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Extended multirate infinitesimal step methods: Derivation of order conditions

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A B S T R A C T

Multirate methods are specially designed for problems with multiple time scales. The multirate infinitesimal step method (MIS) was developed as a generalization of the so called split-explicit Runge–Kutta methods, where the integration of the fast part is conducted analytically. The MIS method was originally evolved for applications related to numerical weather prediction, i.e. the integration of the compressible Euler equation.

In this work, an extension to MIS methods will be presented where an arbitrary Runge–Kutta method (RK) is applied for the integration of the fast component. Furthermore, the order convergence from the original MIS method will be reinvestigated including the derivation of conditions up to order four. Additionally will be presented how well-known methods such as recursive flux splitting multirate method, (Schlegel et al., 2012) partitioned Runge–Kutta method, (Jackiewicz and Vermiglio, 2000) and generalized additive Runge–Kutta method, (Sandu and Günther, 2015) are related to or can be cast as an extended MIS method. An exemplary MIS method of order four with five stages will show that the convergence behaviour not only depends on the applied method for the integration of the fast component. The method will further indicate that the used fast time step plays a significant role.

1. Introduction

Differential equations often exhibit solutions with different time scales. These are often inherited from the physical nature of the solution which for instance contains waves of disparate speed. Another source is the concurrent different processes like advection, diffusion and stiff chemical reactions in one equation. A further example is spacial discretization on anisotropic grids like stretched grids or local regions of refined grids.

Efficient time integration methods have to be taken into account for such phenomena and multirate methods with different time steps play an important role. The main idea in general is based on a splitting of an ordinary differential equation into multiple components. Numerical methods for such problems have been developed and investigated since 1980s, see e.g. [1–3]. However, there are various methods depending on the structure of the differential equation. For Runge–Kutta type methods, there is now existing a vast literature of multirate methods with different type of splitting, see e.g. [4–6]. Another approach is related to local time stepping methods where especially for hyperbolic problems each component is stepped forward with its own individual time step, see e.g. [7].

For an additive splitting of the right hand side, i.e.

$$\dot{y} = f\left(y\right) + g\left(y\right) = F\left(y\right), \qquad (1)$$

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with initial value $y(0) = y_0 \in \mathbb{R}^d$, Knoth and Wolke [8] proposed a multirate method which combines a Runge–Kutta method for the slow time scale and the analytic solution of an auxiliary ordinary differential equation with a frozen linear combination of the slow part. This construction principle inspired the development of further multirate integration methods. Wensch et al. [9] introduced a more general method named multirate infinitesimal step method (MIS) with focus on the solution of the compressible Euler equation in the low Mach number regime. This regime is typical for numerical weather prediction. As a special case, these methods include the so called split-explicit integration scheme, a common time integration scheme in numerical weather prediction [10,11].

In practical implementations, the exact solution has to be replaced by a finite integration scheme which introduces an additional error in the convergence behaviour of the whole algorithm. In [12], a method was presented where the auxiliary ordinary differential equation of the method proposed in [8] is integrated with an explicit Runge–Kutta method. In [13], an extension to the method proposed in [12] was presented where any RK method, explicit or implicit, with the first stage given in an explicit manner, has been used for the integration of the auxiliary ordinary equation. A second generalization was proposed in [14] where the linear combination coefficients of the slow part are defined as time dependent coefficient, i.e. as polynomials in time. Finally, there are other approaches where the slow integration method is of a different type, such as Adams–Bashforth methods, compare [15,16], or Peer-methods, see e.g. [17].

The aim of this paper is the presentation of an extended multirate infinitesimal step method extMIS where the auxiliary ordinary differential equation is solved with any arbitrary Runge–Kutta method (RK). Furthermore, it will be shown how this method is connected or can be reformulated to other existing multirate RK methods, including recursive flux splitting multirate method, partitioned Runge–Kutta method and generalized additive Runge–Kutta method. Following these connections to other methods, they will be used to present a general concept of deriving order conditions for order four or higher. Therefore, an extMIS method of order four will be suggested.

Moreover, it will be presented that the additional order conditions given in [9] are the only additional conditions up to order three. This is justified if the applied RK methods for the integration of the auxiliary ordinary differential equation are at least of the same order as the underlying explicit RK method.

An extend multirate infinitesimal step method of order four with five stages has been found for problems related to the compressible Euler equation. The details will be discussed with the cold bubble downburst benchmark example, see e.g [18,19] and [9,20].

2. Extended multirate infinitesimal step method (extMIS)

The newly derived extended MIS method is specially designed for problems of kind of Eq. (1). However, the method is based on the original MIS method proposed in [9], which is defined by the following algorithm.

$$Z_i(0) = y_n + \sum_{j=1}^{i-1} \left[\alpha_{ij} \left(Y_j - y_n\right)\right] \tag{2a}$$

$$\frac{\mathrm{d}Z_i(\tau)}{\mathrm{d}\tau} = \frac{1}{h} \sum_{j=1}^{i-1} \left[\gamma_{ij} \left(Y_j - y_n\right)\right] + \sum_{j=1}^{i-1} \left[\beta_{ij} f\left(Y_j\right)\right] + d_i g\left(Z_i(\tau)\right) \tag{2b}$$

$$Y_i = Z_i(h) \tag{2c}$$

$$i = 1, \ldots, s+1$$

$$y_{n+1} = Y_{s+1} \tag{2d}$$

In each MIS stage i an initial value problem has to be solved, where Eq. (2a) represents the initial value and Eq. (2b) the ordinary differential equation. The integration length for each MIS stage i is given by h, which is also the time step to advance from y_n to y_{n+1}. Although there are formally $s+1$ initial value problems, only s problems have to be solved in reality. For $i = 1$, $Y_1 = y_n$ since the parameters $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ are strictly lower triangular matrices with $(\boldsymbol{\alpha})_{ij} = \alpha_{ij}$, $(\boldsymbol{\gamma})_{ij} = \gamma_{ij}$ and $(\boldsymbol{\beta})_{ij} = \beta_{ij}$, compare with [9,20]. The method is said to be balanced, if $\mathbf{D}$ is a diagonal matrix storing the sum of rows of $\boldsymbol{\beta}$, i.e. $(\mathbf{D})_{ii} = d_i = \sum_{j=1}^{i-1} \left[\beta_{ij}\right]$, see [9, equation (2.3)].

In the following, the strong assumption that the solution of Eq. (2b) is given by an exact integration will be weakened due to integrating with an arbitrary Runge–Kutta method for each MIS stage i. The chosen RK method can be either an explicit or implicit method.

Throughout this paper, subscripted indices will be used for the stages of the MIS method and superscripted indices for the integration of the auxiliary ordinary methods. The Butcher tableau

$$\begin{array}{c|c} \mathbf{c}_i & \mathbf{A}_i \\ \hline & \mathbf{b}_i^T \end{array}$$

represents the utilized RK in stage i of the MIS method. Furthermore, $\mathbf{A}_i$ is the coefficient matrix and $(\mathbf{A}_i)_{\lambda l} = a_i^{\lambda l}$ represents the RK coefficient of stage l while integrating the auxiliary ordinary equation in stage λ during the integration in MIS stage i. The same holds similarly for $\mathbf{b}_i$, i.e. $(\mathbf{b}_i)_l = b_i^l$ as well as for $\mathbf{c}_i$, i.e. $(\mathbf{c}_i)_\lambda = c_i^\lambda$.

The integration of Eq. (2b) in MIS stage i with an arbitrary RK is then given by the following algorithm

$$\begin{aligned} Z_i(0) &= y_n + \sum_{j=1}^{i-1}\left[\alpha_{ij}\left(Y_j - y_n\right)\right] \\ Z_i^{\lambda} &= Z_i(0) + h\sum_{l=1}^{s_i}\left[a_i^{\lambda l}\left(\frac{1}{h}\sum_{j=1}^{i-1}\left[\gamma_{ij}\left(Y_j - y_n\right)\right] + \sum_{j=1}^{i-1}\left[\beta_{ij} f\left(Y_j\right)\right] + d_i g\left(Z_i^l\right)\right)\right] \\ Z_i(h) &= Z_i(0) + h\sum_{\lambda=1}^{s_i}\left[b_i^{\lambda}\left(\frac{1}{h}\sum_{j=1}^{i-1}\left[\gamma_{ij}\left(Y_j - y_n\right)\right] + \sum_{j=1}^{i-1}\left[\beta_{ij} f\left(Y_j\right)\right] + d_i g\left(Z_i^{\lambda}\right)\right)\right] \end{aligned}$$

with time step h. λ is the index for the number of stages s_i of the applied RK method. This represents the integration of Eq. (2b) in one RK step. Allowing an even smaller time step, i.e. micro time step $h_i = \frac{h}{m_i}$ with m_i being the number of steps in MIS stage i, the algorithm reads then

$$\begin{aligned} Z_i(0) &= y_n + \sum_{j=1}^{i-1}\left[\alpha_{ij}\left(Y_j - y_n\right)\right] \\ Z_i^{\mu,\lambda} &= Z_i\left((\mu-1)\cdot h_i\right) + h_i\sum_{l=1}^{s_i}\left[a_i^{\lambda l}\left(\frac{1}{h}\sum_{j=1}^{i-1}\left[\gamma_{ij}\left(Y_j - y_n\right)\right] + \sum_{j=1}^{i-1}\left[\beta_{ij} f\left(Y_j\right)\right] + d_i g\left(Z_i^{\mu,l}\right)\right)\right] \\ Z_i\left(\mu\cdot h_i\right) &= Z_i\left((\mu-1)\cdot h_i\right) + h_i\sum_{\lambda=1}^{s_i}\left[b_i^{\lambda}\left(\frac{1}{h}\sum_{j=1}^{i-1}\left[\gamma_{ij}\left(Y_j - y_n\right)\right] + \sum_{j=1}^{i-1}\left[\beta_{ij} f\left(Y_j\right)\right] + d_i g\left(Z_i^{\mu,\lambda}\right)\right)\right] \end{aligned}$$

where $\mu = 1, \ldots, m_i$. For writing purpose, a special choice for the RK method has been made, i.e. $a_i^{s_i+1,l} = b_i^l$ or given as Butcher tableau $\mathbf{c}_i \mid \mathbf{A}_i$.

