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# CONTRACTIVE IDEMPOTENTS ON LOCALLY COMPACT QUANTUM GROUPS

MATTHIAS NEUFANG, PEKKA SALMI, ADAM SKALSKI, AND NICO SPRONK

ABSTRACT. A general form of contractive idempotent functionals on coamenable locally compact quantum groups is obtained, generalising the result of Greenleaf on contractive measures on locally compact groups. The image of a convolution operator associated to a contractive idempotent is shown to be a ternary ring of operators. As a consequence a one-to-one correspondence between contractive idempotents and a certain class of ternary rings of operators is established.

*For Lynx is a Fearsome Beast, and when it Doth  
Come Out, the Groups of Quantum Ilk Surrender  
Meekly*

Anonymous

## INTRODUCTION

Idempotent probability measures on locally compact groups arise naturally as limit distributions of random walks and have been studied in this context at least since the 1940s (for the history of the investigations of such idempotent measures we refer to the recent survey [Sal]). They all turn out to arise as Haar measures on compact subgroups of the original group. There are several possible generalisations of the problem of characterising all idempotent probability measures. On one hand one can ask about *all* idempotent measures. The problem turns out to be very hard and so far has only been solved in the case of abelian groups in a celebrated paper by P. Cohen ([Coh60]). In fact, a solution to this problem for discrete groups would solve Kaplansky's idempotent conjecture. Thus, it is significantly easier to analyse all *contractive* (with respect to the total variation norm) idempotent measures – in 1965 F.P. Greenleaf ([Gre65]) showed that they arise as combinations of Haar measures on compact subgroups and characters on those subgroups. On the other hand one can ask about the counterparts of probability measures on locally compact *quantum* groups, usually called idempotent states. The study of the latter was initiated in [FS09<sub>1</sub>] and continued later for example in [SS12] (once again the description of these developments and a full list of references can be located in [Sal]). An important feature of idempotent states is that they need not be *Haar idempotents*, i.e. Haar states on compact quantum subgroups.

In this article we combine the two generalisations and investigate *contractive idempotents on locally compact quantum groups*. These are idempotent functionals on  $C_0(\mathbb{G})$ , the algebra of continuous functions on a locally compact quantum group  $\mathbb{G}$ , and as such can be viewed

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as quantum counterparts of contractive idempotent measures on locally compact groups. In particular they include contractive idempotent functions in  $B(G)$ , the Fourier-Stieltjes algebra of a locally compact group  $G$ . Similarly to the classical case, contractive idempotents on  $\mathbb{G}$  arise as limits of iterated convolutions of a given contractive quantum measure. They also turn out to be of great relevance for the study of the harmonic functions on quantum groups, i.e. fixed points of convolution operators associated with contractive quantum measures (in the case of standard groups and their duals such structures were studied for example in the book [CL02] and, more recently, in [CL06]; for Poisson boundaries associated to *states* on locally compact quantum groups we refer to the recent preprint [KNR] and references therein). For technical reasons we assume that the locally compact quantum groups we consider are *coamenable*. Under this assumption we show that each contractive idempotent  $\omega$  on a quantum group  $\mathbb{G}$  is associated via its polar decomposition to two (possibly identical) idempotent states. This leads to a general characterization of the form of  $\omega$  and can be further simplified in the case when the idempotent states associated to  $\omega$  are Haar idempotents. In the latter case the resulting structure precisely mirrors that obtained by Greenleaf for classical groups: a contractive idempotent  $\omega$  arises from a Haar state on a compact quantum subgroup and a character (i.e. a group-like element) on that subgroup. We show also that in this case the left and right absolute values of  $\omega$  coincide. These results contain in particular the classical theorem of Greenleaf.

In [SS12] (see also [Sal11]) idempotent states on  $\mathbb{G}$  were shown to be closely related to *right invariant expected* subalgebras of  $C_0(\mathbb{G})$ . It turns out that a similar result holds true for contractive idempotents, but subalgebras have to be replaced by *ternary rings of operators (TROs)*. The appearance of the TRO structure is related to the fact that the left convolution operator associated to a contractive idempotent satisfies the algebraic relations characterising the so-called TRO conditional expectations. The TROs arising in this way are invariant under right convolution operators; moreover their *linking algebras* contained in two-by-two matrices over  $C_0(\mathbb{G})$  admit conditional expectations preserving a suitable amplification of the left Haar weight. We show that if  $\mathbb{G}$  is *unimodular*, then in fact there is a bijective correspondence between such TROs in  $C_0(\mathbb{G})$  and contractive idempotents on  $\mathbb{G}$ . We would like to point out that such TRO structures play a fundamental role in the study of the extended Poisson boundaries for contractive quantum measures, which we undertake in the forthcoming work [NSSS].

A detailed plan of the paper is as follows. In Section 1 we set up notation, introduce basic terminology of quantum group theory and recall fundamental results on idempotent states. In Section 2 we characterise the general form of contractive idempotents and analyse their connection to idempotent states. In section 3 we study the properties of the convolution operator associated with a contractive idempotent, showing in particular that its image is a TRO; further in Section 4 we establish a one-to-one correspondence between contractive idempotents and a certain class of TROs. Finally Section 5 shows how our results allow us to obtain quickly the known characterisations for  $\mathbb{G}$  being either a classical group or the dual of a classical group, and presents some examples.

A word of warning is in place – the quantum group notation used in this paper differs from that in [SS12]; it is however consistent with the notation more and more often adopted in the modern literature of the subject.

## 1. PRELIMINARIES

In this section we introduce our notation, prove a few technical results we will need later, and recall basic information on idempotent states on locally compact quantum groups.

**1.1.  $C^*$ -algebra notation.** Let  $A$  be a  $C^*$ -algebra. The state space of  $A$  will be denoted  $S(A)$ , the multiplier algebra of  $A$  by  $M(A)$ . We will often use the strict topology and strict extensions of maps to multiplier algebras without any comment – the detailed arguments can be found in [Lan95] or in Section 1 of [SS12]. The symbol  $\otimes$  will denote the spatial tensor product of  $C^*$ -algebras and  $\bar{\otimes}$  will be reserved for the von Neumann algebraic tensor product. If  $H$  is a Hilbert space then  $K_H$  denotes the algebra of compact operators on  $H$ . If  $K$  is an additional Hilbert space and  $X \subset B(H, K)$ ,  $Y \subset B(K, H)$  we write  $X^* = \{x^* : x \in X\}$  and  $\langle XY \rangle = \overline{\text{span}}\{xy : x \in X, y \in Y\}$ .

If  $\omega \in S(A)$  then we write  $N_\omega = \{a \in A : \omega(a^*a) = 0\}$ .

If  $\omega \in A^*$ , its unique normal extension to a functional on the von Neumann algebra  $A^{**}$  will be sometimes denoted by the same symbol.

We have natural left and right actions of  $A$  on  $A^*$ . They can be extended in a natural sense to actions of  $A^{**}$ : for  $\omega \in A^*$ ,  $x, y \in A^{**}$ , write

$$\omega.x(y) = \omega(xy) \quad \text{and} \quad x.\omega(y) = \omega(yx).$$

By [Tak02], given a functional  $\omega \in A^*$  there is a (unique) polar decomposition

$$\omega = u.\omega|_r$$

where  $|\omega|_r$  is a positive functional in  $A^*$ , called the *right absolute value* of  $\omega$ , and  $u$  is a partial isometry in  $A^{**}$  such that  $u^*u$  is the support of  $|\omega|_r$  (i.e.  $u^*u$  is the minimal projection satisfying  $|\omega|_r(u^*ux) = |\omega|_r(xu^*u) = |\omega|_r(x)$ ). It follows that

$$|\omega|_r = u^*.\omega.$$

Putting  $|\omega|_l = u.\omega|_r.u^*$  we obtain another positive functional  $|\omega|_l$  (the *left absolute value* of  $\omega$ ) such that

$$\omega = |\omega|_l.u, \quad |\omega|_l = \omega.u^*.$$

(Now  $uu^*$  is the support of  $|\omega|_l$ .)

