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Cohesive zone-type delamination in visco-elasticity

To the occasion of the 60th anniversary of Tomaš Roubíček

Marita Thomas[1], Chiara Zanini [2]

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[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: marita.thomas@wias-berlin.de

[2] Department of Mathematical Sciences
Politecnico di Torino
Corso Duca degli Abruzzi 24
I-10129 Torino
Italy
E-Mail: chiara.zanini@polito.it

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We study a model for the rate-independent evolution of cohesive zone delamination in a visco-elastic solid, also exposed to dynamics effects. The main feature of this model, inspired by [OP99], is that the surface energy related to the crack opening depends on the history of the crack separation between the two sides of the crack path, and allows for different responses upon loading and unloading.

Due to the presence of multivalued and unbounded operators featuring non-penetration and the 'memory'-constraint in the strong formulation of the problem, we prove existence of a weaker notion of solution, known as *semistable energetic solution*, pioneered in [Rou09] and refined in [RT15a].

1 Delamination models with a process zone

In the last decade cohesive zone models have received great attention from a mathematical point of view, and different aspects have been considered. The main feature of the cohesive zone models, cf. e.g. [OP99, p. 1268], *pioneered by Dugdale [Dug60], Barenblatt [Bar62], Rice [Ric68], and others– is to regard fracture as a gradual phenomenon in which separation takes place across an extended crack 'tip', or cohesive zone and is resisted by cohesive tractions.* In this contribution we consider the case where crack initiation and propagation are confined to a prescribed interface $\Gamma_{\rm C} \subset \mathbb{R}^{d-1}$ between two parts Ω_+, Ω_- of a visco-elastic solid $\Omega \subset \mathbb{R}^d$. This assumption in combination with cohesive zone fracture, has been employed by several authors, see, e.g., [DMZ07, Cag08, CT11, Alm16, CLO16], and it is also related to the lower semicontinuity of functionals including cohesive surface energies, cf. [BBB95, DMOT15].

Following [OP99] the cohesive phenomenon in the setting of small strains is modeled by the pair $(u(t,x), \zeta(t,x))$, where $u : [0,T] \times \Omega \setminus \Gamma_{\rm C} \to \mathbb{R}^d$ is the displacement and $\zeta : [0,T] \times \Gamma_{\rm C} \to [0,\infty)$ is an internal variable. In cohesive zone models the internal variable $\zeta(\cdot,x)$ has the purpose of 'memory', as it tracks the history of the maximal separation of the two parts of the body in the material point $x \in \Gamma_{\rm C}$ during the time span $[0,T]$. In other words, denoting by $[\![u(t,x)]\!]_{\mathbf{n}} = (u|_{\Omega_+}(t,x) - u|_{\Omega_-}(t,x)) \cdot \mathbf{n}$ the jump of the displacements (=separation) at time $t \in [0,T]$ in the point $x \in \Gamma_{\rm C}$ in direction $\mathbf{n}$ normal to $\Gamma_{\rm C}$, then, ideally, $\zeta(t,x) := \sup_{s\in[0,t]} [\![u(s,x)]\!]_{\mathbf{n}}$. Thus, $\zeta(t,x) = 0$ means that the two parts of the body have not been separated yet in $x \in \Gamma_{\rm C}$ up to time t, while $\zeta(t,x) = \alpha$ for some $\alpha > 0$ tells us that there was a time $s_\alpha \in [0,t]$, when $[\![u(s_\alpha,x)]\!]_{\mathbf{n}} = \alpha$ and that $[\![u(s,x)]\!]_{\mathbf{n}} \le \alpha$ for every $s \in [0,t]$.

Adhesive contact and cohesive zone models: Building on approaches by e.g. [Fré88, Fré02], there is a large amount of analytical results on *adhesive contact* delamination models, see e.g. [MM05, KMR06, BBR08, BBR09, RSZ09, RR11, MRT12, RMP13, RPM13, RR13, VMR14, RKZ14, KPR15, RTP15, Sca15, PMR16, AOR16]. In *adhesive contact* models, the internal variable $z : [0,T] \times \Gamma_{\rm C} \to [0,1]$ has the meaning of *active bonds* in the adhesive that glues the two parts of the body along $\Gamma_{\rm C}$. Hence, $z(t,x) = 1$ indicates that the adhesive is sound in $(t,x) \in [0,T] \times \Gamma_{\rm C}$, while $z(t,x) = 0$ means that it is completely broken.

Due to the different meaning of the internal variables, it turns out that $t \mapsto \zeta(t)$ is increasing in time while, the internal variable $t \mapsto z(t)$ for adhesive contact models is decreasing. Moreover the two classes of models are also equipped with different surface energies. For *adhesive contact* a typical surface energy density is given by

$$\phi_{\rm adh}([\![u]\!]_{\mathbf{n}}, z) := \tfrac{\kappa}{2} z \big|[\![u]\!]_{\mathbf{n}}\big|^2 , \qquad \kappa > 0 , \tag{1.1}$$

i.e., $\phi_{\rm adh}([\![u]\!]_{\mathbf{n}}, \cdot)$ is linear and decreases as z decreases. In contrast, surface energy densities $\phi_{\rm coh}(\cdot)$ in cohesive zone models are nonlinear, bounded, and monotonically increasing wrt. ζ. Typical cohesive zone energy densities $\phi_{\rm coh}$ from engineering literature [OP99, Kre05], are depicted in terms of the solid curves in column A of Fig. 1. The solid curve ϕ_{coh} hereby represents the energy envelope determined by a monotonically increasing (maximal) normal separation ζ. The corresponding traction stresses $\mathrm{D}_\zeta\phi_{\mathrm{coh}}(\zeta)$ are depicted in terms of the solid curves in column B. The works [OP99, PP11, Kre05] also allow for a non-monotone separation behavior. They propose to use linear traction-separation relations according to the dashed lines in Fig. 1 column B, which are active while the separation decreases and subsequently increases till the envelope $\mathrm{D}_\zeta\phi_{\mathrm{coh}}(\zeta)$ (solid curve) is reached again. Then, the traction-separation law follows the solid curve $\mathrm{D}_\zeta\phi_{\mathrm{coh}}(\zeta)$ until the next onset of unloading.

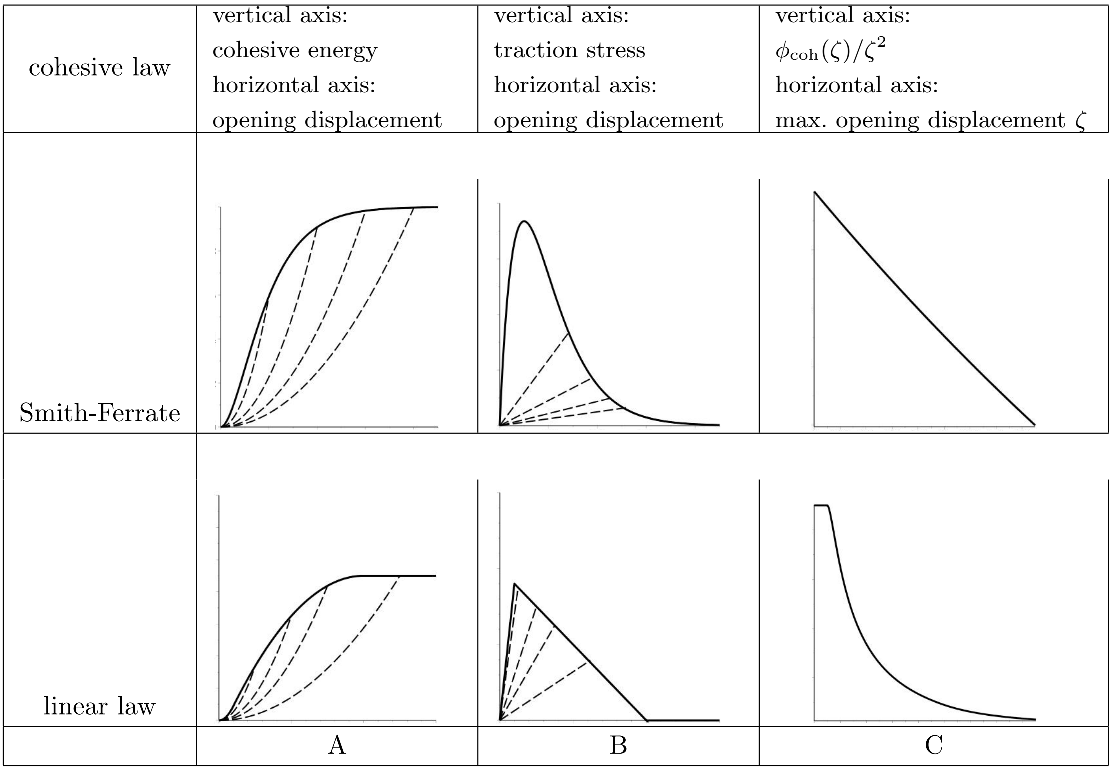

Figure 1: Typical cohesive laws. Solid lines: curves corresponding to maximal loading envelope, dashed lines: loading - unloading rules.

The cohesive zone model discussed in this work: In this work we want to treat a cohesive zone model, in the case of unidirectionality, that is, with $t \mapsto \zeta(t)$ increasing, that also allows for a non-monotone separation behavior. In the same spirit as described above, we incorporate non-monotonous separations $[\![u]\!]_{\mathbf{n}}$ into the model by replacing the cohesive surface energy density ϕ_{coh} by the expression

$$\phi^{\mathrm{coh}}([\![u]\!]_{\mathbf{n}}, \zeta) := \frac{\phi_{\mathrm{coh}}(\zeta)}{2\zeta^2}\left|[\![u]\!]_{\mathbf{n}}\right|^2. \tag{1.2}$$

By the meaning assigned to the internal variable ζ in cohesive zone models, it should always hold $[\![u(t,x)]\!]_{\mathbf{n}} \le \zeta(t,x)$. Therefore, if $[\![u(t,x)]\!]_{\mathbf{n}} < \zeta(t,x)$, the curve $\phi^{\mathrm{coh}}([\![u(t,x)]\!]_{\mathbf{n}}, \zeta(t,x))$ corresponds to a dashed curve in Fig. 1, column A, which is the energy density corresponding to a dashed traction curve in column B. For $[\![u(t,x)]\!]_{\mathbf{n}} = \zeta(t,x)$, the energy envelope $\phi_{\mathrm{coh}}(\zeta)$ is reached, which again corresponds to the energy density due to monotonically increasing (maximal) normal separation ζ. Altogether, with $\phi^{\mathrm{coh}}([\![u]\!]_{\mathbf{n}}, \zeta)$ from (1.2) the surface energy ϕ^{s} in our model will have the form:

$$\phi^{\mathsf{s}}([\![u]\!]_{\mathbf{n}}, \zeta) := \phi^{\mathrm{coh}}([\![u]\!]_{\mathbf{n}}, \zeta) + I_{[0,\zeta]}([\![u]\!]_{\mathbf{n}}) + I_{[0,\zeta^*]}(\zeta) + G(\zeta)\,. \tag{1.3}$$

Hereby, the indicator term

$$I_{[0,\zeta]}(\llbracket u \rrbracket_{\mathbf{n}}) = \begin{cases} 0 & \text{if } \llbracket u\rrbracket_{\mathbf{n}} \in [0,\zeta], \\ \infty & \text{otw.} \end{cases} \tag{1.4}$$

enforces the non-penetration condition $\llbracket u\rrbracket_{\mathbf{n}} = \llbracket u\rrbracket \cdot \mathrm{n} \geq 0$, as well as the 'memory'-constraint $\llbracket u\rrbracket_{\mathbf{n}} \leq \zeta$, which is a relaxation of the characteristic feature $\zeta(t,x) = \sup_{s\in[0,t]} \llbracket u(s,x)\rrbracket_{\mathbf{n}}$ of the cohesive zone internal variable. Observe that $I_{[0,\zeta]}(\cdot)$ imposes a convex, non-smooth constraint on the displacements, which depends on the internal variable ζ. This causes the main challenge in the analysis. In addition ϕ^{s} features the indicator function $I_{[0,\zeta^*]}$ of the fixed interval $[0,\zeta^*]$, for a fixed maximal separation amount ζ^*. With this constraint we prevent that the separation can reach unphysically large values, larger than ζ^*. Finally, the density G induces a regularization for ζ in terms of a Sobolev- or a Sobolev-Slobodeckij seminorm, see (2.17e) for the details.

Relation to adhesive contact: When deciding to drop the 'memory'-aspect $\llbracket u\rrbracket_{\mathbf{n}} \leq \zeta$, then, in view of the monotonically decreasing nature of the function $\phi_{\mathrm{coh}}(\zeta)/(2\zeta^2)$, see Fig. 1, column C, one may set $z := \phi_{\mathrm{coh}}(\zeta)/(\kappa\zeta^2)$ and introduce z as an alternative internal variable. In particular, for Smith-Ferrate's cohesive law, cf. Fig. 1, line 1, also upon renormalization, the cohesive zone surface energy density ϕ^{coh} (without memory), then goes over to the adhesive contact energy ϕ_{adh} from (1.1). An adhesive contact model involving ϕ_{adh} thus can be seen as a cohesive zone model *without the memory of the history of maximal separations*.

Coupled rate-independent/rate-dependent evolution: In their modeling approach [OP99] assume the cohesive internal variable ζ to evolve in a rate-independent, unidirectional way, such that $\dot\zeta \geq 0$ a.e. in $(0,T)$. We incorporate the rate-independent, unidirectional evolution to our model with the aid of the positively 1-homogeneous dissipation potential

$$\mathcal{R}(\dot\zeta(t)) := \int_{\Gamma_{\mathrm{C}}} R(\dot\zeta(t,x))\,\mathrm{d}x, \quad \text{with } R(\dot\zeta) := \begin{cases} \rho\dot\zeta & \text{if } \dot\zeta \geq 0, \text{ with given } \rho > 0, \\ \infty & \text{otherwise.} \end{cases} \tag{1.5}$$

In the bulk domain, with mechanical energy $\mathcal{E}^{\mathrm{bulk}}(t,\cdot)$ we will assume a viscoelastic material response, governed by a viscous dissipation potential $\mathcal{V}(\cdot)$. More precisely, denoting by $e(u) = \frac{1}{2}\big(\nabla u + \nabla u^\top\big)$ the symmetrized strain tensor, we introduce

$$\mathcal{E}^{\mathrm{bulk}}(t,u) := \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \tfrac{1}{2}\mathbb{C}e(u) : e(u)\,\mathrm{d}x - \int_{\Omega\backslash\Gamma_{\mathrm{C}}} f(t)\cdot u\,\mathrm{d}x - \int_{\Gamma_{\mathrm{N}}} h(t)\cdot u\,\mathrm{d}\mathcal{H}^{d-1}.$$

Here, $\Gamma_{\mathrm{N}} \subset \partial\Omega$ denotes the Neumann boundary and $\Gamma_{\mathrm{D}} = \partial\Omega\backslash\Gamma_{\mathrm{N}} \neq \varnothing$ the Dirichlet boundary.
The viscous, quadratic dissipation potential takes the form

$$\mathcal{V}(\dot u) := \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \tfrac{1}{2}\mathbb{D}e(\dot u) : e(\dot u)\,\mathrm{d}x\,. \tag{1.6}$$

We refer to Sec. 2.1 for the precise assumptions on the domain, the tensors $\mathbb{C}$ and $\mathbb{D}$, and the time-dependent loadings $f(t)$ and $h(t)$. In addition, we also account for dynamic effects governed by an acceleration term $\varrho\ddot u$. Thus, the evolution of the pair (u,ζ) is characterized by the system of equations

$$\varrho\ddot u - \mathrm{div}\,\big(\mathbb{C}e(u) + \mathbb{D}e(\dot u)\big) = f(t) \qquad \text{in } \Omega\backslash\Gamma_{\mathrm{C}}\,, \tag{1.7a}$$

$$\llbracket \mathbb{C}e(u) + \mathbb{D}e(\dot u)\rrbracket_{\mathbf{n}} = 0 \qquad \text{on } \Gamma_{\mathrm{C}}\,, \tag{1.7b}$$

$$-\big((\mathbb{C}e(u) + \mathbb{D}e(\dot u))\cdot\mathbf{n} + \mathrm{D}_u\phi^{\mathrm{coh}}(\llbracket u\rrbracket_{\mathbf{n}},\zeta)\big) \in \partial_u I_{[0,\zeta]}(\llbracket u\rrbracket_{\mathbf{n}}) \qquad \text{on } \Gamma_{\mathrm{C}}\,, \tag{1.7c}$$

$$-\big(\mathrm{D}_\zeta\phi^{\mathrm{coh}}(\llbracket u\rrbracket_{\mathbf{n}},\zeta) + \mathrm{D}_\zeta G(\zeta)\big) \in \partial_\zeta I_{[\llbracket u\rrbracket_{\mathbf{n}},\zeta^*]}(\zeta) + \partial_{\dot\zeta}R(\dot\zeta) \qquad \text{on } \Gamma_{\mathrm{C}}\,, \tag{1.7d}$$

$$u = 0 \qquad \text{on } \Gamma_{\mathrm{D}}\,, \tag{1.7e}$$

$$\big(\mathbb{C}e(u) + \mathbb{D}e(\dot u)\big)\mathbf{n} = h(t) \qquad \text{on } \Gamma_{\mathrm{N}}\,. \tag{1.7f}$$

Scope, structure, and challenges of the paper: This contribution deals with the existence analysis for system (1.7). In fact, owing to its coupled rate-dependent/rate-independent character we will not analyse the cohesive zone system in its strong form (1.7), but resort to a weaker notion of solution, which combines concepts from the theory of rate-independent systems [MR15] with the one of (viscous) gradient flows ($\varrho = 0$ in (1.7a)), resp. hyperbolic PDEs ($\varrho > 0$ in (1.7a)). Pioneered in [Rou09] and refined in [RT15a], this concept of *semistable energetic solutions* is well-suited to handle rate-independent evolution, as it replaces the subdifferential inclusion (1.7d) by a minimality condition for ζ combined with an energy-dissipation estimate. We introduce the notion of *semistable energetic solutions* at the beginning of Sec. 2, see Def. 2.1.

Observe that the transmission condition (1.7c) features the subdifferential $\partial_u I_{[0,\zeta]}(\llbracket u \rrbracket_{\mathbf{n}})$ of the combined non-penetration and 'memory'-constraint (1.4). Indeed, to deal with this multivalued term, we will, in a first step, replace it by its corresponding Yosida-approximation $\partial_u I_k(\llbracket u \rrbracket_{\mathbf{n}}, \zeta_k)$. We specify in Sec. 2.1 the analytical setup, all the functionals and function spaces involved. Our results are presented and discussed in Sec. 2.2. Our first result, Thm. 2.4, states the existence of semistable energetic solutions for the Yosida-regularized model, for both cases $\varrho = 0$ and $\varrho > 0$. Our main result, Thm. 2.5, provides the existence of semistable energetic solutions for the unregularized cohesive zone model (featuring $\partial_u I_{[0,\zeta]}(\llbracket u \rrbracket_{\mathbf{n}})$) in the case of gradient flows, i.e. $\varrho = 0$ in (1.7a). Based on Thm. 2.4, whose proof is carried out in Sec. 3, the proof of Thm. 2.5 is obtained in Sec. 4 in terms of an evolutionary Γ-limit passage. Hereby, the main challenge will lie in the fact that $I_{[0,\zeta]}(\llbracket \cdot \rrbracket_{\mathbf{n}})$ encodes the non-smooth, bilateral non-penetration & 'memory'-constraint which, on top, depends on the state ζ. Thus, for the limit passage, also its Yosida-regularization $I_k(\llbracket \cdot \rrbracket_{\mathbf{n}}, \zeta_k)$ will depend on the state ζ_k, the internal-variable-component of a semistable energetic solution pair (u_k, ζ_k) to the regularized model. An evolutionary Γ-limit passage will therefore require the proof of Mosco-convergence for the corresponding functionals, see Prop. 4.3, with a careful design of recovery sequences, taylored to the constraints imposed on the states varying with k.

Comparison with other existing results: The problems attached to the approximation of a state-dependent non-smooth constraint in combination with rate-dependent effects and dynamics have already been experienced in [RT15b, RT16]. Therein, the adhesive contact energy ϕ_{adh} from (1.1) is used as a regularization of the brittle constraint $z|\llbracket u \rrbracket_{\mathbf{n}}| = 0$ a.e. on Γ_{C}. For the limit $\kappa \to \infty$ in the energy term (1.1) the brittle constraint could be re-obtained, also necessitating a sophisticated design of recovery sequences involving the proof of additional fine convergence properties of the semistable internal variables z_k.

The history dependence of the crack opening was addressed also in [DMZ07], where existence of globally stable quasistatic evolution (i.e. energetic solution) was obtained, in the case of a cohesive zone model featuring a general ϕ_{coh} without a regularizing term G for the internal variable ζ. Hence there the main mathematical difficulty was the compactness of the approximating functions $t \mapsto \zeta_k(t)$, solved by introducing a new notion of convergence, which is the counterpart of the notion of convergence of sets introduced in [DMFT05]. More recently vanishing viscosity techniques have been applied in cohesive zone models w.r.t. the internal variable, [CT11, Alm16]. In particular, history dependence of the crack opening was also considered in [CT11] for a cohesive zone model taking into account different responses upon loading and unloading. Existence of a solution obtained by means of vanishing viscosity have been established, by replacing the internal variable by a Young measure. [Cag08, ACFS16] study cohesive zone delamination for a visco-elastic solid without introducing an internal variable and prove existence of solutions as well as a vanishing viscosity limit. Different responses upon loading and unloading, also related to fatigue, have been recently addressed in [CLO16], where the existence of energetic solutions is established. Fatigue effects related to cohesive fracture, more precisely in a gradient damage model coupled with plasticity were discussed in [AMV14], and the authors of [DMOT16] show in a static setting that cohesive hypersurface energy functionals can be obtained as a limit of volume damage coupled with plasticity.