Substituting the integration in Eq. (2) gives the new extMIS method in a recursive form

$$\begin{aligned} Z_i^{\mu,\lambda} = y_n &+ \sum_{j=1}^{i-1}\left[\left(\alpha_{ij} + \frac{h_i}{h}\left(\mu - 1 + c_i^{\lambda}\right)\gamma_{ij}\right)\left(Y_j - y_n\right)\right] + h_i\left(\mu - 1 + c_i^{\lambda}\right)\sum_{j=1}^{i-1}\left[\beta_{ij} f\left(Y_j\right)\right] \\ &+ h_i d_i \sum_{k=1}^{\mu-1}\sum_{l=1}^{s_i+1}\left[b_i^l g\left(Z_i^{k,l}\right)\right] + h_i d_i \sum_{l=1}^{s_i+1}\left[a_i^{\lambda l} g\left(Z_i^{\mu,l}\right)\right] \\ &\qquad \mu = 1, \ldots, m_i, \quad \lambda = 1, \ldots, s_i + 1 \end{aligned} \tag{3a}$$

$$Y_i = Z_i^{m_i, s_i+1} \tag{3b}$$

$$i = 1, \ldots, s+1$$

$$y_{n+1} = Y_{s+1}. \tag{3c}$$

Note that, the term $h_i\left(\mu - 1 + c_i^{\lambda}\right)$ defines the time point of stage $Z_i^{\mu,\lambda}$.

Remark. There are several indices applied in equations or summations throughout this paper. Each index is used only for a specific purpose. i represents the current MIS stage and if i is already used, then j is utilized. Furthermore, μ is the current time point while integrating the auxiliary ordinary differential equation and if μ is already applied, then k represents the current time point. Moreover, λ shows the current stage during the integration of the auxiliary ordinary differential equation and l is used if λ is already applied.

For the derivation of order conditions the recursive formulation of the stages Y_i is replaced with an explicit formulation. This will ease further calculations. Applying a simplified notation for Eq. (3b),

$$Y_i = Z_i^{m_i, s_i+1} = y_n + \sum_{j=1}^{i-1}\left[\left(\alpha_{ij} + \gamma_{ij}\right)\left(Y_j - y_n\right)\right] + h\sum_{j=1}^{i-1}\left[\beta_{ij} f\left(Y_j\right)\right] + h_i d_i \sum_{\mu=1}^{m_i}\sum_{\lambda=1}^{s_i+1}\left[b_i^{\lambda} g\left(Z_i^{\mu,\lambda}\right)\right], \tag{4}$$

through

$$\begin{aligned} U_i &= Y_i - y_n \\ \mathcal{A}_i &= h \sum_{j=1}^{i-1} \left[\beta_{ij} \cdot f\left(Y_j\right) \right] + h_i d_i \sum_{\mu=1}^{m_i} \sum_{\lambda=1}^{s_i+1} \left[b_i^{\lambda} g\left(Z_i^{\mu,\lambda} \right) \right] \\ \sigma_{ij} &= \alpha_{ij} + \gamma_{ij}. \end{aligned}$$

Eq. (4) is then reformulated to

$$U_i = \sum_{j=1}^{i-1} \left[\sigma_{ij} \cdot U_j \right] + \mathcal{A}_i$$

and in vector notation

$$\mathbf{U} = (\boldsymbol{\Sigma} \otimes \mathbf{I}_d)\, \mathbf{U} + \mathcal{A} = \left((\mathbf{I}_{s+1} - \boldsymbol{\Sigma}) \otimes \mathbf{I}_d \right)^{-1} \mathcal{A} = \left(\mathbf{R}^{-1} \otimes \mathbf{I}_d \right)^{-1} \mathcal{A} = (\mathbf{R} \otimes \mathbf{I}_d)\, \mathcal{A},$$

where $\mathbf{R} = (\mathbf{I}_{s+1} - \boldsymbol{\Sigma})^{-1} = (\mathbf{I}_{s+1} - \boldsymbol{\alpha} - \boldsymbol{\gamma})^{-1}$. Hence, an explicit form for Y_i is then given by

$$Y_i = y_n + h \sum_{j=1}^{i-1} \left[a_{ij} f\left(Y_j\right) \right] + \sum_{j=1}^{i} \sum_{\mu=1}^{m_j} \sum_{\lambda=1}^{s_j+1} \left[(\mathbf{RD})_{ij}\, h_j b_j^{\lambda} g\left(Z_j^{\mu,\lambda} \right) \right], \tag{5}$$

with the underlying explicit Runge–Kutta method (eRK) coefficient matrix $\mathbf{A} = \mathbf{R}\boldsymbol{\beta}$ and $\mathbf{b} = \mathbf{e}_{s+1}^T \mathbf{A}$. Note that the underlying eRK method is represented by the coefficients $(\mathbf{A})_{ij} = a_{ij}$.

Remark. For each (extended) MIS method, an eRK method is embedded, i.e. by setting $g \equiv 0$, then Eq. (5) reduces to the standard eRK method formulation. This property was given in [8,9].

Substituting Eq. (5) into Eq. (3a) returns an explicit formulation of the extMIS method from Eq. (3), i.e.

$$\begin{aligned} Z_i^{\mu,\lambda} &= y_n + h \sum_{j=1}^{i-1} \left[\left(\boldsymbol{\alpha}\mathbf{A} + \frac{1}{m_i} \left(\mu - 1 + c_i^{\lambda} \right) (\mathbf{A} - \boldsymbol{\alpha}\mathbf{A}) \right)_{ij} f\left(Y_j\right) \right] \\ &\quad + \sum_{j=1}^{i-1} \sum_{k=1}^{m_j} \sum_{l=1}^{s_j+1} \left[\left(\boldsymbol{\alpha}\mathbf{RD} + \frac{1}{m_i} \left(\mu - 1 + c_i^{\lambda} \right) \boldsymbol{\gamma}\mathbf{RD} \right)_{ij} h_j b_j^l g\left(Z_j^{k,l} \right) \right] + h_i d_i \sum_{k=1}^{\mu-1} \sum_{l=1}^{s_i+1} \left[b_i^l g\left(Z_i^{k,l} \right) \right] \\ &\quad + h_i d_i \sum_{l=1}^{s_i+1} \left[a_i^{\lambda l} g\left(Z_i^{\mu,l} \right) \right] & \text{(6a)} \\ Y_i &= Z_i^{m_i, s_i+1} & \text{(6b)} \\ y_{n+1} &= Y_{s+1}. & \text{(6c)} \end{aligned}$$

For further discussions, the special case with the assumption that the integration of the auxiliary ordinary differential equation in each MIS stage i will be performed with only one time step, i.e. $m_i \equiv 1$, is used. Hence, Eq. (6) reads

$$\begin{aligned} Z_i^{1,\lambda} &= y_n + h \sum_{j=1}^{i-1} \left[\left(\boldsymbol{\alpha}\mathbf{A} + c_i^{\lambda} (\mathbf{A} - \boldsymbol{\alpha}\mathbf{A}) \right)_{ij} f\left(Y_j\right) \right] \\ &\quad + h \sum_{j=1}^{i-1} \sum_{l=1}^{s_j+1} \left[\left(\boldsymbol{\alpha}\mathbf{RD} + c_i^{\lambda} \boldsymbol{\gamma}\mathbf{RD} \right)_{ij} b_j^l g\left(Z_j^{1,l} \right) \right] + h d_i \sum_{l=1}^{s_i+1} \left[a_i^{\lambda l} g\left(Z_i^{1,l} \right) \right] & \text{(7a)} \\ Y_i &= Z_i^{1, s_i+1} & \text{(7b)} \\ y_{n+1} &= Y_{s+1}. & \text{(7c)} \end{aligned}$$

Remark. Conducting the integration of the auxiliary ordinary differential equation m_i times can also be achieved by the composition of the applied Runge–Kutta method, see e.g. [21]. Hence, the number of time steps will then be equal to one.

