Math. Model. Nat. Phenom. 15 (2020) 51
https://doi.org/10.1051/mmnp/2020013

Mathematical Modelling of Natural Phenomena
www.mmnp-journal.org

NEWTON AND BOULIGAND DERIVATIVES OF THE SCALAR PLAY AND STOP OPERATOR

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Abstract. We prove that the play and the stop operator possess Newton and Bouligand derivatives, and exhibit formulas for those derivatives. The remainder estimate is given in a strengthened form, and a corresponding chain rule is developed. The construction of the Newton derivative ensures that the mappings involved are measurable.

Mathematics Subject Classification. 47H30, 47J40, 49J52, 49M15, 58C20.

Received April 16, 2019. Accepted April 6, 2020.

1. Introduction

The aim of this paper is to show that the play and the stop operator possess Newton as well as Bouligand derivatives, and to compute those derivatives. Newton derivatives are needed when one wants to solve equations

$$F(u) = 0$$

for nonsmooth operators F by Newton's method with a better than linear convergence rate. Bouligand derivatives are closely related to Newton derivatives, and can be used to provide sensitivity results as well as optimality conditions for problems involving nonsmooth operators.

The scalar play operator and its twin, the scalar stop operator, act on functions $u : [a, b] \to \mathbb{R}$ and yield functions $w = \mathcal{P}_r[u; z_0]$ and $z = \mathcal{S}_r[u; z_0]$ from $[a, b]$ to $\mathbb{R}$. The number z_0 plays the role of an initial condition. Their formal definition, in the spirit of [11, 12], is given below in Section 6; alternatively, they arise as solution operators of the evolution variational inequality

$$\dot{w}(t) \cdot (\zeta - z(t)) \le 0\,, \quad \text{for all } \zeta \in [-r, r], \tag{1.1a}$$

$$z(t) \in [-r, r]\,, \quad z(a) = z_0 \in [-r, r]\,, \tag{1.1b}$$

$$w(t) + z(t) = u(t)\,. \tag{1.1c}$$

Keywords and phrases: Rate independence, hysteresis operator, Newton derivative, Bouligand derivative, play, stop, sensitivity, maximum functional, variational inequality, measurable selector, semismooth, chain rule.

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The play and the stop operator are rate-independent; in fact, they constitute the simplest nontrivial examples of rate-independent operators [4, 13, 14, 19] if one disregards relays whose nature is inherently discontinuous. Due to (1.1c), their mathematical properties are closely related.

A lot is known about the play and the stop. Viewed as operators between function spaces, their typical regularity is Lipschitz (or less). In particular, they are not differentiable in the classical sense. The question whether weaker derivatives (*e.g.*, directional derivatives) exist was addressed, to the author's knowledge, for the first time in [3] where it was shown that the play and the stop are directionally differentiable from $C[a,b]$ to $L^p(a,b)$ for $p < \infty$. (This is not to be confused with the existence and form of time derivatives of functions like $t \mapsto \mathcal{P}_r[u;z_0](t)$, for which there are many results available.)

The results below serve to narrow the gap between differentiability and non-differentiability of rate-independent operators. Their proofs given here are based on the same idea as used in [3], namely, to locally represent the play as a composition of operators whose main ingredient is the cumulated maximum.

It is natural to ask whether it is possible to prove weak differentiability of the play and the stop operator in the framework of the variational formulation (1.1). Indeed, for elliptic variational inequalities, a large body of literature is available, going back to [15]. In that case, the solution operator is closely linked to the metric projection onto convex sets whose differentiability properties also have been analyzed for a long time. For evolution variational inequalities of parabolic type, we refer to the recent contribution [5] and the literature cited there. For rate independent variational inequalities, corresponding results do not seem to exist, not even for the ODE case given in (1.1).

Our main results are given in Theorem 7.15 for Newton differentiability and Theorem 8.2 for Bouligand differentiability of the play. They are based on corresponding results for the maximum functional (Prop. 3.4) and the cumulated maximum operator (Prop. 4.8). The extension to the parametric play is given in Proposition 9.5.

When attempting to prove Newton differentiability of the play, some issues arise which complicate matters and are, at least in part, responsible for the length of this paper. First, the construction of the Newton derivative of the play leads to a set-valued derivative in a natural manner. Its elements L should have the property that the first order approximations $\delta w = L\delta u$ are measurable functions. Since Newton derivatives are not obtained as limits, and we are dealing with operators between function spaces, measurability becomes an issue. Second, with regard to the form of the remainder, we aim at a somewhat stronger result than standard Newton differentiability, having in mind applications to partial differential equations. Third, we want to treat not only a single play operator, but also a parametric family of play operators, having in mind problems where play operators *e.g.* are distributed continuously over space. Again, the problem of measurability has to be solved.

The proofs of Newton and of Bouligand differentiability are rather similar; for Bouligand derivatives, some of the problems mentioned above do not even arise. Nevertheless, we have chosen to elaborate the proofs for both cases to some extent; the details are somewhat cumbersome and should not be placed too much as a burden on the reader.

2. Notions of derivatives

We collect some established notions of derivatives for mappings

$$F : U \to Y \,, \quad U \subset X \,,$$

where X and Y are normed spaces, and U is an open subset of X. These notions are classical, but the terminology is not uniform in the literature.

Definition 2.1. (i) The limit, if it exists,

$$F'(u;h) := \lim_{\lambda \downarrow 0} \frac{F(u+\lambda h) - F(u)}{\lambda} \,, \quad u \in U \,,\, h \in X \,, \tag{2.1}$$

is called the **directional derivative** of F at u in the direction h. It is an element of Y.
(ii) If the directional derivative satisfies

$$F'(u;h) = \lim_{\lambda \downarrow 0} \frac{F(u + \lambda h + r(\lambda)) - F(u)}{\lambda} \tag{2.2}$$

for all functions $r : (0, \lambda_0) \to X$ with $r(\lambda)/\lambda \to 0$ as $\lambda \to 0$, it is called the **Hadamard derivative** of F at u in the direction h.
(iii) If the directional derivative exists for all $h \in X$ and satisfies

$$\lim_{h \to 0} \frac{\|F(u+h) - F(u) - F'(u;h)\|}{\|h\|} = 0\,, \tag{2.3}$$

it is called the **Bouligand derivative** of F at u in the direction h.
(iv) If the Bouligand derivative has the form $F'(u;h) = Lh$ for some linear continuous mapping $L : X \to Y$, then L is called the **Fréchet derivative** of F at u and denoted as $DF(u)$.
(v) The mapping F is called directionally (Hadamard, Bouligand, Fréchet, resp.) differentiable at u (in U, resp.), if the corresponding derivative exists at u (for all $u \in U$, resp.) for all directions $h \in X$.

In the definition above, it is tacitly understood that the limits are taken in the sense "not equal 0".

We have $F'(u; \lambda h) = \lambda F'(u;h)$ if $\lambda \geq 0$. This as well as the following well-known facts are elementary consequences of the above definitions.

Proposition 2.2. *Let F be directionally differentiable and locally Lipschitz continuous at $u \in U$. Then F is Hadamard differentiable at u. Moreover, if ℓ_u is a local Lipschitz constant for F at u,*

$$\|F'(u;h_1) - F'(u;h_2)\| \leq \ell_u \|h_1 - h_2\| \qquad \forall\, h_1, h_2 \in X\,. \tag{2.4}$$

Consequently, if ℓ is a global Lipschitz constant for F,

$$\|F(u+h) - F(u) - F'(u;h)\| \leq 2\ell \|h\| \qquad \forall\, h \in X\,. \tag{2.5}$$

Corollary 2.3. *If F is locally Lipschitz, then directional and Hadamard differentiability at $u \in U$ are equivalent, and are implied by Bouligand differentiability at u.*

In terms of a remainder function, the definition (2.3) of Bouligand differentiability at u is equivalent to

$$\|F(u+h) - F(u) - F'(u;h)\| \leq \rho_u(\|h\|) \cdot \|h\|\,, \tag{2.6}$$

where $\rho_u(\delta) \downarrow 0$ for $\delta \downarrow 0$. In view of (2.5), we may assume that ρ_u is globally bounded,

$$\rho_u \leq 2\ell \qquad \forall\, u \in U\,, \tag{2.7}$$

if ℓ is a global Lipschitz constant for F.

The notion of a Newton derivative is more recent. A mapping $G : U \to \mathcal{L}(X,Y)$, the space of all linear and continuous mappings from X to Y, is called a Newton derivative of F in U, if

$$\lim_{h \to 0} \frac{\|F(u+h) - F(u) - G(u+h)h\|}{\|h\|} = 0 \tag{2.8}$$

holds for all $u \in U$. It is never unique; for example, modifying G at a single point does not affect the validity of (2.8) in U.

It has turned out to be natural to allow Newton derivatives to be set-valued. For set-valued mappings we write "$f : X \rightrightarrows Y$" instead of "$f : X \to \mathcal{P}(Y) \setminus \emptyset$".

Definition 2.4. A mapping $G : U \rightrightarrows \mathcal{L}(X, Y)$ is called a **Newton derivative** of F in U, if

$$\lim_{h \to 0} \sup_{L \in G(u+h)} \frac{\|F(u+h) - F(u) - Lh\|}{\|h\|} = 0 \tag{2.9}$$

holds for all $u \in U$. G is called **locally bounded** if for every $u \in U$ the sets $\{\|L\| : L \in G(v), \|v - u\| \leq \delta\}$ are bounded for some suitable $\delta = \delta(u)$. G is called **globally bounded** if these bounds can be chosen independently from u.

It is well known that if F is continuously Fréchet differentiable in U, then $G(u) = \{DF(u)\}$ is a single-valued Newton derivative of F in U.

We write (2.9) in remainder form,

$$\sup_{L \in G(u+h)} \|F(u+h) - F(u) - Lh\| \leq \rho_u(\|h\|) \cdot \|h\| \,, \tag{2.10}$$

where $\rho_u(\delta) \downarrow 0$ as $\delta \downarrow 0$. If ℓ is a global Lipschitz constant for F and c_G is a global bound for the norms $\|L\|$ of the elements $L \in G(U)$, we may assume that ρ_u is globally bounded,

$$\rho_u \leq \ell + c_G \qquad \forall\, u \in U \,, \tag{2.11}$$

as in the case of the Bouligand derivative.

If $G : U \rightrightarrows \mathcal{L}(X, Y)$ is a Newton derivative of F in U, then so is every $\tilde{G} : U \rightrightarrows \mathcal{L}(X, Y)$ satisfying $\tilde{G}(u) \subset G(u)$ for all $u \in U$. In particular, every **selector** $S : U \to L(X, Y)$ of G, that is, $S(u) \in G(u)$ for all $u \in U$, yields a single-valued Newton derivative of F in U.

We now consider the following situation. The domain of definition U of F can be represented as

$$U = \bigcup_{n \in \mathbb{N}} U_n \,, \tag{2.12}$$

where $U_n \subset U$ are open sets with $U_n \subset U_{n+1}$ for all n, and $U_0 = \emptyset$. We want to obtain a Newton derivative of F on U from Newton derivatives of F on U_n. This can be done in the following setting. Let $V_n \subset U$ be open sets with

$$\overline{V}_n \subset U_n \cap V_{n+1} \quad \text{for all } n \in \mathbb{N}, \quad \bigcup_{n \in \mathbb{N}} V_n = U \,. \tag{2.13}$$

Proposition 2.5. *Let G_n be a Newton derivative of F on U_n, $n \in \mathbb{N}$, with the remainder $\rho_{n,u}$ according to (2.10). Then in the situation just described above, the definition*

$$G(u) = G_n(u) \,, \quad \text{if } u \in \overline{V}_n \setminus \overline{V}_{n-1}, \tag{2.14}$$

yields a Newton derivative $G : U \rightrightarrows \mathcal{L}(X; Y)$ of F on U with the remainder

$$\rho_u = \max\{\rho_{n,u}, \rho_{n+1,u}\} \quad \text{if } u \in \overline{V}_n \setminus \overline{V}_{n-1}. \tag{2.15}$$

Proof. By construction,

$$U = \bigcup_{n\in\mathbb{N}} \overline{V}_n \setminus \overline{V}_{n-1}\,,$$

the union being disjoint. Let $u \in U$, assume that $u \in \overline{V}_n \setminus \overline{V}_{n-1}$. We choose $\delta > 0$ such that $B_\delta(u) = \{v : \|v - u\| < \delta\}$ satisfies, see (2.13),

$$B_\delta(u) \cap \overline{V}_{n-1} = \emptyset\,, \quad B_\delta(u) \subset U_n \cap V_{n+1}\,.$$

Let $h \in X$, $\|h\| < \delta$, let $L \in G(u+h)$. If $u + h \in \overline{V}_n$, then $u + h \in \overline{V}_n \setminus \overline{V}_{n-1}$, $u + h \in U_n$ and $L \in G_n(u + h)$, so

$$\|F(u + h) - F(u) - Lh\| \le \rho_{n,u}(\|h\|)\|h\|\,.$$

If $u + h \notin \overline{V}_n$, then $u + h \in \overline{V}_{n+1} \setminus \overline{V}_n \subset U_{n+1}$ and $L \in G_{n+1}(u + h)$, so

$$\|F(u + h) - F(u) - Lh\| \le \rho_{n+1,u}(\|h\|)\|h\|\,.$$

This proves the assertions. ☐

Remark 2.6. If we have $G_n(u) \subset G_{n+1}(u)$ for all n and all $u \in U_n$, we may dispense with the sets V_n and simply define a Newton derivative G of F on U by

$$G(u) = G_n(u)\,, \quad \text{if } u \in U_n \setminus U_{n-1}.$$

We refer to Remark 7.14 for a discussion of this issue in the context of the play operator.

The following result ([10], Lem. 8.11) shows that Bouligand and Newton derivatives are closely related.

Proposition 2.7. *Let $F : U \to Y$ possess the single-valued Newton derivative $D^N F : U \to \mathcal{L}(X, Y)$. Then F is Bouligand differentiable at $u \in U$ if and only if the limit $\lim_{\lambda\downarrow 0} D^N F(u + \lambda h)h$ exists uniformly w.r.t. $h \in X$ with $\|h\| = 1$. In this case,*

$$F'(u; h) = \lim_{\lambda\downarrow 0} D^N F(u + \lambda h)h\,. \tag{2.16}$$

3. The maximum functional

We consider $\varphi : C[a, b] \to \mathbb{R}$,

$$\varphi(u) = \max_{s\in[a,b]} u(s)\,. \tag{3.1}$$

The functional φ is convex, positively 1-homogeneous and globally Lipschitz continuous with Lipschitz constant 1, w.r.t. the maximum norm on $C[a, b]$. By convex analysis, it is directionally (and thus, Hadamard) differentiable. An explicit formula for the directional derivative is given by (see *e.g.* [6] for a direct proof)

$$\varphi'(u; h) = \max_{s\in M(u)} h(s)\,, \tag{3.2}$$

where

$$M(u) = \{\tau \in [a, b],\, u(\tau) = \varphi(u)\} \tag{3.3}$$

is the set where u attains its maximum.

Let us denote the dual of $C[a,b]$ by $C[a,b]^*$; it consists of all signed regular Borel measures on $[a,b]$. The subdifferential of φ is defined as usual as the set-valued mapping $\partial\varphi : C[a,b] \rightrightarrows C[a,b]^*$ given by

$$\partial\varphi(u) = \{\mu : \mu \in C[a,b]^*,\ \varphi(v) - \varphi(u) \geq \langle \mu, v-u \rangle \text{ for all } v \in C[a,b]\}\,. \tag{3.4}$$

It is not difficult to check that

$$\partial\varphi(u) = \{\mu : \mu \in C[a,b]^*,\ \operatorname{supp}(\mu) \subset M(u),\ \mu \geq 0,\ \|\mu\| = 1\}\,. \tag{3.5}$$

In particular, if u has a unique maximum at $r \in [a,b]$, that is, $M(u) = \{r\}$, then $\partial\varphi(u) = \{\delta_r\}$, where δ_r denotes the Dirac delta at r.

