Z_1^{(i)}(\eta) &= -\delta_{1i}, \qquad \eta \in \partial\Pi_{\|}, \quad i = 1,2; \end{cases} \tag{2.21}$$

$$\begin{cases} -\Delta_\eta\, Z_{\alpha-m+1}^{(0)}(\eta) &= \begin{cases} 0, & \eta \in \Pi^+ \cup \Pi^-, \\ \lambda_{\alpha-m}, & \eta \in \Pi_{l_1}, \end{cases} \\ \partial_{\eta_1} Z_{\alpha-m+1}^{(0)}(\eta) &= 0, \qquad \eta \in \partial\Pi_{\|}; \end{cases} \tag{2.22}$$

$$\begin{cases} -\Delta_\eta\, Z_2^{(0,0)}(\eta) &= \begin{cases} 0, & \eta \in \Pi^+ \cup \Pi^-, \\ \lambda_1 + \lambda_0 Z_1^{(0)}(\eta), & \eta \in \Pi_{l_1}, \end{cases} \\ \partial_{\eta_1} Z_2^{(0,0)}(\eta) &= 0, \qquad \eta \in \partial\Pi_{\|}; \end{cases} \tag{2.23}$$

$$\begin{cases} -\Delta_\eta \, Z_2^{(1,0)}(\eta) &= \begin{cases} 2\partial_{\eta_1} Z_1^{(0)}(\eta), & \eta \in \Pi^+ \cup \Pi^-, \\ 2\partial_{\eta_1} Z_1^{(0)}(\eta) + \lambda_0 Z_1^{(1)}(\eta), & \eta \in \Pi_{l_1}, \end{cases} \\ \partial_{\eta_1} Z_2^{(1,0)}(\eta) &= -Z_1^{(0)}(\eta), \qquad \eta \in \partial\Pi_{\|}; \end{cases} \tag{2.24}$$

$$\begin{cases} -\Delta_\eta \, Z_2^{(0,1)}(\eta) &= \begin{cases} 0, & \eta \in \Pi^+ \cup \Pi^-, \\ \lambda_0 Z_1^{(2)}(\eta), & \eta \in \Pi_{l_1}, \end{cases} \\ \partial_{\eta_1} Z_2^{(0,1)}(\eta) &= 0, \qquad \eta \in \partial\Pi_{\|}; \end{cases} \tag{2.25}$$

$$\begin{cases} -\Delta_\eta \, Z_2^{(0,2)}(\eta) &= 0, & \eta \in \Pi, \\ \partial_{\eta_1} Z_2^{(0,2)}(\eta) &= 0, & \eta \in \partial\Pi_{\|}; \end{cases} \tag{2.26}$$

$$\begin{cases} -\Delta_\eta \, Z_2^{(1,1)}(\eta) &= 2\partial_{\eta_1} Z_1^{(2)}(\eta), & \eta \in \Pi, \\ \partial_{\eta_1} Z_2^{(1,1)}(\eta) &= -Z_1^{(2)}(\eta), & \eta \in \partial\Pi_{\|}; \end{cases} \tag{2.27}$$

$$\begin{cases} -\Delta_\eta \, Z_2^{(2,0)}(\eta) &= 1 + 2\partial_{\eta_1} Z_1^{(1)}(\eta), & \eta \in \Pi, \\ \partial_{\eta_1} Z_2^{(2,0)}(\eta) &= -Z_1^{(1)}(\eta), & \eta \in \partial\Pi_{\|}. \end{cases} \tag{2.28}$$

The problems for $\{X_{\alpha-m+1}^{(k)}\}$ are the same as problems for $\{Z_1^{(k)}\}$, and problem for $Z_{\alpha-m+1}^{(2)}$ is the same as problem for $Z_1^{(2)}$. Therefore, $X_{\alpha-m+1}^{(k)} \equiv Z_1^{(k)}$, $k = 0, 1, 2$, and $Z_{\alpha-m+1}^{(2)} \equiv Z_1^{(2)}$.

The existence and the main asymptotic relations for solutions of those problems can be obtained from general results about the asymptotic behavior of solutions to elliptic problems in domains with different exits to infinity [24, 25, 26, 27]. However, if a domain, where we consider a boundary-value problem, has some symmetry, then we can define more exactly the asymptotic relations and detect other properties of junction-layer solutions (see Lemma 4.1 and Corollary 4.1 from [34]). Using this approach, one can prove the following lemma.

Lemma 2.1. *There exist solutions* $Z_1^{(i)} \in H^1_{loc,\eta_2}(\Pi)$, $i = 0, 1, 2$, *of the problems* (2.20), (2.21), $Z_2^{(\beta)} \in H^1_{loc,\eta_2}(\Pi)$, $|\beta| \le 2$ *of the problems* (2.23), (2.24), (2.25), (2.26), (2.27), (2.28) *and* $Z_{\alpha-m+1}^{(0)} \in H^1_{loc,\eta_2}(\Pi)$ *of the problem* (2.22), *which have the following differentiable asymptotics*

$$Z_1^{(0)}(\eta) = \begin{cases} \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{4h_1 l_1 \lambda_0}{h_2}\, \eta_2 + C_1^{(0)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.29}$$

$$Z_1^{(1)}(\eta) = \begin{cases} \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \Big(-\eta_1 + \dfrac{1}{2}\Big) + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.30}$$

$$Z_1^{(2)}(\eta) = \begin{cases} \eta_2 + \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{\eta_2}{h_2} + C_1^{(2)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.31}$$

$$Z_{\alpha-m+1}^{(0)}(\eta) = \begin{cases} \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{4h_1 l_1 \lambda_{\alpha-m}}{h_2}\, \eta_2 + C_{\alpha-m+1}^{(0)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.32}$$

$$Z_2^{(0,0)}(\eta) = \begin{cases} \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{4h_1 l_1 \lambda_1 + \varsigma_{(0,0)}}{h_2}\ \eta_2 + C_2^{(0,0)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.33}$$

$$Z_2^{(1,0)}(\eta) = \begin{cases} \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{4h_1 l_1 \lambda_0}{h_2}\ \eta_2\Big(-\eta_1 + \dfrac{1}{2}\Big) + C_2^{(1,0)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases}$$

$$Z_2^{(0,1)}(\eta) = \begin{cases} \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{\lambda_0 \int_{\Pi_{l_1}} Z_1^{(2)}(\eta)d\eta}{h_2}\ \eta_2 + C_2^{(0,1)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases}$$

$$Z_2^{(0,2)}(\eta) = \begin{cases} \eta_2 + \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{\eta_2}{h_2} + C_1^{(2)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases}$$

$$Z_2^{(1,1)}(\eta) = \begin{cases} \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{\eta_2}{h_2}\Big(-\eta_1 + \dfrac{1}{2}\Big) + C_2^{(1,1)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.34}$$

$$Z_2^{(2,0)}(\eta) = \begin{cases} -\dfrac{1}{2}\eta_2^2 + \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{1}{2}T^2(\eta_1) + \dfrac{\varsigma_{(2,0)}}{h_2}\eta_2 + C_2^{(2,0)} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.35}$$

where

$$\varsigma_{(0,0)} = \lambda_0 \int\limits_{\Pi_{l_1}} Z_1^{(0)}(\eta)d\eta, \qquad \varsigma_{(2,0)} = \int\limits_{\Pi_{l_1}\cup\Pi^-} \big(1 + \partial_{\eta_1} Z_1^{(1)}(\eta)\big)\ d\eta. \tag{2.36}$$

Moreover functions $Z_1^{(1)}$, $Z_2^{(1,0)}$, $Z_2^{(1,1)}$ are odd in η_1 with respect to $\frac{1}{2}$; functions $Z_1^{(0)}$, $Z_2^{(2,0)}$, $Z_1^{(2)}$, $Z_{\alpha-m+1}^{(0)}$, $Z_2^{(0,0)}$, $Z_2^{(0,1)}$, and $Z_2^{(0,2)}$ are even in η_1 with respect to $\frac{1}{2}$.

For the proof we refer to our previous papers [1, 2].

2.2 Homogenized problem and correctors

We have formally constructed the leading terms of the asymptotic expansions (2.2), (2.3), (2.4) in three different parts of the junction Ω_ε. Now we apply the method of matching of asymptotic expansions, proposed firstly by Il'in A.M. (see [28, 29, 30] and also [31]), to complete the constructions. Following this method, the asymptotics of the external expansions (2.2) and (2.3) as $x_2 \to \pm 0$ have to coincide with the corresponding asymptotics of the internal expansion (2.4) as $\eta_2 \to \pm\infty$ respectively.

Writing down the Taylor series for v_0^+ and $v_{\alpha-m}^+$ with respect to x_2 in a neighborhood of the point $(x_1, 0)$, where $x_1 \in (0, a)$, and passing to the variables $\eta_2 = \varepsilon^{-1}x_2$, we derive

$$\begin{aligned} v(\varepsilon, x) = v_0^+(x_1, 0) + \varepsilon^{\alpha-m} v_{\alpha-m}^+(x_1, 0) + \dots + \varepsilon\eta_2\partial_{x_2} v_0^+(x_1, 0) + \\ + \varepsilon^{\alpha-m+1}\eta_2\,\partial_{x_2} v_{\alpha-m}^+(x_1, 0) + \dots \end{aligned} \tag{2.37}$$

Bearing in mind the asymptotics of the functions $Z_1^{(k)}$, $X_{\alpha-m+1}^{(k)}$, $(k=0,1,2)$, $Z_{\alpha-m+1}^{(0)}$, $Z_2^{(\beta)}$ $(|\beta|<2)$, as $\eta_2\to+\infty$ (see (2.29)–(2.35)), we write down the asymptotics

$$\begin{aligned} v(\varepsilon,x) = v_0^+(x_1,0) + \varepsilon^{\alpha-m}v_{\alpha-m}^+(x_1,0) + \dots + \varepsilon\eta_2\partial_{x_2}v_0^+(x_1,0) + \\ + \varepsilon^{\alpha-m+1}\eta_2\partial_{x_2}v_{\alpha-m}^+(x_1,0) + \dots \end{aligned} \tag{2.38}$$

Thus, the leading terms in (2.37) and (2.38) coincide at ε^0, $\varepsilon^{\alpha-m}$, ε and $\varepsilon^{\alpha-m+1}$.

To match (2.3) and (2.4) we write down the asymptotics of (2.3) as $x_2\to-0$ and pass to the fast variables; as a result we get

$$\begin{aligned} v(\varepsilon,x) = v_0^-(x_1) + \varepsilon^{\alpha-m}v_{\alpha-m}^-(x_1) + \dots + \varepsilon\Big(\underbrace{\Phi_1(x_1)} + T(\eta_1)\partial_{x_1}v_0^-(x_1)\Big) + \\ + \varepsilon^{\alpha-m+1}\Big(\Phi_{\alpha-m+1}(x_1) + T(\eta_1)\partial_{x_1}v_{\alpha-m}^-(x_1)\Big) + \dots \end{aligned} \tag{2.39}$$

Keeping in mind the asymptotics of the functions $Z_1^{(k)}$, $X_{\alpha-m+1}^{(k)}$, $(k=0,1,2)$, $Z_{\alpha-m+1}^{(0)}$ and $Z_{\alpha-m+1}^{(2)}$ as $\eta_2\to-\infty$, we find the following asymptotics of (2.4):

$$\begin{aligned} v(\varepsilon,x) &= v_0^+(x_1,0) + \varepsilon^{\alpha-m}v_{\alpha-m}^+(x_1,0) + \dots + \\ &+ \varepsilon\Big(T(\eta_1)\partial_{x_1}v_0^+(x_1,0) + \overbrace{\frac{\eta_2}{h_2}\,\partial_{x_2}v_0^+(x_1,0)} + \underbrace{C_1^{(2)}\,\partial_{x_2}v_0^+(x_1,0)} + \\ &+ \overbrace{\frac{4h_1l_1\lambda_0}{h_2}\eta_2\,v_0^+(x_1,0)} + \underbrace{C_1^{(0)}\,v_0^+(x_1,0)}\Big) + \varepsilon^{\alpha-m+1}\Big(\Big(\frac{\eta_2}{h_2} + C_1^{(2)}\Big)\partial_{x_2}v_0^+(x_1,0) + \\ &+ \Big(\frac{4h_1l_1\lambda_{\alpha-m}}{h_2}\eta_2 + C_{\alpha-m+1}^{(0)}\Big)v_0^+(x_1,0) + \Big(\frac{4h_1l_1\lambda_0}{h_2}\eta_2 + C_1^{(0)}\Big)v_{\alpha-m}^+(x_1,0) + \\ &+ T(\eta_1)\partial_{x_1}v_{\alpha-m}^+(x_1,0) + \Big(\frac{\eta_2}{h_2} + C_1^{(2)}\Big)\partial_{x_2}v_{\alpha-m}^+(x_1,0)\Big) + \dots, \end{aligned} \tag{2.40}$$

where $T(\eta_1) = -\eta_1 + \frac{1}{2} + [\eta_1]$ and $[\eta_1]$ is the entire part of the number η_1.

Equating the corresponding coefficients in (2.39) and (2.40) at ε^0 and $\varepsilon^{\alpha-m}$, we get

$$v_0^+(x_1,0) = v_0^-(x_1), \qquad v_{\alpha-m}^+(x_1,0) = v_{\alpha-m}^-(x_1), \quad x_1\in(0,a). \tag{2.41}$$

The same procedure at ε^1 brings us the following relations:

$$\partial_{x_2}v_0^+(x_1,0) + 4h_1l_1\lambda_0\,v_0^+(x_1,0) = 0, \quad x_1\in(0,a), \tag{2.42}$$

for the over-braced terms, and

$$\Phi_1(x_1) = C_1^{(2)}\partial_{x_2}v_0^+(x_1,0) + C_1^{(0)}v_0^+(x_1,0), \quad x_1\in(0,a), \tag{2.43}$$

for the under-braced terms. Moreover, taking (2.14) into account, we have

$$v_1^-(x_1,\tfrac{x_1}{\varepsilon}) = \Phi_1(x_1) + T(\tfrac{x_1}{\varepsilon})\,\partial_{x_1}v_0^+(x_1,0), \qquad x\in G_\varepsilon^{(2)}. \tag{2.44}$$

In analogous way

$$\Phi_{\alpha-m+1}(x_1) = C_1^{(2)}\partial_{x_2}v_{\alpha-m}^+(x_1,0) + C_1^{(0)}v_{\alpha-m}^+(x_1,0) + C_1^{(2)}\partial_{x_2}v_0^+(x_1,0) + C_{\alpha-m+1}^{(0)}v_0^+(x_1,0). \tag{2.45}$$

Moreover, taking (2.15) into account, we have

$$v^-_{\alpha-m+1}(x_1, \tfrac{x_1}{\varepsilon}) = \Phi_{\alpha-m+1}(x_1) + T(\tfrac{x_1}{\varepsilon})\, \partial_{x_1} v^+_{\alpha-m}(x_1, 0), \qquad x \in G^{(2)}_\varepsilon. \tag{2.46}$$

Finally,

$$\partial_{x_2} v^+_{\alpha-m+1}(x_1, 0) = -4h_1 l_1 \lambda_0\, v^+_{\alpha-m}(x_1, 0) - 4h_1 l_1 \lambda_{\alpha-m}\, v^+_0(x_1, 0) - \partial_{x_2} v^+_0(x_1, 0). \tag{2.47}$$

Relation (2.42) completes problem (2.5). Thus, for v^+_0 and the number λ_0 we have the following Steklov problem:

$$\begin{cases} \Delta_x\, v^+_0(x) = 0, & x \in \Omega_0 \\ \partial_\nu\, v^+_0(x) = 0, & x \in \Gamma_2, \\ v^+_0(x) = 0, & x \in \Gamma_1, \\ \partial_{x_2} v^+_0(x_1, 0) = -4h_1 l_1 \lambda_0\, v^+_0(x_1, 0), & x_1 \in (0, a), \end{cases} \tag{2.48}$$

which called *homogenized spectral problem* for problem (1.21).

Recall that the number λ_0 is called an eigenvalue of problem (2.48) if there exists a function $v_0 \in \mathcal{H}_0 := \{u \in H^1(\Omega_0) :\ u|_{\Gamma_1} = 0\}$, $v_0 \neq 0$, which is called an eigenfunction corresponding to λ_0, such that the following integral identity holds :

$$\langle v_0,\, \varphi\rangle_{\mathcal{H}_0} = \lambda_0\, (\mathcal{T}_0 v_0, \mathcal{T}_0 \varphi)_{\mathcal{V}_0} \qquad \forall\ \varphi \in \mathcal{H}_0, \tag{2.49}$$

where $\langle v_0,\, \varphi\rangle_{\mathcal{H}_0} := \int_{\Omega_0} \nabla v_0 \cdot \nabla \varphi\, dx$ is the scalar product in $\mathcal{H}_0$; the space $\mathcal{V}_0$ is the weighted space $L_2(I_0)$ with the following scalar product

$$(\mathcal{T}_0 v_0, \mathcal{T}_0 \varphi)_{\mathcal{V}_0} := 4h_1 l_1\, (\mathcal{T}_0 v_0, \mathcal{T}_0 \varphi)_{L_2(I_0)};$$

and $\mathcal{T}_0 : \mathcal{H}_0 \mapsto \mathcal{V}_0$ is the trace operator.

Let $A_0 \equiv \mathcal{T}_0 \circ \mathcal{T}^*_0 : \mathcal{V}_0 \longmapsto \mathcal{V}_0$, where $\mathcal{T}^*_0$ is the conjugate operator to $\mathcal{T}_0$. It is easy to verify (see for instance [17]) that A_0 is self-adjoint, positive, compact, and the spectral problem (2.48) is equivalent to the spectral problem

$$A_0\, (\mathcal{T}_0 v_0) = \frac{1}{\lambda_0}\, \mathcal{T}_0 v_0 \quad \text{in} \ \ \mathcal{V}_0. \tag{2.50}$$

Thus, the eigenvalues of problem (2.48) form the sequence

$$0 < \lambda^{(1)}_0 < \lambda^{(2)}_0 \leq \ldots \leq \lambda^{(n)}_0 \leq \cdots \to +\infty \quad \text{as} \quad n \to \infty \tag{2.51}$$

with the classical convention of repeated eigenvalues. The respective sequence of the corresponding eigenfunctions $\{v^{+,n}_0\}_{n\in\mathbb{N}}$ can be orthonormalized as follows:

$$4h_1 l_1 \int\limits_{I_0} v^{+,n}_0(x_1, 0)\ v^{+,k}_0(x_1, 0)\, dx_1 = \delta_{n,k}, \quad \{n,\, k\} \in \mathbb{N}. \tag{2.52}$$

Next, let λ_0 be an eigenvalue of problem (2.48), v^+_0 is the corresponding eigenfunction normalized by (2.52).