Further, the RK method for the integration in MIS stage i will be assumed to be performed with time step $h_i = h$, i.e. $m_i = 1$, if not stated otherwise. In this case, the superscript μ of $Z_i^{\mu,\lambda}$ is replaced by 1, i.e. $Z_i^{1,\lambda}$.

3. Comparison to other methods

There are several well known methods, which are related to the extMIS method, e.g. recursive flux splitting multirate method (RFSMR), partitioned Runge–Kutta method (PRK) or generalized additive Runge–Kutta method (GARK). However, some of these connections are only valid for the method proposed in [8] which is a special case of the MIS method by setting

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ 0 & 1 & 0 & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma} = \mathbf{0} \quad \text{and} \quad \mathbf{R} = (\mathbf{I} - \boldsymbol{\alpha} - \boldsymbol{\gamma})^{-1} = \begin{pmatrix} 1 & & & & 0 \\ -1 & 1 & & & \\ 0 & -1 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix}, \tag{8}$$

see [9]. The abbreviation MIS-KW will be applied throughout the paper, whenever the special case with the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ as chosen in Eq. (8) is used.

3.1. extMIS as recursive flux splitting multirate method (RFSMR)

In [12], a method called recursive flux splitting multirate method (RFSMR) was proposed which is based on the MIS-KW method where the auxiliary ordinary differential equation (2b) was integrated with an explicit RK method in one time step, i.e. $m_i \equiv 1$. The same explicit Runge–Kutta method was applied in each MIS stage i, i.e. $\mathbf{A}_i$ is identical for all MIS stages. Substituting these properties into Eq. (7) returns the explicit formulation of the RFSMR method proposed by [22, equation (17)], i.e.

$$\begin{aligned} Z_i^{1,\lambda} &= y_n + h\sum_{j=1}^{i-1}\left[\left(a_{(i-1)j} + c_i^{\lambda}\left(a_{ij} - a_{(i-1)j}\right)\right) f\left(Y_j\right)\right] + h\sum_{j=1}^{i-1}\sum_{l=1}^{s_j}\left[d_j b_j^l g\left(Z_j^{1,l}\right)\right] \\ &\quad + h d_i \sum_{l=1}^{\lambda-1}\left[a_i^{\lambda l} g\left(Z_i^{1,l}\right)\right] && \text{(9a)} \\ Y_i &= Z_i^{1,s_i+1} && \text{(9b)} \\ y_{n+1} &= Y_{s+1}. && \text{(9c)} \end{aligned}$$

Hence, the RFSMR method is a special case of the extMIS method. Additionally, all the properties of the RFSMR method including the possibility to reformulate Eq. (9) as a PRK method are applicable as well as all applications suggested in [12,22,23].

Remark. One of the applications of the RFSMR method is a recursive applying of the same eRK method. Therefore, in case of the extMIS method, the chosen arbitrary eRK method for the integration of the auxiliary ordinary differential equation in each MIS stage is identical with the underlying eRK method of the MIS-KW method.

3.2. extMIS as partitioned Runge–Kutta method (PRK)

Since Eq. (9) may be rewritten as a PRK method, see [12, equation (17-23)], the likewise general validity of this reformulation for the extMIS method will be shown. Hence, the following demonstrates how the general extMIS method can be cast as a partitioned Runge–Kutta method (PRK). A PRK method for a split into two components problem, see Eq. (1), is given by

$$\begin{aligned} Y_q &= y_n + h\sum_{r=1}^{s_p+1}\left[a_{qr}^f f\left(Y_r\right)\right] + h\sum_{r=1}^{s_p+1}\left[a_{qr}^g g\left(Y_r\right)\right] \\ y_{n+1} &= Y_{s_p+1} \end{aligned}$$

with two Butcher tableau, $\mathbf{c}^f \mid \mathbf{A}^f$ and $\mathbf{c}^g \mid \mathbf{A}^g$, as well as number of stages s_p, see e.g. [24]. Note that $\mathbf{b}^f = \mathbf{e}_{s_p+1}^T \mathbf{A}^f$ and $\mathbf{b}^g = \mathbf{e}_{s_p+1}^T \mathbf{A}^g$ as well as $\left(\mathbf{c}^f\right)_{s_p+1} = 1$ and $(\mathbf{c}^g)_{s_p+1} = 1$.

In order to apply the correct size of an extMIS method cast as a PRK method, the number of stages are related by

$$s_p + 1 = \sum_{i=1}^{s+1}\left[m_i\left(s_i + 1\right)\right]. \tag{10}$$

Note that in this case $m_i \geq 1$ is assumed. The coefficient matrices $\mathbf{A}^f$ and $\mathbf{A}^g$ are then given by the following algorithm where the indices q and r are similarly derived to the number of stages, see Eq. (10),

$$(q,r) \longleftrightarrow \left(\sum_{o=1}^{i-1}\left[(s_o+1)\,m_o\right] + (\mu-1)\,(s_i+1)+\lambda,\ \sum_{o=1}^{j-1}\left[(s_o+1)\,m_o\right]+(k-1)\left(s_j+1\right)+l\right) \tag{11a}$$

$$a^f_{qr} = \begin{cases}\left(\boldsymbol{\alpha}\mathbf{A}+\frac{1}{m_i}\left(\mu-1+c_i^\lambda\right)(\mathbf{A}-\boldsymbol{\alpha}\mathbf{A})\right)_{ij} & ,j<i,\ k=m_j,\ l=s_j+1\\ 0 & ,\ \text{otherwise}\end{cases} \tag{11b}$$

$$a^g_{qr} = \begin{cases}\left(\boldsymbol{\alpha}\mathbf{RD}+\frac{1}{m_i}\left(\mu-1+c_i^\lambda\right)\boldsymbol{\gamma}\mathbf{RD}\right)_{ij}\frac{1}{m_j}b^l_j & ,j<i\\ \frac{1}{m_i}d_ib^l_i & ,j=i,\ k<\mu\\ \frac{1}{m_i}d_ia^{\lambda l}_i & ,j=i,\ k=\mu\\ 0 & ,\ \text{otherwise}\end{cases} \tag{11c}$$

Furthermore,

$$\mathbf{b}^f = \mathbf{e}^T_{s_p+1}\mathbf{A}^f = \left(\mathbf{0}^T_{m_1(s_1+1)-1}, b_1, \mathbf{0}^T_{m_2(s_2+1)-1}, b_2, \ldots, \mathbf{0}^T_{m_{s+1}\left(s_{s+1}+1\right)-1}, b_{s+1}\right)^T \tag{12a}$$

$$\mathbf{b}^g = \mathbf{e}^T_{s_p+1}\mathbf{A}^g = \left(\widetilde{b}_1\frac{1}{m_1}\mathbb{1}^T_{m_1}\otimes\mathbf{b}^T_1, \widetilde{b}_2\frac{1}{m_2}\mathbb{1}^T_{m_2}\otimes\mathbf{b}^T_2, \ldots, \widetilde{b}_{s+1}\frac{1}{m_{s+1}}\mathbb{1}^T_{m_{s+1}}\otimes\mathbf{b}^T_{s+1}\right)^T \tag{12b}$$

and

$$c^f_q = \sum_{j=1}^{i-1}\left[\left(\boldsymbol{\alpha}\mathbf{A}+\frac{1}{m_i}\left(\mu-1+c_i^\lambda\right)(\mathbf{A}-\boldsymbol{\alpha}\mathbf{A})\right)_{ij}\right] = \widetilde{c}_i + \frac{1}{m_i}\left(\mu-1+c_i^\lambda\right)(c_i-\widetilde{c}_i)$$

$$\begin{aligned} c^g_q &= \sum_{j=1}^{i-1}\sum_{k=1}^{m_j}\sum_{l=1}^{s_j+1}\left[\left(\boldsymbol{\alpha}\mathbf{RD}+\frac{1}{m_i}\left(\mu-1+c_i^\lambda\right)\boldsymbol{\gamma}\mathbf{RD}\right)_{ij}\frac{1}{m_j}b^l_j\right] + \sum_{k=1}^{\mu-1}\sum_{l=1}^{s_j+1}\left[\frac{1}{m_i}d_ib^l_i\right] + \sum_{l=1}^{s_i+1}\left[\frac{1}{m_i}d_ia^{\lambda l}_i\right]\\ &= \sum_{j=1}^{i-1}\left[\left(\boldsymbol{\alpha}\mathbf{RD}+\frac{1}{m_i}\left(\mu-1+c_i^\lambda\right)\boldsymbol{\gamma}\mathbf{RD}\right)_{ij}\right] + \frac{1}{m_i}d_i\left(\mu-1+c_i^\lambda\right) = \widetilde{c}_i + \frac{1}{m_i}\left(\mu-1+c_i^\lambda\right)(c_i-\widetilde{c}_i)\,,\end{aligned}$$

where $\widetilde{\mathbf{b}} = \mathbf{e}^T_{s+1}\mathbf{RD}$ and $\widetilde{\mathbf{c}} = \boldsymbol{\alpha}\mathbf{c}$ with $(\widetilde{\mathbf{c}})_i = \widetilde{c}_i$, compare [9]. Hence,

$$\mathbf{c}^f = \mathbf{c}^g, \tag{13}$$

which represents internal consistency.