A side remark (we will not use this): the directional derivative is linked to the subdifferential by the "max formula" (see [1], Thm. 17.19, for the Hilbert space case)

$$\varphi'(u;h) = \max_{\mu \in \partial\varphi(u)} \langle \mu, h \rangle\,.$$

The subdifferential is a natural candidate for a Newton derivative of a convex functional. However, the subdifferential of $\varphi : C[a,b] \to \mathbb{R}$ is not a Newton derivative of φ, and φ is not Bouligand differentiable. The following example shows that this is true even if we restrict φ to $W^{1,1}(a,b)$.

Here and in the sequel we use the norm

$$\|u\|_{W^{1,p}} = |u(a)| + \|u'\|_p = |u(a)| + \Big(\int_a^b |u'(s)|^p \,\mathrm{d}s \Big)^{1/p}, \quad 1 \leq p < \infty\,.$$

Example 3.1. Consider $u : [0,1] \to \mathbb{R}$ defined by $u(s) = 1 - s$. We have $\varphi(u) = 1$ and $M(u) = \{0\}$. Define $h_\lambda : [0,1] \to \mathbb{R}$ for $\lambda > 0$ by

$$h_\lambda(s) = \begin{cases} 2s\,, & s \leq \lambda\,, \\ 2\lambda\,, & s > \lambda\,. \end{cases} \tag{3.6}$$

Then the function $u + h_\lambda$ attains its maximum at $s = \lambda$, and

$$\|h_\lambda\|_{1,1} = 2\lambda\,, \quad \varphi(u + h_\lambda) = 1 + \lambda\,, \quad \varphi'(u;h_\lambda) = \max_{s \in M(u)} h_\lambda(s) = h_\lambda(0) = 0\,.$$

Consequently, $\|h_\lambda\|_{1,1} \to 0$ but

$$\frac{|\varphi(u + h_\lambda) - \varphi(u) - \varphi'(u;h_\lambda)|}{\|h_\lambda\|_{1,1}} = \frac{\lambda}{2\lambda} = \frac{1}{2}\,. \tag{3.7}$$

Thus, φ is not Bouligand differentiable at u on $X = W^{1,1}(0,1)$. Moreover, setting $\Phi = (\partial\varphi)|X$ we obtain

$$M(u + h_\lambda) = \{\lambda\}\,, \quad \Phi(u + h_\lambda) = \{\delta_\lambda\}\,, \quad \Phi(u + h_\lambda)h_\lambda = h_\lambda(\lambda) = 2\lambda\,,$$

so

$$\frac{|\varphi(u + h_\lambda) - \varphi(u) - \Phi(u + h_\lambda)h_\lambda|}{\|h_\lambda\|_{1,1}} = \frac{\lambda}{2\lambda} = \frac{1}{2}\,. \tag{3.8}$$

Thus, Φ is not a Newton derivative of φ on $W^{1,1}(0,1)$. As $\|h_\lambda\|_\infty = \|h_\lambda\|_{1,1}$ (or due to the embedding $W^{1,1} \to C$), the same is true on $C[0,1]$.

We will show that Φ is a Newton derivative of φ on $C^{0,\alpha}[a,b]$ for every $\alpha > 0$, endowed with the norm

$$\|u\|_{C^{0,\alpha}} = |u(a)| + |u|_{C^{0,\alpha}}\,, \quad |u|_{C^{0,\alpha}} = \sup_{\substack{t,s\in[a,b]\\ s\neq t}} \frac{|u(t)-u(s)|}{|t-s|^\alpha}\,. \tag{3.9}$$

We set $B_\varepsilon = (-\varepsilon, \varepsilon)$.

Lemma 3.2. *The mapping $M : C[a,b] \rightrightarrows [a,b]$ is upper semicontinuous, that is, for every $u \in C[a,b]$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $h \in C[a,b]$*

$$\|h\|_\infty < \delta \quad \Rightarrow \quad M(u+h) \subset M(u) + B_\varepsilon\,. \tag{3.10}$$

Proof. By contradiction. Assume that $u \in C[a,b]$ and $\varepsilon > 0$ are such that for all $n \in \mathbb{N}$ there exist $h_n \in C[a,b]$ with $\|h_n\|_\infty < \frac{1}{n}$ and $M(u+h_n) \not\subset M(u) + B_\varepsilon$. Let $t_n \in M(u+h_n)$ with $d(t_n, M(u)) \geq \varepsilon$. Passing to a subsequence we get $t_n \to t \in [a,b]$, $t \notin M(u)$. On the other hand, $u(t_n) + h_n(t_n) = \varphi(u+h_n)$. Letting $n \to \infty$ yields $u(t) = \varphi(u)$, so $t \in M(u)$, a contradiction. □

For a function $f : I \to \mathbb{R}$, I being an interval, we denote its oscillation on I by

$$\operatorname*{osc}_I(f) = \sup\{|f(t) - f(s)| : t, s \in I\}\,, \tag{3.11}$$

and its modulus of continuity by

$$\omega_I(f;\varepsilon) = \sup\{|f(t) - f(s)| : t, s \in I\,,\ |t-s| \leq \varepsilon\}\,. \tag{3.12}$$

When $I = [a,b]$, we simply write $\operatorname{osc}(f)$ and $\omega(f;\varepsilon)$.

Lemma 3.3. *Let $u, h \in C[a,b]$, $\mu \in \partial\varphi(u+h)$. Then*

$$\varphi'(u;h) \leq \varphi(u+h) - \varphi(u) \leq \langle \mu, h\rangle\,. \tag{3.13}$$

Let moreover be $\varepsilon > 0$ such that

$$M(u+h) \subset M(u) + B_\varepsilon\,. \tag{3.14}$$

Then we have

$$\langle \mu, h\rangle - \varphi'(u;h) \leq \sup_{|s-r|\leq\varepsilon} |h(r) - h(s)| = \omega(h;\varepsilon)\,. \tag{3.15}$$

Proof. The first inequality in (3.13) holds since φ is convex; as $\varphi(u) - \varphi(u+h) \geq \langle \mu, -h\rangle$, the second inequality follows. Now assume that (3.14) holds. Recalling (3.5), given $r \in \operatorname{supp}(\mu) \subset M(u+h)$ we find an $s_r \in M(u)$ with $|r - s_r| < \varepsilon$, so

$$h(r) - \varphi'(u;h) = h(r) - \max_{s\in M(u)} h(s) \leq h(r) - h(s_r) \leq \omega(h;\varepsilon)\,.$$

Integrating both sides of this inequality over $r \in [a,b]$ with respect to μ yields (3.15). □

For the modulus of continuity, we have

$$\omega(h;\varepsilon) \le |h|_{C^{0,\alpha}}\varepsilon^{\alpha}\,, \quad \omega(h;\varepsilon) \le \|h'\|_{L^p}\varepsilon^{1-1/p}\,. \tag{3.16}$$

Proposition 3.4. *Let $X = C^{0,\alpha}[a,b]$ or $X = W^{1,p}(a,b)$, with $0 < \alpha \le 1$ resp. $1 < p \le \infty$. Then the set-valued mapping $\Phi = (\partial\varphi)|X$ given in (3.5) is a globally bounded Newton derivative of the maximum functional φ on X. In particular, for every $u \in X$ there exists a nondecreasing and bounded $\rho_u : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho_u(\delta) \to 0$ as $\delta \to 0$, ρ_u is bounded independently from u, and*

$$|\varphi(u+h) - \varphi(u) - Lh| \le \begin{cases} \rho_u(\|h\|_\infty)|h|_{C^{0,\alpha}} \\ \rho_u(\|h\|_\infty)\|h'\|_{L^p} \end{cases} \tag{3.17}$$

respectively, for every $h \in X$ and every $L \in \Phi(u+h)$.

Moreover, φ is Bouligand differentiable on X, and for every $u \in X$

$$|\varphi(u+h) - \varphi(u) - \varphi'(u;h)| \le \begin{cases} \rho_u(\|h\|_\infty))|h|_{C^{0,\alpha}} \\ \rho_u(\|h\|_\infty)\|h'\|_{L^p} \end{cases} \tag{3.18}$$

respectively, for every $h \in X$.

Proof. We consider the case $X = C^{0,\alpha}[a,b]$. Let $u \in X$ be given, let

$$\varepsilon_u(\delta) = \inf\{\varepsilon : M(u + B_\delta) \subset M(u) + B_\varepsilon\}$$

for $\delta > 0$. Then ε_u is increasing. As M is upper semicontinous by Lemma 3.2, we have $0 < \varepsilon_u(\delta) \to 0$ as $\delta \to 0$, According to (3.13) and (3.15), for $h \in X$ and $L = \mu \in \Phi(u+h)$ we get

$$|\varphi(u+h) - \varphi(u) - Lh| \le \omega(h; \varepsilon_u(\|h\|_\infty) \le \varepsilon_u(\|h\|_\infty)^\alpha \cdot |h|_{C^{0,\alpha}}\,.$$

Setting $\rho_u(\delta) = (\varepsilon_u(\delta))^\alpha$, (3.17) follows for the Hölder case. Since $\|L\|_{C\to\mathbb{R}} = 1$, we have $\|L\|_{C^{0,\alpha}\to\mathbb{R}} \le c_\alpha$ and

$$|\varphi(u+h) - \varphi(u) - Lh| \le 2\|h\|_\infty \le 2c_\alpha\|h\|_\infty\,,$$

where c_α denotes the norm of the embedding $C^{0,\alpha} \to C$. Thus, c_α is a global bound for Φ, and $2c_\alpha$ furnishes a global bound for ρ_u.

The proof for the case $X = W^{1,p}(a,b)$ is analogous. (One might also refer to Morrey's embedding theorem which implies that $W^{1,p}(a,b)$ is continuously embedded into $C^{0,\alpha}[a,b]$ for $\alpha \le 1 - 1/p$.) □

Note that the estimates (3.17) and (3.18) are slightly stronger than required for Newton and Bouligand differentiability (the factor $\rho_u(\|h\|_X)$ instead of $\rho_u(\|h\|_\infty)$, as well as the norms instead of the seminorms, would suffice). This strenghtening is motivated by applications to partial differential equations.

4. The cumulated maximum

We define the cumulated maximum of a function $u \in C[a,b]$ as

$$\varphi_t(u) = \max_{s\in[a,t]} u(s)\,, \quad t \in [a,b]\,. \tag{4.1}$$

Setting

$$(Fu)(t) = \varphi_t(u) \tag{4.2}$$

we obtain an operator

$$F : C[a,b] \to C[a,b]\,. \tag{4.3}$$

The function Fu is nondecreasing for every $u \in C[a,b]$. Since

$$|\varphi_t(u) - \varphi_t(v)| \le \max_{s\in[a,t]} |u(s) - v(s)|\,, \quad \text{for all } u, v \in C[a,b],$$

we have

$$\|Fu - Fv\|_{\infty,t} \le \|u - v\|_{\infty,t}\,, \quad \text{for all } u, v \in C[a,b],\ t \in [a,b]. \tag{4.4}$$

Here and in the following we use the notation

$$\|u\|_{\infty,t} = \sup_{s\le t} |u(s)|\,. \tag{4.5}$$

For any fixed $t \in [a,b]$, the directional derivative of $\varphi_t : C[a,b] \to \mathbb{R}$ given in (3.2) yields that, for all $u, h \in C[a,b]$,

$$F^{PD}(u;h)(t) := \lim_{\lambda\downarrow 0} \frac{(F(u+\lambda h))(t) - (Fu)(t)}{\lambda} = \varphi_t'(u;h) = \max_{s\in M(u,t)} h(s)\,, \tag{4.6}$$

where

$$M(u,t) = \{\tau : \tau \in [a,t],\ u(\tau) = \varphi_t(u)\} \tag{4.7}$$

is the set where u attains its maximum on $[a,t]$. As in [3], we call **pointwise directional derivative of $\boldsymbol{F}$** the function $F^{PD}(u;h) : [a,b] \to \mathbb{R}$ obtained in this manner.

Example 4.3 in [3] shows that the function $F^{PD}(u;h) : [a,b] \to \mathbb{R}$ does not need to be continuous even though u and h are; so $F : C[a,b] \to C[a,b]$ is not directionally differentiable. When this happens, the difference quotients

$$\frac{F(u+\lambda h) - Fu}{\lambda}$$

do not converge uniformly to $F^{PD}(u;h)$. They do, on the other hand, converge in $L^r(a,b)$ for every $r < \infty$, as they are uniformly bounded by $\|h\|_\infty$. As a consequence, $F : C[a,b] \to L^r(a,b)$ is Hadamard differentiable ([3]). In order to obtain Bouligand or Newton differentiability, as in the case of the maximum functional one has to strengthen the norm in the domain space. Indeed, the functions from Example 3.1 can be used to show that F is not Bouligand differentiable on $C[a,b]$.

Bouligand differentiability of the cumulated maximum. Let again X stand for $C^{0,\alpha}[a,b]$ with $0 < \alpha \le 1$, or for $W^{1,p}(a,b)$ with $1 < p \le \infty$. We want to prove that $F : X \to L^q(a,b)$ is Bouligand differentiable for $1 \le q < \infty$ with the improved remainder estimate as in Proposition 3.4. For this, we have to show that

$$\rho_u^F(\delta) := \sup_{\|h\|_\infty \le \delta} \frac{\|F(u+h) - F(u) - F'(u;h)\|_{L^q}}{\|h\|_X} \to 0 \qquad \text{as } \delta \to 0. \tag{4.8}$$

Proposition 4.1. *The cumulated maximum $F : X \to L^q(a,b)$ is Bouligand differentiable for every $q < \infty$, and $F' = F^{PD}$. Moreover,*

$$\|F(u+h) - F(u) - F'(u;h)\|_{L^q} \le \rho_u^F(\|h\|_\infty) \cdot \|h\|_X\,, \tag{4.9}$$

and $\rho_u^F(\delta) \to 0$ as $\delta \to 0$. In addition, ρ_u^F is bounded uniformly in u.

Proof. Assume that (4.8) does not hold. Then there exists $\varepsilon > 0$ and a sequence $\{h_n\}$ in X with $\|h_n\|_\infty \to 0$ and

$$\varepsilon\|h_n\|_X \le \|F(u+h_n) - F(u) - F^{PD}(u;h_n)\|_{L^q} = \left(\int_a^b d_n(t)^q \,\mathrm{d}t\right)^{1/q}, \tag{4.10}$$

where

$$d_n(t) = |\varphi_t(u+h_n) - \varphi_t(u) - \varphi_t'(u;h_n)|\,.$$

Setting $\rho_n = d_n/\|h_n\|_X$ we have $\rho_n(t) \to 0$ pointwise, because $\varphi_t : X \to \mathbb{R}$ is Bouligand differentiable for every t by Proposition 3.4, with the remainder estimate (3.18). Since moreover $\{\rho_n\}$ is uniformly bounded, by dominated convergence $\|\rho_n\|_{L^q} \to 0$ which contradicts (4.10). Therefore F is Bouligand differentiable and $F' = F^{PD}$. The global bound on ρ_u^F follows from the estimate $\|F(u+h) - F(u) - F'(u;h)\|_\infty \le 2\|h\|_\infty$ combined with the embedding constants. □

Newton differentiability of the cumulated maximum. A Newton derivative of the cumulated maximum is constructed from the Newton derivative of the maximum functional given in the previous section. Its elements L will have the form $(Lh)(t) = \langle \mu^t, h\rangle$, where μ^t belongs to the Newton derivative Φ^t of φ_t. In order that Lh becomes a measurable function, the measures μ^t are constructed from measurable selectors of the family $\{\Phi^t\}$.