Analogously, relation (2.47) completes problem (2.6). Thus, for $v^+_{\alpha-m}$ and $\lambda_{\alpha-m}$ we get the following problem:

$$\begin{cases} \Delta_x\, v^+_{\alpha-m}(x) = 0, & x\in\Omega_0;\\ \partial_\nu\, v^+_{\alpha-m}(x) = 0, \quad x\in\Gamma_2; & v^+_{\alpha-m}(x)=0, \quad x\in\Gamma_1;\\ \partial_{x_2} v^+_{\alpha-m}(x_1,0) = -\,4h_1 l_1\lambda_0\, v^+_{\alpha-m}(x_1,0) - 4h_1 l_1 \lambda_{\alpha-m}\, v^+_0(x_1,0) - \partial_{x_2} v^+_0(x_1,0). \end{cases} \tag{2.53}$$

Since λ_0 is an eigenvalue of problem (2.48), we choose the number $\lambda_{\alpha-m}$ to satisfy the solvability condition for the problem (2.53). Writing down the integral identity (2.49) for problem (2.48) with the test-function $v^+_{\alpha-m}$ and the respective integral identity of problem (2.53) with the test-function v^+_0, then subtracting them and bearing in mind (2.52) and (2.55), we get

$$\lambda_{\alpha-m} = -\frac{1}{4h_1 l_1}\int\limits_{I_0} v^+_0(x_1,0)\ \partial_{x_2} v^+_0(x_1,0)\ dx_1. \tag{2.54}$$

Obviously, the solution to problem (2.53) is not uniquely defined and for the uniqueness we demand the following orthogonality condition:

$$\int\limits_{I_0} v^+_{\alpha-m}(x_1,0)\, v^+_0(x_1,0)\, dx_1 = 0. \tag{2.55}$$

2.3 Global asymptotic approximation in Ω_ε and estimation of its residuals

For any given eigenvalue λ_0 of the homogenized spectral problem (2.48) and the corresponding eigenfunction v^+_0 normalized by (2.52), we can define $\lambda_{\alpha-m}$ with the help of (2.54) and the unique solutions $v^+_{\alpha-m}$ to problem (2.53).

An approximating function R_ε is constructed as the sum of the first terms of outer expansions (2.2), (2.3) and inner expansion (2.4) with the subtraction of the identical terms of their asymptotics (as $x_2 \to \pm 0$ and $\eta_2 \to \pm\infty$ respectively) because they are summed twice. Taking (2.41) into account, we obtain

$$R_\varepsilon(x) = \begin{cases} v^+_0(x) + \varepsilon^{\alpha-m} v^+_{\alpha-m}(x) + \chi_0(x_2)\, \mathcal{N}^+_\varepsilon(x_1, \frac{x}{\varepsilon}), & x\in\Omega_0,\\ v^+_0(x_1,0) + \varepsilon^{\alpha-m} v^+_{\alpha-m}(x_1,0) + \mathcal{N}^-_{1,\varepsilon}(x_1,\frac{x}{\varepsilon}), & x\in G^{(1)}_\varepsilon,\\ v^+_0(x_1,0) + \varepsilon^{\alpha-m} v^+_{\alpha-m}(x_1,0) + \varepsilon\big(\Phi_1(x_1) + T(\frac{x_1}{\varepsilon})\partial_{x_1} v^+_0(x_1,0)\big) & \\ \quad + \varepsilon^{\alpha-m+1}\big(\Phi_{\alpha-m+1}(x_1) + T(\frac{x_1}{\varepsilon})\partial_{x_1} v^+_{\alpha-m}(x_1,0)\big) + \chi_0(x_2)\, \mathcal{N}^-_{2,\varepsilon}(x_1,\frac{x}{\varepsilon}), & x\in G^{(2)}_\varepsilon, \end{cases} \tag{2.56}$$

where χ_0 is a smooth cut-off function such that $\chi_0(x_2) = 1$ for $|x_2| \le \tau_0/2$, and $\chi_0(x_2) = 0$ for

$|x_2| \geq \tau_0$ ($\tau_0 < \min\{\delta_0, l_2\}$ (see subsection 1.1));

$$\mathcal{N}_\varepsilon^+(x_1,\eta) = \varepsilon\Bigg(Z_1^{(0)}(\eta)\, v_0^+(x_1,0) + \sum_{i=1}^{2}\big(Z_1^{(i)}(\eta) - \delta_{i,2}\eta_2\big)\partial_{x_i} v_0^+(x_1,0)\Bigg) + \\ + \varepsilon^{\alpha-m+1}\Bigg(\Big(Z_{\alpha-m+1}^{(0)}(\eta) - \eta_2\Big)v_0^+(x_1,0) + \Big(Z_{\alpha-m+1}^{(2)}(\eta) - \eta_2\Big)\partial_{x_2} v_0^+(x_1,0) + \\ + \Big(X_{\alpha-m+1}^{(0)}(\eta) - \eta_2\Big)v_{\alpha-m}^+(x_1,0) + \sum_{i=1}^{2}\Big(X_{\alpha-m+1}^{(i)}(\eta) - \delta_{i,2}\,\eta_2\Big)\partial_{x_i} v_{\alpha-m}^+(x_1,0)\Bigg), \quad (2.57)$$

$$\mathcal{N}_{1,\varepsilon}^-(x_1,\eta) = \varepsilon\Bigg(Z_1^{(0)}(\eta) v_0^+(x_1,0) + \sum_{i=1}^{2} Z_1^{(i)}(\eta)\partial_{x_i} v_0^+(x_1,0)\Bigg) + \\ + \varepsilon^{\alpha-m+1}\Bigg(Z_{\alpha-m+1}^{(0)}(\eta) v_0^+(x_1,0) + Z_{\alpha-m+1}^{(2)}(\eta)\partial_{x_2} v_0^+(x_1,0) + \\ + X_{\alpha-m+1}^{(0)}(\eta) v_{\alpha-m}^+(x_1,0) + \sum_{i=1}^{2} X_{\alpha-m+1}^{(i)}(\eta)\partial_{x_i} v_{\alpha-m}^+(x_1,0)\Bigg) \quad (2.58)$$

and

$$\mathcal{N}_{2,\varepsilon}^-(x_1,\eta) = \varepsilon\Bigg(\big(Z_1^{(1)}(\eta) - T(\eta_1)\big)\partial_{x_1} v_0^+(x_1,0) + \Big(Z_1^{(2)}(\eta) - \frac{\eta_2}{h_2} - C_1^{(2)}\Big)\partial_{x_2} v_0^+(x_1,0) + \\ + \Big(Z_1^{(0)}(\eta) - \frac{4h_1 l_1 \lambda_0}{h_2}\,\eta_2 - C_1^{(0)}\Big)v_0^+(x_1,0)\Bigg) + \\ + \varepsilon^{\alpha-m+1}\Bigg(\Big(Z_{\alpha-m+1}^{(0)}(\eta) - \frac{4h_1 l_1 \lambda_{\alpha-m}}{h_2}\,\eta_2\Big)v_0^+(x_1,0) + \Big(Z_{\alpha-m+1}^{(2)}(\eta) - \frac{\eta_2}{h_2}\Big)\partial_{x_2} v_0^+(x_1,0) + \\ + \Big(X_{\alpha-m+1}^{(0)}(\eta) - \frac{4h_1 l_1 \lambda_0}{h_2}\,\eta_2\Big)v_{\alpha-m}^+(x_1,0) + \\ + \big(X_{\alpha-m+1}^{(1)}(\eta) - T(\eta_1)\big)\,\partial_{x_1} v_{\alpha-m}^+(x_1,0) + \Big(X_{\alpha-m+1}^{(2)}(\eta) - \frac{\eta_2}{h_2}\Big)\partial_{x_2} v_{\alpha-m}^+(x_1,0)\Bigg). \quad (2.59)$$

Due to (2.43) and (2.45) it is easy to verify that $R_\varepsilon|_{x_2=0+} = R_\varepsilon|_{x_2=0-}$ on Q_ε, i.e., $R_\varepsilon \in H^1(\Omega_\varepsilon;\Gamma_1)$. Also using (2.42) and (2.47), one can verify that

$$\partial_{x_2} R_\varepsilon|_{x_2=0+} = \partial_{x_2} R_\varepsilon|_{x_2=0-} \quad \text{on} \quad Q_\varepsilon. \quad (2.60)$$

2.3.1 Discrepancies in the equation of problem (1.21).

Substituting R_ε and $\lambda_0 + \varepsilon^{\alpha-m}\lambda_{\alpha-m}$ in the differential equation of problem (1.21) instead of $v(\varepsilon,\cdot)$ and $\Lambda(\varepsilon)$ respectively, and calculating discrepancies with regard to problems (2.20)–(2.22) and (2.48) and (2.53), we get

$$\Delta_x R_\varepsilon(x) + \varepsilon^{\alpha-1}\big(\lambda_0 + \varepsilon^{\alpha-m}\lambda_{\alpha-m}\big)R_\varepsilon(x) = \varepsilon^{\alpha-1}\big(\lambda_0 + \varepsilon^{\alpha-m}\lambda_{\alpha-m}\big)R_\varepsilon(x) + \\ + \varepsilon^{-1}\chi_0'(x_2)\big(\partial_{\eta_2}\mathcal{N}_\varepsilon^+(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} + \partial_{x_2}\big(\chi_0'(x_2)\mathcal{N}_\varepsilon^+(x_1,\tfrac{x}{\varepsilon})\big) + \\ + \varepsilon^{-1}\chi_0(x_2)\big(\partial^2_{x_1\eta_1}\mathcal{N}_\varepsilon^+(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} + \chi_0(x_2)\partial_{x_1}\Big(\big(\partial_{x_1}\mathcal{N}_\varepsilon^+(x_1,\eta)\big)\big|_{\eta=x/\varepsilon}\Big) \quad \text{in} \quad \Omega_0; \quad (2.61)$$

$$\Delta_x R_\varepsilon(x) + \varepsilon^{-1}\big(\lambda_0 + \varepsilon^{\alpha-m}\lambda_{\alpha-m}\big) R_\varepsilon(x) = \varepsilon^{\alpha-m}\partial^2_{x_1x_1} v^+_{\alpha-m}(x_1,0) + \varepsilon^{2\alpha-2m-1}\lambda_{\alpha-m} v^+_{\alpha-m}(x_1,0) + \\ + \varepsilon^{-1}\big(\lambda_0 + \varepsilon^{\alpha-m}\lambda_{\alpha-m}\big)\mathcal{N}^-_{1,\varepsilon}(x_1,\tfrac{x}{\varepsilon}) + \varepsilon^{-1}\big(\partial^2_{x_1\eta_1}\mathcal{N}^-_{1,\varepsilon}(x_1,\eta)\big)\big|_{\eta=\frac{x}{\varepsilon}} + \\ + \partial_{x_1}\Big(\big(\partial_{x_1}\mathcal{N}^-_{1,\varepsilon}(x_1,\eta)\big)\big|_{\eta=\frac{x}{\varepsilon}}\Big) \quad \text{in} \quad G^{(1)}_\varepsilon; \quad (2.62)$$

and

$$\Delta_x R_\varepsilon(x) + \varepsilon^{\alpha-1}\big(\lambda_0 + \varepsilon^{\alpha-m}\lambda_{\alpha-m}\big) R_\varepsilon(x) = \\ = \varepsilon\partial_{x_1}\big(\partial_{x_1}\Phi_1(x_1) + T(\tfrac{x_1}{\varepsilon})\partial^2_{x_1x_1} v^+_0(x_1,0)\big) + \varepsilon^{\alpha-m+1}\partial_{x_1}\big(\partial_{x_1}\Phi_{\alpha-m+1}(x_1) + T(\tfrac{x_1}{\varepsilon})\partial^2_{x_1x_1} v^+_{\alpha-m}(x_1,0)\big) + \\ + \varepsilon^{\alpha-1}\big(\lambda_0 + \varepsilon^{\alpha-m}\lambda_{\alpha-m}\big) R_\varepsilon(x) + \varepsilon^{-1}\chi_0'(x_2)\big(\partial_{\eta_2}\mathcal{N}^-_{2,\varepsilon}(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} + \partial_{x_2}\big(\chi_0'(x_2)\mathcal{N}^-_{2,\varepsilon}(x_1,\tfrac{x}{\varepsilon})\big) + \\ + \varepsilon^{-1}\chi_0(x_2)\big(\partial^2_{x_1\eta_1}\mathcal{N}^-_{2,\varepsilon}(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} + \chi_0(x_2)\partial_{x_1}\Big(\big(\partial_{x_1}\mathcal{N}^-_{2,\varepsilon}(x_1,\eta)\big)\big|_{\eta=x/\varepsilon}\Big) \quad \text{in} \quad G^{(2)}_\varepsilon. \quad (2.63)$$

2.3.2 Discrepancies on the boundary.

It is easy to check that $R_\varepsilon = 0$ on Γ_1 and $\partial_\nu R_\varepsilon = 0$ on the whole boundary $\partial\Omega_\varepsilon \setminus \Gamma_1$, except its vertical parts, on which

$$\partial_{x_1} R_\varepsilon(x) = \chi_0(x_2)\big(\partial_{x_1}\mathcal{N}^+_\varepsilon(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} \quad (2.64)$$

on the vertical parts of $\partial\Omega_0$,

$$\partial_{x_1} R_\varepsilon(x) = \big(\partial_{x_1}\mathcal{N}^-_{1,\varepsilon}(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} \quad (2.65)$$

on the vertical parts of $\partial G^{(1)}_\varepsilon$, and

$$\partial_{x_1} R_\varepsilon(x) = \varepsilon\big(\partial_{x_1}\Phi_1(x_1) + T(\tfrac{x_1}{\varepsilon})\partial^2_{x_1x_1} v^+_0(x_1,0)\big) \\ + \varepsilon^{\alpha-m+1}\big(\partial_{x_1}\Phi_{\alpha-m+1}(x_1) + T(\tfrac{x_1}{\varepsilon})\partial^2_{x_1x_1} v^+_{\alpha-m+1}(x_1,0)\big) + \chi_0(x_2)\big(\partial_{x_1}\mathcal{N}^-_{2,\varepsilon}(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} \quad (2.66)$$

on the vertical parts of $\partial G^{(2)}_\varepsilon$.

2.3.3 Discrepancies in the integral identity.

Multiplying (2.61)-(2.63) with arbitrary function $\psi \in \mathcal{H}_\varepsilon$, integrating by parts and taking (2.60) and (2.64)-(2.66) into account, we deduce

$$-\int_{\Omega_\varepsilon} \nabla_x R_\varepsilon \cdot \nabla_x \psi\, dx + \varepsilon^{\alpha-1}\big(\lambda_0 + \varepsilon^{\alpha-m}\lambda_{\alpha-m}\big)\int_{\Omega_0\cup G^{(2)}_\varepsilon} R_\varepsilon\,\psi\, dx + \\ + \varepsilon^{-1}\big(\lambda_0 + \varepsilon^{\alpha-m}\lambda_{\alpha-m}\big)\int_{G^{(1)}_\varepsilon} R_\varepsilon \psi\, dx = \ell_\varepsilon(\psi), \quad (2.67)$$

where the linear functional ℓ_ε is defined as follows:

$$\ell_\varepsilon(\psi) := \varepsilon^{\alpha-1}\big(\lambda_0 + \varepsilon^{\alpha-m}\lambda_{\alpha-m}\big)\int\limits_{\Omega_0} R_\varepsilon\,\psi\, dx +$$

$$+\varepsilon^{2\alpha-2m-1}\lambda_{\alpha-m}\int\limits_{G_\varepsilon^{(1)}} v_{\alpha-m}^+(x_1,0)\,\psi\,dx+\varepsilon^{\alpha-m}\int\limits_{G_\varepsilon^{(1)}}\partial^2_{x_1x_1}v^+_{\alpha-m}(x_1,0)\,\psi\,dx+$$

$$+\varepsilon^{-1}\big(\lambda_0+\varepsilon^{\alpha-m}\lambda_{\alpha-m}\big)\int\limits_{G_\varepsilon^{(1)}}\mathcal{N}^-_{1,\varepsilon}(x_1,\tfrac{x}{\varepsilon})\,\psi\,dx+\varepsilon^{-1}\int\limits_{G_\varepsilon^{(1)}}\big(\partial^2_{x_1\eta_1}\mathcal{N}^-_{1,\varepsilon}(x_1,\eta)\big)|_{\eta=\frac{x}{\varepsilon}}\,\psi\,dx-$$

$$-\varepsilon\int\limits_{G_\varepsilon^{(2)}}\big(\partial_{x_1}\Phi_1(x_1)+T(\tfrac{x_1}{\varepsilon})\partial^2_{x_1x_1}v_0^+(x_1,0)\big)\,\partial_{x_1}\psi\,dx-$$

$$-\varepsilon^{\alpha-m+1}\int\limits_{G_\varepsilon^{(2)}}\big(\partial_{x_1}\Phi_{\alpha-m+1}(x_1)+T(\tfrac{x_1}{\varepsilon})\partial^2_{x_1x_1}v^+_{\alpha-m}(x_1,0)\,\partial_{x_1}\psi\,dx+$$

$$+\varepsilon^{\alpha-1}\big(\lambda_0+\varepsilon^{\alpha-m}\lambda_{\alpha-m}\big)\int\limits_{G_\varepsilon^{(2)}}R_\varepsilon\,\psi\,dx+$$

$$+\varepsilon^{-1}\int\limits_{\Omega_0\cup G_\varepsilon^{(2)}}\chi_0'(x_2)\big(\partial_{\eta_2}\mathcal{N}_\varepsilon(x_1,\eta)\big)|_{\eta=\frac{x}{\varepsilon}}\,\psi\,dx-\int\limits_{\Omega_0\cup G_\varepsilon^{(2)}}\chi_0'(x_2)\mathcal{N}_\varepsilon(x_1,\frac{x}{\varepsilon})\,\partial_{x_2}\psi\,dx+$$

$$+\varepsilon^{-1}\int\limits_{\Omega_0\cup G_\varepsilon^{(2)}}\chi_0(x_2)\big(\partial^2_{x_1\eta_1}\mathcal{N}_\varepsilon(x_1,\eta)\big)|_{\eta=\frac{x}{\varepsilon}}\,\psi\,dx-\int\limits_{\Omega_0\cup G_\varepsilon^{(2)}}\chi_0(x_2)\big(\partial_{x_1}\mathcal{N}_\varepsilon(x_1,\eta)\big)|_{\eta=\frac{x}{\varepsilon}}\,\partial_{x_1}\psi\,dx-$$

$$-\int\limits_{G_\varepsilon^{(1)}}\big(\partial_{x_1}\mathcal{N}^-_{1,\varepsilon}(x_1,\eta)\big)|_{\eta=\frac{x}{\varepsilon}}\,\partial_{x_1}\psi\,dx. \tag{2.68}$$

Here $\mathcal{N}_\varepsilon$ coincides with $\mathcal{N}_\varepsilon^+$ on Ω_0 and with $\mathcal{N}^-_{2,\varepsilon}$ on $G_\varepsilon^{(2)}$.

