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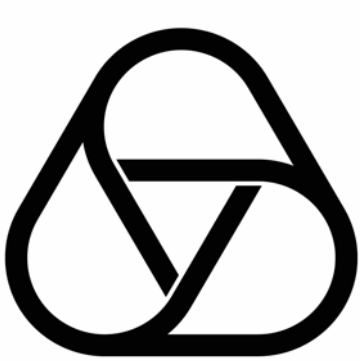

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CRISTIAN LENART AND ANNE SCHILLING

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CRYSTAL ENERGY FUNCTIONS VIA THE CHARGE IN TYPES A AND C

CRISTIAN LENART AND ANNE SCHILLING

Abstract. The Ram–Yip formula for Macdonald polynomials (at $t = 0$) provides a statistic which we call charge. In types A and C it can be defined on tensor products of Kashiwara–Nakashima single column crystals. In this paper we prove that the charge is equal to the (negative of the) energy function on affine crystals. The algorithm for computing charge is much simpler and can be more efficiently computed than the recursive definition of energy in terms of the combinatorial R-matrix.

1. Introduction

The energy function of affine crystals is an important grading used in one-dimensional configuration sums [10, 11] and generalized Kostka polynomials [42, 44, 45]. It is defined by the action of the affine Kashiwara crystal operators through a local combinatorial rule and the R-matrix.

From a computational perspective, the definition of the energy is not very efficient, as it involves a recursive definition of a local energy, and also the combinatorial R-matrix, for which not in all cases efficient algorithms exist. This leads us to the role of the charge statistic, which can be calculated very efficiently, as it only involves the detection of descents and the computation of arm lengths of cells in Young diagrams.

Charge was originally defined in type A by Lascoux and Schützenberger [20] as a statistic on words with partition content. It is calculated by enumerating certain cycles in the given word, see Section 3. Lascoux and Schützenberger showed that the charge can also be defined as the grading in the so-called cyclage graph, and used it to express combinatorially the Kostka–Foulkes polynomials, or Lusztig's q-analogue of weight multiplicities [33], based on their Morris recurrence. In type A, Nakayashiki and Yamada [36] analyzed the subtle combinatorial relationship between charge and the R-matrix, showing that the energy coincides with the charge. In [19] it was observed that the cyclage is related to the action of the crystal operator f_0 on a tensor product of type A columns (by the Kyoto path model, the latter can be identified with an affine Demazure crystal). Thus, the results of Lascoux–Schützenberger and Nakayashiki–Yamada are rederived in a more conceptual way. See also the work of Shimozono [44, 45] for a more extensive discussion of the combinatorics involved in [19], in the more general context of tensor products of type A Kirillov–Reshetikhin (KR) crystals of arbitrary rectangular shapes, as opposed to only column shapes. Charge for KR crystals of rectangular shape (or Littlewood–Richardson tableaux) was also defined in [42] using cyclage.

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Lecouvey [24, 25] extended two approaches to the Lascoux–Schützenberger charge, namely cyclage and catabolism, to types B, C, and D. He thus defined a charge statistic on the corresponding Kashiwara–Nakashima (KN) tableaux [17]. But he was only able to relate his charge to the corresponding Kostka–Foulkes polynomials in very special cases, as the original idea of Lascoux–Schützenberger based on the Morris recurrence, which he pursued, has limited applicability in this case.

In this paper we use a charge statistic coming from the Ram–Yip formula [39] for Macdonald polynomials $P_\mu(x;q,t)$ of arbitrary type [34] at $t=0$. The terms in this formula correspond to certain chains of Weyl group elements which come from the alcove walk model (this was defined in [3, 31, 32], and was then developed in subsequent papers, including [39]). The statistic is defined on the mentioned chains, and describes the powers of q. In [29] it is shown that, in types A and C, the chains are in bijection with elements in a tensor product of KR crystals of the form $B^{k,1}$. It is also shown that, under this bijection, the above statistic can be translated into a statistic on the elements of the mentioned crystal, which we call charge. Thus, we have

$$P_\mu(x;q,0) = \sum_{b \in B^{\mu'_1,1} \otimes B^{\mu'_2,1} \otimes \ldots} q^{\mathrm{charge}(b)} x^{\mathrm{wt}(b)}. \tag{1.1}$$

In type A, one can rewrite this formula as an expansion of the Macdonald P-polynomials in terms of Schur polynomials $s_\lambda(x)$

$$P_\mu(x;q,0) = \sum_\lambda K_{\lambda'\mu'}(q)\; s_\lambda(x)\,; \tag{1.2}$$

here $K_{\lambda'\mu'}(q)$ is the Kostka–Foulkes polynomial and λ' denotes the transpose of the partition λ. A generalization of (1.2) to simply-laced types was given in [12]; in types A and D, this result is sharpened in [41, Section 9.2] by replacing the Kostka–Foulkes polynomials with the corresponding one-dimensional configuration sums (which are generating functions for the energy). Both (1.1) in type A and (1.2) are expressed combinatorially in terms of the Lascoux–Schützenberger charge, whereas the type C charge given by (1.1) is a new statistic. It is worth noting that the main ingredient in these charge constructions is the so-called quantum Bruhat graph [2], which first arose in connection to Chevalley multiplication formulas for the quantum cohomology of flag varieties.

The goal of this paper is to show in an efficient, conceptual way that the charge in [29] coincides with the energy function on the corresponding tensor products of KR crystals. We focus on types A and C, and expect to extend these results to types B and D. There are several reasons for not yet addressing the case of arbitrary types. First of all, the elements of the crystals $B^{k,1}$ of classical types are represented in a very explicit way, by KN columns. In fact, concrete descriptions of certain KR crystals of exceptional type have only been given in special cases, see for example [13, 16, 35, 50]. Furthermore, the so-called stable one-dimensional configuration sums studied in [27] are all of classical type. However, we do expect that the main result of this paper would generalize to arbitrary types, if we were to phrase it in the context of the alcove walk model and the statistic in the Ram–Yip formula mentioned above (see Section 7.3).

We use the recent reinterpretation in [41] of the (global) energy function as the affine grading on a tensor product of KR crystals under "Demazure arrows". In type A, KR crystals are perfect and hence, by the Kyoto path model [14, 15], can be realized as Demazure crystals. By the result of [41], "Demazure arrows" (see Definition 2.7) change energy by 1. Together with the result that charge is well-behaved under crystal operators, this proves the equality of energy and charge. For type C, we use the same approach, but in this case KR crystals are not perfect.

There is still an embedding of a tensor product of KR crystals into an affine highest weight crystal (see Proposition 2.9) by analogy with the Kyoto path model, but now there are several highest weight components in the image, instead of just one. For each of these components we exhibit an explicit path from its highest weight (or ground state) to "type A elements" in the component, using only "Demazure arrows" (see Theorem 5.2). This additional result suffices to establish the equality of energy and charge in type C, based on the corresponding result in type A. As a by-product, we obtain an explicit description of the components that appear in the nonperfect setting of single columns for type C.

Our main result can now be stated as follows.

Theorem 1.1. *Let $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ be a tensor product of KR crystals in type $A^{(1)}_{n-1}$ or type $C^{(1)}_n$ with $r_N \geq r_{N-1} \geq \cdots \geq r_1 \geq 1$. Then for all $b \in B$ we have*

$$D(b) = -\mathrm{charge}(b), \tag{1.3}$$

where $D(b)$ is given in Definition 2.4*, and* $\mathrm{charge}(b)$ *is defined in Section* 3*.*

From a theoretical point of view, the above result is not surprising due to work of Ion [12], which relates Macdonald polynomials at $t = 0$ and affine Demazure characters in simply-laced types. However, this result does not work in type C; in addition, it only gives an equality of polynomials (the generating functions for the statistics and the weights), not of individual terms.

To compare our work with the previous papers on charge and the energy, let us first say that our results apply to arbitrary vertices in a tensor product of KN columns, not just to the highest weight elements (with respect to the nonzero arrows), that are used in the work involving Kostka–Foulkes polynomials. In type A, our approach via affine Demazure crystals comes closest to [19, 44, 45]. However, in addition to the aspect mentioned above, it differs from the approach in these papers because we do not use the so-called cyclage operation, which is based on the corresponding plactic relations, see [24]. These relations are the main cause of the complications in type C, in the work of Lecouvey [24, 25].

The paper is organized as follows. In Section 2 we review the necessary crystal theory and define the energy function. In Section 3 we give the definition of charge both in types A and C. The proof of Theorem 1.1 for type A using the method of Demazure arrows is given in Section 4. In Section 5 we classify the various components under the Demazure arrows in type C in the extension of the Kyoto path model, and show that each ground state is connected to a type A filling. These results are used in Section 6 to prove Theorem 1.1 for type C. We conclude in Section 7 with a discussion of various possible extensions of this work.

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2. Crystals and energy function

In this section we review and set up crystal theory and define the energy function.

2.1. **Crystal generalities.** Crystal bases provide a combinatorial method to study representations of quantum algebras $U_q(\mathfrak{g})$. For a good review on crystal base theory see the book by Hong and Kang [9]. Here $\mathfrak{g}$ is a Lie algebra or affine Kac–Moody Lie algebra with index set I, weight lattice P, and simple roots α_i with $i \in I$. The set of dominant weights is denoted by P_+. For affine Kac–Moody (resp. finite Lie) algebras we denoted the fundamental weights by Λ_i (resp. ω_i) for $i \in I$.

A $\mathfrak{g}$-*crystal* is a nonempty set B together with maps $e_i, f_i : B \to B \cup \{\emptyset\}$ for $i \in I$ and $\mathrm{wt} : B \to P$. For $b \in B$, we set $\varepsilon_i(b) = \max\{k \mid e_i^k(b) \neq \emptyset\}$, $\varphi_i(b) = \max\{k \mid f_i^k(b) \neq \emptyset\}$,

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i.$$

The beauty about crystal theory is that it is well-behaved with respect to taking tensor products. Let B_1 and B_2 be two $\mathfrak{g}$-crystals. As a set $B_1 \otimes B_2$ is the Cartesian product of the two sets. For $b = b_1 \otimes b_2 \in B_1 \otimes B_2$, the weight function is simply $\mathrm{wt}(b) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$. The crystal operators are given by

$$f_i(b_1 \otimes b_2) = \begin{cases} f_i(b_1) \otimes b_2 & \text{if } \varepsilon_i(b_1) \geq \varphi_i(b_2), \\ b_1 \otimes f_i(b_2) & \text{otherwise,} \end{cases}$$

and similarly for $e_i(b)$. This rule can also be expressed combinatorially by the signature rule.