We first analyze the mapping $M : C[a,b] \times [a,b] \rightrightarrows [a,b]$

$$M(u,t) = \{\tau : \tau \in [a,t],\, u(\tau) = \varphi_t(u)\}\,. \tag{4.11}$$

The sets $M(u,t)$ are compact nonempty subsets of $[a,b]$, and $M(u,a) = \{a\}$.

Lemma 4.2. *The set-valued mapping M is upper semicontinuous and measurable.*

Proof. To prove that M is upper semicontinuous according to Definition 10.1, let $A \subset [a,b]$ be closed, and let (u_n, t_n) be a sequence in $M^{-1}(A)$ with $u_n \to u \in C[a,b]$ and $t_n \to t \in [a,b]$. In order to show that $(u,t) \in M^{-1}(A)$, let $\tau_n \in A$ such that $\tau_n \in M(u_n,t_n)$, thus $u_n(\tau_n) = \varphi_{t_n}(u_n)$. Passing to a subsequence we have $\tau_n \to \tau \in A$ since A is closed. Moreover, $\tau \le t$, $u_n(\tau_n) \to u(\tau)$ and

$$\varphi_{t_n}(u_n) = (\varphi_{t_n}(u_n) - \varphi_{t_n}(u)) + \varphi_{t_n}(u) \to \varphi_t(u)$$

by (4.4) and since $t \mapsto \varphi_t(u)$ is continuous. Therefore $u(\tau) = \varphi_t(u)$ and $\tau \in M(u,t)$. Thus M is upper semicontinuous. It now follows from Proposition 6.2.3 in [16] that M is measurable. □

The set-valued mapping M possesses a dense sequence of measurable selectors.

Proposition 4.3. *There exists a sequence $\{f_n\}$ of measurable selectors of M such that*

$$M(u,t) = \overline{\{f_n(u,t) : n \in \mathbb{N}\}}\,, \quad \textit{for all } u \in C[a,b],\ t \in [a,b]\,. \tag{4.12}$$

In particular $\max M(u,t) = \sup_n f_n(u,t)$ *and* $\min M(u,t) = \inf_n f_n(u,t)$ *are measurable selectors of M.*

Proof. This is a consequence of Theorem 6.3.18 in [16], as $[a,b]$ is a complete separable metric space. □

We consider the mapping $\Phi : C[a,b] \times [a,b] \rightrightarrows C[a,b]^*$,

$$\Phi(u,t) = \{\nu \in C[a,b]^* : \mathrm{supp}(\nu) \subset M(u,t), \nu \geq 0, \|\nu\| = 1\}\,. \tag{4.13}$$

The following facts are well known. The closed unit ball K in $C[a,b]^*$, endowed with the weak star topology, is compact (hence complete), metrizable and separable. The sets $\Phi(u,t)$ are nonempty convex and weak star compact subsets of K (note that for $\nu \geq 0$ we have $\|\nu\| = \langle \nu, 1\rangle$). Moreover,

$$\Phi(u,a) = \{\delta_a\}\,, \tag{4.14}$$

$$M(u+c,t) = M(u,t)\,, \quad \Phi(u+c,t) = \Phi(u,t) \quad \text{for all } c \in \mathbb{R}, \tag{4.15}$$

$$(\Phi(u,t))(c) = \{c\} \quad \text{for all } c \in \mathbb{R}. \tag{4.16}$$

Lemma 4.4. *Let $\{u_n\}$, $\{t_n\}$, $\{\nu_n\}$ be sequences in $C[a,b]$, $[a,b]$ and $C[a,b]^*$ respectively, with $u_n \to u$, $t_n \to t$ and $\nu_n \overset{*}{\rightharpoonup} \nu$, let $\mathrm{supp}(\nu_n) \subset M(u_n,t_n)$ for all $n \in \mathbb{N}$. Then $\mathrm{supp}(\nu) \subset M(u,t)$.*

Proof. Let $f \in C_0^\infty(\mathbb{R} \setminus M(u,t))$. We have to show that $\langle \nu, f\rangle = 0$. Let

$$\varepsilon = \inf\{|s-\tau| : s \in \mathrm{supp}(f), \tau \in M(u,t)\}\,.$$

We have $\varepsilon > 0$ because the sets $\mathrm{supp}(f)$ and $M(u,t)$ are disjoint and compact. Since M is upper semicontinuous by Proposition 4.2, we may choose $N \in \mathbb{N}$ such that $M(u_n,t_n) \subset M(u,t) + B_{\varepsilon/2}$ holds for all $n \geq N$. Then $\mathrm{supp}(f) \cap M(u_n,t_n) = \emptyset$ and thus $\langle \nu_n, f\rangle = 0$ for all $n \geq N$. Passing to the limit $n \to \infty$ we arrive at $\langle \nu, f\rangle = 0$. □

Proposition 4.5. *The mapping $\Phi : C[a,b] \times [a,b] \rightrightarrows C[a,b]^*$ defined in (4.13) is upper semicontinuous, thus measurable.*

Proof. Let $A \subset C[a,b]^*$ be weak star closed. We have to show that $\Phi^{-1}(A)$ is closed. To this end, let $\{(u_n,t_n)\}$ be a sequence in $\Phi^{-1}(A)$ with $u_n \to u$ in $C[a,b]$ and $t_n \to t$ in $[a,b]$. Let $\nu_n \in \Phi(u_n,t_n)$, so $\nu_n \in A$ as well as $\nu_n \geq 0$, $\|\nu_n\| = 1$ and $\mathrm{supp}(\nu_n) \subset M(u_n,t_n)$ for all $n \in \mathbb{N}$. For some subsequence, we have $\nu_{n_k} \overset{*}{\rightharpoonup} \nu$ with $\nu \geq 0$, $\|\nu\| = 1$ and $\nu \in A$. By Lemma 4.4, $\mathrm{supp}(\nu) \subset M(u,t)$. Thus, $(u,t) \in \Phi^{-1}(A)$ and the proof is complete. □

Proposition 4.6. *There exists a sequence $\{\mu_n\}$ of measurable selectors of Φ such that*

$$\Phi(u,t) = \overline{\{\mu_n(u,t) : n \in \mathbb{N}\}}\,, \quad \textit{for all } u \in C[a,b],\ t \in [a,b]\,, \tag{4.17}$$

the closure being taken w.r.t. the weak star topology.

Proof. This follows from Theorem 6.3.18 in [16], as the unit ball in $C[a,b]^*$ is a complete separable metrizable space w.r.t. the weak star topology. □

Lemma 4.7. *Let μ be a measurable selector of Φ. Then*

$$(Lh)(t) = \langle \mu(u,t), h\rangle$$

defines an element $L \in \mathcal{L}(C[a,b]; L^\infty(a,b))$ with $\|L\| = 1$ and

$$\|Lh\|_{\infty,t} \leq \|h\|_{\infty,t}\,. \tag{4.18}$$

Proof. For every $u, h \in C[a, b]$, the mapping $t \mapsto \langle \mu(u, t), h \rangle$ is measurable and satisfies $|\langle \mu(u, s), h \rangle| \le \|h\|_{\infty,t}$ for all $s \le t$, since $\mu(u, s)$ has support in $[a, s]$. Thus, L is well-defined, $\|L\| \le 1$ and (4.18) holds. As $\mu \ge 0$ and $L(1) = 1$, we have $\|L\| = 1$. □

Let X again denote any one of the spaces $C^{0,\alpha}[a, b]$ for $0 < \alpha \le 1$ or $W^{1,p}(a, b)$ for $1 < p \le \infty$.

Proposition 4.8. *Let S_Φ be the set of all measurable selectors of Φ, let $q \in [1, \infty)$. The set-valued mapping $G : X \rightrightarrows \mathcal{L}(X, L^q(a, b))$,*

$$G(u) = \{L : (Lh)(t) = \langle \mu(u, t), h \rangle,\ \mu \in S_\Phi\} \tag{4.19}$$

defines a Newton derivative of the cumulated maximum $F : X \to L^q(a, b)$ with

$$\|F(u + h) - F(u) - Lh\|_{L^q} \le \rho_u^G(\|h\|_\infty) \cdot \|h\|_X\,, \tag{4.20}$$

for all $L \in G(u + h)$. Here, $\rho_u^G : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function with $\rho_u^G(\delta) \to 0$ as $\delta \to 0$, bounded independently from u.

Proof. Fix $u \in X$. For $h \in X$ we define

$$d(h, t) = \sup_{\mu^t \in \Phi(u+h,t)} |\varphi_t(u + h) - \varphi_t(u) - \langle \mu^t, h \rangle|\,.$$

Let $\{\mu_k\}$ be a sequence of measurable selectors of Φ according to Proposition 4.6, set

$$d_k(h, t) = |\varphi_t(u + h) - \varphi_t(u) - \langle \mu_k(u + h, t), h \rangle|$$

Then $d(h, t) = \sup_k d_k(h, t)$ by (4.17), and therefore the mapping $t \mapsto d(h, t)$ is measurable. Moreover,

$$\sup_{L \in G(u+h)} \|F(u + h) - F(u) - Lh\|_{L^q} = \left(\int_a^b d(h, t)^q \,\mathrm{d}t\right)^{1/q} =: d^G(h)\,.$$

The remainder of the proof is analogous to that of Proposition 4.1. We define

$$\rho_u^G(\delta) = \sup_{\|h\|_\infty \le \delta} \frac{d^G(h)}{\|h\|_X}\,. \tag{4.21}$$

Assume that $\lim_{\delta \to 0} \rho_u^G(\delta) = 0$ does not hold. Then there exist $\varepsilon > 0$ and a sequence $\{h_n\}$ in X with $\|h_n\|_\infty \to 0$ and

$$\varepsilon \|h_n\|_X \le \left(\int_a^b d(h_n, t)^q \,\mathrm{d}t\right)^{1/q}\,. \tag{4.22}$$

Since $\Phi(\cdot, t)$ is a Newton derivative of φ_t, we have $\rho_n(t) = d(h_n, t)/\|h_n\|_X \to 0$ pointwise in t as $n \to \infty$. Moreover, ρ_n is uniformly bounded. Applying dominated convergence, we arrive at a contradiction to (4.22). The global boundedness of ρ_u^G follows from the estimate $\|F(u + h) - F(u) - Lh\|_\infty \le 2\|h\|_\infty$. □

Proposition 4.10 below shows that the set S_Φ is large enough to approximate the whole range of Φ.

Lemma 4.9. *Let $f : C[a,b] \times [a,b] \to [a,b]$ be a measurable selector of M. Then*

$$\mu(u,t) = \delta_{f(u,t)} \tag{4.23}$$

defines a measurable selector $\mu : C[a,b] \times [a,b] \to C[a,b]^*$ of Φ.

Proof. For each $v \in C[a,b]$, the mapping $s \mapsto v(s) = \langle \delta_s, v\rangle$ is continuous from $[a,b]$ to $\mathbb{R}$. Thus, the mapping $s \mapsto \delta_s$ is weak star continuous from $[a,b]$ to $C[a,b]^*$, and consequently (4.24) defines a measurable mapping. □

Proposition 4.10. *Let $\{f_n\}$ be a sequence of measurable selectors of M such that*

$$M(u,t) = \overline{\{f_n(u,t) : n \in \mathbb{N}\}}\,, \quad \text{for all } u \in C[a,b],\ t \in [a,b]\,. \tag{4.24}$$

Taking all rational convex combinations of the mappings $(u,t) \mapsto \delta_{f_n(u,t)}$ we obtain a sequence $\{\mu_n\}$ of measurable selectors of Φ such that

$$\Phi(u,t) = \overline{\{\mu_n(u,t) : n \in \mathbb{N}\}}\,, \quad \text{for all } u \in C[a,b],\ t \in [a,b]\,, \tag{4.25}$$

the closure being taken w.r.t. the weak star topology.

Proof. Let $u \in C[a,b]$ and $t \in [a,b]$ be given. The set $D = \{f_n(u,t) : n \in \mathbb{N}\}$ is a countable dense subset of $M(u,t)$. The set of all convex combinations with rational coefficients of elements of the set $\{\delta_\tau : \tau \in D\}$ then is dense in $\Phi(u,t)$ w.r.t. the weak star topology. □

5. The chain rule

In the following sections we will see that the play operator can be represented as a finite composition of cumulated maxima and positive part mappings. The Newton differentiability of these mappings will imply Newton differentiability of the play, by virtue of the chain rule. It is a standard result that the chain rule is valid for Newton derivatives, see Proposition A.1 in [7] for the single-valued and Proposition 3.8 in [18] for the set-valued case.

As a result of investigating the maximum and the cumulated maximum, we have seen above that these operators satisfy a slightly stronger version of Newton and Bouligand differentiability. For the cumulated maximum $F : X \to Y$ with $X = W^{1,p}(a,b)$ or $C^{0,\alpha}[a,b]$ and $Y = L^r(a,b)$, we have constructed a Newton derivative $G : X \rightrightarrows \mathcal{L}(X;Y)$ with a remainder estimate

$$\sup_{L \in G(u+h)} \|F(u+h) - F(u) - Lh\|_Y \le \rho_u(\|h\|_{\tilde{X}}) \cdot \|h\|_X\,, \tag{5.1}$$

where $\tilde{X} = C[a,b]$, endowed with the maximum norm. The purpose of this section is to extend the chain rule to this situation, for Newton as well as for Bouligand derivatives.

We consider the following setting.

Assumption 5.1.
(i) X, Y, Z are normed spaces, $U \subset X$ and $V \subset Y$ are open. $F_1 : U \to Y$ and $F_2 : V \to Z$ with $F_1(U) \subset V$ are locally Lipschitz.
(ii) $\tilde{X}$ and $\tilde{Y}$ are normed spaces with continuous embeddings $X \subset \tilde{X}$ and $Y \subset \tilde{Y}$.
(iii) $G_1 : U \rightrightarrows \mathcal{L}(X;Y)$ and $G_2 : V \rightrightarrows \mathcal{L}(Y;Z)$ satisfy, for every $u \in U$ and $v \in V$,

$$\sup_{L_1 \in G_1(u+h)} \|F_1(u+h) - F_1(u) - L_1 h\|_Y \le \rho_{1,u}(\|h\|_{\tilde{X}}) \cdot \|h\|_X \tag{5.2}$$

for every $h \in X$ with $u + h \in U$,

$$\sup_{L_2 \in G_2(v+k)} \|F_2(v+k) - F_2(v) - L_2 k\|_Z \le \rho_{2,v}(\|k\|_{\tilde{Y}}) \cdot \|k\|_Y \tag{5.3}$$

for every $k \in Y$ with $v + k \in V$, with functions $\rho_{1,u}, \rho_{2,v} : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\rho_{1,u}(\delta) \downarrow 0$ and $\rho_{2,v}(\delta) \downarrow 0$ for $\delta \downarrow 0$.
(iv) $F_1 : (U, \|\cdot\|_{\tilde{X}}) \to (V, \|\cdot\|_{\tilde{Y}})$ is continuous.
(v) G_2 is locally bounded on $(V, \|\cdot\|_Y)$.

Since $\rho_{1,u}(\|h\|_{\tilde{X}}) \le \rho_{1,u}(c\|h\|_X)$ for some constant c, part (iii) of the assumption implies that G_1 and G_2 are Newton derivatives for F_1 in U and F_2 in V, respectively. Note also that the assumption "G_2 locally bounded" already implies that F_2 is locally Lipschitz.

In the special case $\tilde{X} = X$ and $\tilde{Y} = Y$, (5.2) and (5.3) reduce to the standard remainder form (2.10), and part (iv) of the assumption is implied by part (i); the following result then reduces to the standard chain rule for Newton derivatives.