Let us estimate $|\ell_\varepsilon(\psi)|$. It is easy to see that the integral in the first line of (2.68) is of order $\mathcal{O}(\varepsilon^{\alpha-1})$.

The integrals in second line can be estimated with the help of the following Friedrichs-type inequality:

$$\varepsilon^{-1}\int_{G_\varepsilon^{(1)}}v^2\,dx\le C_3\int_{\Omega_\varepsilon}|\nabla v|^2\,dx\quad\forall\,v\in\mathcal{H}_\varepsilon, \tag{2.69}$$

proved in [32], by the following way:

$$\varepsilon^{2\alpha-2m-1}\Bigg|\lambda_{\alpha-m}\int\limits_{G_\varepsilon^{(1)}}v^+_{\alpha-m}(x_1,0)\,\psi\,dx\Bigg|\le\varepsilon^{2\alpha-2m-\frac12}C_1\|\psi\|_{L^2(G_\varepsilon^{(1)})}\le\varepsilon^{2(\alpha-m)}C_2\|\psi\|_{\mathcal{H}_\varepsilon}.$$

The main term in the first integral of the third line of (2.68) we bound again with the help of (2.69) as follows:

$$\lambda_0\Big|\int_{G_\varepsilon^{(1)}}Z_1^{(0)}(\tfrac{x}{\varepsilon})\,v_0^+(x_1,0)\,\psi\,dx\Big|\le\sqrt{\varepsilon}C_1\|\psi\|_{\mathcal{H}_\varepsilon}\sqrt{\int_{G_\varepsilon^{(1)}}\Big|Z_1^{(0)}(\tfrac{x}{\varepsilon})\Big|^2\,dx}\le$$

$$\le\varepsilon C_2\|\psi\|_{\mathcal{H}_\varepsilon}\sqrt{\int_{\Pi_{l_1}}\Big|Z_1^{(0)}(\eta)\Big|^2\,d\eta}\le\varepsilon C_3\|\psi\|_{\mathcal{H}_\varepsilon}. \tag{2.70}$$

Similarly we estimate the second integral in this line and it is of order $\mathcal{O}(\varepsilon)$ as well.

One can verify that the integral from the forth line is of order $\mathcal{O}(\varepsilon)$, the integral from the fifth line is of order $\mathcal{O}(\varepsilon^{\alpha-m+1})$, and the integral from the sixth line of (2.68) is of order $\mathcal{O}(\varepsilon^{\alpha-1})$.

Due to the asymptotic relations (2.29)-(2.31), the first integral in the seventh line of (2.68) is exponentially small and the second one is of order $\mathcal{O}(\varepsilon^{\alpha-m+1})$. Thanks to Lemma 3.1 ([14]) the integrals in the eighth line of (2.68) are of order $\mathcal{O}(\varepsilon^{1-\delta})$, where δ is arbitrary positive number.

Obviously, that the integral in the last line is of order $\mathcal{O}(\varepsilon^{\frac{3}{2}})$.

Thus, we have

$$|\ell_\varepsilon(\psi)| \le c_1(\delta)\,\varepsilon^{2(\alpha-m)}\|\psi\|_{\mathcal{H}_\varepsilon} \ \text{ if } \ \alpha\in(m,m+\tfrac{1}{2}), \quad |\ell_\varepsilon(\psi)| \le c_2(\delta)\,\varepsilon^{1-\delta}\|\psi\|_{\mathcal{H}_\varepsilon} \ \text{ if } \ \alpha\in[m+\tfrac{1}{2},m+1). \tag{2.71}$$

With the help of operator $A_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_\varepsilon$ defined in (1.22) we deduce from (2.67) and (2.71) the following inequality:

$$\| \, R_\varepsilon \, - \, (\lambda_0 + \varepsilon^{\alpha-m}\lambda_{\alpha-m})A_\varepsilon R_\varepsilon \, \|_{\mathcal{H}_\varepsilon} \, \le \, c(\delta)\,\varepsilon^{\nu(\alpha)}, \tag{2.72}$$

where $\nu(\alpha)=2(\alpha-m)$ if $\alpha\in(m,m+\frac{1}{2})$, or $\nu(\alpha)=1-\delta$ if $\alpha\in[m+\frac{1}{2},m+1)$; δ is arbitrary positive number small enough.

3 Justification of the asymptotics

To justify the asymptotic approximations constructed above, we use the scheme proposed in [17] for investigation of the asymptotic behavior of the eigenvalues and eigenfunctions of an family of abstract operators $\{A_\epsilon : H_\epsilon \mapsto H_\epsilon\}_{\epsilon>0}$ in the limit passage as $\epsilon\to 0$. This scheme generalizes a procedure of justification of the asymptotic behavior of eigenvalues and eigenfunctions of boundary value problems in perturbed domains that was proposed in [7].

In our case this is the family of operators $\{A_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_\varepsilon\}_{\varepsilon>0}$ defined in (1.22). Recall that the operator A_ε corresponds to problem (1.21).

For thick junctions there exist no extension operators that would be bounded uniformly in ε in the Sobolev space H^1 (see [14]). But as was shown in [14], for eigenfunctions of spectral problems in thick junctions it was possible to construct special extensions that are bounded on each eigenfunction. A such extension operator was constructed for eigenfunctions of problem (1.1) in the case when the parameter $\alpha\in(0,1]$ in our papers [1, 2]. Repeating word for word the proof of Theorem 4.1 (see [1, 2]), we get the following result.

Theorem 3.1. *There exists an extension operator $\mathbf{P}_\varepsilon : \mathcal{H}_\varepsilon \mapsto H^1(\Omega,\Gamma_1)$ such that for any eigenfunction $v_n(\varepsilon,\cdot)$ normalized by (1.24) there exist positive constants C_n and ε_n such that for all values of the parameter ε from the interval $(0,\varepsilon_n)$ the following estimates hold:*

$$\| \, \mathbf{P}_\varepsilon v_n(\varepsilon,\cdot) \, \|_{H^1(\Omega,\Gamma_1)} \le \, C_n \, \| \, v_n(\varepsilon,\cdot) \, \|_{\mathcal{H}_\varepsilon} \le \, C_n', \tag{3.1}$$

where Ω is the interior of the union $\overline{\Omega}_0 \cup \overline{D}_2$.

3.1 Condition $\mathbf{D_1} - \mathbf{D_5}$

For the convenience of readers we write here the conditions of the scheme from paper [17], which are modified under problems (1.21) and (2.48).

Let $N(\frac{1}{\mu}, \mathrm{A}_0)$ denote the proper subspace corresponding to the eigenvalue $\frac{1}{\mu}$ of operator A_0 defined in (2.50) and let $\{(v_n(\varepsilon,\cdot), \Lambda_n(\varepsilon))\}_{\varepsilon>0}$ denote the sequence whose components are the eigenfunction v_n ($\|v_n\|_{\mathcal{V}_\varepsilon} = 1$) and the corresponding characteristic number of operator A_ε.

Condition $\mathbf{D_1}$. There exists a linear operator $\mathrm{S}_\varepsilon : \mathcal{H}_0 \mapsto \mathcal{H}_\varepsilon$ such that

$$\|\mathrm{S}_\varepsilon u\|_{\mathcal{H}_\varepsilon} \le c_1 \|u\|_{\mathcal{H}_0}, \quad \forall u \in \mathcal{H}_0,$$

where the constant c_1 is independent of ε and u.

Condition $\mathbf{D_2}$. There exists a linear operator $\mathrm{P}_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_0$ such that

$$\forall\, n \in \mathbb{N} \;\; \exists\, c_2 > 0 \;\; \exists\, \varepsilon_0 > 0 \;\; \forall \varepsilon \in (0, \varepsilon_0): \quad \|\mathrm{P}_\varepsilon v_n(\varepsilon,\cdot)\|_{\mathcal{H}_0} \le c_2 \, \|v_n(\varepsilon,\cdot)\|_{\mathcal{H}_\varepsilon}.$$

Condition $\mathbf{D_3}$. For an arbitrary sequence $\{(v_n(\varepsilon,\cdot), \Lambda_n(\varepsilon))\}_{\varepsilon>0}$ and any subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$, such that $\mathrm{P}_{\varepsilon'} v_n(\varepsilon',\cdot) \to v_n^0$ weakly in $\mathcal{H}_0$, one has

$$\lim_{\varepsilon' \to 0} \left(v_n(\varepsilon',\cdot), \mathrm{S}_{\varepsilon'}\varphi\right)_{\mathcal{H}_{\varepsilon'}} = \left(v_n^0, \varphi\right)_{\mathcal{H}_0} \quad \forall \varphi \in \mathcal{H}_0.$$

Condition $\mathbf{D_4}$. If for certain functions $w^\varepsilon, v^\varepsilon \in \mathcal{H}_\varepsilon$ one has $\mathrm{P}_\varepsilon w^\varepsilon \to w^0$ and $\mathrm{P}_\varepsilon v^\varepsilon \to v^0$ weakly in $\mathcal{H}_0$ as $\varepsilon \to 0$, then

$$\lim_{\varepsilon \to 0} \left(w^\varepsilon, v^\varepsilon\right)_{\mathcal{V}_\varepsilon} = \left(\mathcal{T}_0 w^0, \mathcal{T}_0 v^0\right)_{\mathcal{V}_0}.$$

If $v \in \mathcal{H}_0$, then $\mathrm{P}_\varepsilon(\mathrm{S}_\varepsilon v) \to v$ weakly in $\mathcal{H}_0$ as $\varepsilon \to 0$.

Condition $\mathbf{D_5}$. There exists a number $\varpi_0 > 0$ such that for any $\frac{1}{\mu} \in \sigma(A_0)$ there exists a linear operator $\mathcal{R}_\varepsilon : N(\frac{1}{\mu}, A_0) \mapsto \mathcal{H}_\varepsilon$ such that for every eigenfunction $v \in N(\frac{1}{\mu}, A_0)$, normalized by $\|\mathcal{T}_0 v\|_{\mathcal{V}_0} = 1$, we have

$$\mathcal{R}_\varepsilon v = \mathrm{S}_\varepsilon v + \mathcal{O}(\varepsilon) \;\text{ in } \mathcal{V}_\varepsilon \quad \text{and} \quad \|\mathcal{R}_\varepsilon v\|_{\mathcal{H}_\varepsilon} = c_v + \mathcal{O}(\varepsilon);$$

in addition, there exist constants c_3, ε_0, ρ, τ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\|\mathcal{R}_\varepsilon v - (\mu + \varepsilon^\tau \rho)\, A_\varepsilon(\mathcal{R}_\varepsilon v)\|_{\mathcal{H}_\varepsilon} \le c_3 \varepsilon^{\varpi_0}.$$

To clarify these conditions, we use the following diagram:

$$\begin{array}{ccc}
\mathcal{H}_\varepsilon & \xrightarrow{J_\varepsilon} & \mathcal{V}_\varepsilon \\
\mathrm{P}_\varepsilon \big\downarrow \quad \big\uparrow \mathrm{S}_\varepsilon & & \\
\mathcal{H}_0 & \xrightarrow{\mathcal{T}_0} & \mathcal{V}_0
\end{array}$$

where operator $J_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{V}_\varepsilon$ is the identical imbedding operator, operator $\mathcal{T}_0 : \mathcal{H}_0 \mapsto \mathcal{V}_0$ is the trace operator (see (2.49)). Conditions $\mathbf{D_1}$ and $\mathbf{D_2}$ are some connection conditions between spaces $\mathcal{H}_\varepsilon$ and $\mathcal{H}_0$ that defined in subsection 1.1 and 2.2 respectively. If conditions $\mathbf{D_3}$ and $\mathbf{D_4}$ are satisfied, then it means that spectral problem (2.48) is the homogenized problem for problem (1.21). Condition $\mathbf{D_5}$ means that it is possible to construct asymptotic approximations near points of the spectrum of operator A_0.

Now let us verify conditions $\mathbf{D_1}$ – $\mathbf{D_5}$ for our problems (1.21) and (2.48). The operator $S_\varepsilon : \mathcal{H}_0 \mapsto \mathcal{H}_\varepsilon$ assigns to each function $v \in \mathcal{H}_0$ its bounded extension Ev to $H^1(\Omega, \Gamma_1)$ and then restricts Ev to Ω_ε, i.e., $S_\varepsilon = (Ev)|_{\Omega_\varepsilon}$. Clearly, S_ε is uniformly bounded with respect to ε. Thus condition $\mathbf{D_1}$ is satisfied.

The operator $\mathrm{P}_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_0$ from condition $\mathbf{D_2}$ is associated with the restriction of the extension operator $\mathbf{P}_\varepsilon$ from Theorem 3.1 to domain Ω_0, i.e. $\mathrm{P}_\varepsilon v_n = (\mathbf{P}_\varepsilon v_n)|_{\Omega_0}$.

Let us verify condition $\mathbf{D_3}$. Consider the sequence $\{v_n(\varepsilon, \cdot)\}_{\varepsilon>0}$ for any fixed index $n \in \mathbb{N}$. Due to Theorem 3.1 there exists some subsequence $\{\varepsilon'\} \subset \{\varepsilon\}$ (again denoted by $\{\varepsilon\}$) such that $\mathbf{P}_\varepsilon v_n(\varepsilon, \cdot) \to v_0$ weakly in $H^1(\Omega, \Gamma_1)$ as $\varepsilon \to 0$. Since

$$\int\limits_{D_2} \chi_{h_2}(\tfrac{x_1}{\varepsilon})\, \partial_{x_2} \mathbf{P}_\varepsilon(v_n(\varepsilon, x))\, \phi(x)\, dx = -\int\limits_{D_2} \chi_{h_2}(\tfrac{x_1}{\varepsilon})\, \mathbf{P}_\varepsilon(v_n(\varepsilon, x))\, \partial_{x_2}\phi\, dx \quad \forall\, \phi \in C_0^\infty(D_2),$$

we get

$$\chi_{h_2}(\tfrac{x_1}{\varepsilon})\partial_{x_2}\mathbf{P}_\varepsilon(v_n(\varepsilon, x)) \to h_2\, \partial_{x_2} v_n^0(x) \quad \text{weakly in } \ L_2(D_2) \quad \text{as} \ \ \varepsilon \to 0. \tag{3.2}$$

Here $\chi_{h_2}(\eta_1)$ ($\eta_1 \in \mathbb{R}$) is 1-periodic function that equals 1 on the interval $\left(\frac{1-h_2}{2},\ \frac{1+h_2}{2}\right)$ and vanishing on the rest of the segment $[0, 1]$.

If we consider the corresponding integral identity for problem (1.21) with the following test function:

$$\psi(x) = \begin{cases} 0, & x \in \Omega_0 \cup G_\varepsilon^{(1)}, \\ \varepsilon T\left(\frac{x_1}{\varepsilon}\right)\phi(x), & x \in G_\varepsilon^{(2)}, \end{cases} \qquad \phi \in C_0^\infty(D_2),$$

where T is defined in (2.40), we get

$$\int\limits_{D_2} \chi_{h_2}(x_1/\varepsilon)\partial_{x_1}\mathbf{P}_\varepsilon(v_n(\varepsilon, x))\, \phi\, dx = \mathcal{O}(\varepsilon) \quad \text{as} \ \ \varepsilon \to 0. \tag{3.3}$$

Due to the second inequality in (3.1), it is easy to verify that

$$\int\limits_{G_\varepsilon^{(1)}} \nabla v_n(\varepsilon, x) \cdot \nabla\varphi(x)\, dx \to 0 \quad \text{as} \ \ \varepsilon \to 0 \quad \forall\, \varphi \in H^1(\Omega, \Gamma_1). \tag{3.4}$$

The corresponding integral identity for problem (1.21) with a test function $\phi \in C_0^\infty(D_2)$ reads as follows:

$$\int\limits_{G_\varepsilon^{(2)}} \nabla v_n(\varepsilon, x) \cdot \nabla\phi(x)\, dx = \varepsilon^{\alpha-1}\Lambda_n(\varepsilon) \int\limits_{G_\varepsilon^{(2)}} v_n(\varepsilon, x)\, \phi(x)\, dx \tag{3.5}$$

for ε small enough. Taking into account limits (3.2), (3.3) and the boundedness of $\Lambda_n(\varepsilon)$ with respect to ε (see Lemma 1.1), we deduce from (3.5) that

$$h_2 \int\limits_{D_2} \partial_{x_2} v_n^0\, \partial_{x_2}\phi\, dx = 0 \quad \forall\, \phi \in C_0^\infty(D_2),$$

i.e., $\partial_{x_2} v_n^0$ is some function of x_1 a.e. in D_2. On the other hand $\partial_{x_2} v_n^0|_{x_2=-l_2} = 0$, because $\partial_{x_2} v_n(\varepsilon,\cdot)|_{x_2=-l_2} = 0$. Therefore, $v_n^0(x) = v_n^0(x_1,0)$ for a.e. $x \in D_2$.

Thus, we ascertain that

$$\lim_{\varepsilon\to 0} \big(v_n(\varepsilon,\cdot), S_\varepsilon\varphi\big)_{\mathcal{H}_\varepsilon} = \lim_{\varepsilon\to 0} \Big(\int\limits_{\Omega_0} \nabla v_n(\varepsilon,x)\cdot\nabla\varphi\,dx + \int\limits_{G_\varepsilon^{(1)}} \nabla v_n(\varepsilon,x)\cdot\nabla(E\varphi)|_{G_\varepsilon^{(1)}}\,dx +$$

$$+ \int\limits_{D_2} \chi_{h_2}(\tfrac{x_1}{\varepsilon})\nabla\big(\partial_{x_2}\mathbf{P}_\varepsilon(v_n(\varepsilon,x))\big)\cdot\nabla(E\varphi)|_{D_2}\,dx\Big) = \int\limits_{\Omega_0} \nabla v_n^0(x)\cdot\nabla\varphi\,dx = \big(v,\varphi\big)_{\mathcal{H}_0} \quad \forall\,\varphi\in\mathcal{H}_0,$$

i.e., condition $\mathbf{D_3}$ is satisfied.