A *highest weight crystal* $B(\lambda)$ of highest weight $\lambda \in P_+$ is a crystal with a unique element u_λ such that $e_i(u_\lambda) = \emptyset$ for all $i \in I$ and $\mathrm{wt}(u_\lambda) = \lambda$. On finite-dimensional highest weight crystals $B(\lambda)$ there exists an involution $S : B(\lambda) \to B(\lambda)$, called the *Lusztig involution*, which is a crystal isomorphism such that

$$S(f_i) = e_{i^*} \quad \text{and} \quad S(e_i) = f_{i^*}.$$

Here i^* is defined through the map $\alpha_i \mapsto \alpha_{i^*} := -w_0(\alpha_i)$ with w_0 the longest element in the Weyl group of $\mathfrak{g}$. Explicitly, we have $i^* = n - i$ for type A_{n-1} and $i^* = i$ for type C_n. Under S the highest weight element goes to the lowest weight element.

In [8], Henriques and Kamnitzer defined a *crystal commutor* on the tensor product of two classically highest weight crystals in terms of Lusztig's involution as

$$\begin{aligned} B(\lambda) \otimes B(\mu) &\to B(\mu) \otimes B(\lambda) \\ b_1 \otimes b_2 &\mapsto S(S(b_2) \otimes S(b_1)). \end{aligned} \tag{2.1}$$

2.2. **Kashiwara–Nakashima columns for type C.** Kashiwara and Nakashima [17] developed general tableaux models for finite-dimensional highest weight crystals for all non-exceptional classical Lie algebras $\mathfrak{g}$. For type C_n, the Kashiwara–Nakashima (KN) columns [17] of height k index the vertices of the fundamental representation $V(\omega_k)$ of the symplectic algebra $\mathfrak{sp}_{2n}(\mathbb{C})$. These columns are filled with entries in $[\overline{n}] := \{1 < 2 < \cdots < n < \overline{n} < \overline{n-1} < \cdots < \overline{1}\}$.

Definition 2.1. *A column-strict filling $b = b(1) \dots b(k)$ with entries in $[\overline{n}]$ is a KN column if there is no pair $(z, \overline{z})$ of letters in b such that:*

$$z = b(p)\,, \qquad \overline{z} = b(q)\,, \qquad q - p \leq k - z\,.$$

We will need a different definition of KN columns, which was proved to be equivalent to the one above in [43].

Definition 2.2. *Let b be a column and $I = \{z_1 > \cdots > z_r\}$ the set of unbarred letters z such that the pair $(z, \overline{z})$ occurs in b. The column b can be split when there exists a set of r unbarred letters $J = \{t_1 > \cdots > t_r\} \subset [n]$ such that:*

- *t_1 is the greatest letter in $[n]$ satisfying: $t_1 < z_1$, $t_1 \notin b$, and $\overline{t_1} \notin b$,*
- *for $i = 2, \ldots, r$, the letter t_i is the greatest one in $[n]$ satisfying $t_i < \min(t_{i-1}, z_i)$, $t_i \notin b$, and $\overline{t_i} \notin b$.*

In this case we write:

- *b^R for the column obtained by changing $\overline{z_i}$ into $\overline{t_i}$ in b for each letter $z_i \in I$, and by reordering if necessary,*
- *b^L for the column obtained by changing z_i into t_i in b for each letter $z_i \in I$, and by reordering if necessary.*

The pair $b^L b^R$ will be called a split column.

Example 2.3. The following is a KN column of height 5 in type C_n for $n \geq 5$, together with the corresponding split column:

$$b = \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline \overline{5} \\ \hline \overline{4} \\ \hline \overline{3} \\ \hline \end{array}, \qquad b^L b^R = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline \overline{5} & \overline{3} \\ \hline \overline{4} & \overline{2} \\ \hline \overline{3} & \overline{1} \\ \hline \end{array}.$$

We used the fact that $\{z_1 > z_2\} = \{5 > 4\}$, so $\{t_1 > t_2\} = \{2 > 1\}$.

We will consider Definition 2.2 as the definition of KN columns.

2.3. **Kirillov–Reshetikhin crystals.** For the definition of the crystal energy function, we need to endow the KN columns with an affine crystal structure. These finite-dimensional affine crystals are called *Kirillov–Reshetikhin (KR) crystals*. Combinatorial models for all non-exceptional types were provided in [5].

Here we only describe the KR crystals $B^{r,1}$ for types $A^{(1)}_{n-1}$ and $C^{(1)}_n$, where $r \in \{1, 2, \ldots, n-1\}$ and $r \in \{1, 2, \ldots, n\}$, respectively. As a classical type A_{n-1} (resp. C_n) crystal, the KR crystal is isomorphic to

$$B^{r,1} \cong B(\omega_r).$$

The crystal operator f_0 is given as follows. Let $b \in B^{k,1}$, represented by a one-column KN tableau. In type A_{n-1}, if b contains the letter n and no 1, $f_0(b)$ is the obtained from b by removing n and adding 1 to the column, leaving all letters in strictly increasing order. Otherwise $f_0(b) = \emptyset$. In type C_n, if b contains the letter $\overline{1}$, then $f_0(b)$ is obtained from b by removing the $\overline{1}$ and adding the letter 1, arranging all letters again in strictly increasing order. Otherwise $f_0(b) = \emptyset$. Note that if b contains $\overline{1}$, then it cannot contain 1 by the KN condition of Definition 2.1.

Similarly, in type A_{n-1}, $e_0(b)$ changes a 1 into n if 1 is in b and n is not, and otherwise $e_0(b) = \emptyset$. In type C_n, $e_0(b)$ is obtained from b by changing a 1 into a $\overline{1}$ if it exists, and $e_0(b) = \emptyset$ otherwise.

2.4. **The D function.** Now we come to the definition of the energy function D on tensor products of KR crystals $B^{r,1}$ of type $A^{(1)}_{n-1}$ or $C^{(1)}_n$. It is defined by summing up combinatorially defined "local" energy contributions using the combinatorial R-matrix.

Let B_1, B_2 be two affine crystals with generators v_1 and v_2, respectively, such that $B_1 \otimes B_2$ is connected and $v_1 \otimes v_2$ lies in a one-dimensional weight space. By [27, Proposition 3.8], this holds for any two KR crystals. The generator $v_{r,s}$ for the KR crystal $B^{r,s}$ is the unique element of classical weight $s\omega_r$.

The *combinatorial R-matrix* [14, Section 4] is the unique crystal isomorphism

$$\sigma : B_2 \otimes B_1 \to B_1 \otimes B_2.$$

By weight considerations, this must satisfy $\sigma(v_2 \otimes v_1) = v_1 \otimes v_2$.

As in [14] and [38, Theorem 2.4], there is a function $H = H_{B_2,B_1} : B_2 \otimes B_1 \to \mathbb{Z}$, unique up to a global additive constant, such that, for all $b_2 \in B_2$ and $b_1 \in B_1$,

$$H(e_i(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} -1 & \text{if } i = 0 \text{ and LL}, \\ 1 & \text{if } i = 0 \text{ and RR}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Here LL (resp. RR) indicates that e_0 acts on the left (resp. right) tensor factor in both $b_2 \otimes b_1$ and $\sigma(b_2 \otimes b_1)$. When B_1 and B_2 are KR crystals, we normalize H_{B_2,B_1} by requiring $H_{B_2,B_1}(v_2 \otimes v_1) = 0$, where v_1 and v_2 are the generators defined above. The map H is called the *local energy function*.

Definition 2.4. *For $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ of type $A^{(1)}_{n-1}$ or $C^{(1)}_n$, set*

$$H^R_{j,i} := H_i \sigma_{i+1} \sigma_{i+2} \cdots \sigma_{j-1} \quad \text{and} \quad H^L_{j,i} := H_{j-1} \sigma_{j-2} \sigma_{j-3} \cdots \sigma_i,$$

where σ_j and H_j act on the j-th and $(j+1)$-st tensor factors. We define a right and left energy function $D^R_B, D^L_B : B \to \mathbb{Z}$ as

$$D^R_B := \sum_{N \geq j > i \geq 1} H^R_{j,i} \quad \text{and} \quad D^L_B := \sum_{N \geq j > i \geq 1} H^L_{j,i}. \tag{2.3}$$

We set $D_B := D^L_B$ and, when there is no confusion, we shorten D_B to simply D; this is referred to as the energy function.

Remark 2.5. Note that the energies D^R_B and D^L_B can be defined for general tensor products of KR crystals. When the KR crystals decompose into several components as classical crystals (unlike in the cases for type $A^{(1)}_{n-1}$ and $C^{(1)}_n$ we consider), there is an extra term in the analogue of (2.3), see [38].

There is a precise relationship between D^R and D^L using the Lusztig involution. To state it, let us introduce the following map

$$\begin{aligned} \tau : \quad & B_N \otimes \cdots \otimes B_1 \to B_1 \otimes \cdots \otimes B_N \\ & b_N \otimes \cdots \otimes b_1 \mapsto S(b_1) \otimes \cdots \otimes S(b_N). \end{aligned}$$

For types $A^{(1)}_{n-1}$ and $C^{(1)}_n$ and $B_i = B^{r_i,1}$, the KR crystal B_i is connected as a classical crystal and under S the classically highest weight element u_i^{highest} maps to the lowest weight element

u_i^{lowest}. It is not hard to show from the explicit description of S, e_0 and f_0 in this case, that the following diagrams commute:

$$\begin{array}{ccc} B^{r,1} & \xrightarrow{f_0} & B^{r,1} \\ \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} \\ B^{r,1} & \xrightarrow[e_0]{} & B^{r,1} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} B & \xrightarrow{f_0} & B \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\ \widetilde{B} & \xrightarrow[e_0]{} & \widetilde{B} \end{array} \tag{2.4}$$

where $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ and $\widetilde{B} = B^{r_1,1} \otimes \cdots \otimes B^{r_N,1}$.

This shows in particular that the crystal commutor (2.1) is lifted to an affine crystal isomorphism in these cases and hence must coincide with the combinatorial R-matrix σ.