Proposition 5.2 (Refined Chain Rule, Newton Derivative)**.**
Let Assumption 5.1 hold. Then

$$\begin{gathered} G : U \rightrightarrows \mathcal{L}(X; Z) \\ G(u) = \{L_2 \circ L_1 : L_1 \in G_1(u), L_2 \in G_2(F_1(u))\} \end{gathered} \tag{5.4}$$

is a Newton derivative of $F = F_2 \circ F_1$ in U which satisfies, for every $u \in U$,

$$\sup_{L \in G(u+h)} \|F(u+h) - F(u) - Lh\|_Z \le \rho_u(\|h\|_{\tilde{X}}) \cdot \|h\|_X \tag{5.5}$$

for every $h \in X$ with $u + h \in U$, where $\rho_u : \mathbb{R}_+ \to \mathbb{R}_+$ is a function with $\rho_u(\delta) \downarrow 0$ for $\delta \downarrow 0$.

Proof. Let $u \in U$, $h \in X$ with $u + h \in U$, set $k = F_1(u+h) - F_1(u)$. Let $L_1 \in G_1(u+h)$, $L_2 \in G_2(F_1(u+h)) = G_2(F_1(u) + k)$. By the triangle inequality,

$$\begin{aligned} &\|(F_2 \circ F_1)(u+h) - (F_2 \circ F_1)(u) - (L_2 \circ L_1)h\|_Z \\ &\qquad \le \|F_2(F_1(u) + k) - F_2(F_1(u)) - L_2 k\|_Z + \|L_2(k - L_1 h)\|_Z \end{aligned} \tag{5.6}$$

Since G_2 is locally bounded, there exists a $C > 0$ such that for sufficiently small $\|h\|_X$ we have $\|L_2\| \le C$ for all $L_2 \in G_2(F_1(u+h))$. Consequently, for all such h and L_2, and for all $L_1 \in G_1(u+h)$ we have by (5.2)

$$\|L_2(k - L_1 h)\|_Z \le C\|F_1(u+h) - F_1(u) - L_1 h\|_Y \le C\rho_{1,u}(\|h\|_{\tilde{X}}) \cdot \|h\|_X \,. \tag{5.7}$$

Moreover, by (5.3)

$$\|F_2(F_1(u) + k) - F_2(F_1(u)) - L_2 k\|_Z \le \rho_{2,F_1(u)}(\|k\|_{\tilde{Y}}) \cdot \|k\|_Y \,. \tag{5.8}$$

Since F_1 is locally Lipschitz, $\|k\|_Y \le C_1\|h\|_X$ for small enough $\|h\|_X$.

Now let us define $\tilde{\rho}_u : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\tilde{\rho}_u(\lambda) = \sup\{\|F_1(u+h) - F_1(u)\|_{\tilde{Y}} : \|h\|_{\tilde{X}} \le \lambda\} \,. \tag{5.9}$$

By part (iv) of Assumption 5.1, $\tilde{\rho}_u(\lambda) \to 0$ as $\lambda \to 0$. Putting together the estimates obtained so far, we get

$$\begin{aligned} &\|(F_2 \circ F_1)(u+h) - (F_2 \circ F_1)(u) - (L_2 \circ L_1)h\|_Z \\ &\qquad \leq \big(C_1 \rho_{2,F_1(u)}(\tilde{\rho}_u(\|h\|_{\tilde{X}})) + C\rho_{1,u}(\|h\|_{\tilde{X}})\big) \cdot \|h\|_X \end{aligned} \tag{5.10}$$

independent from the choice of L_1 and L_2, as long as $\|h\|_X$ is sufficiently small. Setting

$$\rho_u(\lambda) = C_1 \rho_{2,F_1(u)}(\tilde{\rho}_u(\lambda)) + C\rho_{1,u}(\lambda)$$

we have $\rho_u(\lambda) \to 0$ as $\lambda \to 0$. Thus, it follows from (5.10) that (5.5) holds. □

In order to obtain the refined chain rule for Bouligand derivatives, we replace Assumption 5.1(iii) by

F_1 and F_2 are Bouligand differentiable in U and V, respectively. For every $u \in U$, $v \in V$ we have

$$\begin{aligned} \|F_1(u+h) - F_1(u) - F_1'(u;h)\|_Y &\leq \rho_{1,u}(\|h\|_{\tilde{X}}) \cdot \|h\|_X \\ \|F_2(v+k) - F_2(v) - F_2'(v;k)\|_Z &\leq \rho_{2,v}(\|k\|_{\tilde{Y}}) \cdot \|k\|_Y \end{aligned} \tag{5.11}$$

for every $h \in X$ with $u + h \in U$ and every $k \in Y$ with $v + k \in V$.

Lemma 5.3. *If F_1 and F_2 are Hadamard differentiable at u resp. $F_1(u)$, then $F_2 \circ F_1$ is Hadamard differentiable at u, and the chain rule*

$$(F_2 \circ F_1)'(u;h) = F_2'(F_1(u); F_1'(u;h)) \tag{5.12}$$

holds for all $h \in X$.

Proof. See *e.g.* [2], Proposition 2.47. □

Proposition 5.4 (Refined Chain Rule, Bouligand Derivative)**.**
Let (i)–(iv) of Assumption 5.1 hold, with (iii) replaced by (5.11). Then $F = F_2 \circ F_1$ is Bouligand differentiable in U, and

$$F'(u;h) = F_2'(F_1(u); F_1'(u;h))\,. \tag{5.13}$$

Moreover, for every $u \in U$ and $h \in X$ with $u + h \in U$

$$\|F(u+h) - F(u) - F_2'(F_1(u); F_1'(u;h))\|_Z \leq \rho_u(\|h\|_{\tilde{X}})\|h\|_X \tag{5.14}$$

for some $\rho_u : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho_u(\delta) \downarrow 0$ as $\delta \downarrow 0$.

Proof. By Lemma 5.3, F is Hadamard differentiable and the chain rule holds. It remains to show (5.14) for the remainder. Let $u \in U$, $h \in X$ with $u + h \in U$, set $k = F_1(u+h) - F(u)$. We have

$$\begin{aligned} &F_2(F_1(u+h)) - F_2(F_1(u)) - F_2'(F_1(u); F_1'(u;h)) \\ &\qquad = \big(F_2(F_1(u)+k) - F_2(F_1(u)) - F_2'(F_1(u);k))\big) \\ &\qquad\quad + \big(F_2'(F_1(u);k)) - F_2'(F_1(u); F_1'(u;h))\big). \end{aligned} \tag{5.15}$$

Let C_i be local Lipschitz constants for F_i. The inequality

$$C_2\|k - F_1'(u;h))\|_X \leq C_2 \rho_{1,u}(\|h\|_{\tilde{X}})\|h\|_X$$

yields an estimate for the second term on the right side of (5.15); the first term is estimated by

$$\rho_{2,F_1(u)}(\|k\|_{\tilde{Y}}) \cdot \|k\|_Y \,.$$

Since $\|k\|_Y \le C_1\|h\|_X$, we argue as in the proof of Proposition 5.2 and obtain, with $\tilde{\rho}_u$ defined as in (5.9),

$$\begin{aligned}&\|(F_2 \circ F_1)(u+h) - (F_2 \circ F_1)(u) - F_2'(F_1(u); F_1'(u;h))\|_Z \\ &\qquad \le \big(C_1\rho_{2,F_1(u)}(\tilde{\rho}_u(\|h\|_{\tilde{X}})) + C_2\rho_{1,u}(\|h\|_{\tilde{X}})\big) \cdot \|h\|_X \,.\end{aligned}$$

From this, the claim readily follows. □

6. The scalar play and stop operators

The original construction of the play and the stop operators in [11, 12] is based on piecewise monotone input functions. A continuous function $u : [a,b] \to \mathbb{R}$ is called **piecewise monotone**, if the restriction of u to each interval $[t_i, t_{i+1}]$ of a suitably chosen partition $\Delta = \{t_i\}$, $a = t_0 < t_1 < \cdots < t_N = b$, called a **monotonicity partition** of u, is either nondecreasing or nonincreasing. By $C_{pm}[a,b]$ we denote the space of all such functions.

For arbitrary $r \ge 0$, the play operator $\mathcal{P}_r$ and the stop operator $\mathcal{S}_r$ are constructed as follows. (For more details, we refer to Sect. 2.3 of [4].) Given a function $u \in C_{pm}[a,b]$ and an initial value $z_0 \in \mathbb{R}$, we define functions $w, z : [a,b] \to \mathbb{R}$ successively on the intervals $[t_i, t_{i+1}]$, $0 \le i < N$, of a monotonicity partition Δ of u by

$$z(a) = \pi_r(z_0) := \max\{-r, \min\{r, z_0\}\}\,, \quad w(a) = u(a) - z(a)\,, \tag{6.1}$$

and

$$\begin{aligned} w(t) &= \max\{u(t) - r\,,\, \min\{u(t) + r, w(t_i)\}\}\,, \\ z(t) &= u(t) - w(t)\,, \end{aligned} \qquad t_i < t \le t_{i+1}\,. \tag{6.2}$$

In this manner, we obtain operators

$$w = \mathcal{P}_r[u; z_0]\,, \quad z = \mathcal{S}_r[u; z_0]\,, \quad \mathcal{P}_r, \mathcal{S}_r : C_{pm}[a,b] \times \mathbb{R} \to C_{pm}[a,b]\,.$$

By construction,

$$u = w + z = \mathcal{P}_r[u; z_0] + \mathcal{S}_r[u; z_0]\,. \tag{6.3}$$

The play operator satisfies

$$\|\mathcal{P}_r[u; z_0] - \mathcal{P}_r[v; y_0]\| \le \max\{\|u - v\|\,,\, |z_0 - y_0|\} \tag{6.4}$$

for all $u, v \in C_{pm}[a,b]$ and all $z_0, y_0 \in \mathbb{R}$. Therefore, $\mathcal{P}_r$ and $\mathcal{S}_r$ can be uniquely extended to Lipschitz continuous operators

$$\mathcal{P}_r, \mathcal{S}_r : C[a,b] \times \mathbb{R} \to C[a,b]$$

which satisfy (6.4) for all $u, v \in C[a,b]$ and all $z_0, y_0 \in \mathbb{R}$.

In [3], Hadamard derivatives of $\mathcal{P}_r$ and of $\mathcal{S}_r$ have been obtained. We recall some of the terminology used there, as it is also relevant for the present paper.

Let $(u, z_0) \in C[a,b] \times \mathbb{R}$ be given, let $w = \mathcal{P}_r[u; z_0]$, $z = \mathcal{S}_r[u; z_0]$ with $r > 0$. (For $r = 0$, $\mathcal{P}_r[u; z_0] = u$.) The trajectories $\{(u(t), w(t)) : t \in [a,b]\}$ lie within the subset $A = \{|u - w| \le r\}$ of the plane $\mathbb{R}^2$ bounded by the straight lines $u - w = \pm r$. They consist of parts which belong to the interior, the right or the left boundary of A. Correspondingly, the time interval $[a, b]$ decomposes into the three disjoint sets

$$\begin{aligned} I_0 &= \{t \in [a,b] : |u(t) - w(t)| = |z(t)| < r\}, \\ I_{\partial +} &= \{t \in [a,b] : u(t) - w(t) = z(t) = r\}, \\ I_{\partial -} &= \{t \in [a,b] : u(t) - w(t) = z(t) = -r\}. \end{aligned}$$

The set I_0 is an open subset of $[a, b]$, the sets $I_{\partial\pm}$ are compact. As $I_{\partial+}$ and $I_{\partial-}$ are disjoint,

$$\delta_I := \min\{|\tau - \sigma| : \tau \in I_{\partial +}\,,\, \sigma \in I_{\partial -}\} > 0\,. \tag{6.5}$$

Because of this, there exists a finite partition $\Delta(u, z_0) = \{t_k\}$ of $[a, b]$ such that on each partition interval $I_k = [t_{k-1}, t_k]$ we have $z(t) > -r$ for all $t \in I_k$ or $z(t) < r$ for all $t \in I_k$, or both. In the former case, I_k is called a **plus interval**; on I_k the trajectory stays away from the left boundary of A, and $I_k \subset I_0 \cup I_{\partial+}$. In the latter case, I_k is called a **minus interval**; the trajectory stays away from the right boundary of A, and $I_k \subset I_0 \cup I_{\partial-}$. Note that if $I_k \subset I_0$, then I_k is a plus as well as a minus interval.

It has been proved in [3], Lemma 5.1, that on such intervals the play operator behaves like a cumulated maximum resp. minimum. More precisely, on a plus interval I_k,

$$w(t) = \mathcal{P}_r[u; z_0](t) = \max\{w(t_{k-1})\,,\, \max_{s \in [t_{k-1}, t]} (u(s) - r)\} \tag{6.6}$$

holds, no matter whether u is monotone on I_k or not. On a minus interval,

$$w(t) = \mathcal{P}_r[u; z_0](t) = \min\{w(t_{k-1})\,,\, \min_{s \in [t_{k-1}, t]} (u(s) + r)\}\,. \tag{6.7}$$

In particular, $w(t) = w(t_{k-1})$ if $I_k \subset I_0$.

Due to (6.5) and the continuity of $\mathcal{P}_r$, in this manner the play and the stop operator can locally be represented by a finite composition of operators arising from the cumulated maximum resp. minimum. The following result has been proved in [3], Lemma 5.2.

Proposition 6.1. *For every $(u, z_0) \in C[a, b] \times \mathbb{R}$ there exists a partition $\Delta(u, z_0) = \{t_k\}_{0 \le k \le N}$ of $[a, b]$ and a $\delta > 0$ such that every partition interval $[t_{k-1}, t_k]$ of Δ is a plus interval for all $(v, y_0) \in U_\delta \times \mathbb{R}$, or it is a minus interval for all $(v, y_0) \in U_\delta \times \mathbb{R}$. Here,*

$$U_\delta := \{(v, y_0) : \|v - u\|_\infty < \delta,\ |y_0 - z_0| < \delta,\ v \in C[a, b],\ y_0 \in \mathbb{R}\} \tag{6.8}$$

is the δ-neighbourhood of (u, z_0) w.r.t the maximum norm.

As a consequence, invoking the chain rule for Hadamard derivatives, it has been proved in [3] that $\mathcal{P}_r$ and $\mathcal{S}_r$ are Hadamard differentiable on $C[a, b] \times \mathbb{R}$, if $L^q(a, b)$ with $q < \infty$ is chosen as the range space.

7. Newton derivative of the play and the stop

We want to use the approach outlined in the previous section in order to obtain a Newton derivative of $\mathcal{P}_r$, based on the Newton derivative of the cumulated maximum.

We want to construct the Newton derivative such that its dependence upon (u, z_0) becomes measurable in a suitable manner; for this, the local representation of the play obtained from Proposition 6.1 seems to be of very limited value. Instead, we employ properties of the set-valued mappings involved when constructing above the Newton derivative of the cumulated maximum. To this purpose, we turn around the approach of

Proposition 6.1. Instead of finding a suitable partition Δ for a given (u, z_0), for a given partition Δ we consider sets of (u, z_0) for which the play can be "decomposed" by Δ.

Throughout the following, the space X stands for $C^{0,\alpha}[a, b]$ or $W^{1,p}(a, b)$.

Let $\Delta = \{t_k\}$ be a partition of $[a, b]$, $a = t_0 < \cdots < t_N = b$ for some $N \in \mathbb{N}$. We set

$$I_k = [t_{k-1}, t_k]\,, \quad |\Delta| = \max_{1\le k\le N} |I_k| = \max_{1\le k\le N} (t_k - t_{k-1})\,.$$

We define

$$\begin{aligned} C^\Delta &= \{u : u \in C[a, b],\ \operatorname*{osc}_{I_k}(u) < r \text{ for all } k\}\,, \\ X^\Delta &= X \cap C^\Delta\,, \\ Z^\Delta &= C^\Delta \times \mathbb{R} = \{(u, z_0) : u \in C^\Delta,\ z_0 \in \mathbb{R}\}\,. \end{aligned} \tag{7.1}$$

The sets C^Δ, X^Δ and Z^Δ are open subsets of $C[a, b]$, X and $C[a, b] \times \mathbb{R}$, respectively.

