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## Stochastic two-scale convergence and Young measures

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# Stochastic two-scale convergence and Young measures

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## Abstract

In this paper we compare the notion of stochastic two-scale convergence in the mean (by Bourgeat, Mikelić and Wright), the notion of stochastic unfolding (recently introduced by the authors), and the quenched notion of stochastic two-scale convergence (by Zhikov and Pyatnitskii). In particular, we introduce stochastic two-scale Young measures as a tool to compare mean and quenched limits. Moreover, we discuss two examples, which can be naturally analyzed via stochastic unfolding, but which cannot be treated via quenched stochastic two-scale convergence.

## 1 Introduction

In this paper we compare *quenched* stochastic two-scale convergence [38] with the notion of *stochastic unfolding* [30, 19], which is equivalent to stochastic two-scale convergence in the *mean* [6]. In particular, we introduce the concept of stochastic two-scale Young measures to relate quenched stochastic two-scale limits with the mean limit and discuss examples of convex homogenization problems that can be treated with two-scale convergence in the mean, but not conveniently in the quenched setting of two-scale convergence.

Two-scale convergence has been introduced in [32, 1, 25] for homogenization problems (partial differential equations or variational problems) with periodic coefficients. The essence of two-scale convergence is that the two-scale limit of an oscillatory sequence captures oscillations that emerge along the sequence and that are to leading order periodic on a definite microscale, typically denoted by  $\varepsilon > 0$ . It is especially well-suited for problems where oscillations of solutions solely stem from prescribed oscillations of the coefficients or the data. For instance, this is the case for equations with a *monotone* structure or *convex* variational problems. In contrast, problems that feature pattern formation to leading order (e.g., nonconvex variational problems or singular partial differential equations with non-convex domain) typically cannot be conveniently treated with two-scale convergence. Another well established method for periodic homogenization is *periodic unfolding*, see [9, 35, 27, 10] as well as [36, 3] for the periodic modulation method, which is related. These methods build on an isometric operator—the periodic unfolding (or dilation) operator. It allows us to embed oscillatory sequences into a larger two-scale space and to transform an oscillatory problem into an “unfolded” problem on the two-scale space. The latter often features a better separation of macro- and microscopic properties, which often is convenient for the analysis. We refer to [14, 7, 28, 8, 15, 24, 26] for various interesting applications of this method. Both notions are closely linked, since weak convergence of “unfolded” sequence in the two-scale space is equivalent to weak two-scale convergence, see [5].

In this paper we are interested in stochastic homogenization, i.e. problems with random coefficients with a stationary distribution. The first stochastic homogenization result has been obtained by Papanicolaou and Varadhan in [33] (and independently by Kozlov [23]) for linear, elliptic equations with stationary and ergodic random coefficients on  $\mathbb{R}^d$ . In their seminal paper, Papanicolaou and Varadhan

introduce a functional analytic framework, which, by now, is the standard way to model random coefficients. We briefly recall it in the special case of convex integral functionals with quadratic growth: Let  $(\Omega, \mathcal{F}, P)$  denote a probability space of parameter fields  $\omega \in \Omega$  and let  $\tau_x : \Omega \rightarrow \Omega, x \in \mathbb{R}^d$ , denote a measure preserving and ergodic group action, see Assumption 2.1 for details. A standard model for a convex, integral functional with a stationary, ergodic, random microstructure on scale  $\varepsilon > 0$  is then given by the functional  $\mathcal{E}_\varepsilon^\omega : H^1(Q) \rightarrow \mathbb{R} \cup \{\infty\}$ ,

$$\mathcal{E}_\varepsilon^\omega(u) = \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, \nabla u(x)) - f(x)u(x) dx$$

where  $Q \subset \mathbb{R}^d$  denotes an open and bounded domain,  $f \in L^2(Q)$ , and  $V(\omega, F)$  is an integrand that is measurable in  $\omega \in \Omega$ , convex in  $F \in \mathbb{R}^d$ , and satisfies a quadratic growth condition. A classical result [11] shows that in the homogenization limit  $\varepsilon \rightarrow 0$ , the functionals  $\Gamma$ -converge to the homogenized functional  $\mathcal{E}_{\text{hom}} : H^1(Q) \rightarrow \mathbb{R} \cup \{\infty\}$ , given by

$$\mathcal{E}_{\text{hom}}(u) = \int_Q V_{\text{hom}}(\nabla u(x)) - f(x)u(x) dx,$$

where  $V_{\text{hom}}$  is a deterministic, convex integrand and characterized by a homogenization formula, see (31) below. There are different natural choices for the topology when passing to this limit:

- In the *mean* setting, minimizers  $u_\varepsilon^\omega$  of  $\mathcal{E}_\varepsilon^\omega, \omega \in \Omega$ , are viewed as random fields  $(\omega, x) \mapsto u_\varepsilon^\omega(x)$  in  $L^2(\Omega; H^1(Q))$  and one considers  $\Gamma$ -convergence of the averaged functional

$$L^2(\Omega; H^1(Q)) \ni u \mapsto \int_\Omega \mathcal{E}_\varepsilon(u) dP$$

w.r.t. strong convergence in  $L^2(\Omega \times Q)$ . In fact, the first result in stochastic homogenization [33] establishes convergence of solutions in this mean sense.

- In the *quenched* setting, one studies the limiting behavior of a minimizer  $u_\varepsilon \in H^1(Q)$  of  $\mathcal{E}_\varepsilon^\omega$  for fixed  $\omega \in \Omega$ . One then considers  $\Gamma$ -convergence of  $\mathcal{E}_\varepsilon^\omega$  w.r.t. strong convergence in  $L^2(Q)$  for  $\mathbb{P}$ -a.a.  $\omega \in \Omega$ .

Similarly, two variants of stochastic two-scale convergence have been introduced as generalizations of periodic two-scale convergence (for the sake of brevity, we restrict the following review to the Hilbert-space case  $p = 2$ , and note that the following extends to  $L^p(\Omega \times Q)$  with  $p \in (1, \infty)$ ):

- In [6, 2] the *mean* variant has been introduced as follows: We say that a sequence of random fields  $(u_\varepsilon) \subset L^2(\Omega \times Q)$  *stochastically two-scale converges in the mean* to  $u \in L^2(\Omega \times Q)$ , if

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega \times Q} u_\varepsilon(\omega, x) \varphi(\tau_{\frac{x}{\varepsilon}}\omega, x) dP(\omega) dx = \int_{\Omega \times Q} u(\omega, x) \varphi(\omega, x) dP(\omega) dx, \quad (1)$$

for all *admissible* test functions  $\varphi \in L^2(\Omega \times Q)$ , see Remark 3.3 for details.

- More recently, Zhikov and Pyatnitskii introduced in [38] a *quenched* variant: We say that a sequence  $(u_\varepsilon) \subset L^2(Q)$  *quenched stochastically two-scale converges* to  $u \in L^2(\Omega \times Q)$  w.r.t. to a fixed parameter field  $\omega_0 \in \Omega$ , if

$$\lim_{\varepsilon \rightarrow 0} \int_Q u_\varepsilon(x) \varphi(\tau_{\frac{x}{\varepsilon}}\omega_0, x) dx = \int_{\Omega \times Q} u(\omega, x) \varphi(\omega, x) dP(\omega) dx,$$

for all *admissible* test functions  $\varphi \in L^2(\Omega \times Q)$ . Note that the two-scale limit  $u$  a priori depends on  $\omega_0$ . In fact, in [37] (see also [16]) quenched two-scale convergence has been introduced in a very general setting that includes the case of integration against random, rapidly oscillating measures, which naturally emerge when describing coefficients defined relative to random geometries. In this work, we restrict our considerations to the simplest case where the random measure is the Lebesgue measure.

Similarly to the periodic case, stochastic two-scale convergence in the mean can be rephrased with help of a transformation operator, see [30, 19, 34], where the *stochastic unfolding operator*  $\mathcal{T}_\varepsilon : L^2(\Omega \times Q) \rightarrow L^2(\Omega \times Q)$ ,

$$\mathcal{T}_\varepsilon u(\omega, x) = u(\tau_{-\frac{x}{\varepsilon}}\omega, x), \quad (2)$$

has been introduced. As in the periodic case, it is a linear isometry and it turns out that for a bounded sequence  $(u_\varepsilon) \subseteq L^2(\Omega \times Q)$ , stochastic two-scale convergence in the mean is equivalent to weak convergence of the unfolded sequence  $\mathcal{T}_\varepsilon u_\varepsilon$ . As we demonstrate below in Section 4.1, the stochastic unfolding method leads to a very economic and streamlined analysis of convex homogenization problems. Moreover, it allows us to derive two-scale functionals of the form  $\mathcal{E}(u, \chi) = \int_\Omega \int_Q V(\omega, \nabla u(x) + \chi(\omega, x)) dx dP$  as a  $\Gamma$ -limit of  $\mathcal{E}_\varepsilon$ , see Theorem 4.1 for details. In contrast to the periodic case, where the unfolding operator is an isometry from  $L^2(\mathbb{R}^d)$  to  $L^2(\mathcal{Y} \times \mathbb{R}^d)$  (with  $\mathcal{Y}$  denoting the unit torus), in the random case it is not possible to interpret (2) as a continuous operator from  $L^2(Q)$  to  $L^2(\Omega \times Q)$ . Therefore, quenched two-scale convergence cannot be characterized via stochastic unfolding directly.

In the present paper we compare the different notions of stochastic two-scale convergence. Although the mean and quenched notion of two-scale convergence look quite similar, it is non-trivial to relate both. As a main result, we introduce stochastic two-scale Young measures as a tool to compare quenched and mean limits, see Theorem 3.15. The construction invokes a metric characterization of quenched stochastic two-scale convergence, which is a tool of independent interest, see Lemma 3.8. As an application we demonstrate how to lift a mean two-scale homogenization result to a quenched statement, see Section 4.3. Moreover, we present two examples that can only be conveniently treated with the mean notion of two-scale convergence. In the first example, see Section 4.1, the assumption of ergodicity is dropped (as it is natural in the context of periodic representative volume approximation schemes). In the second example we consider a model that invokes a mean field interaction in form of a *variance-type* regularization of a convex integral functional with degenerate growth, see Section 4.2.

**Structure of the paper.** In the following section we present the standard setting for stochastic homogenization. In Section 3 we provide the main properties of the stochastic unfolding method, present the most important facts about quenched two-scale convergence and present our results about Young measures. In Section 4 we present examples of stochastic homogenization and applications of the methods developed in this paper.

## 2 Standard model of random coefficients

In the following we briefly recall the standard setting for stochastic homogenization. Throughout the entire paper we assume the following:

**Assumption 2.1.** *Let  $(\Omega, \mathcal{F}, P)$  be a complete and separable probability space. Let  $\tau = \{\tau_x\}_{x \in \mathbb{R}^d}$  denote a group of invertible measurable mappings  $\tau_x : \Omega \rightarrow \Omega$  such that:*

- (i) (Group property).  $\tau_0 = Id$  and  $\tau_{x+y} = \tau_x \circ \tau_y$  for all  $x, y \in \mathbb{R}^d$ .
- (ii) (Measure preservation).  $P(\tau_{-x}E) = P(E)$  for all  $E \in \mathcal{F}$  and  $x \in \mathbb{R}^d$ .
- (iii) (Measurability).  $(\omega, x) \mapsto \tau_x \omega$  is  $(\mathcal{F} \otimes \mathcal{L}(\mathbb{R}^d), \mathcal{F})$ -measurable, where  $\mathcal{L}(\mathbb{R}^d)$  denotes the Lebesgue  $\sigma$ -algebra.

We write  $\langle \cdot \rangle$  to denote the expectation  $\int_{\Omega} \cdot dP$ . By the separability assumption on the measure space it follows that  $L^p(\Omega)$  is separable for  $p \geq 1$ . The proof of the following lemma is a direct consequence of Assumption 2.1, thus we omit it.

**Lemma 2.2** (Stationary extension). *Let  $\varphi : \Omega \rightarrow \mathbb{R}$  be  $\mathcal{F}$ -measurable. Let  $Q \subset \mathbb{R}^d$  be open and denote by  $\mathcal{L}(Q)$  the corresponding Lebesgue  $\sigma$ -algebra. Then  $S\varphi : \Omega \times Q \rightarrow \mathbb{R}$ ,  $S\varphi(\omega, x) := \varphi(\tau_x \omega)$  defines an  $\mathcal{F} \otimes \mathcal{L}(Q)$ -measurable function – called the stationary extension of  $\varphi$ . Moreover, if  $Q$  is bounded, for all  $1 \leq p < \infty$  the map  $S : L^p(\Omega) \rightarrow L^p(\Omega \times Q)$  is a linear injection satisfying*

$$\|S\varphi\|_{L^p(\Omega \times Q)} = |Q|^{\frac{1}{p}} \|\varphi\|_{L^p(\Omega)}.$$

We say  $(\Omega, \mathcal{F}, P, \tau)$  is *ergodic* ( $\langle \cdot \rangle$  is ergodic), if

every shift invariant  $A \in \mathcal{F}$  (i.e.  $\tau_x A = A$  for all  $x \in \mathbb{R}^d$ ) satisfies  $P(A) \in \{0, 1\}$ .

In this case the celebrated Birkhoff's ergodic theorem applies, which we recall in the following form:

**Theorem 2.3** (Birkhoff's ergodic Theorem [12, Theorem 10.2.II]). *Let  $\langle \cdot \rangle$  be ergodic and  $\varphi : \Omega \rightarrow \mathbb{R}$  be integrable. Then for  $P$ -a.a.  $\omega \in \Omega$  it holds:  $S\varphi(\omega, \cdot)$  is locally integrable and for all open, bounded sets  $Q \subset \mathbb{R}^d$  we have*

$$\lim_{\varepsilon \rightarrow 0} \int_Q S\varphi(\omega, \frac{x}{\varepsilon}) dx = |Q| \langle \varphi \rangle. \quad (3)$$

Furthermore, if  $\varphi \in L^p(\Omega)$  with  $1 \leq p \leq \infty$ , then for  $P$ -a.a.  $\omega \in \Omega$  it holds:  $S\varphi(\omega, \cdot) \in L^p_{loc}(\mathbb{R}^d)$ , and provided  $p < \infty$  it holds  $S\varphi(\omega, \frac{\cdot}{\varepsilon}) \rightharpoonup \langle \varphi \rangle$  weakly in  $L^p_{loc}(\mathbb{R}^d)$  as  $\varepsilon \rightarrow 0$ .