Proposition 2.6. *Let $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ of type $A_{n-1}^{(1)}$ or $C_n^{(1)}$ and $b \in B$. Then*

$$D_B^R(b) = D_B^L(\tau(b)). \tag{2.5}$$

Proof. Note that if $\sigma(b_1 \otimes b_2) = b_2' \otimes b_1'$, then $\sigma(S(b_2) \otimes S(b_1)) = S(b_1') \otimes S(b_2')$ since σ and the commutor (2.1) agree on two tensor factors under the conditions of the proposition by the above arguments. Hence the terms in the sum of D^R are in one-to-one correspondence with terms in D^L. Therefore it suffices to show that the local energy satisfies

$$H(b_2 \otimes b_1) = H(S(b_1) \otimes S(b_2)) \tag{2.6}$$

for $b_2 \otimes b_1 \in B^{r_2,1} \otimes B^{r_1,1}$. Since $S(u_1^{\mathrm{highest}}) \otimes S(u_2^{\mathrm{highest}}) = u_1^{\mathrm{lowest}} \otimes u_2^{\mathrm{lowest}}$ lies in the same component as $u_1^{\mathrm{highest}} \otimes u_2^{\mathrm{highest}}$, we have

$$H(u_2^{\mathrm{highest}} \otimes u_1^{\mathrm{highest}}) = H(S(u_1^{\mathrm{highest}}) \otimes S(u_2^{\mathrm{highest}})) = 0.$$

In the recursion (2.2), if for example e_0 acts LL on $b_2 \otimes b_1$, then f_0 acts RR on $S(b_1) \otimes S(b_2) = S(b_1) \otimes S(b_2)$ by (2.4). Hence H changes by the same amount in the left and right hand side of (2.6). The other cases can be checked analogously, which proves (2.6). □

2.5. D **energy as affine grading.** As suggested in [37, Section 2.5] and proven in [41], the energy D^R is the same as the affine degree grading in the associated highest weight affine crystals up to an overall shift. We will explain this now since it plays a crucial role in the proof of the equality between charge and energy.

We begin with the definition of "Demazure arrows". For this we need constants c_r for $r \in I \setminus \{0\}$ as for example defined in [6]. In the cases of concern to us here, we have $c_r = 1$ for all r in type $A_{n-1}^{(1)}$ and $c_r = 2$ for $1 \le r < n$ and $c_n = 1$ in type $C_n^{(1)}$.

Definition 2.7. *Let $B = B^{r_N,s_N} \otimes \cdots \otimes B^{r_1,s_1}$ be a tensor product of KR crystals and fix an integer ℓ such that $\ell \ge \lceil s_k/c_k \rceil$ for all $1 \le k \le N$. We call such a tensor product a composite KR crystal of level bounded by ℓ.*

An arrow f_i is called an ℓ-Demazure arrow on $b \in B$ if $\varphi_i(b) > 0$ and either $i \in I \setminus \{0\}$ or $i = 0$ and $\varepsilon_0(b) \ge \ell$.

In the setting of this paper, we are only concerned with tensor products of types $A_{n-1}^{(1)}$ and $C_n^{(1)}$ of the form $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$. In this case one can pick $\ell = 1$ and a *Demazure arrow* for B is a 1-Demazure arrow.

Lemma 2.8. *Let $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ of type $A_{n-1}^{(1)}$ or $C_n^{(1)}$ and $b \in B$. Then*

(1) $\varepsilon_0(b) \geq 1$ *implies* $D^R(f_0(b)) = D^R(b) + 1$;
(2) $\varphi_0(b) \geq 1$ *implies* $D^L(e_0(b)) = D^L(b) + 1$.

Proof. Part (1) follows directly from [41, Lemma 7.3]. For part (2), recall that by (2.4) and Proposition 2.6 we have $e_0(\tau(b)) = \tau(f_0(b))$ and $D^L(\tau(b)) = D^R(b)$. Also, setting $\tilde{b} = \tau(b)$, we have $\varphi_0(\tilde{b}) \geq 1$ if $\varepsilon_0(b) \geq 1$; thus, by using part (1), we deduce

$$D^L(e_0(\tilde{b})) = D^L(e_0(\tau(b))) = D^L(\tau(f_0(b))) = D^R(f_0(b)) = D^R(b) + 1 = D^L(\tilde{b}) + 1\,,$$

which proves the claim. □

The proof of the following essentially appeared in [14, Proof of Theorem 4.4.1] and was spelled out in this precise form in [41, Proposition 8.1]. Here

$$P_\ell^+ = \{\lambda \in P^+ \mid \mathrm{lev}(\lambda) = \ell\},$$

where $\mathrm{lev}(\lambda) := \lambda(c)$ is the level of λ and c is the central element $c = \sum_{i \in I} a_i^\vee \alpha_i^\vee$.

Proposition 2.9. *For B a composite KR crystal of level bounded by ℓ,*

$$B \otimes B(\ell\Lambda_0) \cong \bigoplus_{\Lambda'} B(\Lambda'), \tag{2.7}$$

where the sum is over a finite collection of (not necessarily distinct) $\Lambda' \in P_\ell^+$.

In Section 5, we discuss in more detail the sum on the right hand side of (2.7) for type $C_n^{(1)}$ and $\ell = 1$.

Definition 2.10. *For each $b \in B$, let $u_b^{\ell\Lambda_0}$ be the unique element of B such that $u_b^{\ell\Lambda_0} \otimes u_{\ell\Lambda_0}$ is the highest weight in the component from Proposition* 2.9 *containing $b \otimes u_{\ell\Lambda_0}$.*

Define the function deg on a direct sum of highest weight crystals to be the basic grading on each component, with all highest weight elements placed in degree 0.

Corollary 2.11. *Choose an isomorphism $m : B \otimes B(\ell\Lambda_0) \cong \bigoplus_{\Lambda'} B(\Lambda')$. Then for all $b \in B$, we have $D(b) - D(u_b^{\ell\Lambda_0}) = \deg(m(b \otimes u_{\ell\Lambda_0}))$.*

Corollary 2.12. *The minimal number of e_0 in a string of e_i taking b to $u_b^{\ell\Lambda_0}$ is $D(b) - D(u_b^{\ell\Lambda_0})$.*

Remark 2.13. One special case of interest is when B is a tensor product of perfect KR crystals of level ℓ, and Λ also has level exactly ℓ. Then the right side of (2.7) is a single highest weight crystal and the isomorphism in Proposition 2.9 is used in the Kyoto path model [14]. Hence $u_b^{\ell\Lambda_0}$ does not in fact depend on b, simplifying Corollaries 2.11 and 2.12.

3. The charge construction

3.1. **The classical charge.** Let us start by recalling the construction of the classical charge of a word due to Lascoux and Schützenberger [20]. Assume that w is a word with letters in the alphabet $[n] := \{1, \ldots, n\}$ which has partition content, i.e., the number of j's is greater than or equal to the number of $j+1$'s, for each $j = 1, \ldots, n-1$. The statistic charge(w) is calculated as a sum based on the following algorithm. Scan the word starting from its right end, and select the numbers $1, 2, \ldots$ in this order, up to the largest possible k. We always pick the first available entry $j+1$ to the left of the previous entry j. Whenever there is no such entry, we pick the rightmost entry $j+1$, so we start scanning the word from its right end once again; in

this case, we also add $k-j$ to the sum that computes charge(w). At the end of this process, we remove the selected numbers and repeat the whole procedure until the word becomes empty.

Example 3.1. Consider the word $w = 11\mathbf{32214}323$, where the first group of selected numbers is shown in bold. The corresponding contribution to the charge is 1. After removing the bold numbers and another round of selections (again shown in bold), we have $1\mathbf{12}\mathbf{32}3$, so the contribution to the charge is 2. We are left with the word 123, whose contribution to the charge is $2+1=3$. So charge$(w) = 1+2+(2+1) = 6$.

We now reinterpret the classical charge as a statistic on a tensor product of type $A^{(1)}_{n-1}$ KR crystals. Such a crystal indexed by a column of height k is traditionally denoted $B^{k,1}$, and its vertices are indexed by increasing fillings of the mentioned column with integers in $[n]$. Given a partition μ (i.e., a dominant weight in the root system), let

$$B_\mu := \bigotimes_{i=1}^{\mu_1} B^{\mu'_i,1}\,, \tag{3.1}$$

where μ' is the conjugate partition to μ. This is simply the set of column-strict fillings of the Young diagram μ with integers in $[n]$. Note that unlike in Section 2, the tensor factors in (3.1) are ordered in weakly decreasing order.

Fix a filling b in B_μ written as a concatenation of columns $b_1 \dots b_{\mu_1}$. We attach to it a filling $c := \text{circ-ord}(b) = c_1 \dots c_{\mu_1}$ according to the following algorithm, which is based on the circular order $\prec_i$ on $[n]$ starting at i, namely $i \prec_i i+1 \prec_i \cdots \prec_i n \prec_i 1 \prec_i \cdots \prec_i i-1$.

Algorithm 3.2.
let $c_1 := b_1$*;*
for j *from* 2 *to* μ_1 *do*
  for i *from* 1 *to* μ'_j *do*
    let $c_j(i) := \min\left(b_j \setminus \{c_j(1), \dots, c_j(i-1)\},\, \prec_{c_{j-1}(i)}\right)$
  end do;
end do;
return $c := c_1 \dots c_{\mu_1}$.

Example 3.3. Algorithm 3.2 constructs the filling c from the filling b below. The bold entries in c are only relevant in Example 3.5 below.

$$b = \begin{array}{|c|c|c|c|}\hline 3&2&1&2\\\hline 5&3&2\\\cline{1-3} 6&4&4\\\cline{1-3}\end{array} \quad \text{and} \quad c = \begin{array}{|c|c|c|c|}\hline 3&3&\mathbf{4}&2\\\hline \mathbf{5}&2&2\\\cline{1-3} \mathbf{6}&\mathbf{4}&1\\\cline{1-3}\end{array}\,. \tag{3.2}$$

We introduce some terminology in order to reinterpret the classical charge in terms of a statistic on B_μ. Given the considered filling b in B_μ, we define its *charge word* as the biword cw(b) containing a biletter $\binom{k}{j}$ for each entry k in the column b_j of b. We order the biletters in the decreasing order of the k's, and for equal k's, in the decreasing order of j's. The obtained word formed by the lower letters j will be denoted by cw$_2(b)$. We refer to Example 3.5 for an illustration of the charge word. On the other hand, given the filling $c = c_1 \dots c_{\mu_1}$ constructed by Algorithm 3.2, we say that the cell γ in column c_j and row i is a descent if $c_j(i) > c_{j+1}(i)$, assuming that $c_{j+1}(i)$ is defined. Let Des(c) denote the set of descents in c. As usual, we define the arm length arm(γ) of a cell γ as the number of cells to its right.