**1.2. Locally compact quantum groups – general background.** We follow the  $C^*$ -algebraic approach to locally compact quantum groups due to Kustermans and Vaes [KV00].

A *comultiplication* on a  $C^*$ -algebra  $A$  is a nondegenerate  $*$ -homomorphism  $\Delta : A \rightarrow M(A \otimes A)$  that is coassociative:

$$(\text{id}_A \otimes \Delta)\Delta = (\Delta \otimes \text{id}_A)\Delta.$$

A  $C^*$ -algebra  $A$  with a comultiplication  $\Delta$  is *the algebra of continuous functions vanishing at infinity on a locally compact quantum group* if  $(A, \Delta)$  satisfies the quantum cancellation laws

$$(1) \quad \overline{\text{span}} \Delta(A)(A \otimes 1_A) = A \otimes A, \quad \overline{\text{span}} \Delta(A)(1_A \otimes A) = A \otimes A$$

and admits a left Haar weight  $\phi$  and a right Haar weight  $\psi$ . That is,  $\phi$  and  $\psi$  are faithful KMS-weights on  $A$  such that  $\phi$  is *left invariant*

$$\phi((\omega \otimes \text{id})\Delta(a)) = \omega(1_A)\phi(a) \quad (\omega \in A_+^*, a \in A_+, \phi(a) < \infty)$$

and  $\psi$  is *right invariant*

$$\psi((\text{id} \otimes \omega)\Delta(a)) = \omega(1_A)\psi(a) \quad (\omega \in A_+^*, a \in A_+, \psi(a) < \infty)$$

(for the meaning of the KMS property we refer the reader to [KV00]). In such a case we use the notation  $\mathbf{A} = C_0(\mathbb{G})$  and call  $\mathbb{G}$  a *locally compact quantum group*.

If the  $C^*$ -algebra  $C_0(\mathbb{G})$  is unital, we say that the quantum group  $\mathbb{G}$  is *compact* and write  $C(\mathbb{G})$  instead of  $C_0(\mathbb{G})$ . In this case there is a unique *Haar state* of  $\mathbb{G}$  that is both left and right invariant.

A *counit* is a linear functional  $\epsilon$  on  $C_0(\mathbb{G})$  such that

$$(\epsilon \otimes \text{id}_{C_0(\mathbb{G})})\Delta = (\text{id}_{C_0(\mathbb{G})} \otimes \epsilon)\Delta = \text{id}_{C_0(\mathbb{G})}.$$

A quantum group that admits a bounded counit is said to be *coamenable* [BT03].

**Assumption 1.1.** *From now on we assume that all the locally compact quantum groups studied in this paper are coamenable.*

Note that due to the assumption of coamenability, the  $C^*$ -algebra  $C_0(\mathbb{G})$  coincides with the universal  $C^*$ -algebra  $C_0^u(\mathbb{G})$  associated to  $\mathbb{G}$  (as defined by Kustermans [Kus01]). The multiplier algebra of  $C_0(\mathbb{G})$  will be sometimes denoted  $C_b(\mathbb{G})$ .

The GNS representation space for the left Haar weight will be denoted by  $L^2(\mathbb{G})$ . We may in fact assume that  $C_0(\mathbb{G})$  is a nondegenerate subalgebra of  $B(L^2(\mathbb{G}))$ . The Banach space dual of  $C_0(\mathbb{G})$  will be denoted  $M(\mathbb{G})$  and called the *measure algebra* of  $\mathbb{G}$ . It is a Banach algebra with the product defined by

$$\omega \star \mu := (\omega \otimes \mu) \circ \Delta, \quad \omega, \mu \in M(\mathbb{G}),$$

and since  $\mathbb{G}$  is coamenable,  $M(\mathbb{G})$  is unital.

Given  $\omega \in M(\mathbb{G})$  the associated *left convolution* operator  $L_\omega : C_0(\mathbb{G}) \rightarrow C_0(\mathbb{G})$  is defined by the formula

$$L_\omega(a) = (\omega \otimes \text{id}_{C_0(\mathbb{G})})(\Delta(a)), \quad a \in C_0(\mathbb{G}).$$

(the right convolution operators  $R_\omega$  are defined in an analogous way). They are automatically strict and hence extend to maps on  $C_b(\mathbb{G})$ . The following lemma from [SS12] will allow us later to identify the (left) convolution operators.

**Lemma 1.2** (Lemma 1.6 of [SS12]). *Let  $T : C_0(\mathbb{G}) \rightarrow C_0(\mathbb{G})$  be a completely bounded map such that  $T \otimes \text{id}_{C_0(\mathbb{G})} : C_0(\mathbb{G}) \otimes C_0(\mathbb{G}) \rightarrow C_0(\mathbb{G}) \otimes C_0(\mathbb{G})$  is strictly continuous on bounded subsets. Suppose that  $Z \subseteq M(\mathbb{G})$  is weak\*-dense and invariant under the right action of some dense subalgebra  $\mathcal{A}$  of  $C_0(\mathbb{G})$ . Then the following conditions are equivalent:*

- (i)  $T = L_\mu$ , for some  $\mu \in M(\mathbb{G})$ ;
- (ii)  $(T \otimes \text{id}_{\mathcal{A}})\Delta = \Delta T$ ;
- (iii)  $TR_\nu = R_\nu T$  for all  $\nu \in Z$ .

*If the above conditions hold,  $T = R_{\epsilon \circ T}$  is a linear combination of completely positive nondegenerate maps and the equality in (iii) is valid for all  $\nu \in M(\mathbb{G})$ .*

**Definition 1.3.** A  $C^*$ -subalgebra  $C \subseteq C_0(\mathbb{G})$  is said to be *right invariant* if  $R_\mu(C) \subseteq C$  for all  $\mu \in M(\mathbb{G})$ .

Define  $L^1(\mathbb{G})$  to be the collection of those functionals in  $M(\mathbb{G})$  which admit normal extensions to  $B(L^2(\mathbb{G}))$ . The *antipode*  $S$  of  $\mathbb{G}$  is a (possibly unbounded) operator on  $C_0(\mathbb{G})$ ; the set

$$\{(\sigma \otimes \text{id})(V); \sigma \in B(L^2(\mathbb{G}))_*\},$$

where  $V \in B(L^2(\mathbb{G}) \otimes_2 L^2(\mathbb{G}))$  is the *right multiplicative unitary* of  $\mathbb{G}$ , gives a core for  $S$  and

$$S((\sigma \otimes \text{id})(V)) = (\sigma \otimes \text{id})(V^*), \quad \sigma \in B(L^2(\mathbb{G}))_*.$$

For  $\omega \in M(\mathbb{G})$ , define  $\bar{\omega} \in M(\mathbb{G})$  by  $\bar{\omega}(a) = \overline{\omega(a^*)}$ . When  $\bar{\omega} \circ S$  (defined on  $\text{dom}(S)$ ) has a *bounded* extension to all of  $C_0(\mathbb{G})$ , we write  $\omega^\sharp$  for this functional. Then  $L^\sharp_1(\mathbb{G})$  denotes those  $\omega$  in  $L^1(\mathbb{G})$  for which  $\bar{\omega} \circ S$  is bounded.