2 The mathematical model and our results

In this paper we will treat the cohesive zone model and its approximations in the framework of gradient systems (that is, $\varrho = 0$ a.e. on $\Omega \setminus \Gamma_C$ in (1.7a)) and damped inertial systems (that is, $\varrho > 0$ a.e. on $\Omega \setminus \Gamma_C$ in (1.7a)), which consist of:

- two Hilbert spaces
$$\mathbf{V} \text{ and } \mathbf{W},\ \mathbf{W} \text{ identified with its dual } \mathbf{W}^*, \text{ such that } \mathbf{V} \Subset \mathbf{W} \text{ compactly and densely,} \tag{2.1a}$$
so that $\mathbf{V} \subset \mathbf{W} = \mathbf{W}^* \subset \mathbf{V}^*$ (dual of $\mathbf{V}$) continuously and densely, and $\langle w, u\rangle_{\mathbf{V}} = (w, u)_{\mathbf{W}}$ for all $u \in \mathbf{V}$ and $w \in \mathbf{W}$;
- a separable Banach space $\mathbf{Z}$;
- a dissipation potential $\mathcal{V} : \mathbf{V} \to [0, \infty)$ of the form
$$\mathcal{V}(v) = \tfrac{1}{2}a(v, v) \qquad \text{with } a : \mathbf{V} \times \mathbf{V} \to \mathbb{R} \text{ a continuous coercive bilinear form;} \tag{2.1b}$$
- a dissipation potential $\mathcal{R} : \mathbf{Z} \to [0, \infty]$, with domain $\mathrm{dom}(\mathcal{R})$, lower semicontinuous, convex, positively 1-homogeneous and coercive i.e.,
$$\begin{aligned} &\mathcal{R}(\lambda\zeta) = \lambda\mathcal{R}(\zeta) \quad \text{for all } \zeta \in \mathbf{Z} \text{ and } \lambda \geq 0,\\ &\exists\, C_R > 0\ \ \forall\, \zeta \in \mathbf{Z} \qquad \mathcal{R}(\zeta) \geq C_R \|\zeta\|_{\mathbf{Z}}\,; \end{aligned} \tag{2.1c}$$
- a *kinetic energy* $\mathcal{K} : \mathbf{W} \to [0, \infty)$, $\mathcal{K}(v) := \frac{\varrho}{2}\|v\|^2_{\mathbf{W}}$, with $\varrho > 0$ in case of a damped inertial system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$ and $\varrho = 0$ in case of a gradient system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$
- an *energy functional* $\mathcal{E} : [0, T] \times \mathbf{V} \times \mathbf{Z} \to \mathbb{R} \cup \{\infty\}$, with proper domain $\mathrm{dom}(\mathcal{E}) = [0, T] \times \mathrm{dom}_u \times \mathrm{dom}_\zeta$, such that
$$\begin{aligned} &t \mapsto \mathcal{E}(t, u, \zeta) \text{ is differentiable} && \text{for all } (u, \zeta) \in \mathrm{dom}_u \times \mathrm{dom}_\zeta,\\ &(u, \zeta) \mapsto \mathcal{E}(t, u, \zeta) \text{ is lower semicontinuous} && \text{for all } t \in [0, T],\\ &u \mapsto \mathcal{E}(t, u, \zeta) \text{ is convex} && \text{for all } (t, \zeta) \in [0, T] \times \mathrm{dom}_\zeta\,. \end{aligned} \tag{2.1d}$$

In what follows, we shall denote by $\partial_u\mathcal{E} : [0, T] \times \mathbf{V} \times \mathbf{Z} \rightrightarrows \mathbf{V}^*$ the subdifferential of the functional $\mathcal{E}(t, \cdot, \zeta)$ in the sense of convex analysis. We postpone to Section 3 ahead the precise statement of the further conditions on $\mathcal{E}$ required in the existence result from [RT15a] that we shall apply to deduce the existence of solutions to the cohesive zone model and its approximants. Let us only mention here that the assumptions on $\zeta \mapsto \mathcal{E}(t, u, \zeta)$ (cf. the coercivity requirement (3.3) ahead) also involve a second space $\mathbf{X}$ such that
$$\mathbf{X} \text{ is the dual of a separable Banach space and } \mathbf{X} \Subset \mathbf{Z} \text{ compactly.} \tag{2.1e}$$
If $\varrho = 0$ we speak of a gradient system and denote it by the characterizing quintuple $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$; for $\varrho > 0$ we speak of a damped inertial system, denoted by $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$.

A suitable solution concept for systems that couple rate-independent processes with rate-dependent and dynamic ones goes back on the pioneering work [Rou09]. Based on a time-discrete scheme with alternating (decoupled) minimization w.r.t. the variables u and ζ it was recently refined and developed further in [RT15a, Def. 3.1]. Below in (2.2b) we denote by $\mathrm{B}([0, T]; \mathbf{X})$ the set of bounded functions defined on $[0, T]$ with values on $\mathbf{X}$.

Definition 2.1 (Semistable energetic solutions)**.** Consider a gradient system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$, resp. a damped inertial system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$. We call a pair $(u, \zeta) : [0, T] \to \mathbf{V} \times \mathbf{Z}$ a *semistable energetic solution* to the gradient system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$, resp. the damped inertial system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$, if

$$u \in W^{1,1}(0, T; \mathbf{V})\,, \text{and for a damped inertial system } (\varrho > 0) \text{ also } \dot{u} \in L^\infty(0, T; \mathbf{W})\,, \quad \ddot{u} \in L^2(0, T; \mathbf{V}^*)\,, \tag{2.2a}$$

$$\zeta \in \mathrm{B}([0, T]; \mathbf{X}) \cap \mathrm{BV}([0, T]; \mathbf{Z}) \tag{2.2b}$$

fulfill

- the subdifferential inclusion

$$\varrho\ddot{u}(t) + \partial\mathcal{V}(\dot{u}(t)) + \partial_u\mathcal{E}(t, u(t), \zeta(t)) \ni 0 \qquad \text{in } \mathbf{V}^* \quad \text{for a.a. } t \in (0, T), \tag{2.3}$$

- the semistability condition

$$\mathcal{E}(t, u(t), \zeta(t)) \le \mathcal{E}(t, u(t), \tilde{z}) + \mathcal{R}(\tilde{z} - \zeta(t)) \qquad \text{for all } \tilde{z} \in \mathbf{Z} \text{ for all } t \in [0, T]; \tag{2.4}$$

- the energy-dissipation inequality

$$\begin{aligned} &\tfrac{\varrho}{2}\|\dot{u}(t)\|^2_{\mathbf{W}} + \int_0^t 2\mathcal{V}(\dot{u}(s))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}}(\zeta, [0, t]) + \mathcal{E}(t, u(t), \zeta(t)) \\ &\qquad \le \tfrac{\varrho}{2}\|\dot{u}(0)\|^2_{\mathbf{W}} + \mathcal{E}(0, u(0), \zeta(0)) + \int_0^t \partial_t\mathcal{E}(s, u(s), \zeta(s))\,\mathrm{d}s \qquad \text{for all } t \in [0, T]\,, \end{aligned} \tag{2.5}$$

  with $\mathrm{Var}_{\mathcal{R}}$ denoting the total variation induced by the dissipation potential $\mathcal{R}$, i.e.

$$\mathrm{Var}_{\mathcal{R}}(\zeta; [s, t]) := \sup\left\{\sum_{j=1}^N \mathcal{R}(\zeta(r_j) - \zeta(r_{j-1})) : \quad s = r_0 < r_1 < \ldots < r_{N-1} < r_N = t\right\}$$

  for a given subinterval $[s, t] \subset [0, T]$.

Note that for the cohesive zone models, due to the unidirectionality incorporated in the dissipation potential (1.5), the total variation will take the specific form

$$\mathrm{Var}_{\mathcal{R}}(\zeta; [s, t]) = \mathcal{R}(\zeta(t) - \zeta(s))\,. \tag{2.6}$$

2.1 Basic assumptions, spaces, and functionals

2.1.1 Basic assumptions

Assumptions on the reference domain: We suppose that

$$\Omega = \mathrm{int}(\overline{\Omega_+} \cup \overline{\Omega_-}) \subset \mathbb{R}^d,\ d \ge 2, \text{ is bounded, } \Omega_-,\ \Omega_+,\ \Omega \text{ are Lipschitz domains, } \Omega_+ \cap \Omega_- = \emptyset\,, \tag{2.7a}$$

$$\partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}},\ \Gamma_{\mathrm{D}},\ \Gamma_{\mathrm{N}} \text{ open subsets in } \partial\Omega, \tag{2.7b}$$

$$\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset,\ \mathrm{dist}(\overline{\Gamma}_{\mathrm{D}}, \overline{\Gamma}_{\mathrm{C}}) = h > 0,\ \mathcal{H}^{d-1}(\Gamma_{\mathrm{D}} \cap \overline{\Omega}_+) > 0\,,\ \mathcal{H}^{d-1}(\Gamma_{\mathrm{D}} \cap \overline{\Omega}_-) > 0\,, \tag{2.7c}$$

$$\begin{aligned} &\Gamma_{\mathrm{C}} = \overline{\Omega}_+ \cap \overline{\Omega}_- \subset \mathbb{R}^{d-1} \text{ is a “flat” surface, i.e. contained in a hyperplane of } \mathbb{R}^d, \\ &\qquad\qquad \text{such that, in particular, } \mathcal{H}^{d-1}(\Gamma_{\mathrm{C}}) = \mathcal{L}^{d-1}(\Gamma_{\mathrm{C}}) > 0\,, \end{aligned} \tag{2.7d}$$

and Γ_{C} has the normal vector $\mathbf{n}$, defined as the outer unit normal to $\partial\Omega_+ \cap \Gamma_{\mathrm{C}}$.

Here, $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure and $\mathcal{L}^{d-1}$ is the $(d-1)$-dimensional Lebesgue measure.

Assumptions on the given data: For the tensors $\mathbb{C}, \mathbb{D} \in \mathbb{R}^{d\times d\times d\times d}$ in (1.6) & (1.7) and a time-dependent external force $\mathbf{f}$, we require that

$$\begin{aligned}&\mathbb{C}, \mathbb{D} \in \mathbb{R}^{d\times d\times d\times d} \text{ are symmetric and positive definite, i.e.,}\\ &\exists\, C^1_{\mathbb{C}}, C^2_{\mathbb{C}}, C^1_{\mathbb{D}}, C^2_{\mathbb{D}} > 0, \forall\, \eta \in \mathbb{R}^{d\times d}:\ C^1_{\mathbb{C}}|\eta|^2 \le \eta : \mathbb{C}\eta \le C^2_{\mathbb{C}}|\eta|^2 \text{ and } C^1_{\mathbb{D}}|\eta|^2 \le \eta : \mathbb{D}\eta \le C^2_{\mathbb{D}}|\eta|^2\,,\end{aligned} \tag{2.8a}$$

$$\mathbf{f} \in \mathrm{C}^1([0,T];\mathbf{V}^*) \text{ and } \sup_{t\in[0,T]} \big(\|\mathbf{f}(t)\|_{\mathbf{V}^*} + \|\dot{\mathbf{f}}(t)\|_{\mathbf{V}^*}\big) \le C_{\mathbf{f}}\,. \tag{2.8b}$$

Hereby, the external force $\mathbf{f}$ may comprise both the volume force f from (1.7a) and the surface force h from (1.7f). Moreover, to keep notation and arguments simple, we prescribe *homogeneous Dirichlet data* on Γ_{D}, as already indicated in (1.7e).

Assumptions on the cohesive surface energy density: In line of the works [OP99, PP11, Kre05] we assume that

$$\begin{aligned}&\phi_{\mathrm{coh}} : [0,\infty) \to [0,a), \ \text{ with } a>0 \text{ and } \phi_{\mathrm{coh}}(0)=0, \ \text{is continuous, non-decreasing and}\\ &\text{the map } \zeta \mapsto \frac{\phi_{\mathrm{coh}}(\zeta)}{2\zeta^2} \text{ is continuous, monotonically decreasing and bounded by } b>0.\end{aligned} \tag{2.9}$$

In fact, it can be seen in Figure 1, columns A & C, that typical cohesive zone energy densities ϕ_{coh} from engineering literature do comply with (2.9).

2.1.2 Function spaces and traces

Throughout this paper the set of function spaces described in (2.1) will be chosen as follows:

$$\mathbf{V} := H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) := \{v \in H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \,:\, v = 0 \text{ a.e. on } \Gamma_{\mathrm{D}}\}\,, \tag{2.10a}$$

$$\mathbf{W} = L^2(\Omega;\mathbb{R}^d) \text{ endowed with the norm } \|v\|_{\mathbf{W}} := \Big(\int_\Omega |v|^2\,\mathrm{d}x\Big)^{1/2}\,, \tag{2.10b}$$

$$\mathbf{Z} = L^1(\Gamma_{\mathrm{C}})\,, \tag{2.10c}$$

$$\mathbf{X} = \begin{cases} W^{1,r}(\Gamma_{\mathrm{C}}) \text{ with } r > d-1 & \text{in Sec. 3 \& 4,}\\ H^{1/2}(\Gamma_{\mathrm{C}}) & \text{in Sec. 3}\,.\end{cases} \tag{2.10d}$$

The space $\mathbf{X}$ will be related to the (gradient) regularization of the internal variable ζ contributing to the surface energy functionals, see $\Phi^{\mathrm{surf}}, \Phi^{\mathrm{surf}}_k$ in (2.17c) & (2.19b) below. Depending on its choice we will obtain existence results of different quality: The choice $\mathbf{X} = H^{1/2}(\Gamma_{\mathrm{C}})$ is more natural in view of (2.10a), if one has in mind that the purpose of the internal variable in a cohesive zone model is to keep track of the history of the maximal jumps of the displacements across Γ_{C}. Indeed, the choice $\mathbf{X} = H^{1/2}(\Gamma_{\mathrm{C}})$ is suited to obtain existence results both in the dynamic and in the gradient-flow case if one regularizes the ζ-dependent indicator term in (2.17f) by its Yosida-approximation, cf. Thm. 2.4 & Sec. 3. But so far, only the enforcement $\mathbf{X} = W^{1,r}(\Gamma_{\mathrm{C}})$ with $r > d-1$, allows it to pass from the regularized models to a model displaying the unregularized cohesive zone energy (2.17c), cf. Thm. 2.5 & Sec. 4, by heavily exploiting the compact embedding

$$W^{1,r}(\Gamma_{\mathrm{C}}) \Subset C(\overline{\Gamma_{\mathrm{C}}}) \ \text{ for } r > d-1\,. \tag{2.11}$$

Note that, thanks to assumption (2.7d), we have for all $\zeta \in \mathbf{X} = H^{1/2}(\Gamma_{\mathrm{C}})$:

$$\|\zeta\|^2_{\mathbf{X}} = \|\zeta\|^2_{L^2(\Gamma_{\mathrm{C}})} + |\zeta|^2_{H^{1/2}(\Gamma_{\mathrm{C}})} \ \text{ with } \ |\zeta|^2_{H^{1/2}(\Gamma_{\mathrm{C}})} = \int_{\Gamma_{\mathrm{C}}}\int_{\Gamma_{\mathrm{C}}} \frac{|\zeta(x)-\zeta(y)|^2}{|x-y|^d}\,\mathrm{d}x\,\mathrm{d}y\,, \tag{2.12a}$$

and there holds, cf. e.g. [DNPV12, Thm. 7.1],

$$\mathbf{X} = H^{1/2}(\Gamma_{\mathrm{C}}) \Subset L^2(\Gamma_{\mathrm{C}}) \text{ compactly.} \tag{2.12b}$$

For later purpose we verify here that the operation $\max\{\cdot,\cdot\}$ defines a bounded operator from $H^{1/2}(\Gamma_{\mathrm{C}})\times H^{1/2}(\Gamma_{\mathrm{C}})$ to $H^{1/2}(\Gamma_{\mathrm{C}})$.

Lemma 2.2. *Let $\Gamma_{\mathrm{C}} \subset \mathbb{R}^{d-1}$ comply with (2.7). Then, the operator*

$$\begin{aligned} &\max\{\cdot,\cdot\} : H^{1/2}(\Gamma_{\mathrm{C}}) \times H^{1/2}(\Gamma_{\mathrm{C}}) \to H^{1/2}(\Gamma_{\mathrm{C}}) \quad \textit{is bounded with} \\ &|\max\{f,g\}|^2_{H^{1/2}(\Gamma_{\mathrm{C}})} \le |f|^2_{H^{1/2}(\Gamma_{\mathrm{C}})} + |g|^2_{H^{1/2}(\Gamma_{\mathrm{C}})} \quad \textit{for any } f,g \in H^{1/2}(\Gamma_{\mathrm{C}})\,. \end{aligned} \tag{2.13}$$

Proof. Consider two functions $f, g \in H^{1/2}(\Gamma_{\mathrm{C}})$ and observe that for every $x, y \in \Gamma_{\mathrm{C}}$

$$|\max\{f(x), g(x)\} - \max\{f(y), g(y)\}| \le \max\{|f(x) - f(y)|\,, |g(x) - g(y)|\}\,.$$

Introduce $F := \{(x,y) \in \Gamma_{\mathrm{C}} \times \Gamma_{\mathrm{C}},\ |f(x)-f(y)| \ge |g(x)-g(y)|\}$ as well as $G := \{(x,y) \in \Gamma_{\mathrm{C}} \times \Gamma_{\mathrm{C}},\ |g(x)-g(y)| \ge |f(x)-f(y)|\}$. It is easy to check that $\|\max\{f,g\}\|_{L^2(\Gamma_{\mathrm{C}})} \le \|f\|_{L^2(\Gamma_{\mathrm{C}})} + \|g\|_{L^2(\Gamma_{\mathrm{C}})}$. For the $H^{1/2}$-seminorm we see that

$$\begin{aligned} |\max\{f,g\}|^2_{H^{1/2}(\Gamma_{\mathrm{C}})} &= \int_{\Gamma_{\mathrm{C}}}\int_{\Gamma_{\mathrm{C}}} \frac{|\max\{f(x),g(x)\} - \max\{f(y),g(y)\}|^2}{|x-y|^d}\,\mathrm{d}x\,\mathrm{d}y \\ &\le \int_{\Gamma_{\mathrm{C}}}\int_{\Gamma_{\mathrm{C}}} \frac{(\max\{|f(x)-f(y)|\,, |g(x)-g(y)|\})^2}{|x-y|^d}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_F \frac{|f(x)-f(y)|^2}{|x-y|^d}\,\mathrm{d}x\,\mathrm{d}y + \int_G \frac{|g(x)-g(y)|^2}{|x-y|^d}\,\mathrm{d}x\,\mathrm{d}y \\ &\le |f|^2_{H^{1/2}(\Gamma_{\mathrm{C}})} + |g|^2_{H^{1/2}(\Gamma_{\mathrm{C}})}\,, \end{aligned}$$

which concludes the proof. □

2.1.3 The functionals

First and foremost, we denote by $\mathsf{T}_{\Omega_\pm\to\Gamma_{\mathrm{C}}} : H^1(\Omega_\pm) \to H^{1/2}(\Gamma_{\mathrm{C}};\mathbb{R}^d)$ the trace operator from $\Omega_\pm$ to the interface Γ_{C}. With its aid we introduce the operator indicating the jump of functions $u \in H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ across Γ_{C}

$$[\![\,\cdot\,]\!] : H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \to H^{1/2}(\Gamma_{\mathrm{C}};\mathbb{R}^d), \quad [\![u]\!] := \mathsf{T}_{\Omega_+\to\Gamma_{\mathrm{C}}}(u|_{\Omega_+}) - \mathsf{T}_{\Omega_-\to\Gamma_{\mathrm{C}}}(u|_{\Omega_-})\,, \tag{2.14}$$

as well as the operator indicating the jump of functions $u \in H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ across Γ_{C} in normal direction $\mathbf{n}$, cf. (2.7d),

$$[\![\,\cdot\,]\!]_{\mathbf{n}} : H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \to H^{1/2}(\Gamma_{\mathrm{C}}), \quad [\![u]\!]_{\mathbf{n}} := [\![u]\!]\cdot\mathbf{n}\,. \tag{2.15}$$

Thus, for every $u \in \mathbf{V}$, cf. (2.10a), we have $[\![u]\!] \in H^{1/2}(\Gamma_{\mathrm{C}};\mathbb{R}^d)$ and, thanks to (2.7d), also $[\![u]\!]_{\mathbf{n}} \in H^{1/2}(\Gamma_{\mathrm{C}})$. In particular, we find that $\|[\![u]\!]_{\mathbf{n}}\|_{H^{1/2}(\Gamma_{\mathrm{C}})} \le \|[\![u]\!]\|_{H^{1/2}(\Gamma_{\mathrm{C}};\mathbb{R}^d)}$.

Clearly, both $[\![\cdot]\!] : H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \to H^{1/2}(\Gamma_{\mathrm{C}};\mathbb{R}^d)$ and $[\![\cdot]\!]_{\mathbf{n}} : H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \to H^{1/2}(\Gamma_{\mathrm{C}})$ are linear, continuous operators with derivatives

$$\mathrm{D}[\![\,\cdot\,]\!] : H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \to H^{1/2}(\Gamma_{\mathrm{C}};\mathbb{R}^d), \qquad \mathrm{D}[\![u]\!][v] = [\![v]\!]\,, \tag{2.16a}$$

$$\mathrm{D}[\![\,\cdot\,]\!]_{\mathbf{n}} : H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d) \to H^{1/2}(\Gamma_{\mathrm{C}}), \qquad \mathrm{D}[\![u]\!]_{\mathbf{n}}[v] = [\![v]\!]_{\mathbf{n}} \tag{2.16b}$$

for all $u, v \in H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$.