3.3. extMIS as generalized additive Runge–Kutta method (GARK)

The GARK method was proposed in [25] as a more general approach to PRK methods. The connection for MIS as GARK has been suggested in [5]. The focus of that work was on the MIS-KW case.

Using the definition from [25, equation (2.5)], a GARK method for a split into two components problem is then given by

$$Y_i = y_n + h\sum_{i=1}^{s^p+1}\left[a^f_{ij}f\left(Y_j\right)\right] + h\sum_{i=1}^{s^p+1}\left[a^g_{ij}g\left(Y_j\right)\right] \tag{14a}$$

$$y_{n+1} = Y_{s^p+1} \tag{14b}$$

or by

$$Y_i = y_n + h\sum_{j=1}^{s^f}\left[a^{ff}_{ij}f\left(Y_j\right)\right] + h\sum_{j=1}^{s^g}\left[a^{fg}_{ij}g\left(Z_j\right)\right] \tag{15a}$$

$$Z_i = y_n + h\sum_{j=1}^{s^f}\left[a^{gf}_{ij}f\left(Y_j\right)\right] + h\sum_{j=1}^{s^g}\left[a^{gg}_{ij}g\left(Z_j\right)\right] \tag{15b}$$

$$y_{n+1} = y_n + h\sum_{i=1}^{s^f}\left[b^{ff}_if\left(Y_i\right)\right] + h\sum_{i=1}^{s^g}\left[b^{gg}_ig\left(Z_i\right)\right]. \tag{15c}$$

Eq. (14) is exactly the definition of a PRK method, see also [25, theorem 2.4] and compare with Eq. (11). Henceforth, the focus for this section will be on the definition given by Eq. (15). The extended Butcher tableau from [25, equation (2.4)] reads

$$\begin{array}{c|cc} \mathbf{c}^{ff} & \mathbf{A}^{ff} & \mathbf{A}^{fg} \\ \mathbf{c}^{gg} & \mathbf{A}^{gf} & \mathbf{A}^{gg} \\ \hline & \mathbf{b}^{ff} & \mathbf{b}^{gg} \end{array}.$$

The coefficient matrices $\mathbf{A}^{ff}$ and $\mathbf{A}^{gg}$ are for the integration of the f and g component, respectively. On the other hand, the matrices $\mathbf{A}^{fg}$ and $\mathbf{A}^{gf}$ represent the coupling between the f and g components. In each MIS stage i, an arbitrary RK method with time step $h_i = h$ is applied, i.e. $m_i \equiv 1$. Eq. (7) indicates how $\mathbf{A}^{ff} \in \mathbb{R}^{s^f+1\times s^f+1}$, $\mathbf{A}^{fg} \in \mathbb{R}^{s^f+1\times s^g+1}$, $\mathbf{A}^{gf} \in \mathbb{R}^{s^g+1\times s^f+1}$, and $\mathbf{A}^{gg} \in \mathbb{R}^{s^g+1\times s^g+1}$ have to be chosen, i.e.

$$\begin{aligned}
\mathbf{A}^{ff} &= \mathbf{A} \\
\mathbf{A}^{fg} &= \begin{pmatrix} (\mathbf{RD})_{11}\,\mathbf{b}_1^T & & \mathbf{0} \\ \vdots & \ddots & \\ (\mathbf{RD})_{s+1,1}\,\mathbf{b}_1^T & \cdots & (\mathbf{RD})_{s+1,s+1}\,\mathbf{b}_{s+1}^T \end{pmatrix} \\
\mathbf{A}^{gf} &= \begin{pmatrix} (\alpha\mathbf{A})_{11}\,\mathbb{1}_{s_1+1} & & \mathbf{0} \\ \vdots & \ddots & \\ (\alpha\mathbf{A})_{s+1,1}\,\mathbb{1}_{s_{s+1}+1} & \cdots & (\alpha\mathbf{A})_{s+1,s+1}\,\mathbb{1}_{s_{s+1}+1} \end{pmatrix} \\
&\quad + \begin{pmatrix} (\mathbf{A}-\alpha\mathbf{A})_{11}\,\mathbf{c}_1 & & \mathbf{0} \\ \vdots & \ddots & \\ (\mathbf{A}-\alpha\mathbf{A})_{s+1,1}\,\mathbf{c}_{s+1} & \cdots & (\mathbf{A}-\alpha\mathbf{A})_{s+1,s+1}\,\mathbf{c}_{s+1} \end{pmatrix} \\
\mathbf{A}^{gg} &= \begin{pmatrix} \mathbf{0} & & & \mathbf{0} \\ (\alpha\mathbf{RD})_{21}\,\mathbb{1}_{s_2+1}\mathbf{b}_1^T & \mathbf{0} & & \\ \vdots & \ddots & \ddots & \\ (\alpha\mathbf{RD})_{s+1,1}\,\mathbb{1}_{s_{s+1}+1}\mathbf{b}_1^T & \cdots & (\alpha\mathbf{RD})_{s+1,s}\,\mathbb{1}_{s_{s+1}+1}\mathbf{b}_s^T & \mathbf{0} \end{pmatrix} \\
&\quad + \begin{pmatrix} \mathbf{0} & & & \mathbf{0} \\ (\gamma\mathbf{RD})_{21}\,\mathbf{c}_2\mathbf{b}_1^T & \mathbf{0} & & \\ \vdots & \ddots & \ddots & \\ (\gamma\mathbf{RD})_{s+1,1}\,\mathbf{c}_{s+1}\mathbf{b}_1^T & \cdots & (\gamma\mathbf{RD})_{s+1,s}\,\mathbf{c}_{s+1}\mathbf{b}_s^T & \mathbf{0} \end{pmatrix} + \begin{pmatrix} d_1\mathbf{A}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & d_{s+1}\mathbf{A}_{s+1} \end{pmatrix}.
\end{aligned}$$

Remark. The matrices elements are given in the form of a multiplication of a scalar with either a vector or matrix, i.e. $(.)_{ij}$ denotes an element of a matrix.

As indicated by the extended Butcher tableau, the internal consistency is also fulfilled, i.e.

$$\begin{aligned}
\mathbf{c}^{ff} &= \mathbf{A}^{ff}\,\mathbb{1}_{s^f+1} = \mathbf{A}^{fg}\,\mathbb{1}_{s^g+1} = \mathbf{c} \\
\mathbf{c}^{gg} &= \mathbf{A}^{gf}\,\mathbb{1}_{s^f+1} = \mathbf{A}^{gg}\,\mathbb{1}_{s^g+1} = \left(\widetilde{c}_1\mathbb{1}_{s_1+1}^T + (c-\widetilde{c})_1\,\mathbf{c}_1{}^T, \ldots, \widetilde{c}_{s+1}\mathbb{1}_{s_{s+1}+1}^T + (c-\widetilde{c})_{s+1}\,\mathbf{c}_{s+1}{}^T\right)^T,
\end{aligned}$$

compare [25, equation (2.6)]. Finally,

$$\mathbf{b}^{ff} = \mathbf{b} \qquad \text{and} \qquad \mathbf{b}^{gg} = \left(\widetilde{b}_1\mathbf{b}_1^T, \widetilde{b}_2\mathbf{b}_2^T, \ldots, \widetilde{b}_{s+1}\mathbf{b}_{s+1}^T\right)^T.$$

Remark. Although, $m_i \equiv 1$ has been assumed, this was only made for writing purposes. The general extMIS method with $m_i \geq 1$ for stages i can also be cast as GARK method. The coefficient matrices $\mathbf{A}^{fg}$, $\mathbf{A}^{gf}$ and $\mathbf{A}^{gg}$ are then becoming larger and overcrowded.

Remark. Several other methods are related to extMIS methods. Examples are multirate generalized additive Runge–Kutta method (MGARK), [5,26], relaxed MIS methods (RMIS), [13] and multirate infinitesimal GARK schemes (MRI-GARK), [14]. However, some of these connections are only possible for the MIS-KW method.

4. Order conditions for extMIS

In this section, a general approach for the derivation of order conditions of extMIS methods will be presented. The reformulation to PRK methods is applied for the calculation. This is justified since every extMIS method can be represented as a PRK or GARK method. The order theory given in [25] is utilized. This work was based on the order theory using N-trees from [27]. All the conditions up to order four are given by [25, theorem 2.6]. Therefore, the conditions up to order four for the extMIS method will be calculated. Due to internal consistency, see Eq. (13), there are some conditions which are identical to others. This will reduce the number of conditions.

In the following, the abbreviations $f = f(y_n)$ and $g = g(y_n)$ as well as $F = f + g$ will be used. Moreover, the consistency for the utilized RK methods in MIS stage i will always be assumed, i.e.

$$\sum_{\lambda=1}^{s_i+1} \left[b_i^\lambda\right] = \mathbf{b}_i \cdot \mathbb{1}_{s_i+1} = 1.$$

Furthermore, if the applied RK methods are of sufficient order, then the conditions for the extMIS method simplify to the conditions of the general MIS method. Hence, this approach returns also the conditions for the MIS method from [9,20]. However, the conditions for order four have not been derived, yet.