Let for certain functions $u^\varepsilon, v^\varepsilon \in \mathcal{H}_\varepsilon$ one has $\mathbf{P}_\varepsilon u^\varepsilon \to u^0$ and $\mathbf{P}_\varepsilon v^\varepsilon \to v^0$ weakly in $H^1(\Omega,\Gamma_1)$ as $\varepsilon\to 0$. Then

$$\lim_{\varepsilon\to 0}\big(u^\varepsilon, v^\varepsilon\big)_{\mathcal{V}_\varepsilon} = \lim_{\varepsilon\to 0}\varepsilon^{-1}\int\limits_{G_\varepsilon^{(1)}} u^\varepsilon v^\varepsilon\,dx. \tag{3.6}$$

With the help of the inequality

$$\varepsilon^{-1}\int\limits_{G_\varepsilon^{(1)}} \big(\varphi(x)-\varphi(x_1,0)\big)^2\,dx \le \varepsilon\, l_1 \int\limits_{G_\varepsilon^{(1)}} \big(\partial_{x_2}\varphi(x)\big)^2\,dx \quad \forall\,\varphi\in H^1(G_\varepsilon^{(1)}),$$

we deduce that $\lim_{\varepsilon\to 0}\varepsilon^{-1}\int_{G_\varepsilon^{(1)}} u^\varepsilon v^\varepsilon\,dx = 4h_1 l_1 \int_{I_0} u^0(x_1,0)\,v^0(x_1,0)\,dx_1$. This means that the first part of condition $\mathbf{D_4}$ holds. The second part of condition $\mathbf{D_4}$ in our case is obvious.

Condition $\mathbf{D_5}$, in fact, has been verified in subsection 2.3, namely the action of the operator $\mathcal{R}_\varepsilon$ in $\mathbf{D_5}$ is the construction of the approximating function R_ε on the basis of an eigenfunction of the homogenized problem (2.48). Furthermore, the approximating function satisfies inequality (2.58) that is analog of the corresponding inequality in condition $\mathbf{D_5}$.

3.2 The main results

Thus, all conditions $\mathbf{D_1}$–$\mathbf{D_5}$ of the scheme from [17] are satisfied. Applying this scheme and taking into account (1.20), we get the following theorems.

Theorem 3.2. *For any $n\in\mathbb{N}$*

$$\varepsilon^{1-\alpha}\lambda_n(\varepsilon) \to \lambda_0^{(n)} \quad \textit{as} \ \ \varepsilon\to 0,$$

where $\{\lambda_n(\varepsilon)\}_{n\in\mathbb{N}}$ is the ordered sequence (1.2) *of eigenvalues of problem* (1.1)*, $\{\lambda_0^{(n)}\}_{n\in\mathbb{N}}$ is the ordered sequence* (2.51) *of eigenvalues of the homogenized problem* (2.48).

There exists a subsequence of the sequence $\{\varepsilon\}$ (again denoted by $\{\varepsilon\}$) such that

$$\forall\, n\in\mathbb{N} \quad \varepsilon^{-\frac{\alpha-1}{2}}\mathbf{P}_\varepsilon u_n(\varepsilon,\cdot)\to v_0^n \quad \textit{weakly in } H^1(\Omega,\Gamma_1) \quad \textit{as} \ \ \varepsilon\to 0,$$

where

$$v_0^n(x) = \begin{cases} v_0^{+,n}(x), & x\in\ \Omega_0,\\ v_0^{+,n}(x_1,0), & x\in\ D_2=(0,a)\times(-l_2,0),\end{cases}$$

$\{u_n(\varepsilon,\cdot)\}_{n\in\mathbb{N}}$ *is the sequence of eigenfunctions that are orthonormalized with relations* (1.3), $\{v_0^{+,n}\}_{n\in\mathbb{N}}$ *are eigenfunctions of the homogenized problem* (2.48) *that satisfy the following orthonormalized conditions:*

$$\big(v_0^{+,n}, v_0^{+,k}\big)_{\mathcal{V}_0} = 4h_1 l_1 \int\limits_{I_0} v_0^{+,n}(x_1,0)\, v_0^{+,k}(x_1,0)\, dx_1 = \delta_{n,k}, \qquad n,k\in\mathbb{N}.$$

Let $\lambda_0^{(n+1)} = \ldots = \lambda_0^{(n+r)}$ be an $r-$multiple eigenvalue of the homogenized problem (2.48); the corresponding eigenfunctions $v_0^{+,n+1},\ldots,v_0^{+,n+r}$ are orthonormalized in $\mathcal{V}_0$. Using formula (2.54), we successively construct next terms $\varepsilon^{\alpha-m}\lambda_{\alpha-m}^{(n+i)}$, $i=1,\ldots,r$, of the asymptotic expansion (2.1) and define the unique solution $v_{\alpha-1}^{+,n+i}$ to problem (2.53) .

We formulate next theorem under assumption that all $\lambda_{\alpha-m}^{(n+i)}$, $i=1,\ldots,r$, are different and

$$\lambda_{\alpha-m}^{(n+1)} < \lambda_{\alpha-m}^{(n+2)} < \ldots < \lambda_{\alpha-m}^{(n+r)}. \tag{3.7}$$

In general case the formulation is the same as in Theorems 5.4 and 5.6 from our paper [2].

Theorem 3.3. *Let inequalities* (3.7) *are satisfied. Then for any positive δ small enough and for any $i\in\{1,\ldots,r\}$ there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ we have*

$$\left|\varepsilon^{1-\alpha}\lambda_{n+i}(\varepsilon) - \big(\lambda_0^{(n+i)} + \varepsilon^{\alpha-m}\lambda_{\alpha-m}^{(n+i)}\big)\right| \;\le\; C_1(n,\delta)\,\varepsilon^{\nu(\alpha)},$$

and

$$\left\|\varepsilon^{-\frac{\alpha-1}{2}}u_{n+i}(\varepsilon,\cdot) - \frac{R_\varepsilon^{(n+i)}}{\|R_\varepsilon^{(n+i)}\|_{\mathcal{H}_\varepsilon}}\right\|_{H^1(\Omega_\varepsilon)} \le C_2(n,\delta)\;\varepsilon^{\nu(\alpha)}, \tag{3.8}$$

where the value $\nu(\alpha)$ is defined in (2.72), *$R_\varepsilon^{(n+i)}$ is the approximating function constructed by formula* (2.56) *with the help of solutions $v_0^{+,n+i}$, $v_{\alpha-m}^{+,n+i}$ and $Z_1^{(k)}, Z_{\alpha-m+1}^{(0)}, Z_{\alpha-m+1}^{(2)}, X_{\alpha-m+1}^{(k)}$ $k=0,1,2$.*

4 Construction of the asymptotics for $\alpha = m\in\mathbb{N}$, $m\ge 2$

4.1 Formal asymptotics

In this case we seek the main terms of the asymptotics for the eigenvalue $\Lambda_n(\varepsilon)$ and the eigenfunction $v_n(\varepsilon,\cdot)$ in the form (index n is omitted):

$$\Lambda(\varepsilon) \approx \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots \tag{4.1}$$

$$v(\varepsilon,x) \approx v_0^+(x) + \varepsilon v_1^+(x) + \varepsilon^2 v_2^+(x) + \ldots \quad \text{in domain } \Omega_0; \tag{4.2}$$

$$v(\varepsilon,x) \approx v_0^-(x_1,x_2,\frac{x_1}{\varepsilon}-j) + \varepsilon v_1^-(x_1,x_2,\frac{x_1}{\varepsilon}-j) + \varepsilon^2 v_2^-(x_1,x_2,\frac{x_1}{\varepsilon}-j) + \ldots \tag{4.3}$$

in the thin rectangle $G_j^{(2)}(\varepsilon)$ $(j=0,\ldots,N-1)$; and in the junction zone of the body and thin rectangles of both classes (which we call internal expansion) the series of the following type:

$$v(\varepsilon,x) \approx v_0^+(x_1,0) + \varepsilon\bigg(v_1^+(x_1,0) + Z_1^{(0)}(\tfrac{x}{\varepsilon})v_0^+(x_1,0) + \sum_{i=1}^2 Z_1^{(i)}(\tfrac{x}{\varepsilon})\partial_{x_i}v_0^+(x_1,0)\bigg) +$$
$$+\,\varepsilon^2\bigg(X_2^{(0)}(\tfrac{x}{\varepsilon})v_1^+(x_1,0) + \sum_{i=1}^2 X_2^{(i)}(\tfrac{x}{\varepsilon})\partial_{x_i}v_1^+(x_1,0) + \sum_{\beta\in\mathcal{B}} Z_2^{(\beta)}(\eta)D^\beta v_0^+(x_1,0)\bigg) + \ldots, \tag{4.4}$$

where $\mathcal{B} := \{(0,0);(0,1);(1,0);(2,0)\}$.

Substituting (4.1) and (4.2) in problem (1.21) and collecting terms with equal order of ε, we get

$$\begin{cases} -\Delta_x \ v_0^+(x) = 0, \quad x \in \Omega_0, \\ \partial_\nu v_0^+(x)|_{x\in\Gamma_2} = 0, \qquad v_0^+(x)|_{x\in\Gamma_1} = 0. \end{cases} \tag{4.5}$$

Collecting terms of order ε^1, we have

$$\begin{cases} -\Delta_x \ v_1^+(x) = \delta_{2,m}\lambda_0 v_0^+(x), \quad x \in \Omega_0, \\ \partial_\nu v_1^+(x)|_{x\in\Gamma_2} = 0, \qquad v_1^+(x)|_{x\in\Gamma_1} = 0, \end{cases} \tag{4.6}$$

where $\delta_{2,m}$ is the Kronecker symbol. To complete these problems we have to find conditions on I_0; this is done in Subsection 4.2.

4.1.1 Formal asymptotics in each thin rectangle $G_j^{(2)}(\varepsilon)$

Using Taylor series for the functions $\{v_\gamma^-\}$ in (4.3) in a neighborhood of the point $x_1 = \varepsilon(j+\frac{1}{2})$, we get

$$v(\varepsilon, x) \approx W_0^{(j)}(x_2, \eta_1) + \varepsilon W_1^{(j)}(x_2, \eta_1) + \varepsilon^2 W_2^{(j)}(x_2, \eta_1) + \ldots, \tag{4.7}$$

where

$$W_0^{(j)}(x_2, \eta_1) = v_0^-\big(\varepsilon(j+\tfrac{1}{2}), x_2, \eta_1 - j\big), \tag{4.8}$$

$$W_1^{(j)}(x_2, \eta_1) = v_1^-\big(\varepsilon(j+\tfrac{1}{2}), x_2, \eta_1 - j\big) + \big(\eta_1 - j - \tfrac{1}{2}\big)\frac{\partial v_0^-}{\partial x_1}\big(\varepsilon(j+\tfrac{1}{2}), x_2, \eta_1 - j\big) \tag{4.9}$$

and

$$\begin{aligned} W_2^{(j)}(x_2, \eta_1) = v_2^-\big(\varepsilon(j+\tfrac{1}{2}), x_2, \eta_1 - j\big) &+ \big(\eta_1 - j - \tfrac{1}{2}\big)\frac{\partial v_1^-}{\partial x_1}\big(\varepsilon(j+\tfrac{1}{2}), x_2, \eta_1 - j\big) \\ &+ \frac{1}{2}\big(\eta_1 - j - \tfrac{1}{2}\big)^2\frac{\partial^2 v_0^-}{\partial x_1^2}\big(\varepsilon(j+\tfrac{1}{2}), x_2, \eta_1 - j\big). \end{aligned} \tag{4.10}$$

Substituting (4.1) and (4.7) in the problem (1.21) instead of $\Lambda_n(\varepsilon)$ and $v_n(\varepsilon, \cdot)$ respectively, collecting terms with equal powers of ε, we obtain the following boundary-value problems:

$$\begin{cases} -\partial^2_{\eta_1\eta_1} W_\gamma^{(j)}(x_2, \eta_1) \;=\; 0, \quad \eta_1 \in \big(\frac{1-h_2}{2}, \frac{1+h_2}{2}\big), \\ \partial_{\eta_1} W_\gamma^{(j)}(x_2, \frac{1\pm h_2}{2}) \;=\; 0, \end{cases} \tag{4.11}$$

for $\gamma \in \{0,1\}$. Here the variable x_2 is regarded as a parameter. From (4.11) we deduce that the solutions $W_\gamma^{(j)}$, $\gamma \in \{0,1\}$, are independent of η_1.

Then, for $\gamma = 2$ we get the following problem:

$$\begin{cases} -\partial^2_{\eta_1\eta_1} W_2^{(j)}(x_2, \eta_1) \;=\; \partial^2_{x_2x_2} W_0^{(j)}(x_2), \quad \eta_1 \in \big(\frac{1-h_2}{2}, \frac{1+h_2}{2}\big), \\ \partial_{\eta_1} W_2^{(j)}(x_2, \frac{1\pm h_2}{2}) \;=\; 0. \end{cases} \tag{4.12}$$

The solvability condition for (4.12) gives us the relations

$$\partial^2_{x_2x_2} W_0^{(j)}(x_2) = 0, \quad x_2 \in (-l_2, 0).$$

It is the same as

$$\partial^2_{x_2x_2} v_0^-\big(\varepsilon(j+\tfrac{1}{2}), x_2\big) = 0, \quad x_2 \in (-l_2, 0), \tag{4.13}$$

because of (4.8). Bearing in mind the boundary conditions of the original problem at $x_2 = -l_2$, we add the condition $\partial_{x_2} v_0^-\big(\varepsilon(j+\frac{1}{2}), -l_2\big) = 0$ to (4.13). This means that v_0^- is independent of x_2.

If $\gamma = 3$ we obtain

$$\begin{cases} -\partial^2_{\eta_1\eta_1} W_3^{(j)}(x_2,\eta_1) &=& \partial^2_{x_2x_2} W_1^{(j)}(x_2) + \delta_{2,m}\lambda_0 W_0^{(j)}(x_2), \quad \eta_1 \in \big(\frac{1-h_2}{2}, \frac{1+h_2}{2}\big), \\ \partial_{\eta_1} W_3^{(j)}(x_2, \frac{1\pm h_2}{2}) &=& 0. \end{cases} \tag{4.14}$$

The solvability condition for (4.14) gives us the following relations:

$$\partial^2_{x_2x_2} W_1^{(j)}(x_2) = \delta_{2,m}\lambda_0 W_0^{(j)}(x_2), \quad x_2 \in (-l_2, 0).$$

Similarly, but now with regard to (4.9), we get

$$\begin{aligned} v_1^-\big(\varepsilon(j+\tfrac{1}{2}), x_2, \eta_1 - j\big) = \delta_{2,m}(-\frac{1}{2}x_2^2 - l_2x_2)\lambda_0 v_0^-(\varepsilon(j+\frac{1}{2}))+ \\ + T(\eta_1)\frac{\partial v_0^-}{\partial x_1}\big(\varepsilon(j+\tfrac{1}{2})\big) + \Phi_1\big(\varepsilon(j+\tfrac{1}{2})\big), \end{aligned} \tag{4.15}$$

and using (4.10), we derive

$$\begin{aligned} v_2^-\big(\varepsilon(j+\tfrac{1}{2}), x_2, \eta_1 - j\big) = \delta_{3,m}(-\frac{1}{2}x_2^2 - l_2x_2)\lambda_0 v_0^-(\varepsilon(j+\frac{1}{2})) + T(\eta_1)\frac{\partial v_1^-}{\partial x_1}\big(\varepsilon(j+\tfrac{1}{2})\big) + \\ - \frac{1}{2}T^2(\eta_1)\partial^2_{x_1x_1} v_0^-\big(\varepsilon(j+\tfrac{1}{2})\big) + \Phi_2\big(\varepsilon(j+\tfrac{1}{2})\big), \end{aligned} \tag{4.16}$$

where Φ_1, Φ_2 are some functions of x_1. Here $\delta_{k,m}$ is the Kronecker symbol. Since we look only for the leading terms of the asymptotics, we put $\Phi_1 \equiv \Phi_2 \equiv 0$.

Since the points $\{x_1 = \varepsilon(j+\frac{1}{2}) : j = 0, \ldots, N-1\}$ form the ε-net in the interval $(0, a)$, then we extend the relation (4.15), and (4.16) to the whole interval $(0, a)$.

4.1.2 Junction-layer solutions

To find problems for Z, X from (4.4) we calculate

$$\begin{aligned} \partial_{x_1} v(\varepsilon, x) \approx \varepsilon^0 \Big(\partial_{x_1} v_0^+(x_1, 0)\Big[1 + \partial_{\eta_1} Z_1^{(1)}\Big] + \partial_{\eta_1} Z_1^{(0)} v_0^+(x_1, 0) + \partial_{\eta_1} Z_1^{(2)} \partial_{x_2} v_0^+(x_1, 0)\Big) + \\ + \varepsilon^1 \Big(\partial_{\eta_1} X_2^{(0)} v_1^+(x_1, 0) + \partial_{x_1} v_1^+(x_1, 0)\Big[1 + \partial_{\eta_1} X_2^{(1)}\Big] + \\ + \partial_{\eta_1} X_2^{(2)} \partial_{x_2} v_1^+(x_1, 0) + Z_1^{(0)}(\eta)\partial_{x_1} v_0^+(x_1, 0) + \\ + \sum_{i=1}^{2} Z_1^{(i)}(\eta)\partial^2_{x_1x_i} v_0^+(x_1, 0) + \sum_{\beta\in\mathcal{B}} \partial_{\eta_1} Z_2^{(\beta)}(\eta) D^\beta v_0^+(x_1, 0)\Big) + \\ + \varepsilon^2 \Big(\partial_{\eta_1} Z_3^{(0)}(\eta) v_0^+(x_1, 0) + \partial_{\eta_1} Z_3^{(1)}(\eta)\partial_{x_2} v_0^+(x_1, 0)\Big) + \mathcal{O}(\varepsilon^3) \end{aligned} \tag{4.17}$$

and

$$\Delta_x v(\varepsilon,x) \approx \varepsilon^{-1}\Big(\Delta_\eta Z_1^{(0)}(\eta) v_0^+(x_1,0) + \sum_{i=1}^{2} \Delta_\eta Z_1^{(i)}(\eta)\partial_{x_i} v_0^+(x_1,0)\Big) +$$
$$+ \varepsilon^0\Big(\Delta_\eta X_2^{(0)}(\eta) v_1^+(x_1,0) + \sum_{i=1}^{2} \Delta_\eta X_2^{(i)}(\eta)\partial_{x_i} v_1^+(x_1,0) +$$
$$+ \partial^2_{x_1x_1} v_0^+(x_1,0)\Big[1 + 2\partial_{\eta_1} Z_1^{(1)}(\eta)\Big] + 2\partial_{\eta_1} Z_1^{(0)}(\eta)\partial_{x_1} v_0^+(x_1,0) + \tag{4.18}$$
$$+ 2\partial_{\eta_1} Z_1^{(2)}(\eta)\partial^2_{x_1x_2} v_0^+(x_1,0) + \sum_{\beta\in\mathcal{B}} \Delta_\eta Z_2^{(\beta)}(\eta) D^\beta v_0^+(x_1,0)\Big) +$$
$$+ \varepsilon\Big(\Delta_\eta Z_3^{(0)}(\eta) v_0^+(x_1,0) + \Delta_\eta Z_3^{(1)}(\eta)\partial_{x_2} v_0^+(x_1,0)\Big) + \mathcal{O}(\varepsilon^2).$$

Keeping in mind (4.17) and (4.18), substituting the series (4.4) and (4.1) in the problem (1.21) and collecting terms with equal powers of ε, we get problems for $Z_1^{(i)}$, $i = 0,1,2$ (the problems (2.20), (2.21), and (2.23)), $Z_2^{(\beta)}$, $\beta \in \mathcal{B}$ (the problems (2.23), (2.24), (2.25), and (2.28)). The problems for $\{X_2^{(k)}\}$ are the same as problems for $\{Z_1^{(k)}\}$ (the problems (2.20), and (2.21)). Therefore, $X_2^{(k)} \equiv Z_1^{(k)}$, $k = 0,1,2$.