Thus, to model cohesive zone delamination along the interface Γ_{C} in a visco-elastic body, we introduce the **Energy functional** of the following form:

$$\mathcal{E}_\infty : [0,T]\times\mathbf{V}\times\mathbf{Z}\to\mathbb{R}\cup\{\infty\},\ \mathcal{E}_\infty(t,u,\zeta) := \mathcal{E}^{\mathrm{bulk}}(t,u) + \Phi^{\mathrm{surf}}(u,\zeta), \quad \text{with} \tag{2.17a}$$

$$\mathcal{E}^{\mathrm{bulk}} : [0,T]\times\mathbf{V}\to\mathbb{R},\ \mathcal{E}^{\mathrm{bulk}}(t,u) := \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \tfrac{1}{2}\mathbb{C}e(u) : e(u)\,\mathrm{d}x - \langle \mathbf{f}(t), u\rangle_{\mathbf{V}}\,, \tag{2.17b}$$

$$\Phi^{\mathrm{surf}}(\cdot,\cdot) : \mathbf{V}\times\mathbf{Z}\to[0,\infty],\ \Phi^{\mathrm{surf}}(u,\zeta) := \begin{cases} \Phi^{\mathrm{coh}}([\![u]\!]_{\mathbf{n}},\zeta) + \mathcal{J}([\![u]\!]_{\mathbf{n}},\zeta) + \mathcal{G}(\zeta) & \text{if } (u,\zeta)\in\mathbf{V}\times\mathbf{X},\\ \infty & \text{otw.,}\end{cases} \tag{2.17c}$$

$$\Phi^{\mathrm{coh}} : L^2(\Gamma_{\mathrm{C}})\times\mathbf{X}\to[0,\infty],\ \Phi^{\mathrm{coh}}(\mathsf{v},\zeta) := \int_{\Gamma_{\mathrm{C}}} \phi^{\mathrm{coh}}(\mathsf{v},\zeta)\,\mathrm{d}\mathcal{H}^{d-1} = \int_{\Gamma_{\mathrm{C}}} \frac{\phi_{\mathrm{coh}}(\zeta)}{2\zeta^2}\mathsf{v}^2\,\mathrm{d}\mathcal{H}^{d-1}, \tag{2.17d}$$

$$\mathcal{G} : \mathbf{X}\to[0,\infty],\ \mathcal{G}(\zeta) := \int_{\Gamma_{\mathrm{C}}} I_{[0,\zeta^*]}(\zeta)\,\mathrm{d}\mathcal{H}^{d-1} + |\zeta|^r_{\mathbf{X}} \quad \text{with } r\in\begin{cases}\{2\} & \text{if } \mathbf{X}=H^{1/2}(\Gamma_{\mathrm{C}}),\\ [d-1,\infty) & \text{if } \mathbf{X}=W^{1,r}(\Gamma_{\mathrm{C}}),\end{cases} \tag{2.17e}$$

and with ζ^* a positive constant,

$$\mathcal{J}(\cdot,\cdot) : L^2(\Gamma_{\mathrm{C}})\times\mathbf{X}\to[0,\infty],\ \mathcal{J}(\mathsf{v},\zeta) := \int_{\Gamma_{\mathrm{C}}} I_{[0,\zeta]}(\mathsf{v})\,\mathrm{d}\mathcal{H}^{d-1}\,. \tag{2.17f}$$

The regularization $\mathcal{G}$ for the internal variable consists of the indicator function of the interval $[0,\zeta^*]$ and the seminorm of the space $\mathbf{X}$, cf. (2.12) for $\mathbf{X}=H^{1/2}(\Gamma_{\mathrm{C}})$, and $\int_{\Gamma_{\mathrm{C}}}|\nabla\zeta|^r\,\mathrm{d}\mathcal{H}^{d-1}$ for $\mathbf{X}=W^{1,r}(\Gamma_{\mathrm{C}})$. The indicator term confines the values of ζ to the interval $[0,\zeta^*]$ for some $\zeta^*>0$. This is done in view of the properties of ϕ^{coh}, cf. (1.3) and also (2.9): In order to further decrease the surface energy $\Phi^{\mathrm{coh}}([\![u]\!]_{\mathbf{n}},\cdot)$ the model might favor to attain large values of ζ. Having in mind that the internal variable is linked to the jump of the displacements, this may be unnatural. The indicator thus prevents too high values of ζ. Since the internal variable has a rate-independent evolution, governed by the 1-homogeneous dissipation potential $\mathcal{R}$, cf. (1.5), the evolution equation of the variable ζ can be reformulated in terms of the semistability inequality (2.4), which is a minimality property involving only the functionals $\mathcal{E}$ and $\mathcal{R}$, but not their differentials. Hence, here, the non-smooth, unbounded functionals $\mathcal{G},\mathcal{J}([\![u]\!]_{\mathbf{n}},\cdot)$ contributing to Φ^{surf}, can be handled in the realm of calculus of variations. This is in contrast to the rate-dependent, dynamic evolution of the displacement variable u, which is governed by the subdifferential inclusion (2.3) that cannot be reformulated without the differential $\partial_u\mathcal{E}$: Here, the non-smoothness and unboundedness of the functional $\mathcal{J}([\![\cdot]\!]_{\mathbf{n}},\zeta)$ imposes an obstruction to the analysis as the constraint (2.17f) therein implemented in particular depends on the internal variable ζ. Therefore, in order to handle the non-smooth constraint (2.17f) in the rate-dependent setting, we will first treat a regularized problem. For this, we replace the functional $\mathcal{J}$ by its Yosida-approximation

$$\begin{aligned}\forall\, k\in\mathbb{N}:\ \ \mathcal{J}_k : L^2(\Gamma_{\mathrm{C}})\times\mathbf{X}\to\mathbb{R},\ \mathcal{J}_k(\mathsf{v},\zeta) \ &:= \inf_{\mathsf{w}\in L^2(\Gamma_{\mathrm{C}})}\Big(\mathcal{J}(\mathsf{w},\zeta) + \tfrac{k}{2}\big\|\mathsf{v}-\mathsf{w}\big\|^2_{L^2(\Gamma_{\mathrm{C}})}\Big)\\ &= \tfrac{k}{2}\big(\|(\mathsf{v})^-\|^2_{L^2(\Gamma_{\mathrm{C}})} + \|(\mathsf{v}-\zeta)^+\|^2_{L^2(\Gamma_{\mathrm{C}})}\big)\,,\end{aligned} \tag{2.18a}$$

where $(\mathsf{v})^- := -\min\{\mathsf{v},0\}$ and $(\mathsf{v})^+ := \max\{\mathsf{v},0\}$.

With $\mathcal{E}^{\mathrm{bulk}}$ from (2.17b), Φ^{coh} from (2.17d), and $\mathcal{G}$ from (2.17e) this leads to the **Yosida-regularized energy functionals for every** $k\in\mathbb{N}$ **:**

$$\mathcal{E}_k : [0,T]\times\mathbf{V}\times\mathbf{Z}\to\mathbb{R}\cup\{\infty\},\ \mathcal{E}_k(t,u,\zeta) := \mathcal{E}^{\mathrm{bulk}}(t,u) + \Phi^{\mathrm{surf}}_k(u,\zeta) \tag{2.19a}$$

$$\Phi^{\mathrm{surf}}_k(\cdot,\cdot) : \mathbf{V}\times\mathbf{Z}\to[0,\infty], \qquad \Phi^{\mathrm{surf}}_k(u,\zeta) := \begin{cases}\Phi^{\mathrm{coh}}([\![u]\!]_{\mathbf{n}},\zeta) + \mathcal{J}_k([\![u]\!]_{\mathbf{n}},\zeta) + \mathcal{G}(\zeta) & \text{if } (u,v)\in\mathbf{V}\times\mathbf{X},\\ \infty & \text{otw..}\end{cases} \tag{2.19b}$$

We note that the domain $\mathrm{dom}(\mathcal{E}_k)$ of the functionals $\mathcal{E}_k$ is given by

$$\begin{aligned}&\mathrm{dom}(\mathcal{E}_k) = [0,T] \times \mathrm{dom}_u \times \mathrm{dom}_\zeta \quad \text{with}\\ &\mathrm{dom}_u = \mathbf{V} \quad \text{and} \quad \mathrm{dom}_\zeta = \mathbf{X} \cap \{\zeta \in L^\infty(\Gamma_\mathrm{C}),\, 0 \le \zeta \le \zeta^* \text{ a.e. in } \Gamma_\mathrm{C}\}\,.\end{aligned} \tag{2.20}$$

Remark 2.3 (Interpretation of the surface functionals and their derivatives). For given $\zeta \in \mathbf{X}$, (2.19b) composes the cohesive functional $\Phi^{\mathrm{coh}}(\cdot,\zeta) : L^2(\Gamma_\mathrm{C}) \to [0,\infty)$ and the Yosida-regularization functional $\mathcal{J}_k(\cdot,\zeta) : L^2(\Gamma_\mathrm{C}) \to [0,\infty)$ with the normal jump $[\![\cdot]\!]_\mathbf{n} : \mathbf{V} \to H^{1/2}(\Gamma_\mathrm{C})$, i.e., $\Phi^{\mathrm{coh}}([\![\cdot]\!]_\mathbf{n},\zeta) = \Phi^{\mathrm{coh}}(\cdot,\zeta)\circ[\![\cdot]\!]_\mathbf{n} : \mathbf{V} \to [0,\infty)$, resp. $\mathcal{J}_k([\![\cdot]\!]_\mathbf{n},\zeta) = \mathcal{J}_k(\cdot,\zeta) \circ [\![\cdot]\!]_\mathbf{n} : \mathbf{V} \to [0,\infty)$. Both $\Phi^{\mathrm{coh}}(\cdot,\zeta) : L^2(\Gamma_\mathrm{C}) \to [0,\infty)$ and $\mathcal{J}_k(\cdot,\zeta) : L^2(\Gamma_\mathrm{C}) \to [0,\infty)$ are convex and continuous in all points of $H^{1/2}(\Gamma_\mathrm{C})$. This is the image of the linear operator $[\![\cdot]\!]_\mathbf{n} : \mathbf{V} \to H^{1/2}(\Gamma_\mathrm{C})$ with its derivative given by (2.16b). Thus, thanks to the chain rule for the composition of convex (non-smooth) functionals with linear ones, cf. e.g. [IT79, Thm. 2, p. 201], we conclude that for all $u, v \in \mathbf{V}$ and $\zeta \in \mathbf{X}$,

$$\mathrm{D}_u\Phi^{\mathrm{coh}}([\![\,\cdot\,]\!]_\mathbf{n},\zeta) : \mathbf{V} \to \mathbf{V}^*, \qquad \begin{aligned}\langle \mathrm{D}_u\Phi^{\mathrm{coh}}([\![u]\!]_\mathbf{n},\zeta), v\rangle_\mathbf{V} &= \langle \mathrm{D}_{[\![u]\!]_\mathbf{n}}\Phi^{\mathrm{coh}}([\![u]\!]_\mathbf{n},\zeta), [\![v]\!]_\mathbf{n}\rangle_{H^{1/2}(\Gamma_\mathrm{C})}\\ &= \int_{\Gamma_\mathrm{C}} \tfrac{1}{\zeta^2}\mathrm{D}_u\phi_{\mathrm{coh}}([\![u]\!]_\mathbf{n},\zeta)[\![v]\!]_\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}\,,\end{aligned} \tag{2.21a}$$

$$\mathrm{D}_u\mathcal{J}_k([\![\,\cdot\,]\!]_\mathbf{n},\zeta) : \mathbf{V} \to \mathbf{V}^*, \qquad \begin{aligned}\langle \mathrm{D}_u\mathcal{J}_k([\![u]\!]_\mathbf{n},\zeta), v\rangle_\mathbf{V} &= \langle \mathrm{D}_{[\![u]\!]_\mathbf{n}}\mathcal{J}_k([\![u]\!]_\mathbf{n},\zeta), [\![v]\!]_\mathbf{n}\rangle_{H^{1/2}(\Gamma_\mathrm{C})}\\ &= \int_{\Gamma_\mathrm{C}} \mathrm{D}_{[\![u]\!]_\mathbf{n}}I_k([\![u]\!]_\mathbf{n},\zeta)[\![v]\!]_\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}\,.\end{aligned} \tag{2.21b}$$

Hereby, $I_k(\mathsf{v},\zeta) = \inf_{\mathsf{w}\in\mathbb{R}}\big(I_{[0,\zeta]}(\mathsf{w}) + \frac{k}{2}|\mathsf{v}-\mathsf{w}|^2\big) = \frac{k}{2}\big(|(\mathsf{v})^-|^2 + |(\mathsf{v}-\zeta)^+|^2\big)$ denotes the pointwise density of the functional $\mathcal{J}_k : L^2(\Gamma_\mathrm{C}) \to [0,\infty)$.

Then, cf. e.g. [Bré73, Example 2.8.2 p.46], the Fréchet-derivative $\mathrm{D}_\mathsf{v}\mathcal{J}_k(\cdot,\zeta)$, which is the Yosida-regularization to the subdifferential $\partial_\mathsf{v}\mathcal{J}(\cdot,\zeta)$ of $\mathcal{J}(\cdot,\zeta) : L^2(\Gamma_\mathrm{C}) \to [0,\infty]$ is given by

$$\mathrm{D}_\mathsf{v}\mathcal{J}_k(\cdot,\zeta) = k\big(\mathrm{Id} - P_{[0,\zeta]}\big), \tag{2.22}$$

with $\mathrm{Id} : L^2(\Gamma_\mathrm{C}) \to L^2(\Gamma_\mathrm{C})$ the identity and $P_{[0,\zeta]} : L^2(\Gamma_\mathrm{C}) \to L^2(\Gamma_\mathrm{C})$ the projection operator, i.e.,

$$P_{[0,\zeta]}(\mathsf{v}) = \begin{cases} 0 & \text{if } \mathsf{v} < 0,\\ \mathsf{v} & \text{if } \mathsf{v} \in [0,\zeta],\\ \zeta & \text{if } \mathsf{v} > \zeta,\end{cases}\quad. \tag{2.23}$$

Thus, the pointwise equivalent to (2.22) computes as

$$\mathrm{D}_\mathsf{v}I_k(\mathsf{v},\zeta) = k\big(-(\mathsf{v})^- + (\mathsf{v}-\zeta)^+\big)\,, \text{ i.e., } \mathrm{D}_\mathsf{v}I_k(\mathsf{v},\zeta) = \begin{cases} k\mathsf{v} & \text{if } \mathsf{v} < 0,\\ 0 & \text{if } \mathsf{v} \in [0,\zeta],\\ k(\mathsf{v}-\zeta) & \text{if } \mathsf{v} > \zeta,\end{cases} \tag{2.24}$$

In view of this, setting $\phi_k^\mathsf{s}([\![u]\!]_\mathbf{n},\zeta) := \phi^{\mathrm{coh}}([\![u]\!]_\mathbf{n},\zeta) + I_k([\![u]\!]_\mathbf{n},\zeta) + I_{[0,\zeta^*]}(\zeta) + G(\zeta)$, we get

$$\mathrm{D}_{[\![u]\!]_\mathbf{n}}\phi_k^\mathsf{s}([\![u]\!]_\mathbf{n},\zeta) = \frac{\phi_{\mathrm{coh}}(\zeta)}{\zeta^2}[\![u]\!]_\mathbf{n} + \mathrm{D}_{[\![u]\!]_\mathbf{n}}I_k([\![u]\!]_\mathbf{n},\zeta)$$

and, moreover, for every $(t,u,\zeta) \in \mathrm{dom}(\mathcal{E}_k)$, the functional $\mathcal{E}_k(t,\cdot,\zeta)$ is Gâteaux-differentiable with its derivative given by

$$\begin{aligned}&\langle \mathrm{D}_u\mathcal{E}_k(t,u,\zeta), v\rangle_\mathbf{V}\\ &= \int_{\Omega\backslash\Gamma_\mathrm{C}} \mathbb{C}e(u) : e(v)\,\mathrm{d}x - \langle \mathbf{f}(t), v\rangle_\mathbf{V} + \int_{\Gamma_\mathrm{C}} \Big(\frac{\phi_{\mathrm{coh}}(\zeta)}{\zeta^2}[\![u]\!]_\mathbf{n} + k\big(-([\![u]\!]_\mathbf{n})^- + ([\![u]\!]_\mathbf{n}-\zeta)^+\big)\Big)[\![v]\!]_\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}\,.\end{aligned} \tag{2.25}$$

Revisiting (2.21), by the validity of the chain rule due to the continuity of the composed functionals, the differentials can be represented in $L^2(\Gamma_{\rm C})$. However, this is no longer true for the non-smooth functional $\mathcal{J}(\cdot,\zeta) : H^{1/2}(\Gamma_{\rm C}) \to H^{1/2}(\Gamma_{\rm C})^*$, which is discontinuous on $H^{1/2}(\Gamma_{\rm C})$, the image of $[\![\cdot]\!]_{\mathbf{n}}$. For $\tilde{\xi} \in \partial_u \mathcal{J}([\![u]\!]_{\mathbf{n}},\zeta)$ we have here the representation $\tilde{\xi} = \xi \circ [\![\cdot]\!]_{\mathbf{n}}$ with $\xi \in \partial_{[\![u]\!]_{\mathbf{n}}} \mathcal{J}([\![u]\!]_{\mathbf{n}},\zeta)$ i.e.,

$$\langle \tilde{\xi}, v\rangle_{\mathbf{V}} = \langle \xi, [\![v]\!]_{\mathbf{n}}\rangle_{H^{1/2}(\Gamma_{\rm C})}\,, \tag{2.26}$$

but it cannot be concluded that $\xi \in L^2(\Gamma_{\rm C})$. Therefore the limit passage $k \to \infty$, i.e. from the regularized models to the cohesive zone model, has to be carried out in the topology of $\mathbf{V}$, or equivalently in $H^{1/2}(\Gamma_{\rm C})$, which is much stronger than the $L^2(\Gamma_{\rm C})$-topology. Exactly this strengthening of the topology for the non-smooth constraint and its regularizations features the main challenge in the analysis of the limit passage.

2.2 Our results for the Yosida-regularized model and its cohesive limit

The Yosida-regularized models $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E}_k)$, resp. $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E}_k)$, fall into the class of *gradient systems*, resp. *damped inertial systems* introduced at the beginning of Sec. 2. Based on abtract existence results proved in [RT15a] we will establish the existence of semistable energetic solutions for the systems $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E}_k)$, resp. $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E}_k)$, in Sec. 3.

Theorem 2.4 (Existence of semistable energetic solutions to the regularized systems)**.** *Let the assumptions* (2.7)–(2.8) *hold true. Then, for every $k \in \mathbb{N}$ fixed, for every initial datum*

(G) $(u_0,\zeta_0) \in \mathrm{dom}_u \times \mathrm{dom}_\zeta$ for the gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E}_k)$, $\varrho = 0$,

(I) $(u_0,u_1,\zeta_0) \in \mathrm{dom}_u \times \mathbf{W} \times \mathrm{dom}_\zeta$ for the damped inertial system $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E}_k)$, $\varrho > 0$,

with (u_0,ζ_0) satisfying (2.4) *for $\mathcal{E}_k$ and $\mathcal{R}$, the*

(G) gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E}_k)$, $\varrho = 0$,

(I) damped inertial system $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E}_k)$, $\varrho > 0$,

given by (2.10), (2.19), (1.5), *admits a semistable energetic solution (u_k,ζ_k) in the sense of Def. 2.1 of regularity*

$$u_k \in H^1(0,T;\mathbf{V})\,, \tag{2.27}$$

and such that

(G) $(u(0),\zeta(0)) = (u_0,\zeta_0)$ for the gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E}_k)$, $\varrho = 0$,

(I) $(u(0),\dot{u}(0),\zeta(0)) = (u_0,u_1,\zeta_0)$ for the damped inertial system $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E}_k)$, $\varrho > 0$.

Moreover, the energy-dissipation inequality (2.5) *even holds as a balance*

$$\begin{aligned} &\tfrac{\varrho}{2}\|\dot{u}_k(t)\|^2_{\mathbf{W}} + \int_0^t 2\mathcal{V}(\dot{u}_k(s))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}}(\zeta_k,[0,t]) + \mathcal{E}_k(t,u_k(t),\zeta_k(t)) \\ &\qquad = \tfrac{\varrho}{2}\|\dot{u}_k(0)\|^2_{\mathbf{W}} + \mathcal{E}_k(0,u_k(0),\zeta_k(0)) + \int_0^t \partial_t\mathcal{E}_k(s,u_k(s),\zeta_k(s))\,\mathrm{d}s \end{aligned} \tag{2.28}$$

for all $t \in [0,T]$, where $\varrho = 0$ for $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E}_k)$ and $\varrho > 0$ for $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E}_k)$.