4.1. Order $p = 1$

For order one, only two conditions are appearing. Using Eqs. (12) and (13), the following is gained

$$\begin{aligned}
&\langle f\rangle && \mathbf{b}^f \cdot \mathbb{1}_{s+1} = \mathbf{b} \cdot \mathbb{1}_{s+1} &&= 1\\
&\langle g\rangle && \mathbf{b}^g \cdot \mathbb{1}_{s+1} = \sum_{i=1}^{s+1}\left[\widetilde{b}_i \mathbf{b}_i \cdot \mathbb{1}_{s_i+1}\right] = \sum_{i=1}^{s+1}\left[\widetilde{b}_i\right] = \widetilde{\mathbf{b}} \cdot \mathbb{1}_{s+1} = \mathbf{e}_{s+1}^T \mathbf{RD} \mathbb{1}_{s+1} = \mathbf{e}_{s+1}^T \mathbf{c} = c_{s+1} &&= 1.
\end{aligned}$$

Both conditions are representing the same equation since by definition $c_{s+1} = \mathbf{b} \cdot \mathbb{1}_{s+1}$. Therefore, the remaining condition is the classical order one condition for an arbitrary RK method.

4.2. Order $p = 2$

Like for order one, only two conditions appear for order two. Using Eqs. (12) and (13), the following is achieved

$$\begin{aligned}
&f'\langle F\rangle && \mathbf{b}^f \cdot \mathbf{c}^f = \sum_{i=1}^{s+1}\left[b_i\left(\widetilde{c}_i + \tfrac{1}{m_i}\left(m_i - 1 + c_i^{s_i+1}\right)(c_i - \widetilde{c}_i)\right)\right] = \mathbf{b} \cdot \mathbf{c} &&= \tfrac{1}{2}\\
&g'\langle F\rangle && \mathbf{b}^g \cdot \mathbf{c}^g = \sum_{i=1}^{s+1}\sum_{\mu=1}^{m_i}\sum_{\lambda=1}^{s_i}\left[\widetilde{b}_i \tfrac{1}{m_i} b_i^\lambda \left(\widetilde{c}_i + \tfrac{1}{m_i}\left(\mu - 1 + c_i^\lambda\right)(c_i - \widetilde{c}_i)\right)\right] &&=\\
&&& \tfrac{1}{2}\widetilde{\mathbf{b}}^T(\mathbf{c} + \widetilde{\mathbf{c}}) + \sum_{i=1}^{s+1}\left[\widetilde{b}_i \tfrac{1}{m_i}\left(\mathbf{b}_i \cdot \mathbf{c}_i - \tfrac{1}{2}\right)(c_i - \widetilde{c}_i)\right] &&= \tfrac{1}{2}.
\end{aligned}$$

If $\mathbf{b}_i \cdot \mathbf{c}_i = \frac{1}{2}$ for every RK method in MIS stage i, which is the standard order condition for a RK method, the last condition reduces to the additional MIS condition for order two, see [9, equation 3.7], i.e.

$$\widetilde{\mathbf{b}}^T(\mathbf{c} + \widetilde{\mathbf{c}}) = 1.$$

Hence, for order two, the exact integration of the auxiliary ordinary differential equation of the original MIS method is not required, if every RK method in stage i is of order two.

Note that if a RK method of only order one in MIS stage i is chosen, then the arbitrary number of steps m_i linearly influences the method parameters $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$.

4.3. Order $p = 3$

For order three, the internal consistency, see Eq. (13), allows a reduction of the number of order conditions. Table 1 shows how the order conditions from a standard PRK method are modified.

Therefore, only six conditions remain,

$$\mathbf{b}^f \cdot \mathbf{c}^{f\,2} = \mathbf{b} \cdot \mathbf{c}^2 = \frac{1}{3} \tag{16a}$$

$$\begin{aligned}
\mathbf{b}^g \cdot \mathbf{c}^{g\,2} = {} & \frac{1}{3}\widetilde{\mathbf{b}}^T\left(\mathbf{c}^2 + \mathbf{c}\widetilde{\mathbf{c}} + \widetilde{\mathbf{c}}^2\right) + \sum_{i=1}^{s+1}\left[\widetilde{b}_i \frac{1}{m_i}\left(\mathbf{b}_i \cdot \mathbf{c}_i - \frac{1}{2}\right)\left(c_i^2 - \widetilde{c}_i^2\right)\right]\\
& + \sum_{i=1}^{s+1}\left[\widetilde{b}_i \frac{1}{m_i^2}\left(\mathbf{b}_i \cdot \mathbf{c}_i^2 - \mathbf{b}_i \cdot \mathbf{c}_i + \frac{1}{6}\right)(c_i - \widetilde{c}_i)^2\right] = \frac{1}{3}
\end{aligned} \tag{16b}$$

$$\mathbf{b}^f \cdot \mathbf{A}^f \mathbf{c}^f = \mathbf{b} \cdot \mathbf{A}\mathbf{c} = \frac{1}{6} \tag{16c}$$

$$\mathbf{b}^f \cdot \mathbf{A}^g\mathbf{c}^g = \frac{1}{2}\mathbf{b}^T\mathbf{RD}\left(\mathbf{c}+\widetilde{\mathbf{c}}\right) + \sum_{i=1}^{s+1}\left[\left(\mathbf{b}^T\mathbf{RD}\right)_i \frac{1}{m_i}\left(\mathbf{b}_i\cdot\mathbf{c}_i - \frac{1}{2}\right)\left(c_i-\widetilde{c}_i\right)\right] = \frac{1}{6} \tag{16d}$$

$$\mathbf{b}^g \cdot \mathbf{A}^f\mathbf{c}^f = \frac{1}{2}\widetilde{\mathbf{b}}^T\left(\mathbf{I}+\boldsymbol{\alpha}\right)\mathbf{Ac} + \sum_{i=1}^{s+1}\left[\widetilde{b}_i\frac{1}{m_i}\left(\mathbf{b}_i\cdot\mathbf{c}_i - \frac{1}{2}\right)\left(\mathbf{Ac}-\boldsymbol{\alpha}\mathbf{Ac}\right)_i\right] = \frac{1}{6} \tag{16e}$$

$$\begin{aligned}\mathbf{b}^g \cdot \mathbf{A}^g\mathbf{c}^g = {} & \frac{1}{2}\widetilde{\mathbf{b}}^T\left(\boldsymbol{\alpha}+\frac{\boldsymbol{\gamma}}{2}\right)\mathbf{RD}\left(\mathbf{c}+\widetilde{\mathbf{c}}\right) + \frac{1}{6}\widetilde{\mathbf{b}}^T\mathbf{D}\left(\mathbf{c}+2\widetilde{\mathbf{c}}\right)\\ & + \sum_{i=1}^{s+1}\sum_{j=1}^{i-1}\left[\widetilde{b}_i\frac{1}{m_j}\left(\mathbf{b}_j\cdot\mathbf{c}_j-\frac{1}{2}\right)\left(\left(\boldsymbol{\alpha}\mathbf{RD}\right)_{ij}+\frac{1}{2}\left(\boldsymbol{\gamma}\mathbf{RD}\right)_{ij}\right)\left(c_j-\widetilde{c}_j\right)\right]\\ & + \frac{1}{2}\sum_{i=1}^{s+1}\sum_{j=1}^{i-1}\left[\widetilde{b}_i\frac{1}{m_i}\left(\mathbf{b}_i\cdot\mathbf{c}_i-\frac{1}{2}\right)\left(\boldsymbol{\gamma}\mathbf{RD}\right)_{ij}\left(c_j+\widetilde{c}_j\right)\right]\\ & + \sum_{i=1}^{s+1}\sum_{j=1}^{i-1}\left[\widetilde{b}_i\frac{1}{m_i}\left(\mathbf{b}_i\cdot\mathbf{c}_i-\frac{1}{2}\right)\frac{1}{m_j}\left(\mathbf{b}_j\cdot\mathbf{c}_j-\frac{1}{2}\right)\left(\boldsymbol{\gamma}\mathbf{RD}\right)_{ij}\left(c_j-\widetilde{c}_j\right)\right]\\ & + \sum_{i=1}^{s+1}\left[\left(\widetilde{\mathbf{b}}^T\mathbf{D}\right)_i\frac{1}{m_i}\left(\mathbf{b}_i\cdot\mathbf{c}_i-\frac{1}{2}\right)c_i\right]\\ & + \sum_{i=1}^{s+1}\left[\left(\widetilde{\mathbf{b}}^T\mathbf{D}\right)_i\frac{1}{m_i^2}\left(\frac{1}{3}-\mathbf{b}_i\cdot\mathbf{c}_i+\mathbf{b}_i\cdot\mathbf{A}_i\mathbf{c}_i\right)\left(c_i-\widetilde{c}_i\right)\right] = \frac{1}{6}.\end{aligned} \tag{16f}$$

Remark. A full derivation of the conditions can be found in the supplement.

