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A Real Algebra Perspective on Multivariate Tight
Wavelet Frames

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A real algebra perspective on multivariate tight wavelet frames

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Abstract

Recent results from real algebraic geometry and the theory of polynomial optimization are related in a new framework to the existence question of multivariate tight wavelet frames whose generators have at least one vanishing moment. Namely, several equivalent formulations of the so-called Unitary Extension Principle (UEP) from [33] are interpreted in terms of hermitian sums of squares of certain nongenerative trigonometric polynomials and in terms of semi-definite programming. The latter together with the results in [31, 35] answer affirmatively the long standing open question of the existence of such tight wavelet frames in dimension $d = 2$; we also provide numerically efficient methods for checking their existence and actual construction in any dimension. We exhibit a class of counterexamples in dimension $d = 3$ showing that, in general, the UEP property is not sufficient for the existence of tight wavelet frames. On the other hand we provide stronger sufficient conditions for the existence of tight wavelet frames in dimension $d \geq 3$ and illustrate our results by several examples.

Keywords: multivariate wavelet frame, real algebraic geometry, torus, hermitian square, polynomial optimization, trigonometric polynomial.

Math. Sci. Classification 2000: 65T60, 12D15, 90C26, 90C22.

1 Introduction

Several fundamental results due to two groups of authors (I. Daubechies, B. Han, A. Ron, Z. Shen [16] and respectively C. Chui, W. He, J. Stöckler [8, 9]) lay at the foundation of the theory of tight wavelet frames and also provide their characterizations. These characterizations allow on one hand to establish the connection between frame constructions and the challenging algebraic problem of existence of sums of squares representations (sos) of non-negative

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trigonometric polynomials. On the other hand, the same characterizations provide methods, however unsatisfactory from the practical point of view, for constructing tight wavelet frames.

The existence and effective construction of tight frames, together with good estimates on the number of frame generators, are still open problems. One can easily be discouraged by a general result by Scheiderer in [36], which implies that not all non-negative trigonometric polynomials in the dimension $d \geq 3$ possess sos representations. However, the main focus is on dimension $d = 2$ and on special non-negative trigonometric polynomials. This motivates us to pursue the issue of existence of sos representations further.

It has been observed in [14] that redundancy of wavelet frames has advantages for applications in signal denoising - if the data is redundant, then losing some data during transmission does not necessarily affect the reconstruction of the original signal. Shen et al. [18] use the tight wavelet frame decomposition to recover a clear image from a single motion-blurred image. In [1] the authors show how to use multiresolution wavelet filters p and q_j to construct irreducible representations for the Cuntz algebra and, conversely, how to recover wavelet filters from these representations. Wavelet and frame decompositions for subdivision surfaces are one of the basic tools, e.g., for progressive compression of $3 - d$ meshes or interactive surface viewing [5, 28, 32]. Adaptive numerical methods based on wavelet frame discretizations have produced very promising results [11, 12] when applied to a large class of operator equations, in particular, PDE's and integral equations. We list some existing constructions of compactly supported MRA wavelet tight frames of $L_2(\mathbb{R}^d)$ [7, 10, 16, 23, 29, 33, 38] that employ the Unitary Extension Principle. For any dimension and in the case of a general expansive dilation matrix, the existence of tight wavelet frames is always ensured by [3, 4], if the coefficients of the associated refinement equation are real and nonnegative. A few other compactly supported multi-wavelet tight frames are circulating nowadays, see [3, 5, 21].

The main goal of this paper is to relate the existence of multivariate tight wavelet frames to recent advances in real algebraic geometry and the theory of moment problems. The starting point of our study is the so-called Unitary Extension Principle (UEP) from [33], a special case of the above mentioned characterizations in [8, 9, 16]. In section 3 we first list several equivalent well-known formulations of the UEP from wavelet and frame literature, but use the novel algebraic terminology to state them. It has been already observed in [29] that a sufficient condition for the existence of tight wavelet frames satisfying UEP can be expressed in terms of sums of squares representations of a certain nonnegative trigonometric polynomial. In [29, Theorem 3.4], the authors also provide an algorithm for the actual construction of the corresponding frame generators. In subsection 3.2, we extend the result of [29, Theorem 3.4] and obtain another equivalent formulation of UEP, which combined with the results from [35] guarantees the existence of UEP tight wavelet frames in the two-dimensional case, see subsection 4.1. We also exhibit there a class of three-dimensional counterexamples showing that, in general, the UEP conditions are not sufficient for the existence of tight wavelet frames. In those examples, however, we make a rather strong assumption on the underlying refinable function, which leaves hope that in certain other cases we will be able to show the existence of such tight wavelet frames. The novel, purely algebraic equivalent formulation of the UEP in Theorem 3.10 is aimed at better understanding the structure of tight wavelet frames. The constructive method in [29, Theorem 3.4] yields frame generators of support twice as large as the one of the underlying refinable function. Theorem 3.10 leads to a numerically efficient method for frame constructions with no such restriction on the size of their support. Namely,