**Stochastic gradient.** For  $p \in (1, \infty)$  consider the group of isometric operators  $\{U_x : x \in \mathbb{R}^d\}$  on  $L^p(\Omega)$  defined by  $U_x \varphi(\omega) = \varphi(\tau_x \omega)$ . This group is strongly continuous (see [22, Section 7.1]). For  $i = 1, \dots, d$ , we consider the 1-parameter group of operators  $\{U_{he_i} : h \in \mathbb{R}\}$  and its infinitesimal generator  $D_i : \mathcal{D}_i \subset L^p(\Omega) \rightarrow L^p(\Omega)$

$$D_i \varphi = \lim_{h \rightarrow 0} \frac{U_{he_i} \varphi - \varphi}{h},$$

which we refer to as *stochastic derivative*.  $D_i$  is a linear and closed operator and its domain  $\mathcal{D}_i$  is dense in  $L^p(\Omega)$ . We set  $W^{1,p}(\Omega) = \cap_{i=1}^d \mathcal{D}_i$  and define for  $\varphi \in W^{1,p}(\Omega)$  the stochastic gradient as  $D\varphi = (D_1 \varphi, \dots, D_d \varphi)$ . In this way, we obtain a linear, closed and densely defined operator  $D : W^{1,p}(\Omega) \rightarrow L^p(\Omega)^d$ , and we denote by

$$L^p_{\text{pot}}(\Omega) := \overline{\mathcal{R}(D)} \subset L^p(\Omega)^d \quad (4)$$

the closure of the range of  $D$  in  $L^p(\Omega)^d$ . We denote the adjoint of  $D$  by  $D^* : \mathcal{D}^* \subset L^q(\Omega)^d \rightarrow L^q(\Omega)$  where here and below  $q := \frac{p}{p-1}$  denotes the dual exponent. It is a linear, closed and densely defined operator ( $\mathcal{D}^*$  is the domain of  $D^*$ ). We define the subspace of shift invariant functions in  $L^p(\Omega)$  by

$$L^p_{\text{inv}}(\Omega) = \left\{ \varphi \in L^p(\Omega) : U_x \varphi = \varphi \text{ for all } x \in \mathbb{R}^d \right\},$$

and denote by  $P_{\text{inv}} : L^p(\Omega) \rightarrow L^p_{\text{inv}}(\Omega)$  the conditional expectation with respect to the  $\sigma$ -algebra of shift invariant sets  $\{A \in \mathcal{F} : \tau_x A = A \text{ for all } x \in \mathbb{R}^d\}$ .  $P_{\text{inv}}$  a contractive projection and for  $p = 2$  it coincides with the orthogonal projection onto  $L^2_{\text{inv}}(\Omega)$ . The following well-known equivalence holds:

$$\langle \cdot \rangle \text{ is ergodic} \Leftrightarrow L^p_{\text{inv}}(\Omega) \simeq \mathbb{R} \Leftrightarrow P_{\text{inv}} f = \langle f \rangle.$$

**Random fields.** We introduce function spaces for functions defined on  $\Omega \times Q$  as follows: For closed subspaces  $X \subset L^p(\Omega)$  and  $Y \subset L^p(Q)$ , we denote by  $X \otimes Y$  the closure of

$$X \overset{a}{\otimes} Y := \left\{ \sum_{i=1}^n \varphi_i \eta_i : \varphi_i \in X, \eta_i \in Y, n \in \mathbb{N} \right\}$$

in  $L^p(\Omega \times Q)$ . Note that in the case  $X = L^p(\Omega)$  and  $Y = L^p(Q)$ , we have  $X \otimes Y = L^p(\Omega \times Q)$ . Up to isometric isomorphisms, we may identify  $L^p(\Omega \times Q)$  with the Bochner spaces  $L^p(\Omega; L^p(Q))$  and  $L^p(Q; L^p(\Omega))$ . Slightly abusing the notation, for closed subspaces  $X \subset L^p(\Omega)$  and  $Y \subset W^{1,p}(Q)$ , we denote by  $X \otimes Y$  the closure of

$$X \overset{a}{\otimes} Y := \left\{ \sum_{i=1}^n \varphi_i \eta_i : \varphi_i \in X, \eta_i \in Y, n \in \mathbb{N} \right\}$$

in  $L^p(\Omega; W^{1,p}(Q))$ . In this regard, we may identify  $u \in L^p(\Omega) \otimes W^{1,p}(Q)$  with the pair  $(u, \nabla u) \in L^p(\Omega \times Q)^{1+d}$ . We mostly focus on the space  $L^p(\Omega \times Q)$  and the above notation is convenient for keeping track of its various subspaces.

### 3 Stochastic two-scale convergence, unfolding and Young measures

In the following we first discuss two notions of stochastic two-scale convergence and their connection through Young measures. In particular, Section 3.1 is devoted to the introduction of the stochastic unfolding operator and its most important properties. In Section 3.2 we discuss quenched two-scale convergence and its properties. Section 3.3 presents the results about Young measures.

#### 3.1 Stochastic unfolding and two-scale convergence in the mean

In the following we briefly introduce the stochastic unfolding operator and provide its main properties, for the proofs and detailed studies we refer to [30, 19, 34, 31].

**Lemma 3.1** ([19, Lemma 3.1]). *Let  $\varepsilon > 0$ ,  $1 < p < \infty$ ,  $q = \frac{p}{p-1}$ , and  $Q \subset \mathbb{R}^d$  be open. There exists a unique linear isometric isomorphism*

$$\mathcal{T}_\varepsilon : L^p(\Omega \times Q) \rightarrow L^p(\Omega \times Q)$$

such that

$$\forall u \in L^p(\Omega) \overset{a}{\otimes} L^p(Q) : \quad (\mathcal{T}_\varepsilon u)(\omega, x) = u(\tau_{-\frac{x}{\varepsilon}} \omega, x) \quad \text{a.e. in } \Omega \times Q.$$

Moreover, its adjoint is the unique linear isometric isomorphism  $\mathcal{T}_\varepsilon^* : L^q(\Omega \times Q) \rightarrow L^q(\Omega \times Q)$  that satisfies  $(\mathcal{T}_\varepsilon^* u)(\omega, x) = u(\tau_{\frac{x}{\varepsilon}} \omega, x)$  a.e. in  $\Omega \times Q$  for all  $u \in L^q(\Omega) \overset{a}{\otimes} L^q(Q)$ ,  $q := \frac{p}{p-1}$ .

**Definition 3.2** (Unfolding and two-scale convergence in the mean). *The operator  $\mathcal{T}_\varepsilon : L^p(\Omega \times Q) \rightarrow L^p(\Omega \times Q)$  in Lemma 3.1 is called the stochastic unfolding operator. We say that a sequence  $(u_\varepsilon) \subset L^p(\Omega \times Q)$  weakly (strongly) two-scale converges in the mean in  $L^p(\Omega \times Q)$  to  $u \in L^p(\Omega \times Q)$  if (as  $\varepsilon \rightarrow 0$ )*

$$\mathcal{T}_\varepsilon u_\varepsilon \rightarrow u \quad \text{weakly (strongly) in } L^p(\Omega \times Q).$$

In this case we write  $u_\varepsilon \xrightarrow{2} u$  ( $u_\varepsilon \xrightarrow{2} u$ ) in  $L^p(\Omega \times Q)$ .

**Remark 3.3** (Equivalence to stochastic two-scale convergence in the mean). Stochastic two-scale convergence in the mean was introduced in [6]. In particular, it is said that a sequence of random fields  $u_\varepsilon \in L^p(\Omega \times Q)$  stochastically two-scale converges in the mean if

$$\lim_{\varepsilon \rightarrow 0} \left\langle \int_Q u_\varepsilon(\omega, x) \varphi(\tau_{\frac{x}{\varepsilon}} \omega, x) dx \right\rangle = \left\langle \int_Q u(\omega, x) \varphi(\omega, x) dx \right\rangle, \quad (5)$$

for any  $\varphi \in L^q(\Omega \times Q)$ ,  $q = \frac{p}{p-1}$ , that is admissible, i.e., in the sense that the transformation  $(\omega, x) \mapsto \varphi(\tau_{\frac{x}{\varepsilon}} \omega, x)$  is well-defined. For a bounded sequence  $u_\varepsilon \in L^p(\Omega \times Q)$ , (5) is equivalent to  $\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u$  weakly in  $L^p(\Omega \times Q)$ , i.e., to weak stochastic two-scale convergence in the mean. Indeed, with help of  $\mathcal{T}_\varepsilon$  (and its adjoint) we might rephrase the integral on the left-hand side in (5) as

$$\left\langle \int_Q u_\varepsilon(\mathcal{T}_\varepsilon^* \varphi) dx \right\rangle = \left\langle \int_Q (\mathcal{T}_\varepsilon u_\varepsilon) \varphi dx \right\rangle, \quad (6)$$

which proves the equivalence.

We summarize some of the main properties:

**Proposition 3.4** (Main properties). *Let  $p \in (1, \infty)$ ,  $q = \frac{p}{p-1}$  and  $Q \subset \mathbb{R}^d$  be open.*

- (i) (Compactness, [19, Lemma 3.4].) *If  $\limsup_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^p(\Omega \times Q)} < \infty$ , then there exists a subsequence  $\varepsilon'$  and  $u \in L^p(\Omega \times Q)$  such that  $u_{\varepsilon'} \xrightarrow{2} u$  in  $L^p(\Omega \times Q)$ .*
- (ii) (Limits of gradients, [19, Proposition 3.7]) *Let  $(u_\varepsilon)$  be a bounded sequence in  $L^p(\Omega) \otimes W^{1,p}(Q)$ . Then, there exist  $u \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}(Q)$  and  $\chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(Q)$  such that (up to a subsequence)*

$$u_\varepsilon \xrightarrow{2} u \text{ in } L^p(\Omega \times Q), \quad \nabla u_\varepsilon \xrightarrow{2} \nabla u + \chi \text{ in } L^p(\Omega \times Q)^d. \quad (7)$$

*If, additionally,  $\langle \cdot \rangle$  is ergodic, then  $u = P_{\text{inv}} u = \langle u \rangle \in W^{1,p}(Q)$  and  $\langle u_\varepsilon \rangle \rightharpoonup u$  weakly in  $W^{1,p}(Q)$ .*

- (iii) (Recovery sequences, [19, Lemma 4.3]) *Let  $u \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}(Q)$  and  $\chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(Q)$ . There exists  $u_\varepsilon \in L^p(\Omega) \otimes W^{1,p}(Q)$  such that*

$$u_\varepsilon \xrightarrow{2} u, \quad \nabla u_\varepsilon \xrightarrow{2} \nabla u + \chi \text{ in } L^p(\Omega \times Q).$$

*If additionally  $u \in L^p_{\text{inv}}(\Omega) \otimes W_0^{1,p}(Q)$ , we have  $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)$ .*

## 3.2 Quenched two-scale convergence

In this section, we recall the concept of quenched stochastic two-scale convergence (see [38, 16]). The notion of quenched stochastic two-scale convergence is based on the individual ergodic theorem, see Theorem 2.3. We thus assume throughout this section that

$$\langle \cdot \rangle \text{ is ergodic.}$$



Moreover, throughout this section we fix exponents  $p \in (1, \infty)$ ,  $q := \frac{p}{p-1}$ , and an open and bounded domain  $Q \subset \mathbb{R}^d$ . We denote by  $(\mathcal{B}^p, \|\cdot\|_{\mathcal{B}^p})$  the Banach space  $L^p(\Omega \times Q)$  and the associated norm, and we write  $(\mathcal{B}^p)^*$  for the dual space. For the definition of quenched two-scale convergence we need to specify a suitable space of test-functions in  $\mathcal{B}^q$  that is countably generated. To that end we fix sets  $\mathcal{D}_\Omega$  and  $\mathcal{D}_Q$  such that

- $\mathcal{D}_\Omega$  is a countable set of bounded, measurable functions on  $(\Omega, \mathcal{F})$  that contains the identity  $\mathbf{1}_\Omega \equiv 1$  and is dense in  $L^1(\Omega)$  (and thus in  $L^r(\Omega)$  for any  $1 \leq r < \infty$ ).
- $\mathcal{D}_Q \subset C(\overline{Q})$  is a countable set that contains the identity  $\mathbf{1}_Q \equiv 1$  and is dense in  $L^1(Q)$  (and thus in  $L^r(Q)$  for any  $1 \leq r < \infty$ ).

We denote by

$$\mathcal{A} := \{\varphi(\omega, x) = \varphi_\Omega(\omega)\varphi_Q(x) : \varphi_\Omega \in \mathcal{D}_\Omega, \varphi_Q \in \mathcal{D}_Q\}$$

the set of simple tensor products (a countable set), and by  $\mathcal{D}_0$  the  $\mathbb{Q}$ -linear span of  $\mathcal{A}$ , i.e.

$$\mathcal{D}_0 := \left\{ \sum_{j=1}^m \lambda_j \varphi_j : m \in \mathbb{N}, \lambda_1, \dots, \lambda_m \in \mathbb{Q}, \varphi_1, \dots, \varphi_m \in \mathcal{A} \right\}.$$

We finally set

$$\mathcal{D} := \text{span} \mathcal{A} = \text{span} \mathcal{D}_0 \quad \text{and} \quad \overline{\mathcal{D}} := \text{span}(\mathcal{D}_Q)$$

(the span of  $\mathcal{D}_Q$  seen as a subspace of  $\mathcal{D}$ ), and note that  $\mathcal{D}$  and  $\mathcal{D}_0$  are dense subsets of  $\mathcal{B}^q$ , while the closure of  $\overline{\mathcal{D}}$  in  $\mathcal{B}^q$  is isometrically isomorphic to  $L^q(Q)$ . Let us anticipate that  $\mathcal{D}$  serves as our space of test-functions for stochastic two-scale convergence. As opposed to two-scale convergence in the mean, “quenched” stochastic two-scale convergence is defined relative to a fixed “admissible” realization  $\omega_0 \in \Omega$ . Throughout this section we denote by

$$\Omega_0 \quad \text{the set of admissible realizations;}$$

it is a set of full measure determined by the following lemma:

**Lemma 3.5.** *There exists a measurable set  $\Omega_0 \subset \Omega$  with  $P(\Omega_0) = 1$  s.t. for all  $\varphi, \varphi' \in \mathcal{A}$ , all  $\omega_0 \in \Omega_0$ , and  $r \in \{p, q\}$  we have with  $(\mathcal{T}_\varepsilon^* \varphi)(\omega, x) := \varphi(\tau_{\frac{x}{\varepsilon}} \omega, x)$ ,*

$$\limsup_{\varepsilon \rightarrow 0} \|(\mathcal{T}_\varepsilon^* \varphi)(\omega_0, \cdot)\|_{L^r(Q)} \leq \|\varphi\|_{\mathcal{B}^r}$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_Q \mathcal{T}_\varepsilon^*(\varphi\varphi')(\omega_0, x) dx = \left\langle \int_Q (\varphi\varphi')(\omega_0, x) dx \right\rangle.$$

*Proof.* This is a simple consequence of Theorem 2.3 and the fact that  $\mathcal{A}$  is countable. □

For the rest of the section  $\Omega_0$  is fixed according to Lemma 3.5.

The idea of quenched stochastic two-scale convergence is similar to periodic two-scale convergence: We associate with a bounded sequence  $(u_\varepsilon) \subset L^p(Q)$  and  $\omega_0 \in \Omega_0$ , a sequence of linear functionals  $(U_\varepsilon)$  defined on  $\mathcal{D}$ . We can pass (up to a subsequence) to a pointwise limit  $U$ , which is again a linear functional on  $\mathcal{D}$  and which (thanks to Lemma 3.5) can be uniquely extended to a bounded linear functional on  $\mathcal{B}^q$ . We then define the *weak quenched  $\omega_0$ -two-scale limit* of  $(u_\varepsilon)$  as the Riesz-representation  $u \in \mathcal{B}^p$  of  $U \in (\mathcal{B}^q)^*$ .

**Definition 3.6** (quenched two-scale limit, cf. [38, 17]). Let  $(u_\varepsilon)$  be a sequence in  $L^p(Q)$ , and let  $\omega_0 \in \Omega_0$  be fixed. We say that  $u_\varepsilon$  converges (weakly, quenched)  $\omega_0$ -two-scale to  $u \in \mathcal{B}^p$ , and write  $u_\varepsilon \xrightarrow{2}_{\omega_0} u$ , if the sequence  $u_\varepsilon$  is bounded in  $L^p(Q)$ , and for all  $\varphi \in \mathcal{D}$  we have

$$\lim_{\varepsilon \rightarrow 0} \int_Q u_\varepsilon(x) (\mathcal{T}_\varepsilon^* \varphi)(\omega_0, x) dx = \int_\Omega \int_Q u(x, \omega) \varphi(\omega, x) dx dP(\omega). \quad (8)$$

**Lemma 3.7** (Compactness). Let  $(u_\varepsilon)$  be a bounded sequence in  $L^p(Q)$  and  $\omega_0 \in \Omega_0$ . Then there exists a subsequence (still denoted by  $\varepsilon$ ) and  $u \in \mathcal{B}^p$  such that  $u_\varepsilon \xrightarrow{2}_{\omega_0} u$  and

$$\|u\|_{\mathcal{B}^p} \leq \liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^p(Q)}, \quad (9)$$

and  $u_\varepsilon \rightharpoonup \langle u \rangle$  weakly in  $L^p(Q)$ .

(For the proof see Section 3.2.1).

For our purpose it is convenient to have a metric characterization of two-scale convergence.