### 1.3. Idempotent states on locally compact quantum groups.

*Definition 1.4.* An *idempotent functional* in  $M(\mathbb{G})$  is a non-zero functional  $\omega \in M(\mathbb{G})$  such that  $\omega \star \omega = \omega$ .

Idempotent states on  $M(\mathbb{G})$  were studied in [SS12] (see also [FST<sub>1</sub>] for the compact case). Here we quote the main results and some auxiliary lemmas we will use later. First we need some further definitions.

*Definition 1.5.* Assume that  $\mathbb{G}$  is a coamenable locally compact quantum group and  $\mathbb{H}$  a compact quantum group. Then  $\mathbb{H}$  is a *closed quantum subgroup* of  $\mathbb{G}$  if there exists a surjective  $*$ -homomorphism  $\pi_{\mathbb{H}} : C_0(\mathbb{G}) \rightarrow C(\mathbb{H})$  intertwining the respective coproducts (sometimes we shorten this to write that  $(\mathbb{H}, \pi)$  is a compact quantum subgroup of  $\mathbb{G}$ ).

Note that there are various possible definitions of locally compact quantum subgroups; all of them however coincide when the quantum subgroup in question is compact (see [DKSS12] for a complete discussion).

*Definition 1.6.* An idempotent state  $\omega \in M(\mathbb{G})$  is a *Haar idempotent* if there exists a compact quantum subgroup  $\mathbb{H}$  of  $\mathbb{G}$  such that  $\omega = h_{\mathbb{H}} \circ \pi_{\mathbb{H}}$ , where  $h_{\mathbb{H}}$  denotes the Haar state on  $\mathbb{H}$ .

**Theorem 1.7** (Theorem 3.7 of [SS12]). *Let  $\omega$  be an idempotent state in  $M(\mathbb{G})$ . Then the following are equivalent:*

- (i)  $\omega$  is a Haar idempotent;
- (ii)  $N_\omega$  is an ideal (equivalently a  $*$ -subspace).

## 2. GENERAL FORM OF CONTRACTIVE IDEMPOTENT FUNCTIONALS

In this section we describe the general form of contractive idempotents in  $M(\mathbb{G})$ . Note that a contractive idempotent must necessarily have norm 1. We begin by showing how one can construct them using idempotent states and elements in  $C_b(\mathbb{G})$  satisfying a version of the group-like property.

**Proposition 2.1.** *Suppose that  $\sigma$  is an idempotent state in  $M(\mathbb{G})$ , and  $u \in C_b(\mathbb{G})$  satisfies*

$$\Delta(u) - u \otimes u \in N_{\sigma \otimes \sigma} \subseteq M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G})).$$

*Then either  $\sigma(u^*u) = 1$  or  $\sigma(u^*u) = 0$ . In the former case  $u \cdot \sigma \cdot u^*$  is an idempotent state and  $u \cdot \sigma$  and  $\sigma \cdot u^*$  are contractive idempotents.*

*Proof.* The Cauchy–Schwarz inequality implies that for every  $X$  in  $M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$

$$(\sigma \otimes \sigma)((u^* \otimes u^*)X) = (\sigma \otimes \sigma)(\Delta(u^*)X) \quad \text{and} \quad (\sigma \otimes \sigma)(X(u \otimes u)) = (\sigma \otimes \sigma)(X\Delta(u)).$$

In particular, using the assumption that  $\sigma$  is idempotent, we see that  $\sigma(u^*u)^2 = \sigma(u^*u)$ , so  $\sigma(u^*u)$  is either 1 or 0. (Note that if  $\sigma(u^*u) = 0$ , then  $u \cdot \sigma = \sigma \cdot u^* = u \cdot \sigma \cdot u^* = 0$ .)

Moreover,

$$(u \cdot \sigma \star u \cdot \sigma)(a) = (\sigma \otimes \sigma)(\Delta(a)(u \otimes u)) = (\sigma \otimes \sigma)(\Delta(au)) = u \cdot \sigma(a)$$

for every  $a$  in  $C_0(\mathbb{G})$ . Therefore  $u \cdot \sigma$  is a contractive idempotent when  $\sigma(u^*u) \neq 0$ . The other cases are similar.  $\square$

We will now show that in fact all contractive idempotents in  $M(\mathbb{G})$  have the form described above. We begin with the following lemma, which is a generalisation to quantum groups of Theorem 2.1.2 in [Gre65]. Recall the polar decomposition of functionals described in Subsection 1.1.

**Lemma 2.2.** *Let  $\omega_1, \omega_2 \in M(\mathbb{G})$  be such that  $\|\omega_1 \star \omega_2\| = \|\omega_1\| \|\omega_2\|$ . Then  $|\omega_1 \star \omega_2|_r = |\omega_1|_r \star |\omega_2|_r$ .*

*Proof.* Let  $u_1, u_2$  and  $u_{12}$  denote the partial isometries in  $C_0(\mathbb{G})^{**}$  associated with the polar decompositions of  $\omega_1, \omega_2$  and  $\omega_1 \star \omega_2$ , respectively. Then, for every  $x$  in  $C_0(\mathbb{G})^{**}$ ,

$$\begin{aligned} |\omega_1 \star \omega_2|_r(x) &= \omega_1 \star \omega_2(xu_{12}^*) = (\omega_1 \otimes \omega_2)(\Delta(xu_{12}^*)) \\ &= (|\omega_1|_r \otimes |\omega_2|_r)(\Delta(x)\Delta(u_{12}^*)(u_1 \otimes u_2)). \end{aligned}$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned} \left| |\omega_1 \star \omega_2|_r(x) \right|^2 &\leq (|\omega_1|_r \otimes |\omega_2|_r)(\Delta(xx^*)) (|\omega_1|_r \otimes |\omega_2|_r)((u_1^* \otimes u_2^*)\Delta(u_{12}u_{12}^*)(u_1 \otimes u_2)) \\ &\leq (|\omega_1|_r \star |\omega_2|_r)(xx^*) (|\omega_1|_r \otimes |\omega_2|_r)(u_1^*u_1 \otimes u_2^*u_2) \\ &= \|\omega_1\| \|\omega_2\| (|\omega_1|_r \star |\omega_2|_r)(xx^*) = \|\omega_1 \star \omega_2\| (|\omega_1|_r \star |\omega_2|_r)(xx^*) \end{aligned}$$

(the last identity is due to the hypothesis). It then follows from the uniqueness of the absolute value  $|\omega_1 \star \omega_2|_r$  (Proposition 4.6 of [Tak02]) that

$$|\omega_1 \star \omega_2|_r = |\omega_1|_r \star |\omega_2|_r. \quad \square$$

The next theorem is the main result of this section. Comparing the theorem with Proposition 2.1 shows that we have a characterisation of contractive idempotents. We shall use the notation set up in the theorem for the rest of the paper; that is, given a contractive idempotent  $\omega$ , we let  $v$  denote an element satisfying the following theorem.

**Theorem 2.3.** *Let  $\omega$  be a contractive idempotent in  $M(\mathbb{G})$ . Then the absolute values  $|\omega|_r$  and  $|\omega|_l$  are idempotent states and there exists  $v \in C_0(\mathbb{G})$  such that  $\omega = |\omega|_l.v = v.|\omega|_r$  and*

$$\Delta(v) - v \otimes v \in N_{|\omega|_r \otimes |\omega|_r} \cap (N_{|\omega|_l \otimes |\omega|_l})^*.$$

*Proof.* Note that  $\|\omega\| = 1$ . By Lemma 2.2

$$|\omega|_r \star |\omega|_r = |\omega|_r$$

so  $|\omega|_r$  is an idempotent state.