in subsection 3.3, we show how to reformulate Theorem 3.10 equivalently as a problem of semi-definite programming. This establishes a connection between constructions of tight wavelet frames and moment problems, see [24, 30, 31] for details.

In section 4.2, we give sufficient conditions for the existence of tight wavelet frames in dimension $d \geq 3$ and illustrate our results by several examples of three-dimensional subdivision. In section 4.3, we discuss an elegant method that sometimes simplifies the frame construction and allows to determine the frame generators analytically. We illustrate this method on the example of the so-called butterfly scheme from [19].

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2 Background and Notation

2.1 Real algebraic geometry

Let $d \in \mathbb{N}$, let T denote the d -dimensional anisotropic real (algebraic) torus, and let $\mathbb{R}[T]$ denote the (real) affine coordinate ring of T

$$\mathbb{R}[T] = \mathbb{R}[x_j, y_j : j = 1, \dots, d] / (x_j^2 + y_j^2 - 1 : j = 1, \dots, d).$$

In other words, T is the subset of \mathbb{R}^{2d} defined by the equations $x_j^2 + y_j^2 = 1, 1 \leq j \leq d$, and endowed with additional algebraic structure which will become apparent in the following pages. Rather than working with the above description, we will mostly employ the complexification of T , together with its affine coordinate ring $\mathbb{C}[T] = \mathbb{R}[T] \otimes_{\mathbb{R}} \mathbb{C}$. This coordinate ring comes with a natural involution $*$ on $\mathbb{C}[T]$, induced by complex conjugation. Namely,

$$\mathbb{C}[T] = \mathbb{C}[z_1^{\pm 1}, \dots, z_d^{\pm 1}]$$

is the ring of complex Laurent polynomials, and $*$ sends z_j to z_j^{-1} and is complex conjugation on coefficients. The real coordinate ring $\mathbb{R}[T]$ consists of the $*$ -invariant polynomials in $\mathbb{C}[T]$, i.e. $p = \sum_{\alpha \in \mathbb{Z}^d} p(\alpha) z^\alpha \in \mathbb{R}[T]$ if and only if $p(-\alpha) = \overline{p(\alpha)}$.

The group of \mathbb{C} -points of T is $T(\mathbb{C}) = (\mathbb{C}^*)^d = \mathbb{C}^* \times \dots \times \mathbb{C}^*$. In this paper we often denote the group of \mathbb{R} -points of T by \mathbb{T}^d . Therefore,

$$\mathbb{T}^d = T(\mathbb{R}) = \{(z_1, \dots, z_d) \in (\mathbb{C}^*)^d : |z_1| = \dots = |z_d| = 1\}$$

is the direct product of d copies of the circle group S^1 . The neutral element of this group we denote by $\mathbf{1} = (1, \dots, 1)$.

Via the exponential map \exp , the coordinate ring $\mathbb{C}[T] = \mathbb{C}[z_1^{\pm 1}, \dots, z_d^{\pm 1}]$ of T is identified with the algebra of (complex) trigonometric polynomials. Namely, \exp identifies (z_1, \dots, z_d) with $e^{-i\omega} := (e^{-i\omega_1}, \dots, e^{-i\omega_d})$. In the same way, the real coordinate ring $\mathbb{R}[T]$ is identified with the ring of real trigonometric polynomials, i.e. polynomials with real coefficients in $\cos(\omega_j)$ and $\sin(\omega_j)$, $j = 1, \dots, d$.