**Lemma 3.8** (Metric characterization). (i) Let  $\{\varphi_j\}_{j \in \mathbb{N}}$  denote an enumeration of the countable set  $\{\frac{\varphi}{\|\varphi\|_{\mathcal{B}^q}} : \varphi \in \mathcal{D}_0\}$ . The vector space  $\text{Lin}(\mathcal{D}) := \{U : \mathcal{D} \rightarrow \mathbb{R} \text{ linear}\}$  endowed with the metric

$$d(U, V; \text{Lin}(\mathcal{D})) := \sum_{j \in \mathbb{N}} 2^{-j} \frac{|U(\varphi_j) - V(\varphi_j)|}{|U(\varphi_j) - V(\varphi_j)| + 1}$$

is complete and separable.

(ii) Let  $\omega_0 \in \Omega_0$ . Consider the maps

$$J_\varepsilon^{\omega_0} : L^p(Q) \rightarrow \text{Lin}(\mathcal{D}), \quad (J_\varepsilon^{\omega_0} u)(\varphi) := \int_Q u(x) (\mathcal{T}_\varepsilon^* \varphi)(\omega_0, x) dx,$$

$$J_0 : \mathcal{B}^p \rightarrow \text{Lin}(\mathcal{D}), \quad (J_0 u)(\varphi) := \left\langle \int_Q u \varphi \right\rangle.$$

Then for any bounded sequence  $u_\varepsilon$  in  $L^p(Q)$  and any  $u \in \mathcal{B}^p$  we have  $u_\varepsilon \xrightarrow{2}_{\omega_0} u$  if and only if  $J_\varepsilon^{\omega_0} u_\varepsilon \rightarrow J_0 u$  in  $\text{Lin}(\mathcal{D})$ .

(For the proof see Section 3.2.1).

**Remark 3.9.** Convergence in the metric space  $(\text{Lin}(\mathcal{D}), d(\cdot, \cdot, \text{Lin}(\mathcal{D})))$  is equivalent to pointwise convergence.  $(\mathcal{B}^q)^*$  is naturally embedded into the metric space by means of the restriction  $J : (\mathcal{B}^q)^* \rightarrow \text{Lin}(\mathcal{D})$ ,  $JU = U|_{\mathcal{D}}$ . In particular, we deduce that for a bounded sequences  $(U_k)$  in  $(\mathcal{B}^q)^*$  we have  $U_k \xrightarrow{*} U$  if and only if  $JU_k \rightarrow JU$  in the metric space. Likewise,  $\mathcal{B}^p$  (resp.  $L^p(Q)$ ) can be embedded into the metric space  $\text{Lin}(\mathcal{D})$  via  $J_0$  (resp.  $J_\varepsilon^{\omega_0}$  with  $\varepsilon > 0$  and  $\omega_0 \in \Omega_0$  arbitrary but fixed), and for a bounded sequence  $(u_k)$  in  $\mathcal{B}^p$  (resp.  $L^p(Q)$ ) weak convergence in  $\mathcal{B}^p$  (resp.  $L^p(Q)$ ) is equivalent to convergence of  $(J_0 u_k)$  (resp.  $(J_\varepsilon^{\omega_0} u_k)$ ) in the metric space.

**Lemma 3.10** (Strong convergence implies quenched two-scale convergence). Let  $(u_\varepsilon)$  be a strongly convergent sequence in  $L^p(Q)$  with limit  $u \in L^p(Q)$ . Then for all  $\omega_0 \in \Omega_0$  we have  $u_\varepsilon \xrightarrow{2}_{\omega_0} u$ .

(For the proof see Section 3.2.1).

**Definition 3.11** (set of quenched two-scale cluster points). For a bounded sequence  $(u_\varepsilon)$  in  $L^p(Q)$  and  $\omega_0 \in \Omega_0$  we denote by  $\mathcal{CP}(\omega_0, (u_\varepsilon))$  the set of all  $\omega_0$ -two-scale cluster points, i.e. the set of  $u \in \mathcal{B}^p$  with  $J_0 u \in \bigcap_{k=1}^{\infty} \overline{\{J_\varepsilon^{\omega_0} u_\varepsilon : \varepsilon < \frac{1}{k}\}}$  where the closure is taken in the metric space  $(\text{Lin}(\mathcal{D}), d(\cdot, \cdot; \text{Lin}(\mathcal{D})))$ .

We conclude this section with two elementary results on quenched stochastic two-scale convergence:

**Lemma 3.12** (Approximation of two-scale limits). Let  $u \in \mathcal{B}^p$ . Then for all  $\omega_0 \in \Omega_0$ , there exists a sequence  $u_\varepsilon \in L^p(Q)$  such that  $u_\varepsilon \xrightarrow{2}_{\omega_0} u$  as  $\varepsilon \rightarrow 0$ .

(For the proof see Section 3.2.1).

Similar to the slightly different setting in [17] one can prove the following result:

**Lemma 3.13** (Two-scale limits of gradients). Let  $(u_\varepsilon)$  be a sequence in  $W^{1,p}(Q)$  and  $\omega_0 \in \Omega_0$ . Then there exist a subsequence (not relabeled) and functions  $u \in W^{1,p}(Q)$  and  $\chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(Q)$  such that  $u_\varepsilon \rightharpoonup u$  weakly in  $W^{1,p}(Q)$  and

$$u_\varepsilon \xrightarrow{2}_{\omega_0} u \quad \text{and} \quad \nabla u_\varepsilon \xrightarrow{2}_{\omega_0} \nabla u + \chi \quad \text{as } \varepsilon \rightarrow 0.$$

### 3.2.1 Proofs

*Proof of Lemma 3.7.* Set  $C_0 := \limsup_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^p(Q)}$  and note that  $C_0 < \infty$ . By passing to a subsequence (not relabeled) we may assume that  $C_0 = \liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^p(Q)}$ . Fix  $\omega_0 \in \Omega_0$ . Define linear functionals  $U_\varepsilon \in \text{Lin}(\mathcal{D})$  via

$$U_\varepsilon(\varphi) := \int_Q u_\varepsilon(x) (\mathcal{T}_\varepsilon^* \varphi)(\omega_0, x) dx.$$

Note that for all  $\varphi \in \mathcal{A}$ ,  $(U_\varepsilon(\varphi))$  is a bounded sequence in  $\mathbb{R}$ . Indeed, by Hölder's inequality and Lemma 3.5,

$$\limsup_{\varepsilon \rightarrow 0} |U_\varepsilon(\varphi)| \leq \limsup_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^p(Q)} \|\mathcal{T}_\varepsilon^* \varphi(\omega_0, \cdot)\|_{L^q(Q)} \leq C_0 \|\varphi\|_{\mathcal{B}^q}. \quad (10)$$

Since  $\mathcal{A}$  is countable we can pass to a subsequence (not relabeled) such that  $U_\varepsilon(\varphi)$  converges for all  $\varphi \in \mathcal{A}$ . By linearity and since  $\mathcal{D} = \text{span}(\mathcal{A})$ , we conclude that  $U_\varepsilon(\varphi)$  converges for all  $\varphi \in \mathcal{D}$ , and  $U(\varphi) := \lim_{\varepsilon \rightarrow 0} U_\varepsilon(\varphi)$  defines a linear functional on  $\mathcal{D}$ . In view of (10) we have  $|U(\varphi)| \leq C_0 \|\varphi\|_{\mathcal{B}^q}$ , and thus  $U$  admits a unique extension to a linear functional in  $(\mathcal{B}^q)^*$ . Let  $u \in \mathcal{B}^p$  denote its Riesz-representation. Then  $u_\varepsilon \xrightarrow{2}_{\omega_0} u$ , and

$$\|u\|_{\mathcal{B}^p} = \|U\|_{(\mathcal{B}^q)^*} \leq C_0 = \liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^p(Q)}.$$

Since  $\mathbf{1}_\Omega \in \mathcal{D}_\Omega$  we conclude that for all  $\varphi_Q \in \mathcal{D}_Q$  we have

$$\int_Q u_\varepsilon \varphi_Q dx = U_\varepsilon(\mathbf{1}_\Omega \varphi_Q) \rightarrow U(\mathbf{1}_\Omega \varphi_Q) = \left\langle \int_Q u(\omega, x) \varphi_Q(x) dx \right\rangle = \int_Q \langle u(x) \rangle \varphi_Q(x) dx.$$

Since  $(u_\varepsilon)$  is bounded in  $L^p(Q)$ , and  $\mathcal{D}_Q \subset L^p(Q)$  is dense, we conclude that  $u_\varepsilon \rightharpoonup \langle u \rangle$  weakly in  $L^p(Q)$ .  $\square$

*Proof of Lemma 3.8.* We use the following notation in this proof  $\mathcal{A}_1 := \left\{ \frac{\varphi}{\|\varphi\|_{\mathcal{B}^q}} : \varphi \in \mathcal{D}_0 \right\}$ .

(i) *Argument for completeness:* If  $(U_j)$  is a Cauchy sequence in  $\text{Lin}(\mathcal{D})$ , then for all  $\varphi \in \mathcal{A}_1$ ,  $(U_j(\varphi))$  is a Cauchy sequence in  $\mathbb{R}$ . By linearity of the  $U_j$ 's this implies that  $(U_j(\varphi))$  is Cauchy in  $\mathbb{R}$  for all  $\varphi \in \mathcal{D}$ . Hence,  $U_j \rightarrow U$  pointwise in  $\mathcal{D}$  and it is easy to check that  $U$  is linear. Furthermore,  $U_j \rightarrow U$  pointwise in  $\mathcal{A}_1$  implies  $U_j \rightarrow U$  in the metric space.

*Argument for separability:* Consider the (injective) map  $J : (\mathcal{B}^q)^* \rightarrow \text{Lin}(\mathcal{D})$  where  $J(U)$  denotes the restriction of  $U$  to  $\mathcal{D}$ . The map  $J$  is continuous, since for all  $U, V \in (\mathcal{B}^q)^*$  and  $\varphi \in \mathcal{A}_1$  we have  $|(JU)(\varphi) - (JV)(\varphi)| \leq \|U - V\|_{(\mathcal{B}^q)^*} \|\varphi\|_{\mathcal{B}^q} = \|U - V\|_{(\mathcal{B}^q)^*}$  (recall that the test functions in  $\mathcal{A}_1$  are normalized). Since  $(\mathcal{B}^q)^*$  is separable (as a consequence of the assumption that  $\mathcal{F}$  is countably generated), it suffices to show that the range  $\mathcal{R}(J)$  of  $J$  is dense in  $\text{Lin}(\mathcal{D})$ . To that end let  $U \in \text{Lin}(\mathcal{D})$ . For  $k \in \mathbb{N}$  we denote by  $U_k \in (\mathcal{B}^q)^*$  the unique linear functional that is equal to  $U$  on the finite dimensional (and thus closed) subspace  $\text{span}\{\varphi_1, \dots, \varphi_k\} \subset \mathcal{B}^q$  (where  $\{\varphi_j\}$  denotes the enumeration of  $\mathcal{A}_1$ ), and zero on the orthogonal complement in  $\mathcal{B}^q$ . Then a direct calculation shows that  $d(U, J(U_k); \text{Lin}(\mathcal{D})) \leq \sum_{j>k} 2^{-j} = 2^{-k}$ . Since  $k \in \mathbb{N}$  is arbitrary, we conclude that  $\mathcal{R}(J) \subset \text{Lin}(\mathcal{D})$  is dense.

(ii) Let  $u_\varepsilon$  denote a bounded sequence in  $L^p(Q)$  and  $u \in \mathcal{B}^p$ . Then by definition,  $u_\varepsilon \xrightarrow{2}_{\omega_0} u$  is equivalent to  $J_\varepsilon^{\omega_0} u_\varepsilon \rightarrow J_0 u$  pointwise in  $\mathcal{D}$ , and the latter is equivalent to convergence in the metric space  $\text{Lin}(\mathcal{D})$ .  $\square$

*Proof of Lemma 3.10.* This follows from Hölder's inequality and Lemma 3.5, which imply for all  $\varphi \in \mathcal{A}$  the estimate

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \int_Q |(u_\varepsilon(x) - u(x)) \mathcal{T}_\varepsilon^* \varphi(\omega_0, x)| dx \\ \leq \limsup_{\varepsilon \rightarrow 0} \left( \|u_\varepsilon - u\|_{L^p(Q)} \left( \int_Q |\mathcal{T}_\varepsilon^* \varphi(\omega_0, x)|^q dx \right)^{\frac{1}{q}} \right) = 0. \end{aligned}$$

$\square$

*Proof of Lemma 3.12.* Since  $\mathcal{D}$  (defined as in Lemma 3.8) is dense in  $\mathcal{B}^p$ , for any  $\delta > 0$  there exists  $v_\delta \in \mathcal{D}_0$  with  $\|u - v_\delta\|_{\mathcal{B}^p} \leq \delta$ . Define  $v_{\delta, \varepsilon}(x) := \mathcal{T}_\varepsilon^* v_\delta(\omega_0, x)$ . Let  $\varphi \in \mathcal{D}$ . Since  $v_\delta$  and  $\varphi$  (resp.  $v_\delta \varphi$ ) are by definition linear combinations of functions (resp. products of functions) in  $\mathcal{A}$ , we deduce from Lemma 3.5 that  $(v_{\delta, \varepsilon})_\varepsilon$  is bounded in  $L^p(Q)$ , and that

$$\int_Q v_{\delta, \varepsilon} \mathcal{T}_\varepsilon^* \varphi(\omega_0, x) = \int_Q \mathcal{T}_\varepsilon^* (v_\delta \varphi)(\omega_0, x) \rightarrow \left\langle \int_Q v_\delta \varphi \right\rangle.$$

By appealing to the metric characterization, we can rephrase the last convergence statement as  $d(J_\varepsilon^{\omega_0} v_{\delta, \varepsilon}, J_0 v_\delta; \text{Lin}(\mathcal{D})) \rightarrow 0$ . By the triangle inequality we have

$$d(J_\varepsilon^{\omega_0} v_{\delta, \varepsilon}, J_0 u; \text{Lin}(\mathcal{D})) \leq d(J_\varepsilon^{\omega_0} v_{\delta, \varepsilon}, J_0 v_\delta; \text{Lin}(\mathcal{D})) + d(J_0 v_\delta, J_0 u; \text{Lin}(\mathcal{D})).$$

The second term is bounded by  $\|v_\delta - u\|_{\mathcal{B}^p} \leq \delta$ , while the first term vanishes for  $\varepsilon \downarrow 0$ . Hence, there exists a diagonal sequence  $u_\varepsilon := v_{\delta(\varepsilon), \varepsilon}$  (bounded in  $L^p(Q)$ ) such that there holds the convergence  $d(J_\varepsilon^{\omega_0} u_\varepsilon, J_0 u; \text{Lin}(\mathcal{D})) \rightarrow 0$ . The latter implies  $u_\varepsilon \xrightarrow{2}_{\omega_0} u$  by Lemma 3.8.  $\square$

### 3.3 Young measures generated by two-scale convergence

In this section we establish a relation between quenched two-scale convergence and two-scale convergence in the mean (in the sense of Definition 3.2). The relation is established by Young measures: We show that any bounded sequence  $u_\varepsilon$  in  $\mathcal{B}^p$  – viewed as a functional acting on test-functions of the form  $\mathcal{T}_\varepsilon^* \varphi$  – generates (up to a subsequence) a Young measure on  $\mathcal{B}^p$  that (a) concentrates on the quenched two-scale cluster points of  $u_\varepsilon$ , and (b) allows to represent the two-scale limit (in the mean) of  $u_\varepsilon$ . In entire Section 3.3 we assume that

$$\langle \cdot \rangle \text{ is ergodic.}$$

Also, throughout this section we fix exponents  $p \in (1, \infty)$ ,  $q := \frac{p}{p-1}$ , and an open and bounded domain  $Q \subset \mathbb{R}^d$ . Furthermore, we frequently use the objects and notations introduced in Section 3.2.

**Definition 3.14.** We say  $\nu := \{\nu_\omega\}_{\omega \in \Omega}$  is a Young measure on  $\mathcal{B}^p$ , if for all  $\omega \in \Omega$ ,  $\nu_\omega$  is a Borel probability measure on  $\mathcal{B}^p$  (equipped with the weak topology) and

$$\omega \mapsto \nu_\omega(B) \text{ is measurable for all } B \in \mathcal{B}(\mathcal{B}^p),$$

where  $\mathcal{B}(\mathcal{B}^p)$  denotes the Borel- $\sigma$ -algebra on  $\mathcal{B}^p$  (equipped with the weak topology).