Let  $u \in C_0(\mathbb{G})^{**}$  be the partial isometry associated with the polar decomposition of  $\omega$ . Fix  $a \in C_0(\mathbb{G})$  such that  $\|L_\omega(a)\| = 1$  (such  $a$  exists because otherwise  $\omega = 0$ ). Choose  $\nu \in C_0(\mathbb{G})^*$  such that  $\|\nu\| \leq 1$  and  $\nu(L_\omega(a)) = 1$ . Put  $v = R_\nu(L_\omega(a))^*$ , so  $\|v\| \leq 1$ . Since  $\omega$  is an idempotent,

$$1 = \nu(L_\omega(a)) = \omega(R_\nu(L_\omega(a))) = \omega(v^*).$$

Hence  $\|v\| = 1$ . Moreover,

$$\begin{aligned} |\omega|_r((v-u)^*(v-u)) &= |\omega|_r(v^*v) - |\omega|_r(v^*u) - |\omega|_r(u^*v) + |\omega|_r(u^*u) \\ &= |\omega|_r(v^*v) - \omega(v^*) - \overline{\omega(v^*)} + |\omega|_r(u^*u) \leq \|v\|^2 - 1 - 1 + 1 = 0. \end{aligned}$$

It follows that  $\omega = v.|\omega|_r$  and  $|\omega|_r(v^*v) = 1$ .



Now

$$\begin{aligned}
 & (|\omega|_r \otimes |\omega|_r)((\Delta(v) - v \otimes v)^*(\Delta(v) - v \otimes v)) \\
 &= (|\omega|_r \otimes |\omega|_r)(\Delta(v^*v)) - (|\omega|_r \otimes |\omega|_r)(\Delta(v^*)(v \otimes v)) \\
 &\quad - (|\omega|_r \otimes |\omega|_r)((v^* \otimes v^*)\Delta(v)) + (|\omega|_r \otimes |\omega|_r)(v^*v \otimes v^*v) \\
 &= |\omega|_r(v^*v) - (\omega \otimes \omega)(\Delta(v^*)) - \overline{(\omega \otimes \omega)(\Delta(v^*))} + |\omega|_r(v^*v)^2 \\
 &= 2 - \omega(v^*) - \overline{\omega(v^*)} = 0.
 \end{aligned}$$

Hence  $\Delta(v) - v \otimes v \in N_{|\omega|_r \otimes |\omega|_r}$ .

Finally,  $v \cdot |\omega|_r \cdot v^* = u \cdot |\omega|_r \cdot u^* = |\omega|_l$ , so also  $|\omega|_l$  is an idempotent state by Proposition 2.1. Using similar arguments as those applied to  $|\omega|_r$ , we obtain that  $\omega = |\omega|_l \cdot v$  and, moreover,  $\Delta(v) - v \otimes v \in (N_{|\omega|_l \otimes |\omega|_l})^*$ .  $\square$

*Remark 2.4.* We know from the classical case due to Greenleaf [Gre65] that every contractive idempotent measure on a locally compact group  $G$  is of the form  $d\mu(s) = \chi(s) dm_H(s)$  where  $m_H$  is the Haar measure of a compact subgroup  $H$  of  $G$  and  $\chi$  is a continuous character of  $H$ . This characterisation follows also from Theorem 2.3 (see Proposition 5.1 below);  $|\omega|_r = m_H$  and the character  $\chi$  is just the restriction of  $v \in C_0(G)$  to  $H$ .

Now consider the dual case when  $C_0(\mathbb{G}) = C^*(\Gamma)$  for a locally compact amenable group  $\Gamma$  (sometimes one writes then  $\mathbb{G} = \hat{\Gamma}$ ). Ilie and Spronk [IS05, Theorem 2.1] showed that a contractive idempotent in the Fourier–Stieltjes algebra  $B(\Gamma) \cong C^*(\Gamma)^*$  is a characteristic function  $1_C$  of an open coset  $C$  of  $\Gamma$  (even when  $\Gamma$  is not amenable). The most natural factorisation of  $1_C$  is certainly  $1_C = 1_H(\cdot s) = \lambda(s) \cdot 1_H$  where  $H$  is an open subgroup of  $\Gamma$  with  $C = Hs^{-1}$ . However,  $\lambda(s)$  is in the multiplier algebra  $M(C^*(\Gamma))$  and not in general in  $C^*(\Gamma)$ . A factorisation that does fit into the description of Theorem 2.3 is given by  $1_C = \lambda(f) \cdot 1_H$  where  $f \in L^1(\Gamma)$  is such that  $\text{supp } f \subseteq sH$  and  $\int_G f(t) dt = 1$ . We note that also the description of contractive idempotents in  $B(\Gamma)$  (for amenable  $\Gamma$ ) as characteristic functions of open cosets follows from our results; see Proposition 5.2.

**Theorem 2.5.** *Let  $\omega$  be a contractive idempotent in  $M(\mathbb{G})$ . Suppose that  $|\omega|_r$  is a Haar idempotent and let  $(\mathbb{H}, \pi)$  be the associated compact quantum subgroup (so that  $\pi: C_0(\mathbb{G}) \rightarrow C(\mathbb{H})$  is the morphism). Then there exists a group-like unitary  $u \in C(\mathbb{H})$  such that*

$$\omega = h_{\mathbb{H}}(\pi(\cdot)u).$$

Moreover,  $|\omega|_r = |\omega|_l$ .

*Proof.* Let  $v$  be as in Theorem 2.3 and write  $u = \pi(v)$ . Let  $h_{\mathbb{H}}$  be the Haar state of  $\mathbb{H}$  so that  $|\omega|_r = h_{\mathbb{H}} \circ \pi$ . Since  $N_{|\omega|_r \otimes |\omega|_r} = \ker \pi \otimes \pi$  (because  $|\omega|_r \otimes |\omega|_r = (h_{\mathbb{H}} \otimes h_{\mathbb{H}}) \circ (\pi \otimes \pi)$  and  $h_{\mathbb{H}} \otimes h_{\mathbb{H}}$  is faithful), we have

$$\Delta_{\mathbb{H}}(u) = u \otimes u$$

by Theorem 2.3. Then

$$h_{\mathbb{H}}(uu^*)1 = (h_{\mathbb{H}} \otimes \text{id})\Delta_{\mathbb{H}}(uu^*) = h_{\mathbb{H}}(uu^*)uu^*.$$

Since  $h_{\mathbb{H}}$  is faithful and  $u \neq 0$  (because  $\omega \neq 0$ ), it follows that  $uu^* = 1$ . Similarly  $u^*u = 1$ , so  $u$  is a group-like unitary in  $C(\mathbb{H})$ .

It remains to prove that  $|\omega|_l = |\omega|_r$ . Consider the state  $h' := u \cdot h_{\mathbb{H}} \cdot u^* \in C(\mathbb{H})^*$ . It is easy to check that  $h'$  is an idempotent state (see Proposition 2.1). As it is faithful on  $C(\mathbb{H})$  it must

be in fact equal to  $h_{\mathbb{H}}$  (see Lemma 2.1 of [Wor98]). Let then  $a \in C_0(\mathbb{G})$  and compute:

$$\begin{aligned} |\omega|_l(a) &= |\omega|_r(v^*av) = h_{\mathbb{H}}(\pi(v^*av)) = h_{\mathbb{H}}(u^*\pi(a)u) \\ &= h'(\pi(a)) = h_{\mathbb{H}}(\pi(a)) = |\omega|_r(a). \end{aligned}$$

□

*Remark 2.6.* Naturally the above theorem has also a ‘left’ version – one can begin by assuming that  $|\omega|_l$  is a Haar idempotent (the latter is clearly true if and only if  $|\omega|_r$  is a Haar idempotent).