Let $M \in \mathbb{Z}^{d \times d}$ be a matrix with $\det(M) \neq 0$, and write $m := |\det M|$. The finite abelian group

$$G := 2\pi M^{-T} \mathbb{Z}^d / 2\pi \mathbb{Z}^d \quad (1)$$

is (via \exp) a subgroup of $\mathbb{T}^d = T(\mathbb{R})$. It is isomorphic to $\mathbb{Z}^d / M^T \mathbb{Z}^d$ and has order $|G| = m$. Its character group is $G' = \mathbb{Z}^d / M \mathbb{Z}^d$, via the natural pairing

$$G \times G' \rightarrow \mathbb{C}^*, \quad \langle \sigma, \chi \rangle = e^{i\sigma \cdot \chi}, \quad \sigma \in G, \quad \chi \in G'.$$

Here $\sigma \cdot \chi$ is the ordinary inner product on \mathbb{R}^d , and $\langle \sigma, \chi \rangle$ is a root of unity of order dividing m . Note that the group G acts on the coordinate ring $\mathbb{C}[T]$ via multiplication on the torus

$$p \mapsto p^\sigma(e^{-i\omega}) := p(e^{-i(\omega + \sigma)}), \quad \sigma \in G, \quad \omega \in \mathbb{R}^d. \quad (2)$$

The group action commutes with the involution $*$, that is, $(p^*)^\sigma = (p^\sigma)^*$ holds for $p \in \mathbb{C}[T]$ and $\sigma \in G$.

From the action of the group G we get an associated direct sum decomposition of $\mathbb{C}[T]$ into the eigenspaces of this action

$$\mathbb{C}[T] = \bigoplus_{\chi \in G'} \mathbb{C}[T]_\chi,$$

where $\mathbb{C}[T]_\chi$ consists of all $p \in \mathbb{C}[T]$ satisfying $p^\sigma = \langle \sigma, \chi \rangle p$ for all $\sigma \in G$. For $\chi \in G'$ and $p \in \mathbb{C}[T]$, we denote by p_χ the weight χ isotypical component of p . Thus,

$$p_\chi = \frac{1}{m} \sum_{\sigma \in G} \langle \sigma, \chi \rangle p^\sigma$$

lies in $\mathbb{C}[T]_\chi$, and we have $p = \sum_{\chi \in G'} p_\chi$. For every $\chi \in G'$, we choose a lift $\alpha_\chi \in \mathbb{Z}^d$ such that

$$\tilde{p}_\chi := z^{-\alpha_\chi} p_\chi \quad (3)$$

is G -invariant. The components \tilde{p}_χ are called *polyphase components* of p , see [39].

2.2 Wavelet tight frames

A wavelet tight frame is a structured system of functions that has some special group structure and is defined by the actions of translates and dilates on a finite set of functions $\psi_j \in L_2(\mathbb{R}^d)$, $1 \leq j \leq N$. More precisely, let $M \in \mathbb{Z}^{d \times d}$ be a general expansive matrix, i.e. $\rho(M^{-1}) < 1$, or equivalently, all eigenvalues of M are strictly larger than 1 in modulus, and let $m = |\det M|$. We define translation operators T_α on $L_2(\mathbb{R}^d)$ by $T_\alpha f = f(\cdot - \alpha)$, $\alpha \in \mathbb{Z}^d$, and dilation (homothety) U_M by $U_M f = m^{1/2} f(M \cdot)$. Note that these operators are isometries on $L_2(\mathbb{R}^d)$.

Definition 2.1. Let $\{\psi_j : 1 \leq j \leq N\} \subseteq L_2(\mathbb{R}^d)$. The family

$$\Psi = \{U_M^\ell T_\alpha \psi_j : 1 \leq j \leq N, \ell \in \mathbb{Z}, \alpha \in \mathbb{Z}^d\}$$

is a wavelet tight frame of $L_2(\mathbb{R}^d)$, if

$$\|f\|_{L_2}^2 = \sum_{\substack{1 \leq j \leq N, \ell \in \mathbb{Z}, \\ \alpha \in \mathbb{Z}^d}} |\langle f, U_M^\ell T_\alpha \psi_j \rangle|^2 \quad \text{for all } f \in L_2(\mathbb{R}^d). \quad (4)$$

The foundation for the construction of multiresolution wavelet basis or wavelet tight frame is a compactly supported function $\phi \in L_2(\mathbb{R}^d)$ with the following properties.