**Theorem 3.15.** Let  $u_\varepsilon$  denote a bounded sequence in  $\mathcal{B}^p$ . Then there exists a subsequence (still denoted by  $\varepsilon$ ) and a Young measure  $\nu$  on  $\mathcal{B}^p$  such that for all  $\omega_0 \in \Omega_0$ ,

$$\nu_{\omega_0} \text{ is concentrated on } \mathcal{CP}(\omega_0, (u_\varepsilon(\omega_0, \cdot))),$$

and

$$\liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{\mathcal{B}^p}^p \geq \int_{\Omega} \left( \int_{\mathcal{B}^p} \|v\|_{\mathcal{B}^p}^p d\nu_\omega(v) \right) dP(\omega).$$

Moreover, we have

$$u_\varepsilon \xrightarrow{2} u \quad \text{where } u := \int_{\Omega} \int_{\mathcal{B}^p} v d\nu_\omega(v) dP(\omega).$$

Finally, if there exists  $v : \Omega \rightarrow \mathcal{B}^p$  measurable and  $\nu_\omega = \delta_{v(\omega)}$  for  $P$ -a.a.  $\omega \in \Omega$ , then up to extraction of a further subsequence (still denoted by  $\varepsilon$ ) we have

$$u_\varepsilon(\omega) \xrightarrow{2} v(\omega) \quad \text{for } P\text{-a.a. } \omega \in \Omega.$$

(For the proof see Section 3.3.1).

In the opposite direction we observe that quenched two-scale convergence implies two-scale convergence in the mean in the following sense:

**Lemma 3.16.** Consider a family  $\{(u_\varepsilon^\omega)\}_{\omega \in \Omega}$  of sequences  $(u_\varepsilon^\omega)$  in  $L^p(Q)$  and suppose that:

- (i) There exists  $u \in \mathcal{B}^p$  s.t. for  $P$ -a.a.  $\omega \in \Omega$  we have  $u_\varepsilon^\omega \xrightarrow{2} u$ .
- (ii) There exists a sequence  $(\tilde{u}_\varepsilon)$  s.t.  $u_\varepsilon^\omega(x) = \tilde{u}_\varepsilon(\omega, x)$  for a.a.  $(\omega, x) \in \Omega \times Q$ .
- (iii) There exists a bounded sequence  $(\chi_\varepsilon)$  in  $L^p(\Omega)$  such that  $\|u_\varepsilon^\omega\|_{L^p(Q)} \leq \chi_\varepsilon(\omega)$  for a.a.  $\omega \in \Omega$ .

Then  $\tilde{u}_\varepsilon \xrightarrow{2} u$  weakly two-scale (in the mean).

(For the proof see Section 3.3.1).

To compare homogenization of convex integral functionals w.r.t. stochastic two-scale convergence in the mean and in the quenched sense, we appeal to the following result:

**Lemma 3.17.** *Let  $h : \Omega \times Q \times \mathbb{R}^d \rightarrow \mathbb{R}$  be such that for all  $\xi \in \mathbb{R}^d$ ,  $h(\cdot, \cdot, \xi)$  is  $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable and for a.a.  $(\omega, x) \in \Omega \times Q$ ,  $h(\omega, x, \cdot)$  is convex. Let  $(u_\varepsilon)$  denote a bounded sequence in  $\mathcal{B}^p$  that generates a Young measure  $\nu$  on  $\mathcal{B}^p$  in the sense of Theorem 3.15. Suppose that  $h_\varepsilon : \Omega \rightarrow \mathbb{R}$ ,  $h_\varepsilon(\omega) := - \int_Q \min \{0, h(\tau_{\frac{x}{\varepsilon}}\omega, x, u_\varepsilon(\omega, x))\} dx$  is uniformly integrable. Then*

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} \int_Q h(\tau_{\frac{x}{\varepsilon}}\omega, x, u_\varepsilon(\omega, x)) dx dP(\omega) \\ \geq \int_{\Omega} \int_{\mathcal{B}^p} \left( \int_{\Omega} \int_Q h(\tilde{\omega}, x, v(\tilde{\omega}, x)) dx dP(\tilde{\omega}) \right) d\nu_\omega(v) dP(\omega). \end{aligned} \quad (11)$$

(For the proof see Section 3.3.1).

**Remark 3.18.** In [18, Lemma 5.1] it is shown that  $h$  satisfying the assumptions of Lemma 3.17 satisfies the following property: For  $P$ -a.a.  $\omega_0 \in \Omega_0$  we have: For any sequence  $(u_\varepsilon)$  in  $L^p(Q)$  it holds

$$u_\varepsilon \xrightarrow{2} \omega_0 u \quad \Rightarrow \quad \liminf_{\varepsilon \rightarrow 0} \int_Q h(\tau_{\frac{x}{\varepsilon}}\omega_0, x, u_\varepsilon(x)) dx \geq \int_{\Omega} \int_Q h(\omega, x, u(\omega, x)) dx dP(\omega). \quad (12)$$

### 3.3.1 Proof of Theorem 3.15 and Lemmas 3.17 and 3.16

We first recall some notions and results of Balder's theory for Young measures [4]. Throughout this section  $\mathcal{M}$  is assumed to be a separable, complete metric space with metric  $d(\cdot, \cdot; \mathcal{M})$ .

**Definition 3.19.** ■ We say a function  $s : \Omega \rightarrow \mathcal{M}$  is measurable, if it is  $\mathcal{F} - \mathcal{B}(\mathcal{M})$ -measurable where  $\mathcal{B}(\mathcal{M})$  denotes the Borel- $\sigma$ -algebra in  $\mathcal{M}$ .

- A function  $h : \Omega \times \mathcal{M} \rightarrow (-\infty, +\infty]$  is called a normal integrand, if  $h$  is  $\mathcal{F} \otimes \mathcal{B}(\mathcal{M})$ -measurable, and for all  $\omega \in \Omega$  the function  $h(\omega, \cdot) : \mathcal{M} \rightarrow (-\infty, +\infty]$  is lower semicontinuous.
- A sequence  $s_\varepsilon$  of measurable functions  $s_\varepsilon : \Omega \rightarrow \mathcal{M}$  is called tight, if there exists a normal integrand  $h$  such that for every  $\omega \in \Omega$  the function  $h(\omega, \cdot)$  has compact sublevels in  $\mathcal{M}$  and  $\limsup_{\varepsilon \rightarrow 0} \int_{\Omega} h(\omega, s_\varepsilon(\omega)) dP(\omega) < \infty$ .
- A Young measure in  $\mathcal{M}$  is a family  $\mu := \{\mu_\omega\}_{\omega \in \Omega}$  of Borel probability measures on  $\mathcal{M}$  such that for all  $B \in \mathcal{B}(\mathcal{M})$  the map  $\Omega \ni \omega \mapsto \mu_\omega(B) \in \mathbb{R}$  is  $\mathcal{F}$ -measurable.

**Theorem 3.20** ([4, Theorem I]). *Let  $s_\varepsilon : \Omega \rightarrow \mathcal{M}$  denote a tight sequence of measurable functions. Then there exists a subsequence, still indexed by  $\varepsilon$ , and a Young measure  $\mu : \Omega \rightarrow \mathcal{M}$  such that for every normal integrand  $h : \Omega \times \mathcal{M} \rightarrow (-\infty, +\infty]$  we have*

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} h(\omega, s_\varepsilon(\omega)) dP(\omega) \geq \int_{\Omega} \int_{\mathcal{M}} h(\omega, \xi) d\mu_\omega(\xi) dP(\omega), \quad (13)$$

provided that the negative part  $h_\varepsilon^-(\cdot) = |\min\{0, h(\cdot, s_\varepsilon(\cdot))\}|$  is uniformly integrable. Moreover, for  $P$ -a.a.  $\omega \in \Omega_0$  the measure  $\mu_\omega$  is supported in the set of all cluster points of  $s_\varepsilon(\omega)$ , i.e. in  $\bigcup_{k=1}^{\infty} \overline{\{s_\varepsilon(\omega) : \varepsilon < \frac{1}{k}\}}$  (where the closure is taken in  $\mathcal{M}$ ).

In order to apply the above theorem we require an appropriate metric space in which two-scale convergent sequences and their limits embed:

**Lemma 3.21.** (i) We denote by  $\mathcal{M}$  the set of all triples  $(U, \varepsilon, r)$  with  $U \in \text{Lin}(\mathcal{D})$ ,  $\varepsilon \geq 0$ ,  $r \geq 0$ .  $\mathcal{M}$  endowed with the metric

$$d((U_1, \varepsilon_1, r_1), (U_2, \varepsilon_2, r_2); \mathcal{M}) := d(U_1, U_2; \text{Lin}(\mathcal{D})) + |\varepsilon_1 - \varepsilon_2| + |r_1 - r_2|$$

is a complete, separable metric space.

(ii) For  $\omega_0 \in \Omega_0$  we denote by  $\mathcal{M}^{\omega_0}$  the set of all triples  $(U, \varepsilon, r) \in \mathcal{M}$  such that

$$U = \begin{cases} J_\varepsilon^{\omega_0} u & \text{for some } u \in L^p(Q) \text{ with } \|u\|_{L^p(Q)} \leq r \text{ in the case } \varepsilon > 0, \\ J_0 u & \text{for some } u \in \mathcal{B}^p \text{ with } \|u\|_{\mathcal{B}^p} \leq r \text{ in the case } \varepsilon = 0. \end{cases} \quad (14)$$

Then  $\mathcal{M}^{\omega_0}$  is a closed subspace of  $\mathcal{M}$ .

(iii) Let  $\omega_0 \in \Omega_0$ , and  $(U, \varepsilon, r) \in \mathcal{M}^{\omega_0}$ . Then the function  $u$  in the representation (14) of  $U$  is unique, and

$$\begin{cases} \|u\|_{L^p(Q)} = \sup_{\varphi \in \overline{\mathcal{D}}, \|\varphi\|_{\mathcal{B}^q} \leq 1} |U(\varphi)| & \text{if } \varepsilon > 0, \\ \|u\|_{\mathcal{B}^p} = \sup_{\varphi \in \mathcal{D}, \|\varphi\|_{\mathcal{B}^q} \leq 1} |U(\varphi)| & \text{if } \varepsilon = 0. \end{cases} \quad (15)$$

(iv) For  $\omega_0 \in \Omega_0$  the function  $\|\cdot\|_{\omega_0} : \mathcal{M}^{\omega_0} \rightarrow [0, \infty)$ ,

$$\|(U, \varepsilon, r)\|_{\omega_0} := \begin{cases} \left( \sup_{\varphi \in \overline{\mathcal{D}}, \|\varphi\|_{\mathcal{B}^q} \leq 1} |U(\varphi)|^p + \varepsilon + r^p \right)^{\frac{1}{p}} & \text{if } (U, \varepsilon, r) \in \mathcal{M}^{\omega_0}, \varepsilon > 0, \\ \left( \sup_{\varphi \in \mathcal{D}, \|\varphi\|_{\mathcal{B}^q} \leq 1} |U(\varphi)|^p + r^p \right)^{\frac{1}{p}} & \text{if } (U, \varepsilon, r) \in \mathcal{M}^{\omega_0}, \varepsilon = 0, \end{cases}$$

is lower semicontinuous on  $\mathcal{M}^{\omega_0}$ .

(v) For all  $(u, \varepsilon)$  with  $u \in L^p(Q)$  and  $\varepsilon > 0$  we have  $s := (J_\varepsilon^{\omega_0} u, \varepsilon, \|u\|_{L^p(Q)}) \in \mathcal{M}^{\omega_0}$  and  $\|s\|_{\omega_0} = (2\|u\|_{L^p(Q)}^p + \varepsilon)^{\frac{1}{p}}$ . Likewise, for all  $(u, \varepsilon)$  with  $u \in \mathcal{B}^p$  and  $\varepsilon = 0$  we have  $s = (J_0 u, \varepsilon, \|u\|_{\mathcal{B}^p})$  and  $\|s\|_{\omega_0} = 2^{\frac{1}{p}} \|u\|_{\mathcal{B}^p}$ .

(vi) For all  $R < \infty$  the set  $\{(U, \varepsilon, r) \in \mathcal{M}^{\omega_0} : \|(U, \varepsilon, r)\|_{\omega_0} \leq R\}$  is compact in  $\mathcal{M}$ .

(vii) Let  $\omega_0 \in \Omega_0$  and let  $u_\varepsilon$  denote a bounded sequence in  $L^p(Q)$ . Then the triple  $s_\varepsilon$  given by  $s_\varepsilon := (J_\varepsilon^{\omega_0} u_\varepsilon, \varepsilon, \|u_\varepsilon\|_{L^p(Q)})$  defines a sequence in  $\mathcal{M}^{\omega_0}$ . Moreover, we have  $s_\varepsilon \rightarrow s_0$  in  $\mathcal{M}$  as  $\varepsilon \rightarrow 0$  if and only if  $s_0 = (J_0 u, 0, r)$  for some  $u \in \mathcal{B}^p$ ,  $r \geq \|u\|_{\mathcal{B}^p}$ , and  $u_\varepsilon \xrightarrow{2}_{\omega_0} u$ .

*Proof.* (i) This is a direct consequence of Lemma 3.8 (i) and the fact that the product of complete, separable metric spaces remains complete and separable.

(ii) Let  $s_k := (U_k, \varepsilon_k, r_k)$  denote a sequence in  $\mathcal{M}^{\omega_0}$  that converges in  $\mathcal{M}$  to some  $s_0 = (U_0, \varepsilon_0, r_0)$ . We need to show that  $s_0 \in \mathcal{M}^{\omega_0}$ . By passing to a subsequence, it suffices to study the following three cases:  $\varepsilon_k > 0$  for all  $k \in \mathbb{N}_0$ ,  $\varepsilon_k = 0$  for all  $k \in \mathbb{N}_0$ , and  $\varepsilon_0 = 0$  while  $\varepsilon_k > 0$  for all  $k \in \mathbb{N}$ .

Case 1:  $\varepsilon_k > 0$  for all  $k \in \mathbb{N}_0$ .

W.l.o.g. we may assume that  $\inf_k \varepsilon_k > 0$ . Hence, there exist  $u_k \in L^p(Q)$  with  $U_k = J_{\varepsilon_k}^{\omega_0} u_k$ .

Since  $r_k \rightarrow r$ , we conclude that  $(u_k)$  is bounded in  $L^p(\Omega)$ . We thus may pass to a subsequence (not relabeled) such that  $u_k \rightharpoonup u_0$  weakly in  $L^p(Q)$ , and

$$\|u_0\|_{L^p(Q)} \leq \liminf_k \|u_k\|_{L^p(Q)} \leq \lim_k r_k = r_0. \tag{16}$$

Moreover,  $U_k \rightarrow U$  in the metric space  $\text{Lin}(\mathcal{D})$  implies pointwise convergence on  $\mathcal{D}$ , and thus for all  $\varphi_Q \in \mathcal{D}_Q$  we have  $U_k(\mathbf{1}_\Omega \varphi_Q) = \int_Q u_k \varphi_Q \rightarrow \int_Q u_0 \varphi_Q$ . We thus conclude that  $U_0(\mathbf{1}_\Omega \varphi_Q) = \int_Q u_0 \varphi_Q$ . Since  $\mathcal{D}_Q \subset L^q(Q)$  dense, we deduce that  $u_k \rightharpoonup u_0$  weakly in  $L^p(Q)$  for the entire sequence. On the other hand the properties of the shift  $\tau$  imply that for any  $\varphi_\Omega \in \mathcal{D}_\Omega$  we have  $\varphi_\Omega(\tau_{\frac{\cdot}{\varepsilon_k}} \omega_0) \rightarrow \varphi_\Omega(\tau_{\frac{\cdot}{\varepsilon_0}} \omega_0)$  in  $L^q(Q)$ . Hence, for any  $\varphi_\Omega \in \mathcal{D}_\Omega$  and  $\varphi_Q \in \mathcal{D}_Q$  we have

$$\begin{aligned} U_k(\varphi_\Omega \varphi_Q) &= \int_Q u_k(x) \varphi_Q(x) \varphi_\Omega(\tau_{\frac{x}{\varepsilon_k}} \omega_0) dx \\ &\rightarrow \int_Q u_0(x) \varphi_Q(x) \varphi_\Omega(\tau_{\frac{x}{\varepsilon_0}} \omega_0) dx = J_{\varepsilon_0}^{\omega_0}(\varphi_\Omega \varphi_Q) \end{aligned}$$

and thus (by linearity)  $U_0 = J_{\varepsilon_0}^{\omega_0} u_0$ .