It is not hard to see that Algorithm 3.2 for constructing c from b translates precisely into the selection algorithm which computes charge($\mathrm{cw}_2(b)$). More precisely, consider the ith sequence $1, 2, \ldots$ extracted from $\mathrm{cw}_2(b)$ (which turns out to have length μ_i), and the letter j in this sequence; then the top letter paired with the mentioned letter j in $\mathrm{cw}(b)$ is precisely the entry $c_j(i)$ in row i and column j of the filling c. In particular, the steps to the right in the ith iteration of the charge computation correspond precisely to the descents in the ith row of c, while the corresponding charge contributions and arm lengths coincide. We conclude that

$$\sum_{\gamma \in \mathrm{Des}(c)} \mathrm{arm}(\gamma) = \mathrm{charge}(\mathrm{cw}_2(b)) \,. \tag{3.3}$$

For simplicity, we set $\mathrm{charge}(b) := \mathrm{charge}(\mathrm{cw}_2(b))$.

Remark 3.4. In [29] we showed that the charge statistic on B_μ can be derived from the Ram–Yip formula [39] for the corresponding Macdonald polynomial at $t = 0$. In fact, we showed that Algorithm 3.2 is closely related to the corresponding quantum Bruhat graph (see, e.g., [2]). So we can conclude that this graph explains the charge construction itself. The mentioned idea was extended to type C, and it led to the definition of a type C charge, that we describe in Section 3.2.

Example 3.5. Note that $\mathrm{cw}_2(b)$ for b in Example 3.3 is precisely the word w in Example 3.1. In fact, the full biword $\mathrm{cw}(b)$ is shown below, using the order on the biletters specified above. The index attached to a lower letter is the number of the iteration in which the given letter is selected in the process of computing charge(b):

$$\mathrm{cw}(b) = \begin{pmatrix} 6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\ 1_3 & 1_2 & 3_1 & 2_3 & 2_1 & 1_1 & 4_1 & 3_2 & 2_2 & 3_3 \end{pmatrix} .$$

One can note the parallel between the mentioned selection process and the construction of c from b in Example 3.3. The entries in the cells of $\mathrm{Des}(c)$ are shown in bold in (3.2).

3.2. **The type C charge.** In this section we recall from [29] the construction of the type C charge. We start by fixing a dominant weight μ in the root system of type C_n. Let

$$B_\mu := \bigotimes_{i=1}^{\mu_1} B^{\mu'_i,1} \,, \tag{3.4}$$

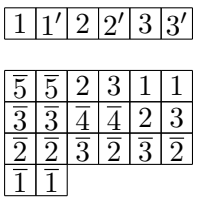
where $B^{k,1}$ is the type $C_n^{(1)}$ KR crystal indexed by a column of height k. Note that B_μ is the set of fillings $b = b_1 \ldots b_{\mu_1}$ of the shape μ with integers in $[\overline{n}]$ whose columns b_j are KN columns; indeed, the KN columns of height k label the vertices of $B^{k,1}$. As mentioned above, it will be more useful to represent b_j in the split form $b_j^L b_j^R$; in this case, b becomes a filling $b_1^L b_1^R \ldots b_{\mu_1}^L b_{\mu_1}^R$ of the shape 2μ.

Now fix a filling b in B_μ, represented with split columns, which are labeled from left to right by $1, 1', 2, 2', \ldots$. We can apply a slight modification of Algorithm 3.2 to b and obtain a filling $c = c_1^L c_1^R \ldots c_{\mu_1}^L c_{\mu_1}^R = \mathrm{circ\text{-}ord}(b)$ of 2μ; namely, we start by setting $c_1^L := b_1^L$, and then consider the (doubled) columns of b from left to right. We use the circular order on $[\overline{n}]$ starting at various values i, which we still denote by $\prec_i$.

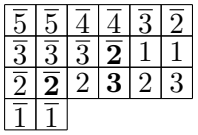
Example 3.6. Consider the following tensor product of KN columns:

$$\begin{array}{|c|}\hline \overline{5} \\ \hline \overline{3} \\ \hline \overline{2} \\ \hline \overline{1} \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 3 \\ \hline \overline{4} \\ \hline \overline{3} \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \overline{3} \\ \hline \end{array} .$$

This is represented with split columns as the following filling b of the shape $2\mu = (6,6,6,2)$:



where the top row consists of the column labels. The corresponding filling c is



Define the charge word $\mathrm{cw}(b)$ of b by analogy with type A, as the biword containing a biletter $\binom{k}{j}$ for each entry k in column j of b; here j and k belong to the alphabets $\{1 < 1' < 2 < 2' < \ldots\}$ and $[\overline{n}]$, respectively. We order the biletters as in the type A case (in the decreasing order of the k's, and for equal k's, in the decreasing order of j's), and define $\mathrm{cw}_2(b)$ in the same way (as the word formed by the lower letters j).

The modification of Algorithm 3.2 for constructing c from b can be rephrased in terms of $\mathrm{cw}(b)$, as explained below; we will refer to this rephrasing as the charge algorithm. We start by scanning $\mathrm{cw}_2(b)$ from right to left and by selecting the entries $1, 1', 2, 2', \ldots, \mu_1, (\mu_1)'$ in this order, according to the following rule: always pick the first available entry to the left, but if the desired entry is not available then scan the word from its right end once again. As in type A, we can see that the sequence of top letters paired with $1, 1', 2, 2', \ldots, \mu_1, (\mu_1)'$ is the first row of the filling c (read from right to left). We then remove the selected entries from $\mathrm{cw}_2(b)$ and repeat the above procedure, which will now give the other rows of c, from top to bottom. It was shown in [29] that we always go left from j to j', but we can go right from j' to $j+1$.

Example 3.7. This is a continuation of Example 3.6. The charge word $\mathrm{cw}(b)$, with the order on the biletters indicated above, is

$$\begin{pmatrix} \overline{1} & \overline{1} & \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{3} & \overline{3} & \overline{3} & \overline{3} & \overline{4} & \overline{4} & \overline{5} & \overline{5} & 3 & 3 & 2 & 2 & 1 & 1 \\ 1'_4 & 1_4 & 3'_1 & 2'_2 & 1'_3 & 1_3 & 3_1 & 2_2 & 1'_2 & 1_2 & 2'_1 & 2_1 & 1'_1 & 1_1 & 3'_3 & 2'_3 & 3_3 & 2_3 & 3'_2 & 3_2 \end{pmatrix}.$$

The index attached to a lower letter is the number of the iteration in which the given letter is selected by the charge algorithm.

Descents are defined as usual, cf. Section 3.1. It is easy to see that the descents in c correspond to the steps to the right in the charge algorithm applied to $\mathrm{cw}_2(b)$. By an observation made above, we only have descents of the form $c_j^R(i) > c_{j+1}^L(i)$. We are led to the following definition of the type C charge.

Definition 3.8. *Consider a word w with letters in the alphabet $1, 1', 2, 2', \ldots$, containing as many letters j as j', and at least as many letters j as $j+1$. Apply the charge algorithm to w, and assume that a selected entry j' is always to the left of the previously selected j. Let* $\mathrm{charge}(w)$ *be the sum of $k-j$ for each selected entry $j+1$ to the right of the previously selected j', where the selected entries in the given iteration are $1, 1', \ldots, k, k'$.*

The above discussion leads to the following result:

$$\frac{1}{2}\sum_{\gamma\in \mathrm{Des}(c)} \mathrm{arm}(\gamma) = \mathrm{charge}(\mathrm{cw}_2(b))\,. \tag{3.5}$$

For simplicity, we again set $\mathrm{charge}(b) := \mathrm{charge}(\mathrm{cw}_2(b))$.

Example 3.9. This is still a continuation of Example 3.6. The entries in the descents of c are shown above in bold. Correspondingly, the charge algorithm applied to $\mathrm{cw}_2(b)$ makes one step to the right in the second iteration (from $2'$ to 3), and two steps to the right in the third iteration (from $1'$ to 2 and from $2'$ to 3). Thus, $\mathrm{charge}(b) = 1 + (2 + 1) = 4$.

4. Energy and charge in type A

In this section we rederive the result of Nakayashiki–Yamada [36] showing the equality of the energy function and charge in type A_{n-1}. We do this in a more conceptual way, by using the method of "Demazure arrows" (the proof in [36] is based on subtle combinatorics of Young tableaux). Furthermore, we work with all the crystal vertices in a tensor product of columns, not just the highest weight vertices considered in [36]. Note that the setup in the mentioned paper is that of the right energy and a tensor product of columns of increasing heights which, by Proposition 2.6, is equivalent to the setup in this paper.

We start by studying the behavior of the type A charge with respect to the crystal operators.

Proposition 4.1. *The type A_{n-1} charge is preserved by the crystal operators $f_1, \ldots, f_{n-1}$.*

Proof. Let b be a tensor product of columns in some B_μ (see Section 3.1 and the terminology therein, which we use freely). It is known that the word $\mathrm{cw}_2(f_i(b))$ is in the same plactic equivalence class (see, e.g., [21]) as $\mathrm{cw}_2(b)$. More precisely, the former is obtained from the latter by considering its subword formed by the letters x corresponding to biletters $\binom{i+1}{x}$ or $\binom{i}{x}$ in $\mathrm{cw}(b)$, by viewing this subword as (the column word of) a skew tableau with two columns, and by using jeu de taquin to move a letter from the right column to the left one. This is explained in detail in [22, Section 2], based on the notion of "double crystal graphs". Then we use the well-known fact that the classical charge is preserved by the plactic relations (see, e.g., [21, Lemma 6.6.6 (ii)]). □

Proposition 4.2. *Let $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ be of type $A^{(1)}_{n-1}$ with $r_N \geq r_{N-1} \geq \cdots \geq r_1$ and $b \in B$. If $\varphi_0(b) \geq 1$ and $\varepsilon_0(b) \geq 1$, then the type A_{n-1} charge satisfies*

$$\mathrm{charge}(e_0(b)) = \mathrm{charge}(b) - 1\,.$$

Proof. Let $b = b_1 \ldots b_{\mu_1}$, and assume that e_0 changes the entry 1 in b_j to n. The condition $\varphi_0(b) \geq 1$ implies $j > 1$. Let $c := \text{circ-ord}(b)$ with $c = c_1 \ldots c_{\mu_1}$, where $c_j(i) = 1$, and $d := \text{circ-ord}(e_0(b))$ with $d = d_1 \ldots d_{\mu_1}$ (recall that the map circ-ord is defined by Algorithm 3.2).