- (i) ϕ is refinable, i.e. there exists a finitely supported sequence $p = (p(\alpha))_{\alpha \in \mathbb{Z}^d}$, $p(\alpha) \in \mathbb{C}$, such that ϕ satisfies

$$\phi(x) = m^{1/2} \sum_{\alpha \in \mathbb{Z}^d} p(\alpha) U_M T_\alpha \phi(x), \quad x \in \mathbb{R}^d. \quad (5)$$

Taking the Fourier-Transform

$$\widehat{\phi}(\omega) = \int_{\mathbb{R}^d} \phi(x) e^{-i\omega \cdot x} dx$$

of both sides of (5) leads to its equivalent form

$$\widehat{\phi}(M^T \omega) = p(e^{-i\omega}) \widehat{\phi}(\omega), \quad \omega \in \mathbb{R}^d, \quad (6)$$

where the trigonometric polynomial $p \in \mathbb{C}[T]$ is given by

$$p(e^{-i\omega}) = \sum_{\alpha \in \mathbb{Z}^d} p(\alpha) e^{-i\alpha \omega}, \quad \omega \in \mathbb{R}^d.$$

The isotypical components p_χ of p are given by

$$p_\chi(e^{-i\omega}) = \sum_{\alpha \equiv \chi \pmod{M\mathbb{Z}^d}} p(\alpha) e^{-i\alpha \omega}, \quad \chi \in G'. \quad (7)$$

- (ii) One usually assumes that $\widehat{\phi}(0) = 1$ by proper normalization. This assumption on $\widehat{\phi}$ and (6) allow us to read all properties of ϕ from the polynomial p , since the refinement equation (6) then implies

$$\widehat{\phi}(\omega) = \prod_{\ell=1}^{\infty} p(e^{-i(M^T)^{-\ell} \omega}), \quad \omega \in \mathbb{R}^d.$$

The uniform convergence of this infinite product on compact sets is guaranteed by $p(\mathbf{1}) = 1$.

- (iii) One of the approximation properties of ϕ is the requirement that the translates $T_\alpha \phi$, $\alpha \in \mathbb{Z}^d$, form a partition of unity. Then

$$p_\chi(\mathbf{1}) = m^{-1}, \quad \chi \in G'. \quad (8)$$

The functions ψ_j , $j = 1, \dots, N$, are assumed to be of the form

$$\widehat{\psi}_j(M^T \omega) = q_j(e^{-i\omega}) \widehat{\phi}(\omega), \quad (9)$$

where $q_j \in \mathbb{C}[T]$. These assumptions imply that ψ_j have compact support and, as in (5), are finite linear combinations of $U_M T_\alpha \phi$.

3 Equivalent formulations of UEP

In this section we first recall the method called UEP (unitary extension principle) that allows us to determine the trigonometric polynomials q_j , $1 \leq j \leq N$, such that the family Ψ in Definition 2.1 is a wavelet tight frame of $L_2(\mathbb{R}^d)$, see [16, 33]. We also give several equivalent formulations of UEP to link frame constructions with problems in algebraic geometry and semi-definite programming.

We assume throughout this section that $\phi \in L_2(\mathbb{R}^d)$ is a refinable function with respect to the expansive matrix $M \in \mathbb{Z}^{d \times d}$, with trigonometric polynomial p in (6) and $\hat{\phi}(0) = 1$, and the functions ψ_j are defined as in (9). We also make use of the definitions (1) for G and (2) for p^σ , $\sigma \in G$.

3.1 Formulations of UEP in wavelet frame literature

Most formulations of the UEP are given in terms of identities for trigonometric polynomials, see [16, 33].

Theorem 3.1. (UEP) *Let the trigonometric polynomial $p \in \mathbb{C}[T]$ satisfy $p(\mathbf{1}) = 1$. If the trigonometric polynomials $q_j \in \mathbb{C}[T]$, $1 \leq j \leq N$, satisfy the identities*

$$\delta_{\sigma,\tau} - p^{\sigma*} p^\tau = \sum_{j=1}^N q_j^{\sigma*} q_j^\tau, \quad \sigma, \tau \in G, \quad (10)$$

then the family Ψ is a wavelet tight frame of $L_2(\mathbb{R}^d)$.

We next state another equivalent formulation of the Unitary Extension Principle in Theorem 3.1 in terms of the isotypical components $p_\chi, q_{j,\chi}$ of the polynomials p, q_j . In the wavelet and frame literature, see e.g. [39], this equivalent formulation of UEP is usually given in terms of the polyphase components in (3) of p and q_j . The proof we present here serves as an illustration of the algebraic structure behind wavelet and tight wavelet frame constructions.