### 3. TROs ASSOCIATED WITH CONTRACTIVE IDEMPOTENTS

In this section we show that to each contractive idempotent in  $M(\mathbb{G})$  one can associate in a natural way a ternary ring of operators in  $C_0(\mathbb{G})$ . We begin by introducing some definitions.

A *ternary ring of operators (TRO)* acting between Hilbert spaces  $H$  and  $K$  is a closed subspace  $T$  of  $B(H; K)$  that is closed under the ternary product:

$$ab^*c \in T \quad \text{whenever } a, b, c \in T.$$

If  $X \subseteq T$  is a sub-TRO (i.e. a closed subspace that is closed under the ternary product), then a contractive linear map  $P$  from  $T$  onto  $X$  is called a *TRO conditional expectation* if

$$P(ax^*y) = P(a)x^*y, \quad P(xa^*y) = xP(a)^*y, \quad P(xy^*a) = xy^*P(a),$$

for every  $a \in T$  and  $x, y \in X$ . TRO conditional expectations are automatically completely contractive ([EOR01]).

We shall show that the closed right invariant subspace  $C_\omega = L_\omega(C_0(\mathbb{G}))$  associated to a contractive idempotent  $\omega$  is a sub-TRO of  $C_0(\mathbb{G})$  and that  $L_\omega: C_0(\mathbb{G}) \rightarrow C_\omega$  is a TRO conditional expectation.

**Lemma 3.1.** *Let  $\omega$  be a contractive idempotent on  $C_0(\mathbb{G})$  and let  $a, b \in C_0(\mathbb{G})$ . Then*

- (1)  $L_\omega(L_\omega(a)b) = L_\omega(a)L_{|\omega|_l}(b)$ ;
- (2)  $L_{|\omega|_l}(L_\omega(a)^*b) = L_\omega(a)^*L_\omega(b)$ ;
- (3)  $L_\omega(aL_\omega(b)) = L_{|\omega|_r}(a)L_\omega(b)$ ;
- (4)  $L_{|\omega|_r}(aL_\omega(b)^*) = L_\omega(a)L_\omega(b)^*$ .

*Proof.* Let  $a, b \in C_0(\mathbb{G})$ . Denote by  $\Delta^{(2)}: C_0(\mathbb{G}) \rightarrow M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$  the map given by  $(\Delta \otimes \text{id}) \circ \Delta$  (equivalently  $(\text{id} \otimes \Delta) \circ \Delta$ ). Then

$$\begin{aligned} L_\omega(L_\omega(a)b) &= (\omega \otimes \text{id})((\omega \otimes \text{id} \otimes \text{id})(\Delta^{(2)}(a))\Delta(b)) \\ &= (|\omega|_l \otimes |\omega|_l \otimes \text{id})((v \otimes v \otimes 1)(\Delta^{(2)}(a))(1 \otimes \Delta(b))) \\ &= (|\omega|_l \otimes |\omega|_l \otimes \text{id})((\Delta(v) \otimes 1)(\Delta^{(2)}(a))(1 \otimes \Delta(b))) \end{aligned}$$

because  $\Delta(v) - v \otimes v \in (N_{|\omega|_l \otimes |\omega|_l})^*$ . Continuing the calculation we have

$$\begin{aligned} L_\omega(L_\omega(a)b) &= (|\omega|_l \otimes |\omega|_l \otimes \text{id})((\Delta \otimes \text{id})((v \otimes 1)\Delta(a))(1 \otimes \Delta(b))) \\ &= (|\omega|_l \otimes \text{id})((L_{|\omega|_l} \otimes \text{id})((v \otimes 1)\Delta(a))\Delta(b)) \\ &= (|\omega|_l \otimes \text{id})((L_{|\omega|_l} \otimes \text{id})((v \otimes 1)\Delta(a)))(|\omega|_l \otimes \text{id})(\Delta(b)) \end{aligned}$$

because the multiplicative domain of the idempotent state  $|\omega|_l$  contains  $L_{|\omega|_l}(C_0(\mathbb{G}))$  by Lemma 2.5 of [SS12]. Rewriting the last expression, we obtain the first statement:

$$L_\omega(L_\omega(a)b) = L_\omega(a)L_{|\omega|_l}(b).$$

Next we confirm the second statement, using similar kind of manipulations as above:

$$\begin{aligned} L_{|\omega|_l}(L_\omega(a)^*b) &= (|\omega|_l \otimes \text{id})(\Delta(b^*)\Delta(L_\omega(a)))^* \\ &= (\omega \otimes \text{id})((v^* \otimes 1)\Delta(b^*)\Delta(L_\omega(a)))^* \\ &= (|\omega|_r \otimes \text{id})((v^* \otimes 1)\Delta(b^*)\Delta(L_\omega(a))(v \otimes 1))^* \\ &= (|\omega|_r \otimes |\omega|_r \otimes \text{id})((1 \otimes v^* \otimes 1)(1 \otimes \Delta(b^*))\Delta^{(2)}(a)(v \otimes v \otimes 1))^* \\ &= (|\omega|_r \otimes \text{id})((v^* \otimes 1)\Delta(b^*)(L_{|\omega|_r} \otimes \text{id})(\Delta(a)(v \otimes 1)))^* \\ &= (|\omega|_r \otimes \text{id})(\Delta(a)(v \otimes 1))^*(|\omega|_r \otimes \text{id})((v^* \otimes 1)\Delta(b^*))^* \\ &= L_\omega(a)^*L_\omega(b). \end{aligned}$$

The last two statements are right-hand analogues of the first two and follow if the first two statements are applied to the contractive idempotent  $\bar{\omega}$ , which is defined by  $\bar{\omega}(a) = \omega(a^*)^*$ . (Note that  $|\bar{\omega}|_r = |\omega|_l$ .)  $\square$

Before we formulate the main theorem of this section we present one consequence of the above formulas: contractive idempotents are automatically invariant under the adjoint map  $\sharp$ .

**Corollary 3.2.** *If  $\omega \in M(\mathbb{G})$  is a contractive idempotent then  $\omega^\sharp = \omega$ .*

*Proof.* The proof is similar to the proof of Proposition 2.6 of [SS12] concerning idempotent states. However, since  $L_\omega$  is no longer a conditional expectation, we sketch a proof.