The main case. We start by assuming that $c_{j-1}(i) > 1$ and that the column c_{j+1}, if it exists, does not contain n in a row $k \geq i$. We claim that, in this case, $d_l(k) = c_l(k)$ for all k, l with the exception of $d_j(i) = n$. Indeed, all we need to check are the following, in this order: (1) the entries $d_j(k)$ for $k = 1, \ldots, i-1$ are as claimed, since the alternative, namely that one of them is n, would lead to a contradiction; (2) the value of $d_j(i)$ follows from the fact that $c_{j-1}(i) \neq 1$;

(3) the value of $d_{j+1}(i)$, if this entry exists, follows from the above condition on the column c_{j+1}. By the above facts, we have the same descents in the fillings c and d, with the exception of the descent $c_{j-1}(i) > c_j(i) = 1$, which corresponds to $d_j(i) = n > d_{j+1}(i)$, assuming that $d_{j+1}(i)$ exists. The difference between the arm lengths of the mentioned descents is 1, which concludes the proof by (3.3). The remaining part of the proof treats the exceptions to this case.

Exception 1. Assume that $c_{j-1}(i) = 1$. Then the column b_{j-1} must contain n (otherwise the entry 1 in b_j would not be the leftmost unpaired 1). Assuming that $c_{j-1}(k) = n$, we must have $k > i$, by Algorithm 3.2 and the fact that $c_j(i) = 1$. The condition $\varphi_0(b) \geq 1$ implies $j > 2$, and we have $c_{j-2}(i) = 1$ (otherwise $c_{j-1}(i) = 1$ and $c_{j-1}(k) = n$ for $k > i$ are in contradiction with the way in which Algorithm 3.2 reorders the column b_{j-1}). This reasoning can be continued indefinitely, so this case is impossible.

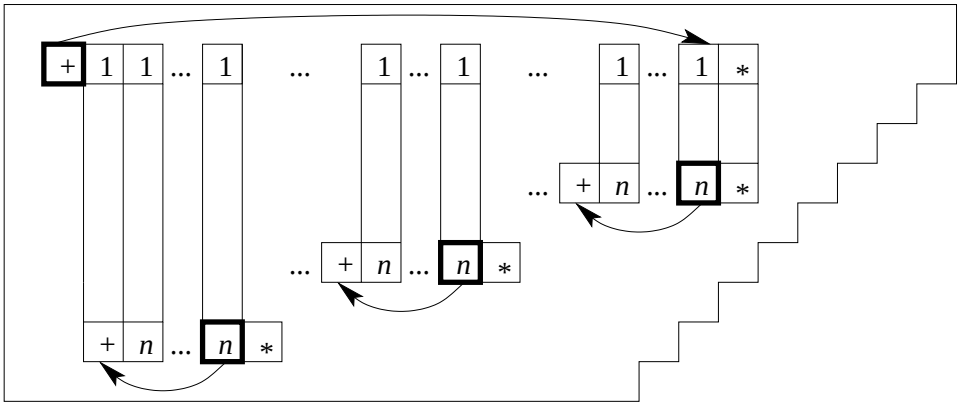
Exception 2. Assume that $c_{j-1}(i) > 1$, but $c_{j+1}(k) = n$ for some $k \geq i$. Then b_{j+1} must contain 1 as well, namely $c_{j+1}(l) = 1$ (otherwise the entry 1 in b_j would be paired with the entry n in b_{j+1}). We must have $l \leq i$, by Algorithm 3.2 and the fact that $c_j(i) = 1$. The above facts imply that $l < k$.

Exception 2.1. Assume that $l < i$. The fact that $c_j(l) > 1$, $c_{j+1}(l) = 1$, and $c_{j+1}(k) = n$ for $k > l$ are in contradiction with the way in which Algorithm 3.2 reorders the column b_{j+1}. So this case is impossible.

Exception 2.2. The only possibility left is that $c_{j+1}(i) = 1$ and $c_{j+1}(k) = n$ for $k = k_1 > i$. Let us assume for the moment that the second condition in Exception 2 holds for the column c_{j+2}, namely $c_{j+2}(k_2) = n$ for some $k_2 \geq i$. By the same reasoning as above, we deduce that $c_{j+2}(i) = 1$, so $k_2 > i$. In fact, we also have $k_1 \geq k_2$, by Algorithm 3.2. By continuing this reasoning, we obtain

$$c_{j+1}(i) = \ldots = c_{j+p}(i) = 1\,, \qquad c_{j+1}(k_1) = \ldots = c_{j+p}(k_p) = n\,, \quad \text{where } k_1 \geq \ldots \geq k_p\,;$$

on the other hand, we can assume that, if the column c_{j+p+1} exists, then it does not contain n in rows $i, i+1, \ldots$. This information about the filling c is represented in the figure below. The column c_j is the leftmost column with an entry 1 displayed, while $+$ and $*$ stand for entries different from 1 and n, respectively. The boxes shown in bold represent descents. By a reasoning similar to the main case above, we deduce that the only difference between the fillings c and d consists of the entries 1 and n in the figure changing to n and 1 in d, respectively. This leads to the marked descents moving to the boxes indicated by the arrows. It is now easy to see that the sum of the arm lengths of descents decreases by 1 when passing from c to d.



□

Proof of Theorem 1.1 *in type A.* The proof is immediate based on Corollary 2.11 and Propositions 4.1, 4.2 using the fact that KR crystals of type $A^{(1)}_{n-1}$ are perfect. □

5. Kyoto path model for nonperfect type C

In this section, we make Proposition 2.9 more explicit in the case of $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ and $\ell = 1$ for type $C^{(1)}_n$, by providing a correspondence between highest weight elements (or ground states) in $B \otimes B(\Lambda_0)$ and elements in $B(\Lambda')$ in the sum on the right hand side of (2.7), which are of type A. This will help in the next section to prove Theorem 1.1 for type $C^{(1)}_n$.

We call the highest weight elements in $B \otimes B(\Lambda_0)$ *ground state paths*. There is a recursive construction for them, which starts by listing all elements $b_1 \in B^{r_1,1}$ such that $\varepsilon(b_1) = \Lambda_0$. Suppose $b_k \otimes \cdots \otimes b_1 \in B^{r_k,1} \otimes \cdots \otimes B^{r_1,1}$ are already constructed. Then $b_{k+1} \in B^{r_{k+1},1}$ can be any of the elements such that $\varepsilon(b_{k+1}) = \varphi(b_k)$. The weight of the ground state is $\varphi(b_N)$, which is some fundamental weight Λ_h. For perfect crystals there are unique elements $b_N, \ldots, b_1$ with the described properties. However, in type $C^{(1)}_n$ the crystals $B^{r,1}$ are not perfect and the above construction gives a tree of ground state elements.

Example 5.1. Take $B = B^{1,1} \otimes B^{2,1} \otimes B^{2,1} \otimes B^{3,1}$ of type $C^{(1)}_3$. Then b_1 is the column 321 and b_2 the column $\overline{23}$. For b_3 there are two choices, namely the columns 32 or $\overline{2}2$. In the first case b_4 is $\overline{3}$, and in the second case b_4 can be 2 or $\overline{1}$. In summary the three ground states are

$$\boxed{\overline{3}} \otimes \boxed{\begin{array}{c}2\\3\end{array}} \otimes \boxed{\begin{array}{c}\overline{3}\\\overline{2}\end{array}} \otimes \boxed{\begin{array}{c}1\\2\\3\end{array}} \otimes u_{\Lambda_0} \qquad \boxed{2} \otimes \boxed{\begin{array}{c}2\\\overline{2}\end{array}} \otimes \boxed{\begin{array}{c}\overline{3}\\\overline{2}\end{array}} \otimes \boxed{\begin{array}{c}1\\2\\3\end{array}} \otimes u_{\Lambda_0} \qquad \boxed{\overline{1}} \otimes \boxed{\begin{array}{c}2\\\overline{2}\end{array}} \otimes \boxed{\begin{array}{c}\overline{3}\\\overline{2}\end{array}} \otimes \boxed{\begin{array}{c}1\\2\\3\end{array}} \otimes u_{\Lambda_0}\ .$$

The weights are Λ_2, Λ_2, and Λ_0, respectively.

Theorem 5.2. *Let $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ of type $C^{(1)}_n$. From each ground state $u \otimes u_{\Lambda_0} \in B \otimes B(\Lambda_0)$ there exists a sequence of Demazure arrows f_i (see Definition* 2.7*), which ends at an element $b \otimes u_{\Lambda_0}$ such that b does not contain any barred letter.*

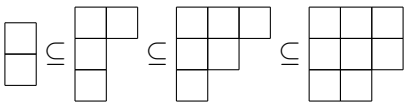
Note that in Theorem 5.2 there is no assumption on the order of the r_i.

In order to provide a proof of Theorem 5.2, we describe the explicit sequence of f_i satisfying the required conditions. For this we recursively define the following objects.

- Let $\lambda_0 \subseteq \lambda_1 \subseteq \cdots \subseteq \lambda_s$ (s is given below) be a sequence of shapes, where λ_0 is a single column of height h if the weight of the ground state path $u \otimes u_{\Lambda_0}$ is Λ_h. The other shapes are all of the form $\lambda(k, h_2, h_1)$, a partition with k columns of height n followed by two columns of heights $n > h_2 \geq h_1 \geq 0$, respectively. If $\lambda_j = \lambda(k, h_2, h_1)$, then $\lambda_{j+1} = \lambda(k, h_2 + 1, h_1 + 1)$, where we identify $\lambda(k, h_2 + 1, h_1 + 1) \cong \lambda(k + 1, h_1 + 1, 0)$ if $h_2 + 1 = n$ and $h_1 + 1 < n$, and $\lambda(k + 2, 0, 0)$ if $h_2 + 1 = h_1 + 1 = n$. This adds one horizontal domino in the consecutive rightmost columns up to height n.
- We recursively define
$$v_{j+1} := F_j(v_j)$$
with $v_0 = u$ the ground state and F_j the sequence
$$F_j := f_0^{k+1} f_1^{2k+2} \cdots f_{h_1}^{2k+2} f_{h_1+1}^{2k+1} \cdots f_{h_2}^{2k+1} f_{h_2+1}^{2k} \cdots f_{n-1}^{2k} f_n^k\,,$$

if $\lambda_j = \lambda(k, h_2, h_1)$. We continue doing this as long as possible, in other words, until $F_s(v_s)$ is undefined.