Theorem 3.2. *Let the trigonometric polynomial $p \in \mathbb{C}[T]$ satisfy $p(\mathbf{1}) = 1$. The identities (10) are equivalent to*

$$\begin{aligned} p_\chi^* p_\chi + \sum_{j=1}^N q_{j,\chi}^* q_{j,\chi} &= m^{-1}, \quad \chi \in G', \\ p_\chi^* p_\eta + \sum_{j=1}^N q_{j,\chi}^* q_{j,\eta} &= 0 \quad \chi, \eta \in G', \quad \chi \neq \eta. \end{aligned} \quad (11)$$

Proof. Recall that $p = \sum_{\chi \in G'} p_\chi$ and $p_\chi = m^{-1} \sum_{\sigma \in G} \langle \sigma, \chi \rangle p^\sigma$ imply

$$p^* = \sum_{\chi \in G'} p_\chi^* \quad \text{and} \quad p^{\sigma*} = \sum_{\chi \in G'} (p_\chi^*)^\sigma = \sum_{\chi \in G'} \langle \sigma, \chi \rangle p_\chi^*.$$

Thus, with $\eta' = \chi + \eta$ in the next identity, we get

$$p^{\sigma*}p = \sum_{\chi, \eta' \in G'} \langle \sigma, \chi \rangle p_{\chi}^* p_{\eta'} = \sum_{\eta \in G'} \sum_{\chi \in G'} \langle \sigma, \chi \rangle p_{\chi}^* p_{\chi+\eta}.$$

Note that the isotypical components of $p^{\sigma*}p$ are given by

$$(p^{\sigma*}p)_{\eta} = \sum_{\chi \in G'} \langle \sigma, \chi \rangle p_{\chi}^* p_{\chi+\eta}, \quad \eta \in G'. \quad (12)$$

Similarly for q_j . Therefore, we get that the identities (10) for $\tau = 0$ are equivalent to

$$\sum_{\chi \in G'} \langle \sigma, \chi \rangle \left(p_{\chi}^* p_{\chi+\eta} + \sum_{j=1}^N q_{j,\chi}^* q_{j,\chi+\eta} \right) = \delta_{\sigma,0} \delta_{\eta,0}, \quad \eta \in G', \quad \sigma \in G. \quad (13)$$

Note that the identities (10) for $\tau \in G$ are redundant and it suffices to consider only those for $\tau = 0$. For fixed $\eta \in G'$, (13) is a system of m equations indexed by $\sigma \in G$ in m unknowns $p_{\chi}^* p_{\chi+\eta} + \sum_{j=1}^N q_{j,\chi}^* q_{j,\chi+\eta}$, $\chi \in G'$. The corresponding system matrix $A = (\langle \sigma, \chi \rangle)_{\sigma \in G, \chi \in G'}$ is invertible and $A^{-1} = m^{-1}A^*$. Thus, (13) is equivalent to (11). \square

It is easy to see that the identities in Theorem 3.1 and in Theorem 3.2 have equivalent matrix formulations.

Theorem 3.3. *The identities (10) are equivalent to*

$$U^*U = I_m \quad (14)$$

with

$$U^* = [p^{\sigma*} \quad q_1^{\sigma*} \quad \cdots \quad q_N^{\sigma*}]_{\sigma \in G} \in M_{m \times (N+1)}(\mathbb{C}[T]),$$

and are also equivalent to

$$\tilde{U}^* \tilde{U} = m^{-1} I_m, \quad (15)$$

with

$$\tilde{U}^* = [\tilde{p}_{\chi}^* \quad \tilde{q}_{1,\chi}^* \quad \cdots \quad \tilde{q}_{N,\chi}^*]_{\chi \in G'} \in M_{m \times (N+1)}(\mathbb{C}[T]).$$

Remark 3.4. The identities (14) and (15) connect the construction of q_1, \dots, q_N to the following matrix extension problem: extend the first row $(p^{\sigma})_{\sigma \in G}$ of the polynomial matrix U (or $(\tilde{p}_{\chi})_{\chi \in G'}$ of \tilde{U}) to a rectangular $(N+1) \times m$ polynomial matrix satisfying (14) (or (15)). There are two major differences between the identities (14) and (15). While the first column (p, q_1, \dots, q_N) of U determines all other columns of U as well, the columns of the matrix \tilde{U} can be chosen independently, see [39]. All entries of \tilde{U} , however, are forced to be G -invariant trigonometric polynomials.

The following simple consequence of the above results provides a necessary condition for the existence of UEP tight wavelet frames.