Case 2:  $\varepsilon_k = 0$  for all  $k \in \mathbb{N}_0$ .

In this case there exist a bounded sequence  $u_k$  in  $\mathcal{B}^p$  with  $U_k = J_0 u_k$  for  $k \in \mathbb{N}$ . By passing to a subsequence we may assume that  $u_k \rightharpoonup u_0$  weakly in  $\mathcal{B}^p$  for some  $u_0 \in \mathcal{B}^p$  with

$$\|u_0\|_{\mathcal{B}^p} \leq \liminf_k \|u_{\varepsilon_k}\|_{\mathcal{B}^p} \leq r_0. \tag{17}$$

This implies that  $U_k = J_0 u_k \rightarrow J_0 u_0$  in  $\text{Lin}(\mathcal{D})$ . Hence,  $U_0 = J_0 u_0$  and we conclude that  $s_0 \in \mathcal{M}^{\omega_0}$ .

Case 3:  $\varepsilon_k > 0$  for all  $k \in \mathbb{N}$  and  $\varepsilon_0 = 0$ .

There exists a bounded sequence  $u_k$  in  $L^p(Q)$ . Thanks to Lemma 3.7, by passing to a subsequence we may assume that  $u_k \xrightarrow{2} \omega_0 u_0$  for some  $u \in \mathcal{B}^p$  with

$$\|u_0\|_{\mathcal{B}^p} \leq \liminf_k \|u_k\|_{L^p(Q)} \leq r_0. \tag{18}$$

Furthermore, Lemma 3.8 implies that  $J_{\varepsilon_k}^{\omega_0} u_k \rightarrow J_0 u_0$  in  $\text{Lin}(\mathcal{D})$ , and thus  $U_0 = J_0 u_0$ . We conclude that  $s_0 \in \mathcal{M}^{\omega_0}$ .

- (iii) We first argue that the representation (14) of  $U$  by  $u$  is unique. In the case  $\varepsilon > 0$  suppose that  $u, v \in L^p(Q)$  satisfy  $U = J_\varepsilon^{\omega_0} u = J_\varepsilon^{\omega_0} v$ . Then for all  $\varphi_Q \in \mathcal{D}_Q$  we have  $\int_Q (u - v) \varphi_Q = J_\varepsilon^{\omega_0} u(\mathbf{1}_\Omega \varphi_Q) - J_\varepsilon^{\omega_0} v(\mathbf{1}_\Omega \varphi_Q) = U(\mathbf{1}_\Omega \varphi_Q) - U(\mathbf{1}_\Omega \varphi_Q) = 0$ , and since  $\mathcal{D}_Q \subset L^q(Q)$  dense, we conclude that  $u = v$ . In the case  $\varepsilon = 0$  the statement follows by a similar argument from the fact that  $\mathcal{D}$  is dense  $\mathcal{B}^q$ .

To see (15) let  $u$  and  $U$  be related by (14). Since  $\overline{\mathcal{D}}$  (resp.  $\mathcal{D}$ ) is dense in  $L^q(Q)$  (resp.  $\mathcal{B}^q$ ), we have

$$\begin{cases} \|u\|_{L^p(Q)} = \sup_{\varphi \in \overline{\mathcal{D}}, \|\varphi\|_{\mathcal{B}^q} \leq 1} \left| \int_Q u \varphi dx dP \right| = \sup_{\varphi \in \overline{\mathcal{D}}, \|\varphi\|_{\mathcal{B}^q} \leq 1} |U(\varphi)| & \text{if } \varepsilon > 0, \\ \|u\|_{\mathcal{B}^p} = \sup_{\varphi \in \mathcal{D}, \|\varphi\|_{\mathcal{B}^q} \leq 1} \left| \int_Q \int_Q u \varphi dx dP \right| = \sup_{\varphi \in \mathcal{D}, \|\varphi\|_{\mathcal{B}^q} \leq 1} |U(\varphi)| & \text{if } \varepsilon = 0. \end{cases}$$



- (iv) Let  $s_k = (U_k, \varepsilon_k, r_k)$  denote a sequence in  $\mathcal{M}^{\omega_0}$  that converges in  $\mathcal{M}$  to a limit  $s_0 = (U_0, \varepsilon_0, r_0)$ . By (ii) we have  $s_0 \in \mathcal{M}^{\omega_0}$ . For  $k \in \mathbb{N}_0$  let  $u_k$  in  $L^p(Q)$  or  $\mathcal{B}^p$  denote the representation of  $U_k$  in the sense of (14). We may pass to a subsequence such that one of the three cases in (ii) applies and (as in (ii)) either  $u_k$  weakly converges to  $u_0$  (in  $L^p(Q)$  or  $\mathcal{B}^p$ ), or  $u_k \xrightarrow{2}_{\omega_0} u_0$ . In any of these cases the claimed lower semicontinuity of  $\|\cdot\|_{\omega_0}$  follows from  $\varepsilon_k \rightarrow \varepsilon_0$ ,  $r_k \rightarrow r_0$ , and (15) in connection with one of the lower semicontinuity estimates (16) – (18).
- (v) This follows from the definition and duality argument (15).
- (vi) Let  $s_k$  denote a sequence in  $\mathcal{M}^{\omega_0}$ . Let  $u_k$  in  $L^p(Q)$  or  $\mathcal{B}^p$  denote the (unique) representative of  $U_k$  in the sense of (14). Suppose that  $\|s_k\|_{\omega_0} \leq R$ . Then  $(r_k)$  and  $(\varepsilon_k)$  are bounded sequences in  $\mathbb{R}_{\geq 0}$ , and  $\sup_k \|u_k\| \leq \sup_k r_k < \infty$  (where  $\|\cdot\|$  stands short for either  $\|\cdot\|_{L^p(Q)}$  or  $\|\cdot\|_{\mathcal{B}^p}$ ). Thus we may pass to a subsequence such that  $r_k \rightarrow r_0$ ,  $\varepsilon_k \rightarrow \varepsilon_0$ , and one of the following three cases applies:
  - Case 1:  $\inf_{k \in \mathbb{N}_0} \varepsilon_k > 0$ . In that case we conclude (after passing to a further subsequence) that  $u_k \rightharpoonup u_0$  weakly in  $L^p(Q)$ , and thus  $U_k \rightarrow U_0 = J_{\varepsilon_0}^{\omega_0} u_0$  in  $\text{Lin}(\mathcal{D})$ .
  - Case 2:  $\varepsilon_k = 0$  for all  $k \in \mathbb{N}_0$ . In that case we conclude (after passing to a further subsequence) that  $u_k \rightharpoonup u_0$  weakly in  $\mathcal{B}^p(Q)$ , and thus  $U_k \rightarrow U_0 = J_0 u_0$  in  $\text{Lin}(\mathcal{D})$ .
  - Case 3:  $\varepsilon_k > 0$  for all  $k \in \mathbb{N}$  and  $\varepsilon_0 = 0$ . In that case we conclude (after passing to a further subsequence) that  $u_k \xrightarrow{2}_{\omega_0} u_0$ , and thus  $U_k \rightarrow U_0 = J_0 u_0$  in  $\text{Lin}(\mathcal{D})$ .

In all of these cases we deduce that  $s_0 = (U_0, \varepsilon_0, r_0) \in \mathcal{M}^{\omega_0}$ , and  $s_k \rightarrow s_0$  in  $\mathcal{M}$ .

- (vii) This is a direct consequence of (ii) – (vi), and Lemma 3.8.

□

Now we are in position to prove Theorem 3.15

*Proof of Theorem 3.15.* Let  $\mathcal{M}$ ,  $\mathcal{M}^{\omega_0}$ ,  $J_{\varepsilon}^{\omega_0}$  etc. be defined as in Lemma 3.21.

*Step 1. (Identification of  $(u_{\varepsilon})$  with a tight  $\mathcal{M}$ -valued sequence).* Since  $u_{\varepsilon} \in \mathcal{B}^p$ , by Fubini’s theorem, we have  $u_{\varepsilon}(\omega, \cdot) \in L^p(Q)$  for  $P$ -a.a.  $\omega \in \Omega$ . By modifying  $u_{\varepsilon}$  on a null-set in  $\Omega \times Q$  (which does not alter two-scale limits in the mean), we may assume w.l.o.g. that  $u_{\varepsilon}(\omega, \cdot) \in L^p(Q)$  for all  $\omega \in \Omega$ . Consider the measurable function  $s_{\varepsilon} : \Omega \rightarrow \mathcal{M}$  defined as

$$s_{\varepsilon}(\omega) := \begin{cases} (J_{\varepsilon}^{\omega} u_{\varepsilon}(\omega, \cdot), \varepsilon, \|u_{\varepsilon}(\omega, \cdot)\|_{L^p(Q)}) & \text{if } \omega \in \Omega_0 \\ (0, 0, 0) & \text{else.} \end{cases}$$

We claim that  $(s_{\varepsilon})$  is tight. To that end consider the integrand  $h : \Omega \times \mathcal{M} \rightarrow (-\infty, +\infty]$  defined by

$$h(\omega, (U, \varepsilon, r)) := \begin{cases} \|(U, \varepsilon, r)\|_{\omega}^p & \text{if } \omega \in \Omega_0 \text{ and } (U, \varepsilon, r) \in \mathcal{M}^{\omega}, \\ +\infty & \text{else.} \end{cases}$$

From Lemma 3.21 (iv) and (vi) we deduce that  $h$  is a normal integrand and  $h(\omega, \cdot)$  has compact sublevels for all  $\omega \in \Omega$ . Moreover, for all  $\omega_0 \in \Omega_0$  we have  $s_{\varepsilon}(\omega_0) \in \mathcal{M}^{\omega_0}$  and thus  $h(\omega_0, s_{\varepsilon}(\omega_0)) = 2\|u_{\varepsilon}(\omega_0, \cdot)\|_{L^p(Q)}^p + \varepsilon$ . Hence,

$$\int_{\Omega} h(\omega, s_{\varepsilon}(\omega)) dP(\omega) = 2\|u_{\varepsilon}\|_{\mathcal{B}^p}^p + \varepsilon.$$

We conclude that  $(s_\varepsilon)$  is tight.

*Step 2. (Compactness and definition of  $\nu$ ).* By appealing to Theorem 3.20 there exists a subsequence (still denoted by  $\varepsilon$ ) and a Young measure  $\mu$  that is generated by  $(s_\varepsilon)$ . Let  $\mu_1$  denote the first component of  $\mu$ , i.e. the Young measure on  $\text{Lin}(\mathcal{D})$  characterized for  $\omega \in \Omega$  by

$$\int_{\text{Lin}(\mathcal{D})} f(\xi) d\mu_{1,\omega}(\xi) = \int_{\mathcal{M}} f(\xi_1) d\mu_\omega(\xi),$$

for all  $f : \text{Lin}(\mathcal{D}) \rightarrow \mathbb{R}$  continuous and bounded, where  $\mathcal{M} \ni \xi = (\xi_1, \xi_2, \xi_3) \rightarrow \xi_1 \in \text{Lin}(\mathcal{D})$  denotes the projection to the first component. By Balder's theorem,  $\mu_\omega$  is concentrated on the limit points of  $(s_\varepsilon(\omega))$ . By Lemma 3.21 we deduce that for all  $\omega \in \Omega_0$  any limit point  $s_0(\omega)$  of  $s_\varepsilon(\omega)$  has the form  $s_0(\omega) = (J_0 u, 0, r)$  where  $0 \leq r < \infty$  and  $u \in \mathcal{B}^p$  is a  $\omega$ -two-scale limit of a subsequence of  $u_\varepsilon(\omega, \cdot)$ . Thus,  $\mu_{1,\omega}$  is supported on  $\{J_0 u : u \in \mathcal{CP}(\omega, (u_\varepsilon(\omega, \cdot)))\}$  which in particular is a subset of  $(\mathcal{B}^q)^*$ . Since  $J_0 : \mathcal{B}^p \rightarrow (\mathcal{B}^q)^*$  is an isometric isomorphism (by the Riesz-Frechet theorem), we conclude that  $\nu = \{\nu_\omega\}_{\omega \in \Omega}$ ,  $\nu_\omega(B) := \mu_{1,\omega}(J_0 B)$  (for all Borel sets  $B \subset \mathcal{B}^p$  where  $\mathcal{B}^p$  is equipped with the weak topology) defines a Young measure on  $\mathcal{B}^p$ , and for all  $\omega \in \Omega_0$ ,  $\nu_\omega$  is supported on  $\mathcal{CP}(\omega, (u_\varepsilon(\omega, \cdot)))$ .

*Step 3. (Lower semicontinuity estimate).* Note that  $h : \Omega \times \mathcal{M} \rightarrow [0, +\infty]$ ,

$$h(\omega, (U, \varepsilon, r)) := \begin{cases} \sup_{\varphi \in \overline{\mathcal{D}}, \|\varphi\|_{\mathcal{B}^q} \leq 1} |U(\varphi)|^p & \text{if } \omega \in \Omega_0 \text{ and } (U, \varepsilon, r) \in \mathcal{M}^\omega, \varepsilon > 0, \\ \sup_{\varphi \in \mathcal{D}, \|\varphi\|_{\mathcal{B}^q} \leq 1} |U(\varphi)|^p & \text{if } \omega \in \Omega_0 \text{ and } (U, \varepsilon, r) \in \mathcal{M}^\omega, \varepsilon = 0, \\ +\infty & \text{else.} \end{cases}$$

defines a normal integrand, as can be seen as in the proof of Lemma 3.21. Thus Theorem 3.20 implies that

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} h(\omega, s_\varepsilon(\omega)) dP(\omega) \geq \int_{\Omega} \int_{\mathcal{M}} h(\omega, \xi) d\mu_\omega(\xi) dP(\omega).$$

In view of Lemma 3.21 we have  $\sup_{\varphi \in \overline{\mathcal{D}}, \|\varphi\|_{\mathcal{B}^q} \leq 1} |(J_\varepsilon^\omega u_\varepsilon)(\omega, \cdot)(\varphi)| = \|u_\varepsilon(\omega, \cdot)\|_{L^p(Q)}$  for  $\omega \in \Omega_0$ , and thus the left-hand side turns into  $\liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{\mathcal{B}^p}^p$ . Thanks to the definition of  $\nu$  the right-hand side turns into  $\int_{\Omega} \int_{\mathcal{B}^p} \|v\|_{\mathcal{B}^p}^p d\nu_\omega(v) dP(\omega)$ .

*Step 4. (Identification of the two-scale limit in the mean).* Let  $\varphi \in \mathcal{D}_0$ . Then  $h : \Omega \times \mathcal{M} \rightarrow [0, +\infty]$ ,

$$h(\omega, (U, \varepsilon, r)) := \begin{cases} U(\varphi) & \text{if } \omega \in \Omega_0, (U, \varepsilon, r) \in \mathcal{M}^\omega, \\ +\infty & \text{else.} \end{cases}$$

defines a normal integrand. Since  $h(\omega, s_\varepsilon(\omega)) = \int_Q u_\varepsilon(\omega, x) (\mathcal{T}_\varepsilon^* \varphi)(\omega, x) dx$  for  $P$ -a.a.  $\omega \in \Omega$ , we deduce that  $|h(\cdot, s_\varepsilon(\cdot))|$  is uniformly integrable. Thus, (13) applied to  $\pm h$  and the definition of  $\nu$  imply that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \int_Q u_\varepsilon(\omega, x) (\mathcal{T}_\varepsilon^* \varphi)(\omega, x) dx dP(\omega) &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} h(\omega, s_\varepsilon(\omega)) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{B}^p} h(\omega, v) d\nu_\omega(v) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{B}^p} \left\langle \int_Q v \varphi \right\rangle d\nu_\omega(v) dP(\omega). \end{aligned}$$

Set  $u := \int_{\Omega} \int_{\mathcal{B}^p} v d\nu_\omega(v) dP(\omega) \in \mathcal{B}^p$ . Then Fubini's theorem yields

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \int_Q u_\varepsilon(\omega, x) (\mathcal{T}_\varepsilon^* \varphi)(\omega, x) dx dP(\omega) = \left\langle \int_Q u \varphi \right\rangle.$$

Since  $\text{span}(\mathcal{D}_0) \subset \mathcal{B}^q$  dense, we conclude that  $u_\varepsilon \xrightarrow{2} u$ .