The dynamics on an interval for small input oscillation.

It turns out below in Proposition 7.5 that an interval $I \subset [a, b]$ is a plus or a minus interval for the play if the oscillation of u on I is less than r.

Let $I = [t_*, t^*] \subset [a, b]$, $u \in C(I)$. We denote the cumulated maximum of u on I and the sets where it is attained by

$$\begin{aligned} (F^I u)(t) &= \max_{s\in I, s\le t} u(s)\,, \quad t \in I\,, \\ M^I(u, t) &= \{s : s \in I, s \le t, u(s) = (F^I u)(t)\}\,. \end{aligned} \tag{7.2}$$

As above, $F^I : C(I) \to C(I)$, $F^I u$ is nondecreasing and $F^I(u + c) = F^I(u) + c$ if c is a constant. Moreover,

$$\operatorname*{osc}_I(F^I u) = (F^I u)(t^*) - (F^I u)(t_*) \le \operatorname*{osc}_I(u)\,, \tag{7.3}$$

$$0 \le F^I u - u \le \operatorname*{osc}_I u \quad \text{on } I, \tag{7.4}$$

and consequently

$$\operatorname*{osc}_I(F^I u - u) \le \operatorname*{osc}_I u\,. \tag{7.5}$$

The cumulated minimum of u on I can be written as

$$\min_{s\in I, s\le t} u(s) = -(F^I(-u))(t)\,, \quad t \in I\,. \tag{7.6}$$

The corresponding sets of minima are given by $M^I(-u, t)$.

For $u \in C(I)$, $p \in \mathbb{R}$ and $r > 0$ we define the functions (here and in the following, the max and the min are taken pointwise in t)

$$\begin{aligned} w_+ &= \max\{p, F^I(u - r)\}\,, \quad z_+ = u - w_+\,, \\ w_- &= \min\{p, -F^I(-u - r)\}\,, \quad z_- = u - w_-\,. \end{aligned} \tag{7.7}$$

This corresponds to the operations in (6.6) and (6.7). We have $w_+, w_-, z_+, z_- \in C(I)$. Obviously $w_- \le w_+$, $z_+ \le z_-$.

Since $p \le w_+ = u - z_+$ and $p \ge w_- = u - z_-$, we have

$$z_+ \le u - p \le z_- \,. \tag{7.8}$$

Lemma 7.1. *Let $u \in C(I)$, $p \in \mathbb{R}$, $r > 0$.*
(i) We have $z_+ \le r$ on I. If $z_+(t) = r$ for some $t \in I$, then $u(t) = (F^I u)(t) \ge p + r$.
(ii) We have $z_- \ge -r$ on I. If $z_-(t) = -r$ for some $t \in I$, then $u(t) = -(F^I(-u))(t) \le p - r$.

Proof. To obtain (i), we use the estimate

$$z_+ = u - w_+ = u - \max\{p, F^I(u-r)\} \le u - F^I(u-r) \le r \,.$$

If $z_+(t) = r$, equality holds everywhere, so $F^I(u-r)(t) \ge p$ and $u(t) = (F^I u)(t)$. The proof of (ii) is analogous. □

We consider inputs in $C(I)$ whose oscillation is smaller than r. We define

$$\begin{aligned} Z^I &= \{(u,p) : u \in C(I), p \in \mathbb{R}, \operatorname*{osc}_I u < r\} \\ Z^I_+ &= \{(u,p) : u \in C(I), p \in \mathbb{R}, \operatorname*{osc}_I u < r, z_+ > -r \text{ on } I\} \\ Z^I_- &= \{(u,p) : u \in C(I), p \in \mathbb{R}, \operatorname*{osc}_I u < r, z_- < r \text{ on } I.\} \end{aligned} \tag{7.9}$$

The sets Z^I_+ and Z^I_- are open subsets of Z^I in $C(I) \times \mathbb{R}$; we will see that they are related to plus and minus intervals for the play.

Lemma 7.2.
(i) If $(u,p) \in Z^I_-$ then $F^I(u - r - p) < 0$ and $w_+ = p$ on I.
(ii) If $(u,p) \in Z^I_+$ then $F^I(-u - r + p) < 0$ and $w_- = p$ on I.
(iii) If $(u,p) \in Z^I_- \cap Z^I_+$ then $w_+ = w_- = p$ and $z_+ = z_- = u - p$ on I.

Proof. If $(u,p) \in Z^I_-$ then $u - p - r \le z_- - r < 0$ by (7.8), so $F^I(u-r) - p < 0$, so $w_+ = p$. If $(u,p) \in Z^I_+$ then $-u + p - r \le -z_+ - r < 0$ by (7.8), so $F^I(-u-r) + p < 0$, so $w_- = p$. □

Lemma 7.3. *Let $u \in C(I)$, $\operatorname{osc}_I(u) < r$, $p \in \mathbb{R}$. Then*

$$\min\{u - p, 0\} \le z_+ \le z_- \le \max\{u - p, 0\} \,. \tag{7.10}$$

Proof. We have

$$-z_+ = w_+ - u = \max\{p - u, F^I(u) - r - u\} \le \max\{p - u, 0\} \,,$$

since $F^I u - u \le \operatorname{osc}_I u < r$ by (7.4). Analogously,

$$-z_- = w_- - u = \min\{p - u, -F^I(-u-r) - u\} \ge \min\{p - u, 0\} \,,$$

since $F^I(-u) - (-u) \le \operatorname{osc}_I(-u) < r$ by (7.4). □

Lemma 7.4. *We have $Z^I = Z^I_+ \cup Z^I_-$.*

Proof. Let $(u,p) \in Z^I$, assume that $(u,p) \notin Z^I_+$. Then $z_+(t) \le -r$ for some $t \in I$. By (7.10), $u(t) - p \le -r$. As $\operatorname{osc}_I(u) < r$, we have $u - p \le 0$ on I. By (7.10), $z_- \le 0$ on I, so $(u,p) \in Z^I_-$. □

We define $P_+^I : Z_+^I \to C(I)$ and $P_-^I : Z_-^I \to C(I)$ by

$$\begin{aligned} P_+^I(u,p) &= p + \max\{0, F^I(u-r-p)\}\,, \\ P_-^I(u,p) &= p - \max\{0, F^I(-u-r+p)\}\,. \end{aligned} \tag{7.11}$$

Therefore, in view of (7.7),

$$u - P_+^I(u,p) = u - w_+ = z_+ > -r \quad \text{on } I \quad \Leftrightarrow \quad (u,p) \in Z_+^I, \tag{7.12}$$

$$u - P_-^I(u,p) = u - w_- = z_- < r \quad \text{on } I \quad \Leftrightarrow \quad (u,p) \in Z_-^I\,. \tag{7.13}$$

On $Z_+^I \cap Z_-^I$ both expressions simplify to $P_\pm^I(u,p) = p$ by Lemma 7.2. Therefore,

$$P^I(u,p) = P_\pm^I(u,p)\,, \quad \text{if } (u,p) \in Z_\pm^I \tag{7.14}$$

yields a well-defined Lipschitz continuous mapping $P^I : Z^I \to C(I)$.

The next result states that for $u \in C^\Delta$ the intervals I_k yield a decomposition of the play operator. This is the analogue of Proposition 6.1.

Proposition 7.5. *Let $u \in C^\Delta$ and $z_0 \in \mathbb{R}$, set $p = \mathcal{P}_r[u; z_0](t_{k-1})$, $k \geq 1$. Then*

$$w(t) = \mathcal{P}_r[u; z_0](t) = P^{I_k}(u,p)(t)\,, \quad \textit{for all } t \in I_k. \tag{7.15}$$

Moreover,

$$\begin{aligned} I_k \textit{ is a plus interval} \quad &\Leftrightarrow \quad (u,p) \in Z_+^{I_k}\,, \\ I_k \textit{ is a minus interval} \quad &\Leftrightarrow \quad (u,p) \in Z_-^{I_k}\,. \end{aligned} \tag{7.16}$$

Proof. For $u \in C^\Delta$ and $z_0 \in \mathbb{R}$, we set $w = \mathcal{P}_r[u; z_0]$, $z = u - w$, and denote by w_+ etc. the functions defined in (7.7) with $I = I_k$.

Let I_k be a plus interval. Thus, on I_k we have $u - w = z > -r$ and, by (6.6), $w = P_+^{I_k}(u,p) = w_+$, so $-r < u - w_+ = z_+$. This shows that (7.15) holds and that $(u,p) \in Z_+^{I_k}$.

To prove the converse, we first assume that u is piecewise monotone on I_k. Let $t_{k-1} = \tau_0 < \cdots < \tau_N = t_k$ be a monotonicity partition for u on I_k. It follows from (6.2) that

$$w(t) \leq \max\{u(t) - r\,,\, w(\tau_{j-1})\}\,, \quad t \in [\tau_{j-1}, \tau_j]\,.$$

By induction, $w \leq \max\{F^{I_k}(u-r), p\} = w_+$ on I_k. Therefore, $z = u - w \geq u - w_+ = z_+ > -r$ as $(u,p) \in Z_+^{I_k}$, so I_k is a plus interval. If u is not piecewise monotone, let $\{u_n\}$ be a sequence in $C[a,b]$ with $u_n = u$ on $[0, t_{k-1}]$, u_n piecewise monotone on I_k and $u_n \to u$ uniformly. Since $z_{n,+} \to z_+$ uniformly, for n large enough we have $(u_n, p) \in Z_+^{I_k}$. By what we have just proved, $z_n \geq z_{n,+} > -r$ on I_k. As $z_n \to z$ uniformly, we have $z > -r$ on I_k, so I_k is a plus interval.

By the above, we have shown the first equivalence in (7.16) and that (7.15) holds in this case. With an analogous proof one obtains the second equivalence and that (7.15) holds in that case, too. Since $Z^{I_k} = Z_+^{I_k} \cup Z_-^{I_k}$ by Lemma 7.4, all pairs $(u, z_0) \in C^\Delta \times \mathbb{R}$ are covered and the proof is complete. ∎

A Newton derivative on an interval of small input oscillation. We want to obtain a Newton derivative for $P^I : Z^I \cap (X \times \mathbb{R}) \to L^q(I)$, where $I = [t_*, t^*] \subset [a, b]$. The mapping P^I_+ decomposes into

$$P^I_+(u, p) = p + F_{pp}(\tilde{F}^I(u, p)) \,. \tag{7.17}$$

Here, $\tilde{F}^I : Z^I \cap (X \times \mathbb{R}) \to C(I)$ is defined as

$$\tilde{F}^I(u, p) = F^I(u - p - r) \,, \tag{7.18}$$

and F_{pp} denotes the positive part mapping

$$(F_{pp}u)(t) = \max\{0, u(t)\} \,. \tag{7.19}$$

We first analyze the mapping $\tilde{F}^I$. We expect to obtain a Newton derivative $\tilde{G}^I$ of $\tilde{F}^I$ if we choose elements $L \in \tilde{G}^I(u, p)$ of the form

$$(L(h, \eta))(t) = \langle \mu^I(u, t) \,,\, h - \eta \rangle = \langle \mu^I(u, t) \,,\, h \rangle - \eta \,, \tag{7.20}$$

where $\mu^I(u, t)$ are probability measures arising from the Newton derivative of the cumulated maximum on I. More precisely, let Φ^I be the mapping defined in (4.13) with $[a, b]$ and M replaced with I and M^I from (7.2), that is,

$$\Phi^I(u, t) = \{\nu \in C(I)^* : \operatorname{supp}(\nu) \subset M^I(u, t), \nu \geq 0, \|\nu\| = 1\} \,. \tag{7.21}$$

Let S^I_Φ be the set of all measurable selectors of Φ^I. We also consider mappings $\tilde{\mu} : Z^I \times I \to (C(I) \times \mathbb{R})^*$ and the set

$$\tilde{S}^I_\Phi = \{\tilde{\mu} : \tilde{\mu}(u, p, t) = \mu^I(u, t) \circ \pi_1 - \pi_2, \mu^I \in S^I_\Phi\} \,, \tag{7.22}$$

where $\pi_1 : C(I) \times \mathbb{R} \to C(I)$ and $\pi_2 : C(I) \times \mathbb{R} \to \mathbb{R}$ denote the projections $\pi_1(h, \eta) = h$ and $\pi_2(h, \eta) = \eta$. (Actually, the elements of $\tilde{S}^I_\Phi$ do not depend on p.) This enables us to rewrite (7.20) in the form

$$(L(h, \eta))(t) = \langle \tilde{\mu}(u, p, t) \,,\, (h, \eta) \rangle \tag{7.23}$$

with $\tilde{\mu} \in \tilde{S}^I_\Phi$.

Proposition 7.6. *The elements of $\tilde{S}^I_\Phi$ are measurable functions $\tilde{\mu} : Z^I \times I \to (C(I) \times \mathbb{R})^*$. For $\tilde{F}^I : Z^I \cap (X \times \mathbb{R}) \to L^{\tilde{q}}(I)$ with $\tilde{q} < \infty$, a Newton derivative $\tilde{G}^I$ is given by*

$$\begin{aligned} \tilde{G}^I &: Z^I \cap (X \times \mathbb{R}) \rightrightarrows \mathcal{L}(X \times \mathbb{R}, L^{\tilde{q}}(I)) \\ \tilde{G}^I(u, p) &= \{L : L \textit{ has the form } (7.20) \textit{ with } \mu^I \in S^I_\Phi\} \\ &= \{L : L \textit{ has the form } (7.23) \textit{ with } \tilde{\mu} \in \tilde{S}^I_\Phi\} \,. \end{aligned} \tag{7.24}$$

The elements L of $\tilde{G}^I(u, p)$ satisfy

$$\|L(h, \eta)\|_{\infty,t} \leq \|h\|_{\infty,t} + |\eta| \tag{7.25}$$

for all $h \in C(I)$, $\eta \in \mathbb{R}$, $t \in I$. Moreover, the remainder estimate

$$\begin{aligned}\sup_{L \in \tilde{G}^I(u+p,h+\eta)} &\|\tilde{F}^I(u+h,p+\eta) - \tilde{F}^I(u,p) - L(h,\eta)\|_{L^{\tilde{q}}(I)} \\ &\le \rho_{(u,p)}(\|h\|_\infty + |\eta|)\|(h,\eta)\|_{X\times\mathbb{R}}\end{aligned} \tag{7.26}$$

holds. The remainder term $\rho_{(u,p)}$ satisfies $\rho_{(u,p)}(\delta) \downarrow 0$ as $\delta \downarrow 0$ and is uniformly bounded in (u,p).

Proof. The elements of $\tilde{S}^I_\Phi$ are measurable as compositions of measurable functions. As F^I has a Newton derivative given by Proposition 4.8 and $\tilde{F}^I(u,p) = F^I(u-p-r)$, setting $\tilde{X} = \tilde{Y} = C(I)$ and $Z = L^{\tilde{q}}(I)$ we check that the assumptions of the refined chain rule, Proposition 5.2, are satisfied. Therefore, $\tilde{G}^I$ is a Newton derivative of $\tilde{F}^I$ and (7.26) holds. (7.25) is a consequence of (7.20) and (4.18). Since $\tilde{F}$ is globally Lipschitz w.r.t. the maximum norm, together with (7.25) the final assertion follows. □

Since $\Phi^I(v,t_*) = \{\delta_{t_*}\}$ for all v, by (7.22) we have for all $\tilde{\mu} \in \tilde{S}^I_\Phi$

$$\langle \tilde{\mu}(u,p,t_*), (h,\eta)\rangle = h(t_*) - \eta \tag{7.27}$$

for all $(u,p) \in Z^I$ and all $(h,\eta) \in C(I) \times I$.