In view of (2.24), the surface energy functional $\Phi_k^{\rm surf}$, cf. (2.19b), due to the Yosida-regularization $\mathcal{J}_k$, satisfies the following k-dependent (sub)gradient estimate, evaluated along solutions (u_k,ζ_k)

$$\|\partial_u\Phi_k^{\rm surf}(u_k,\zeta_k)\|_{L^2(0,T;\mathbf{V}^*)} \le c\sqrt{k} + C\,, \tag{2.29}$$

see Sec. 3 for the details of the calculation. Nevertheless, for gradient systems $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E}_k)$, i.e. $\varrho = 0$ in (2.28), a uniform bound on $\|\partial_u\Phi_k^{\rm surf}(u_k,\zeta_k)\|_{L^2(0,T;\mathbf{V}^*)}$ can be obtained by comparison in the k-momentum balance. This will be used in Sec. 4 to prove for the cohesive zone delamination model $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E}_\infty)$

the existence of semistable energetic solutions by performing an evolutionary Γ-limit as $k \to \infty$ from the regularized systems $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_k)$ to the cohesive limit system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_\infty)$. The main challenge in this limit analysis lies in the fact that the non-smooth constraint incorporated in $\mathcal{J}([\![\cdot]\!]_\mathbf{n}, \zeta)$, cf. (2.17f), and approximated by $\mathcal{J}_k([\![\cdot]\!]_\mathbf{n}, \zeta_k)$, cf. (2.18a), depends on the internal variable itself. This will require to prove the Mosco-convergence of the functionals $(\mathcal{J}_k([\![\cdot]\!]_\mathbf{n}, \zeta_k))_k$ in the topology $\mathbf{V} = H^1(\Omega\backslash\Gamma_\mathrm{C})$, cf. Prop. 4.3. We now state the main result of this work: The existence of semistable energetic solutions to the gradient cohesive zone system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_\infty)$ obtained via evolutionary Γ-convergence:

Theorem 2.5 (Evolutionary Γ-convergence of the gradient systems, $\varrho = 0$)**.** *Let the assumptions of Theorem 2.4 hold true. In addition assume that $\mathbf{X} = W^{1,r}(\Gamma_\mathrm{C})$ with $r > d-1$ and that the initial datum complies with the following assertion:*

$$\exists\, \zeta_* \text{ with } 0 < \zeta_* \ll 1 : \ \zeta_0 \geq \zeta_* \quad \text{a.e. on } \Gamma_\mathrm{C}\,. \tag{2.30}$$

Moreover, assume that the initial data are well-prepared, i.e., $(u_k(0), \zeta_k(0)) = (u_0^k, \zeta_0^k)$ are semistable for $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_k)$ at $t = 0$, $\mathcal{E}_k(0, u_0^k, \zeta_0^k) \to \mathcal{E}_\infty(0, u_0, \zeta_0)$, and $\mathcal{J}([\![u_0]\!]_\mathbf{n}, \zeta_0) = 0$. For each $k \in \mathbb{N}$, let (u_k, ζ_k) be a semistable energetic solution to the regularized system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_k)$. Then, there exists a (not relabeled) subsequence $(u_k, \zeta_k)_k$ and a limit pair (u, ζ) of regularity

$$u \in H^1(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{V}) \quad \text{and} \quad \zeta \in L^\infty(0,T;\mathbf{X})\,, \tag{2.31}$$

such that the following convergences hold true

$$u_k \rightharpoonup u \text{ in } H^1(0,T;\mathbf{V})\,, \tag{2.32a}$$
$$\zeta_k(t) \rightharpoonup \zeta(t) \text{ in } \mathbf{X} \text{ for all } t \in [0,T]\,, \tag{2.32b}$$
$$\zeta_k(t) \to \zeta(t) \text{ uniformly in } \Gamma_\mathrm{C} \text{ for all } t \in [0,T]\,, \tag{2.32c}$$
$$\zeta_k \to \zeta \text{ in } L^q(0,T;L^q(\Gamma_\mathrm{C})) \quad \text{for all } q \in [1,\infty)\,, \tag{2.32d}$$
$$\mathrm{D}_u\mathcal{J}_k(\left[\!\left[u_k\right]\!\right]_\mathbf{n}, \zeta_k) \rightharpoonup \tilde{\xi} \quad \text{in } L^2(0,T;\mathbf{V}^*). \tag{2.32e}$$

The limit pair (u, ζ) is a semistable energetic solution of the cohesive zone system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_\infty)$, cf. Def. 2.1. Moreover, the following statements hold true:

1. *For a.e. $t \in (0,T)$ the momentum balance takes the following form*

$$\langle \mathrm{D}_u\mathcal{E}^{\mathrm{bulk}}(u(t)) + \mathrm{D}_u\Phi^{\mathrm{coh}}(\left[\!\left[u(t)\right]\!\right]_\mathbf{n}, \zeta(t)) + \tilde{\xi}(t) + \mathrm{D}_{\dot{u}}\mathcal{V}(\dot{u}(t)), v\rangle_\mathbf{V} = 0 \tag{2.33}$$

   for every $v \in \mathbf{V}$, with $\tilde{\xi}(t) \in \partial_u\mathcal{J}([\![u(t)]\!]_\mathbf{n}, \zeta(t))$ for a.e. $t \in (0,T)$.

2. *For all $t \in [0,T]$ the energy dissipation inequality is satisfied:*

$$\mathcal{E}_k(t, u(t), \zeta(t)) + \int_0^t 2\mathcal{V}(\dot{u}(\tau))\,\mathrm{d}\tau + \mathrm{Var}_\mathcal{R}(\zeta, [0,t]) \leq \mathcal{E}(0, u(0), \zeta(0)) + \int_0^t \partial_t\mathcal{E}(\tau, u(\tau), \zeta(\tau))\,\mathrm{d}\tau\,. \tag{2.34}$$

Remark 2.6 (Condition on the initial datum (2.30))*.* The strictly positive bound from below (2.30) on the initial datum means that the bonding along Γ_C has already experienced an opening in normal direction of at least ζ_* in the past, and at initial time of the monitoring of the loading experiment, the maximal opening ever experienced before in the point $x \in \Gamma_\mathrm{C}$ takes the value $\zeta_0(x) \geq \zeta_*$. Furthermore, observe that the unidirectionality of the 1-homogeneous dissipation potential $\mathcal{R}$ together with the initial condition $\zeta(0) = \zeta_0 \geq \zeta_*$ ensures that $\zeta(t,x) \geq \zeta_0(x) \geq \zeta_*$ a.e..

Remark 2.7 (Limit passage in case of damped inertial systems). For the systems $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E}_k)$, a comparison argument in the subdifferential inclusion (2.3) only provides uniform bounds, independent of $k \in \mathbb{N}$, on the sum $\varrho \ddot{u}_k(t) + \mathrm{D}_u \mathcal{J}_k([\![u_k(t)]\!]_{\mathbf{n}}, \zeta_k(t))$, but not on the two terms separately. Moreover, in view of (2.29) one obtains an analogous k-dependent bound on the inertial term:

$$\varrho \|\ddot{u}_k\|_{L^2(0,T;\mathbf{V}^*)} \leq \tilde{c}\sqrt{k} + \widetilde{C}\,. \tag{2.35}$$

Since the bounds (2.29)&(2.35) blow up as $k \to \infty$, one cannot use the notion of semistable energetic solutions from Def. 2.1 to carry out an evolutionary Γ-limit passage for the damped inertial systems $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E}_k)$. A promising ansatz would be to follow an alternative approach established in the recent series of works [BRSS15, SS15, Sca15] that successfully handle the analysis of problems combining unilateral, non-smooth constraints with the visco-elasto*dynamic* setting. Therein, the key to circumvent the problems caused by the blow-up of the bounds (2.29)&(2.35) is to carry out the limit passage from a regularized to the original model in a notion of solution that uses a weak formulation of the momentum balance in Bochner spaces $H^1(0, T, \mathbf{V})$ instead of the pointwise-in-time formulation (2.3). But let us stress that their approach so far has proved successful only for non-smooth unilateral constraints that are *independent* of the state variables. This is in clear contrast to our model, where the constraint strongly depends on the internal variable ζ. Exactly here lies the limitation of this approach: our models combine the rate-dependent (and dynamic) evolution of the variable u with a *rate-independent* evolution of the internal variable. As can be seen from (2.2) the internal variable has much less temporal regularity than the displacement variable. Therefore, at the moment, it seems to be out of reach to exploit a weak formulation of the momentum balance in $(H^1(0, T, \mathbf{V}))^*$ in order to pass to the cohesive zone limit. The problem can be phrased more precisely: Passing from a pointwise-in-time formulation of (2.3) (equivalently, from a formulation in $L^2(0, T; \mathbf{V}^*)$) to a formulation in $(H^1(0, T, \mathbf{V}))^*$ will provide uniform bounds on $(\mathrm{D}_u \mathcal{J}_k([\![u_k(t)]\!]_{\mathbf{n}}, \zeta_k(t)))_k$ in $(H^1(0, T, \mathbf{V}))^*$ and necessitate the identification of the limit as an element of the subdifferential $\partial_u \int_0^T \mathcal{J}([\![u(t)]\!]_{\mathbf{n}}, \zeta(t))\,\mathrm{d}t \subset (H^1(0, T, \mathbf{V}))^*$. This is at the price that Mosco-convergence of the functionals $(\int_0^T \mathcal{J}_k([\![u_k(t)]\!]_{\mathbf{n}}, \zeta_k(t))\,\mathrm{d}t)_k$ has to be established in $H^1(0, T, \mathbf{V})$. But since the internal variable ζ_k is expected to be only of BV-regularity in time, the contruction of a recovery sequence for the displacements that depends on ζ_k and converges strongly in $H^1(0, T, \mathbf{V})$ seems to be out of reach.

In other words, the mismatch between the temporal regularity of the diplacements and the one of the internal variable seems to inhibit the Γ-limit passage in case of damped inertial systems. This problem might be overcome if one assumes that the internal variable has a rate-dependent evolution governed by a viscous dissipation potential, which will be the scope of a follow-up work.

3 Proof of Theorem 2.4 – Existence of semistable energetic solutions ($k \in \mathbb{N}$ fixed)

In this section we address the existence of semistable energetic solutions for the Yosida-regularized systems $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_k)$, resp. $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E}_k)$, with $k \in \mathbb{N}$ fixed, by resorting to the abstract existence results proved in [RT15a], cf. [RT15a, Thm. 4.9] for gradient systems, resp. [RT15a, Thm. 5.6] for damped inertial systems. In what follows, for the reader's convenience we shall first revisit the prerequisites on an abstract gradient system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$, resp. an abstract damped inertial system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$, underlying the existence results in [RT15a], and then verify that the systems $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_k)$, resp.

$(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E}_k)$, do comply with them, thus deducing the existence of semistable energetic solutions for the Yosida-regularized systems.

Let $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ be a gradient system, resp. $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E})$ be a damped inertial system, complying with the basic conditions (2.1). In line with the direct method of the calculus of variations and with tools from rate-independent and gradient systems, [RT15a, Thm. 4.9 & Thm. 5.6] put the following additional requirements on the functionals $\mathcal{V}:\mathbf{V}\to[0,\infty)$, $\mathcal{R}:\mathbf{Z}\to[0,\infty]$, and $\mathcal{E}:[0,T]\times\mathbf{V}\times\mathbf{Z}\to\mathbb{R}\cup\{\infty\}$:

Boundedness from below & Weak lower semicontinuity:

$$\mathcal{E}\text{ is bounded from below: }\exists\, C_0>0,\ \forall\,(t,u,\zeta)\subset\mathrm{dom}(\mathcal{E}):\ \mathcal{E}(t,u,\zeta)\geq C_0\,; \tag{3.1a}$$

$$\text{for all } t\in[0,T],\ \mathcal{E}(t,\cdot,\cdot)\text{ is weakly sequentially lower semicontinuous on }\mathbf{V}\times\mathbf{Z}\,, \tag{3.1b}$$

indeed, if $\mathcal{E}$ is bounded from below, up to a shift we can assume that it is bounded by a positive constant.

Temporal regularity and power control:

$$\left.\begin{array}{l}\forall\,(u,\zeta)\in\mathrm{dom}_u\times\mathrm{dom}_\zeta,\ \text{the map } t\mapsto\mathcal{E}(t,u,\zeta)\text{ is differentiable with derivative }\partial_t\mathcal{E}(t,u,\zeta)\text{ s.t.}\\ \exists\, C_1,C_2>0,\forall\,(t,u,\zeta)\in\mathrm{dom}(\mathcal{E}):\ |\partial_t\mathcal{E}(t,u,\zeta)|\leq C_1(\mathcal{E}(t,u,\zeta)+C_2)\text{ and fulfilling}\\ \text{for all sequences } t_n\to t,\ u_n\to u\text{ in }\mathbf{V},\ \zeta_n\to\zeta\text{ in }\mathbf{Z}\text{ with }\sup_n\mathcal{E}(t_n,u_n,\zeta_n)\leq C\text{ that}\\ \limsup_{n\to\infty}\partial_t\mathcal{E}(t_n,u_n,\zeta_n)\leq\partial_t\mathcal{E}(t,u,\zeta)\,.\end{array}\right\} \tag{3.2}$$

Coercivity:

$$\left.\begin{array}{l}\text{there exist }\tau_o>0\text{ such that for all }(t,u_o,\zeta_o)\in[0,T]\times\mathbf{V}\times\mathbf{Z}\\ \text{the map }(u,\zeta)\mapsto\mathcal{E}(t,u,\zeta)+\tau_o\mathcal{V}\big(\tfrac{u-u_o}{\tau_o}\big)+\mathcal{R}(\zeta-\zeta_o)\text{ has sublevels bounded in }\mathbf{V}\times\mathbf{X}\,.\end{array}\right\} \tag{3.3}$$

Mutual recovery sequence condition ensuring the closedness of of stable sets:

$$\left.\begin{array}{l}\text{Let }(t_n,u_n,\zeta_n)_n\subset\mathrm{dom}(\mathcal{E})\text{ for every }n\in\mathbb{N}\text{ satisfy semistability condition (2.4),}\\ \text{let }t_n\to t,(u_n,\zeta_n)\rightharpoonup(u,\zeta)\text{ in }\mathbf{V}\times\mathbf{Z}\text{ with }\sup_n\mathcal{E}(t,u_n,\zeta_n)\leq C\text{ for all }t\in[0,T].\\ \text{Then, for every }\tilde\zeta\in\mathbf{Z}\text{ there exists }\tilde\zeta_n\rightharpoonup\tilde\zeta\text{ in }\mathbf{Z}\text{ such that}\\ \limsup_{n\to\infty}\big(\mathcal{E}(t_n,u_n,\tilde\zeta_n)+\mathcal{R}(\tilde\zeta_n-\zeta_n)-\mathcal{E}(t_n,u_n,\zeta_n)\big)\leq\mathcal{E}(t,u,\tilde\zeta)+\mathcal{R}(\tilde\zeta-\zeta)-\mathcal{E}(t,u,\zeta)\,.\end{array}\right\} \tag{3.4}$$

As previouly mentioned, the existence results in [RT15a] allow for a non-smooth and even non-convex (in lower order terms) dependence $u\mapsto\mathcal{E}(t,u,\zeta)$. However, since the energies $\mathcal{E}_k(t,\cdot,\zeta)$ from (2.19) are convex and Gâteaux-differentiable, we will confine the discussion to energies with this property and denote by $\partial_u\mathcal{E}(t,\cdot,\zeta)$ the Gâteaux-differential of the convex functional $\mathcal{E}(t,\cdot,\zeta)$. Following [RT15a, Thm. 4.9 & Thm. 5.6], we need to impose a suitable condition on the differentials $\partial_u\mathcal{E}$ in the spirit of Minty's trick:

Continuity:

For all sequences $(\mathrm{t}_n)_n,\mathrm{t}_n:[0,T]\to[0,T],(u_n)_n\subset L^\infty(0,T;\mathbf{V})\cap H^1(0,T;\mathbf{V})$,

$(\zeta_n)_n\subset L^\infty(0,T;\mathbf{X})\cap\mathrm{BV}([0,T];\mathbf{Z}),(\partial_u\mathcal{E}(\mathrm{t}_n,u_n,\zeta_n))_n\subset L^2(0,T;\mathbf{V}^*)$ s.t.

$\exists\, C>0,\ \forall\, n\in\mathbb{N},\ \forall\, t\in[0,T]:\ \mathcal{E}(t,u_n(t),\zeta_n(t))\leq C$ and

$$\left\{\begin{array}{l}\mathrm{t}_n\to\mathrm{t}\text{ pointwise a.e. in }(0,T)\,,\\ u_n\overset{*}{\rightharpoonup}u\text{ in }L^\infty(0,T;\mathbf{V})\cap H^1(0,T;\mathbf{V})\,,\\ \zeta_n\overset{*}{\rightharpoonup}\zeta\text{ in }L^\infty(0,T;\mathbf{X})\,,\ \zeta_n(t)\overset{*}{\rightharpoonup}\zeta(t)\text{ in }\mathbf{X}\text{ for all }t\in[0,T]\,,\\ \partial_u\mathcal{E}(\mathrm{t}_n,u_n,\zeta_n)\rightharpoonup\xi\text{ in }L^2(0,T;\mathbf{V}^*)\,,\ \limsup_{n\to\infty}\int_0^T\langle\partial_u\mathcal{E}(\mathrm{t}_n,u_n,\zeta_n),u_n\rangle_{\mathbf{V}}\,\mathrm{d}t\leq\int_0^T\langle\xi,u\rangle_{\mathbf{V}}\,\mathrm{d}t\,,\end{array}\right\}$$

then there holds $\xi(t)=\partial_u\mathcal{E}(\mathrm{t}(t),u(t),\zeta(t))$ for a.a. $t\in(0,T)$.

$$\tag{3.5}$$

Finally, to find a bound on the inertial term, a further requirement of [RT15a, Thm. 4] is the following

Subgradient estimate:

$$\begin{aligned}&\text{There exists constants } C_3, C_4, C_5 > 0 \text{ and } \sigma \in [1,\infty) \text{ such that}\\ &\forall\,(t,u,\zeta)\in\mathrm{dom}(\mathcal{E}):\quad \|\partial_u\mathcal{E}(t,u,\zeta)\|^\sigma_{\mathbf{V}^*} \le C_3\mathcal{E}(t,u,\zeta)+C_4\|u\|_{\mathbf{V}}+C_5\,.\end{aligned}\tag{3.6}$$

We are now in the position to recall the existence result from [RT15a] for damped inertial systems $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E})$.

Theorem 3.1 ([RT15a, Thm. 5.6])**.** *Let* $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E})$ *fulfill* (2.1) *&* (3.1)–(3.6).

Then for every $(u_0,u_1,\zeta_0)\in\mathrm{dom}_u\times\mathbf{W}\times\mathrm{dom}_\zeta$ *fulfilling the semistability* (2.4) *at* $t=0$*, i.e.*

$$\mathcal{E}(0,u_0,\zeta_0)\le\mathcal{E}(0,u_0,\tilde\zeta)+\mathcal{R}(\tilde\zeta-\zeta_0)\qquad\textit{for all } \tilde\zeta\in\mathbf{Z}\tag{3.7}$$

there exists a semistable energetic solution (u,ζ) *(in the sense of Definition 2.1) to the damped inertial system* $(\mathbf{V},\mathbf{W},\mathbf{Z},\mathcal{V},\mathcal{K},\mathcal{R},\mathcal{E})$ *satisfying the Cauchy condition* $(u(0),\dot u(0),\zeta(0))=(u_0,u_1,\zeta_0)$*.*

Moreover, the abstract existence result from [RT15a] for gradient systems $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ reads as follows:

Theorem 3.2 ([RT15a, Thm. 4.9])**.** *Let* $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$ *fulfill* (2.1) *&* (3.1)–(3.5).

Then, for every pair $(u_0,\zeta_0)\in\mathrm{dom}_u\times\mathrm{dom}_\zeta$ *complying with* (3.7)*, there exists a semistable energetic solution* (u,ζ) *(in the sense of Definition 2.1) to the gradient system* $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E})$*, satisfying the initial condition* $(u(0),\zeta(0))=(u_0,\zeta_0)$*.*

Proof of Theorem 2.4:

As a first step we will prove the mutual recovery sequence condition (3.4) as a separate lemma. The proof of this lemma uses the approach developed in [TM10, Tho13], but is here for the first time adapted to the character of the Sobolev-Slobodeckij seminorm. Recall from (2.17e) & (2.20) that for any triple (t,u,ζ) belonging to the domain $\mathrm{dom}(\mathcal{E}_k)$, it holds in particular that $\zeta\in[0,\zeta^*]$ a.e. on Γ_{C}.

Lemma 3.3 (Mutual recovery sequence condition (3.4))**.** *Let* $k\in\mathbb{N}$ *fixed and consider a sequence* $(t_n,u_n,\zeta_n)_n\subset\mathrm{dom}(\mathcal{E}_k)$ *such that* $t_n\to t$ *in* $[0,T]$*,* $u_n\rightharpoonup u$ *in* $\mathbf{V}$ *and* $\zeta_n\rightharpoonup\zeta$ *in* $\mathbf{X}$*, as well as an element* $\hat\zeta\in\mathbf{X}$*. Then the sequence* $(\hat\zeta_n)_n\subset\mathbf{X}$*, given by*

$$\hat\zeta_n:=\min\{\zeta^*,\max\{\zeta_n,\hat\zeta+\delta_n\}\}\quad\textit{with}\quad\delta_n:=\max\big\{\|\zeta_n-\zeta\|^{1/2}_{L^2(\Gamma_{\mathrm{C}})},1/n\big\}\,,\tag{3.8}$$

serves as a mutual recovery sequence for the functionals $\mathcal{E}_k(t_n,u_n,\cdot)$ *&* $\mathcal{R}$*, i.e., Condition* (3.4) *is satisfied. In particular, for* $\hat\zeta\in\mathbf{X}$ *such that* $\hat\zeta\ge\zeta$ *a.e. in* Γ_{C}*, the following relations hold true with* $r=2$ *for* $\mathbf{X}=H^{1/2}(\Gamma_{\mathrm{C}})$ *and* $r>d-1$ *for* $\mathbf{X}=W^{1,r}(\Gamma_{\mathrm{C}})$*:*

$$\hat\zeta_n\rightharpoonup\hat\zeta\quad\textit{in }\mathbf{X}\,,\tag{3.9a}$$

$$\limsup_{n\to\infty}\Big(|\hat\zeta_n|^r_{\mathbf{X}}-|\zeta_n|^r_{\mathbf{X}}\Big)\le|\hat\zeta|^r_{\mathbf{X}}-|\zeta|^r_{\mathbf{X}}\,,\tag{3.9b}$$

$$\lim_{n\to\infty}\mathcal{R}(\hat\zeta_n-\zeta_n)=\mathcal{R}(\hat\zeta-\zeta)\,,\tag{3.9c}$$

$$\limsup_{n\to\infty}\Big(\Phi^{\mathrm{coh}}([\![u_n]\!]_{\mathbf{n}},\hat\zeta_n)-\Phi^{\mathrm{coh}}([\![u_n]\!]_{\mathbf{n}},\zeta_n)\Big)=\Phi^{\mathrm{coh}}([\![u]\!]_{\mathbf{n}},\hat\zeta)-\Phi^{\mathrm{coh}}([\![u]\!]_{\mathbf{n}},\zeta)\,,\tag{3.9d}$$

$$\Big(\mathcal{J}_k([\![u_n]\!]_{\mathbf{n}},\hat\zeta_n)-\mathcal{J}_k([\![u_n]\!]_{\mathbf{n}},\zeta_n)\Big)\le 0\quad\textit{for every } n\in\mathbb{N}\,,\tag{3.9e}$$

$$\limsup_{n\to\infty}\Big(\mathcal{J}_k([\![u_n]\!]_{\mathbf{n}},\hat\zeta_n)-\mathcal{J}_k([\![u_n]\!]_{\mathbf{n}},\zeta_n)\Big)=\mathcal{J}_k([\![u]\!]_{\mathbf{n}},\hat\zeta)-\mathcal{J}_k([\![u]\!]_{\mathbf{n}},\zeta)\,.\tag{3.9f}$$

Proof. In the case $\mathbf{X} = W^{1,r}(\Gamma_\mathrm{C})$ it is straighforward to adapt the arguments of [TM10, Thm. 3.14] to the present situation.