*Step 5. Recovery of quenched two-scale convergence.* Suppose that  $\nu_\omega$  is a delta distribution on  $\mathcal{B}^p$ , say  $\nu_\omega = \delta_{v(\omega)}$  for some measurable  $v : \Omega \rightarrow \mathcal{B}^p$ . Note that  $h : \Omega \times \mathcal{M} \rightarrow [0, +\infty]$ ,

$$h(\omega, (U, \varepsilon, r)) := -d(U, J_0 v(\omega); \text{Lin}(\mathcal{D}))$$

is a normal integrand and  $|h(\cdot, s_\varepsilon(\cdot))|$  is uniformly integrable. Thus, (13) yields

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} d(J_\varepsilon^\omega u_\varepsilon(\omega, \cdot), J_0 v(\omega); \text{Lin}(\mathcal{D})) dP(\omega) \\ &= - \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} h(\omega, s_\varepsilon(\omega)) dP(\omega) \\ &\leq - \int_{\Omega} \int_{\mathcal{B}^p} h(\omega, J_0 v) d\nu_\omega(v) dP(\omega) = - \int_{\Omega} h(\omega, J_0 v(\omega)) dP(\omega) = 0. \end{aligned}$$

Thus, there exists a subsequence (not relabeled) such that  $d(J_\varepsilon^\omega u_\varepsilon(\omega, \cdot), J_0 v(\omega); \text{Lin}(\mathcal{D})) \rightarrow 0$  for a.a.  $\omega \in \Omega_0$ . In view of Lemma 3.8 this implies that  $u_\varepsilon \xrightarrow{2} v(\omega)$  for a.a.  $\omega \in \Omega_0$ .  $\square$

*Proof of Lemma 3.17. Step 1. Representation of the functional by a lower semicontinuous integrand on  $\mathcal{M}$ .*

For all  $\omega_0 \in \Omega_0$  and  $s = (U, \varepsilon, r) \in \mathcal{M}^{\omega_0}$  we write  $\pi^{\omega_0}(s)$  for the unique representation  $u$  in  $\mathcal{B}^p$  (resp.  $L^p(Q)$ ) of  $U$  in the sense of (14). We thus may define for  $\omega \in \Omega_0$  and  $s \in \mathcal{M}^{\omega_0}$  the integrand

$$\bar{h}(\omega_0, s) := \begin{cases} \int_Q h(\tau_{\frac{x}{\varepsilon}} \omega, x, (\pi^{\omega_0} s)(x)) dx & \text{if } s = (U, \varepsilon, s) \text{ with } \varepsilon > 0, \\ \int_{\Omega} \int_Q h(\omega, x, (\pi^{\omega_0} s)(x)) dx dP(\omega) & \text{if } s = (U, \varepsilon, s) \text{ with } \varepsilon = 0. \end{cases}$$

We extend  $\bar{h}(\omega_0, \cdot)$  to  $\mathcal{M}$  by  $+\infty$ , and define  $\bar{h}(\omega, \cdot) \equiv 0$  for  $\omega \in \Omega \setminus \Omega_0$ . We claim that  $\bar{h}(\omega, \cdot) : \mathcal{M} \rightarrow (-\infty, +\infty]$  is lower semicontinuous for all  $\omega \in \Omega$ . It suffices to consider  $\omega_0 \in \Omega_0$  and a convergent sequence  $s_k = (U_k, \varepsilon_k, r_k)$  in  $\mathcal{M}^{\omega_0}$ . For brevity we only consider the (interesting) case when  $\varepsilon_k \downarrow \varepsilon_0 = 0$ . Set  $u_k := \pi^{\omega_0}(s_k)$ . By construction we have

$$\bar{h}(\omega_0, s_k) = \int_Q h(\tau_{\frac{x}{\varepsilon_k}} \omega_0, u_k(\omega_0, x)) dx,$$

and

$$\bar{h}(\omega_0, s_0) = \int_{\Omega} \int_Q h(\omega, x, u_0(\omega, x)) dx dP(\omega).$$

Since  $s_k \rightarrow s_0$  and  $\varepsilon_k \rightarrow 0$ , Lemma 3.21 (vi) implies that  $u_k \xrightarrow{2} u_0$ , and since  $h$  satisfies 12 from Remark 3.18, we conclude that  $\liminf_k \bar{h}(\omega_0, s_k) \geq \bar{h}(\omega_0, s_0)$ , and thus  $\bar{h}$  is a normal integrand.

*Step 2. Conclusion.*

As in Step 1 of the proof of Theorem 3.15 we may associate with the sequence  $(u_\varepsilon)$  a sequence of measurable functions  $s_\varepsilon : \Omega \rightarrow \mathcal{M}$  that (after passing to a subsequence that we do not relabel) generates a Young measure  $\mu$  on  $\mathcal{M}$ . Since by assumption  $u_\varepsilon$  generates the Young measure  $\nu$  on  $\mathcal{B}^p$ , we deduce that the first component  $\mu_1$  satisfies  $\nu_\omega(B) = \mu_\omega(J_0 B)$  for any Borel set  $B$ . Applying

(13) to the integrand  $\bar{h}$  of Step 1, yields

$$\begin{aligned}
& \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} \int_Q h(\tau_{\varepsilon} \omega_0, u_{\varepsilon}(\omega_0, x)) dx dP(\omega) \\
&= \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} \bar{h}(\omega, s_{\varepsilon}(\omega)) dP(\omega) \\
&\geq \int_{\Omega} \int_{\mathcal{M}} \bar{h}(\omega, \xi) d\mu_{\omega}(\xi) dP(\omega) \\
&= \int_{\Omega} \int_{\mathcal{B}^p} \left( \int_{\Omega} \int_Q h(\tilde{\omega}, x, v(\tilde{\omega}, x)) dx dP(\tilde{\omega}) \right) d\nu_{\omega}(v) dP(\omega).
\end{aligned}$$

□

*Proof of Lemma 3.16.* By (b) and (c) the sequence  $(\tilde{u}_{\varepsilon})$  is bounded in  $\mathcal{B}^p$  and thus we can pass to a subsequence such that  $(\tilde{u}_{\varepsilon})$  generates a Young measure  $\nu$ . Set  $\tilde{u} := \int_{\Omega} \int_{\mathcal{B}^p} v d\nu_{\omega}(v) dP(\omega)$  and note that Theorem 3.15 implies that  $\tilde{u}_{\varepsilon} \xrightarrow{2} \tilde{u}$  weakly two-scale in the mean. On the other hand the theorem implies that  $\nu_{\omega}$  concentrates on the quenched two-scale cluster points of  $(u_{\varepsilon}^{\omega})$  (for a.a.  $\omega \in \Omega$ ). Hence, in view of (a) we conclude that for a.a.  $\omega \in \Omega$  the measure  $\nu_{\omega}$  is a Dirac measure concentrated on  $u$ , and thus  $\tilde{u} = u$  a.e. in  $\Omega \times Q$ . □

## 4 Convex homogenization via stochastic unfolding

In this section we revisit a standard model example of stochastic homogenization of integral functionals from the viewpoint of stochastic two-scale convergence and unfolding. In particular, we discuss two examples of convex homogenization problems that can be treated with stochastic two-scale convergence in the *mean*, but not with the *quenched* variant. In the first example in Section 4.1 the randomness is *nonergodic* and thus quenched two-scale convergence does not apply. In the second example, in Section 4.2, we consider a *variance-regularization* to treat a convex minimization problem with degenerate growth conditions. In these two examples we also demonstrate the simplicity of using the stochastic unfolding operator. Furthermore, in Section 4.3 we use the results of Section 3.3 to further reveal the structure of the previously obtained limits in the classical ergodic case with non-degenerate growth with help of Young measures. In particular, we show how to lift mean homogenization results to quenched statements.

### 4.1 Nonergodic case

In this section we consider a nonergodic stationary medium. Such random ensembles arise naturally, e.g., in the context of periodic representative volume element (RVE) approximations, see [13]. For example, we may consider a family of i.i.d. random variables  $\{\bar{\omega}(z)\}_{z \in \mathbb{Z}^d}$ . A realization of a stationary and ergodic random checkerboard is given by

$$\omega : \mathbb{R}^d \rightarrow \mathbb{R}, \quad \omega(x) = \sum_{i \in \mathbb{Z}^d} \mathbf{1}_{i+y+\square}(x) \bar{\omega}(\lfloor x \rfloor),$$

where  $\lfloor x \rfloor \in \mathbb{Z}^d$  is the integer part of  $x$  and  $y \in \square$  is the center of the checkerboard chosen uniformly from  $\square = [0, 1)^d$ . For  $L \in \mathbb{N}$ , we may consider the map  $\pi_L : \omega \mapsto \omega_L$  given by  $\pi_L \omega(x) = \omega(x)$  for  $x \in [0, L)^d$  and  $\pi_L \omega$  is  $L$ -periodically extended. The push forward of the map  $\pi_L$  defines a

stationary and nonergodic probability measure, that is a starting point in the periodic RVE method. Another standard example of a nonergodic structure may be obtained by considering a medium with a *noncompatible* quasiperiodic microstructure, see [38, Example 1.2].

In this section we consider the following situation. Let  $p \in (1, \infty)$  and  $Q \subset \mathbb{R}^d$  be open and bounded. We consider  $V : \Omega \times Q \times \mathbb{R}^d \rightarrow \mathbb{R}$  and assume:

(A1)  $V(\cdot, \cdot, F)$  is  $\mathcal{F} \otimes \mathcal{L}(Q)$ -measurable for all  $F \in \mathbb{R}^d$ .

(A2)  $V(\omega, x, \cdot)$  is convex for a.a.  $(\omega, x) \in \Omega \times Q$ .

(A3) There exists  $C > 0$  such that

$$\frac{1}{C}|F|^p - C \leq V(\omega, x, F) \leq C(|F|^p + 1)$$

for a.a.  $(\omega, x) \in \Omega \times Q$  and all  $F \in \mathbb{R}^d$ .

We consider the functional

$$\mathcal{E}_\varepsilon : L^p(\Omega) \otimes W_0^{1,p}(Q) \rightarrow \mathbb{R}, \quad \mathcal{E}_\varepsilon(u) = \left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, \nabla u(\omega, x)) dx \right\rangle. \quad (19)$$

Under assumptions (A1)-(A3), in the limit  $\varepsilon \rightarrow 0$  we obtain the two-scale functional

$$\begin{aligned} \mathcal{E}_0 : (L_{\text{inv}}^p(\Omega) \otimes W_0^{1,p}(Q)) \times (L_{\text{pot}}^p(\Omega) \otimes L^p(Q)) &\rightarrow \mathbb{R}, \\ \mathcal{E}_0(u, \chi) &= \left\langle \int_Q V(\omega, x, \nabla u(\omega, x) + \chi(\omega, x)) dx \right\rangle. \end{aligned} \quad (20)$$

**Theorem 4.1** (Two-scale homogenization). *Let  $p \in (1, \infty)$  and  $Q \subset \mathbb{R}^d$  be open and bounded. Assume (A1)-(A3).*

(i) *(Compactness and liminf inequality.) Let  $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)$  with  $\limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) < \infty$ . There exist  $(u, \chi) \in (L_{\text{inv}}^p(\Omega) \otimes W_0^{1,p}(Q)) \times (L_{\text{pot}}^p(\Omega) \otimes L^p(Q))$  and a subsequence (not relabeled) such that*

$$u_\varepsilon \xrightarrow{2} u \text{ in } L^p(\Omega \times Q), \quad \nabla u_\varepsilon \xrightarrow{2} \nabla u + \chi \text{ in } L^p(\Omega \times Q), \quad (21)$$

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_0(u, \chi). \quad (22)$$

(ii) *(Limsup inequality.) Let  $(u, \chi) \in (L_{\text{inv}}^p(\Omega) \otimes W_0^{1,p}(Q)) \times (L_{\text{pot}}^p(\Omega) \otimes L^p(Q))$ . There exists a sequence  $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)$  such that*

$$u_\varepsilon \xrightarrow{2} u \text{ in } L^p(\Omega \times Q), \quad \nabla u_\varepsilon \xrightarrow{2} \nabla u + \chi \text{ in } L^p(\Omega \times Q), \quad (23)$$

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) \leq \mathcal{E}_0(u, \chi). \quad (24)$$

*Proof of Theorem 4.1.* (i) The Poincaré inequality and (A3) imply that  $u_\varepsilon$  is bounded in  $L^p(\Omega) \otimes W^{1,p}(Q)$ . By Proposition 3.4 (ii) there exist  $u \in L_{\text{inv}}^p(\Omega) \otimes W^{1,p}(Q)$  and  $\chi \in L_{\text{pot}}^p(\Omega) \otimes L^p(Q)$  such that (21) holds. Also, note that  $P_{\text{inv}} u_\varepsilon \rightharpoonup u$  weakly in  $L^p(\Omega) \otimes W^{1,p}(Q)$  and  $P_{\text{inv}} u_\varepsilon \in L_{\text{inv}}^p(\Omega) \otimes$

$W_0^{1,p}(Q)$ , which implies that  $u$  also has 0 boundary values, i.e.,  $u \in L_{\text{inv}}^p(\Omega) \otimes W_0^{1,p}(Q)$ . Finally, we note that, see [19, Proposition 3.5 (i)],

$$\left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, v(\omega, x)) \right\rangle = \left\langle \int_Q V(\omega, x, \mathcal{T}_\varepsilon v(\omega, x)) \right\rangle \quad \text{for any } v \in L^p(\Omega \times Q), \quad (25)$$

and thus using the convexity of  $V$  we conclude

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon \rightarrow 0} \left\langle \int_Q V(\omega, x, \mathcal{T}_\varepsilon \nabla u_\varepsilon) \right\rangle \geq \mathcal{E}_0(u, \chi).$$

(ii) The existence of a sequence  $u_\varepsilon$  with (23) follows from Proposition 3.4 (iii). Furthermore, (25) and the growth assumption (A3) yield

$$\lim_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \lim_{\varepsilon \rightarrow 0} \left\langle \int_Q V(\omega, x, \mathcal{T}_\varepsilon \nabla u_\varepsilon) \right\rangle = \mathcal{E}_0(u, \chi).$$

This concludes the claim, in particular, we even show a stronger result stating convergence of the energy. □

*Remark 4.2 (Convergence of minimizers).* We consider the setting of Theorem 4.1. Let  $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)$  be a minimizer of the functional

$$\mathcal{I}_\varepsilon : L^p(\Omega) \otimes W_0^{1,p}(Q) \rightarrow \mathbb{R}, \quad \mathcal{I}_\varepsilon(u) = \mathcal{E}_\varepsilon(u) - \left\langle \int_Q u_\varepsilon f_\varepsilon dx \right\rangle,$$

where  $f_\varepsilon \in L^q(\Omega \times Q)$  and  $f_\varepsilon \xrightarrow{2} f$  with  $f \in L^q(Q)$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . By a standard argument from the theory of  $\Gamma$ -convergence Theorem 4.1 (cf. [34, Corollary 7.2]) implies that there exist a subsequence (not relabeled),  $u \in L_{\text{inv}}^p(\Omega) \times W_0^{1,p}(Q)$ , and  $\chi \in L_{\text{pot}}^p(\Omega) \otimes L^p(Q)$  such that  $u_\varepsilon \xrightarrow{2} u$  in  $L^p(\Omega \times Q)$ ,  $\nabla u_\varepsilon \xrightarrow{2} \nabla u + \chi$  in  $L^p(\Omega \times Q)$ , and

$$\lim_{\varepsilon \rightarrow 0} \min \mathcal{I}_\varepsilon = \lim_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon(u_\varepsilon) = \mathcal{I}_0(u, \chi) = \min \mathcal{I}_0,$$

where  $\mathcal{I}_0 : L_{\text{inv}}^p(\Omega) \otimes W_0^{1,p}(Q) \rightarrow \mathbb{R}$  is given by  $\mathcal{I}_0(u) = \mathcal{E}_0(u) - \int_Q f u dx$ . This, in particular, rigorously justifies the formal two-scale expansion  $\nabla u_\varepsilon(x) \approx \nabla u_0(\omega, x) + \chi(\tau_{\frac{x}{\varepsilon}}\omega, x)$ .