For functions $u : I \to \mathbb{R}$, we consider the positive part mapping F_{pp} defined by

$$(F_{pp}u)(t) = \max\{0, u(t)\}\,, \tag{7.28}$$

which maps $L^q(I)$ as well as $C(I)$ into itself. Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be the set-valued Heaviside function

$$H(x) = \begin{cases} 0\,, & x < 0\,, \\ [0,1]\,, & x = 0\,, \\ 1\,, & x > 0\,. \end{cases} \tag{7.29}$$

The mapping H is usc. By

$$S_H = \{\lambda_H : \lambda_H \text{ selector of } H,\ \lambda_H(0) \in \mathbb{Q}\} \tag{7.30}$$

we define a countable family of measurable selectors of H whose values are dense in the range of H. We then define

$$\begin{aligned} G_{pp} : L^{\tilde{q}}(I) &\rightrightarrows L(L^{\tilde{q}}(I), L^q(I)) \\ G_{pp}(u) &= \{L : L(h) = (\lambda_H \circ u)\cdot h,\ \lambda_H \in S_H\}\,. \end{aligned} \tag{7.31}$$

Lemma 7.7. *The mapping G_{pp} is a Newton derivative of $F_{pp} : L^{\tilde{q}}(I) \to L^q(I)$ for $1 \le q < \tilde{q} \le \infty$.*

Proof. This is a well-known result, see Proposition 3.49 in [18] or Example 8.14 in [10]. □

We have now all ingredients to define a Newton derivative G^I_+ of P^I_+. The elements of $G^I_+(u,p)$ involve the composition of elements of $\tilde{G}^I(u,p)$, given in (7.24), and of elements of $G_{pp}(\tilde{F}^I(u,p))$. Indeed, $L^I_+ \in G^I_+(u,p)$ is expected to have the form

$$L^I_+(h,\eta)(t) = \eta + \lambda_H(\tilde{F}^I(u,p)(t)) \cdot \langle \mu^I(u,t), h - \eta\rangle \tag{7.32}$$

with functions $\lambda_H \in S_H$ and measures $\mu^I \in S^I_\Phi$. Again, we want to write this as

$$L^I_+(h,\eta)(t) = \langle \tilde{\nu}(u,p,t)\,,\,(h,\eta)\rangle \tag{7.33}$$

with $\tilde{\nu} : Z^I \times I \to (C(I)\times\mathbb{R})^*$. This we achieve by setting

$$\tilde{S}^I_{pp} = \{\tilde{\lambda} : \tilde{\lambda}(u,p,t) = \lambda_H(\tilde{F}^I(u,p)(t)),\, \lambda_H \in S_H\} \tag{7.34}$$

and

$$S^I_+ = \{\tilde{\nu} : \tilde{\nu}(u,p,t) = \pi_2 + \tilde{\lambda}(u,p,t)\tilde{\mu}(u,p,t),\, \tilde{\mu} \in \tilde{S}^I_\Phi,\, \tilde{\lambda} \in \tilde{S}^I_{pp}\}\,, \tag{7.35}$$

where again $\pi_2 : C(I)\times\mathbb{R} \to \mathbb{R}$ denotes the projection on the second component, $\pi_2(h,\eta) = \eta$.

Proposition 7.8. *The set S^I_+ given in (7.35) consists of measurable functions $\tilde{\nu} : Z^I \times I \to (C(I)\times\mathbb{R})^*$. The mapping*

$$\begin{aligned} G^I_+ &: Z^I_+ \cap (X\times\mathbb{R}) \rightrightarrows \mathcal{L}(X\times\mathbb{R}, L^q(I)) \\ G^I_+(u,p) &= \{L^I_+ : L^I_+ \text{ given by (7.32) with } \lambda_H \in S_H,\, \mu^I \in S^I_\Phi\} \\ &= \{L^I_+ : L^I_+ \text{ given by (7.33) with } \tilde{\nu} \in S^I_+\} \end{aligned} \tag{7.36}$$

is a Newton derivative of $P^I_+ : Z^I_+ \cap (X\times\mathbb{R}) \to L^q(I)$ for every $q < \infty$. The elements L^I_+ of $G^I_+(u,p)$ satisfy, for all $(u,p) \in Z^I_+ \cap (X\times\mathbb{R})$,

$$\|L^I_+(h,\eta)\|_{\infty,t} \le \max\{\|h\|_{\infty,t}, |\eta|\} \tag{7.37}$$

for all $h \in C(I)$, $\eta \in \mathbb{R}$. Moreover, for all such (u,p) the remainder estimate

$$\begin{aligned} \sup_{L^I_+ \in G^I_+(u+h,p+\eta)} &\|P^I_+(u+h,p+\eta) - P^I_+(u,p) - L^I_+(h,\eta)\|_{L^q(I)} \\ &\le \rho_{(u,p)}(\|h\|_\infty + |\eta|)\|(h,\eta)\|_{X\times\mathbb{R}} \end{aligned} \tag{7.38}$$

holds for all $h \in X$ with $u+h \in Z^I_+$ and all $\eta \in \mathbb{R}$. The remainder term $\rho_{(u,p)}$ satisfies $\rho_{(u,p)}(\delta) \downarrow 0$ as $\delta \downarrow 0$ and is uniformly bounded in (u,p).

Proof. The elements of S^I_+ are measurable as compositions, sums and products of measurable functions.

Due to Proposition 7.6 and Lemma 7.7, the assumptions of the refined chain rule, Proposition 5.2, are satisfied with $\tilde{X} = C(I)$, $Y = \tilde{Y} = L^{\tilde{q}}(I)$ for some $\infty > \tilde{q} > q$, $Z = L^q(I)$. This proves (7.38). The estimate (7.37) follows from (7.32) and (4.18), as λ_H takes values in $[0,1]$ and, setting $\lambda_t = \lambda_H(\tilde{F}^I(u,p)(t))$,

$$L^I_+(h,\eta)(t) = (1-\lambda_t)\eta + \lambda_t\langle \mu^I(u,t), h\rangle\,.$$

Since P^I_+ is global Lipschitz continuous w.r.t. the maximum norm, the final assertion, too, follows in view of (7.37). ☐

We also need a variant of the preceding proposition. For $I = [t_*, t^*]$ we define

$$P^I_{+,*} : Z^I_+ \to \mathbb{R}\,, \quad P^I_{+,*}(u,p) = P^I_+(u,p)(t^*)\,. \tag{7.39}$$

According to (7.32), setting

$$L^I_{+,*}(h,y) = L^I_+(h,y)(t^*)\,, \quad L^I_+ \in G^I_+(u,p)\,, \tag{7.40}$$

yields a well-defined element $L^I_{+,*} \in (C(I)\times\mathbb{R})^*$.

Proposition 7.9. *The mapping*

$$\begin{gathered} G^I_{+,*} : Z^I_+ \cap (X\times\mathbb{R}) \rightrightarrows (X\times\mathbb{R})^* \\ G^I_{+,*}(u,p) = \{L^I_{+,*} : L^I_{+,*} \text{ given by (7.40)}\} \end{gathered} \tag{7.41}$$

is a Newton derivative of $P^I_{+,*} : Z^I_+ \cap (X\times\mathbb{R}) \to \mathbb{R}$. The elements $L^I_{+,*}$ of $G^I_{+,*}(u,p)$ satisfy, for all $(u,p) \in Z^I_+ \cap (X\times\mathbb{R})$,

$$|L^I_{+,*}(h,\eta)| \le \max\{\|h\|_{\infty,t^*}, |\eta|\} \tag{7.42}$$

for all $h \in C(I)$, $\eta\in\mathbb{R}$. Moreover, for all such (u,p) the remainder estimate

$$\begin{aligned} \sup_{L^I_{+,*}\in G^I_{+,*}(u+h,p+\eta)} & |P^I_{+,*}(u+h,p+\eta) - P^I_{+,*}(u,p) - L^I_{+,*}(h,\eta)| \\ & \le \rho_{(u,p)}(\|h\|_\infty + |\eta|)\|(h,\eta)\|_{X\times\mathbb{R}} \end{aligned} \tag{7.43}$$

holds for all $h\in X$ with $u+h\in Z^I_+$ and all $\eta\in\mathbb{R}$. The remainder term $\rho_{(u,p)}$ satisfies $\rho_{(u,p)}(\delta)\downarrow 0$ as $\delta\downarrow 0$ and is uniformly bounded in (u,p).

Proof. We proceed in a manner analogous to the proof of Proposition 7.8. We apply Proposition 5.2 to the decomposition

$$P^I_{+,*}(u,p) = p + \max\{0, \max_I (u-p-r)\}$$

The Newton derivative of the inner maximum satisfies the refined remainder estimate given in Proposition 3.4. The outer maximum is just the positive part mapping on $\mathbb{R}$. □

A Newton derivative G^I_- of the mapping

$$P^I_-(u,p) = p - F_{pp}(\tilde F^I(-u,-p)) \tag{7.44}$$

is obtained with analogous computations. Its elements $L^I_- \in G^I_-(u,p)$ have the form

$$L^I_-(h,\eta)(t) = \eta - \lambda_H(\tilde F^I(-u,-p)(t))\cdot\langle\mu^I(-u,t), -h+\eta\rangle \tag{7.45}$$

with functions $\lambda_H\in S_H$ and measures $\mu^I\in S^I_\Phi$. The associated set S^I_- of measurable mappings $\tilde\nu : Z^I\times I\to (C(I)\times\mathbb{R})^*$ is given by

$$S^I_- = \{\tilde\nu : \tilde\nu(u,p,t) = \pi_2 + \tilde\lambda(-u,-p,t)\tilde\mu(-u,-p,t),\ \tilde\mu\in\tilde S^I_\Phi,\ \tilde\lambda\in\tilde S^I_{pp}\}\,. \tag{7.46}$$

The analogue of Proposition 7.9 also holds on minus intervals.

We combine $G^I_\pm$ and $\Psi^I_\pm$ into mappings G^I and Ψ^I. Indeed, on $Z^I_+\cap Z^I_-$, we have $\tilde F^I(u,p) < 0$ and $\tilde F^I(-u,-p) < 0$ by Lemma 7.2. Consequently, the argument of λ_H in the representations (7.32) and (7.45) is negative, therefore $L^I_\pm(h,\eta)(t) = \eta$ on I. As the sets $Z^I_\pm$ are open subsets of Z^I, from Proposition 7.8 and the corresponding result for P^I_- we get the following result.

Proposition 7.10. *The set*

$$S^I = \{\nu : \nu(u,p,t) = \tilde{\nu}_\pm(u,p,t) \text{ with } \tilde{\nu}_\pm \in S^I_\pm,\ (u,p,t) \in Z^I_\pm \times I\} \tag{7.47}$$

consists of measurable mappings $\nu : Z^I \times I \to (C(I)\times\mathbb{R})^*$. The mapping $G^I : Z^I \cap (X\times\mathbb{R}) \rightrightarrows \mathcal{L}(X\times\mathbb{R}, L^q(I))$ given by $G^I = G^I_\pm$ on $Z^I_\pm \cap (X\times\mathbb{R})$ is well-defined and is a Newton derivative of $P^I : Z^I \cap (X\times\mathbb{R}) \to L^q(I)$. The estimates (7.37) and (7.38) hold with G^I, P^I, Z^I and L^I in place of G^I_+, P^I_+, Z^I_+ and L^I_+, respectively. The remainder term $\rho_{(u,p)}$ satisfies $\rho_{(u,p)}(\delta) \downarrow 0$ as $\delta \downarrow 0$ and is uniformly bounded in (u,p).

The initial value. According to (6.1), the initial value of the play is given by

$$w_0(u,z_0) = u(a) - \pi_r(z_0) = u(a) - \max\{-r, \min\{r, z_0\}\}\,. \tag{7.48}$$

It is well known that the mapping $R : \mathbb{R} \rightrightarrows \mathbb{R}$,

$$R(x) = \begin{cases} 0\,, & |x| > r\,, \\ [0,1]\,, & |x| = r\,, \\ 1\,, & |x| < r \end{cases} \tag{7.49}$$

is a Newton derivative of π_r and that R is usc. Then

$$S_R = \{\lambda_0 : \lambda_0 \text{ selector of } R,\ \lambda_0(\pm r) \in \mathbb{Q}\} \tag{7.50}$$

defines a countable family of measurable selectors of R.

Lemma 7.11. *A Newton derivative of $w_0 : C[a,b]\times\mathbb{R} \to \mathbb{R}$ is given by*

$$\begin{gathered} G_0 : C[a,b]\times\mathbb{R} \rightrightarrows (C[a,b]\times\mathbb{R})^*\,, \\ G_0(u,z_0) = \{L : L(h,y) = h(a) - \lambda_0(z_0)y\,, \lambda_0 \in S_R\}\,. \end{gathered} \tag{7.51}$$

We have

$$|L(h,y)| \le \|h\|_\infty + |y| \tag{7.52}$$

for all $L \in G_0(u,z_0)$ and all $(u,z_0) \in C[a,b]\times\mathbb{R}$.

Proof. Let $L \in G_0(u,z_0)$. Then for all $(h,y) \in C[a,b]\times\mathbb{R}$ we have

$$\begin{aligned} |w_0(u+h, z_0+y) - w_0(u,z_0) - L(h,y)| =& |(\pi_r(z_0+y) - \pi_r(z_0) - \lambda_0(z_0)y)| \\ \le& \rho(|y|)|y| \end{aligned} \tag{7.53}$$

with some $\rho(\delta) \downarrow 0$ as $\delta \downarrow 0$, since R is a Newton derivative of π_r. ☐

A Newton derivative on a partition for small input oscillations.
Let $\Delta = \{t_k\}_{0\le k\le N}$ be a partition of $[a,b]$. According to Proposition 7.5, on the set Z^Δ of small input oscillations, see (7.1), the play can be written as a composition of the mappings P^{I_k} which belong to the partition intervals $I_k = [t_{k-1}, t_k]$. Consequently, we obtain a Newton derivative of the play on Z^Δ as a composition of the Newton derivatives of P^{I_k} as follows.

We define $w_k^\Delta : Z^\Delta \to \mathbb{R}$ and $P_k^\Delta : Z^\Delta \to C(I_k)$, setting $w_0^\Delta = w_0$ from (7.48), and for $k \geq 1$

$$\begin{aligned} w_k^\Delta(u, z_0) &= P^{I_k}(u, w_{k-1}^\Delta(u, z_0))(t_k)\,, \\ P_k^\Delta(u, z_0)(t) &= P^{I_k}(u, w_{k-1}^\Delta(u, z_0))(t)\,, \quad t \in I_k\,. \end{aligned} \tag{7.54}$$

Using Lemma 7.5 successively we see that $P_k^\Delta(u, z_0) = \mathcal{P}_r[u; z_0]$ on I_k.

Denoting by π_1 the projection $\pi_1 : C[a,b] \times \mathbb{R} \to C(I_k)$, $\pi_1(u, z_0) = u|I_k$, we define the sets $S_0^\Delta = S_R$,

$$\begin{aligned} S_k^\Delta = \{\mu_k^\Delta : \mu_k^\Delta(u, z_0, t) = \nu(u, p, t) \circ (\pi_1, \mu_{k-1}^\Delta(u, z_0, t_{k-1})), \\ \nu \in S^{I_k},\, p = w_{k-1}^\Delta(u, z_0),\, \mu_{k-1}^\Delta \in S_{k-1}^\Delta\}\,, \quad k \geq 1\,. \end{aligned} \tag{7.55}$$

The elements of S_Δ^k are measurable functions $\mu_k^\Delta : Z^\Delta \times I_k \to (C[a,b] \times \mathbb{R})^*$.