Example 5.3. Let $u \otimes u_{\Lambda_0}$ be the second ground state from Example 5.1. Then the sequence of $\lambda_0 \subseteq \lambda_1 \subseteq \cdots \subseteq \lambda_s$ is



with

$$F_0 = f_0 f_1 f_2$$
$$F_1 = f_0^2 f_1^3 f_2^2 f_3$$
$$F_2 = f_0^2 f_1^4 f_2^3 f_3.$$

Let $\lambda_j = (\lambda_{j1} \geq \lambda_{j2} \geq \ldots)$, and consider the conjugate partition $\lambda'_j = (\lambda'_{j1} \geq \lambda'_{j2} \geq \ldots)$. We will now explain how to represent an element v_j by a collection of non-crossing lattice paths (which might touch each other), see Figure 1. The paths have the following types of steps: down (southeast), up (northeast), and horizontal (east). More precisely, the paths in the collection $P_j = \{p_1, \ldots, p_l\}$ representing v_j correspond to the $l = \lambda_{j1}$ columns of λ_j, and they satisfy the conditions below.

(1) The endpoints of $p_1, \ldots, p_l$ are aligned at height 0 from right to left, and p_i starts at height λ'_{ji}.
(2) The paths $p_2, \ldots, p_l$ and the segment of p_1 before the end of p_2 (in case p_2 exists) consist entirely of down and horizontal steps. Two up or down steps never lie below one another, and neither do only horizontal steps.
(3) The path p_i starts after p_{i+2} ends, for $i = 1, \ldots, l-2$.

The terms "before" and "after" refer here to the order of the corresponding x-coordinates, with equality allowed. Each down (resp. up) step starting at height i is labeled by a letter i (resp. $\overline{i}$). We define word(P_j) as the word obtained by reading the labels on the paths in P_j from left to right (recall that two labels never lie below one another).

Let us now explain the construction of the collections P_j. We start by defining P_0 as consisting of a single path: the one having the same word as $u = v_0$, ending at height 0, and having no horizontal steps; the word of u, denoted word(u), is defined as usual, by reading its columns from left to right, bottom to top. We construct the collections P_j recursively, via the following transformation rule $P \mapsto \mathrm{up}(P)$ on a collection of paths $P = (p_1, \ldots, p_l)$ satisfying conditions (1)-(3) above.

Rule 5.4.

- *Locate the leftmost up step in p_1, and let $\overline{y}$ be its label. Replace it with a horizontal step at height y and a down step below it ending at height* 0.
- *Shift up by* 1 *the segment of p_1 to the left of the position where the above change occurred, to connect it to the tail of p_1. Shift up by* 1 *all the other paths.*
- *Replace any down step above height n by a down step below it, ending at height* 0.
- *Consider the down steps ending at height* 0 *from right to left, excluding the rightmost one. Match them with the shifts of $p_2, \ldots, p_l$, in this order, and connect the matched pairs by horizontal lines of height* 1.

Note that, in the last step of the rule, the last one or two down steps might have no match, so they start new paths. For simplicity, any horizontal steps at the beginning of a path are ignored. It is easy to see that, since P satisfies conditions (1)-(3) above, the rule can be applied, and $\mathrm{up}(P)$ satisfies the same conditions. Thus, we can recursively define $P_{j+1} := \mathrm{up}(P_j)$ as long as there are up steps in P_j.

We claim that applying the rule to P_j corresponds to applying F_j to v_j. To make this precise, we introduce some notation. Let r be the collection of columns of $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$, i.e., the tuple $(r_N, \ldots, r_1)$. Given a word w of length $r_1 + \cdots + r_N$, we define a rearrangement of it $\mathrm{ord}_r(w)$ by slicing w into segments of length $r_N, \ldots, r_1$ in this order, and by reordering each segment decreasingly. We will show below that, for all $j = 0, \ldots, s$, we have

$$\mathrm{ord}_r(\mathrm{word}(P_j)) = \mathrm{word}(v_j)\,. \tag{5.1}$$

We will see that the reordering of the segments of $\mathrm{word}(P_j)$ done by ord_r, if any, is very simple: a segment is a concatenation of two decreasing segments which can be swapped to give a decreasing sequence. Based on the above discussion, one can see that λ_j is the weight of v_j, and that this element is of highest weight in its classical (non-affine) component.

Example 5.5. The collections of paths P_j corresponding to Example 5.3 is shown in Figure 1.

Proof of Theorem 5.2. It essentially suffices to show that $F_j(v_j)$ is defined if and only if $\mathrm{up}(P_j)$ is defined and that (5.1) holds, for all j (the only extra fact to check is that all arrows corresponding to F_j are Demazure arrows, see below). Indeed, Rule 5.4 can be applied as long as there are up steps in P_j, so the final element v_s will have only positive entries. We will first show that Rule 5.4 precisely describes the action of F_j on the word of P_j, viewed as an element of the tensor product $(B^{1,1})^{\otimes(r_1+\cdots+r_N)}$; more precisely, we have

$$F_j(\mathrm{word}(P_j)) = \mathrm{word}(P_{j+1})\,. \tag{5.2}$$

We will then show that the above action of F_j is completely similar to that on v_j, which will prove (5.1).

Let us describe the action of F_j on $\mathrm{word}(P_j)$. First note that the connected components of the paths in P_j correspond to bracketed units in the tensor product $(B^{1,1})^{\otimes(r_1+\cdots+r_N)}$ according to the signature rule for the application of the Kashiwara operator on tensor products of crystals. This means that any given operator f_i can only operate on the starting points of the connected paths in P_j. The application of f_n^k lifts all starting points of value n to $\overline{n}$. Note that by the weight $\lambda_j = \lambda(k, h_2, h_1)$ of v_j there are precisely k of them. Next f_{n-1}^{2k} lifts all k just created $\overline{n}$ to $\overline{n-1}$ and the k letters $n-1$ to the left of the leftmost up-step to n that were previously bracketed with the just lifted n etc.. Potentially there are two more connected path components with starting points at h_1 and h_2, hence the exponents of f_i with $i \leq h_1$ and $i \leq h_2$ are increased accordingly. In summary, the starting points n of path components are eventually lifted to $\overline{1}$ and then turned into 1 by f_0; the other positive letters to the left of the leftmost up step are all raised by one. In the rightmost path, this lifting process eventually reaches the leftmost up step $\overline{y}$ which is lifted to $\overline{y-1}$ by f_{y-1}. The steps to the right of the leftmost up step $\overline{y}$ are either $\overline{y+1}$ or y. Both would need an f_y to be lifted. Since the f_i in F_j are applied in decreasing order of the indices i, this cannot happen and hence the leftmost up step is lifted to $\overline{1}$ and then turned into 1 by f_0. Altogether, the changes are precisely as described in Rule 5.4.

Let us now compare the action of F_j on $\mathrm{word}(P_j)$, i.e., on $(B^{1,1})^{\otimes(r_1+\cdots+r_N)}$, with that on v_j, i.e., on $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$. Note that by the action of f_0 on columns rather than tensor

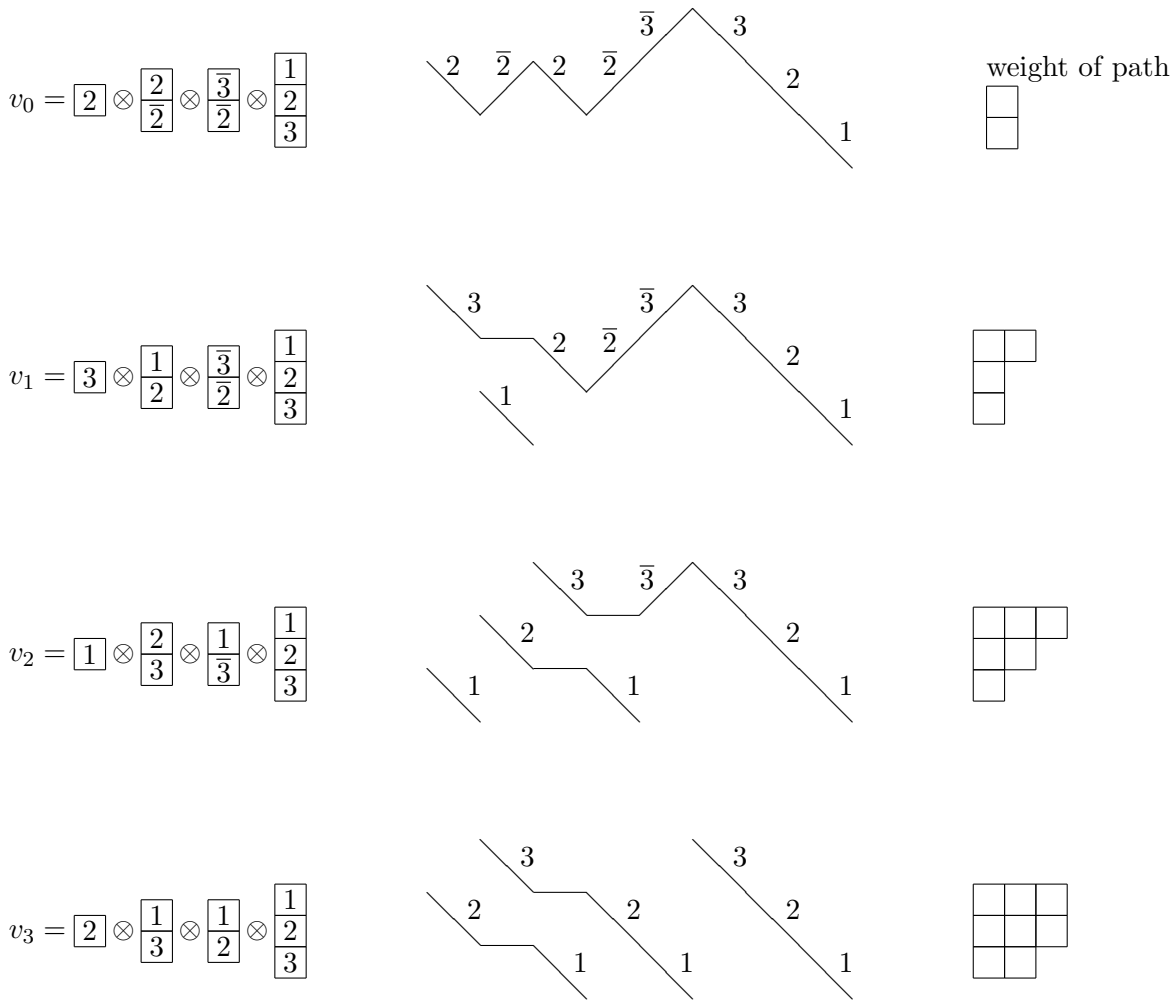


FIGURE 1. Paths for Example 5.5

products of single boxes, the word word(P_j) of a path is a concatenation of cyclically shifted columns of v_j. We claim that F_j acts on precisely the same entries in the two cases. In the expression for F_j the f_i with larger indices i act first and hence lift up the upper or barred portion of the paths. This is the case before or after reordering. f_0 changes a $\overline{1}$ into a 1 in both cases. Hence the action commutes with the reordering.