Let now $\mathbf{X} = H^{1/2}(\Gamma_\mathrm{C})$. First of all, by Lemma 2.2 we convince ourselves that $\hat\zeta_n$ obtained by the superposition of the Lipschitz-functions max and min with the functions $\zeta_n, (\hat\zeta+\delta_n) \in \mathbf{X}$ and the constant ζ^* according to (3.8) has a bounded $H^{1/2}$-seminorm:

$$|\hat\zeta_n|^2_{H^{1/2}(\Gamma_\mathrm{C})} \le |\zeta_n|^2_{H^{1/2}(\Gamma_\mathrm{C})} + |\hat\zeta + 1|^2_{H^{1/2}(\Gamma_\mathrm{C})} + |\zeta^*|^2_{H^{1/2}(\Gamma_\mathrm{C})}$$

for n sufficiently large such that $\delta_n \le 1$. Moreover, by $\hat\zeta_n \le \hat\zeta + \delta_n + \zeta_n$ a.e. in Γ_C, we find that also $\|\hat\zeta_n\|_{L^2(\Gamma_\mathrm{C})} \le \|\zeta^*\|_{L^2(\Gamma_\mathrm{C})} + \|\hat\zeta + 1\|_{L^2(\Gamma_\mathrm{C})} + \|\zeta_n\|_{L^2(\Gamma_\mathrm{C})}$. Hence, $\hat\zeta_n \in \mathbf{X}$ for all $n \in \mathbb{N}$.

Recall now the definition of $\mathcal{R}$ from (1.5). If $\hat\zeta \in \mathbf{X}$ is such that $\mathcal{L}^{d-1}([\hat\zeta < \zeta]) \neq 0$, then $\mathcal{R}(\hat\zeta - \zeta) = \infty$ and thus, condition (3.4) is trivially satisfied. Therefore, let us assume from now on that $\hat\zeta \ge \zeta$ a.e. on Γ_C.

Ad (3.9a)**:** In view of (3.8) we have the uniform bound $\|\hat\zeta_n\|_\mathbf{X} \le \|\hat\zeta+1\|_\mathbf{X} + \sup_n \|\zeta_n\|_\mathbf{X} + \|\zeta^*\|_{L^2(\Gamma_\mathrm{C})} \le C$, since $\zeta_n \rightharpoonup \zeta$ in $\mathbf{X}$ by assumption. Therefore, $\hat\zeta_n \rightharpoonup \tilde\zeta$ in $\mathbf{X}$ for some $\tilde\zeta \in \mathbf{X}$. Moreover, from (3.8) we see that $\hat\zeta_n \to \hat\zeta$ pointwise a.e. in Γ_C. Hence we conclude that, indeed, $\tilde\zeta = \hat\zeta$.

Ad (3.9b)**:** We introduce the following notation and note that

$$\begin{aligned} A_n &:= [\zeta^* > \hat\zeta + \delta_n \ge \zeta_n]\,,\ \text{hence } \hat\zeta_n = \hat\zeta + \delta_n \text{ on } A_n\,, && (3.10a)\\ B_n &:= [\hat\zeta + \delta_n < \zeta_n]\,,\ \text{hence } \hat\zeta_n = \zeta_n \text{ on } B_n\,, && (3.10b)\\ C_n &:= \Gamma_\mathrm{C} \setminus (A_n \cup B_n) = [\zeta_n \le \zeta^* \le \hat\zeta + \delta_n]\,,\ \text{hence } \hat\zeta_n = \zeta^* \text{ on } C_n\,, && (3.10c) \end{aligned}$$

with the short-hand $[a \le b] := \{x \in \Gamma_\mathrm{C},\, a(x) \le b(x) \text{ for a.e. } x \in \Gamma_\mathrm{C}\}$. For the sets B_n we observe that

$$B_n = [\hat\zeta + \delta_n < \zeta_n] \subset [\hat\zeta + \delta_n \le \zeta_n] = [\delta_n \le \zeta_n - \hat\zeta] \subset [\delta_n \le \zeta_n - \zeta] \subset [\delta_n \le |\zeta_n - \zeta|]\,, \tag{3.11}$$

because $\hat\zeta \ge \zeta$ by assumption. Moreover, since $\zeta_n \rightharpoonup \zeta$ in $\mathbf{X}$ by assumption, hence $\zeta_n \to \zeta$ strongly in $L^2(\Gamma_\mathrm{C})$, we find that

$$\mathcal{L}^{d-1}(B_n) \le \mathcal{L}^{d-1}([\delta_n \le |\zeta_n - \zeta|]) \le \frac{1}{\delta_n^2}\|\zeta_n - \zeta\|^2_{L^2} \le \|\zeta_n - \zeta\|_{L^2} \to 0\,, \tag{3.12}$$

where we used Markoff's inequality to obtain the second estimate. The last estimate follows from the very definition of $\delta_n := \max\big\{\|\zeta_n - \zeta\|^{1/2}_{L^2}, 1/n\big\}$, cf. (3.8). Thus, we have that

$$\mathcal{L}^{d-1}(B_n) \to 0 \quad \text{and} \quad \mathcal{L}^{d-1}(\Gamma_\mathrm{C}\backslash(A_n \cup C_n)) \to 0 \quad \text{as } n \to \infty\,. \tag{3.13}$$

We use that $\Gamma_\mathrm{C} = (A_n \cup C_n) \cup B_n$ and in view of the definition of $(\hat\zeta_n)_n$ from (3.8) the term involving the Sobolev-Slobodeckij seminorms rewrites and estimates as follows:

$$\begin{aligned} &\big|\hat\zeta_n\big|^2_{H^{1/2}(\Gamma_\mathrm{C})} - \big|\zeta_n\big|^2_{H^{1/2}(\Gamma_\mathrm{C})} = \sum_{j=1}^{9} I_j^n\,, \quad \text{with} && (3.14a)\\ &I_1^n := \big|\hat\zeta + \delta_n\big|^2_{H^{1/2}(A_n)} - \big|\zeta_n\big|^2_{H^{1/2}(A_n)}\,, && (3.14b)\\ &I_2^n := \big|\hat\zeta_n\big|^2_{H^{1/2}(B_n)} - \big|\zeta_n\big|^2_{H^{1/2}(B_n)} = 0\,, && (3.14c)\\ &I_3^n := \int_{x\in B_n}\int_{y\in A_n} \tfrac{1}{|x-y|^d}\Big(-2\zeta_n(x)(\hat\zeta(y)+\delta_n) + |\hat\zeta(y)+\delta_n|^2 + 2\zeta_n(x)\zeta_n(y) - |\zeta_n(y)|^2\Big)\,\mathrm{d}x\,\mathrm{d}y\,, && (3.14d)\\ &I_4^n := \int_{x\in A_n}\int_{y\in B_n} \tfrac{1}{|x-y|^d}\Big(-2\zeta_n(y)(\hat\zeta(x)+\delta_n) + |\hat\zeta(x)+\delta_n|^2 + 2\zeta_n(x)\zeta_n(y) - |\zeta_n(x)|^2\Big)\,\mathrm{d}x\,\mathrm{d}y\,, && (3.14e)\\ &I_5^n := \big|\hat\zeta_n\big|^2_{H^{1/2}(C_n)} - \big|\zeta_n\big|^2_{H^{1/2}(C_n)} = -\big|\zeta_n\big|^2_{H^{1/2}(C_n)}\,, && (3.14f) \end{aligned}$$

$$
\begin{aligned}
I_6^n &:= \int_{x\in A_n}\int_{y\in C_n} \frac{|\zeta^*-\hat\zeta(x)-\delta_n|^2}{|x-y|^d} - \int_{x\in A_n}\int_{y\in C_n} \frac{|\zeta_n(x)-\zeta_n(y)|^2}{|x-y|^d} \\
&\le \int_{x\in A_n}\int_{y\in C_n} \frac{|\hat\zeta(y)+\delta_n-\hat\zeta(x)-\delta_n|^2}{|x-y|^d} - \int_{x\in A_n}\int_{y\in C_n} \frac{|\zeta_n(x)-\zeta_n(y)|^2}{|x-y|^d}\,, \qquad (3.14g)\\
I_7^n &:= \int_{x\in C_n}\int_{y\in A_n} \frac{|\zeta^*-\hat\zeta(y)-\delta_n|^2}{|x-y|^d} - \int_{x\in C_n}\int_{y\in A_n} \frac{|\zeta_n(x)-\zeta_n(y)|^2}{|x-y|^d} \\
&\le \int_{x\in C_n}\int_{y\in A_n} \frac{|\hat\zeta(x)+\delta_n-\hat\zeta(y)-\delta_n|^2}{|x-y|^d} - \int_{x\in C_n}\int_{y\in A_n} \frac{|\zeta_n(x)-\zeta_n(y)|^2}{|x-y|^d}\,, \qquad (3.14h)\\
I_8^n &:= \int_{x\in B_n}\int_{y\in C_n} \frac{|\zeta_n(x)-\zeta^*|^2}{|x-y|^d} - \int_{x\in B_n}\int_{y\in C_n} \frac{|\zeta_n(x)-\zeta_n(y)|^2}{|x-y|^d} \\
&\le \int_{x\in B_n}\int_{y\in C_n} \frac{|\hat\zeta(x)+\delta_n-\hat\zeta(y)-\delta_n|^2}{|x-y|^d} - 0\,, \qquad (3.14i)\\
I_9^n &:= \int_{x\in C_n}\int_{y\in B_n} \frac{|\zeta_n(y)-\zeta^*|^2}{|x-y|^d} - \int_{x\in C_n}\int_{y\in B_n} \frac{|\zeta_n(x)-\zeta_n(y)|^2}{|x-y|^d} \\
&\le \int_{x\in C_n}\int_{y\in B_n} \frac{|\hat\zeta(x)+\delta_n-\hat\zeta(y)-\delta_n|^2}{|x-y|^d} - 0\,. \qquad (3.14j)
\end{aligned}
$$

Our aim is to further process the above terms in such a way that (3.14a) can be estimated from above by

$$
\big|\hat\zeta_n\big|^2_{H^{1/2}(\Gamma_C)} - \big|\zeta_n\big|^2_{H^{1/2}(\Gamma_C)} \le \sum_{j=1}^{9} I_j^n \le \big|\hat\zeta+\delta_n\big|^2_{H^{1/2}(\Gamma_C)} - \big|\zeta_n\big|^2_{H^{1/2}(A_n\cup C_n)}\,. \qquad (3.15)
$$

We see that the terms I_1^n, I_2^n and I_5^n–I_9^n will already nicely contribute to this estimate. But it remains to suitably estimate the mixed integrals (3.14d) & (3.14e). For this, we want to make use of the following estimate for (the integrand of) I_n^3:

$$
-2\zeta_n(x)(\hat\zeta(y)+\delta_n)+|\hat\zeta(y)+\delta_n|^2+2\zeta_n(x)\zeta_n(y)-|\zeta_n(y)|^2 \le |\hat\zeta(x)+\delta_n|^2-2(\hat\zeta(x)+\delta_n)(\hat\zeta(y)+\delta_n)+|\hat\zeta(y)+\delta_n|^2\,. \qquad (3.16)
$$

Verifying (3.16) is equivalent to showing that

$$
0 \le |\hat\zeta(x)+\delta_n|^2 + |\zeta_n(y)|^2 - 2(\hat\zeta(x)+\delta_n)(\hat\zeta(y)+\delta_n) + 2\zeta_n(x)(\hat\zeta(y)+\delta_n) - 2\zeta_n(x)\zeta_n(y)\,. \qquad (3.17)
$$

Using that $|\hat\zeta(x)+\delta_n|^2+|\zeta_n(y)|^2 \ge 2\zeta_n(y)(\hat\zeta(x)+\delta_n)$ and keeping in mind that $x\in B_n$, whereas $y\in A_n$, we can further estimate the right-hand side of (3.17) from below as follows

$$
(\hat\zeta(x)+\delta_n)\big(\zeta_n(y)-(\hat\zeta(y)+\delta_n)\big)+\zeta_n(x)\big((\hat\zeta(y)+\delta_n)-\zeta_n(y)\big) = \big((\hat\zeta(y)+\delta_n)-\zeta_n(y)\big)\big(\zeta_n(x)-(\hat\zeta(x)+\delta_n)\big) \ge 0\,, \qquad (3.18)
$$

i.e., (3.17), resp. (3.16), is verified. With the same arguments we can deduce an analogous estimate for the (integrand of) I_n^4, essentially by swapping the meaning of the variables x and y in (3.17), resp. (3.16). Putting these findings together, we see that

$$
I_n^3 \le \int_{A_n}\int_{B_n} \frac{|\hat\zeta(x)+\delta_n-\hat\zeta(y)-\delta_n|^2}{|x-y|^d}\,\mathrm{d}x\,\mathrm{d}y \quad\text{and}\quad I_n^4 \le \int_{B_n}\int_{A_n} \frac{|\hat\zeta(x)+\delta_n-\hat\zeta(y)-\delta_n|^2}{|x-y|^d}\,\mathrm{d}x\,\mathrm{d}y\,, \qquad (3.19)
$$

so that indeed, (3.15) holds true. Hence, we deduce that

$$\begin{aligned}
&\limsup_{n\to\infty}\Big(\big|\hat\zeta_n\big|^2_{H^{1/2}(\Gamma_{\rm C})}-\big|\zeta_n\big|^2_{H^{1/2}(\Gamma_{\rm C})}\Big)=\limsup_{n\to\infty}\sum_{j=1}^{9}I_j^n\\
&\le\limsup_{n\to\infty}\Big(\big|\hat\zeta+\delta_n\big|^2_{H^{1/2}(\Gamma_{\rm C})}-\big|\zeta_n\big|^2_{H^{1/2}(A_n\cup C_n)}\Big)\\
&\le\big|\hat\zeta\big|^2_{H^{1/2}(\Gamma_{\rm C})}-\liminf_{n\to\infty}\big|\zeta_n\big|^2_{H^{1/2}(A_n\cup C_n)}\,.
\end{aligned}$$

In view of this, it remains to show that

$$-\liminf_{n\to\infty}\big|\zeta_n\big|^2_{H^{1/2}(A_n\cup C_n)}\le-\big|\zeta\big|^2_{H^{1/2}(\Gamma_{\rm C})}\,. \tag{3.20}$$

For this purpose, we choose a (not relabeled) subsequence $(\zeta_n)_n$ such that $\liminf_{n\to\infty}\big|\zeta_n\big|^2_{H^{1/2}(A_n\cup C_n)}=\lim_{n\to\infty}\big|\zeta_n\big|^2_{H^{1/2}(A_n\cup C_n)}$. Moreover, in view of (3.13), we may choose a further (not relabeled) subsequence $(\zeta_n)_n$ such that $\sum_{n=1}^\infty\mathcal{L}^{d-1}(B_n)<\infty$. For the sets $\cup_{n=N}^\infty B_n$ we thus find that

$$\lim_{N\to\infty}\mathcal{L}^{d-1}\big(\cup_{n=N}^\infty B_n\big)\le\lim_{N\to\infty}\sum_{n=N}^\infty\mathcal{L}^{d-1}(B_n)=0\,.$$

Hence, the complements $U_N:=\Gamma_{\rm C}\backslash\cup_{n=N}^\infty B_n$ satisfy

$$\forall\,n\ge N\in\mathbb{N}:\ U_N:=\Gamma_{\rm C}\backslash\cup_{n=N}^\infty B_n\subset(A_n\cup C_n)\quad\text{and}\quad\lim_{N\to\infty}\mathcal{L}^{d-1}((U_N\backslash\Gamma_{\rm C})\cup(\Gamma_{\rm C}\backslash U_N))=0\,. \tag{3.21}$$

We may now use the sets U_N to further estimate the left-hand side of (3.20); i.e., for $N\in\mathbb{N}$ fixed, for every $n\ge N$ we have

$$-\liminf_{n\to\infty}\big|\zeta_n\big|^2_{H^{1/2}(A_n\cup C_n)}\le-\liminf_{n\to\infty}\big|\zeta_n\big|^2_{H^{1/2}(U_N)}\le-\big|\zeta\big|^2_{H^{1/2}(U_N)}\,,$$

due to the weak lower semicontinuity of $|\cdot|^2_{H^{1/2}(U_N)}$ and the fact that $\zeta_n\rightharpoonup\zeta$ in $\mathbf{X}$. For $N\to\infty$ we then conclude (3.20). Thus, (3.9b) is proven.

Ad (3.9c)**:** This convergence result now follows from the continuous embedding $L^2(\Gamma_{\rm C})\subset L^1(\Gamma_{\rm C})$.

Ad (3.9d)**:** In order to verify the convergence result for the cohesive surface energy $\Phi^{\rm coh}$ we choose further (not relabeled) subsequences $(u_n)_n,(\hat\zeta_n)_n,(\zeta_n)_n$ such that $\limsup_{n\to\infty}\big(\Phi^{\rm coh}([\![u_n]\!]_{\mathbf{n}},\hat\zeta_n)-\Phi^{\rm coh}([\![u_n]\!]_{\mathbf{n}},\zeta_n)\big)=\lim_{n\to\infty}\big(\Phi^{\rm coh}([\![u_n]\!]_{\mathbf{n}},\hat\zeta_n)-\Phi^{\rm coh}([\![u_n]\!]_{\mathbf{n}},\zeta_n)\big)$ and $\hat\zeta_n\to\hat\zeta$ as well as $\zeta_n\to\zeta$ pointwise a.e. in $\Gamma_{\rm C}$. With z as a placeholder for $\hat\zeta_n$ and ζ_n we verify for the cohesive surface density $\phi^{\rm coh}$ that

$$|\phi^{\rm coh}(\,[\![u_n]\!]_{\mathbf{n}},z)|\le b\big|\,[\![u_n]\!]_{\mathbf{n}}\big|^2\,, \tag{3.22}$$

where we made use of the assumptions (2.9). The term on the right-hand side is integrable with $\int_{\Gamma_{\rm C}}b\big|[\![u_n]\!]_{\mathbf{n}}\big|^2\,\mathrm{d}x\to\int_{\Gamma_{\rm C}}b\big|[\![u]\!]_{\mathbf{n}}\big|^2\,\mathrm{d}x$ thanks to the compactness of the trace operator from $H^1(\Omega\backslash\Gamma_{\rm C})$ to $L^2(\Gamma_{\rm C})$. Thus, we are entitled to conclude (3.9d) with the aid of the dominated convergence theorem.

Ad (3.9e)**:** This estimate can be verified by arguing pointwise on the density I_k of the Yosida-regularization, i.e., $I_k(r,z)=\frac{k}{2}|(r)^-|^2+\frac{k}{2}|(r-z)^+|^2$. Taking into account that $\hat\zeta_n\ge\zeta_n$ a.e. in $\Gamma_{\rm C}$ according to (3.8), we see that $([\![u_n]\!]_{\mathbf{n}}-\hat\zeta_n)^+\le([\![u_n]\!]_{\mathbf{n}}-\zeta_n)^+$ a.e. in $\Gamma_{\rm C}$. Hence, (3.9e) follows.

Ad (3.9f)**:** This convergence result follows also by the dominated convergence theorem, using that the density I_k of the Yosida-regularization satisfies the following estimate

$$I_k(\,[\![u_n]\!]_{\mathbf{n}},z)=\tfrac{k}{2}\big|\big(\,[\![u_n]\!]_{\mathbf{n}}\big)^-\big|^2+\tfrac{k}{2}\big|\big(\,[\![u_n-z]\!]_{\mathbf{n}}\big)^+\big|^2\le\tfrac{k}{2}\big|\,[\![u_n]\!]_{\mathbf{n}}\big|^2+\tfrac{k}{2}\big|\,[\![u_n-z]\!]_{\mathbf{n}}\big|^2\,, \tag{3.23}$$

where the terms on the right-hand side are integrable and the integrals converge. Above, z is again a placeholder for $\hat\zeta_n$, resp. ζ_n. $\square$

With the next lemma we verify the remaining conditions (2.1)& (3.1)–(3.3), (3.5) & (3.6) ensuring the existence of semistable energetic solutions for the Yosida-regularized systems.

Lemma 3.4. *Keep $k \in \mathbb{N}$ fixed and let the assumptions of Theorem 2.4 be satisfied. Then the gradient systems $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_k)$ comply with the conditions (2.1)& (3.1)–(3.3) & (3.5) and the damped inertial systems $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E}_k)$ comply with the conditions (2.1)& (3.1)–(3.3), (3.5) & (3.6).*

Proof. **Ad** (2.1)**:** In view of (1.5), (1.6), and (2.8a) conditions (2.1b) and (2.1c) on $\mathcal{R}_k$ and $\mathcal{V}$ are verified.