Take the right multiplicative unitary  $V$  of  $\mathbb{G}$  and consider  $p = (\text{id} \otimes \omega)V$ . Then  $(\text{id} \otimes L_\omega)(V) = (p \otimes 1)V$ . By Lemma 3.1 (4), used in the third identity,

$$\begin{aligned} pp^* \otimes 1 &= (p \otimes 1)VV^*(p^* \otimes 1) = (\text{id} \otimes L_\omega)(V)(\text{id} \otimes L_\omega)(V)^* = (\text{id} \otimes L_{|\omega|_r})(V(\text{id} \otimes L_\omega)(V)^*) \\ &= (\text{id} \otimes L_{|\omega|_r})(VV^*(p^* \otimes 1)) = p^* \otimes 1. \end{aligned}$$

It follows that  $p = p^*$  and so

$$\omega((\sigma \otimes \text{id})(V)) = \sigma(p) = \sigma(p^*) = \bar{\omega}((\sigma \otimes \text{id})(V^*)) = \omega^\sharp((\sigma \otimes \text{id})(V))$$

for every  $\sigma \in B(L^2(\mathbb{G}))_*$ .  $\square$

**Theorem 3.3.** *Let  $\omega$  be a contractive idempotent on  $C_0(\mathbb{G})$ . Then  $C_\omega = L_\omega(C_0(\mathbb{G}))$  is a TRO and  $L_\omega$  is a TRO conditional expectation.*

*Proof.* Both statements follow from the identities

$$\begin{aligned} L_\omega(a)L_\omega(b)^*L_\omega(c) &= L_\omega(L_\omega(a)L_\omega(b)^*c) \\ &= L_\omega(L_\omega(a)b^*L_\omega(c)) \\ &= L_\omega(aL_\omega(b)^*L_\omega(c)) \end{aligned}$$

where  $a, b, c \in C_0(\mathbb{G})$ . We now check the first identity, the other two being of similar flavour (in fact, the first identity already implies the other two by the general theory of completely contractive projections; see Corollary 3 of [You83]). By Lemma 3.1 (1) and (2) we have

$$L_\omega(L_\omega(a)L_\omega(b)^*c) = L_\omega(a)L_{|\omega|_l}(L_\omega(b)^*c) = L_\omega(a)L_\omega(b)^*L_\omega(c).$$

$\square$

Every TRO  $T \subset B(\mathbf{H}; \mathbf{K})$  is associated with three  $C^*$ -algebras: the *left linking algebra*  $\langle TT^* \rangle \subset B(\mathbf{K})$ , the *right linking algebra*  $\langle T^*T \rangle \subset B(\mathbf{H})$  and the *linking algebra*

$$A_T = \begin{bmatrix} \langle TT^* \rangle & T \\ T^* & \langle T^*T \rangle \end{bmatrix} \subset B(\mathbf{K} \oplus \mathbf{H}).$$

We say that a sub-TRO  $X \subseteq T$  is *nondegenerate* if

$$\langle XT^*T \rangle = T \quad \text{and} \quad \langle TT^*X \rangle = T.$$

Equivalently, the linking  $C^*$ -algebra  $A_X$  is nondegenerate in  $A_T$ . We note that for every contractive idempotent  $\omega$  the sub-TRO  $L_\omega(C_0(\mathbb{G}))$  is nondegenerate in  $C_0(\mathbb{G})$ . Indeed,

$$L_\omega(C_0(\mathbb{G}))C_0(\mathbb{G}) = (\omega \otimes \text{id})(\Delta(C_0(\mathbb{G}))(1 \otimes C_0(\mathbb{G})))$$

and thus it follows from the quantum cancellation laws (1) that  $L_\omega(C_0(\mathbb{G}))C_0(\mathbb{G})$  is linearly dense in  $C_0(\mathbb{G})$ .

Theorem 2.1 of [SS] shows that a TRO conditional expectation  $P: T \rightarrow X$  onto a nondegenerate sub-TRO  $X \subseteq T$  extends (in a unique way) to a  $C^*$ -algebra conditional expectation  $E: A_T \rightarrow A_X$  between the linking algebras. Explicitly,

$$E = \begin{bmatrix} PP^\dagger & P \\ P^\dagger & P^\dagger P \end{bmatrix}$$

where

$$P^\dagger(a) = P(a^*)^* \quad PP^\dagger(ax^*) = P(a)x^* \quad P^\dagger P(x^*a) = x^*P(a)$$

for  $a \in T$  and  $x \in X$ . It then follows from Lemma 3.1 that the extension of  $L_\omega$  to a conditional expectation  $E: M_2(C_0(\mathbb{G})) \rightarrow A_{L_\omega(C_0(\mathbb{G}))}$  is equal to

$$E = \begin{bmatrix} L_{|\omega|_r} & L_\omega \\ L_{\bar{\omega}} & L_{|\omega|_l} \end{bmatrix}.$$

#### 4. ABSTRACT CHARACTERISATION OF THE SUBSPACES OF $C_0(\mathbb{G})$ RELATED TO CONTRACTIVE IDEMPOTENTS

In this section we generalise results of Sections 3 and 4 in [SS12] (see also [FS09<sub>2</sub>] and [Sal11]) describing a natural correspondence between the idempotent states on  $M(\mathbb{G})$  and certain subalgebras of  $C_0(\mathbb{G})$ .

Denote the left Haar weight of  $\mathbb{G}$  by  $\phi$  and the right Haar weight of  $\mathbb{G}$  by  $\psi$ . Consider the map  $\psi^{(2)}: M_2(C_0(\mathbb{G}))_+ \rightarrow [0, +\infty]$  defined by

$$\psi^{(2)} \left( \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \right) = \psi(a_1) + \psi(a_4).$$

It is easy to check that  $\psi^{(2)}$  is a densely defined, faithful, lower semicontinuous weight on  $M_2(C_0(\mathbb{G}))$ . Moreover, for  $a_1, a_2, a_3, a_4 \in C_0(\mathbb{G})$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \mathfrak{N}_{\psi^{(2)}} \iff a_1, a_2, a_3, a_4 \in \mathfrak{N}_\psi,$$

where  $\mathfrak{N}_\psi = \{x \in C_0(\mathbb{G}) : \psi(x^*x) < \infty\}$ ,  $\mathfrak{N}_{\psi^{(2)}} = \{x \in M_2(C_0(\mathbb{G})) : \psi^{(2)}(x^*x) < \infty\}$ . The weight  $\phi^{(2)}$  is defined in an analogous way. A subset  $F$  of  $M_2(C_0(\mathbb{G}))$  will be called *right invariant* if for all  $\nu \in C_0(\mathbb{G})^*$  the matrix-lifted right shift map  $R_\nu^{(2)}: M_2(C_0(\mathbb{G})) \rightarrow M_2(C_0(\mathbb{G}))$  leaves  $F$  invariant.

If  $\eta$  is a weight on a  $C^*$ -algebra  $A$ , and  $B$  is a  $C^*$ -subalgebra of  $A$  such that there exists a conditional expectation from  $A$  onto  $B$  which preserves  $\eta$  (i.e.  $\eta(P(a)) = \eta(a)$  where  $P$  is the conditional expectation and  $a \in A_+$  with  $\eta(a) < \infty$ ), then we call  $B$  an  $\eta$ -expected subalgebra.