Finally, the rightmost tensor factor in v_0 is the column $c = r_1 \cdots 1 \in B^{r_1,1}$. Note that this column is never changed during the algorithm and satisfies $\varepsilon_0(c) = 1$. By the tensor product rules this implies that $\varepsilon_0(v_j) \geq 1$ for all $0 \leq j < s$, so that all arrows are Demazure arrows. □

There is a more direct way of constructing the type A elements b in Theorem 5.2 from the ground state $u \otimes u_{\Lambda_0}$. For $i = 1, 2, \ldots, N$, place the letter i in column 1 for each unbarred letter in u_i and in column 2 for each barred letter, where $u = u_N \otimes \cdots \otimes u_1$. Note that, due to the

fact that u is a ground state, the difference in height between the first and second column is at most n. Now cut the columns at heights $n, 2n, 3n, \ldots$ and put all pieces next to each other aligned at height 0. The letters in row i record the tensor factors of b in Theorem 5.2 which contain the letter i. Comparing this with the algorithm in the proof of Theorem 5.2, it is not hard to see that it gives the correct answer.

Example 5.6. Continuing Examples 5.3 and 5.5, the chosen ground state yields the columns

$$\begin{array}{|c|c|}\hline 4 & \\ \hline 3 & \\ \hline 1 & 3 \\ \hline 1 & 2 \\ \hline 1 & 2 \\ \hline \end{array} \qquad \text{and, after the cut,} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & \\ \hline 1 & 2 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}\,.$$

This tells us that the letter 1 appears in b_1, b_2, b_3, the letter 2 appears in b_1, b_2, b_4, and the letter 3 appear in b_1 and b_3. This agrees with Figure 1.

6. Energy and charge in type C

The purpose of this section is to provide the proof of Theorem 1.1 for type C.

6.1. **The energy for type A fillings.** We start with a brief discussion of the combinatorial R-matrix $B_1 \otimes B_2 \to B_2 \otimes B_1$ for a tensor product of two type C columns. By the type C Pieri rule [48], the decomposition of $B_1 \otimes B_2$ into classical (non-affine) components is multiplicity free. On the other hand, it was proved in [23] that the type C jeu de taquin due to Sheats [43] is compatible with the classical crystal operators. We conclude that the mentioned jeu de taquin, when applied to our two-column situation, realizes the corresponding combinatorial R-matrix. Note that, in all situations considered below, the type C jeu de taquin is governed by the simpler rules of the classical one, in type A (see, e.g., [7]).

Lemma 6.1. *Let $b = b_1 \otimes b_2$ be a tensor product of two columns with all entries in $[n]$, where the height of b_2 is at most that of b_1. Then the type $A^{(1)}_{n-1}$ and $C^{(1)}_n$ (local) energies of b coincide.*

Proof. Consider the crystal $B^{h_1,1} \otimes B^{h_2,1}$ of type $C^{(1)}_n$, where $n \geq h_1 \geq h_2 \geq 1$. By the type C Pieri rule, the classical components of this crystal containing fillings with all entries in $[n]$ have highest weights $\mu_i = (h_1 + i, h_2 - i)'$, for $i = 0, \ldots, \min(h_2, n - h_1)$. We will calculate the (local) energy on these components based on its definition (2.2), by setting the energy to 0 on the component of highest weight μ_0 (the usual normalization).

For $i = 1, \ldots, \min(h_2, n - h_1)$, consider $b_{1i} \otimes b_{2i}$ in $B^{h_1,1} \otimes B^{h_2,1}$, where

$$b_{1i} = \begin{array}{|c|}\hline n - h_1 + 1 \\ \hline \vdots \\ \hline n \\ \hline \end{array}\,, \qquad b_{2i} = \begin{array}{|c|}\hline 1 \\ \hline n - h_1 - i + 2 \\ \hline \vdots \\ \hline n - h_1 \\ \hline n - h_2 + 1 \\ \hline \vdots \\ \hline n - i \\ \hline \end{array}\,.$$

The image $b'_{2i} \otimes b'_{1i}$ of $b_{1i} \otimes b_{2i}$ under the combinatorial R-matrix, constructed with jeu de taquin, is given by

$$b'_{2i} = \begin{array}{|c|}\hline n-h_2+1 \\ \hline \vdots \\ \hline n \\ \hline \end{array}, \qquad b'_{1i} = \begin{array}{|c|}\hline 1 \\ \hline n-h_1-i+2 \\ \hline \vdots \\ \hline n-i \\ \hline \end{array}.$$

Clearly, e_0 acts RR on $b_{1i} \otimes b_{2i}$ and $b'_{2i} \otimes b'_{1i}$, by changing 1 to $\overline{1}$. By type C insertion [23], which in this case is essentially just type A insertion (see, e.g., [7]), it is easy to see that $b_{1i} \otimes b_{2i}$ and $e_0(b_{1i} \otimes b_{2i})$ lie in the components of highest weights μ_i and μ_{i-1}, respectively. Therefore, the energy on the component of highest weight μ_i is $-i$. This coincides with the type A energy (calculated via a similar procedure or via the type A charge). □

Proposition 6.2. *If b is a tensor product of columns with all entries in $[n]$, then the type A_{n-1} and C_n energies of b coincide. Furthermore, if the columns have weakly decreasing heights, they equal the (type A_{n-1} or C_n) charge of b.*

Proof. When all entries are in $[n]$, the jeu de taquin algorithms in types A_{n-1} and C_n (realizing the corresponding combinatorial R-matrices) work identically. Since the corresponding local energies coincide by Lemma 6.1, the global energies coincide as well. But the type A_{n-1} energy of b is computed by the type A_{n-1} charge when the heights of the columns are weakly decreasing by Theorem 1.1 which was proven in Section 4, which clearly coincides with the corresponding type C_n charge. □

6.2. **The conclusion of the proof.** We start by studying the behavior of the type C charge with respect to the crystal operators and state a result in [30].

Proposition 6.3. [30] *The type C_n charge is preserved by the crystal operators $f_1, \ldots, f_n$.*

Proposition 6.4. *Let $B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ be of type C_n with $r_N \geq r_{N-1} \geq \cdots \geq r_1$ and $b \in B$. If $\varphi_0(b) \geq 1$ and $\varepsilon_0(b) \geq 1$, then the type C_n charge satisfies*

$$\mathrm{charge}(e_0(b)) = \mathrm{charge}(b) - 1\,.$$

Proof. Let $b = b_1 \ldots b_{\mu_1}$, and note that none of these columns contain both 1 and $\overline{1}$. Assume that e_0 changes the entry 1 in b_j to $\overline{1}$. The condition $\varphi_0(b) \geq 1$ implies $j > 1$. Note that the column b_{j-1}, and thus b^R_{j-1}, cannot contain 1 (otherwise, the entry 1 in b_j would not be the leftmost unpaired 1). Similarly, the column b_{j+1}, and thus b^L_{j+1}, if they exist, cannot contain $\overline{1}$ (otherwise, the entry 1 in b_j would be paired with the mentioned $\overline{1}$).

Let $c := \text{circ-ord}(b)$ with $c = c^L_1 c^R_1 \ldots c^L_{\mu_1} c^R_{\mu_1}$, and $d := \text{circ-ord}(e_0(b))$ with $d = d^L_1 d^R_1 \ldots d^L_{\mu_1} d^R_{\mu_1}$ (recall that the map circ-ord is defined by a slight modification of Algorithm 3.2). Assume that $c^L_j(i) = 1$, which implies $c^R_j(i) = 1$ (as we have no descents in the pair $c^L_j c^R_j$, see Section 3.2). We claim that $d^L_l(k) = c^L_l(k)$ and $d^R_l(k) = c^R_l(k)$ for all k, l with the exception of $d^L_j(i) = d^R_j(i) = \overline{1}$. Indeed, all we need to check are the following, in this order: (1) the entries $d^L_j(k)$ and $d^R_j(k)$ for $k = 1, \ldots, i-1$ are as claimed, since the alternative, namely that one of them is $\overline{1}$, would lead to a contradiction; (2) the value of $d^L_j(i)$ follows from the fact that b^R_{j-1} contains no 1; (3) the value of $d^L_{j+1}(i)$, if this entry exists, follows from the fact that b^L_{j+1} contains no $\overline{1}$.

By the above facts, we have the same descents in the fillings c and d, with the exception of the descent $c^R_{j-1}(i) > c^L_j(i) = 1$, which corresponds to $d^R_j(i) = \overline{1} > d^L_{j+1}(i)$, assuming that

$d^L_{j+1}(i)$ exists. The difference between the arm lengths of the mentioned descents is 2, which concludes the proof by (3.5). □

Proof of Theorem 1.1 *in type C.* The proof is immediate based on Lemma 2.8, Theorem 5.2, and Propositions 6.2, 6.3, 6.4. □

7. Open problems

In this section we discuss several directions of research stemming from the results in this paper, which we intend to pursue in the future.

7.1. **Columns of types B and D.** We believe that, in types B_n and D_n, the statistic in the Ram–Yip formula for Macdonald polynomials at $t = 0$ can also be translated into a charge statistic on the corresponding tensor product of KR crystals $B^{k,1}$ of types $B^{(1)}_n$ and $D^{(1)}_n$ (represented with KN columns), thus extending the results in [29]. We conjecture that the type B_n and D_n charge also agrees with the corresponding energy function, which would extend the results in this paper. It should be possible to use a proof technique similar to the one in this paper, based on Lemma 2.8 from [41], to prove the conjecture; the proof would include the generalization of the results in [30] to types B_n and D_n. These claims are supported by [41, Corollary 9.5] expressing a type D_n Macdonald polynomial at $t = 0$ in terms of the corresponding energy function.

From one point of view, the case of types B_n and D_n is easier than the one of type C_n, because the corresponding level 1 KR crystals are perfect. However, the construction of charge in types B_n and D_n displays additional complexity, due to some new aspects, that we now describe. We start by referring to type B_n, as type D_n has all the complexity of type B_n plus additional one. For the corresponding KN columns, indexing the vertices of the crystals $B(\omega_i)$ corresponding to the fundamental representations $V(\omega_i)$, we refer to [17].

The first new aspect in type B_n is related to the splitting of the KR crystal $B^{k,1}$ upon removing the 0-arrows as the following direct sum of (classical) crystals: $B(\omega_k) \oplus B(\omega_{k-2}) \oplus \cdots$. This phenomenon manifests itself in the existence of descents between the left and the right columns of a split KN column, which was not the case in type C_n. We illustrate this based on the following example.