4.2 Homogenized problem and correctors

We have formally constructed the leading terms of the asymptotic expansions (4.2), (4.3), (4.4) in three different parts of the junction Ω_ε. Following the method of matching of asymptotic expansions, we equate the asymptotics of the external expansions (4.2) and (4.3) as $x_2 \to \pm 0$ and the corresponding asymptotics of the internal expansion (4.4) as $\eta_2 \to \pm\infty$ respectively.

Writing down the Taylor series for v_0^+, v_1^+ and v_2^+ with respect to x_2 in a neighborhood of the point $(x_1, 0)$, where $x_1 \in (0, a)$, and passing to the variables $\eta_2 = \varepsilon^{-1}x_2$, we derive

$$v(\varepsilon,x) = v_0^+(x_1,0) + \varepsilon^1\left(v_1^+(x_1,0) + \eta_2\partial_{x_2} v_0^+(x_1,0)\right) +$$
$$+ \varepsilon^2\left(\eta_2\,\partial_{x_2} v_1^+(x_1,0) + \frac{1}{2}\eta_2^2\,\partial^2_{x_2x_2} v_0^+(x_1,0)\right) + \ldots \tag{4.19}$$

Bearing in mind the asymptotics of the functions $Z_1^{(k)}$, $X_2^{(k)}$ $(k = 0,1,2)$, $Z_2^{(\beta)}$ $(\beta \in \mathcal{B})$, as $\eta_2 \to +\infty$ (see (2.29)–(2.35)), we write down the asymptotics

$$v(\varepsilon,x) = v_0^+(x_1,0) + \varepsilon^1\left(v_1^+(x_1,0) + \eta_2\partial_{x_2} v_0^+(x_1,0)\right) +$$
$$+ \varepsilon^2\left(\eta_2\,\partial_{x_2} v_1^+(x_1,0) - \frac{1}{2}\eta_2^2\,\partial^2_{x_1x_1} v_0^+(x_1,0)\right) + \ldots \tag{4.20}$$

Since $\Delta v_0^+ = 0$, the leading terms in (4.19) and (4.20) coincide at ε^0, ε^1, and ε^2.

To match (4.3) and (4.4) we write down the asymptotics of (4.3) as $x_2 \to -0$ and pass to the fast variables; as a result we get

$$v(\varepsilon,x) = v_0^-(x_1) + \varepsilon\,\underbrace{v_1^-(x_1,0,\eta_1)} + \varepsilon^2\Big(v_2^-(x_1,0,\eta_1) + \delta_{2,m}\partial^2_{x_2} v_1^-(x_1,0,\eta_1)\Big) + \ldots \tag{4.21}$$

Keeping in mind the asymptotics of the functions $Z_1^{(k)}$, $X_2^{(k)}$ $(k = 0, 1, 2)$, and $Z_2^{(\beta)}$ $(\beta \in \mathcal{B})$ as $\eta_2 \to -\infty$, we find the following asymptotics of (4.4):

$$
\begin{aligned}
v(\varepsilon, x) = {} & v_0^+(x_1, 0) + \\
& + \varepsilon \Bigg(\underbrace{v_1^+(x_1,0) + T(\eta_1)\partial_{x_1} v_0^+(x_1,0)} + \overbrace{\frac{\eta_2}{h_2}\, \partial_{x_2} v_0^+(x_1,0)} + \underbrace{C_1^{(2)}\, \partial_{x_2} v_0^+(x_1,0)} + \\
& + \overbrace{\frac{4h_1 l_1 \lambda_0}{h_2} \eta_2\, v_0^+(x_1,0)} + \underbrace{C_1^{(0)}\, v_0^+(x_1,0)} \Bigg) + \varepsilon^2 \Bigg(\Big(\frac{4h_1 l_1 \lambda_0 \eta_2}{h_2} + C_1^{(0)} \Big) v_1^+(x_1,0) + \\
& + T(\eta_1)\partial_{x_1} v_1^+(x_1,0) + \Big(\frac{\eta_2}{h_2} + C_1^{(2)} \Big) \partial_{x_2} v_1^+(x_1,0) + \\
& + \Big(\frac{4h_1 l_1 \lambda_1 + \varsigma_{(0,0)}}{h_2} \eta_2 + C_2^{(0,0)} \Big) v_0^+(x_1,0) + \\
& + \Big(\frac{\lambda_0 \int_{\Pi_{l_1}} Z_1^{(2)}(\eta) d\eta}{h_2}\, \eta_2 + C_2^{(0,1)} \Big) \partial_{x_2} v_0^+(x_1,0) + \\
& + \Big(\frac{4h_1 l_1 \lambda_0}{h_2}\, \eta_2 T(\eta_1) + C_2^{(1,0)} \Big) \partial_{x_1} v_0^+(x_1,0) + \\
& + \Big(\frac{1}{2} T^2(\eta_1) + \frac{\varsigma_{(2,0)} \eta_2}{h_2} + C_2^{(2,0)} \Big) \partial^2_{x_1 x_1} v_0^+(x_1,0) \Bigg) + \ldots
\end{aligned}
\tag{4.22}
$$

Equating the corresponding coefficients in (4.21) and (4.22) at ε^0, we get

$$
v_0^+(x_1, 0) = v_0^-(x_1), \quad x_1 \in (0, a). \tag{4.23}
$$

The same procedure at ε^1 brings us the following relations:

$$
\partial_{x_2} v_0^+(x_1, 0) + 4h_1 l_1 \lambda_0\, v_0^+(x_1, 0) = 0, \quad x_1 \in (0, a), \tag{4.24}
$$

for the over-braced terms, and

$$
\begin{aligned}
v_1^-(x_1, 0, \eta_1) = {} & v_1^+(x_1, 0) + T(\eta_1)\partial_{x_1} v_0^+(x_1, 0) + \\
& + C_1^{(2)} \partial_{x_2} v_0^+(x_1, 0) + C_1^{(0)} v_0^+(x_1, 0), \quad x_1 \in (0, a),
\end{aligned}
\tag{4.25}
$$

for the under-braced terms, or in terms of a jump

$$
\big[v_1\big](x_1, 0, \tfrac{x_1}{\varepsilon}) = -T(\tfrac{x_1}{\varepsilon})\partial_{x_1} v_0^+(x_1, 0) - C_1^{(2)} \partial_{x_2} v_0^+(x_1, 0) - C_1^{(0)} v_0^+(x_1, 0), \quad x_1 \in (0, a), \tag{4.26}
$$

for all $m \geq 2$.

Relation (4.24) completes problem (4.5). Thus, v_0^+ and the number λ_0 are an eigenfunction and the corresponding eigenvalue of the Steklov problem (2.48).

Next, let λ_0 be an eigenvalue of problem (2.48), v_0^+ is the corresponding eigenfunction normalized by (2.52).

Equating terms in (4.21) and (4.22) of order $\varepsilon^2 \eta_2$, we have

$$
\begin{aligned}
& \frac{1}{h_2} \partial_{x_2} v_1^+(x_1, 0) + \frac{4h_1 l_1 \lambda_1 + \varsigma_{(0,0)}}{h_2} v_0^+(x_1, 0) + \frac{\lambda_0 \int_{\Pi_{l_1}} Z_1^{(2)}(\eta) d\eta}{h_2} \partial_{x_2} v_0^+(x_1, 0) + \\
& \quad + \frac{\varsigma_{(2,0)}}{h_2} \partial^2_{x_1 x_1} v_0^+(x_1, 0) + \frac{4h_1 l_1 \lambda_0}{h_2} T(\eta_1) \partial_{x_1} v_0^+(x_1, 0) + \frac{4h_1 l_1 \lambda_0}{h_2} v_1^+(x_1, 0) = \\
& \qquad = \delta_{2,m} \partial_{x_2} v_1^-(x_1, 0, \eta_1)
\end{aligned}
\tag{4.27}
$$

or in terms of a jump

$$\partial_{x_2} v_1^+(x_1,0) - h_2\delta_{2,m}\partial_{x_2} v_1^-(x_1,0,\eta_1) = -4h_1l_1\lambda_0 v_1^+(x_1,0) - (4h_1l_1\lambda_1 + \varsigma_{(0,0)})v_0^+(x_1,0) - \\ -\lambda_0\int_{\Pi_{l_1}} Z_1^{(2)}(\eta)\,d\eta\,\partial_{x_2} v_0^+(x_1,0) - \varsigma_{(2,0)}\partial^2_{x_1x_1} v_0^+(x_1,0) - 4h_1l_1\lambda_0 T(\eta_1)\partial_{x_1} v_0^+(x_1,0). \tag{4.28}$$

Continuing the matching procedure, we equate $v_2^-(x_1,0,\eta_1)$ to the remaining terms in (4.22).

In the case $\boldsymbol{m \geq 3}$ the function $v_1^-(x_1,\eta_1)$ does not depend on x_2 and therefore formula (4.28) gives us the boundary condition for the function v_1^+, which is a function of x and η_1. Hence, we have the following problem:

$$\begin{cases} \left(\Delta_x\ v_1^+(x,\eta_1)\right)\big|_{\eta_1=x_1/\varepsilon} = 0, & x\in\Omega_0; \\ \left(\partial_\nu\ v_1^+(x,\eta_1)\right)\big|_{\eta_1=x_1/\varepsilon} = 0, \quad x\in\Gamma_2; & v_1^+(x,\eta_1)=0, \quad x\in\Gamma_1; \\ \partial_{x_2} v_1^+(x_1,0,\tfrac{x_1}{\varepsilon}) = -\,4h_1l_1\lambda_0\, v_1^+(x_1,0,\tfrac{x_1}{\varepsilon}) - (4h_1l_1\lambda_1+\varsigma_{(0,0)})v_0^+(x_1,0) - \\ \qquad -\,\lambda_0\int_{\Pi_{l_1}} Z_1^{(2)}(\eta)\,d\eta\,\partial_{x_2} v_0^+(x_1,0) - \varsigma_{(2,0)}\partial^2_{x_1x_1} v_0^+(x_1,0) - \\ \qquad -\,4h_1l_1\lambda_0 T(\tfrac{x_1}{\varepsilon})\partial_{x_1} v_0^+(x_1,0). \end{cases} \tag{4.29}$$

The solvability condition leads to the following formula:

$$\lambda_1(\varepsilon) = -\frac{\varsigma_{(0,0)}}{4h_1l_1} - \lambda_0\int_{\Pi_{l_1}} Z_1^{(2)}(\eta)\,d\eta\int_{I_0}\partial_{x_2} v_0^+(x_1,0)\,v_0^+(x_1,0)\,dx_1 - \\ -\,\varsigma_{(2,0)}\int_{I_0}\partial^2_{x_1x_1} v_0^+(x_1,0)\,v_0^+(x_1,0)\,dx_1 - \\ -\,4h_1l_1\lambda_0\int_{I_0} T(\tfrac{x_1}{\varepsilon})\partial_{x_1} v_0^+(x_1,0)\,v_0^+(x_1,0)\,dx_1. \tag{4.30}$$

Using Lemma 1.6 from [7, §1] (see also Lemma 1.1 from [35, §1.4]), we derive

$$\left|\int_{I_0} T(\tfrac{x_1}{\varepsilon})\partial_{x_1} v_0^+(x_1,0)\,v_0^+(x_1,0)\,dx_1\right| \leq K\varepsilon. \tag{4.31}$$

Hence,

$$\lambda_1(\varepsilon) = \lambda_1 + k_1\varepsilon, \qquad k_1 = \text{const}, \tag{4.32}$$

where

$$\lambda_1 = -\frac{\varsigma_{(0,0)}}{4h_1l_1} - \varsigma_{(2,0)}\int_{I_0}\partial^2_{x_1x_1} v_0^+(x_1,0)\,v_0^+(x_1,0)\,dx_1 \\ -\,\lambda_0\int_{\Pi_{l_1}} Z_1^{(2)}(\eta)\,d\eta\int_{I_0} v_0^+(x_1,0)\,\partial_{x_2} v_0^+(x_1,0)\,dx_1. \tag{4.33}$$

Let us represent the solution to problem (4.29) in the form

$$v_1^+(x,\tfrac{x_1}{\varepsilon}) = v_1^+(x) + \ldots, \tag{4.34}$$

where $v_1^+(x)$ is a solution of the following problem:

$$\begin{cases} \Delta_x \ v_1^+(x) = 0, \qquad x \in \Omega_0; \\ \partial_\nu \ v_1^+(x) = 0, \quad x \in \Gamma_2; \qquad v_1^+(x) = 0, \quad x \in \Gamma_1; \\ \partial_{x_2} v_1^+(x_1, 0) = -\,4h_1 l_1 \lambda_0 \, v_1^+(x_1, 0) - (4h_1 l_1 \lambda_1 + \varsigma_{(0,0)}) v_0^+(x_1, 0) - \\ \qquad - \lambda_0 \int_{\Pi_{l_1}} Z_1^{(2)}(\eta) \, d\eta \, \partial_{x_2} v_0^+(x_1, 0) - \varsigma_{(2,0)} \partial^2_{x_1 x_1} v_0^+(x_1, 0), \quad x_1 \in (0, a). \end{cases} \tag{4.35}$$

Due to (4.33) the solvability condition for problem (4.35) is satisfied. It is easy to see that the solutions to both problems (4.35) and (4.29)) are defined up to an additive term $\varkappa\, v_0^+(x)$, and $\varkappa$ is an arbitrary constant. For the uniqueness of the solution to problem (4.35) we demand the following orthogonality condition:

$$\int\limits_{I_0} v_1^+(x_1, 0) \, v_0^+(x_1, 0) \, dx_1 = 0, \tag{4.36}$$

i.e. we found and fixed the constant $\varkappa$. Now we can estimate the difference between the solutions of problems (4.35) and (4.29) with fixed constant $\varkappa$, in the Sobolev space. Due to (4.31) we have the estimate

$$\|v_1^+(x, \tfrac{x_1}{\varepsilon}) - v_1^+(x)\|_{H^1(\Omega_0)} \le C\varepsilon. \tag{4.37}$$

In the case $\boldsymbol{m = 2}$ formula (4.26) is the first transmission condition and formula (4.28) gives us the second transmission condition for the functions $v_1^-(x_1, \eta_1)$, $v_1^+(x, \eta_1)$. Hence, we have

$$\begin{cases} -(\Delta_x \ v_1^+(x, \eta_1))\big|_{\eta_1 = x_1/\varepsilon} = \lambda_0 v_0^+(x), \qquad x \in \Omega_0, \\ -(\partial^2_{x_2 x_2} v_1^-(x, \eta_1))\big|_{\eta_1 = x_1/\varepsilon} = \lambda_0 v_0^-(x_1), \qquad x \in D_2, \\ (\partial_\nu \ v_1^+(x, \eta_1))\big|_{\eta_1 = x_1/\varepsilon} = 0, \quad x \in \Gamma_2; \qquad v_1^+(x, \tfrac{x_1}{\varepsilon}) = 0, \quad x \in \Gamma_1, \\ \partial_{x_2} v_1^-(x_1, -l_2, \tfrac{x_1}{\varepsilon}) = 0, \qquad x_1 \in (0, a), \\ \big[v_1\big](x_1, 0, \tfrac{x_1}{\varepsilon}) = -\,T(\tfrac{x_1}{\varepsilon}) \partial_{x_1} v_0^+(x_1, 0) - C_1^{(2)} \partial_{x_2} v_0^+(x_1, 0) - \\ \qquad - C_1^{(0)} v_0^+(x_1, 0), \quad x_1 \in (0, a), \\ \partial_{x_2} v_1^+(x_1, 0, \tfrac{x_1}{\varepsilon}) - h_2 \partial_{x_2} v_1^-(x_1, 0, \tfrac{x_1}{\varepsilon}) = -\,4h_1 l_1 \lambda_0 \, v_1^+(x_1, 0, \tfrac{x_1}{\varepsilon}) - (4h_1 l_1 \lambda_1 + \varsigma_{(0,0)}) v_0^+(x_1, 0) - \\ \qquad - \lambda_0 \int_{\Pi_{l_1}} Z_1^{(2)}(\eta) \, d\eta \ \partial_{x_2} v_0^+(x_1, 0) - \\ \qquad - \varsigma_{(2,0)} \partial^2_{x_1 x_1} v_0^+(x_1, 0) - \\ \qquad - 4h_1 l_1 \lambda_0 T(\tfrac{x_1}{\varepsilon}) \partial_{x_1} v_0^+(x_1, 0), \quad x_1 \in (0, a). \end{cases} \tag{4.38}$$

Solving problem in D_2, and bearing in mind (4.15), we get the boundary value problem to define v_1^+ in the following form:

$$\begin{cases} -(\Delta_x\ v_1^+(x,\eta_1))\big|_{\eta_1=x_1/\varepsilon} = \lambda_0 v_0^+(x), \qquad x\in\Omega_0, \\ \quad (\partial_\nu\ v_1^+(x,\eta_1))\big|_{\eta_1=x_1/\varepsilon} = 0, \quad x\in\Gamma_2; \qquad v_1^+(x,\tfrac{x_1}{\varepsilon}) = 0, \quad x\in\Gamma_1, \\ \qquad \partial_{x_2} v_1^+(x_1,0,\tfrac{x_1}{\varepsilon}) = -\,4h_1l_1\lambda_0\, v_1^+(x_1,0,\tfrac{x_1}{\varepsilon}) - (4h_1l_1\lambda_1 + \varsigma_{(0,0)})v_0^+(x_1,0)- \\ \qquad\qquad -\,\lambda_0 \displaystyle\int_{\Pi_{l_1}} Z_1^{(2)}(\eta)\,d\eta\,\partial_{x_2} v_0^+(x_1,0) - \varsigma_{(2,0)}\partial^2_{x_1x_1} v_0^+(x_1,0)- \\ \qquad\qquad -\,4h_1l_1\lambda_0 T(\tfrac{x_1}{\varepsilon})\partial_{x_1} v_0^+(x_1,0)- \\ \qquad\qquad -\,l_2h_2\lambda_0 v_0^+(x_1,0), \quad x_1\in(0,a). \end{cases} \tag{4.39}$$