Corollary 3.5. *Let the trigonometric polynomial $p \in \mathbb{C}[T]$ satisfy $p(\mathbf{1}) = 1$. For the existence of trigonometric polynomials q_j satisfying (10), it is necessary that the sub-QMF condition*

$$1 - \sum_{\sigma \in G} p^{\sigma*} p^{\sigma} \geq 0 \quad (16)$$

*holds on \mathbb{T}^d . In particular, it is necessary that $1 - p^*p$ is non-negative on \mathbb{T}^d .*

Next, we give an example of a trigonometric polynomial p satisfying $p(\mathbf{1}) = 1$, but for which the corresponding polynomial f is negative for some $\omega \in \mathbb{R}^3$.

Example 3.6. Consider

$$\begin{aligned} p(z_1, z_2, z_3) = & 6z_1z_2z_3 \left(\frac{1+z_1}{2}\right)^2 \left(\frac{1+z_2}{2}\right)^2 \left(\frac{1+z_3}{2}\right)^2 \left(\frac{1+z_1z_2z_3}{2}\right)^2 - \\ & \frac{5}{4}z_1 \left(\frac{1+z_1}{2}\right) \left(\frac{1+z_2}{2}\right)^3 \left(\frac{1+z_3}{2}\right)^3 \left(\frac{1+z_1z_2z_3}{2}\right)^3 - \\ & \frac{5}{4}z_2 \left(\frac{1+z_1}{2}\right)^3 \left(\frac{1+z_2}{2}\right) \left(\frac{1+z_3}{2}\right)^3 \left(\frac{1+z_1z_2z_3}{2}\right)^3 - \\ & \frac{5}{4}z_3 \left(\frac{1+z_1}{2}\right)^3 \left(\frac{1+z_2}{2}\right)^3 \left(\frac{1+z_3}{2}\right) \left(\frac{1+z_1z_2z_3}{2}\right)^3 - \\ & \frac{5}{4}z_1z_2z_3 \left(\frac{1+z_1}{2}\right)^3 \left(\frac{1+z_2}{2}\right)^3 \left(\frac{1+z_3}{2}\right)^3 \left(\frac{1+z_1z_2z_3}{2}\right). \end{aligned}$$

The associated refinable function is continuous as the corresponding subdivision scheme is uniformly convergent, but p does not satisfy the sub-QMF condition, as

$$1 - \sum_{\sigma \in G} |p^{\sigma}(e^{-i\omega})|^2 < 0 \quad \text{for } \omega = \left(\frac{\pi}{6}, 0, 0\right).$$

3.2 Sums of squares

Next we give another equivalent formulation of the UEP in terms of a sums of squares problem for the Laurent polynomial

$$f := 1 - \sum_{\sigma \in G} p^{\sigma*} p^{\sigma}. \quad (17)$$

We say that $f \in C[T]$ is a *sum of hermitian squares*, if there exist $h_1, \dots, h_r \in \mathbb{C}[T]$ such that $f = \sum_{j=1}^r h_j^* h_j$. We start with the following auxiliary lemma.

Lemma 3.7. *Let $p \in \mathbb{C}[T]$ with isotypical components p_{χ} , $\chi \in G'$.*

$$(a) \quad \sum_{\sigma \in G} p^{\sigma*} p^{\sigma} = m \cdot \sum_{\chi \in G'} p_{\chi}^* p_{\chi} \text{ is a } G\text{-invariant Laurent polynomial in } \mathbb{R}[T].$$

(b) If f in (17) is a sum of hermitian squares

$$f = \sum_{j=1}^r h_j^* h_j, \quad (18)$$

with $h_j \in \mathbb{C}[T]$, then

$$f = \sum_{j=1}^r \sum_{\chi \in G'} \tilde{h}_{j,\chi}^* \tilde{h}_{j,\chi}, \quad (19)$$

with the G -invariant polyphase components $\tilde{h}_{j,\chi} \in \mathbb{C}[T]$.