Ad (3.1)**:** To check (3.1a), we calculate in view of (2.17e) & (2.19), using Korn's and Young's inequality:

$$\begin{aligned} \mathcal{E}_k(t,u,z) \geq\ & \tfrac{C_{\mathbb{C}}^1}{2}\|e(u)\|_{L^2}^2 - \|\mathbf{f}(t)\|_{\mathbf{V}^*}\|u\|_{\mathbf{V}} + |\zeta|_{\mathbf{X}}^r \\ \geq\ & \tfrac{C_{\mathbb{C}}^1 C_K^2}{2}\|u\|_{\mathbf{V}}^2 - \tfrac{C_{\mathbb{C}}^1 C_K^2}{4}\|u\|_{\mathbf{V}}^2 - \tfrac{1}{C_{\mathbb{C}}^1 C_K^2}\|\mathbf{f}(t)\|_{\mathbf{V}^*}^2 + \|\zeta\|_{\mathbf{X}}^r - \zeta^*\mathcal{L}^{d-1}(\Gamma_{\mathrm{C}}) \\ \geq\ & -c_* C_{\mathbf{f}}^2 - \zeta^*\mathcal{L}^{d-1}(\Gamma_{\mathrm{C}})\,, \end{aligned} \tag{3.24}$$

where we used that $\|\mathbf{f}(t)\|_{\mathbf{V}^*} \leq C_{\mathbf{f}}$, the bound on ζ imposed by the indicator of the interval $[0,\zeta^*]$, cf. (2.17e), and we set $\frac{1}{C_{\mathbb{C}}^1 C_K^2} = c_*$. This proves (3.1a). For $\mathcal{E}_k(t,u,z) \leq E$ we then find that

$$\|u\|_{\mathbf{V}}^2 \leq \tfrac{4}{C_{\mathbb{C}}^1 C_K^2}(E + c_* C_{\mathbf{f}}^2 + \zeta^*\mathcal{L}^{d-1}(\Gamma_{\mathrm{C}})) \ \text{ and } \ \|z\|_{\mathbf{X}}^2 \leq (E + c_* C_{\mathbf{f}}^2 + \zeta^*\mathcal{L}^{d-1}(\Gamma_{\mathrm{C}}))\,. \tag{3.25}$$

The weak lower semicontinuity of $\mathcal{E}(t,\cdot,\cdot)$ follows by the fact that $\mathcal{E}_k(t,\cdot,\cdot)$ is separately convex and strongly lower semicontinuous on $\mathbf{V} \times \mathbf{Z}$.

Ad (3.2)**:** Observe that $\partial_t\mathcal{E}(t,u,z) = -\langle \dot{\mathbf{f}}(t), u\rangle_{\mathbf{V}}$. In view of the regularity assumption (2.8b) we have $\dot{\mathbf{f}}(t_n) \to \dot{\mathbf{f}}(t)$ in $\mathbf{V}^*$ for $t_n \to t$ in $[0,T]$, which immediately gives the upper semicontinuity property of the powers. In view of (3.25) and Young's inequality we find the following power-control estimate:

$$\begin{aligned} |\partial_t\mathcal{E}_k(t,u,z)| \ \leq\ & C_{\mathbf{f}}\|u\|_{\mathbf{V}} \leq \tfrac12 C_{\mathbf{f}}^2 + \tfrac12\|u\|_{\mathbf{V}}^2 \\ \leq\ & \tfrac12 C_{\mathbf{f}}^2 + \tfrac{2}{C_{\mathbb{C}}^1 C_K^2}(\mathcal{E}_k(t,u,z) + c_* C_{\mathbf{f}}^2 + \zeta^*\mathcal{L}^{d-1}(\Gamma_{\mathrm{C}}))\,. \end{aligned}$$

Ad (3.3)**:** The coercivity assumption on the sum of $\mathcal{E}_k, \mathcal{V}$, and $\mathcal{R}$ directly follows from the the coercivity of $\mathcal{V}$ and $\mathcal{R}$ combined with the just deduced coercivity estimate (3.24) for $\mathcal{E}_k(t,\cdot,\cdot)$.

Ad (3.4)**:** cf. the preceeding Lemma 3.3.

Ad (3.5)**:** For every $(t,u,\zeta) \in \mathrm{dom}(\mathcal{E}_k)$, the regularized functional $\mathcal{E}_k(t,\cdot,\zeta)$ is Gâteaux-differentiable with derivative $\mathrm{D}_u\mathcal{E}_k(t,\cdot,\zeta)$ given by (2.25). Now, due to the quadratic nature of $\mathcal{E}_k(t,\cdot,\zeta)$, resp. the linear nature of $\mathrm{D}_u\mathcal{E}_k(t,\cdot,\zeta)$ the closedness condition (3.5) ensues.

Ad (3.6)**:** In order to verify the subgradient estimate (3.6) for the energy functionals $\mathcal{E}_k$, cf. (2.35), i.e.,

$$\forall\,(t,u,z) \in \mathrm{dom}(\mathcal{E}) : \quad \|\partial_u\mathcal{E}_k(t,u,z)\|_{\mathbf{V}^*}^\sigma \leq C_3\mathcal{E}_k(t,u,z) + C_4\|u\|_{\mathbf{V}} + C_5\,, \tag{3.26}$$

we will check the respective estimate for each of the contributions to $\mathcal{E}_k$ separately. For the bulk energy (2.17b) we verify with standard arguments relying on assumptions (2.8):

$$\begin{aligned} |\langle \mathrm{D}_u\mathcal{E}^{\mathrm{bulk}}(t,u), v\rangle_{\mathbf{V}}| &\leq (C_{\mathbb{C}}^2\|e(u)\|_{L^2} + C_{\mathbf{f}})\|v\|_{\mathbf{W}} \\ &\leq \Big(\big(\tfrac{2}{C_{\mathbb{C}}^1}\int_{\Omega\setminus\Gamma_{\mathrm{C}}} \tfrac12\mathbb{C}e(u){:}e(u)\,\mathrm{d}x + 1\big)^{1/2} + \tfrac{C_{\mathbf{f}}}{C_{\mathbb{C}}^2}\Big)C_{\mathbb{C}}^2\|v\|_{\mathbf{V}} \\ &\leq \Big(\tfrac{2}{C_{\mathbb{C}}^1}\mathcal{E}^{\mathrm{bulk}}(t,u) + \tfrac{2}{C_{\mathbb{C}}^1}C_{\mathbf{f}}\|u\|_{\mathbf{V}} + 1 + \tfrac{C_{\mathbf{f}}}{C_{\mathbb{C}}^2}\Big)C_{\mathbb{C}}^2\|v\|_{\mathbf{V}}\,. \end{aligned} \tag{3.27}$$

Thanks to assumption (2.9) we find for the cohesive energy (2.17d)

$$|\langle \mathrm{D}_u\Phi^{\mathrm{coh}}(\llbracket u\rrbracket_{\mathbf{n}},\zeta), \llbracket v\rrbracket_{\mathbf{n}}\rangle_{L^2(\Gamma_{\mathrm{C}})}| = \Big|\int_{\Gamma_{\mathrm{C}}} \frac{\phi_{\mathrm{coh}}(\zeta)}{2\zeta^2}\llbracket u\rrbracket_{\mathbf{n}}\cdot\llbracket v\rrbracket_{\mathbf{n}}\,\mathrm{d}\mathcal{H}^{d-1}\Big| \leq b\Big(\int_{\Gamma_{\mathrm{C}}} \frac{\phi_{\mathrm{coh}}(\zeta)}{2\zeta^2}\big|\llbracket u\rrbracket_{\mathbf{n}}\big|^2\,\mathrm{d}\mathcal{H}^{d-1} + 1\Big)\|v\|_{\mathbf{V}}\,. \tag{3.28}$$

For the Yosida-regularization term we find

$$
\begin{aligned}
|\langle \mathrm{D}_u \mathcal{J}_k(\llbracket u \rrbracket_{\mathbf{n}}, \zeta), \llbracket v \rrbracket_{\mathbf{n}} \rangle_{L^2(\Gamma_{\mathrm{C}})}| &= \Big| \int_{\Gamma_{\mathrm{C}}} k\big(\big(-\llbracket u \rrbracket_{\mathbf{n}}\big)^- + \big(\llbracket u \rrbracket_{\mathbf{n}} - \zeta\big)^+\big) \cdot \llbracket v \rrbracket_{\mathbf{n}} \,\mathrm{d}\mathcal{H}^{d-1} \Big| \\
&\leq \sqrt{k} \int_{\Gamma_{\mathrm{C}}} \big(k \big| \big(-\llbracket u \rrbracket_{\mathbf{n}}\big)^- \big|^2 + k \big| \big(\llbracket u \rrbracket_{\mathbf{n}} - \zeta\big)^+ \big|^2 + 1 \big) \,\mathrm{d}\mathcal{H}^{d-1} \|v\|_{\mathbf{V}} \\
&\leq \sqrt{k} \big(\mathcal{J}_k(\llbracket u \rrbracket, \zeta) + 1 \big) \|v\|_{\mathbf{V}} \,.
\end{aligned}
\tag{3.29}
$$

□

To conclude the proof of Theorem 2.4 it remains to verify the validity of the energy dissipation *balance* (2.28). For this, we will make use of a general result, cf. Thm. 3.5 below, drawn from [RT15a]. In fact, the proof of Thm. 3.5 below, cf. [RT15a, Thm. 3.6], provides the following integral chain-rule inequality

$$
\begin{aligned}
&\int_0^t \langle \xi(s), \dot u(s) \rangle_{\mathbf{V}} \,\mathrm{d}s \\
&\leq \mathcal{E}(t, u(t), \zeta(t)) - \mathcal{E}(0, u(0), \zeta(0)) - \int_0^t \partial_t \mathcal{E}(s, u(s), \zeta(s)) \,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}}(\zeta, [0,t]) \text{ for all } t \in [0,T],
\end{aligned}
\tag{3.30}
$$

in the case the map $u \mapsto \mathcal{E}(t,u,\zeta)$ is Gâteaux-differentiable, from the semistability condition (2.4). This is achieved by mimicking the Riemann-sum procedure from the proof of [Rou09, Prop. 5.4], see also [Rou10, Prop. 4.3], which in turn is based on the argument first developed in [DMFT05].

Theorem 3.5 ([RT15a, Thm. 3.6])**.** *Assume that the functionals $\mathcal{E}, \mathcal{V}$, and $\mathcal{R}$ satisfy conditions* (2.1)*,* (3.1) *&* (3.2)*. Moreover, assume that for every $(t,\zeta) \in [0,T] \times \mathrm{dom}_\zeta$ the map $u \mapsto \mathcal{E}(t,u,\zeta)$ is Gâteaux-differentiable, and that*

$$
\begin{aligned}
&\forall M > 0 \ \exists S > 0 \ \ \forall t \in [0,T], \ \forall u, \, u_1, \, u_2 \in \mathrm{dom}_u, \ \bar\zeta \in \mathrm{dom}_\zeta \ : \\
&\quad \begin{cases} (u, \bar\zeta) \in \mathcal{S}_M \ \Rightarrow \ \|\mathrm{D}_u \mathcal{E}(t,u,\bar\zeta)\|_{\mathbf{V}^*} \leq S, \\ (u_1, \bar\zeta), \, (u_2, \bar\zeta) \in \mathcal{S}_M \ \Rightarrow \ \|\mathrm{D}_u \mathcal{E}(t,u_1,\bar\zeta) - \mathrm{D}_u \mathcal{E}(t,u_2,\bar\zeta)\|_{\mathbf{V}^*} \leq S \|u_1 - u_2\|_{\mathbf{V}}, \end{cases}
\end{aligned}
\tag{3.31a}
$$

with $\mathcal{S}_M := \{(u,\zeta) \in \mathrm{dom}_u \times \mathrm{dom}_\zeta, \, \sup_{t \in [0,T]} \mathcal{E}(t,u,\zeta) \leq M\}$ the energy sublevel with $M \in \mathbb{R}$, and that $\partial_t \mathcal{E}$ satisfies analogous Lipschitz estimates, i.e.

$$
\begin{aligned}
&\forall \widetilde M > 0 \ \exists \widetilde S > 0 \ \ \forall t_1, t_2 \in [0,T], \ \forall u_1, \, u_2 \in \mathrm{dom}_u, \ \bar\zeta \in \mathrm{dom}_\zeta : \\
&\quad (u_1, \bar\zeta), \, (u_2, \bar\zeta) \in \mathcal{S}_{\widetilde M} \ \Rightarrow \ \begin{cases} \big|\partial_t \mathcal{E}(t_1, u_1, \bar\zeta) - \partial_t \mathcal{E}(t_2, u_1, \bar\zeta)\big| \leq \widetilde S |t_1 - t_2|, \\ \big|\partial_t \mathcal{E}(t_1, u_1, \bar\zeta) - \partial_t \mathcal{E}(t_1, u_2, \bar\zeta)\big| \leq \widetilde S \|u_1 - u_2\|_{\mathbf{V}}. \end{cases}
\end{aligned}
\tag{3.31b}
$$

Let (u,ζ) be a semistable energetic solution to the gradient system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E})$, resp. to the damped inertial system $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E})$. Then, (u,ζ) complies with (3.30) *and the energy-dissipation inequality* (2.5) *holds as an identity.*

Observe that the conditions (2.1), (3.1) & (3.2) are already verified as a part of Lemma 3.4. Thus, in view of the above statement we now check that the Yosida-regularized energy functional $\mathcal{E}_k$ from (2.19) satisfies the conditions (3.31).

Lemma 3.6. *Let the assumptions of Theorem 2.4 hold true and keep $k \in \mathbb{N}$ fixed. Then, the energy functional $\mathcal{E}_k$ from* (2.19) *complies with condition* (3.31)*. Hence, for every $k \in \mathbb{N}$, the energy dissipation balance* (2.28) *holds true for the Yosida-regularized system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_k)$, resp. $(\mathbf{V}, \mathbf{W}, \mathbf{Z}, \mathcal{V}, \mathcal{K}, \mathcal{R}, \mathcal{E}_k)$.*

Proof. **Ad** (3.31a)**:** We observe that the first statement of (3.31a) is a direct consequence of the subgradient estimate (3.6), which has been already verified for the functional $\mathcal{E}_k$ along with Lemma 3.4. Moreover, for the second statement of (3.31a), we recall the form of $\mathrm{D}_u\mathcal{E}_k(t,\cdot,z)$ from (2.25), and observe that the bulk contribution $\mathrm{D}_u\mathcal{E}^{\mathrm{bulk}}(t,\cdot)$ and the contribution arising from $\Phi^{\mathrm{coh}}(\cdot,z)$, thanks to their linear character, directly comply with the second of (3.31a). We now check that also the term arising from $\mathrm{D}_u\mathcal{J}_k(\cdot,z)$ complies with the second of (3.31a). For this, we observe that the function $(\,\cdot\,)^-$, resp. $(\,\cdot\,)^+$, is Lipschitz-continuous, such that $|(\llbracket u_1\rrbracket_{\mathbf{n}})^- - (\llbracket u_2\rrbracket_{\mathbf{n}})^-| \le |\llbracket u_1\rrbracket_{\mathbf{n}} - \llbracket u_2\rrbracket_{\mathbf{n}}|$, resp. $|(\llbracket u_1\rrbracket_{\mathbf{n}} - \bar{z})^- + (\llbracket u_2\rrbracket_{\mathbf{n}} - \bar{z})^+| \le |\llbracket u_1\rrbracket_{\mathbf{n}} - \llbracket u_2\rrbracket_{\mathbf{n}}|$. Because of this, the second of (3.31a) also holds true for the term arising from the Yosida-regularization. Thus, (3.31a) is verified for $\mathcal{E}_k$.

Ad (3.31b)**:** Recall that $\partial_t\mathcal{E}(t,u,z) = \langle\dot{\mathbf{f}}(t),u\rangle_{\mathbf{V}}$, with $\mathbf{f}$ of regularity (2.8b). Hence, (3.31b) holds true. $\square$

4 Limit passage for gradient systems – Proof of Theorem 2.5

In this section we carry out the evolutionary Γ-limit passage from the Yosida-regularized gradient systems $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E}_k)$ to the cohesive zone gradient system $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E}_\infty)$. After deducing the compactness properties (2.32) here below in Prop. 4.1, we establish the limit passage in the semistable inequality in Sec. 4.1 and the passage in the momentum & energy inequality in Sec. 4.2, respectively.

Proposition 4.1 (Compactness (2.32))**.** *Let the assumptions of Thm. 2.5 be satisfied. Then there exists a (not relabeled) subsequence $(u_k,\zeta_k)_k$ of semistable energetic solutions to the regularized systems $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E}_k)$ and a limit pair (u,ζ) of regularity* (2.31) *such that convergences* (2.32) *hold true.*

Proof. First of all, from (2.28), and recalling (2.6), we get the following uniform bounds along the semistable energetic solutions $(u_k,\zeta_k)_k$ of the systems $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E}_k)$:

$$\sup_{t\in[0,T]}|\mathcal{E}_k(t,u_k(t),\zeta_k(t))| + \int_0^T\mathcal{V}(\dot{u}_k(s))\,\mathrm{d}s + \mathcal{R}(\zeta_k(T)-\zeta_0) \le C\,. \tag{4.1}$$

In view of the assumptions (2.8), cf. also (3.25), this amounts to:

$$\|u_k\|_{H^1(0,T;\mathbf{V})} \le C \tag{4.2a}$$

$$\|u_k(t)\|_{\mathbf{V}} \le C \quad \text{for all } t\in[0,T]\,, \tag{4.2b}$$

$$\|\zeta_k(t)\|_{\mathbf{X}} \le C \quad \text{for all } t\in[0,T]\,, \tag{4.2c}$$

$$\|\zeta_k(t)\|_{C(\overline{\Gamma_{\mathrm{C}}})} \le \zeta^* \quad \text{for all } t\in[0,T]\,. \tag{4.2d}$$

Moreover, a comparison in the time-integrated momentum balance

$$\int_0^T\langle\mathrm{D}_u\mathcal{E}^{\mathrm{bulk}}(t,u_k(t)) + \mathrm{D}_u\Phi^{\mathrm{coh}}(\llbracket u_k(t)\rrbracket_{\mathbf{n}},\zeta_k(t)) + \mathrm{D}_u\mathcal{J}_k(\llbracket u_k(t)\rrbracket_{\mathbf{n}},\zeta_k(t)) + \mathrm{D}_{\dot{u}}\mathcal{V}(\dot{u}_k(t)),v(t)\rangle_{\mathbf{V}}\,\mathrm{d}t = 0$$

for every $v\in L^2(0,T;\mathbf{V})$, provides

$$\sup_{v\in L^2(0,T;\mathbf{V}),\ \|v\|_{L^2(0,T;\mathbf{V})}=1}\int_0^T\langle\mathrm{D}_u\mathcal{J}_k(\llbracket u_k(t)\rrbracket_{\mathbf{n}},\zeta_k(t)),v\rangle_{\mathbf{V}}\,\mathrm{d}t \le C\,. \tag{4.2e}$$

The existence of a subsequence complying with (2.32a) is a direct consequence of the uniform bound (4.2a). Moreover, from the uniform bound on $\mathcal{R}(\zeta_k(t)-\zeta_0) \le \mathcal{R}(\zeta_k(T)-\zeta_0) \le C$ together with (4.2c) &

(4.2d), using a generalized version of Helly's selection principle, cf. [MR15, Thm. 2.1.24], the existence of a subsequence satisfying (2.32b) ensues. Consequently, from the embedding $\mathbf{X} = W^{1,r}(\Gamma_{\mathrm{C}}) \Subset C(\overline{\Gamma_{\mathrm{C}}})$ for $r > d-1$ convergences (2.32c) and (2.32d) follow. Moreover, (2.32e) is concluded from the bound (4.2e). $\square$

4.1 Limit passage in the semistability inequality

In order to show that the limit pair (u, ζ) extracted by the convergences (2.32) complies with the semistability inequality for the cohesive zone system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_\infty)$, cf. Def. 2.1, we will make use of a mutual recovery sequence condition akin to (3.4). More precisely, given a sequence $(u_k, \zeta_k)_k \in \mathbf{V} \times \mathbf{Z}$, with $\sup_k \mathcal{E}_k(t, u_k, \zeta_k) \leq C$ for all $t \in [0, T]$ and, for every $k \in \mathbb{N}$ (u_k, ζ_k) being semistable for the functionals $\mathcal{E}_k, \mathcal{R}$, and such that $(u_k, \zeta_k) \rightharpoonup (u, \zeta)$ in $\mathbf{V} \times \mathbf{Z}$, and for every $\hat{\zeta} \in \mathbf{Z}$ there exists a sequence $(\hat{\zeta}_k)_k \subset \mathbf{Z}$, $\hat{\zeta}_k \rightharpoonup \hat{\zeta}$ in $\mathbf{Z}$ such that

$$\limsup_{k\to\infty} \Big(\mathcal{E}_k(t, u_k, \hat{\zeta}_k) - \mathcal{E}_k(t, u_k, \zeta_k) + \mathcal{R}(\hat{\zeta}_k - \zeta_k) \Big) \leq \mathcal{E}_\infty(t, u, \hat{\zeta}) - \mathcal{E}_\infty(t, u, \zeta) + \mathcal{R}(\hat{\zeta} - \zeta)\,. \tag{4.3}$$

Proposition 4.2 (Semistability of the system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_\infty)$)**.** *Let the assumptions of Theorem 2.5 hold true. Then, for each $t \in [0, T]$ fixed, the sequence $(u_k, \zeta_k)_k$ of semistable energetic solutions to the systems $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_k)_k$ and the limit pair (u, ζ) extracted by convergences* (2.32) *satisfy the mutual recovery condition* (4.3)*. Hence, (u, ζ) is semistable for the cohesive zone delamination system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_\infty)$.*

Moreover, $\zeta_k(t) \to \zeta(t)$ even strongly in $\mathbf{X}$ for all $t \in [0, T]$.

Proof. Taking into account the properties of $\mathcal{E}_\infty$ and of $\mathcal{R}$ from (1.5) and (2.17), we see that the right-hand side of (4.3) is finite if and only if $\hat{\zeta} \in \mathbf{X}$ such that

$$\zeta^* \geq \hat{\zeta} \geq \zeta \geq [\![u]\!]_{\mathbf{n}} \qquad \text{a.e. in } \Gamma_{\mathrm{C}}\,. \tag{4.4}$$

If this is not the case, then (4.3) is trivially satisfied. Hence, assume from now on that the competitor $\hat{\zeta}$ complies with (4.4).