The solvability condition leads to the formula

$$\begin{aligned} \lambda_1(\varepsilon) = &-\frac{(l_2h_2\lambda_0+\varsigma_{(0,0)})}{4h_1l_1} - \lambda_0\int_{\Pi_{l_1}} Z_1^{(2)}(\eta)\,d\eta \int_{I_0}\partial_{x_2}v_0^+(x_1,0)\,v_0^+(x_1,0)\,dx_1- \\ &-\varsigma_{(2,0)}\int_{I_0}\partial^2_{x_1x_1}v_0^+(x_1,0)\,v_0^+(x_1,0)\,dx_1 + \lambda_0\int_{\Omega_0}\big(v_0^+(x)\big)^2dx- \\ &-4h_1l_1\lambda_0\int_{I_0} T(\tfrac{x_1}{\varepsilon})\partial_{x_1}v_0^+(x_1,0)\,v_0^+(x_1,0)\,dx_1. \end{aligned} \tag{4.40}$$

Repeating the procedure as in the case $m\ge 3$, we deduce

$$\lambda_1(\varepsilon) = \lambda_1 + k_1\varepsilon, \qquad k_1 = \text{const}, \tag{4.41}$$

where

$$\begin{aligned} \lambda_1 = &-\frac{\lambda_0}{4h_1l_1}\left(h_2l_2 + \int\limits_{\Omega_0}(v_0^+)^2\,dx\right) - \frac{\varsigma_{(0,0)}}{4h_1l_1} - \varsigma_{(2,0)}\int_{I_0}\partial^2_{x_1x_1}v_0^+(x_1,0)\,v_0^+(x_1,0)\,dx_1 \\ &-\lambda_0\int_{\Pi_{l_1}} Z_1^{(2)}(\eta)\,d\eta\int_{I_0} v_0^+(x_1,0)\,\partial_{x_2}v_0^+(x_1,0)\,dx_1 \end{aligned} \tag{4.42}$$

and we represent $v_1^+(x,\frac{x_1}{\varepsilon}) = v_1^+(x)+\dots$, where $v_1^+(x)$ is a solution of the following problem:

$$\begin{cases} -\Delta_x\ v_1^+(x) = \lambda_0 v_0^+(x), \qquad x\in\Omega_0, \\ \quad \partial_\nu\ v_1^+(x) = 0, \quad x\in\Gamma_2; \qquad v_1^+(x) = 0, \quad x\in\Gamma_1, \\ \partial_{x_2}v_1^+(x_1,0) = -\,4h_1l_1\lambda_0\,v_1^+(x_1,0) - (4h_1l_1\lambda_1+\varsigma_{(0,0)})v_0^+(x_1,0)- \\ \qquad -\,\lambda_0\displaystyle\int_{\Pi_{l_1}} Z_1^{(2)}(\eta)\,d\eta\,\partial_{x_2}v_0^+(x_1,0) - \varsigma_{(2,0)}\partial^2_{x_1x_1}v_0^+(x_1,0)- \\ \qquad -\,l_2h_2\lambda_0 v_0^+(x_1,0), \quad x_1\in(0,a), \end{cases} \tag{4.43}$$

which can be chosen to satisfy

$$\|v_1^+(x,\tfrac{x_1}{\varepsilon}) - v_1^+(x)\|_{H^1(\Omega_0)} \le C\varepsilon. \tag{4.44}$$

4.3 Global asymptotic approximation in Ω_ε and estimation of its residuals

For any given eigenvalue λ_0 of the homogenized spectral problem (2.48) and the corresponding eigenfunction v_0^+ normalized by (2.52), we can define λ_1 with the help of (4.33) for $m \geq 3$ (respectively (4.42) for $m = 2$) and the unique solutions v_1^+ to problem (4.35) for $m \geq 3$ (respectively (4.43) for $m = 2$).

An approximating function R_ε is constructed as the sum of the first terms of outer expansions (4.2), (4.3) and inner expansion (4.4) with the subtraction of the identical terms of their asymptotics (as $x_2 \to \pm 0$ and $\eta_2 \to \pm\infty$ respectively), since they are summed twice. Taking (4.23) into account, we obtain

$$R_\varepsilon(x) = \begin{cases} v_0^+(x) + \varepsilon v_1^+(x) + \chi(\frac{x_2}{\sqrt{\varepsilon}})\, \mathcal{N}_\varepsilon^+(x_1, \frac{x}{\varepsilon}), & x \in \Omega_0, \\ v_0^+(x_1, 0) + \varepsilon v_1^+(x_1, 0) + \mathcal{N}_{1,\varepsilon}^-(x_1, \frac{x}{\varepsilon}), & x \in G_\varepsilon^{(1)}, \\ v_0^+(x_1, 0) + \varepsilon v_1^-(x, \frac{x_1}{\varepsilon}) + \varepsilon^2 T(\frac{x_1}{\varepsilon})\partial_{x_1} v_1^-(x, \frac{x_1}{\varepsilon}) + \chi(\frac{x_2}{\sqrt{\varepsilon}})\, \mathcal{N}_{2,\varepsilon}^-(x_1, \frac{x}{\varepsilon}), & x \in G_\varepsilon^{(2)}, \end{cases} \tag{4.45}$$

where χ is a smooth cut-off function such that $\chi(s) = 1$ for $|s| \leq 1/2$; function v_1^- is a solution of problem (4.43), if $m = 2$ and is a solution of problem (4.35), if $m \geq 3$;

$$\begin{aligned} \mathcal{N}_\varepsilon^+(x_1, \eta) = \varepsilon\Big(& Z_1^{(0)}(\eta)\, v_0^+(x_1, 0) + \sum_{i=1}^{2} \big(Z_1^{(i)}(\eta) - \delta_{i,2}\eta_2\big)\partial_{x_i} v_1^+(x_1, 0) + \\ & + \varepsilon^2\Big(X_2^{(0)}(\eta)\, v_1^+(x_1, 0) + \sum_{i=1}^{2} \big(X_2^{(i)}(\eta) - \delta_{i,2}\eta_2\big)\partial_{x_i} v_1^+(x_1, 0) + \\ & + Z_2^{(0,0)}(\eta) v_0^+(x_1, 0) + Z_2^{(1,0)}(\eta)\partial_{x_1} v_0^+(x_1, 0) + Z_2^{(0,1)}(\eta)\partial_{x_2} v_0^+(x_1, 0) + \\ & + \Big(Z_2^{(2,0)}(\eta) + \frac{1}{2}\eta_2^2\Big)\partial^2_{x_1 x_1} v_0^+(x_1, 0)\Big), \end{aligned} \tag{4.46}$$

$$\begin{aligned} \mathcal{N}_{1,\varepsilon}^-(x_1, \eta) = \varepsilon\Big(& Z_1^{(0)}(\eta) v_0^+(x_1, 0) + \sum_{i=1}^{2} Z_1^{(i)}(\eta)\partial_{x_i} v_0^+(x_1, 0)\Big) + \\ & + \varepsilon^2\Big(X_2^{(0)}(\eta) v_1^+(x_1, 0) + \sum_{i=1}^{2} X_2^{(i)}(\eta)\partial_{x_i} v_1^+(x_1, 0) + Z_2^{(0,0)}(\eta) v_0^+(x_1, 0) + \\ & + Z_2^{(1,0)}(\eta)\partial_{x_1} v_0^+(x_1, 0) + Z_2^{(0,1)}(\eta)\partial_{x_2} v_0^+(x_1, 0) + Z_2^{(2,0)}(\eta)\partial^2_{x_1 x_1} v_0^+(x_1, 0)\Big), \end{aligned} \tag{4.47}$$

$$\begin{aligned} \mathcal{N}_{2,\varepsilon}^-(x_1, \eta) = \varepsilon\Big(& Z_1^{(1)}(\eta)\, \partial_{x_1} v_0^+(x_1, 0) + \Big(Z_1^{(2)}(\eta) - \frac{\eta_2}{h_2} - C_1^{(2)}\Big)\partial_{x_2} v_0^+(x_1, 0) + \\ & + \Big(Z_1^{(0)}(\eta) - \frac{4h_1 l_1 \lambda_0}{h_2}\, \eta_2 - C_1^{(0)}\Big) v_0^+(x_1, 0)\Big) + \\ + \varepsilon^2\Big(& X_2^{(1)}(\eta)\, \partial_{x_1} v_1^+(x_1, 0) - T(\eta_1)\, \partial_{x_1} v_1^-(x_1, 0, \eta_1) + \Big(X_2^{(2)}(\eta) - \frac{\eta_2}{h_2}\Big)\partial_{x_2} v_1^+(x_1, 0) + \end{aligned}$$

$$+\Big(X_2^{(0)}(\eta) - \frac{4h_1 l_1 \lambda_0}{h_2}\,\eta_2\Big) v_1^+(x_1,0) + \Big(Z_2^{(0,0)}(\eta) - \frac{4h_1 l_1 \lambda_1 + \varsigma_{(0,0)}}{h_2}\,\eta_2\Big) v_0^+(x_1,0) +$$
$$+\Big(Z_2^{(1,0)}(\eta) - \frac{4h_1 l_1 \lambda_0}{h_2}\,T(\eta_1)\,\eta_2\Big)\partial_{x_1} v_0^+(x_1,0) +$$
$$+\Big(Z_2^{(0,1)}(\eta) - \frac{\lambda_0 \int_{\Pi_{l_1}} Z_1^{(2)}(\eta)\,d\eta}{h_2}\,\eta_2\Big)\partial_{x_2} v_0^+(x_1,0) + \Big(Z_2^{(2,0)}(\eta) - \frac{\varsigma_{(2,0)}}{h_2}\,\eta_2\Big)\partial^2_{x_1x_1} v_0^+(x_1,0)\Big). \quad (4.48)$$

It is easy to verify that $R_\varepsilon|_{x_2=0+} = R_\varepsilon|_{x_2=0-}$ on Q_ε, i.e., $R_\varepsilon \in H^1(\Omega_\varepsilon;\Gamma_1)$. Also using (4.24), (4.35) (for $m \geq 3$) and (4.43) (for $m = 2$), one can verify that

$$\partial_{x_2} R_\varepsilon|_{x_2=0+} - \partial_{x_2} R_\varepsilon|_{x_2=0-} = -\varepsilon\Big(\frac{4h_1 l_1 \lambda_0}{h_2} T(\tfrac{x_1}{\varepsilon})\partial_{x_1} v_0^+(x_1,0)\Big) \quad \text{on} \quad Q_\varepsilon. \qquad (4.49)$$

4.3.1 Discrepancies in the equation of problem (1.21).

Substituting R_ε and $\lambda_0 + \varepsilon\lambda_1$ in the differential equation of problem (1.21) instead of $v(\varepsilon,\cdot)$ and $\Lambda(\varepsilon)$ respectively and calculating discrepancies with regard of problems (2.20)–(2.21), (2.23)–(2.28) and (2.48) and (4.35) for $m \geq 3$ (respectively (4.43) for $m = 2$), we get

$$\Delta_x R_\varepsilon(x) + \varepsilon^{m-1}\big(\lambda_0 + \varepsilon\lambda_1\big) R_\varepsilon(x) = -\varepsilon\delta_{2,m}\lambda_0 v_0^+(x) + \varepsilon^{m-1}\big(\lambda_0 + \varepsilon\lambda_1\big) R_\varepsilon(x)$$
$$+ \varepsilon^{-\frac{3}{2}} \chi_s'(s)\Big|_{s=\frac{x_2}{\sqrt{\varepsilon}}} \big(\partial_{\eta_2}\mathcal{N}_\varepsilon^+(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} + \varepsilon^{-\frac{1}{2}}\partial_{x_2}\big(\chi_s'(s)\Big|_{s=\frac{x_2}{\sqrt{\varepsilon}}} \mathcal{N}_\varepsilon^+(x_1,\tfrac{x}{\varepsilon})\big)$$
$$+ \varepsilon^{-1}\chi(\tfrac{x_2}{\sqrt{\varepsilon}})\big(\partial^2_{x_1\eta_1}\mathcal{N}_\varepsilon^+(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} + \chi(\tfrac{x_2}{\sqrt{\varepsilon}})\partial_{x_1}\big(\big(\partial_{x_1}\mathcal{N}_\varepsilon^+(x_1,\eta)\big)\big|_{\eta=x/\varepsilon}\big) +$$
$$+ \varepsilon^{-2}\chi(\tfrac{x_2}{\sqrt{\varepsilon}})\big(\Delta_\eta \mathcal{N}_\varepsilon^+(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} \quad \text{in} \quad \Omega_0; \quad (4.50)$$

$$\Delta_x R_\varepsilon(x) + \varepsilon^{-1}\big(\lambda_0 + \varepsilon\lambda_1\big) R_\varepsilon(x) = -\varepsilon\delta_{2,m}\lambda_0 v_0^+(x) + \varepsilon\lambda_1 v_1^+(x_1,0) +$$
$$+ \varepsilon^{-1}\big(\lambda_0 + \varepsilon\lambda_1\big)\mathcal{N}_{1,\varepsilon}^-(x_1,\tfrac{x}{\varepsilon}) + \varepsilon^{-1}\big(\partial^2_{x_1\eta_1}\mathcal{N}_{1,\varepsilon}^-(x_1,\eta)\big)\big|_{\eta=\frac{x}{\varepsilon}}$$
$$+ \partial_{x_1}\big(\partial_{x_1}\mathcal{N}_{1,\varepsilon}^-(x_1,\eta)\big)\big|_{\eta=\frac{x}{\varepsilon}} - \lambda_0 Z_1^{(0)}(\tfrac{x}{\varepsilon}) v_0^+(x_1,0) - 2\partial\eta_1 Z_1^{(0)}(\tfrac{x}{\varepsilon})\partial_{x_1} v_0^+(x_1,0) -$$
$$- \lambda_0 Z_1^{(1)}(\tfrac{x}{\varepsilon})\partial_{x_1} v_0^+(x_1,0) - \lambda_0 Z_1^{(2)}(\tfrac{x}{\varepsilon})\partial_{x_2} v_0^+(x_1,0) - 2\partial\eta_1 Z_1^{(1)}(\tfrac{x}{\varepsilon})\partial^2_{x_1x_1} v_0^+(x_1,0) \quad \text{in} \quad G_\varepsilon^{(1)}; \quad (4.51)$$

and

$$\Delta_x R_\varepsilon(x) + \varepsilon^{m-1}\big(\lambda_0 + \varepsilon\lambda_1\big) R_\varepsilon(x) = -\varepsilon T(\tfrac{x_1}{\varepsilon})\partial^3_{x_1x_1x_1} v_0^+(x_1,0) - \varepsilon\delta_{2,m}\lambda_0 v_0^+(x) +$$
$$+ \varepsilon^2\Big(\partial_{x_1}\big(T(\tfrac{x_1}{\varepsilon})\,\partial^2_{x_1x_1} v_1^-(x,\eta_1)\big)\Big)\Big|_{\eta_1=x_1/\varepsilon} + \varepsilon^2 T(\tfrac{x_1}{\varepsilon})\,\partial^2_{x_2x_2} v_1^-(x,\tfrac{x_1}{\varepsilon}) +$$
$$+ \varepsilon^{m-1}\big(\lambda_0 + \varepsilon\lambda_1\big) R_\varepsilon(x) + \varepsilon^{-\frac{3}{2}}\chi_s'(s)\Big|_{s=\frac{x_2}{\sqrt{\varepsilon}}}\big(\partial_{\eta_2}\mathcal{N}_{2,\varepsilon}^-(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} +$$
$$+ \varepsilon^{-\frac{1}{2}}\partial_{x_2}\big(\chi_s'(s)\Big|_{s=\frac{x_2}{\sqrt{\varepsilon}}}\mathcal{N}_{2,\varepsilon}^-(x_1,\tfrac{x}{\varepsilon})\big) + \varepsilon^{-1}\chi(\tfrac{x_2}{\sqrt{\varepsilon}})\big(\partial^2_{x_1\eta_1}\mathcal{N}_{2,\varepsilon}^-(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} +$$
$$+ \chi(\tfrac{x_2}{\sqrt{\varepsilon}})\partial_{x_1}\big(\big(\partial_{x_1}\mathcal{N}_{2,\varepsilon}^-(x_1,\eta)\big)\big|_{\eta=x/\varepsilon}\big) + \varepsilon^{-2}\chi(\tfrac{x_2}{\sqrt{\varepsilon}})\big(\Delta_\eta\mathcal{N}_{2,\varepsilon}^-(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} \quad \text{in} \quad G_\varepsilon^{(2)}. \quad (4.52)$$

Due to (2.20), (2.21), (2.23) – (2.28) we have

$$
\begin{aligned}
\varepsilon^{-2}\chi(\tfrac{x_2}{\sqrt{\varepsilon}})\big(\Delta_\eta \mathcal{N}^+_\varepsilon(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} &= -\chi(\tfrac{x_2}{\sqrt{\varepsilon}})\Big(2\Big(\big(\partial_{\eta_1} Z_1^{(0)}(\eta)\big)\big|_{\eta=x/\varepsilon}\partial_{x_1} v_0^+(x_1,0)+ \\
&\qquad + \big(\partial_{\eta_1} Z_1^{(1)}(\eta)\big)\big|_{\eta=x/\varepsilon}\partial^2_{x_1x_1} v_0^+(x_1,0)\Big) \quad \text{in} \quad \Omega_0; \\
\varepsilon^{-2}\chi(\tfrac{x_2}{\sqrt{\varepsilon}})\big(\Delta_\eta \mathcal{N}^-_{2,\varepsilon}(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} &= -\chi(\tfrac{x_2}{\sqrt{\varepsilon}})\Big(2\Big(\big(\partial_{\eta_1} Z_1^{(0)}(\eta)\big)\big|_{\eta=x/\varepsilon}\partial_{x_1} v_0^+(x_1,0)+ \\
&\qquad + \big(\partial_{\eta_1} Z_1^{(1)}(\eta)\big)\big|_{\eta=x/\varepsilon}\partial^2_{x_1x_1} v_0^+(x_1,0)\Big) \quad \text{in} \quad G_\varepsilon^{(2)}.
\end{aligned} \tag{4.53}
$$

4.3.2 Discrepancies on the boundary.

It easy to checked that $R_\varepsilon = 0$ on Γ_1 and $\partial_\nu R_\varepsilon = 0$ on the whole boundary $\partial\Omega_\varepsilon \setminus \Gamma_1$, except its vertical parts, on which

$$
\partial_{x_1} R_\varepsilon(x) = \chi(\tfrac{x_2}{\sqrt{\varepsilon}})\big(\partial_{x_1}\mathcal{N}^+_\varepsilon(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} \tag{4.54}
$$

on the vertical parts of $\partial\Omega_0$,

$$
\partial_{x_1} R_\varepsilon(x) = \big(\partial_{x_1}\mathcal{N}^-_{1,\varepsilon}(x_1,\eta)\big)\big|_{\eta=x/\varepsilon} \tag{4.55}
$$

on the vertical parts of $\partial G_\varepsilon^{(1)}$, and keeping in mind problems (2.20), (2.21) and (2.23) – (2.28), we get

$$
\begin{aligned}
\partial_{x_1} R_\varepsilon(x) = -\varepsilon\Big(Z_1^{(0)}(\tfrac{x}{\varepsilon})\,\partial_{x_1} v_0^+(x_1,0) + Z_1^{(1)}(\tfrac{x}{\varepsilon})\,\partial^2_{x_1x_1} v_0^+(x_1,0)\Big)+ \\
+\,\varepsilon^2\big(T(\tfrac{x_1}{\varepsilon})\,\partial^2_{x_1x_1} v_1^-(x,\eta_1)\big)\Big|_{\eta_1=x_1/\varepsilon} + \chi(\tfrac{x_2}{\sqrt{\varepsilon}})\big(\partial_{x_1}\mathcal{N}^-_{2,\varepsilon}(x_1,\eta)\big)\big|_{\eta=x/\varepsilon}
\end{aligned} \tag{4.56}
$$

on the vertical parts of $\partial G_\varepsilon^{(2)}$.