*Remark 4.3 (Uniqueness).* If  $V(\omega, x, \cdot)$  is strictly convex the minimizers are unique and the convergence in the above remark holds for the entire sequence.

## 4.2 Variance-regularization applied to degenerate growth

In this section we consider homogenization of convex functionals with degenerate growth. More precisely, we consider an integrand  $V$  that satisfies (A1), (A2) and the following assumption (as a replacement of (A3)):

(A3') There exists  $C > 0$  and a random variable  $\lambda \in L^1(\Omega)$  such that

$$\left\langle \lambda^{-\frac{1}{p-1}} \right\rangle^{p-1} < C \quad (26)$$

and

$$\lambda(\omega)|F|^p - C \leq V(\omega, x, F) \leq C(\lambda(\omega)|F|^p + 1)$$

for a.a.  $(\omega, x) \in \Omega \times Q$  and all  $F \in \mathbb{R}^d$ .

Moreover, we assume that  $\langle \cdot \rangle$  is ergodic. For  $\varepsilon > 0$  we consider the following functional

$$\mathcal{E}_\varepsilon : L^1(\Omega \times Q) \rightarrow \mathbb{R} \cup \{\infty\}, \quad \mathcal{E}_\varepsilon(u) = \left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, \nabla u) dx \right\rangle,$$

for  $u \in X_\varepsilon$  and  $\mathcal{E}_\varepsilon(u) = \infty$  otherwise. Here  $X_\varepsilon$  denotes the closure of  $\{u \in L^p(\Omega) \otimes W_0^{1,p}(Q)\}$  w.r.t. the weighted norm

$$\|u\|_{\lambda_\varepsilon} := \left\langle \int_Q \lambda(\tau_{\frac{x}{\varepsilon}}\omega) |\nabla u|^p dx \right\rangle^{\frac{1}{p}}.$$

Recently, in [29, 20, 21] it is shown that  $\mathcal{E}_\varepsilon$  Mosco-converges to the functional

$$\mathcal{E}_{\text{hom}} : L^1(Q) \rightarrow \mathbb{R} \cup \{\infty\}, \quad \mathcal{E}_{\text{hom}}(u) := \int_Q V_{\text{hom}}(x, \nabla u(x)) dx,$$

for  $u \in W_0^{1,p}(Q)$  and  $\mathcal{E}_{\text{hom}}(u) = \infty$  otherwise, where  $V_{\text{hom}} : Q \times \mathbb{R}^d \rightarrow \mathbb{R}$  is given by the homogenization formula,

$$V_{\text{hom}}(x, F) = \inf_{\chi \in L_{\text{pot}}^p(\Omega)} \langle V(\omega, x, F + \chi(\omega)) \rangle, \tag{27}$$

for  $x \in Q$  and  $F \in \mathbb{R}^d$ . Moreover, it is shown that  $V_{\text{hom}}$  is a normal convex integrand that satisfies a standard  $p$ -growth condition. Note that the assumption (A3') in comparison to (A3) makes a genuine difference in regard to the homogenization formula (27). In particular, in the setting of assumption (A3) minimizers are attained due to the coercivity of the underlying functional in  $L_{\text{pot}}^p(\Omega)$ . It is thus easy to see that the homogenized integrand satisfies  $p$ -growth condition as well, see Section 4.3 below. On the other hand, in the setting of this section assuming (A3'), (27) is a degenerate minimization problem and a priori minimizers will only have finite first moments. An additional argument is required to infer that  $V_{\text{hom}}$  in (27) is non-degenerate, in particular, in [29, Theorem 3.1] it is shown that there exists a constant  $C' > 0$  such that for all  $x \in Q$  and  $F \in \mathbb{R}^d$  we have

$$\frac{1}{C'} |F|^p - C' \leq V_{\text{hom}}(x, F) \leq C' (|F|^p + 1). \tag{28}$$

One of the difficulties in the proof of the homogenization result for  $\mathcal{E}_\varepsilon$  is due to the fact that the domain of the functionals are  $\varepsilon$ -dependent. Moreover, assumption (A3') only yields equicoercivity in  $L^1(\Omega) \otimes W_0^{1,1}(Q)$ , while the limit  $\mathcal{E}_{\text{hom}}$  is properly defined on  $W^{1,p}(Q)$ . Therefore, in practice it is convenient to regularize the problem: For  $\delta > 0$  we consider the regularized homogenization formula

$$V_{\text{hom},\delta}(x, F) = \inf_{\chi \in L_{\text{pot}}^p(\Omega)} \langle V(\omega, x, F + \chi(\omega)) + \delta |\chi(\omega)|^p \rangle.$$

It is simple to show that the infimum on the right-hand side is attained by a unique minimizer. We also consider the corresponding regularized homogenized integral functional

$$\mathcal{E}_{\text{hom},\delta} : L^1(Q) \rightarrow \mathbb{R} \cup \{\infty\}, \quad \mathcal{E}_{\text{hom},\delta}(u) := \int_Q V_{\text{hom},\delta}(\nabla u) dx,$$

for  $u \in W_0^{1,p}(Q)$  and  $\mathcal{E}_{\text{hom},\delta}(u) = \infty$  otherwise. Furthermore, thanks to (A3'), it is relatively easy to see that this regularization is consistent:

**Lemma 4.4.** *Let  $p \in (1, \infty)$  and  $Q \subset \mathbb{R}^d$  be open and bounded. Assume (A1), (A2) and (A3'). Then, for all  $x \in Q$  and  $F \in \mathbb{R}^d$ , we have*

$$\lim_{\delta \rightarrow 0} V_{\text{hom},\delta}(x, F) = V_{\text{hom}}(x, F). \tag{29}$$

Moreover,  $\mathcal{E}_{\text{hom},\delta}$  Mosco converges to  $\mathcal{E}_{\text{hom}}$  as  $\delta \rightarrow 0$ , i.e., the following statements hold:

(i) If  $u_\delta \rightharpoonup u$  weakly in  $L^1(Q)$ , then

$$\liminf_{\delta \rightarrow 0} \mathcal{E}_{\text{hom},\delta}(u_\delta) \geq \mathcal{E}_{\text{hom}}(u).$$

(ii) For any  $u \in L^1(Q)$  there exists a sequence  $u_\delta \in L^1(Q)$  such that

$$u_\delta \rightarrow u \text{ strongly in } L^1(Q), \quad \mathcal{E}_{\text{hom},\delta}(u_\delta) \rightarrow \mathcal{E}_{\text{hom}}(u).$$

*Proof.* Let  $F \in \mathbb{R}^d$  and  $x \in Q$ . Since  $\delta > 0$ , we have  $V_{\text{hom},\delta}(x, F) \geq V_{\text{hom}}(x, F)$ . On the other hand, we consider a minimizing sequence  $\chi_\eta \in L^p_{\text{pot}}(\Omega)$  in (27), e.g.,

$$\langle V(\omega, x, F + \chi_\eta) \rangle \leq V_{\text{hom}}(x, F) + \eta.$$

We have

$$V_{\text{hom},\delta}(x, F) \leq \langle V(\omega, x, F + \chi_\eta) + \delta|\chi_\eta|^p \rangle \leq V_{\text{hom}}(x, F) + \eta + \delta \langle |\chi_\eta|^p \rangle.$$

Letting first  $\delta \rightarrow 0$  and then  $\eta \rightarrow 0$ , we conclude (29).

We further consider a sequence  $u_\delta$  such that  $u_\delta \rightharpoonup u$  weakly in  $L^1(Q)$  as  $\delta \rightarrow 0$ . We assume without loss of generality that  $\limsup_{\delta \rightarrow 0} \mathcal{E}_{\text{hom},\delta}(u_\delta) < \infty$ . This, in particular, with the help of (28) and the Poincaré inequality implies that  $\limsup_{\delta \rightarrow 0} \|u_\delta\|_{W_0^{1,p}(Q)} < \infty$ . Thus, up to a subsequence, we have  $u_\delta \rightharpoonup u$  weakly in  $W_0^{1,p}(Q)$ . Using this, we obtain

$$\liminf_{\delta \rightarrow 0} \mathcal{E}_{\text{hom},\delta}(u_\delta) \geq \liminf_{\delta \rightarrow 0} \mathcal{E}_{\text{hom}}(u_\delta) \geq \mathcal{E}_{\text{hom}}(u).$$

The first inequality follows by (29) and the second is a consequence of the fact that  $V_{\text{hom}}(x, \cdot)$  is convex and of Fatou's Lemma. We conclude that (i) holds.

If  $u \notin \text{dom}(\mathcal{E}_{\text{hom}})$ , we simply choose  $u_\delta = u$ . On the other hand, for  $u \in \text{dom}(\mathcal{E}_{\text{hom}}) = W_0^{1,p}(Q)$ , (29) and the dominated convergence theorem yield

$$\lim_{\delta \rightarrow 0} \mathcal{E}_{\text{hom},\delta}(u) = \mathcal{E}_{\text{hom}}(u).$$

This means that (ii) holds. □

In the following we introduce a variance regularization of the original functional  $\mathcal{E}_\varepsilon$  that removes the degeneracy of the problem and thus can be analyzed by the standard strategy of Section 4.1. For  $\delta > 0$ , we consider

$$\mathcal{E}_{\varepsilon,\delta} : L^1(\Omega \times Q) \rightarrow \mathbb{R}, \quad \mathcal{E}_{\varepsilon,\delta}(u) = \left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, \nabla u(x)) + \delta |\nabla u(x) - \langle \nabla u(x) \rangle|^p dx \right\rangle, \quad (30)$$

for  $u \in L^p(\Omega) \otimes W_0^{1,p}(Q)$  and  $\mathcal{E}_{\varepsilon,\delta} = \infty$  otherwise. Due to the structure of the additional term, we call it a variance-regularization and we note that it only becomes active for non-deterministic functions. For fixed  $\delta > 0$ , the functional  $\mathcal{E}_{\varepsilon,\delta}$  is equicoercive on  $L^p(\Omega) \otimes W_0^{1,p}(Q)$ :

**Lemma 4.5.** *Let  $p \in (1, \infty)$  and  $Q \subset \mathbb{R}^d$  be open and bounded. Assume (A1) and (A3'). Then there exists  $C = C(Q, p) > 0$  such that, for all  $u \in L^p(\Omega) \otimes W_0^{1,p}(Q)$ , it holds*

$$\left\langle \int_Q |\nabla u|^p \right\rangle + \delta \left\langle \int_Q |\nabla u|^p \right\rangle \leq C(\mathcal{E}_{\varepsilon,\delta}(u) + 1).$$



*Proof.* By Jensen's and Hölder's inequalities we have

$$\left\langle \int_Q |\nabla u| dx \right\rangle^p \leq |Q|^{p-1} \int_Q \langle |\nabla u| \rangle^p \leq |Q|^{p-1} \left\langle \lambda_\varepsilon^{-\frac{1}{p-1}} \right\rangle^{p-1} \left\langle \int_Q \lambda_\varepsilon |\nabla u|^p \right\rangle,$$

where we use the notation  $\lambda_\varepsilon(x, \omega) = \lambda(\tau_{\frac{x}{\varepsilon}}\omega)$ . Furthermore, using (A3'), we conclude that

$$\left\langle \int_Q |\nabla u| dx \right\rangle^p \leq C(Q, p) (\mathcal{E}_{\varepsilon, \delta}(u) + 1).$$

In the end, using the variance-regularization we obtain

$$\begin{aligned} 2^{-p} \left\langle \int_Q |\nabla u|^p \right\rangle &\leq \left\langle \int_Q |\nabla u - \langle \nabla u \rangle|^p \right\rangle + \int_Q \langle |\nabla u| \rangle^p \\ &\leq \frac{C}{\delta} (\mathcal{E}_{\varepsilon, \delta}(u) + 1) + C(\mathcal{E}_{\varepsilon, \delta}(u) + 1). \end{aligned}$$

This concludes the proof. □

The regularization on the  $\varepsilon$ -level is also consistent. In particular, we show that in the limit  $\delta \rightarrow 0$ , we recover  $\mathcal{E}_\varepsilon$ . We discuss the mean functionals  $\mathcal{E}_{\varepsilon, \delta}$  and  $\mathcal{E}_\varepsilon$ , since the former does not admit a well-defined pointwise evaluation in  $\omega$  for the reason of the nonlocal variance term. Also, for the same reason the quenched version of stochastic two-scale convergence is not suitable for this setting and we apply the unfolding procedure. On the other hand, the homogenization of  $\mathcal{E}_\varepsilon$  can be conducted on the level of typical realizations, that was in fact studied in [29, 20, 21].

**Lemma 4.6.** *Let  $p \in (1, \infty)$  and  $Q \subset \mathbb{R}^d$  be open and bounded. Assume (A1), (A2) and (A3'). Then,  $\mathcal{E}_{\varepsilon, \delta}$  Mosco converges to  $\mathcal{E}_\varepsilon$  as  $\delta \rightarrow 0$  i.e., the following statements hold:*

(i) *If  $u_\delta \rightharpoonup u$  weakly in  $L^1(\Omega \times Q)$ , then*

$$\liminf_{\delta \rightarrow 0} \mathcal{E}_{\varepsilon, \delta}(u_\delta) \geq \mathcal{E}_\varepsilon(u).$$

(ii) *For any  $u \in L^1(\Omega \times Q)$  there exists a sequence  $u_\delta \in L^1(\Omega \times Q)$  such that*

$$u_\delta \rightarrow u \text{ strongly in } L^1(\Omega \times Q), \quad \mathcal{E}_{\varepsilon, \delta}(u_\delta) \rightarrow \mathcal{E}_\varepsilon(u).$$

*Proof.* (i) Let  $u_\delta$  be a sequence such that  $u_\delta \rightharpoonup u$  weakly in  $L^1(\Omega \times Q)$ . Without loss of generality we assume that  $\limsup_{\delta \rightarrow 0} \mathcal{E}_{\varepsilon, \delta}(u_\delta) < \infty$ . This and the proof of Lemma 4.5 imply that the sequence  $\lambda_\varepsilon^{\frac{1}{p}} \nabla u_\delta$  is bounded in  $L^p(\Omega \times Q)$  with the notation  $\lambda_\varepsilon(x, \omega) = \lambda(\tau_{\frac{x}{\varepsilon}}\omega)$ . This means that, up to a subsequence, we have  $\lambda_\varepsilon^{\frac{1}{p}} \nabla u_\delta \rightharpoonup \psi$  weakly in  $L^p(\Omega \times Q)$  for some  $\psi \in L^p(\Omega \times Q)$ . Thus, for an arbitrary  $\eta \in L^\infty(\Omega \times Q)$ , we have

$$\left\langle \int_Q \nabla u_\delta \eta dx \right\rangle = \left\langle \int_Q \lambda_\varepsilon^{\frac{1}{p}} \nabla u_\delta \lambda_\varepsilon^{-\frac{1}{p}} \eta dx \right\rangle \rightarrow \left\langle \int_Q \psi \lambda_\varepsilon^{-\frac{1}{p}} \eta dx \right\rangle \text{ as } \varepsilon \rightarrow 0.$$

This means that  $\nabla u_\delta$  converges weakly in  $L^1(\Omega \times Q)$  and since  $u_\delta \rightharpoonup u$  weakly in  $L^1(\Omega \times Q)$  we may conclude that  $\nabla u_\delta \rightharpoonup \nabla u$  weakly in  $L^1(\Omega \times Q)$ . This yields

$$\liminf_{\delta \rightarrow 0} \mathcal{E}_{\varepsilon, \delta}(u_\delta) \geq \liminf_{\delta \rightarrow 0} \mathcal{E}_\varepsilon(u_\delta) \geq \mathcal{E}_\varepsilon(u).$$

(ii) For an arbitrary  $u \in \text{dom}(\mathcal{E}_\varepsilon) \subset X_\varepsilon$ , we find a sequence  $u_\eta \in L^p(\Omega) \otimes W_0^{1,p}(Q)$  such that, for  $\eta \rightarrow 0$ ,

$$u_\eta \rightarrow u \quad \text{strongly in } L^1(\Omega) \otimes W_0^{1,1}(Q), \quad \left\langle \int_Q \lambda_\varepsilon |\nabla u_\eta - \nabla u|^p dx \right\rangle \rightarrow 0.$$

Using this and the dominated convergence theorem, we conclude that

$$\lim_{\eta \rightarrow 0} \mathcal{E}_\varepsilon(u_\eta) = \mathcal{E}_\varepsilon(u).$$

This in turn yields

$$\limsup_{\eta \rightarrow 0} \limsup_{\delta \rightarrow 0} |\mathcal{E}_{\varepsilon,\delta}(u_\eta) - \mathcal{E}_\varepsilon(u)| = 0.$$

We extract a diagonal sequence  $\eta(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$  such that  $u_\delta := u_{\eta(\delta)}$  satisfies  $u_\delta \rightarrow u$  strongly in  $L^1(\Omega \times Q)$  and  $\mathcal{E}_{\varepsilon,\delta}(u_\delta) \rightarrow \mathcal{E}_\varepsilon(u)$ . This concludes the proof.  $\square$

The homogenization of the regularized functional  $\mathcal{E}_{\varepsilon,\delta}$  boils down to a very similar simple argumentation as in Section 4.1.