Since $\mathbf{X} = W^{1,r}(\Gamma_{\mathrm{C}})$ with $r > d-1$, due to the compact embedding $\mathbf{X} = W^{1,r}(\Gamma_{\mathrm{C}}) \Subset C(\overline{\Gamma_{\mathrm{C}}})$, it can be verified that $\hat{\zeta}_k := \min\{\zeta^*, \hat{\zeta} + \|\zeta_k - \zeta\|_{C(\overline{\Gamma_{\mathrm{C}}})}\}$ is a suitable recovery sequence for $\hat{\zeta} \geq \zeta$ that satisfies $\zeta_k \leq \hat{\zeta}_k \leq \zeta^*$ and additionally converges strongly in $\mathbf{X}$. To see the strong convergence $\hat{\zeta}_k \to \hat{\zeta}$ in $\mathbf{X}$ it has to be used that the superposition operator $\min\{\zeta^*, \cdot\} : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous. Then, [MM79, Sec. 2] provides that the superposition of a $W^{1,r}(\Gamma_{\mathrm{C}})$-function with $\min\{\zeta^*, \cdot\}$ yields a $W^{1,r}$-function, and that the operator $\min\{\zeta^*, \cdot\} : W^{1,r}(\Gamma_{\mathrm{C}}) \to W^{1,r}(\Gamma_{\mathrm{C}})$ is continuous. Further note that, by construction, property (3.9e) holds true here as well. Thus, by the strong convergence of $(\hat{\zeta}_k)_k$, and the lower semicontinuity of the $W^{1,r}$-seminorm it can be verified that

$$\begin{aligned}
&\limsup_{k\to\infty} \big(\mathcal{E}_k(t, u_k, \hat{\zeta}_k) - \mathcal{E}_k(t, u_k, \zeta_k) + \mathcal{R}(\hat{\zeta}_k - \zeta_k) \big) \\
&\leq \limsup_{k\to\infty} \big(\Phi^{\mathrm{coh}}([\![u_k]\!]_{\mathbf{n}}, \hat{\zeta}_k) + |\hat{\zeta}_k|^r_{\mathbf{X}} - \Phi^{\mathrm{coh}}([\![u_k]\!]_{\mathbf{n}}, \zeta_k) - |\zeta_k|^r_{\mathbf{X}} + \mathcal{R}(\hat{\zeta}_k - \zeta_k) \big) \\
&\leq \mathcal{E}_\infty(t, u, \hat{\zeta}) - \mathcal{E}_\infty(t, u, \zeta) + \mathcal{R}(\hat{\zeta} - \zeta)\,.
\end{aligned}$$

This proves the mutual recovery condition (4.3).

To verify the strong convergence of the semistable sequence $(\zeta_k(t))_k$, we observe that, by the monotonicity assumption (2.9) and by (3.9e), the semistability inequality can be further reduced to the following expression

$$|\zeta_k(t)|^r_{\mathbf{X}} \leq |\hat{\zeta}_k|^r_{\mathbf{X}} + \mathcal{R}(\hat{\zeta}_k - \zeta_k(t))\,. \tag{4.5}$$

Applying the construction to $\hat{\zeta} = \zeta(t)$, i.e., $\hat{\zeta}_k := \min\{\zeta^*, \zeta(t) + \|\zeta_k(t) - \zeta(t)\|_{C(\overline{\Gamma_C})}\}$, and inserting this in (4.5), allows us to verify that $\limsup_{k\to\infty} |\zeta_k(t)|^r_{\mathbf{X}} \le |\zeta(t)|^r_{\mathbf{X}}$. Together with the weak convergence $\zeta_k(t) \rightharpoonup \zeta(t)$ in $\mathbf{X}$ from (2.32b) and the weak lower semicontinuity of the norm this yields $\zeta_k \to \zeta$ strongly in $\mathbf{X}$. □

4.2 Limit passage in the momentum balance and in the energy-dissipation inequality

The momentum balance of the regularized systems $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_k)$, integrated over $(0,T)$, is given by:

$$\int_0^T \langle \mathrm{D}_u \mathcal{E}^{\mathrm{bulk}}(t, u_k(t)) + \mathrm{D}_u \Phi^{\mathrm{coh}}(\llbracket u_k(t) \rrbracket_{\mathbf{n}}, \zeta_k(t)) + \mathrm{D}_u \mathcal{J}_k(\llbracket u_k(t) \rrbracket_{\mathbf{n}}, \zeta_k(t)) + \mathrm{D}_{\dot u} \mathcal{V}(\dot u_k(t)), v(t) \rangle_{\mathbf{V}} \, \mathrm{d}t = 0 \tag{4.6}$$

for every $v \in L^2(0,T;\mathbf{V})$. Due to the well-preparedness of the initial data it is $(u_k(0), \zeta_k(0)) = (u_0, \zeta_0)$ for every $k \in \mathbb{N}$ and thanks to convergences (2.32), together with (4.2d)-(4.2e), we have

$$\zeta_k \overset{*}{\rightharpoonup} \zeta \quad \text{in } L^\infty(0,T;\mathbf{X})\,, \tag{4.7a}$$

$$\zeta_k(t) \rightharpoonup \zeta(t) \quad \text{in } \mathbf{X} \text{ for all } t \in [0,T]\,, \tag{4.7b}$$

$$e(u_k(t)) \rightharpoonup e(u(t)) \quad \text{in } L^2(\Omega;\mathbb{R}^{d\times d}) \text{ for a.a. } t \in (0,T)\,, \tag{4.7c}$$

$$e(\dot u_k) \rightharpoonup e(\dot u) \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))\,, \tag{4.7d}$$

$$\mathrm{D}_u \mathcal{J}_k(\llbracket u_k \rrbracket_{\mathbf{n}}, \zeta_k) \rightharpoonup \tilde{\xi} \quad \text{in } L^2(0,T;\mathbf{V}^*)\,. \tag{4.7e}$$

This will allow us in Prop. 4.4 below to pass to the limit $k \to \infty$ in (4.6) and thus to find for the cohesive zone system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_\infty)$ the following time-integrated momentum balance:

$$\int_0^T \langle \mathrm{D}_u \mathcal{E}^{\mathrm{bulk}}(t, u(t)) + \mathrm{D}_u \Phi^{\mathrm{coh}}(\llbracket u(t) \rrbracket_{\mathbf{n}}, \zeta(t)) + \tilde{\xi}(t) + \mathrm{D}_{\dot u} \mathcal{V}(\dot u(t)), v(t) \rangle_{\mathbf{V}} \, \mathrm{d}t = 0\,. \tag{4.8}$$

Its corresponding pointwise-in-time version can then be concluded by arguing via the fundamental lemma of calculus of variations using specific test functions $\phi = v\eta \in L^2(0,T;\mathbf{V})$ with $v \in \mathbf{V}$ and $\eta \in L^2(0,T)$. In this way, we find for a.e. $t \in (0,T)$, for all $v \in \mathbf{V}$:

$$\langle \mathrm{D}_u \mathcal{E}^{\mathrm{bulk}}(t, u(t)) + \mathrm{D}_u \Phi^{\mathrm{coh}}(\llbracket u(t) \rrbracket_{\mathbf{n}}, \zeta(t)) + \tilde{\xi}(t) + \mathrm{D}_{\dot u} \mathcal{V}(\dot u(t)), v \rangle_{\mathbf{V}} = 0\,. \tag{4.9}$$

The information that $\tilde{\xi}(t) \in \partial_u \mathcal{J}(\llbracket u(t) \rrbracket_{\mathbf{n}}, \zeta(t))$ for a.a. $t \in (0,T)$ can be established by passing to the limit in the definition of the subdifferential of the functional $\widetilde{\mathcal{J}}_k^T(u_k, \zeta_k)$ defined below. More precisely, we argue on the time-integrated versions of the functionals $\mathcal{J}_k$ & $\mathcal{J}$ from (2.18a) & (2.17f) which are defined by:

$$\widetilde{\mathcal{J}}_k^T(\llbracket \cdot \rrbracket_{\mathbf{n}}, \zeta_k) : L^2(0,T;\mathbf{V}) \to [0,\infty)\,, \qquad \widetilde{\mathcal{J}}_k^T(\llbracket w \rrbracket_{\mathbf{n}}, \zeta_k) := \int_0^T \mathcal{J}_k(\llbracket w(s) \rrbracket_{\mathbf{n}}, \zeta_k(s))\, \mathrm{d}s\,, \tag{4.10a}$$

$$\widetilde{\mathcal{J}}^T(\llbracket \cdot \rrbracket_{\mathbf{n}}, \zeta_k) : L^2(0,T;\mathbf{V}) \to \{0,\infty\}\,, \qquad \widetilde{\mathcal{J}}^T(\llbracket w \rrbracket_{\mathbf{n}}, \zeta_k) := \int_0^T \mathcal{J}(\llbracket w(s) \rrbracket_{\mathbf{n}}, \zeta(s))\, \mathrm{d}s\,. \tag{4.10b}$$

In other words, we have to pass to the limit in the inequality:

$$\widetilde{\mathcal{J}}_k^T(\llbracket v_k \rrbracket_{\mathbf{n}}, \zeta_k) - \widetilde{\mathcal{J}}_k^T(\llbracket u_k \rrbracket_{\mathbf{n}}, \zeta_k) \ge \langle \mathrm{D}_u \widetilde{\mathcal{J}}_k^T(\llbracket u_k \rrbracket_{\mathbf{n}}, \zeta_k), v_k - u_k \rangle_{L^2(0,T;\mathbf{V})}\,. \tag{4.11}$$

Supposed that this limit passage is successful, we can then argue for the limit $\tilde{\xi}$ from (4.7e) that $\tilde{\xi} \in \partial_u \widetilde{\mathcal{J}}^T(\llbracket u \rrbracket_{\mathbf{n}}, \zeta)$ if and only if $\tilde{\xi}(t) \in \partial_u \mathcal{J}(\llbracket u(t) \rrbracket_{\mathbf{n}}, \zeta(t))$ for a.e. $t \in (0,T)$, by the fact that $(L^2(0,T;\mathbf{V}))^* \cong$

$L^2(0,T;\mathbf{V}^*)$. However, the above limit passage makes it necessary on the one hand, as already indicated, for any $v \in L^2(0,T;\mathbf{V})$ with $v(t) \leq \zeta(t)$ a.e. in Γ_{C} for a.e. $t \in (0,T)$, to construct a recovery sequence $(v_k)_k \subset L^2(0,T;\mathbf{V})$ that prevents the blow-up of $\widetilde{\mathcal{J}}_k^T([\![v_k]\!]_{\mathbf{n}},\zeta_k)$. To pass to the limit on the right-hand side of (4.11), in particular to find that $\liminf_{k\to\infty}\langle \mathrm{D}_u\widetilde{\mathcal{J}}_k^T([\![u_k]\!]_{\mathbf{n}},\zeta_k),v_k\rangle_{L^2(0,T;\mathbf{V})} \geq \langle\tilde{\xi},v\rangle_{L^2(0,T;\mathbf{V})}$, in view of the weak convergences (4.7e), requires that (a subsequence of) the constructed recovery sequence $(v_k)_k$ converges *strongly* in $L^2(0,T;\mathbf{V})$. The term $\langle \mathrm{D}_u\widetilde{\mathcal{J}}_k^T([\![u_k]\!]_{\mathbf{n}},\zeta_k),-u_k\rangle_{L^2(0,T;\mathbf{V})}$ can standardly be handled by verifying the following lim sup-estimate

$$\limsup_{k\to\infty}\langle \mathrm{D}_u\widetilde{\mathcal{J}}_k^T(\,[\![u_k]\!]_{\mathbf{n}},\zeta_k),u_k\rangle_{L^2(0,T;\mathbf{V})} \leq \langle\tilde{\xi},u\rangle_{L^2(0,T;\mathbf{V})}\,. \tag{4.12}$$

We also refer to [Att84, Sec. 3] for more details. But let us stress here that interpreting the Yosida-approximants $\mathcal{J}_k(\cdot,\zeta_k(t))$ as functionals restricted to the traces of functions from $\mathbf{V}$ instead of defining them on the full $L^2(\Gamma_{\mathrm{C}})$, now requires to prove the strong convergence of recovery sequences $v_k \to v$ in $\mathbf{V} = H^1(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$. In other words, it has to be shown that the functionals $\mathcal{J}_k([\![\cdot]\!]_{\mathbf{n}},\zeta_k(t))$ Mosco-converge in $\mathbf{V}$ to $\mathcal{J}([\![\cdot]\!]_{\mathbf{n}},\zeta(t))$. But since $\mathcal{J}_k([\![\cdot]\!]_{\mathbf{n}},\zeta_k(t))$ strongly depends on $\zeta_k \in \mathbf{X}$, in order to find the convergence $\mathcal{J}_k([\![v_k]\!]_{\mathbf{n}},\zeta_k(t)) \to 0$ the construction of v_k involves ζ_k and hence, the convergence properties of $(v_k)_k$ depend on those of semistable sequences $(\zeta_k)_k$. In view of Prop. 4.2, the desired Mosco-convergence result for $\mathcal{J}_k([\![\cdot]\!]_{\mathbf{n}},\zeta_k(t))$ and, in turn, for $\widetilde{\mathcal{J}}_k^T([\![\cdot]\!]_{\mathbf{n}},\zeta_k)$ can be established thanks to $\mathbf{X} = W^{1,r}(\Gamma_{\mathrm{C}})$ with $r \geq d-1$.

Proposition 4.3 (Mosco-convergence of $(\mathcal{J}_k([\![\cdot]\!]_{\mathbf{n}},\zeta_k(t)))_k$ and $(\widetilde{\mathcal{J}}_k^T([\![\cdot]\!]_{\mathbf{n}},\zeta_k))_k$)**.** *Let the assumptions of Thm. 2.5 be satisfied. Assume that $\zeta_k(s) \rightharpoonup \zeta(s)$ in $\mathbf{X} = W^{1,r}(\Gamma_{\mathrm{C}})$ for all $s \in [0,T]$ and $\zeta_k \overset{*}{\rightharpoonup} \zeta$ in $L^\infty(0,T;\mathbf{X})$ with $\zeta_k(s),\zeta(s) \geq \zeta_*$ a.e. in Γ_{C} for a.e. $s \in (0,T)$. Then, the functionals*

$$\mathcal{J}_k(\cdot,\zeta_k(t)) : \mathbf{V} \to [0,\infty) \ \ \textit{Mosco-converge to}\ \mathcal{J}(\cdot,\zeta(t)) : \mathbf{V} \to \{0,\infty\}\,, \tag{4.13a}$$

$$\widetilde{\mathcal{J}}_k^T(\,[\![\,\cdot\,]\!]_{\mathbf{n}},\zeta_k) : L^2(0,T;\mathbf{V}) \to [0,\infty) \ \ \textit{Mosco-converge to}\ \widetilde{\mathcal{J}}^T(\,[\![\,\cdot\,]\!]_{\mathbf{n}},\zeta) : L^2(0,T;\mathbf{V}) \to \{0,\infty\}\,. \tag{4.13b}$$

Proof. In a first step we verify the Mosco-convergence result (4.13a), which will then be carried over to the time-integrated functionals in a second step.

Ad (4.13a)**:** To verify the Mosco-convergence of $(\mathcal{J}_k([\![\cdot]\!]_{\mathbf{n}},\zeta_k(t)))_k$ we first show the Γ-lim inf-relation and secondly prove the Γ-lim sup-relation to hold with a recovery sequence that converges strongly in $\mathbf{V}$.

Ad Γ-lim inf for (4.13a)**:** Consider a sequence $v_k \rightharpoonup v$ in $\mathbf{V}$. Denoting their normal jumps as $[\![v_k]\!]_{\mathbf{n}} := \mathsf{v}_k$, $[\![v]\!]_{\mathbf{n}} := \mathsf{v}$ we thus have $\mathsf{v}_k \rightharpoonup \mathsf{v}$ in $H^{1/2}(\Gamma_{\mathrm{C}})$. By the compact embedding (2.12) it holds $\mathsf{v}_k \to \mathsf{v}$ in $L^2(\Gamma_{\mathrm{C}})$ and, for a (not relabeled) subsequence, $\mathsf{v}_k \to \mathsf{v}$ pointwise a.e. in Γ_{C} and almost uniformly.

If $\mathcal{J}(\mathsf{v},\zeta(t)) = 0$, due to $\mathcal{J}_k(\mathsf{v}_k,\zeta_k(t)) \geq 0$ for all $k \in \mathbb{N}$, it is $\liminf_{k\to\infty}\mathcal{J}_k(\mathsf{v}_k,\zeta_k(t)) \geq \mathcal{J}(\mathsf{v},\zeta(t))$.

If $\mathcal{J}(\mathsf{v},\zeta(t)) = \infty$, then either $\mathcal{H}^{d-1}([\mathsf{v} > \zeta(t)]) > 0$ or $\mathcal{H}^{d-1}([\mathsf{v} < 0]) > 0$, or both (the notation for the sets is the same introduced in the proof of Lemma 2.2). We consider a subsequence $(\mathsf{v}_{k_l},\zeta_{k_l}(t))_l \subset (\mathsf{v}_k,\zeta_k(t))_k$, such that $\mathsf{v}_{k_l} \to \mathsf{v}$ almost uniformly and which additionally realizes the lim inf, i.e. $\liminf_{k\to\infty}\mathcal{J}_k(\mathsf{v}_k,\zeta_k(t)) = \lim_{l\to\infty}\mathcal{J}_{k_l}(\mathsf{v}_{k_l},\zeta_{k_l}(t))$. Observe that the existence of an almost uniformly converging subsequence of $(\mathsf{v}_k)_k$ is implied by the compact embedding $H^{1/2}(\Gamma_{\mathrm{C}}) \Subset L^2(\Gamma_{\mathrm{C}})$.

First, assume that $\mathcal{H}^{d-1}([\mathsf{v} < 0]) > 0$. Then, we find $\varepsilon > 0$ such that also $\mathcal{H}^{d-1}([\mathsf{v}+2\varepsilon < 0]) > 0$. By the almost uniform convergence of the sequence, for $\varepsilon,\delta > 0$ we find an index $l(\varepsilon,\delta)$ as well as a set $B_\delta \subset B := [\mathsf{v}+2\varepsilon < 0] \subset \Gamma_{\mathrm{C}}$ of measure $\mathcal{H}^{d-1}(B_\delta) \leq \delta$, such that, for all $l \geq l(\varepsilon,\delta)$ we have $|\mathsf{v}_{k_l} - \mathsf{v}| < \varepsilon$ and hence $\mathsf{v}-\varepsilon < \mathsf{v}_{k_l} < \mathsf{v}+\varepsilon < -\varepsilon$ on $B_\delta^{\mathrm{c}} := B\backslash B_\delta$. Then, $\mathcal{J}_k(\mathsf{v}_{k_l},\zeta_{k_l}(t)) \geq \frac{k}{2}\int_{B_\delta^{\mathrm{c}}}|([\![v_{k_l}]\!]_{\mathbf{n}})^-|^2\,\mathrm{d}\mathcal{H}^{d-1} \geq \frac{k}{2}\int_{B_\delta^{\mathrm{c}}}|(-\varepsilon)^-|^2\,\mathrm{d}\mathcal{H}^{d-1} \to \infty$ as $k \to \infty$.

Now, assume that $\mathcal{H}^{d-1}([\mathsf{v} > \zeta(t)]) > 0$. Again we find $\varepsilon > 0$ such that $\mathcal{H}^{d-1}([\mathsf{v}-\zeta(t) > 2\varepsilon]) > 0$ and, thanks to almost uniform convergence, for $\varepsilon,\delta > 0$ there is an index $l(\varepsilon,\delta)$ as well as a set $B_\delta \subset [\mathsf{v}-\zeta > 2\varepsilon]$

of measure $\mathcal{H}^{d-1}(B_\delta) \leq \delta$, such that, for all $l \geq l(\varepsilon, \delta)$ we have on B_δ^{c} that $|\mathsf{v}_{k_l} - \zeta_{k_l}(t) - (\mathsf{v} - \zeta(t))| < \varepsilon$, hence $\varepsilon < \mathsf{v} - \zeta(t) - \varepsilon < \mathsf{v}_{k_l} - \zeta_{k_l}(t) < \mathsf{v} - \zeta(t) + \varepsilon$. This yields $\mathcal{J}_k(\mathsf{v}_{k_l}, \zeta_{k_l}(t)) \geq \frac{k}{2} \int_{B_\delta^{\mathrm{c}}} |(\mathsf{v}_{k_l} - \zeta_{k_l}(t))^+|^2 \,\mathrm{d}\mathcal{H}^{d-1} \geq \frac{k}{2} \int_{B_\delta^{\mathrm{c}}} |(\varepsilon)^+|^2 \,\mathrm{d}\mathcal{H}^{d-1} \to \infty$ as $k \to \infty$.

Ad $\exists$ recovery sequence for (4.13a)**:** First, consider $v \in \mathbf{V}$ with $\mathcal{J}([\![v]\!]_{\mathbf{n}}, \zeta(t)) = \infty$. Then, we may choose the constant sequence $v_k = v$ for all $k \in \mathbb{N}$ to find that $\limsup_{k\to\infty} \mathcal{J}_k([\![v_k]\!]_{\mathbf{n}}, \zeta_k(t)) \leq \mathcal{J}([\![v]\!]_{\mathbf{n}}, \zeta(t))$.