4.3.3 Discrepancies in the integral identity.

Multiplying (4.50) – (4.52) with arbitrary function $\psi \in \mathcal{H}_\varepsilon$, integrating by parts and taking (4.49) and (4.54)-(4.56) into account, we deduce

$$
\begin{aligned}
-\int_{\Omega_\varepsilon} \nabla_x R_\varepsilon \cdot \nabla_x \psi\, dx + \varepsilon^{m-1}\big(\lambda_0 + \varepsilon\lambda_1\big)\int_{\Omega_0\cup G_\varepsilon^{(2)}} R_\varepsilon\, \psi\, dx+ \\
+\,\varepsilon^{-1}\big(\lambda_0+\varepsilon\lambda_1\big)\int_{G_\varepsilon^{(1)}} R_\varepsilon \psi\, dx = \ell_\varepsilon(\psi),
\end{aligned} \tag{4.57}
$$

where the linear functional ℓ_ε is defined as follows

$$
\begin{aligned}
\ell_\varepsilon(\psi) := \varepsilon^{m-1}\big(\lambda_0 + \varepsilon\lambda_1\big) \int\limits_{\Omega_0\cup G_\varepsilon^{(2)}} R_\varepsilon\, \psi\, dx - \varepsilon\lambda_0\delta_{2,m} \int\limits_{\Omega_0\cup G_\varepsilon^{(2)}} v_0^+\, \psi\, dx+ \\
+\varepsilon \int\limits_{G_\varepsilon^{(1)}} \partial^2_{x_1x_1} v_1^+(x_1,0)\,\psi\, dx + \varepsilon\lambda_1 \int\limits_{G_\varepsilon^{(1)}} v_1^+(x_1,0)\,\psi\, dx-
\end{aligned}
$$

$$
\begin{aligned}
&-\varepsilon\frac{4h_1l_1\lambda_0}{h_2}\int\limits_{I_0} T(\tfrac{x_1}{\varepsilon})\partial_{x_1}v_0^+(x_1,0)\psi\,dx_1-\varepsilon\int\limits_{G_\varepsilon^{(2)}} T(\tfrac{x_1}{\varepsilon})\partial^3_{x_1x_1x_1}v_0^+(x_1,0)\psi\,dx+\\
&+\varepsilon^{-1}\big(\lambda_0+\varepsilon\lambda_1\big)\int\limits_{G_\varepsilon^{(1)}}\mathcal{N}^-_{1,\varepsilon}(x_1,\tfrac{x}{\varepsilon})\,\psi\,dx-\lambda_0\int\limits_{G_\varepsilon^{(1)}} Z_1^{(0)}(\tfrac{x}{\varepsilon})v_0^+(x_1,0)\,\psi\,dx-\\
&-\lambda_0\int\limits_{G_\varepsilon^{(1)}} Z_1^{(1)}(\tfrac{x}{\varepsilon})\partial_{x_1}v_0^+(x_1,0)\,\psi\,dx-\lambda_0\int\limits_{G_\varepsilon^{(1)}} Z_1^{(2)}(\tfrac{x}{\varepsilon})\partial_{x_2}v_0^+(x_1,0)\,\psi\,dx-\\
&-\varepsilon^2\int\limits_{G_\varepsilon^{(2)}} T(\tfrac{x_1}{\varepsilon})\,\big(\partial^2_{x_1x_1}v_1^-(x,\eta_1)\big)\big|_{\eta_1=x_1/\varepsilon}\,\partial_{x_1}\psi\,dx+\varepsilon^2\int\limits_{G_\varepsilon^{(2)}} T(\tfrac{x_1}{\varepsilon})\,\partial^2_{x_2x_2}v_1^-(x,\tfrac{x_1}{\varepsilon})\,\psi\,dx+\\
&\underbrace{+\,\varepsilon^{-1}\int\limits_{G_\varepsilon^{(1)}}\big(\partial^2_{x_1\eta_1}\mathcal{N}^-_{1,\varepsilon}(x_1,\eta)\big)|_{\eta=\frac{x}{\varepsilon}}\,\psi\,dx\,-}\\
&\underbrace{-\,2\int\limits_{G_\varepsilon^{(1)}}\partial\eta_1 Z_1^{(0)}(\eta)|_{\eta=\frac{x}{\varepsilon}}\,\partial_{x_1}v_0^+(x_1,0)\,\psi\,dx-2\int\limits_{G_\varepsilon^{(1)}}\partial\eta_1 Z_1^{(1)}(\eta)|_{\eta=\frac{x}{\varepsilon}}\,\partial^2_{x_1x_1}v_0^+(x_1,0)\,\psi\,dx\,+}\\
&+\varepsilon^{-\frac{3}{2}}\int\limits_{\Omega_0\cup G_\varepsilon^{(2)}}\chi'_s(s)\Big|_{s=\frac{x_2}{\sqrt{\varepsilon}}}\big(\partial_{\eta_2}\mathcal{N}_\varepsilon(x_1,\eta)\big)|_{\eta=\frac{x}{\varepsilon}}\,\psi\,dx-\varepsilon^{-\frac{1}{2}}\int\limits_{\Omega_0}\chi'_s(s)\Big|_{s=\frac{x_2}{\sqrt{\varepsilon}}}\mathcal{N}_\varepsilon(x_1,\frac{x}{\varepsilon})\,\partial_{x_2}\psi\,dx-\\
&-\varepsilon^{-\frac{1}{2}}\int\limits_{G_\varepsilon^{(2)}}\chi'_s(s)\Big|_{s=\frac{x_2}{\sqrt{\varepsilon}}}\mathcal{N}_\varepsilon(x_1,\frac{x}{\varepsilon})\,\partial_{x_2}\psi\,dx+\varepsilon^{\frac{1}{2}}\int\limits_{G_\varepsilon^{(2)}}\chi'_s(s)\Big|_{s=\frac{x_2}{\sqrt{\varepsilon}}}T(\tfrac{x_1}{\varepsilon})\partial_{x_1}v_0^+(x_1,0)\,\partial_{x_2}\psi\,dx-\\
&-\varepsilon^{\frac{1}{2}}\int\limits_{G_\varepsilon^{(2)}}\chi'_s(s)\Big|_{s=\frac{x_2}{\sqrt{\varepsilon}}}T(\tfrac{x_1}{\varepsilon})\partial_{x_1}v_0^+(x_1,0)\,\partial_{x_2}\psi\,dx+\\
&\overbrace{+\,\varepsilon^{-1}\int\limits_{\Omega_0\cup G_\varepsilon^{(2)}}\chi(\tfrac{x_2}{\sqrt{\varepsilon}})\big(\partial^2_{x_1\eta_1}\mathcal{N}_\varepsilon(x_1,\eta)\big)|_{\eta=\frac{x}{\varepsilon}}\,\psi\,dx\,-}\\
&\overbrace{-\,2\int\limits_{\Omega_0\cup G_\varepsilon^{(2)}}\chi(\tfrac{x_2}{\sqrt{\varepsilon}})\Big(\big(\partial_{\eta_1}Z_1^{(0)}(\eta)\big)\Big|_{\eta=x/\varepsilon}\partial_{x_1}v_0^+(x_1,0)+\big(\partial_{\eta_1}Z_1^{(1)}(\eta)\big)\Big|_{\eta=x/\varepsilon}\partial^2_{x_1x_1}v_0^+(x_1,0)\Big)\psi\,dx}\,-\\
&-\int\limits_{\Omega_0\cup G_\varepsilon^{(2)}}\chi(\tfrac{x_2}{\sqrt{\varepsilon}})\big(\partial_{x_1}\mathcal{N}_\varepsilon(x_1,\eta)\big)|_{\eta=\frac{x}{\varepsilon}}\,\partial_{x_1}\psi\,dx-\int\limits_{G_\varepsilon^{(1)}}\big(\partial_{x_1}\mathcal{N}^-_{1,\varepsilon}(x_1,\eta)\big)|_{\eta=\frac{x}{\varepsilon}}\,\partial_{x_1}\psi\,dx.
\end{aligned}
\tag{4.58}
$$

Here $\mathcal{N}_\varepsilon$ coincides with $\mathcal{N}_\varepsilon^+$ on Ω_0 and with $\mathcal{N}^-_{2,\varepsilon}$ on $G_\varepsilon^{(2)}$.

Let us estimate $|\ell_\varepsilon(\psi)|$. It is easy to see that the integrals in the first line of (4.58) is of order $\mathcal{O}(\varepsilon^{m-1})$, if $m\geq 3$ and $\mathcal{O}(\varepsilon^2)$, if $m=2$.

The integrals in second line can be estimated with the help of the Friedrichs-type inequality (2.69) in the following way:

$$\varepsilon\Big|\int\limits_{G_\varepsilon^{(1)}} \partial^2_{x_1x_1}v_1^+(x_1,0)\,\psi\,dx\Big| \le \varepsilon^2 C_1\|\psi\|_{\mathcal{H}_\varepsilon}, \qquad \varepsilon\lambda_1\Big|\int\limits_{G_\varepsilon^{(1)}} v_1^+(x_1,0)\,\psi\,dx\Big| \le \varepsilon^2 C_1\|\psi\|_{\mathcal{H}_\varepsilon}.$$

The integrals in the third line can be estimated by means of Lemma 1.6 from [7, §1] and they are of order $\mathcal{O}(\varepsilon^2)$. The main term in the integrals of the fourth and the fifth lines of (4.58) we bound again with the help of (2.69) as follows

$$\varepsilon\lambda_1\Big|\int_{G_\varepsilon^{(1)}} Z_1^{(0)}(\tfrac{x}{\varepsilon})\,v_0^+(x_1,0)\,\psi\,dx\Big| \le \varepsilon^{\frac32}C_1\|\psi\|_{\mathcal{H}_\varepsilon}\sqrt{\int_{G_\varepsilon^{(1)}}\Big|Z_1^{(0)}(\tfrac{x}{\varepsilon})\Big|^2dx}$$
$$\le \varepsilon^2 C_2\|\psi\|_{\mathcal{H}_\varepsilon}\sqrt{\int_{\Pi_{l_1}}\Big|Z_1^{(0)}(\eta)\Big|^2 d\eta} \le \varepsilon^2 C_3\|\psi\|_{\mathcal{H}_\varepsilon}. \quad (4.59)$$

Similarly we estimate the underbraced integrals and it is of order $\mathcal{O}(\varepsilon^2)$ as well. The integrals in the sixth line due to Lemma 1.6 from [7, §1], are of order $\mathcal{O}(\varepsilon^3)$.

Using the asymptotic relations (2.29) – (2.31), we conclude that the integrals in the ninth line of (4.58) are exponentially small, the integrals in the tenth line are together of order $\mathcal{O}(\varepsilon^2)$ and the integral in the eleventh line is of order $\mathcal{O}(\varepsilon^2)$ due to Lemma 1.6 from [7, §1]. Thanks to Lemma 3.1 ([14]) the overbraced integrals in (4.58) are of order $\mathcal{O}(\varepsilon^{2-\delta})$, where δ is arbitrary positive number.

The first integral in the last line is of order $\mathcal{O}(\varepsilon^{\frac32})$ and the last integral in the last line is of order $\mathcal{O}(\varepsilon^2)$.

Thus, we have

$$|\ell_\varepsilon(\psi)| \le c_2\,\varepsilon^{\frac32}\|\psi\|_{\mathcal{H}_\varepsilon}. \quad (4.60)$$

With the help of operator $A_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_\varepsilon$ defined in (1.22) we deduce from (4.57) and (4.60) the following inequality

$$\|\, R_\varepsilon \,-\, (\lambda_0+\varepsilon\lambda_1)A_\varepsilon R_\varepsilon\,\|_{\mathcal{H}_\varepsilon} \,\le\, c\,\varepsilon^{\frac32}. \quad (4.61)$$

4.4 The main results

The justification of the asymptotics can be provided in the same way as in Section 3.

In this case Theorem 3.2 holds true without changes.

Let $\lambda_0^{(n+1)} = \ldots = \lambda_0^{(n+r)}$ be an $r-$multiple eigenvalue of the homogenized problem (2.48); the corresponding eigenfunctions $v_0^{+,n+1},\ldots,v_0^{+,n+r}$ are orthonormalized in $\mathcal{V}_0$. Using formula (4.33) for $m\ge 3$ ((4.42) for $m=2$), we successively construct next terms $\varepsilon\lambda_1^{(n+i)}$, $i=1,\ldots,r$, of the asymptotic expansion (4.1) and define the unique solution $v_1^{+,n+i}$ to problem (4.35) for $m\ge 3$ ((4.43) for $m=2$).

We formulate next theorem under assumption that all $\lambda_1^{(n+i)}$, $i=1,\ldots,r$, are different and

$$\lambda_1^{(n+1)} < \lambda_1^{(n+2)} < \ldots < \lambda_1^{(n+r)}. \quad (4.62)$$

Theorem 4.1. *Let inequalities* (4.62) *are satisfied. Then for any positive* δ *small enough and for any* $i \in \{1, \dots, r\}$ *there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *we have*

$$\Big|\varepsilon^{1-\alpha}\lambda_{n+i}(\varepsilon) - \big(\lambda_0^{(n+i)} + \varepsilon\lambda_1^{(n+i)}\big)\Big| \;\le\; C_1(n)\,\varepsilon^{\frac{3}{2}},$$

and

$$\Big\|\varepsilon^{-\frac{\alpha-1}{2}}u_{n+i}(\varepsilon,\cdot) - \frac{R_\varepsilon^{(n+i)}}{\|R_\varepsilon^{(n+i)}\|_{\mathcal{H}_\varepsilon}}\Big\|_{H^1(\Omega_\varepsilon)} \le C_2(n)\;\varepsilon^{\frac{3}{2}}, \tag{4.63}$$

where $R_\varepsilon^{(n+i)}$ *is the approximating function constructed by formula* (4.45) *with the help of solutions* $v_0^{+,\,n+i}$, $v_1^{-,\,n+i}$, $v_1^{+,\,n+i}$ *and* $Z_1^{(k)}, X_2^{(k)}$ $k = 0, 1, 2$ *and* $Z_2^{(\beta)}$, $\beta \in \mathcal{B}$.

5 High-frequency cell-vibrations

It is known that for spectral problems with concentrated masses there are low- and high-frequency vibrations (see [6, 8, 9, 10, 12, 7, 11]). In [11] it was showed, that frequencies of high-vibrations can be presented only on the following eigenfrequency range $[\mathcal{T}, +\infty)$, where $\mathcal{T} = \sup_{n\in\mathbb{N}} \limsup_{\varepsilon\to 0} \lambda_n(\varepsilon)$ is the threshold of low eigenvalues.

For our problem (1.1) (see Theorem 5.2 in [2], and Lemma 1.1) we have

$$\mathcal{T} = \begin{cases} \left(\dfrac{\pi}{2\,l_2}\right)^2 & \text{in the case } \alpha \in (0, 1], \\ 0 & \text{in the case } \alpha \in (1, +\infty). \end{cases}$$

This magnitude again indicates the qualitative difference in the asymptotic behaviour of the eigenvalues and eigenfunctions of problem (1.1) for different values of the parameter α.

In [2] we have studied some kinds of high-frequency vibrations in problem (1.1) in the case $\alpha \in (0, 1]$. Here we show that there is a new kind of high-frequency vibrations in problem (1.1) for any $\alpha \in (0, +\infty)$. Usually this kind of high-frequency vibrations is connected with the corresponding spectral local problem. In our case it corresponds to the following *spectral cell-problem*:

$$\begin{cases} -\Delta_\eta\,\mathcal{Z}(\eta) &= \begin{cases} 0, & \eta \in \Pi^+ \cup \Pi^-, \\ \Lambda\,\mathcal{Z}, & \eta \in \Pi_{l_1}, \end{cases} \\ \partial_{\eta_1}\mathcal{Z}(\eta) &= 0, \qquad\qquad \eta \in \partial\Pi_{\|}; \\ \partial^s_{\eta_1}\mathcal{Z}(0,\eta_2) &= \partial^s_{\eta_1}\mathcal{Z}(1,\eta_2), \quad \eta_2 > 0, \quad s = 0, 1; \\ \partial_{\eta_2}\mathcal{Z}(\eta_1, 0) &= 0, \qquad\qquad (\eta_1, 0) \in \partial\Pi, \\ \partial_{\eta_2}\mathcal{Z}(\eta_1, -l_1) &= 0, \qquad\qquad (\eta_1, -l_1) \in \partial\Pi. \end{cases} \tag{5.1}$$

Let $\widehat{C}_0^\infty(\overline{\Pi})$ be a space of infinitely differentiable functions in $\overline{\Pi}$ that satisfy the periodicity conditions (2.18) and are finite in η_2, i.e.,

$$\forall v \in \widehat{C}_0^\infty(\overline{\Pi}) \quad \exists R > 0 \quad \forall \eta \in \overline{\Pi} \quad |\eta_2| \ge R \,:\, v(\eta) = 0\,.$$

Let $\mathcal{H}$ be the completion of the space $\widehat{C}_0^\infty(\overline{\Pi})$ with respect to the norm

$$\|u\|_{\mathcal{H}} = \left(\|\nabla_\eta u\|^2_{L_2(\Pi)} + \|\rho u\|^2_{L_2(\Pi)}\right)^{1/2},$$

where $\rho(\eta_2) = (1 + |\eta_2|)^{-1}\,, \eta_2 \in \mathbb{R}\,.$

A number Λ is an eigenvalue of problem (5.1) if there exists a function $\mathcal{Z} \in \mathcal{H} \setminus \{0\}$ such that the following integral identity holds:

$$\int_{\Pi} \nabla_\eta \mathcal{Z} \cdot \nabla_\eta v \, d\eta = \Lambda \int_{\Pi_{l_1}} \mathcal{Z}\, v \, d\eta \qquad \forall v \in \mathcal{H}. \tag{5.2}$$

With the help of Hardy's inequality

$$\int_0^{+\infty} (1+\eta_2)^{-2} \phi^2(\eta_2)\, d\eta_2 \le 4 \int_0^{+\infty} |\partial_{\eta_2} \phi|^2 \, d\eta_2\,, \quad \forall \phi \in C^1([0,+\infty))\,, \ \ \phi(0) = 0,$$

we can show (see for instance Lemma 3.1 in [15]) that problem (5.1) is equivalent to a spectral problem for some positive, self-adjoint, compact operator. Thus, the eigenvalues of problem (5.1) form the sequence

$$0 = \Lambda_0 < \Lambda_1 < \Lambda_2 \le \ldots \le \Lambda_n \le \cdots \to +\infty \quad \text{as} \quad n \to \infty,$$

with the classical convention of repeated eigenvalues. The respective sequence of the corresponding eigenfunctions $\{\mathcal{Z}_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{H}$ can be orthonormalized as follows

$$\int_{\Pi_{l_1}} \mathcal{Z}_n\, \mathcal{Z}_k \, d\eta = \delta_{n,k}, \quad \{n,\, k\} \in \mathbb{Z}_+. \tag{5.3}$$

Also, it follows from the results of §3.1([15]), that the eigenfunctions have the asymptotics $\mathcal{Z}_n(\eta) = \mathcal{O}(\exp(-2\pi\eta_2))$ as $\eta_2 \to +\infty$ in Π^+, and $\mathcal{Z}_n(\eta) = C_n(h_2) + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2))$ as $\eta_2 \to -\infty$ in $\eta \in \Pi^-$, for $n \in \mathbb{N}$. But now taking into account the harmonicity of $\mathcal{Z}_n(\eta)$ in $\Pi^+ \cup \Pi^-$, we can state that the constant $C_n(h_2) = 0$ and in addition $\mathcal{Z}_n(\eta) = 0$ for $\eta \in \Pi^-$ and there exists $\varrho_0 > 0$ such that for all $\eta \in \Pi^+$, $\eta_2 \ge \varrho_0$ we have $\mathcal{Z}_n(\eta) = 0$.