Note that if a RK method of only order one in MIS stage i is chosen, then the method parameters $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ are influenced both linearly and quadratically by the arbitrary number of steps m_i, compare Eq. (16). In case of order two, only two conditions are quadratically manipulated by the number of steps m_i, see Eqs. (16b) and (16f), i.e.

$$\begin{aligned}\mathbf{b}^f\cdot\mathbf{c}^{f\,2} &= & \mathbf{b}^T\mathbf{c}^2 &= \tfrac{1}{3}\\ \mathbf{b}^g\cdot\mathbf{c}^{g\,2} &= & \tfrac{1}{3}\widetilde{\mathbf{b}}^T\left(\mathbf{c}^2+\mathbf{c}\widetilde{\mathbf{c}}+\widetilde{\mathbf{c}}^2\right)+\sum_{i=1}^{s+1}\left[\widetilde{b}_i\tfrac{1}{m_i^2}\left(\mathbf{b}_i\cdot\mathbf{c}_i^2-\tfrac{1}{3}\right)\left(c_i-\widetilde{c}_i\right)^2\right] &= \tfrac{1}{3}\\ \mathbf{b}^f\cdot\mathbf{A}^f\mathbf{c}^f &= & \mathbf{b}^T\mathbf{Ac} &= \tfrac{1}{6}\\ \mathbf{b}^f\cdot\mathbf{A}^g\mathbf{c}^g &= & \mathbf{b}^T\mathbf{RD}\left(\mathbf{c}+\widetilde{\mathbf{c}}\right) &= \tfrac{1}{3}\\ \mathbf{b}^g\cdot\mathbf{A}^f\mathbf{c}^f &= & \widetilde{\mathbf{b}}^T\left(\mathbf{I}+\boldsymbol{\alpha}\right)\mathbf{Ac} &= \tfrac{1}{3}\\ \mathbf{b}^g\cdot\mathbf{A}^g\mathbf{c}^g &= & \tfrac{1}{2}\widetilde{\mathbf{b}}^T\left(\boldsymbol{\alpha}+\tfrac{\boldsymbol{\gamma}}{2}\right)\mathbf{RD}\left(\mathbf{c}+\widetilde{\mathbf{c}}\right)+\tfrac{1}{6}\widetilde{\mathbf{b}}^T\mathbf{D}\left(\mathbf{c}+2\widetilde{\mathbf{c}}\right) & \\ & & +\sum_{i=1}^{s+1}\left[\left(\widetilde{\mathbf{b}}^T\mathbf{D}\right)_i\tfrac{1}{m_i^2}\left(\mathbf{b}_i\cdot\mathbf{A}_i\mathbf{c}_i-\tfrac{1}{6}\right)\left(c_i-\widetilde{c}_i\right)\right] &= \tfrac{1}{6}.\end{aligned} \tag{17}$$

To some extend, with a significantly large number of time steps m_i in MIS stage i, the overall quadratic influence towards the MIS method parameters is very low. Furthermore, if every RK method in MIS stage i is at least of order three, then the remaining conditions are

$$\begin{aligned}\mathbf{b}^T\mathbf{c}^2 &= \tfrac{1}{3}\\ \mathbf{b}^T\mathbf{Ac} &= \tfrac{1}{6}\\ \widetilde{\mathbf{b}}^T\left(\widetilde{\mathbf{c}}^2+\widetilde{\mathbf{c}}\mathbf{c}+\mathbf{c}^2\right) &= 1\\ \mathbf{b}^T\mathbf{RD}\left(\mathbf{c}+\widetilde{\mathbf{c}}\right) &= \tfrac{1}{3}\\ \widetilde{\mathbf{b}}^T\left(\mathbf{I}+\boldsymbol{\alpha}\right)\mathbf{Ac} &= \tfrac{1}{3}\\ 3\widetilde{\mathbf{b}}^T\left(\boldsymbol{\alpha}+\tfrac{\boldsymbol{\gamma}}{2}\right)\mathbf{RD}\left(\mathbf{c}+\widetilde{\mathbf{c}}\right)+\widetilde{\mathbf{b}}^T\mathbf{D}\left(\mathbf{c}+2\widetilde{\mathbf{c}}\right) &= 1.\end{aligned}$$

These conditions were also derived in [9] for the general MIS method with an analytical integration of the auxiliary ordinary differential equation. Therefore, the analytical integration is not required if RK methods of order three are chosen for the inner stages i.

4.4. Order $p = 4$

For order four, the procedure from order three for the derivation of the conditions is repeated.

Table 1
Differentials and conditions for order three for a standard PRK method in comparison with the differentials and conditions for an extMIS method.

PRK			extMIS	
Differential	Order conditions		Differential	Order conditions
$f''\langle f,f\rangle$	$\frac{1}{3} = \mathbf{b}^f \cdot \mathbf{c}^{f\,2}$			
$f''\langle f,g\rangle$	$\frac{1}{3} = \mathbf{b}^f \cdot \mathbf{c}^f\mathbf{c}^g$	$\Rightarrow$	$f''\langle F,F\rangle$	$\mathbf{b}^f \cdot \mathbf{c}^{f\,2} = \frac{1}{3}$
$f''\langle g,f\rangle$	$\frac{1}{3} = \mathbf{b}^f \cdot \mathbf{c}^g\mathbf{c}^f$			
$f''\langle g,g\rangle$	$\frac{1}{3} = \mathbf{b}^f \cdot \mathbf{c}^{g\,2}$			
$g''\langle f,f\rangle$	$\frac{1}{3} = \mathbf{b}^g \cdot \mathbf{c}^{f\,2}$			
$g''\langle f,g\rangle$	$\frac{1}{3} = \mathbf{b}^g \cdot \mathbf{c}^f\mathbf{c}^g$	$\Rightarrow$	$g''\langle F,F\rangle$	$\mathbf{b}^g \cdot \mathbf{c}^{g\,2} = \frac{1}{3}$
$g''\langle g,f\rangle$	$\frac{1}{3} = \mathbf{b}^g \cdot \mathbf{c}^g\mathbf{c}^f$			
$g''\langle g,g\rangle$	$\frac{1}{3} = \mathbf{b}^g \cdot \mathbf{c}^{g\,2}$			
$f'\langle f'\langle f\rangle\rangle$	$\frac{1}{6} = \mathbf{b}^f \cdot \mathbf{A}^f\mathbf{c}^f$	$\Rightarrow$	$f'\langle f'\langle F\rangle\rangle$	$\mathbf{b}^f \cdot \mathbf{A}^f\mathbf{c}^f = \frac{1}{6}$
$f'\langle f'\langle g\rangle\rangle$	$\frac{1}{6} = \mathbf{b}^f \cdot \mathbf{A}^f\mathbf{c}^g$			
$f'\langle g'\langle f\rangle\rangle$	$\frac{1}{6} = \mathbf{b}^f \cdot \mathbf{A}^g\mathbf{c}^f$	$\Rightarrow$	$f'\langle g'\langle F\rangle\rangle$	$\mathbf{b}^f \cdot \mathbf{A}^g\mathbf{c}^g = \frac{1}{6}$
$f'\langle g'\langle g\rangle\rangle$	$\frac{1}{6} = \mathbf{b}^f \cdot \mathbf{A}^g\mathbf{c}^g$			
$g'\langle f'\langle f\rangle\rangle$	$\frac{1}{6} = \mathbf{b}^g \cdot \mathbf{A}^f\mathbf{c}^f$	$\Rightarrow$	$g'\langle f'\langle F\rangle\rangle$	$\mathbf{b}^g \cdot \mathbf{A}^f\mathbf{c}^f = \frac{1}{6}$
$g'\langle f'\langle g\rangle\rangle$	$\frac{1}{6} = \mathbf{b}^g \cdot \mathbf{A}^f\mathbf{c}^g$			
$g'\langle g'\langle f\rangle\rangle$	$\frac{1}{6} = \mathbf{b}^g \cdot \mathbf{A}^g\mathbf{c}^f$	$\Rightarrow$	$g'\langle g'\langle F\rangle\rangle$	$\mathbf{b}^g \cdot \mathbf{A}^g\mathbf{c}^g = \frac{1}{6}$
$g'\langle g'\langle g\rangle\rangle$	$\frac{1}{6} = \mathbf{b}^g \cdot \mathbf{A}^g\mathbf{c}^g$			

Remark. The table showing the reduction of the number of order conditions is given in the supplement.