We define $W_0^\Delta = G_0$ and inductively for $k \geq 1$

$$\begin{aligned} W_k^\Delta &: Z^\Delta \rightrightarrows (C[a,b] \times \mathbb{R})^*\,, \\ W_k^\Delta(u, z_0) &= \{L_k^w : L_k^w = \mu_k^\Delta(u, z_0, t_k) \text{ with } \mu_k^\Delta \in S_k^\Delta\}\,. \end{aligned} \tag{7.56}$$

The elements $L_k^w \in W_k^\Delta(u, z_0)$ satisfy

$$\begin{aligned} L_k^w(h, y) = L^{I_k}(h, L_{k-1}^w(h, y))(t_k)\,, \\ L^{I_k} \in G^{I_k}(u, w_{k-1}(u, z_0))\,, \quad L_{k-1}^w \in W_{k-1}^\Delta(u, z_0)\,. \end{aligned} \tag{7.57}$$

We define $G_0^\Delta = G_0$ and inductively for $k \geq 1$

$$\begin{aligned} G_k^\Delta &: Z^\Delta \rightrightarrows \mathcal{L}(C[a,b] \times \mathbb{R}, L^\infty(I_k))\,, \\ G_k^\Delta(u, z_0) &= \{L_k^\Delta : L_k^\Delta \text{ satisfies (7.59) for some } \mu_k^\Delta \in S_k^\Delta\}\,, \end{aligned} \tag{7.58}$$

$$L_k^\Delta(h, y)(t) = \langle \mu_k^\Delta(u, z_0, t)\,,\, (h, y)\rangle\,. \tag{7.59}$$

The mappings L_k^Δ satisfy

$$\begin{aligned} L_k^\Delta(h, y)(t) = L^{I_k}(h, L_{k-1}^w(h, y))(t)\,, \\ L^{I_k} \in G^{I_k}(u, w_{k-1}(u, z_0))\,, \quad L_{k-1}^w \in W_{k-1}^\Delta(u, z_0)\,. \end{aligned} \tag{7.60}$$

Proposition 7.12. *Let $0 \leq k \leq N$, $1 \leq q < \infty$.*
(i) The sets S_Δ^k consist of measurable functions $\mu_k^\Delta : Z^\Delta \times I_k \to (C[a,b] \times \mathbb{R})^*$.
(ii) The mapping $W_k^\Delta : Z^\Delta \cap (X \times \mathbb{R}) \rightrightarrows (C[a,b] \times \mathbb{R})^*$ is a Newton derivative of $w_k^\Delta : Z^\Delta \cap (X \times \mathbb{R}) \to \mathbb{R}$. The elements L_k^w of W_k^Δ satisfy the estimate

$$|L_k^w(h, y)| \leq \|h\|_{\infty, t_k} + |y| \tag{7.61}$$

for all $h \in C[a,b]$ and $y \in \mathbb{R}$, uniformly in (u, z_0). Moreover, for all such (u, z_0) the remainder estimate

$$\begin{aligned} \sup_{L_k^w \in W_k^\Delta(u+h, z_0+y)} &|w_k^\Delta(u+h, z_0+y) - w_k^\Delta(u, z_0) - L_k^w(h, y)| \\ &\leq \rho_{(u,z_0)}(\|h\|_\infty + |y|)\|(h, y)\|_{X \times \mathbb{R}} \end{aligned} \tag{7.62}$$

holds for all $h \in X$ *with* $u + h \in Z_+^I$ *and all* $y \in \mathbb{R}$. *The remainder term* $\rho_{(u,z_0)}$ *satisfies* $\rho_{(u,z_0)}(\delta) \downarrow 0$ *as* $\delta \downarrow 0$ *and is uniformly bounded in* (u, z_0).
(iii) The mapping $G_k^\Delta : Z^\Delta \cap (X \times \mathbb{R}) \rightrightarrows \mathcal{L}(X \times \mathbb{R}, L^q(I_k))$ *is a Newton derivative of* P_k^Δ. *The elements* L_k^Δ *of* $G_k^\Delta(u, z_0)$ *satisfy the estimate*

$$\|L_k^\Delta(h, y)\|_{\infty,t} \le \|h\|_{\infty,t} + |y| \tag{7.63}$$

for all $h \in C[a, b]$ *and* $y \in \mathbb{R}$, *uniformly in* (u, z_0). *Moreover, for all such* (u, z_0) *the remainder estimate*

$$\begin{aligned} \sup_{L_k^w \in W_k^\Delta(u+h,z_0+y)} & |P_k^\Delta(u + h, z_0 + y) - P_k^\Delta(u, z_0) - L_k^\Delta(h, y)| \\ & \le \rho_{(u,z_0)}(\|h\|_\infty + |y|)\|(h, y)\|_{X\times\mathbb{R}} \end{aligned} \tag{7.64}$$

holds for all $h \in X$ *with* $u + h \in Z_+^I$ *and all* $y \in \mathbb{R}$. *The remainder term* $\rho_{(u,z_0)}$ *satisfies* $\rho_{(u,z_0)}(\delta) \downarrow 0$ *as* $\delta \downarrow 0$ *and is uniformly bounded in* (u, z_0).

Proof. We proceed by induction over k. The case $k = 0$ is treated in Lemma 7.11. Now assume the result is proved for $k - 1$.
(i) This follows immediately from the definition of S_k^Δ.
(ii) We apply Proposition 5.2 to the decomposition

$$(u, z_0) \mapsto (u, w_{k-1}^\Delta(u, z_0)) \mapsto w_k^\Delta(u, z_0)$$

given by the first equation in (7.54). Its assumptions are satisfied by the induction hypothesis and by Proposition 7.9. (7.61) follows from the estimate

$$\begin{aligned} |L_k^w(h, y)| &\le \|L^{I_k}(h, L_{k-1}^w(h, y))\|_{\infty,t_k} \le \max\{\|h\|_{\infty,t_k}, \|h\|_{\infty,t_k} + |y|\} \\ &= \|h\|_{\infty,t_k} + |y| \,. \end{aligned}$$

(iii) This follows as in (ii), using Proposition 7.10 instead of Proposition 7.9, as well as the estimate

$$\begin{aligned} \|L_k^\Delta(h, y)\|_{\infty,t} &= \|L^{I_k}(h, L_{k-1}^w(h, y))\|_{\infty,t} \le \max\{\|h\|_{\infty,t}, \|h\|_{\infty,t} + |y|\} \\ &= \|h\|_{\infty,t} + |y| \,. \end{aligned}$$

□

From the Newton derivatives G_k^Δ of P_k^Δ we now obtain a Newton derivative G^Δ of the play on $Z^\Delta \cap (X \times \mathbb{R})$. We define the set S^Δ consisting of mappings $\mu^\Delta : Z^\Delta \times [a, b] \to (C[a, b] \times \mathbb{R})^*$ by

$$\begin{aligned} S^\Delta = \{\mu^\Delta : \mu^\Delta = \mu_1^\Delta \text{ on } Z^\Delta \times [t_0, t_1], & \\ \mu^\Delta = \mu_k^\Delta \text{ on } Z^\Delta \times (t_{k-1}, t_k] \text{ for } k > 1,\ \mu_k^\Delta \in S_k^\Delta\} \,. \end{aligned} \tag{7.65}$$

We define

$$\begin{aligned} G^\Delta : Z^\Delta &\rightrightarrows \mathcal{L}(C[a, b] \times \mathbb{R}, L^\infty(a, b)) \,, \\ G^\Delta(u, z_0) &= \{L^\Delta : L^\Delta(h, y)(t) = \langle \mu^\Delta(u, z_0, t), (h, y)\rangle \\ &\qquad \text{on } [a, b] \text{ for some } \mu^\Delta \in S^\Delta\} \,. \end{aligned} \tag{7.66}$$

Proposition 7.13. *Let $1 \le q < \infty$.*
The mapping $G^\Delta : Z^\Delta \cap (X \times \mathbb{R}) \rightrightarrows \mathcal{L}(X \times \mathbb{R}, L^q(a,b))$ is a Newton derivative of the play $\mathcal{P}_r : Z^\Delta \cap (X \times \mathbb{R}) \to L^q(a,b)$. The elements L^Δ of $G^\Delta(u, z_0)$ satisfy the estimate

$$\|L^\Delta(h,y)\|_{\infty,t} \le \|h\|_{\infty,t} + |y| \tag{7.67}$$

for all $h \in C[a,b]$ and $y \in \mathbb{R}$, uniformly in (u, z_0). Moreover, for all such (u, z_0) the remainder estimate

$$\begin{aligned} \sup_{L^\Delta \in G^\Delta(u+h, z_0+y)} & \|\mathcal{P}_r[u+h; z_0+y] - \mathcal{P}_r[u; z_0] - L^\Delta(h,y)\|_{L^q(a,b)} \\ & \le \rho_{(u,z_0)}(\|h\|_\infty + |y|)\|(h,y)\|_{X \times \mathbb{R}} \end{aligned} \tag{7.68}$$

holds for all $h \in X$ with $u + h \in Z_+^I$ and all $y \in \mathbb{R}$, where $\rho_{(u,z_0)}(\delta) \downarrow 0$ as $\delta \downarrow 0$ and $\rho_{(u,z_0)}$ is uniformly bounded in (u, z_0).

Proof. Let $(u, z_0) \in Z^\Delta \cap (X \times \mathbb{R})$, $(h, y) \in X \times \mathbb{R}$ with $\|h\|_X$ small enough, and $L^\Delta \in G^\Delta(u+h, z_0+y)$. Since $\mathcal{P}_r[u; z_0] = P_k^\Delta(u, z_0)$ and $\mathcal{P}_r[u+h; z_0+y] = P_k^\Delta(u+h, z_0+y)$ on I_k, we have in view of the definition of S^Δ, S_k^Δ and G_k^Δ

$$\begin{aligned} & \|\mathcal{P}_r[u+h; z_0+y] - \mathcal{P}_r[u; z_0] - L^\Delta(h,y)\|_{L^q(a,b)}^q \\ & \quad = \sum_{k=1}^N \|P_k^\Delta(u+h, z_0+y) - P_k^\Delta(u, z_0) - L_k^\Delta(h,y)\|_{L^q(I_k)}^q \end{aligned}$$

for some $L_k^\Delta \in G_k^\Delta(u+h, z_0+y)$. As G_k^Δ is a Newton derivative of P_k^Δ by Proposition 7.12, (7.63) holds for L_k^Δ, and (7.64) holds for the remainder, the claim follows. ☐

A Newton derivative of the play on the whole space $X \times \mathbb{R}$.
Let $\{\Delta_n\}$ be a sequence of partitions of $[a,b]$ such that $|\Delta_n| \to 0$ as $n \to \infty$ and that Δ_{n+1} is obtained from Δ_n by adding a single point $t \notin \Delta_n$, starting from $\Delta_1 = \{a, b\}$. We have

$$Z^{\Delta_n} \subset Z^{\Delta_{n+1}}\,, \quad C[a,b] \times \mathbb{R} = \bigcup_{n \in \mathbb{N}} Z^{\Delta_n} \tag{7.69}$$

and consequently

$$X \times \mathbb{R} = \bigcup_{n \in \mathbb{N}} (Z^{\Delta_n} \cap (X \times \mathbb{R})) = \bigcup_{n \in \mathbb{N}} (X^{\Delta_n} \times \mathbb{R})\,. \tag{7.70}$$

We construct a Newton derivative G^{P_r} of the play $\mathcal{P}_r$ on $X \times \mathbb{R}$ from the Newton derivatives G^{Δ_n} obtained in Proposition 7.13.

Remark 7.14. According to Remark 2.6, if $G^{\Delta_n}(u, z_0) \subset G^{\Delta_{n+1}}(u, z_0)$ would hold on the domain Z^{Δ_n} of G^{Δ_n}, we might simply set $G^{P_r} = G^{\Delta_n}$ on $Z^{\Delta_n} \setminus Z^{\Delta_{n-1}}$. This leads to the following problem. Let $I = [t_*, t^*]$ be the interval of Δ_n which is partitioned into $I = I' \cup I''$ in Δ_{n+1}. If the inclusion holds, all functions $\mu^{\Delta_n}|I'$ and $\mu^{\Delta_n}|I''$ with $\mu^{\Delta_n} \in S^{\Delta_n}$ must be representable as $\mu^{\Delta_{n+1}}|I'$ and $\mu^{\Delta_{n+1}}|I''$ for some $\mu^{\Delta_{n+1}} \in S^{\Delta_{n+1}}$. While this is true on I', it is a nontrivial question whether it is true on I'': the construction of $S^{\Delta_n}|I''$ directly refers to $S^{\Delta_n}|[0, t_*]$, the construction of $S^{\Delta_{n+1}}|I''$ indirectly refers to $S^{\Delta_{n+1}}|[0, t_*] = S^{\Delta_n}|[0, t_*]$ via the detour over $S^{\Delta_{n+1}}|I'$; thus, the relations between the maximum sets M^I, $M^{I'}$ and $M^{I''}$ play a role. The author does not know the answer but conjectures that it is "no" for certain pairs (u, z_0).

In order to utilize Proposition 2.5, we set

$$U = X \times \mathbb{R}\,, \quad U_n = X^{\Delta_n} \times \mathbb{R}\,.$$

Let $0 < r_1 < r_2 < \dots$ be an increasing sequence of positive numbers with $r_n < r$ for all n. Let $\{I_{n,k}\}$ be the partition intervals of Δ_n. We define

$$V_n = \{(u, z_0) : u \in X,\ z_0 \in \mathbb{R},\ \operatorname*{osc}_{I_{n,k}} u < r_n \text{ for all } k\}\,. \tag{7.71}$$

Since $r_n < r_{n+1} < r$, we have $\overline{V}_n \subset U_n \cap V_{n+1}$. Moreover, by (7.70)

$$\bigcup_n V_n = X \times \mathbb{R} = U\,, \quad \text{because } |\Delta_n| \to 0 \text{ as } n \to \infty.$$

Thus, all assumptions of Proposition 2.5 are satisfied.

We finally arrive at the main result.

Theorem 7.15. *Let $1 \le q < \infty$.*
The mapping $G^{P_r} : X \times \mathbb{R} \rightrightarrows \mathcal{L}(X \times \mathbb{R}, L^q(a,b))$ defined by

$$G^{P_r}(u, z_0) = G^{\Delta_n}(u, z_0) \quad \text{if } (u, z_0) \in \overline{V}_n \setminus \overline{V}_{n-1}\,, \tag{7.72}$$

is a Newton derivative of the play $\mathcal{P}_r : X \times \mathbb{R} \to L^q(a,b)$ with the remainder estimate

$$\begin{aligned} \sup_{L^{\mathcal{P}_r} \in G^{P_r}(u+h, z_0+y)} &\|\mathcal{P}_r[u+h; z_0+y] - \mathcal{P}_r[u; z_0] - L^{\mathcal{P}_r}(h,y)\|_{L^q(a,b)} \\ &\le \rho_{(u,z_0)}(\|h\|_\infty + |y|)\|(h,y)\|_{X\times\mathbb{R}} \end{aligned} \tag{7.73}$$

where $\rho_{(u,z_0)}(\delta) \downarrow 0$ as $\delta \downarrow 0$. The elements L^{P_r} of $G^{P_r}(u, z_0)$ satisfy the estimate

$$\|L^{P_r}(h,y)\|_{\infty,t} \le \|h\|_{\infty,t} + |y| \tag{7.74}$$

for all $h \in X$ and $y \in \mathbb{R}$, uniformly in (u, z_0). They have the form

$$L^{P_r}(h,y)(t) = \langle \mu^{P_r}(u, z_0, t), (h,y)\rangle\,, \quad t \in (a,b)\,, \tag{7.75}$$

with

$$\mu^{P_r}(u, z_0, t) = \mu^{\Delta_n}(u, z_0, t) \quad \text{if } (u, z_0) \in \overline{V}_n \setminus \overline{V}_{n-1}\,, \quad \mu^{\Delta_n} \in S^{\Delta_n}\,. \tag{7.76}$$

The functions $\mu^{P_r} : C[a,b] \times \mathbb{R} \times [a,b] \to (C[a,b] \times \mathbb{R})^*$ are measurable.