We now verify the Γ-lim sup-estimate for functions $v \in \mathbf{V}$ with $\mathcal{J}([\![v]\!]_{\mathbf{n}}, \zeta(t)) = 0$, i.e., the constraint $0 \leq [\![v]\!]_{\mathbf{n}} \leq \zeta(t)$ is satisfied a.e. in Γ_{C}. We have to find a sequence $(v_k)_k \subset \mathbf{V}$ with $0 \leq [\![v_k]\!]_{\mathbf{n}} \leq \zeta_k(t)$ a.e. in Γ_{C} and such that $v_k \to v$ strongly in $\mathbf{V}$. For this, observe that $\zeta_k(t) \to \zeta(t)$ uniformly on Γ_{C} thanks to $\zeta_k(t) \to \zeta(t)$ in $W^{1,r}(\Gamma_{\mathrm{C}}) \Subset C(\overline{\Gamma_{\mathrm{C}}})$ with $r > d-1$, cf. Prop. 4.2. Moreover, $\zeta(t), \zeta_k(t) \geq \zeta_*$ in Γ_{C} by (2.30) and the unidirectionality of $\mathcal{R}$. Thus, the quotient $\zeta_k(t)/\zeta(t)$ is well-defined in Γ_{C}. We introduce the set $B_k := [\zeta(t) \geq \zeta_k(t)]$ and find

$$\gamma_k := \inf_{x \in B_k} \left| \frac{\zeta_k(t,x)}{\zeta(t,x)} \right| \leq 1 \quad \text{and} \quad \gamma_k \to 1 \,. \tag{4.14}$$

The convergence can be seen from the fact that $1 \geq \gamma_k = \inf_{x\in B_k} |1 - \frac{\zeta(t,x) - \zeta_k(t,x)}{\zeta(t)}| \geq \inf_{x \in B_k} |1 - \frac{\varepsilon_k}{\zeta_*}|$ for every $k \geq k(\zeta_*)$ with $k(\zeta_*)$ such that $\varepsilon_k < \zeta_*$, and $\sup_{x\in\Gamma_{\mathrm{C}}} |\zeta(x) - \zeta_k(x)| =: \varepsilon_k \to 0$. In view of (4.14) we have that

$$0 \leq \gamma_k [\![v]\!]_{\mathbf{n}} \leq \gamma_k \zeta(t) \leq \zeta_k(t) \quad \text{and} \quad \gamma_k v \in \mathbf{V} \text{ with } \gamma_k v \to v \text{ strongly in } \mathbf{V} \,. \tag{4.15}$$

We conclude that $v_k := \gamma_k v$ is a suitable recovery sequence.

Ad (4.13b)**:** In order to deduce the Γ-lim inf-relation, consider a sequence $v_k \rightharpoonup v$ in $L^2(0,T;\mathbf{V})$, i.e., for all $\mu \in (L^2(0,T;\mathbf{V}))^* \cong L^2(0,T;\mathbf{V}^*)$ we have $\int_0^T \langle \mu(t), v_k(t) \rangle_{\mathbf{V}} \,\mathrm{d}t \to \int_0^T \langle \mu(t), v(t) \rangle_{\mathbf{V}} \,\mathrm{d}t$. Choose now $\mu := \eta\hat{\mu} \in L^2(0,T;\mathbf{V}^*)$ with $\hat{\mu} \in \mathbf{V}^*$ and $\eta \in C_0^\infty(0,T)$. Then, the fundamental lemma of calculus of variations implies that $\langle \hat{\mu}, v_k(t) \rangle_{\mathbf{V}} \to \langle \hat{\mu}, v(t) \rangle_{\mathbf{V}}$, i.e., $v_k(t) \rightharpoonup v(t)$ in $\mathbf{V}$ for a.e. $t \in (0,T)$. This allows us to carry over the arguments from the proof of (4.13a). As for the Γ-lim sup-relation, due to $\sup_{t\in[0,T]} \gamma_k(t) \leq C$ by (4.2d) and (2.30) we see by dominated convergence that the pointwise construction of the recovery sequence from (4.13a) is applicable as well. □

Thanks to the Mosco-convergence result we are now in the position to verify the momentum balance of the limit system (4.9) to hold with $\tilde{\xi}(t) \in \partial_u \mathcal{J}([\![u(t)]\!]_{\mathbf{n}}, \zeta(t))$ for a.e. $t \in (0,T)$ and to deduce enhanced convergence of solutions $(u_k)_k$.

Proposition 4.4 (Momentum balance of the cohesive zone system $(\mathbf{V}, \mathbf{Z}, \mathcal{V}, \mathcal{R}, \mathcal{E}_\infty)$)**.** *Let the assumptions of Thm. 2.5 hold true. Then, in addition to convergences* (4.7) *the following statements are satisfied:*

1. *The following convergences hold true:*

$$\begin{aligned} \mathrm{D}_u\mathcal{E}^{\mathrm{bulk}}(t, u_k(t)) &\rightharpoonup \mathrm{D}_u\mathcal{E}^{\mathrm{bulk}}(t, u(t)) \quad \textit{in } \mathbf{V}^* \textit{ for all } t \in [0,T] \,, && (4.16\mathrm{a}) \\ \mathrm{D}_u\Phi^{\mathrm{coh}}([\![u_k(t)]\!]_{\mathbf{n}}, \zeta_k(t)) &\rightharpoonup \mathrm{D}_u\Phi^{\mathrm{coh}}([\![u(t)]\!]_{\mathbf{n}}, \zeta(t)) \quad \textit{in } \mathbf{V}^* \textit{ for all } t \in [0,T] \,, && (4.16\mathrm{b}) \\ \mathrm{D}_{\dot{u}}\mathcal{V}(\dot{u}_k) &\rightharpoonup \mathrm{D}_{\dot{u}}\mathcal{V}(\dot{u}) \quad \textit{in } L^2(0,T;\mathbf{V}^*) \,. && (4.16\mathrm{c}) \end{aligned}$$

2. *For a.a. $t \in (0,T)$ the limit pair $(u,\zeta) \in \mathbf{V} \times \mathbf{Z}$ extracted by convergences* (2.32) *satisfies momentum balance* (4.8) *&* (4.9)*;*
3. (4.12) *is satisfied, hence, $\tilde{\xi}(t) \in \partial_u \mathcal{J}([\![u(t)]\!]_{\mathbf{n}}, \zeta(t))$ for a.a. $t \in (0,T)$.*

Proof. **Ad 1.:** Convergences (4.16a) & (4.16c) directly ensue from (4.7c) & (4.7d). In order to deduce (4.16b) we will invoke the dominated convergence theorem. For this, we observe that convergences (4.7b) & (4.7c) ensure that $\zeta_k(t) \to \zeta(t)$ in $L^2(\Gamma_{\mathrm{C}})$ as well as $[\![u_k(t)]\!]_{\mathbf{n}} \to [\![u(t)]\!]_{\mathbf{n}}$ in $L^2(\Gamma_{\mathrm{C}})$, thus, convergence in measure. Moreover, by (2.9), for any $v \in \mathbf{V}$, it is $\frac{\phi_{\mathrm{coh}}(\zeta_k(t))}{\zeta_k^2(t)}[\![u_k(t)]\!]_{\mathbf{n}}[\![v(t)]\!]_{\mathbf{n}} \le b\big|[\![u_k(t)]\!]_{\mathbf{n}}[\![v(t)]\!]_{\mathbf{n}}\big| \le b\big|[\![u_k(t)]\!]_{\mathbf{n}}\big|\big|[\![v(t)]\!]_{\mathbf{n}}\big|$. Hence, $\int_{\Gamma_{\mathrm{C}}} \frac{\phi_{\mathrm{coh}}(\zeta_k(t))}{|\zeta_k(t)|^2}[\![u_k(t)]\!]_{\mathbf{n}}[\![v]\!]_{\mathbf{n}}\,\mathrm{d}\mathcal{H}^{d-1} \to \int_{\Gamma_{\mathrm{C}}} \frac{\phi_{\mathrm{coh}}(\zeta(t))}{|\zeta(t)|^2}[\![u(t)]\!]_{\mathbf{n}}[\![v]\!]_{\mathbf{n}}\,\mathrm{d}\mathcal{H}^{d-1}$ by the dominated convergence theorem, i.e. (4.16b).

Ad 2.: Thanks to convergences (4.16) we can pass to the limit in (4.6) and conclude (4.8); the pointwise-in-time balance (4.9) is then obtained as described along with (4.8)-(4.9).

Ad 3.: To prove (4.12) we pass through the time-integrated k-momentum balance and exploit convergences (4.7e) in combination with the lower semicontinuity of time-integrated versions of the functionals $\mathcal{E}^{\mathrm{bulk}}, \mathcal{V}, \Phi^{\mathrm{coh}}$. The latter holds true by Ioffe's semicontinuity theorem, cf. e.g. [Iof77, Thm. 1]. Then, starting from (4.6) and invoking the already deduced momentum balance (4.9) of the limit system, we find the following chain of inequalities

$$
\begin{aligned}
&\limsup_{k\to\infty}\langle \mathrm{D}_u\widetilde{\mathcal{J}}_k^T([\![u_k]\!]_{\mathbf{n}},\zeta_k),u_k\rangle_{L^2(0,T;\mathbf{V})}\\
&=\limsup_{k\to\infty}\Big(-\big(\langle \mathrm{D}_u\mathcal{E}^{\mathrm{bulk}}(\cdot,u_k)+\mathrm{D}_{\dot u}\mathcal{V}(\dot u_k),u_k\rangle_{L^2(0,T;\mathbf{V})}+\langle \mathrm{D}_u\Phi^{\mathrm{coh}}([\![u_k]\!]_{\mathbf{n}},\zeta_k),u_k\rangle_{L^2(0,T;\mathbf{V})}\big)\Big)\\
&\le-\liminf_{k\to\infty}\int_0^T\int_{\Omega\backslash\Gamma_{\mathrm{C}}}\mathbb{C}e(u_k(s)):e(u_k(s))\,\mathrm{d}x-\langle \mathbf{f}(s),u_k(s)\rangle_{\mathbf{V}}\,\mathrm{d}s-\liminf_{k\to\infty}\big(\mathcal{V}(u_k(t))-\mathcal{V}(u_k(0))\big)\\
&\qquad-\liminf_{k\to\infty}\int_0^t 2\Phi^{\mathrm{coh}}([\![u_k(s)]\!]_{\mathbf{n}},\zeta_k(s))\,\mathrm{d}s\\
&\le-\int_0^T\int_{\Omega\backslash\Gamma_{\mathrm{C}}}\mathbb{C}e(u(s)):e(u(s))\,\mathrm{d}x-\langle \mathbf{f}(s),u(s)\rangle_{\mathbf{V}}\,\mathrm{d}s-\big(\mathcal{V}(u(t))-\mathcal{V}(u(0))\big)-\int_0^T 2\Phi^{\mathrm{coh}}([\![u(s)]\!]_{\mathbf{n}},\zeta(s))\,\mathrm{d}s\\
&=\Big(-\big(\langle \mathrm{D}_u\mathcal{E}^{\mathrm{bulk}}(\cdot,u)+\mathrm{D}_{\dot u}\mathcal{V}(\dot u),u\rangle_{L^2(0,T;\mathbf{V})}+\langle \mathrm{D}_u\Phi^{\mathrm{coh}}([\![u]\!]_{\mathbf{n}},\zeta),u\rangle_{L^2(0,T;\mathbf{V})}\big)\Big)\\
&=\langle\tilde\xi,u\rangle_{L^2(0,T;\mathbf{V})}
\end{aligned}
$$

which proves (4.12). Thanks to this, as outlined along with (4.11), cf. also [Att84, Lemma 3.57]) for more details, we are entitled to conclude that $\tilde\xi \in \partial_u\widetilde{\mathcal{J}}^T([\![u]\!]_{\mathbf{n}},\zeta)$, which yields that $\tilde\xi(t) \in \partial_u\mathcal{J}([\![u(t)]\!]_{\mathbf{n}},\zeta(t))$, for a.a. $t \in (0,T)$. $\square$

From the lower Γ-limit provided by the Mosco-convergence of the functionals $(\mathcal{J}_k([\![\cdot]\!]_{\mathbf{n}},\zeta_k(t)))_k$ and the uniform bound on $(\mathcal{E}_k(t,u_k(t),\zeta_k(t)))_k$ it can be concluded that indeed $0 = \mathcal{J}([\![u(t)]\!]_{\mathbf{n}},\zeta(t))$. Also exploiting the lower semicontinuity properties of $\mathcal{E}^{\mathrm{bulk}}$ and Φ^{coh} in combination with convergences (2.32) we obtain the energy-dissipation inequality (2.34) of the limit systems $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E}_\infty)$.

Corollary 4.5 (Energy-dissipation estimate for $(\mathbf{V},\mathbf{Z},\mathcal{V},\mathcal{R},\mathcal{E}_\infty)$)**.** *Let the assumptions of Thm. 2.5 be satisfied. Then the limit pair (u,z) extracted by convergences* (2.32) *satisfies the energy-dissipation inequality*

$$
\mathcal{E}_k(t,u(t),\zeta(t))+\int_0^t 2\mathcal{V}(\dot u(\tau))\,\mathrm{d}\tau+\mathrm{Var}_{\mathcal{R}}(\zeta,[s,t])\le\mathcal{E}(0,u(0),\zeta(0))+\int_0^t\partial_t\mathcal{E}(\tau,u(\tau),\zeta(\tau))\,\mathrm{d}\tau\,. \tag{4.17}
$$

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References

[ACFS16] M. Artina, F. Cagnetti, M. Fornasier, and F. Solombrino. Linearly constrained evolutions of critical points and an application to cohesive fractures. *arXiv-Preprint no. 1508.02965*, 2016.

[Alm16] S. Almi. Energy release rate and quasistatic evolution via vanishing viscosity in a cohesive fracture model with an activation threshold. *ESAIM: Control Optim. Calc. Var.*, 2016. published online.

[AMV14] R. Alessi, J.-J. Marigo, and S. Vidoli. Gradient damage models coupled with plasticity and nucleation of cohesive cracks. *Arch. Ration. Mech. Anal.*, 214(2):575–615, 2014.

[AOR16] L. Adam, J.V. Outrata, and T. Roubíček. Identification of some nonsmooth evolution systems with illustration on adhesive contacts at small strains. *arXiv-Preprint no. 1411.4903*, 2016.

[Att84] H. Attouch. *Variational convergence for functions and operators.* Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[Bar62] G.I. Barenblatt. The mathematical theory of equilibrium of cracks in brittle fracture. *Adv. Appl. Mech.*, 7:55–129, 1962.

[BBB95] G. Bouchitté, A. Braides, and G. Buttazzo. Relaxation results for some free discontinuity problems. *J. Reine Angew. Math.*, 458:1–18, 1995.

[BBR08] E. Bonetti, G. Bonfanti, and R. Rossi. Global existence for a contact problem with adhesion. *Math. Meth. Appl. Sci.*, 31:1029–1064, 2008.

[BBR09] E. Bonetti, G. Bonfanti, and R. Rossi. Thermal effects in adhesive contact: Modelling and analysis. *Nonlinearity*, pages 2697–2731, 2009.

[Bré73] H. Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).

[BRSS15] E. Bonetti, E. Rocca, R. Scala, and G. Schimperna. On the strongly damped wave equation with constraint. *WIAS-Preprint 2094*, 2015.

[Cag08] F. Cagnetti. A vanishing viscosity approach to fracture growth in a cohesive zone model with prescribed crack path. *Math. Models Methods Appl. Sci.*, 18(7):1027–1071, 2008.

[CLO16] V. Crismale, G. Lazzaroni, and G. Orlando. Cohesive fracture with irreversibility: quasistatic evolution for a model subject to fatigue. *SISSA-Preprint SISSA 40/2016/MATE 1508.02965*, 2016.

[CT11] F. Cagnetti and R. Toader. Quasistatic crack evolution for a cohesive zone model with different response to loading and unloading: a Young measures approach. *ESAIM Control Optim. Calc. Var.*, 17(1):1–27, 2011.

[DMFT05] G. Dal Maso, G. Francfort, and R. Toader. Quasistatic crack growth in nonlinear elasticity. *Arch. Rat. Mech. Anal.*, 176:165–225, 2005.

[DMOT15] G. Dal Maso, G. Orlando, and R. Toader. Lower semicontinuity of a class of integral functionals on the space of functions of bounded deformation. *Adv. Cal. Var., Published online*, 2015.

[DMOT16] G. Dal Maso, G. Orlando, and R. Toader. Fracture models for elasto-plastic materials as limits of gradient damage models coupled with plasticity: the anticline case. *Calc. Var. Partial Differential Equations*, 55(3), 2016.

[DMZ07] G. Dal Maso and C. Zanini. Quasi-static crack growth for a cohesive zone model with prescribed crack path. *Proc. Roy. Soc. Edinburgh Sect. A*, 137(2):253–279, 2007.

[DNPV12] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhikers guide to the fractional sobolev spaces. *Bulletin des Sciences Mathématiques*, 136:521–573, 2012.

[Dug60] D.S. Dugdale. Yielding of steel sheets containing clits. *J. Mech. Phys. Solids*, 8:100–104, 1960.

[Fré88] M. Frémond. Contact with adhesion. In J.J. Moreau, P.D. Panagiotopoulos, and G. Strang, editors, *Topics in Nonsmooth Mechanics*, pages 157–186. Birkhäuser, 1988.

[Fré02] M. Frémond. *Non-Smooth Thermomechanics*. Springer-Verlag Berlin Heidelberg, 2002.

[Iof77] A.D. Ioffe. On lower semicontinuity of integral functionals. I. *SIAM J. Control Optimization*, 15(4):521–538, 1977.

[IT79] A. D. Ioffe and V. M. Tihomirov. *Theory of extremal problems*, volume 6 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1979. Translated from the Russian by Karol Makowski.

[KMR06] M. Kočvara, A. Mielke, and T. Roubíček. A rate-independent approach to the delamination problem. *Math. Mech. Solids*, 11:423–447, 2006.

[KPR15] M. Kružík, C.G. Panagiotopoulos, and T. Roubíček. Quasistatic adhesive contact delaminating in mixed mode and its numerical treatment. *Math. Mech. Solids*, 20(5):582–599, 2015.

[Kre05] R. Kregting. Cohesive zone models towards a robust implementation of irreversible behavior. Technical Report MT05.11, TU Eindhoven, Materials Technology, 2005.

[MM79] M. Marcus and V.J. Mizel. Every superposition operator mapping one Sobolev space into another is continuous. *J. Functional Analysis*, 33:217–229, 1979.

[MM05] A. Mainik and A. Mielke. Existence results for energetic models for rate-independent systems. *Calc. Var. Partial Differential Equations*, 22(1):73–99, 2005.

[MR15] A. Mielke and T. Roubíček. *Rate-independent Systems: Theory and Application*, volume 193 of *Applied Mathematical Sciences*. Springer, 2015.

[MRT12] A. Mielke, T. Roubíček, and M. Thomas. From damage to delamination in nonlinearly elastic materials at small strains. *J. Elasticity*, 109(2):235–273, 2012.

[OP99] M. Ortiz and A. Pandolfi. Finite-deformation irreversible cohesive elements for three-dimensional crack-propagation analysis. *International Journal for Numerical Methods in Engineering*, 44:1267–1282, 1999.

[PMR16] C.G. Panagiotopoulos, V. Mantič, and T. Roubíček. Two adhesive-contact models for quasistatic mixed-mode delamination problems. *arXiv-Preprint no. 1511.08873*, 2016.

[PP11] K. Park and G.H. Paulino. Cohesive Zone Models: A Critical Review of Traction-Separation Relationships Across Fracture Surfaces. *Applied Mechanics Reviews*, 64, November 2011.

[Ric68] J.R. Rice. *Fracture*, chapter Mathematical analysis in the mechanics of fracture, pages 191–311. Academic Press, New York, 1968.

[RKZ14] T. Roubíček, M. Kružík, and J. Zeman. Delamination and adhesive contact models and their mathematical analysis and numerical treatment. In *Mathematical methods and models in composites*, volume 5 of *Comput. Exp. Methods Struct.*, pages 349–400. Imp. Coll. Press, London, 2014.

[RMP13] T. Roubíček, V. Mantič, and C.G. Panagiotopoulos. A quasistatic mixed-mode delamination model. *Discrete Contin. Dyn. Syst. Ser. S*, 6(2):591–610, 2013.

[Rou09] T. Roubíček. Rate-independent processes in viscous solids at small strains. *Math. Methods Appl. Sci.*, 32(7):825–862, 2009.

[Rou10] T. Roubíček. Thermodynamics of rate-independent processes in viscous solids at small strains. *SIAM J. Math. Anal.*, 42(1):256–297, 2010.

[RPM13] T. Roubíček, C.G. Panagiotopoulos, and V. Mantič. Quasistatic adhesive contact of visco-elastic bodies and its numerical treatment for very small viscosity. *ZAMM Z. Angew. Math. Mech.*, 93(10-11):823–840, 2013.

[RR11] R. Rossi and T. Roubíček. Thermodynamics and analysis of rate-independent adhesive contact at small strains. *Nonlinear Anal.*, 74(10):3159–3190, 2011.

[RR13] R. Rossi and T. Roubíček. Adhesive contact delaminating at mixed mode, its thermodynamics and analysis. *Interfaces Free Bound.*, 15(1):1–37, 2013.

[RSZ09] T. Roubíček, L. Scardia, and C. Zanini. Quasistatic delamination problem. *Continuum Mech. Thermodynam.*, 21(3):223–235, 2009.

[RT15a] R. Rossi and M. Thomas. Coupling rate-independent and rate-dependent processes: Existence results. *WIAS-Preprint 2113*, 2015.

[RT15b] R. Rossi and M. Thomas. From an adhesive to a brittle delamination model in thermo-visco-elasticity. *ESAIM Control Optim. Calc. Var.*, 21:1–59, 2015.

[RT16] R. Rossi and M. Thomas. From adhesive to brittle delamination in visco-elastodynamics. *WIAS-Preprint 2259*, 2016.

[RTP15] T. Roubíček, M. Thomas, and C.G. Panagiotopoulos. Stress-driven local-solution approach to quasistatic brittle delamination. *Nonlinear Anal. Real World Appl.*, 22:645–663, 2015.

[Sca15] R. Scala. A weak formulation for a rate-independent delamination evolution with inertial and viscosity effects subjected to unilateral constraint. *WIAS-Preprint 2172*, 2015.

[SS15] R. Scala and G. Schimperna. A contact problem for viscoelastic bodies with inertial effects and unilateral boundary constraints,. *WIAS-Preprint 2147*, 2015.

[Tho13] M. Thomas. Quasistatic damage evolution with spatial BV-regularization. *Discrete Contin. Dyn. Syst. Ser. S*, 6(1):235–255, 2013.

[TM10] M. Thomas and A. Mielke. Damage of nonlinearly elastic materials at small strain: existence and regularity results. *Zeit. angew. Math. Mech.*, 90(2):88–112, 2010.

[VMR14] R. Vodička, V. Mantič, and T. Roubíček. Energetic versus maximally-dissipative local solutions of a quasi-static rate-independent mixed-mode delamination model. *Meccanica*, 49(12):2933–2963, 2014.