Proof. Similar computations as in the proof of Theorem 3.2 yield the identity in (a). The G -invariance and invariance by involution are obvious. For (b) we observe that the left-hand side of (18) is G -invariant as well. Therefore, (18) implies

$$1 - \sum_{\sigma \in G} p^{\sigma*} p^\sigma = m^{-1} \sum_{j=1}^r \sum_{\sigma \in G} h_j^{\sigma*} h_j^\sigma.$$

Using the result in (a) we get

$$m^{-1} \sum_{j=1}^r \sum_{\sigma \in G} h_j^{\sigma*} h_j^\sigma = \sum_{j=1}^r \sum_{\chi \in G'} h_{j,\chi}^* h_{j,\chi}.$$

The polyphase component $\tilde{h}_{j,\chi} = z^{-\alpha_\chi} h_{j,\chi}$, with $\alpha_\chi \in \mathbb{Z}^d$ and $\alpha_\chi \equiv \chi \pmod{M\mathbb{Z}^d}$, is G -invariant and satisfies $\tilde{h}_{j,\chi}^* \tilde{h}_{j,\chi} = h_{j,\chi}^* h_{j,\chi}$. \square

The results in [29] imply that having a sum of hermitian squares decomposition of

$$f = 1 - \sum_{\sigma \in G} p^{\sigma*} p^\sigma = \sum_{j=1}^r h_j^* h_j \in \mathbb{R}[T],$$

with G -invariant polynomials $h_j \in \mathbb{C}[T]$, is sufficient for the existence of the polynomials q_j in Theorem 3.1. The authors in [29] also provide a method for the construction of q_j from a sum of squares decomposition of the trigonometric polynomial f . Lemma 3.7 shows that one does not need to require G -invariance of h_j in (17). Moreover, it is not mentioned in [29], that the existence of the sos decomposition of f is also a necessary condition, and, therefore, provides another equivalent formulation of the UEP conditions (10). We state the following extension of [29, Theorem 3.4].

Theorem 3.8. *For any $p \in \mathbb{C}[T]$, with $p(\mathbf{1}) = 1$, the following conditions are equivalent.*

- (i) *There exist trigonometric polynomials $h_1, \dots, h_r \in \mathbb{C}[T]$ satisfying (17).*
- (ii) *There exist trigonometric polynomials $q_1, \dots, q_N \in \mathbb{C}[T]$ satisfying (10).*

Proof. Assume that (i) is satisfied. Let χ_k be the elements of $G' \simeq \{\chi_1, \dots, \chi_m\}$. For $1 \leq j \leq r$ and $1 \leq k \leq m$, we define the polyphase components \tilde{h}_{j,χ_k} of h_j and set $\alpha_\chi \in \mathbb{Z}^d$, $\alpha_\chi \equiv \chi \pmod{M\mathbb{Z}^d}$, as in Lemma 3.7. The constructive method in the proof of [29, Theorem 3.4] yields the explicit form of q_1, \dots, q_N , with $N = m(r+1)$, satisfying (10), namely

$$q_k = m^{-1/2} z^{\alpha_{\chi_k}} (1 - mpp_{\chi_k}^*), \quad 1 \leq k \leq m, \quad (20)$$

$$q_{m+j+k} = p\tilde{h}_{j,\chi_k}^*, \quad 1 \leq k \leq m, \quad 1 \leq j \leq r. \quad (21)$$

Conversely, if (ii) is satisfied, we obtain from (14)

$$I_m - \begin{bmatrix} \vdots \\ p^{\sigma^*} \\ \vdots \end{bmatrix}_{\sigma \in G} \cdot [\cdots p^\sigma \cdots]_{\sigma \in G} = \sum_{j=1}^N \begin{bmatrix} \vdots \\ q_j^{\sigma^*} \\ \vdots \end{bmatrix}_{\sigma \in G} \cdot [\cdots q_j^\sigma \cdots]_{\sigma \in G}.$$

The determinant of the matrix on the left-hand side is equal to f in (17), and, by the Cauchy-Binet formula, the determinant of the matrix on the right-hand side is a sum of squares. \square

Remark 3.9. The constructive method in [29] yields $N = m(r+1)$ trigonometric polynomials q_j in (10), where r is the number of trigonometric polynomials h_j in (17). Moreover, the degree of some q_j in (20) and (21) is at least twice as high as the degree of p .