Proof. This follows from Proposition 2.5 and Proposition 7.13 when we choose

$$\rho_{u,z_0} = \max\{\rho_{n,u,z_0}, \rho_{n+1,u,z_0}\} \quad \text{if } (u, z_0) \in \overline{V}_n \setminus \overline{V}_{n-1}$$

with the remainder terms ρ_{n,u,z_0} belonging to G^{Δ_n}. □

Since the stop operator is related to the play operator by the formula $\mathcal{S}_r[u; z_0] = u - \mathcal{P}_r[u; z_0]$, it also has a Newton derivative.

Corollary 7.16. *The stop operator*

$$\mathcal{S}_r : X \times \mathbb{R} \to L^q(a,b)\,, \quad 1 \le q < \infty\,, \tag{7.77}$$

has a Newton derivative given by

$$G^{S_r}(u, z_0) = \pi_1 - G^{P_r}(u, z_0) \tag{7.78}$$

with elements

$$L^{S_r}(h, y) = h - L^{P_r}(h, y)\,. \tag{7.79}$$

Here, G^{P_r} and L^{P_r} have the form and properties described in Theorem 7.15, and π_1 denotes the projection $\pi_1(h, y) = h$.

8. Bouligand derivative of the play and the stop

We intend to prove that the play and the stop operator are Bouligand differentiable from $X \times \mathbb{R}$ to L^q, $1 \le q < \infty$. It suffices to show that $\mathcal{P}_r, \mathcal{S}_r : X^\Delta \to L^q(a,b)$ are Bouligand differentiable for arbitrary partitions Δ, as the sets $X^\Delta \subset X \times \mathbb{R}$ are open and their union covers $X \times \mathbb{R}$.

In the previous section we explained how, on X^Δ, the play can be represented as a finite composition of the positive part F_{pp}, the cumulated maximum F^I and continuous linear mappings. By virtue of the chain rule, it therefore suffices to show that F_{pp} and F^I are Bouligand differentiable, and that the function spaces involved in the composition are fitting.

The positive part mapping $\beta : \mathbb{R} \to \mathbb{R}$, $\beta(x) = \max\{x, 0\}$ has the directional (in fact, Bouligand) derivative

$$\beta'(x; y) = \begin{cases} 0\,, & x < 0 \text{ or } x = 0\,,\ y \le 0\,, \\ y\,, & x > 0 \text{ or } x = 0\,,\ y > 0\,. \end{cases} \tag{8.1}$$

Lemma 8.1. *Let $I \subset [a,b]$ be a closed interval, $1 \le q < \tilde{q} \le \infty$. The mapping $F_{pp} : L^{\tilde{q}}(I) \to L^q(I)$, $F_{pp}(u) = \max\{u, 0\}$, is Bouligand differentiable, and*

$$F'_{pp}(u; h)(t) = \beta'(u(t); h(t))\,. \tag{8.2}$$

Proof. See Examples 8.12 and 8.14 in [10]. □

It has already be proved in Proposition 4.1 that the cumulated maximum $F^I : X \to L^{\tilde{q}}(I)$ is Bouligand differentiable for every $\tilde{q} < \infty$, and that

$$(F^I)'(u; h)(t) = \max_{s \in M^I(u,t)} h(s)\,. \tag{8.3}$$

By the chain rule, the mapping $P^I_+ : Z^I \cap (X \times \mathbb{R}) \to L^q(I)$,

$$P^I_+(u, p) = p + F_{pp}(F^I(u - p - r))$$

has the Bouligand derivative

$$(P^I_+)'((u, p); (h, \eta))(t) = \eta + \beta'(\max_{s \in I, s \le t}(u(s) - r - p); \max_{s \in M^I(u,t)}(h(s) - \eta))\,. \tag{8.4}$$

An analogous formula holds for the Bouligand derivative of P_-^I. Applying the chain rule to (7.54), we obtain the Bouligand derivative of the play recursively as

$$\begin{aligned}(w_k^\Delta)'((u,z_0);(h,y)) &= (P_k^I)'((u,w_{k-1}(u,z_0));(h,(w_{k-1}^\Delta)'((u,z_0);(h,y))))(t_k)\,,\\ \mathcal{P}_r'[[u;z_0];[h;y]](t) &= (P_k^I)'((u,w_{k-1}(u,z_0));(h,(w_{k-1}^\Delta)'((u,z_0);(h,y))))(t)\,.\end{aligned} \tag{8.5}$$

We also obtain the refined remainder estimate.

Theorem 8.2. *The Bouligand derivative of the play operator $\mathcal{P}_r$ given in (8.5) satisfies, for all $(u,z_0)\in X\times\mathbb{R}$, the remainder estimate*

$$\begin{aligned}&\|\mathcal{P}_r[u+h;z_0+y]-\mathcal{P}_r[u;z_0]-\mathcal{P}_r'[[u;z_0];[h;y]]\|_{L^q(a,b)}\\ &\quad\le \rho_{u,z_0}(\|h\|_\infty+|y|)\|(h,y)\|_{X\times\mathbb{R}}\end{aligned} \tag{8.6}$$

for all $h\in X$, $y\in\mathbb{R}$. Here, $\rho_{(u,z_0)}(\delta)\downarrow 0$ as $\delta\downarrow 0$ and $\rho_{(u,z_0)}$ is uniformly bounded in (u,z_0).

Proof. The proof is analogous to that for the Newton derivative, using Proposition 5.4 instead of Proposition 5.2. □

9. The parametric play operator

Instead of a single play operator acting on a function $u=u(t)$, we now want to consider a family of play operators acting on a function $u=u(x,t)$, where x plays the role of a parameter. This has been developed in [19] in order to solve boundary value problems for partial differential equations with hysteresis. Here, we are concerned with parametrizing the Newton derivative of the play.

For a given measurable space Ω (that is, a set Ω equipped with a sigma algebra), we want to define the **parametric play operator** $\mathcal{P}_r^\Omega$ by

$$\mathcal{P}_r^\Omega[u;z_0](x,t)=\mathcal{P}_r[u(x,\cdot);z_0(x)](t) \tag{9.1}$$

for functions $u:\Omega\times[a,b]\to\mathbb{R}$, $z_0:\Omega\to\mathbb{R}$. The parametric play operator thus represents a parametric family of play operators.

For a given metric space X, equipped with the Borel sigma algebra, let $\mathcal{M}(\Omega;X)$ denote the space of all measurable functions from Ω to X.

Lemma 9.1. *Formula (9.1) defines an operator*

$$\mathcal{P}_r^\Omega:\mathcal{M}(\Omega;C[a,b])\times\mathcal{M}(\Omega;\mathbb{R})\to\mathcal{M}(\Omega;C[a,b])\,. \tag{9.2}$$

Proof. The assignment $x\mapsto(u(x,\cdot),z_0(x))\mapsto\mathcal{P}_r[u(x,\cdot),z_0(x)]$ defines a mapping $\Omega\to C[a,b]\times\mathbb{R}\to C[a,b]$ which is measurable since $\mathcal{P}_r$ is continuous. □

We define the **parametric cumulated maximum** (that is, the parametric family of cumulated maxima) F^Ω for functions $u:\Omega\to C[a,b]$ by

$$(F^\Omega u)(x)=F(u(x,\cdot))\,,\quad x\in\Omega\,. \tag{9.3}$$

Lemma 9.2. *We have*

$$\begin{aligned}&F^\Omega:\mathcal{M}(\Omega;C[a,b])\to\mathcal{M}(\Omega;C[a,b])\,,\\ &F^\Omega:L^p(\Omega;C[a,b])\to L^p(\Omega;C[a,b])\,,\quad 1\le p\le\infty\,.\end{aligned} \tag{9.4}$$

Proof. If $u : \Omega \to C[a,b]$ is measurable, the composition $x \mapsto u(x,\cdot) \mapsto F(u(x,\cdot))$ defines a measurable mapping since $F : C[a,b] \to C[a,b]$ is continuous. As $\|(F^\Omega u)(x)\|_\infty \le \|u(x,\cdot)\|_\infty$ and because $u(x,\cdot) = v(x,\cdot)$ a.e. in x implies that $F^\Omega u = F^\Omega v$ a.e. in x, the second assertion in (9.4) follows. ☐

The corresponding set-valued mappings M^Ω and Φ^Ω are given by

$$M^\Omega(u,t,x) = M(u(x,\cdot),t)\,, \quad \Phi^\Omega(u,t,x) = \Phi(u(x,\cdot),t)\,. \tag{9.5}$$

For any given function $u : \Omega \to C[a,b]$, these formulas define set-valued mappings

$$\begin{aligned} (x,t) &\mapsto M(u(x,\cdot),t) = M^\Omega(u,t,x)\,, \quad \Omega\times[a,b] \rightrightarrows [a,b]\,, \\ (x,t) &\mapsto \Phi(u(x,\cdot),t) = \Phi^\Omega(u,t,x)\,, \quad \Omega\times[a,b] \rightrightarrows C[a,b]^*\,. \end{aligned} \tag{9.6}$$

Lemma 9.3. *Let $u \in \mathcal{M}(\Omega; C[a,b])$. Then the mappings defined in (9.6) are measurable.*

Proof. The mappings arise as compositions

$$\begin{aligned} (x,t) \mapsto (u(x,\cdot),t) \mapsto M(u(x,\cdot),t)\,, &\quad \Omega\times[a,b] \to C[a,b]\times[a,b] \rightrightarrows [a,b]\,, \\ (x,t) \mapsto (u(x,\cdot),t) \mapsto \Phi(u(x,\cdot),t)\,, &\quad \Omega\times[a,b] \to C[a,b]\times[a,b] \rightrightarrows C[a,b]^*\,. \end{aligned}$$

Due to Propositions 4.2 and 4.5, the assertion follows. ☐

In Proposition 4.8, a Newton derivative G of the cumulated maximum F has been constructed from measurable selectors μ of Φ. Any such $\mu \in S_\Phi$ defines an element of $G(u(x,\cdot))$. More precisely, given $u \in \mathcal{M}(\Omega; C[a,b])$ and $x \in \Omega$ we set

$$[(L^\Omega(x))v](t) = \langle \mu(u(x,\cdot),t), v\rangle\,, \quad v \in C[a,b]\,. \tag{9.7}$$

Proposition 9.4. *Let μ be a measurable selector of Φ, let $u \in \mathcal{M}(\Omega; C[a,b])$. Then (9.7) defines a mapping*

$$L^\Omega : \Omega \to \mathcal{L}(C[a,b]; L^\infty(a,b)) \tag{9.8}$$

with the property

$$L^\Omega(x) \in G(u(x,\cdot)) \quad \text{for all } x \in \Omega. \tag{9.9}$$

Let moreover $h \in \mathcal{M}(\Omega; C[a,b])$. Then

$$(x,t) \mapsto [(L^\Omega(x))h(x,\cdot)](t) = \langle \mu(u(x,\cdot),t), h(x,\cdot)\rangle \tag{9.10}$$

defines a measurable function from $\Omega\times[a,b]$ to $\mathbb{R}$.

Proof. Proposition 4.8 yields (9.8) and (9.9). The composition $(x,t) \mapsto (u(x,\cdot),t) \mapsto \mu(u(x,\cdot),t)$ defines a measurable mapping from $\Omega\times[a,b]$ to $C[a,b]^*$, since $\mu : C[a,b]\times[a,b] \to C[a,b]^*$ is measurable. As the mapping $x \mapsto h(x,\cdot)$ is measurable and the mapping $(\nu,v) \mapsto \langle \nu, v\rangle$ is continuous, (9.10) defines a measurable function. ☐

We define

$$\begin{aligned} G^\Omega &: \mathcal{M}(\Omega; C[a,b]) \rightrightarrows \mathrm{Map}(\Omega; \mathcal{L}(C[a,b]; L^\infty(a,b))) \\ G^\Omega(u) &= \{L^\Omega : L^\Omega \text{ satisfies (9.7) and (9.8) for some } \mu \in S_\Phi\}\,, \end{aligned} \tag{9.11}$$

a parametric family of Newton derivatives of the parametric family of cumulated maxima F^Ω. It is **not** a Newton derivative of F^Ω. (Here, $\mathrm{Map}(A;B)$ stands for the set of all mappings from a set A to a set B.)

For the parametric play $\mathcal{P}_r^\Omega$ we proceed in the same manner. According to Theorem 7.15, the Newton derivative G^{P_r} of $\mathcal{P}_r$ constructed there has, when evaluated at (u, z_0), elements of the form

$$L^{P_r}(h,y)(t) = \langle \mu^{P_r}(u, z_0, t), (h,y)\rangle\,, \quad t \in (a,b)\,, \tag{9.12}$$

for some μ^{P_r} as given in (7.76). We define

$$\begin{gathered} L_r^\Omega : \Omega \to \mathcal{L}(C[a,b] \times \mathbb{R}; L^\infty(a,b)) \\ [(L_r^\Omega(x))(v, y_0)](t) = \langle \mu^{P_r}(u(x,\cdot), z_0(x), t), (v, y_0)\rangle \end{gathered} \tag{9.13}$$

for $(v, y_0) \in C[a,b] \times \mathbb{R}$.

Proposition 9.5. *Let μ^{P_r} be as given in (7.76), let $u \in \mathcal{M}(\Omega; C[a,b])$ and $z_0 \in \mathcal{M}(\Omega;\mathbb{R})$. Then L_r^Ω as given in (9.13) satisfies*

$$L_r^\Omega(x) \in G^{P_r}(u(x,\cdot), z_0(x)) \quad \textit{for all } x \in \Omega. \tag{9.14}$$

Let moreover $h \in \mathcal{M}(\Omega; C[a,b])$, $y \in \mathcal{M}(\Omega;\mathbb{R})$. Then

$$(x,t) \mapsto [(L_r^\Omega(x))(h(x,\cdot), y(x)](t) = \langle \mu^{P_r}(u(x,\cdot), z_0(x), t), (h(x,\cdot), y(x))\rangle \tag{9.15}$$

defines a measurable function from $\Omega \times [a,b]$ to $\mathbb{R}$.

Proof. Analogous to that of Proposition 9.4. □

We may define $G_r^\Omega(u, z_0)$ as the set of all such mappings L_r^Ω and view G_r^Ω as a parametric Newton derivative of the parametric play $\mathcal{P}_r^\Omega$.

10. Appendix: Set-valued mappings

In this section, we recall some basic terminology from set-valued analysis, given *e.g.* in [16].

Let $\Psi : X \rightrightarrows Y$. We generally assume that $\Psi(u) \neq \emptyset$ for every $u \in X$.

Definition 10.1. Let X, Y be Hausdorff topological spaces, let $\Psi : X \rightrightarrows Y$. We say that Ψ is **upper semicontinuous** (or **usc** for short), if

$$\Psi^{-1}(A) := \{u : u \in X,\ \Psi(u) \cap A \neq \emptyset\} \tag{10.1}$$

is closed for every closed subset A of Y. We say that Ψ is **measurable** if $\Psi^{-1}(V)$ is measurable for all open $V \subset Y$. A mapping $\psi : X \to Y$ is called a **measurable selector** of Ψ if ψ is measurable and $\psi(u) \in \Psi(u)$ for every $u \in X$.

Lemma 10.2. *Let X, Y be Hausdorff topological spaces. A mapping $\Psi : X \rightrightarrows Y$ is usc if and only if for every $u \in X$ and every open set V with $\Psi(u) \subset V \subset Y$ there exists an open set $U \subset X$ with $u \in U$ and $\Psi(U) \subset V$.*

Proof. See Proposition 6.1.3 in [16]. □

Obviously, a single-valued mapping is continuous in the usual sense if and only if it is usc in the sense above.

Acknowledgements. The author thanks Michael Ulbrich in particular for pointing out the line of argument used in the proof of Propositions 4.1 and 4.8, and him as well as Constantin Christof, Michael Hintermüller, Pavel Krejčí, Karl Kunisch and Gerd Wachsmuth for valuable discussions.

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