Hence, the standard conditions for the underlying explicit RK method are

$$
\begin{array}{lrcl}
f'''\langle F,F,F\rangle & \mathbf{b}^T\mathbf{c}^3 & = & \frac{1}{4}\\
f''\langle F,f'\langle F\rangle\rangle & \mathbf{b}^T diag\,(\mathbf{c})\,\mathbf{A}\mathbf{c} & = & \frac{1}{8}\\
f'\langle f''\langle F,F\rangle\rangle & \mathbf{b}^T\mathbf{A}\mathbf{c}^2 & = & \frac{1}{12}\\
f'\langle f'\langle f'\langle F\rangle\rangle\rangle & \mathbf{b}^T\mathbf{A}\mathbf{A}\mathbf{c} & = & \frac{1}{24}
\end{array}
$$

and assuming that the auxiliary ordinary differential equations are solved with arbitrary RK methods of at least order four, there are 14 additional conditions

$$
\begin{array}{lrcl}
g'''\langle F,F,F\rangle & \widetilde{\mathbf{b}}^T diag\left(\mathbf{c}^2+\widetilde{\mathbf{c}}^2\right)(\mathbf{c}+\widetilde{\mathbf{c}}) & = & 1\\
f''\langle F,g'\langle F\rangle\rangle & \mathbf{b}^T diag\,(\mathbf{c})\,\mathbf{R}\mathbf{D}\,(\mathbf{c}+\widetilde{\mathbf{c}}) & = & \frac{1}{4}\\
g''\langle F,f'\langle F\rangle\rangle & \widetilde{\mathbf{b}}^T\left(diag\,(2\mathbf{c}+\widetilde{\mathbf{c}})\,\mathbf{A}+diag\,(\mathbf{c}+2\widetilde{\mathbf{c}})\,\alpha\mathbf{A}\right)\mathbf{c} & = & \frac{3}{4}\\
g''\langle F,g'\langle F\rangle\rangle & 2\widetilde{\mathbf{b}}^T diag\,(\mathbf{c}+\widetilde{\mathbf{c}})\,\alpha\mathbf{R}\mathbf{D}\,(\mathbf{c}+\widetilde{\mathbf{c}})+\frac{4}{3}\widetilde{\mathbf{b}}^T diag\left(\mathbf{c}+\frac{1}{2}\widetilde{\mathbf{c}}\right)\gamma\mathbf{R}\mathbf{D}\,(\mathbf{c}+\widetilde{\mathbf{c}}) & & \\
 & +\widetilde{\mathbf{b}}^T diag\left(\mathbf{c}+\frac{1}{3}\widetilde{\mathbf{c}}\right)\mathbf{D}\mathbf{c}+\widetilde{\mathbf{b}}^T diag\left(\frac{5}{3}\mathbf{c}+\widetilde{\mathbf{c}}\right)\mathbf{D}\widetilde{\mathbf{c}} & = & 1\\
f'\langle g''\langle F,F\rangle\rangle & \mathbf{b}^T\mathbf{R}\mathbf{D}\left(\mathbf{c}^2+\widetilde{\mathbf{c}}\mathbf{c}+\widetilde{\mathbf{c}}^2\right) & = & \frac{1}{4}\\
g'\langle f''\langle F,F\rangle\rangle & \widetilde{\mathbf{b}}^T\,(\mathbf{I}+\alpha)\,\mathbf{A}\mathbf{c}^2 & = & \frac{1}{6}\\
g'\langle g''\langle F,F\rangle\rangle & 4\widetilde{\mathbf{b}}^T\left(\alpha+\frac{1}{2}\gamma\right)\mathbf{R}\mathbf{D}\left(\mathbf{c}^2+\widetilde{\mathbf{c}}\mathbf{c}+\widetilde{\mathbf{c}}^2\right)+\widetilde{\mathbf{b}}^T\mathbf{D}\left(\mathbf{c}^2+2\widetilde{\mathbf{c}}\mathbf{c}+3\widetilde{\mathbf{c}}^2\right) & = & 1\\
f'\langle f'\langle g'\langle F\rangle\rangle\rangle & \mathbf{b}^T\mathbf{A}\mathbf{R}\mathbf{D}\,(\mathbf{c}+\widetilde{\mathbf{c}}) & = & \frac{1}{12}\\
f'\langle g'\langle f'\langle F\rangle\rangle\rangle & \mathbf{b}^T\mathbf{R}\mathbf{D}\,(\mathbf{I}+\alpha)\,\mathbf{A}\mathbf{c} & = & \frac{1}{12}\\
f'\langle g'\langle g'\langle F\rangle\rangle\rangle & 3\mathbf{b}^T\mathbf{R}\mathbf{D}\left(\alpha+\frac{1}{2}\gamma\right)\mathbf{R}\mathbf{D}\,(\mathbf{c}+\widetilde{\mathbf{c}})+\mathbf{b}^T\mathbf{R}\mathbf{D}\mathbf{D}\,(\mathbf{c}+2\widetilde{\mathbf{c}}) & = & \frac{1}{4}\\
g'\langle f'\langle f'\langle F\rangle\rangle\rangle & \widetilde{\mathbf{b}}^T\,(\mathbf{I}+\alpha)\,\mathbf{A}\mathbf{A}\mathbf{c} & = & \frac{1}{12}\\
g'\langle f'\langle g'\langle F\rangle\rangle\rangle & \widetilde{\mathbf{b}}^T\,(\mathbf{I}+\alpha)\,\mathbf{A}\mathbf{R}\mathbf{D}\,(\mathbf{c}+\widetilde{\mathbf{c}}) & = & \frac{1}{6}\\
g'\langle g'\langle f'\langle F\rangle\rangle\rangle & 3\widetilde{\mathbf{b}}^T\left(\alpha+\frac{1}{2}\gamma\right)\mathbf{R}\mathbf{D}\,(\mathbf{I}+\alpha)\,\mathbf{A}\mathbf{c}+\widetilde{\mathbf{b}}^T\mathbf{D}\,(\mathbf{I}+2\alpha)\,\mathbf{A}\mathbf{c} & = & \frac{1}{4}\\
g'\langle g'\langle g'\langle F\rangle\rangle\rangle & \frac{1}{2}\widetilde{\mathbf{b}}^T\left(\alpha+\frac{1}{2}\gamma\right)\mathbf{R}\mathbf{D}\left(\alpha+\frac{1}{2}\gamma\right)\mathbf{R}\mathbf{D}\,(\mathbf{c}+\widetilde{\mathbf{c}})+\frac{1}{6}\widetilde{\mathbf{b}}^T\left(\alpha+\frac{1}{2}\gamma\right)\mathbf{R}\mathbf{D}\mathbf{D}\,(\mathbf{c}+2\widetilde{\mathbf{c}}) & & \\
 & +\frac{1}{4}\widetilde{\mathbf{b}}^T\mathbf{D}\left(\alpha+\frac{1}{3}\gamma\right)\mathbf{R}\mathbf{D}\,(\mathbf{c}+\widetilde{\mathbf{c}})+\frac{1}{24}\widetilde{\mathbf{b}}^T\mathbf{D}\mathbf{D}\,(\mathbf{c}+3\widetilde{\mathbf{c}}) & = & \frac{1}{24}.
\end{array}
\tag{18}
$$

Note that the additional conditions are also recovered if the number of steps in each MIS stage i $m_i \to \infty$, which can be interpreted as an exact integration. Hence, these conditions are also valid for a standard MIS method of order four.

Table 2
MIS method parameters with 5 stages and order four, (MIS54).

α_{ij}		γ_{ij}		β_{ij}	
α_{21}	−0.056843003311023	γ_{21}	0.168489083931286	β_{21}	0.219579314792533
α_{31}	0.071035715986068	γ_{31}	−0.025097850341834	β_{31}	−0.032864918414060
α_{32}	0.050143439731979	γ_{32}	0.025515704040468	β_{32}	0.634699918767414
α_{41}	0.021491523917140	γ_{41}	0.106139356407192	β_{41}	−0.241761887431829
α_{42}	0.287530720188756	γ_{42}	0.264445452990869	β_{42}	−0.120631540663984
α_{43}	0.239030810792355	γ_{43}	0.402246482358727	β_{43}	0.374686620841487
α_{51}	0.027558616966568	γ_{51}	−0.031464053194458	β_{51}	−0.058474324094343
α_{52}	0.382675659910308	γ_{52}	−0.068258296801680	β_{52}	0.351217252190521
α_{53}	0.177185696263246	γ_{53}	0.027558616966568	β_{53}	0.309657030167295
α_{54}	−0.314894383613333	γ_{54}	0.015830368641068	β_{54}	0.168604799122988
α_{61}	0.065158401284120	γ_{61}	0.150547662349659	β_{61}	−0.056205055946158
α_{62}	0.079591607322196	γ_{62}	0.088610905686011	β_{62}	−0.068390330952311
α_{63}	0.459806401597571	γ_{63}	0.067880982803316	β_{63}	−0.086209210260269
α_{64}	0.086725275506356	γ_{64}	−0.297416190393485	β_{64}	0.034904705602768
α_{65}	0.439945196292364	γ_{65}	0.148246909195494	β_{65}	0.448964988009822

However, the solution of Eq. (2b) must not be given by an exact integrator. An arbitrary RK method for MIS stage i may be applied as long as it is of the same order as the MIS method.