Now let us take any positive eigenvalue Λ of problem (5.1) and the corresponding eigenfunction $\mathcal{Z}$ that is even in η_1 with respect to $\frac{1}{2}$ (due to the symmetry of the domain Π with respect to the line $\{\eta : \ \eta_1 = \frac{1}{2}\}$ a such egenfunction always does exist). Then we extend it periodically in the direction $O\eta_1$. It should be noted here that thanks to (5.3) we have

$$\big\|\mathcal{Z}(\tfrac{\cdot}{\varepsilon})\big\|_\varepsilon = \sqrt{(\mathcal{Z},\mathcal{Z})_{\mathcal{H}_\varepsilon}} \ \sim \ c\,\Lambda^{\frac{1}{2}} \quad \text{as} \quad \varepsilon \to 0. \tag{5.4}$$

Substituting $\mathcal{Z}(\frac{\cdot}{\varepsilon})$ and $\varepsilon^{\alpha-2}\Lambda$ in the differential equation of problem (1.1) instead of $u(\varepsilon,\cdot)$ and $\lambda(\varepsilon)$ respectively and taking into account properties of $\mathcal{Z}$ mentioned above, we get

$$\begin{gathered} \Delta_x \left(\mathcal{Z}(\tfrac{x}{\varepsilon})\right) + \varepsilon^{\alpha-2}\Lambda\mathcal{Z}(\tfrac{x}{\varepsilon}) = \varepsilon^{\alpha-2}\Lambda\mathcal{Z}(\tfrac{x}{\varepsilon}) \quad \text{in} \quad \Omega_0; \\ \Delta_x \left(\mathcal{Z}(\tfrac{x}{\varepsilon})\right) + \varepsilon^{\alpha-2}\Lambda\mathcal{Z}(\tfrac{x}{\varepsilon}) = 0 \quad \text{in} \quad G_\varepsilon^{(2)}; \\ \Delta_x \left(\mathcal{Z}(\tfrac{x}{\varepsilon})\right) + \varepsilon^{-\alpha}\varepsilon^{\alpha-2}\Lambda\mathcal{Z}(\tfrac{x}{\varepsilon}) = 0 \quad \text{in} \quad G_\varepsilon^{(1)}; \end{gathered}$$

and $\mathcal{Z}(\frac{\cdot}{\varepsilon})$ satisfies all boundary conditions of problem (1.1). As a result, we have

$$\left(\mathcal{Z}(\tfrac{\cdot}{\varepsilon}), v\right)_{\mathcal{H}_\varepsilon} - \varepsilon^{\alpha-2}\Lambda \left(\mathcal{A}_\varepsilon \mathcal{Z}(\tfrac{\cdot}{\varepsilon}), v\right)_{\mathcal{H}_\varepsilon} = \varepsilon^{\alpha-2}\Lambda \int_{\Omega_0} \mathcal{Z}(\tfrac{\cdot}{\varepsilon})\, v \, dx \quad \forall v \in \mathcal{H}_\varepsilon, \tag{5.5}$$

where $\mathcal{A}_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_\varepsilon$ is the corresponding operator to problem (1.1) and it defined by the following equality

$$(\mathcal{A}_\varepsilon u, v)_{\mathcal{H}_\varepsilon} = \int\limits_{\Omega_0 \cup G_\varepsilon^{(2)}} u\, v\, dx + \varepsilon^{-\alpha} \int\limits_{G_\varepsilon^{(1)}} u\, v\, dx \quad \forall\, u, v \in \mathcal{H}_\varepsilon.$$

Since

$$\left| \int_{\Omega_0} \mathcal{Z}(\tfrac{\cdot}{\varepsilon})\, v\, dx \right| = \left| \int_{\Omega_0^\varepsilon} \mathcal{Z}(\tfrac{\cdot}{\varepsilon})\, v\, dx \right| \le \sqrt{\int_{\Omega_0^\varepsilon} |\mathcal{Z}(\tfrac{\cdot}{\varepsilon})|^2\, dx}\ \sqrt{\int_{\Omega_0^\varepsilon} |v|^2\, dx}$$

$$\le \sqrt{\varepsilon}\ \sqrt{\int_{\Pi^+} |\mathcal{Z}(\eta)|^2\, d\eta}\ \ \varepsilon^{\frac{1}{2}-\delta}\, \|v\|_{H^1(\Omega_0)} \le C_0\, \varepsilon^{1-\delta}\, \|v\|_\varepsilon \quad (5.6)$$

(in the last line we used Lemma 1.5 from [7]; here $\Omega_0^\varepsilon := \Omega_0 \cap \{x : \ x_2 \in (0, \varepsilon\, \varrho_0)\}$), with the help of the first statement of the Vishik–Lyusternik Lemma 12 ([36]) and (5.4), we deduce

$$\min_{n\in\mathbb{N}} \left| \frac{1}{\varepsilon^{\alpha-2}\Lambda} - \frac{1}{\lambda_n(\varepsilon)} \right| \le \|\mathcal{Z}(\tfrac{\cdot}{\varepsilon})\|_\varepsilon^{-1} \left\| \mathcal{A}_\varepsilon \mathcal{Z}(\tfrac{\cdot}{\varepsilon}) \ - \ \frac{1}{\varepsilon^{\alpha-2}\Lambda} \mathcal{Z}(\tfrac{\cdot}{\varepsilon}) \right\|_\varepsilon \le C_1 \varepsilon^{1-\delta}, \qquad (5.7)$$

where δ is arbitrary number from the interval $(0, 1)$.

Taking into account the second statement of the Vishik–Lyusternik Lemma 12 ([36]), we prove the following theorem.

Theorem 5.1. *For any positive eigenvalue* Λ *of problem* (5.1) *there exists an eigenvalue* $\lambda_{n(\varepsilon)}(\varepsilon)$ *of problem* (1.1) ($n(\varepsilon) \to +\infty$ *as* $\varepsilon \to 0$), *such that*

$$\left| \frac{1}{\varepsilon^{\alpha-2}\Lambda} - \frac{1}{\lambda_{n(\varepsilon)}(\varepsilon)} \right| \le C_1 \varepsilon^{1-\delta}. \qquad (5.8)$$

In additon, for any $\delta \in (0, \frac{1}{2})$ *there exists a finite linear combination*

$$\widetilde{U}_\varepsilon(x) = \sum_{i=0}^{p(\varepsilon)} d_i(\varepsilon) u_{k(\varepsilon)+i}(\varepsilon, x), \quad x \in \Omega_\varepsilon \quad \left(\ \|\widetilde{u}_\varepsilon\|_\varepsilon^2 = 1 = \sum_{i=0}^{p(\varepsilon)} d_i^2(\varepsilon) \lambda_{k(\varepsilon)+i}(\varepsilon) \ \right),$$

of eigenfunctions corresponding respectively to all eigenvalues $\lambda_{k(\varepsilon)}^{-1}(\varepsilon), \lambda_{k(\varepsilon)+1}^{-1}(\varepsilon), \ldots, \lambda_{k(\varepsilon)+p(\varepsilon)}^{-1}(\varepsilon)$ *of the operator* $\mathcal{A}_\varepsilon$ *from the segment*

$$\left[\frac{1}{\varepsilon^{\alpha-2}\Lambda} - C_1 \varepsilon^\delta, \frac{1}{\varepsilon^{\alpha-2}\Lambda} + C_1 \varepsilon^\delta \right],$$

such that

$$\left\| \frac{\mathcal{Z}(\frac{\cdot}{\varepsilon})}{\|\mathcal{Z}(\frac{\cdot}{\varepsilon})\|_\varepsilon} - \widetilde{U}_\varepsilon \right\|_\varepsilon \ \le \ 2\varepsilon^{1-2\delta}. \qquad (5.9)$$

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References

[1] G.A. Chechkin, T.A. Mel'nyk, *Asymptotic Behavior of the Eigenvalues and Eigenfunctions to a Spectral Problem in a Thick Cascade Junction with Concentrated Masses.* Oberwolfach Preprints (OWP), Preprint **2011 - 12,** Oberwolfach - Walke, MFO, Germany, 2011, ISSN 1864-7596.

[2] G.A. Chechkin, T.A. Mel'nyk, *Asymptotics of Eigenelements to Spectral Problem in Thick Cascade Junction with Concentrated Masses.* Applicable Analysis. **91**, No 6, (2012): 1055–1095; DOI: 10.1080/00036811.2011.602634.

[3] G.A. Chechkin, T.A. Mel'nyk, *Spatial-Skin Effect for a Spectral Problem with "Slightly Heavy" Concentrated Masses in a Thick Cascade Junction.* Oberwolfach preprints (OWP), Preprint **2012 - 10,** Oberwolfach - Walke, MFO, Germany, 2012, ISSN 1864-7596.

[4] T.A. Mel'nyk, G.A. Chechkin, *On New Types of Vibrations of Thick Cascade Junctions with Concentrated Masses.* Doklady Mathematics **87**, No 1, (2013): 102–106 (translated from Doklady Akademii Nauk **448**, No. 6 (2013): 642–647).

[5] G.A. Chechkin, T.A. Mel'nyk, *Spatial-Skin Effect for Eigenvibrations of a Thick Cascade Junction with "Heavy" Concentrated Masses.* Mathematical Methods in Applied Sciences (M2AS). **2013**; DOI: 10.1002/mma.2785

[6] E. Sánchez-Palencia, *Perturbation of eigenvalues in thermo-elasticity and vibration of systems with concentrated masses,* Trends and Applications of Pure Mathematics to Mechanics. Lecture Notes in Phys. **195** (Springer Verlag, 1984) 346–368.

[7] O.A. Oleinik, G.A. Yosifian and A.S. Shamaev, **Mathematical Problems in Elasticity and Homogenization,** Amsterdam: North-Holland, 1992. (Russian edition: Moscow: Izd-stvo MGU. - 1990).

[8] T. A. Mel'nyk, S. A. Nazarov, *The asymptotic structure of the spectrum in the problem of harmonic oscillations of a hub with heavy spokes,* Russian Acad. Sci. Dokl. Math. **48**, No. 3, (1994) 428-432; (Translated from Doklady RAN, **333**, no.1, (1993) 13-15).

[9] M. Lobo, M.E. Pérez, *On Vibrations of a Body With Many Concentrated Masses Near the Boundary.* Math. Models and Methods in Appl. Sci, **3,** 2. (1993) 249–273.

[10] M.Lobo, M.E. Pérez, *A Skin Effect for Systems with Many Concentrated Masses.* C.R. Acad. Sci. Paris. Série II b, **327** (1999) 771–776.

[11] T. A. Mel'nyk, *Vibrations of a thick periodic junction with concentrated masses,* Mathematical Models and Methods in Applied Sciences, **11,** no. 6, (2001) 1001–1029.

[12] M. Lobo, M.E. Pérez, *Local problems for vibrating systems with concentrated masses: a review.* C.R. Acad. Sci. Paris. Mecanique 2003; **331**: 303–317.

[13] T.A. Mel'nyk, S.A. Nazarov, *Asymptotic structure of the spectrum of the Neumann problem in a thin comb-like domain,* C.R. Acad. Sci.,Paris. **319**, Serie 1, (1994) 1343-1348.

[14] T.A. Mel'nyk, S.A. Nazarov, *Asymptotics of the Neumann spectral problem solution in a domain of "thick comb" type,* Journal of Math. Sci. **85**, no. 6, (1997) 2326–2346; (Translated from Trudy Seminara imeni I.G. Petrovskogo. **19** (1996): 138–173).

[15] T. A. Mel'nyk, *Asymptotic analysis of a spectral problem in a periodic thick junction of type 3 : 2 : 1,* Mathematical Methods in the Applied Sciences, **23,** (2000) 321–346.

[16] T.A. Mel'nyk, *Asymptotic behavior of eigenvalues and eigenfunctions of the Steklov problem in a thick periodic junction.* Nonlinear oscillations, **4,** no. 1, (2001) 91–105.

[17] T.A. Mel'nyk, *Scheme of investigation of the spectrum of a family of perturbed operators and its application to spectral problems in thick junctions.* Nonlinear oscillations, **6,** no. 2 (2003) 232–249.

[18] C. Leal and E. Sanchez-Palencia, *Perturbation of the eigenvalues of a membrane with concentrated mass, Quart. Appl. Math.* **47** (1989) 93-103.

[19] O. A. Oleinik, J. Sánchez-Hubert, and G. A. Yosifian, *On vibration of a membrane with concentrated masses, Bull. Sci. Math.* **115**, No. 2, (1991) 1-27.

[20] M. Lobo and M.E. Pérez, *Vibrations of a membrane with many concentrated masses near the boundary, Math. Models Meth. Appl. Sci.* **5**, No. 5, (1995) 565-585.

[21] S.A. Nazarov, *Interaction of concentrated masses in a harmonically oscillating spatial body with Neumann boundary conditions, Model. Math. Anal. Numer.* **27**, No.6, (1993) 777-799.

[22] G.A. Chechkin, *Asymptotic expansion of eigenvalues and eigenfunctions of an elliptic operator in a domain with many "light" concentrated masses situated on the boundary. Two-dimensional case.* Izvestia: Mathematics 2005; **69**(4): 805–846 (Translated from *Izvestia RAN. Ser. Mat.* 2005; **69**(4): 161–204).

[23] G.A. Chechkin, *Asymptotic expansion of eigenelements of the Laplace operator in a domain with a large number of "light" concentrated masses sparsely situated on the boundary. Two-dimensional case.* Trans. Moscow Math. Soc. 2009; **70**: 71–134 (Translated from *Trudy Moskovskogo Matematicheskogo Obshchestva.* 2009; **70**: 102–182).

[24] V.A. Kondrat'ev, O.A. Oleinik, *Boundary-value problems for partial differential equations in non-smooth domains,* Russian Math. Survays, **38,** no. 2, (1983), 1–86; (Translated from Uspekhi Mat. Nauk, **38,** no. 2, (1983), 3–76).

[25] E.M. Landis, G.P. Panasenko, A variant of a theorem of Phragmen-Lindelof type for elliptic equations with coefficients that are periodic in all variables but one. (Russian) Trudy Sem. Petrovsk. No. 5 (1979), 105–136.

[26] E.M. Landis, G.P. Panasenko, *A theorem on the asymptotics of solutions of elliptic equations with coefficients periodic in all variables except one.* Soviet Mathematics, **18**, no. 4, (1977), 1140–1143; (Translated from Dokl. Akad. Nauk SSSR **235**, no. 6, (1977), 1253–1255.)

[27] S.A. Nazarov, B.A. Plamenevskii, **Elliptic problems in domains with piecewise smooth boundaries**, Berlin: Walter de Gruyter, 1994.

[28] Il'in A.M. A Boundary–value Problem for an Elliptic Equation of Second Order in a Domain with a Narrow Slit. I. The Two-Dimensional Case, Mathematics of the USSR - Sbornik, **28**: 4 (1976), 459–480; (Translated from Mat. Sb. (N.S.) **99(141)** (4, 1976): 514–537).

[29] Il'in A.M. A Boundary–value Problem for an Elliptic Equation of Second Order in a Domain with a Narrow Slit. II. Domain with a Small Opening. Mathematics of the USSR - Sbornik, **32**: 2 (1977), 227–244; (Translated from Mat. Sb. (N.S.) **103(145)** (2, 1977): 265–284.

[30] Il'in A.M. Study of the Asymptotic Behavior of the Solution of an Elliptic Boundar–Value Problem in a Domain with a Small Hole (Russian). *Trudy Seminara I.G. Petrovskogo.* **6** (1981): 57–82.

[31] Il'in A.M., **Matching of asymptotic expansions of solutions of boundary value problems. Translations of Mathematical Monographs, 102.** American Mathematical Society, Providence, RI, 1992.

[32] G.A. Chechkin, T.A. Mel'nyk, *Homogenization of a boundary-value problem in a thick cascade junction,* J. Math. Sci. **154** (1, 2008) 50-77, (Translated from Problemy Mat. Analiza **37** (2008) 47-72).

[33] G.A. Chechkin, T.A. Mel'nyk, *Homogenization of a boundary-value problem in thick three-dimensional multi-level junction,* Sbornik: Mathematics **200** (3, 2009) 357–383 (Translated from Mathem. Sbornik **200** (3, 2009) 49–74).

[34] T.A. Mel'nyk, *Homogenization of the Poisson equation in a thick periodic junction,* Zeitschrift für Analysis und ihre Anwendungen, **18,** no. 4, (1999) 953–975.

[35] Chechkin, G.A., Piatnitski, A.L., and Shamaev A.S. **Homogenization. Methods and Applications**. Translations of Mathematical Monographs **234**. American Mathematical Society, Providence, RI. 2007.

[36] M.I. Vishik and L.A. Lyusternik, *Regular degeneration and boundary layer for linear differential equations with small parameter,* Amer. Math. Soc. Transl. (2) **20** (1962), 239-364; (Translated from Uspekhi Mat. Nauk. **12**, no. 5, (1957) 3–122, (in Russian).