Example 7.1. Let $\mu = \omega_4$ in B_5. The KR crystal $B^{4,1}$ contains a vertex indexed by the KN column

$$b = \begin{array}{|c|}\hline 3 \\ \hline \overline{3} \\ \hline \end{array} \in B(\omega_2) \subset B^{4,1} \simeq B(\omega_4) \oplus B(\omega_2) \oplus B(\omega_0)\,.$$

Instead of constructing the split column, we first construct from b the following "extended" KN column of height 4, in order to make the columns corresponding to the components of $B^{4,1}$ of the same height:

$$\widehat{b} = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \overline{3} \\ \hline \overline{1} \\ \hline \end{array};$$

this construction is based on a procedure in [40, Section 3.4]. Only at this point we construct the split column. We claim that the doubling procedure in Definition 2.2 can be extended, but we still match the entries in $I = \{3 > 1\}$ with certain entries "preceding" them; the only

difference is that this now requires us to go counterclockwise around the circle. So we obtain $J = \{2, \overline{4}\}$, which gives the following splitting of $\widehat{b}$ by the usual rule:

$$\widehat{b}^L\widehat{b}^R = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline \overline{4} & 3 \\ \hline \overline{3} & 4 \\ \hline \overline{1} & \overline{2} \\ \hline \end{array}.$$

Finally, we construct $c = c^L c^R = \text{circ-ord}(\widehat{b}^L\widehat{b}^R)$ by the usual rule (see Section 3.2), where $c^L = \widehat{b}^L$:

$$c = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \overline{4} & \overline{2} \\ \hline \overline{3} & 1 \\ \hline \overline{1} & 4 \\ \hline \end{array}.$$

The energy of b in $B^{4,1}$ is -1. This agrees with the charge of $\widehat{b}^L\widehat{b}^R$, computed via the sum in (3.5); indeed, there are two descents in c, whose arm lengths are 1, so $\text{charge}(\widehat{b}^L\widehat{b}^R) = \frac{1}{2}(1+1) = 1$.

The second new aspect in type B_n is the fact that not always a descent in the filling obtained via the procedure "circ-ord" contributes half its arm length to the charge. We illustrate this with another example.

Example 7.2. Let $\mu = 2\omega_2$ in type B_3, so $B_\mu = B^{2,1} \otimes B^{2,1}$. Consider the pair of KN columns

$$b_1 b_2 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \overline{2} & 2 \\ \hline \end{array}$$

in B_μ. The corresponding fillings with split columns $b = b_1^L b_1^R b_2^L b_2^R$ and $c = \text{circ-ord}(b) = c_1^L c_1^R c_2^L c_2^R$ coincide, and they are

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline \overline{2} & \overline{2} & 2 & 2 \\ \hline \end{array}.$$

The corresponding energy is -2, but in c we have only one descent of arm length 2, so we cannot use the sum in (3.5) to compute the charge. This problem occurs because, for all the entries i between $\overline{2}$ and 2 in circular order (namely $i \in \{\overline{1}, 1\}$), either i or $\bar{\imath}$ are above $\overline{2}$ in c_1^R. This forces the descent $\overline{2} > 2$ to be double counted. A more illuminating explanation can be given in the context of deriving the charge from the statistic in the Ram–Yip formula, see [29, Section 9].

The above example suggests that in type B_n we need to modify the definition of charge by the sum in (3.5) as follows. We use the same notation $c = c_1^L c_1^R \cdots c_{\mu_1}^L c_{\mu_1}^R$ as above. Let $\text{Des}'(c)$ denote the descents of the form $\overline{m} = c_j^R(i) > c_{j+1}^L(i) = m$ such that, for any $k = 1, \ldots, m-1$, we have either k or $\overline{k}$ in $c_j^R[1, i-1]$. We claim that the appropriate definition of the type B_n charge is given by the following formula:

$$\frac{1}{2} \sum_{\gamma \in \text{Des}(c) \setminus \text{Des}'(c)} \text{arm}(\gamma) + \sum_{\gamma \in \text{Des}'(c)} \text{arm}(\gamma)\,. \tag{7.1}$$

We said above that in type D_n we have additional complexity still. We explain this using another example.

Example 7.3. Let $\mu = 2\omega_1$ in type D_3. The filling with split columns $c = b = b_1^L b_1^R b_2^L b_2^R = \begin{array}{|c|c|c|c|}\hline 3 & 3 & \overline{3} & \overline{3} \\ \hline \end{array}$ in B_μ has no descents, but the corresponding energy is -1. The reason for this is that the values 3 and $\overline{3}$ are incomparable in type D_3, so the pair $\begin{array}{|c|c|}\hline 3 & \overline{3} \\ \hline \end{array}$ in c needs to contribute 1

to the charge. In fact, the definition of charge needs to be adjusted even more, as the following more subtle example shows.

Let $\mu = 2\omega_2$ in D_4. Consider the following filling with split columns in B_μ:

$$c = b = b_1^L b_1^R b_2^L b_2^R = \begin{array}{|c|c|c|c|}\hline 3 & 3 & \overline{4} & \overline{4} \\ \hline \overline{4} & \overline{4} & \overline{3} & \overline{3} \\ \hline \end{array}\,.$$

This filling has only ascents or equal entries next to each other in a row. However, the corresponding energy is -1, so the charge needs to be 1. The reason for which the charge is not 0, meaning that b is not a split KN tableau, is that the two middle columns in b represent a forbidden configuration in type D_4. (Recall from [17] that, in type D, in addition to the row monotonicity condition for the split tableau in types B and C, there are extra conditions for a sequence of KN columns to form a KN tableau, and these are given in terms of certain forbidden configurations for a pair formed by a right column and the next left column.)

7.2. **Rows of types B, D, and C.** In type A, Nakayashiki and Yamada [36] showed that the Lascoux–Schützenberger charge statistic is related to the energy function on tensor products of both rows and columns. In particular, in [36, Proposition 3.23] they provide an explicit relation between the energy function on rows and columns by giving a bijection of each set to the set of semistandard Young tableaux which preserves the statistics.

We expect a similar result to hold in the other classical types (with slight modifications). This claim is motivated by [27, Theorem 10.10], which relates the one-dimensional configuration sum in the large rank limit for columns to the one for rows in the dual type. More explicitly, denote by $X^{Y_n}_{\lambda,B}(q)$ the one-dimensional configuration sum, that is, the sum of $q^{D(b)}$ over all highest weight elements b in the crystal B of weight λ and type Y_n, graded by the energy $D(b)$. Then up to an overall power of q, $X^{B_n}_{\lambda,B}(q)$ and $X^{C_n}_{\lambda',B'}(q^{-1})$ are equal for large rank n. Here, if $B = B^{r_1,s_1} \otimes \cdots \otimes B^{r_L,s_L}$ then $B' = B^{s_1,r_1} \otimes \cdots \otimes B^{s_L,r_L}$ and λ' is the transpose of λ (where we identify dominant weights with partitions). Note that $X^{B_n}_{\lambda,B}(q) = X^{D_n}_{\lambda,B}(q)$ when n is large [27, 49].

In particular, if the charge for columns of types B_n or D_n were known as outlined in Section 7.1 then, using $B = B^{r_1,1} \otimes \cdots \otimes B^{r_L,1}$, the relation between $X^{B_n}_{\lambda,B}(q)$ and $X^{C_n}_{\lambda',B'}(q^{-1})$ would provide a way to obtain the charge in type C_n also for single rows (as opposed to single columns as treated in this paper). Computer experiments using Sage [46] indicate that this duality not only holds on the level of configuration sums but also on the level of individual terms. We expect that the highest weight crystal elements for both columns and rows are in bijection with certain oscillating tableaux, as introduced in [1], due to the fact that the recording tableaux for the Robinson–Schensted algorithm for types B_n, C_n, D_n as introduced in [23, 26] are given by these tableaux. If this conjecture were to be true, then we would be able to obtain the charge for single rows from the charge for single columns via this bijection in the stable range (that is, large n).

7.3. **Arbitrary order of columns and arbitrary types.** Consider a composition $r = (r_N, \ldots, r_1)$ and the corresponding partition $\mu(r)$, obtained by ordering the parts r_i decreasingly. We refer to the crystal $B := B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ in types A and C, although we will shortly refer to arbitrary types, as well. If we want to compute the energy function on an element b in B, we can clearly apply the combinatorial R-matrix successively, in order to map b to an element of the crystal corresponding to $\mu(r)$; since the R-matrix preserves the energy, we can

use the charge statistic in this paper, which is defined on the latter crystal. We now propose a way to compute the charge directly on B.

The Ram–Yip formula for the Macdonald polynomial $P_{\mu(r)}(x;q,0)$ can be written in a way that is compatible with the composition r rather than the partition $\mu(r)$. Recall from [29, Proposition 2.7] that the terms in the above formula (for arbitrary types) correspond to chains of Weyl group elements giving rise to paths in the quantum Bruhat graph; these chains are partitioned into segments corresponding to the parts of the composition r. Like in the case of partitions, cf. [29], we have a bijection between the mentioned chains and the elements of the crystal B, realized as concatenations of KN columns; this leads to the obvious generalization of (1.1). The only difference is that, in general, the mentioned fillings do not retain enough information from the corresponding chains so that we can redefine the Ram–Yip statistic on them. Therefore, we need to work with the latter statistic. Note that its definition involves counting, with a certain multiplicity, the down steps in Bruhat order in a chain of Weyl group elements. It is this statistic that we now call charge. We conjecture that it agrees with the energy function on B, which would generalize the result in this paper.

Since the Ram–Yip formula works in arbitrary types, we can phrase the entire discussion above, including the conjecture, in this most general context. We believe that we can still use the general approach in this paper, based on Lemma 2.8 from [41]. However, the proof needs to be done in the setup of the alcove walk model rather than the KN tableau model, and this would increase its complexity.

Concretely, the following main facts would need to be addressed:

(1) there is a bijection between the elements of the crystal B and the relevant paths in the corresponding quantum Bruhat graph (this is a generalization of the results in [29]);
(2) generalizing the alcove model by defining the classical crystal operators, as well as the Demazure arrows (see Definition 2.7) on the above paths via the above bijection (this would generalize the results in [30]);
(3) the classical crystal operators preserve the Ram–Yip statistic, whereas the Demazure arrows change it as specified by Lemma 2.8 (this would generalize the results in Sections 4 and 6.2);
(4) the issue of non-perfect crystals needs to be addressed case by case, as in Section 5.

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