**Theorem 4.7.** *Let  $p \in (1, \infty)$  and  $Q \subset \mathbb{R}^d$  be open and bounded. Assume (A1), (A2) and (A3'). For all  $\delta > 0$ , as  $\varepsilon \rightarrow 0$ ,  $\mathcal{E}_{\varepsilon,\delta}$  Mosco converges to  $\mathcal{E}_{\text{hom},\delta}$  in the following sense:*

(i) *Let  $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)$  be such that  $\limsup_{\varepsilon \rightarrow 0} \mathcal{E}_{\varepsilon,\delta}(u_\varepsilon) < \infty$ . Then there exist  $(u, \chi) \in W_0^{1,p}(Q) \times (L_{\text{pot}}^p(\Omega) \otimes L^p(Q))$  and a subsequence (not relabeled) such that*

$$u_\varepsilon \xrightarrow{2} u \text{ in } L^p(\Omega \times Q), \quad \nabla u_\varepsilon \xrightarrow{2} \nabla u + \chi \text{ in } L^p(\Omega \times Q).$$

(ii) *If  $u_\varepsilon \in L^1(\Omega \times Q)$ ,  $u \in L^1(Q)$  and  $\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u$  weakly in  $L^1(\Omega \times Q)$ , then*

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{E}_{\varepsilon,\delta}(u_\varepsilon) \geq \mathcal{E}_{\text{hom},\delta}(u).$$

(iii) *For any  $u \in L^1(Q)$ , there exists a sequence  $u_\varepsilon \in L^1(\Omega \times Q)$  such that*

$$\mathcal{T}_\varepsilon u_\varepsilon \rightarrow u \quad \text{strongly in } L^1(\Omega \times Q), \quad \mathcal{E}_{\varepsilon,\delta}(u_\varepsilon) \rightarrow \mathcal{E}_{\text{hom},\delta}(u).$$

*Proof.* (i) The statement follows analogously to the proof of Theorem 4.1 (i).

(ii) Let  $\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u$  weakly in  $L^1(\Omega \times Q)$ . We may assume w.l.o.g that  $\limsup_{\varepsilon \rightarrow 0} \mathcal{E}_{\varepsilon,\delta}(u_\varepsilon) < \infty$ . In this case, Lemma 4.5 implies that  $u_\varepsilon$  is bounded in  $L^p(\Omega) \otimes W_0^{1,p}(Q)$ . We may proceed analogously to Theorem 4.1 and Remark 4.3 to obtain

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{E}_{\varepsilon,\delta}(u_\varepsilon) \geq \mathcal{E}_{\text{hom},\delta}(u).$$

(ii) This part is analogous to Theorem 4.1 and Remark 4.3.  $\square$

The results of Lemmas (4.4) and (4.6), Theorem (4.7) and [29, 20, 21] can be summarized in the following commutative diagram:

$$\begin{array}{ccc} \mathcal{E}_{\varepsilon,\delta} & \xrightarrow{(\delta \rightarrow 0)} & \mathcal{E}_\varepsilon \\ (\varepsilon \rightarrow 0) \downarrow & & \downarrow (\varepsilon \rightarrow 0) \\ \mathcal{E}_{\text{hom},\delta} & \xrightarrow{(\delta \rightarrow 0)} & \mathcal{E}_{\text{hom}} \end{array}$$

The arrows denote Mosco convergence in the corresponding convergence regimes.

### 4.3 Quenched homogenization of convex functionals

In this section we demonstrate how to lift homogenization results w.r.t. two-scale convergence in the mean to quenched statements at the example of Section 4.1. Throughout this section we assume that  $\langle \cdot \rangle$  is ergodic. For  $\omega \in \Omega$  we define  $\mathcal{E}_\varepsilon^\omega : W_0^{1,p}(Q) \rightarrow \mathbb{R}$ ,

$$\mathcal{E}_\varepsilon^\omega(u) := \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, \nabla u(x)) dx,$$

with  $V$  satisfying (A1)-(A3). The goal of this section is to relate two-scale limits of “mean”-minimizers, i.e. functions  $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)$  that minimize  $\mathcal{E}_\varepsilon$ , with limits of “quenched”-minimizers, i.e. families  $\{u_\varepsilon(\omega)\}_{\omega \in \Omega}$  of minimizers to  $\mathcal{E}_\varepsilon^\omega$  in  $W_0^{1,p}(Q)$ . We also remark that if  $V(\omega, x, \cdot)$  is strictly convex  $u_\varepsilon$  and  $\{u_\varepsilon(\omega)\}_{\omega \in \Omega}$  may be identified since minimizers of both functionals  $\mathcal{E}_\varepsilon$  and  $\mathcal{E}_\varepsilon^\omega$  are unique.

Before presenting the main result of this section, we remark that in the ergodic case, the limit functional (20) reduces to a single-scale energy

$$\mathcal{E}_{\text{hom}} : W_0^{1,p}(Q) \rightarrow \mathbb{R}, \quad \mathcal{E}_{\text{hom}}(u) = \int_Q V_{\text{hom}}(x, \nabla u(x)) dx,$$

where the homogenized integrand  $V_{\text{hom}}$  is given for  $x \in \mathbb{R}^d$  and  $F \in \mathbb{R}^d$  by

$$V_{\text{hom}}(x, F) = \inf_{\chi \in L^p_{\text{pot}}(\Omega)} \langle V(\omega, x, F + \chi(\omega)) \rangle. \tag{31}$$

In particular, we may obtain an analogous statement to Theorem 4.1 where we replace  $\mathcal{E}_0$  with  $\mathcal{E}_{\text{hom}}$ . The proof of this follows analogously with the only difference that in the construction of the recovery sequence we first need to find  $\chi$  such that  $\mathcal{E}_0(u, \chi) = \mathcal{E}_{\text{hom}}(u)$ . This is done by a usual measurable selection argument, cf. [34, Theorem 7.6].

**Theorem 4.8.** *Let  $p \in (1, \infty)$ ,  $Q \subset \mathbb{R}^d$  be open and bounded, and  $\langle \cdot \rangle$  be ergodic. Assume (A1)-(A3). Let  $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)$  be a minimizer of  $\mathcal{E}_\varepsilon$ . Then there exists a subsequence such that  $(u_\varepsilon, \nabla u_\varepsilon)$  generates a Young measure  $\nu$  in  $\mathcal{B} := (\mathcal{B}^p)^{1+d}$  in the sense of Theorem 3.15, and for  $P$ -a.a.  $\omega \in \Omega$ ,  $\nu_\omega$  concentrates on the set  $\{(u, \nabla u + \chi) : \mathcal{E}_0(u, \chi) = \min \mathcal{E}_0\}$  of minimizers of the limit functional. Moreover, if  $V(\omega, x, \cdot)$  is strictly convex for all  $x \in Q$  and  $P$ -a.a.  $\omega \in \Omega$ , then the minimizer  $u_\varepsilon$  of  $\mathcal{E}_\varepsilon$  and the minimizer  $(u, \chi)$  of  $\mathcal{E}_0$  are unique, and for  $P$ -a.a.  $\omega \in \Omega$  we have (for a not relabeled subsequence)*

$$u_\varepsilon(\omega, \cdot) \rightharpoonup u \text{ weakly in } W^{1,p}(Q), \quad u_\varepsilon(\omega, \cdot) \xrightarrow{2}_\omega u, \quad \nabla u_\varepsilon(\omega, \cdot) \xrightarrow{2}_\omega \nabla u + \chi, \\ \text{and } \min \mathcal{E}_\varepsilon^\omega = \mathcal{E}_\varepsilon^\omega(u_\varepsilon(\omega, \cdot)) \rightarrow \mathcal{E}_0(u, \chi) = \min \mathcal{E}_0.$$

*Remark 4.9* (Identification of quenched two-scale cluster points). If we combine Theorem 4.8 with the identification of the support of the Young measure in Theorem 3.15 we conclude the following: There exists a subsequence such that  $(u_\varepsilon, \nabla u_\varepsilon)$  two-scale converges in the mean to a limit of the form  $(u_0, \nabla u_0 + \chi_0)$  with  $\mathcal{E}_0(u_0, \chi_0) = \min \mathcal{E}_0$ , and for a.a.  $\omega \in \Omega$  the set of quenched  $\omega$ -two-scale cluster points  $\mathcal{CP}(\omega, (u_\varepsilon(\omega, \cdot), \nabla u_\varepsilon(\omega, \cdot)))$  is contained in  $\{(u, \nabla u + \chi) : \mathcal{E}_0(u, \chi) = \min \mathcal{E}_0\}$ . In the strictly convex case we further obtain that  $\mathcal{CP}(\omega, (u_\varepsilon(\omega, \cdot), \nabla u_\varepsilon(\omega, \cdot))) = \{(u, \nabla u + \chi)\}$  where  $(u, \chi)$  is the unique minimizer to  $\mathcal{E}_0$ . Note, however, that our argument (that extracts quenched two-scale limits from the sequence of “mean” minimizers) involves an exceptional  $P$ -null-set that a priori depends on the selected subsequence. This is in contrast to the classical result in [11] which is based on a subadditive ergodic theorem and states that there exists a set of full measure  $\Omega'$  such that for all  $\omega \in \Omega'$  the minimizer  $u_\varepsilon^\omega$  to  $\mathcal{E}_\varepsilon^\omega$  weakly converges in  $W^{1,p}(Q)$  to the deterministic minimizer  $u$  of the reduced functional  $\mathcal{E}_{\text{hom}}$  for any sequence  $\varepsilon \rightarrow 0$ .

In the proof of Theorem 4.8 we combine homogenization in the mean in form of Theorem 4.1, the connection to quenched two-scale limits via Young measures in form of Theorem 3.15, and a recent result described in Remark 3.18 by Nesenenko and the first author.

*Proof of Theorem 4.8. Step 1. (Identification of the support of  $\nu$ ).*

Since  $u_\varepsilon$  is a sequence of minimizers, by Corollary 4.2 there exists a subsequence (not relabeled) and minimizers  $(u, \chi) \in W_0^{1,p}(Q) \times (L^p_{\text{pot}}(\Omega) \otimes L^p(Q))$  of  $\mathcal{E}_0$  such that that  $u_\varepsilon \xrightarrow{2} u$  in  $L^p(\Omega \times Q)$ ,  $\nabla u_\varepsilon \xrightarrow{2} \nabla u + \chi$  in  $L^p(\Omega \times Q)^d$ , and

$$\lim_{\varepsilon \rightarrow 0} \min \mathcal{E}_\varepsilon = \lim_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_0(u, \chi) = \min \mathcal{E}_0. \tag{32}$$

In particular, the sequence  $(u_\varepsilon, \nabla u_\varepsilon)$  is bounded in  $\mathcal{B}$ . By Theorem 3.15 we may pass to a further subsequence (not relabeled) such that  $(u_\varepsilon, \nabla u_\varepsilon)$  generates a Young measure  $\nu$  on  $\mathcal{B}$ . Since  $\nu_\omega$  is supported on the set of quenched  $\omega$ -two-scale cluster points of  $(u_\varepsilon(\omega, \cdot), \nabla u_\varepsilon(\omega, \cdot))$ , we deduce from Lemma 3.13 that the support of  $\nu_\omega$  is contained in  $\mathcal{B}_0 := \{\xi = (\xi_1, \xi_2) = (u', \nabla u' + \chi') : u' \in W_0^{1,p}(Q), \chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(Q)\}$  which is a closed subspace of  $\mathcal{B}$ . Moreover, thanks to the relation of the generated Young measure and stochastic two-scale convergence in the mean, we have  $(u, \chi) = \int_\Omega \int_{\mathcal{B}_0} (\xi_1, \xi_2 - \nabla \xi_1) \nu_\omega(d\xi) dP(\omega)$ . Furthermore, Lemma 3.17 implies that

$$\lim_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \int_\Omega \int_{\mathcal{B}} \left( \int_\Omega \int_Q V(\tilde{\omega}, x, \xi_2) dx dP(\tilde{\omega}) \right) \nu_\omega(d\xi) dP(\omega).$$

In view of (32) and the fact that  $\nu_\omega$  is supported in  $\mathcal{B}_0$ , we conclude that

$$\min \mathcal{E}_0 \geq \int_\Omega \int_{\mathcal{B}_0} \mathcal{E}_0(\xi_1, \xi_2 - \nabla \xi_1) \nu_\omega(d\xi) dP(\omega) \geq \min \mathcal{E}_0 \int_\Omega \int_{\mathcal{B}_0} \nu_\omega(d\xi) dP(\omega).$$

Since  $\int_\Omega \int_{\mathcal{B}_0} \nu_\omega(d\xi) dP(\omega) = 1$ , we have  $\int_\Omega \int_{\mathcal{B}_0} |\mathcal{E}_0(\xi_1, \xi_2 - \nabla \xi_1) - \min \mathcal{E}_0| \nu_\omega(d\xi) dP(\omega) = 0$ , and thus we conclude that for  $P$ -a.a.  $\omega \in \Omega_0$ ,  $\nu_\omega$  concentrates on  $\{(u, \nabla u + \chi) : \mathcal{E}_0(u, \chi) = \min \mathcal{E}_0\}$ .

*Step 2. (The strictly convex case).*

The uniqueness of  $u_\varepsilon$  and  $(u, \chi)$  is clear. From Step 1 we thus conclude that  $\nu_\omega = \delta_\xi$  where  $\xi = (u, \nabla u + \chi)$ . Theorem 3.15 implies that  $(u_\varepsilon(\omega, \cdot), \nabla u_\varepsilon(\omega, \cdot)) \xrightarrow{2}_\omega (u, \nabla u + \chi)$  (for  $P$ -a.a.  $\omega \in \Omega$ ). By Lemma 3.17 we have for  $P$ -a.a.  $\omega \in \Omega$ ,

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon^\omega(u_\varepsilon(\omega, \cdot)) \geq \mathcal{E}_0(u, \chi) = \min \mathcal{E}_0.$$

On the other hand, since  $u_\varepsilon(\omega, \cdot)$  minimizes  $\mathcal{E}_\varepsilon^\omega$ , we deduce by a standard argument that for  $P$ -a.a.  $\omega \in \Omega$ ,

$$\lim_{\varepsilon \rightarrow 0} \min \mathcal{E}_\varepsilon^\omega = \lim_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon^\omega(u_\varepsilon(\omega, \cdot)) = \mathcal{E}_0(u, \chi) = \min \mathcal{E}_0.$$

□

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