Moreover, for a MIS-KW method, i.e. utilizing parameter from Eq. (8), the 14 additional conditions from Eq. (18) are reduced to five additional conditions with the parameters from Eq. (8),

$$\begin{array}{lrcl}
g''\left\langle F, f'\left\langle F\right\rangle\right\rangle & \widetilde{\mathbf{b}}^T diag\left(2\mathbf{c}+\widetilde{\mathbf{c}}\right)\mathbf{Ac}+\widetilde{\mathbf{b}}^T diag\left(\mathbf{c}+2\widetilde{\mathbf{c}}\right)\boldsymbol{\alpha}\mathbf{Ac} & = & \frac{3}{4} \\
f'\left\langle g''\left\langle F, F\right\rangle\right\rangle & \widetilde{\mathbf{b}}^T\left(\mathbf{I}+\boldsymbol{\alpha}\right)\mathbf{Ac}^2 & = & \frac{1}{6} \\
f'\left\langle g'\left\langle f'\left\langle F\right\rangle\right\rangle\right\rangle & \mathbf{b}^T\mathbf{RD}\left(\mathbf{I}+\boldsymbol{\alpha}\right)\mathbf{Ac} & = & \frac{1}{12} \\
g'\left\langle f'\left\langle f'\left\langle F\right\rangle\right\rangle\right\rangle & \widetilde{\mathbf{b}}^T\left(\mathbf{I}+\boldsymbol{\alpha}\right)\mathbf{AAc} & = & \frac{1}{12} \\
g'\left\langle g'\left\langle f'\left\langle F\right\rangle\right\rangle\right\rangle & 3\widetilde{\mathbf{b}}^T\boldsymbol{\alpha}\mathbf{RD}\left(\mathbf{I}+\boldsymbol{\alpha}\right)\mathbf{Ac}+\widetilde{\mathbf{b}}^T\mathbf{D}\left(\mathbf{I}+2\boldsymbol{\alpha}\right)\mathbf{Ac} & = & \frac{1}{4}.
\end{array}$$

By straight forward calculation can be shown that all other conditions from Eq. (18) are met.

The applied procedure for the derivation of the order conditions can also be repeated for higher order convergence. However, the number of conditions additional to a standard RK method increases drastically.

In the future, it is of interest to develop a theory based on trees for the derivation of the order conditions of the (extended) MIS method. Although, the order conditions from the PRK method were applied, a tree structure has not been recognized, yet.

5. Example of MIS method of order four for the compressible Euler equation

In the previous sections, the extMIS method has been introduced including the conditions up to order four. An example of a MIS method of order four with five stages (MIS54) for the compressible Euler equation will be presented in this section, extending the set of already known MIS methods from [20]. However, there was no method of order four suggested.

In order to estimate a reasonably good method, the stability concept for the linear acoustic problem as given in [9] has been utilized as a linear representative example for the compressible Euler equation. The set of equation is given by

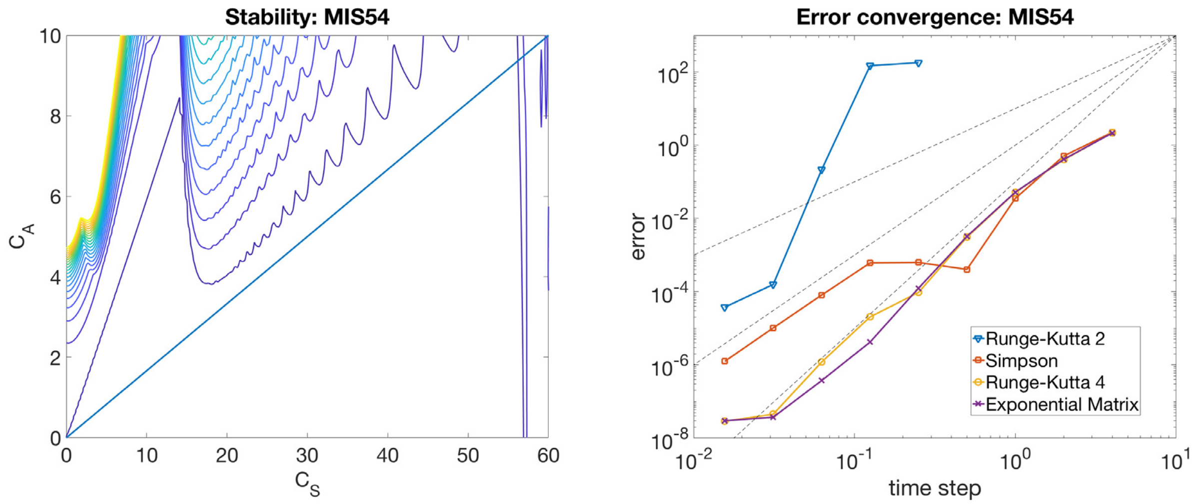
$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}=-c_S\frac{\partial \Pi}{\partial x} \qquad \text{and} \qquad \frac{\partial \Pi}{\partial t}+U\frac{\partial \Pi}{\partial x}=-c_S\frac{\partial u}{\partial x},$$

see [9] or [20, equation (10)] with x-horizontal component of flow velocity (u), Exner function (Π) as well as speed of sound (c_S) and a constant advection velocity (U). This equation is a linearized continuous one-dimensional compressible inviscid equation. The stability region was calculated accordingly to [20].

To obtain a finite dimensional test problem, all spatial differential operators are replaced by finite differences and then transformed in to Fourier space. The pressure and divergence terms are discretized by central differences, the advection term by third order up-winding. The stability is tested for the CFL range $U\Delta t < \frac{1}{6}c_S\Delta t$ and 40 Fourier modes. Therefore, the method is tailored for this special application where the spectra lies near the imaginary axis. For other applications the standard way is to resort to the test equation $y' = \lambda_E y + \lambda_F y$ and a suitable domain for λ_E and λ_F.

Table 2 gives the parameters $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ for the MIS54 method with five stages and order four.

In [20], the MIS methods were derived by solving an optimization problem which has combined the stability properties with the small integration time interval $\sum_{i=1}^{s+1}[d_i]$. The aim was additionally to good stability the minimization of this time interval. However, the MIS54 method was chosen from a set of more than 100 different methods of order four by visualizing the stability region. All these various methods were calculated by solving the non-linear system of equations of all order conditions derived up to order four with random initial values for the MATLAB function fsolve. The step and function tolerance values were chosen to be 10^{-10} and 10^{-12}, respectively. The corresponding stability region is given in



(a) Stability region (b) Error convergence

Fig. 1. Results of MIS54 method.

Fig. 1a. The stability region of this method has considerably expanded the regions given in [20]. The main advantage of this specific method compared to all other calculated methods of order four is the large CFL number for speed of sound (C_S) with a given Mach number or ratio of $\frac{C_A}{C_S} < \frac{1}{6}$ with CFL number for advection (C_A). Even a C_S as high as 50 is possible for a Mach number as low as $\frac{1}{6}$.

Fig. 1b shows the order convergence of the MIS54 method, where the auxiliary ordinary differential equation was solved with various explicit Runge–Kutta methods. The cold bubble downburst benchmark example as described in [20] was used for the calculation of the error convergence. The calculation was performed for time steps ranging from 2^{-6} till 2^2 s. The integration was conducted for 900 s. No background wind was applied. The error was derived by the comparison with a simulation run using an eRK method of order four with a time step of 0.0001 s. Although each time step was applied for each numerical test, the Runge–Kutta method with order two only converged for a time step of 0.25 s or less.

Furthermore, for some time steps, the method converges with a higher order, see e.g. Runge–Kutta method with order two and time step 0.0625 s and Simpson's method with time step 0.5 s. This is related to the number of fast integration time steps applied for each MIS stage. Eq. (17) states that the number of fast integration time steps influences quadratically the order conditions for order three and RK method applied for the fast integration of order two. However, if this number is significantly large enough then there is almost no influence. Moreover, with a further decrease of the time step, the number of fast integration time steps were also reduced. Therefore, the test then converges more slowly. This can also be seen by the fast integration with the Simpson method.

This characteristic allows the application of a RK method of reduced order for fast integration without an increase in error as long as the number of fast time steps is large enough. This is exemplarily justified by the application of the Simpson's method for time steps larger than 0.5 s.

Furthermore, Fig. 1b also shows that the MIS54 method converges with order four by utilizing an exponential integrator for the auxiliary ordinary differential equation.

Remark. During this test, there has been no change of the RK method applied for the integration in each MIS stage. However, this is not required by the order condition from the previous section.

6. Conclusions and outlook

In this work, an extension for the general MIS method has been presented. This allows an integration of the auxiliary ordinary differential equation for the MIS stages with arbitrary RK methods. Furthermore, the suggested extMIS method has been set into relation with other existing multirate RK methods such as RFSMR, PRK and GARK methods. Some of these known relations are only valid for the special case of the extMIS-KW method. However, these connections gave the opportunity to apply the known order theory from the GARK method to the extMIS method. Therefore, a strategy to derive conditions for higher order convergence was presented, including the conditions up to order four. Furthermore, it was shown that by applying RK methods of sufficiently high order for the integration of the auxiliary ordinary differential equation in each MIS stage, the order conditions are also valid for the general MIS method.

Finally, a MIS method of order four was presented as well as the corresponding stability region for a linear acoustic problem. The order of this method has been numerically shown with the cold bubble downburst benchmark test. The observed error behaviour from the different RK methods indicates that for the integration of the auxiliary ordinary differential equation a method of sufficient error convergence will result in a similar convergence behaviour. This implies a need in further research in the error convergence of the MIS method.

Moreover, a theory based on trees for the derivation of the order conditions of the (extended) MIS method will enhance the development of MIS methods. Additionally, the combination of the extMIS method with the class of MRI-GARK methods suggested in [14] will be of interest. Currently, the MRI-GARK methods are derived on the basis of the MIS-KW methods. The result would be an even more generalized class of multirate infinitesimal step methods for additive split problems as given in Eq. (1).

Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.cam.2019.112541.

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