**Theorem 4.1.** *Suppose that  $X$  is a non-degenerate TRO in  $C_0(\mathbb{G})$ . If  $A_X \subset M_2(C_0(\mathbb{G}))$  is a right invariant,  $\psi^{(2)}$ -expected subalgebra, then there exists a contractive idempotent functional  $\omega \in M(\mathbb{G})$  such that  $X = L_\omega(C_0(\mathbb{G}))$ . Moreover  $A_X \subset M_2(C_0(\mathbb{G}))$  is also a  $\phi^{(2)}$ -expected subalgebra.*

*Proof.* Assume that  $X$  satisfies the assumptions of the theorem. Let  $E : M_2(C_0(\mathbb{G})) \rightarrow A_X$  be the corresponding  $\psi^{(2)}$ -preserving conditional expectation. Note first that the right invariance of  $A_X$  means that each of the spaces  $\langle XX^* \rangle$ ,  $\langle X^*X \rangle$  and  $X$  are right invariant (as subsets of  $C_0(\mathbb{G})$ ). This in turn allows us to deduce that for any  $\nu \in M_2(C_0(\mathbb{G}))^*$  the map

$$R_\nu := \begin{bmatrix} R_{\nu_{11}} & R_{\nu_{12}} \\ R_{\nu_{21}} & R_{\nu_{22}} \end{bmatrix}$$

leaves  $A_X$  invariant (we use the natural identification  $M_2(C_0(\mathbb{G}))^* = M_2(C_0(\mathbb{G})^*)$ ). In the next step we show that in fact for all  $\nu \in M_2(C_0(\mathbb{G}))^*$

$$(2) \quad ER_\nu = R_\nu E.$$

The method will be similar to that used in Proposition 3.3 of [SS12]. Note first that by Lemma 1.2 it suffices to show the above equality for  $\nu$  in a weak\*-dense subset of  $M_2(C_0(\mathbb{G}))^*$ . Assume then that

$$\nu = \begin{bmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{bmatrix} \in M_2(L_\#^1(\mathbb{G}))$$

and let  $\rho_{ij} \in L_\#^1(\mathbb{G})$  be such that  $\rho_{ij} = \nu_{ij}^\#$  for  $i, j = 1, 2$ . Further write  $\rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$  and recall that it follows from the computations in the proof of Proposition 3.3 in [SS12] that for any  $a, b \in \mathfrak{N}_\psi$  the following equality holds:  $\psi(a^*R_{\nu_{ij}}(c)) = \psi(R_{\rho_{ij}}(a)^*c)$ .

As  $R_\nu(A_X) \subset A_X$ , by Lemma 1.3 of [SS12] to prove equality (2) it suffices to show that  $R_\nu((A_X)^\perp) \subset (A_X)^\perp$ , where

$$(A_X)^\perp = \{ \mathbf{a} \in M_2(C_0(\mathbb{G})) \cap \mathfrak{N}_{\psi^{(2)}}; \psi^{(2)}(\mathbf{c}^*\mathbf{a}) = 0 \text{ for all } \mathbf{c} \in A_X \cap \mathfrak{N}_{\psi^{(2)}} \}.$$

Let then  $\mathbf{a} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in A_X \cap \mathfrak{N}_{\psi^{(2)}}$  and  $\mathbf{c} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \in (A_X)^\perp$ . We thus have, by the formulas above,

$$\begin{aligned} \psi^{(2)}(\mathbf{a}^*R_\nu(\mathbf{c})) &= \psi(a_{11}^*R_{\nu_{11}}(c_{11}) + a_{21}^*R_{\nu_{21}}(c_{21})) + \psi(a_{12}^*R_{\nu_{12}}(c_{12}) + a_{22}^*R_{\nu_{22}}(c_{22})) \\ &= \psi(R_{\rho_{11}}(a_{11})^*c_{11} + R_{\rho_{21}}(a_{21})^*c_{21}) + \psi(R_{\rho_{12}}(a_{12})^*c_{12} + R_{\rho_{22}}(a_{22})^*c_{22}) \\ &= \psi^{(2)}(R_\rho(\mathbf{a})^*\mathbf{c}) = 0, \end{aligned}$$

where the last formula follows from the definition of  $(A_X)^\perp$ . This finishes the proof of equality (2).

The first consequence of (2) is the following. Consider  $\epsilon_{11} := \begin{bmatrix} \epsilon & 0 \\ 0 & 0 \end{bmatrix} \in M_2(C_0(\mathbb{G}))^*$ , where  $\epsilon \in M(\mathbb{G})$  denotes the counit. It is easy to see that  $R_{\epsilon_{11}} = \begin{bmatrix} \text{id}_{C_0(\mathbb{G})} & 0 \\ 0 & 0 \end{bmatrix}$ , so the analysis of the

commutation relations of  $E$  with  $R_{\epsilon_{11}}$  (and its analogues) implies that  $E$  is a so-called *Schur map*, i.e.

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},$$

where  $E_{ij} : C_0(\mathbb{G}) \rightarrow C_0(\mathbb{G})$  are completely contractive projections; in particular  $E_{12}$  is a completely contractive projection onto  $X$ . It is further easy to see that each of the maps  $E_{ij}$  commutes with all the maps  $R_\nu$ ,  $\nu \in C_0(\mathbb{G})^*$ . Hence by Lemma 1.2 there exist contractive functionals  $\omega_{ij} \in C_0(\mathbb{G})^*$  such that  $E_{ij} = L_{\omega_{ij}}$  for  $i, j = 1, 2$ . Each  $\omega_{ij}$  is a contractive idempotent functional. It suffices to put  $\omega := \omega_{12}$  to end the first part of the proof.

The second part follows because in Section 3 we showed that if  $X = L_\omega(C_0(\mathbb{G}))$ , then the map

$$E' := \begin{bmatrix} L_{|\omega|_r} & L_\omega \\ L_{\bar{\omega}} & L_{|\omega|_l} \end{bmatrix}$$

is a conditional expectation onto  $A_X$ ; it is easy to check that it preserves  $\phi^{(2)}$ . Moreover,  $E' = E$  by the uniqueness of the extension of the TRO expectation  $L_\omega$  (Theorem 2.1 of [SS]).  $\square$

Recall that  $\mathbb{G}$  is said to be *unimodular* if  $\phi = \psi$ . If  $\mathbb{G}$  is unimodular and  $A$  is a  $\psi$ -expected  $C^*$ -subalgebra of  $C_0(\mathbb{G})$ , then we simply call it *expected*.

**Corollary 4.2.** *Assume that  $\mathbb{G}$  is unimodular. There is a bijective correspondence between*

- (i) *contractive idempotent functionals  $\omega \in C_0(\mathbb{G})^*$ ;*
- (ii) *nondegenerate TROs  $X$  in  $C_0(\mathbb{G})$  such that their associated  $C^*$ -algebra  $A_X \subset M_2(C_0(\mathbb{G}))$  is right invariant and expected.*

*In that case the conditional expectation  $E : M_2(C_0(\mathbb{G})) \rightarrow A_X$  preserving the invariant weight is given by the formula*

$$E = \begin{bmatrix} L_{|\omega|_r} & L_\omega \\ L_{\bar{\omega}} & L_{|\omega|_l} \end{bmatrix}.$$

*Proof.* An easy consequence of the last theorem and the results of Section 2.  $\square$

We say that a TRO  $X$  contained in a  $C^*$ -algebra  $A$  is *balanced* if  $\langle X^*X \rangle = \langle XX^* \rangle$ . A right invariant  $C^*$ -subalgebra  $C$  of  $C_0(\mathbb{G})$  (such as  $\langle X^*X \rangle$  or  $\langle XX^* \rangle$ ) is said to be *symmetric* if  $V^*(1 \otimes c)V$  is in  $M(K_{L^2(\mathbb{G})} \otimes C)$  for every  $c \in C$ . The symmetry condition originates from [Tom07] and [Sal11].

**Corollary 4.3.** *There is a bijective correspondence between*

- (i) *contractive idempotent functionals  $\omega \in C_0(\mathbb{G})^*$  whose left (equivalently, right; equivalently left and right) absolute values are Haar idempotents;*
- (ii) *nondegenerate balanced TROs in  $C_0(\mathbb{G})$  such that their associated  $C^*$ -algebra  $A_X \subset M_2(C_0(\mathbb{G}))$  is right-invariant and  $\psi^{(2)}$ -expected (equivalently, expected) and the algebra  $\langle XX^* \rangle = \langle X^*X \rangle$  is symmetric.*

(In fact, if either  $\langle XX^* \rangle$  or  $\langle X^*X \rangle$  is symmetric, then they are equal.)

*Proof.* Let  $\omega \in C_0(\mathbb{G})^*$  be a contractive idempotent and put  $X = L_\omega(C_0(\mathbb{G}))$ . Theorem 3.7 of [SS12] implies that  $|\omega|_l$  is a Haar idempotent if and only if  $L_{|\omega|_l}(C_0(\mathbb{G})) = \langle X^*X \rangle$  is

symmetric. Moreover in that case  $|\omega|_l = |\omega|_r$  by Theorem 2.5 (so also  $\langle X^*X \rangle = \langle XX^* \rangle$ ), and the conditional expectation

$$E = \begin{bmatrix} L_{|\omega|_r} & L_\omega \\ L_{\bar{\omega}} & L_{|\omega|_r} \end{bmatrix}$$

onto  $A_X$  preserves both  $\psi^{(2)}$  and  $\phi^{(2)}$  in view of Proposition 3.12 of [SS12].

Conversely, if  $X$  is a nondegenerate TRO and  $A_X$  is  $\psi^{(2)}$ -expected, then Theorem 4.1 yields a contractive idempotent  $\omega \in C_0(\mathbb{G})^*$  such that  $X = L_\omega(C_0(\mathbb{G}))$ . Moreover the fact that  $\langle X^*X \rangle = L_{|\omega|_l}$  is symmetric implies that  $|\omega|_l$  is a Haar idempotent, so  $|\omega|_l = |\omega|_r$  and  $L_{|\omega|_l}$  preserves both invariant weights. Hence  $A_X$  is expected.  $\square$

## 5. EXAMPLES OF CONTRACTIVE IDEMPOTENTS AND SOME SPECIAL CASES

In this section we collect certain examples and special cases of the results obtained in the previous sections.

**Proposition 5.1** ([Gre65]). *Let  $G$  be a locally compact group. Then every contractive idempotent measure  $\mu$  on  $G$  is of the form*

$$d\mu(s) = \chi(s) dm_H(s)$$

where  $m_H$  is the Haar measure of a compact subgroup  $H$  of  $G$  and  $\chi$  is a continuous character of  $H$ .

*Proof.* Every idempotent state on  $G$  is a Haar idempotent, so Theorem 2.5 applies. Moreover,  $|\mu|_r = |\mu| = m_H$  for some compact subgroup  $H$  of  $G$ . By Theorem 2.5,  $u$  is a continuous character in  $C(H)$ , so we are done.  $\square$

In view of the above proposition it is natural to ask what do contractive idempotent functionals look like in the dual case, i.e. when  $C_0(\mathbb{G}) = C^*(\Gamma)$  for a locally compact amenable group  $\Gamma$  (sometimes one writes then  $\mathbb{G} = \hat{\Gamma}$ ). The answer was given by Theorem 2.1 (i) of [IS05] (in a more general context where  $\Gamma$  need not be amenable); here we show how the special case of that result can be quickly deduced via the techniques developed in this paper.

**Proposition 5.2** ([IS05]). *Let  $G$  be an amenable locally compact group and let  $\omega \in C^*(G)^* = B(G)$ . Then  $\omega$  is a contractive idempotent if and only if it is of the form  $1_C$ , where  $C$  is an open coset of  $G$ . Moreover  $|\omega|_r$  (equivalently,  $|\omega|_l$ ) is a Haar idempotent if and only if  $C^{-1}C$  is a normal subgroup of  $G$ .*

*Proof.* (Sketch) Let  $\omega \in B(G)$ . It is easy to check that the map  $L_\omega : M(C^*(G)) \rightarrow M(C^*(G))$  is then the Schur multiplier corresponding to the function  $\omega$ . Thus the fact that  $L_\omega$  is an idempotent implies that  $\omega$  is a characteristic function. Due to continuity it must be a characteristic function of a clopen set, say  $C$ . Further if  $\omega$  is a contractive idempotent, (strict extensions of) the equalities displayed in the proof of Theorem 3.3 applied to generating unitaries in  $M(C^*(G))$  imply that if  $g_1, g_2, g_3 \in C$ , then also  $g_1 g_2^{-1} g_3 \in C$ . By Proposition 1.1 of [IS05]  $C$  must then be a coset of  $G$ . The other implication is easy to check, for example using the mentioned before fact that TRO conditional expectations are necessarily contractive.

Finally the last statement in the proposition is a consequence of the results in Section 7 of [Sal11]; it can be also checked directly.  $\square$

In fact our results allow us also to produce in some cases a complete list of contractive idempotents on a given (locally) compact quantum group  $\mathbb{G}$ . The next proposition presents a result of that type for  $SU_q(2)$  (see [Wor87] for the definition of the latter quantum group).

**Proposition 5.3.** *Let  $q \in (0, 1)$ . The contractive idempotents on  $SU_q(2)$  are the Haar state and the Haar idempotents associated with the subgroups  $\mathbf{T}$  and  $\mathbf{Z}_n$  combined with the characters of those subgroups.*

*Proof.* By Theorem 5.1 of [FST<sub>1</sub>], all idempotent states of  $SU_q(2)$  are Haar idempotents associated with the subgroups listed in the statement. Then the statement follows from Theorem 2.5 once we note that the algebra  $C(SU_q(2))$  itself does not have any nontrivial group-like unitaries (i.e.  $SU_q(2)$  does not have any non-trivial one-dimensional representations).  $\square$

In the case presented above there exist relatively few contractive idempotents and all of them fit in the scheme described in Theorem 2.5. Below we discuss briefly an example of a genuinely quantum (i.e. neither commutative nor cocommutative) group where there exist non-Haar idempotent states and thus also more complicated contractive idempotent functionals.

*Example 5.4.* Consider the quantum group  $SU_{-1}(2)$ , which is of Kac type. It is well-known that  $SU_{-1}(2)$  admits the dual of  $D_\infty$  (the infinite dihedral group) as a compact quantum subgroup, see for example [Pod95] or [FST<sub>2</sub>]. In the latter notes a relevant surjective \*-homomorphism  $\pi : C(SU_{-1}(2)) \rightarrow C^*(D_\infty)$  exchanging the comultiplications is shown to be determined by the conditions

$$\pi(u_{11}) = \pi(u_{22}) = \frac{1}{2}t(1+z), \pi(u_{12}) = \pi(u_{21}) = \frac{1}{2}t(1-z),$$

where  $U = (u_{ij})_{i,j=1}^2 \in M_n(C(SU_{-1}(2)))$  is the defining fundamental representation of  $SU_{-1}(2)$  and  $t, z$  are the canonical unitary generators of  $C^*(D_\infty)$  satisfying the commutation relations  $t^2 = 1$  and  $ztz = t$ . Thus choosing a contractive idempotent functional  $\mu$  on  $C^*(D_\infty)$  whose left or right absolute value is not a Haar idempotent (i.e., by Proposition 5.2, choosing a coset of a non-normal subgroup of  $D_\infty$ ) and composing it with  $\pi$  yields a contractive idempotent functional on  $C(SU_{-1}(2))$  whose absolute value is not a Haar idempotent. A concrete example can be obtained by putting  $C = \{t, tz\}$  – note we identify the unitary generators of  $C^*(D_\infty)$  with elements of  $D_\infty$ . In fact the results of [FST<sub>2</sub>] and [Tom08] make it possible in principle to provide a full list of the contractive idempotent functionals on  $C(SU_{-1}(2))$ .

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