Next, we give an equivalent formulation of the UEP condition in terms of hermitian sums of squares, derived from the identities (11) in Theorem 3.2. Our goal is to improve the constructive method in [29] and to give an algebraic equivalent formulation that directly delivers the trigonometric polynomials q_j in Theorem 3.1, avoiding any extra computations as in (20) and (21). To this end we write $A = \mathbb{C}[T]$ and consider

$$A \otimes_{\mathbb{C}} A = \mathbb{C}[T \times T].$$

So A is the ring of Laurent polynomials in d variables z_1, \dots, z_d . We may identify $A \otimes A$ with the ring of Laurent polynomials in $2d$ variables u_1, \dots, u_d and v_1, \dots, v_d , where $u_j = z_j \otimes 1$ and $v_j = 1 \otimes z_j$, $j = 1, \dots, d$. On A we have already introduced the G' -grading $A = \bigoplus_{\chi \in G'} A_\chi$ and the involution $*$ satisfying $z_j^* = z_j^{-1}$. On $A \otimes A$ we consider the involution $*$ defined by $(p \otimes q)^* = q^* \otimes p^*$ for $p, q \in A$. Thus $u_j^* = v_j^{-1}$ and $v_j^* = u_j^{-1}$. An element $f \in A \otimes A$ will be called *hermitian* if $f = f^*$. We say that f is a sum of hermitian squares if there are finitely many $q_1, \dots, q_r \in A$ with $f = \sum_{k=1}^r q_k^* \otimes q_k$. On $A \otimes A$ we consider the grading

$$A \otimes A = \bigoplus_{\chi, \eta \in G'} A_\chi \otimes A_\eta.$$

So $A_\chi \otimes A_\eta$ is spanned by the monomials $u^\alpha v^\beta$ with $\alpha + M\mathbb{Z}^d = \chi$ and $\beta + M\mathbb{Z}^d = \eta$. Note that $(A_\chi \otimes A_\eta)^* = A_{-\eta} \otimes A_{-\chi}$.

The multiplication homomorphism $\mu: A \otimes A \rightarrow A$ (with $\mu(p \otimes q) = pq$) is compatible with the involutions. Let $I = \ker(\mu)$, the ideal in $A \otimes A$ that is generated by $u_j - v_j$ with $j = 1, \dots, d$. We also need to consider the smaller ideal

$$J := \bigoplus_{\chi, \eta \in G'} \left(I \cap (A_\chi \otimes A_\eta) \right)$$

of $A \otimes A$. The ideal J is $*$ -invariant. Note that the inclusion $J \subseteq I$ is proper since, for example, $u_j - v_j \notin J$.

Theorem 3.10. *Let $p \in A = \mathbb{C}[T]$ satisfy $p(\mathbf{1}) = 1$. The following conditions are equivalent.*

(i) *The Laurent polynomial*

$$f = 1 - \sum_{\sigma \in G} p^{\sigma*} p^{\sigma}$$

is a sum of hermitian squares in A ; that is, there exist $h_1, \dots, h_r \in A$ with $f = \sum_{j=1}^r h_j^ h_j$.*

(ii) *For any hermitian elements $h_{\chi} = h_{\chi}^*$ in $A_{-\chi} \otimes A_{\chi}$, with $\mu(h_{\chi}) = \frac{1}{m}$ for all $\chi \in G'$, the element*

$$g := \sum_{\chi \in G'} h_{\chi} - p^* \otimes p$$

is a sum of hermitian squares in $A \otimes A$ modulo J ; that is, there exist $q_1, \dots, q_N \in A$ with $g - \sum_{j=1}^N q_j^ q_j \in J$.*

(iii) *p satisfies the UEP condition (10) for suitable $q_1, \dots, q_N \in A$.*

Proof. By Theorem 3.8, (i) is equivalent to (iii). In (ii), let hermitian elements $h_{\chi} \in A_{-\chi, \chi}$ be given with $\mu(h_{\chi}) = \frac{1}{m}$. Then (ii) is equivalent to the existence of $q_1, \dots, q_N \in A$ with

$$\sum_{\chi \in G'} h_{\chi} - p^* \otimes p - \sum_{j=1}^N q_j^* \otimes q_j \in J. \quad (22)$$

We write $p = \sum_{\chi \in G'} p_{\chi}$ and q_j as the sum of its isotypical components and observe that (22) is equivalent to

$$\delta_{\chi, \eta} h_{\chi} - p_{\chi}^* \otimes p_{\eta} - \sum_{j=1}^N q_{j, \chi}^* \otimes q_{j, \eta} \in I \quad \text{for all } \chi, \eta \in G'. \quad (23)$$

Due to $\mu(h_{\chi}) = \frac{1}{m}$, the relation (23) is an equivalent reformulation of equations (11) in Theorem 3.2, and therefore equivalent to equations (10). \square

Remark 3.11. (i) The proof of Theorem 3.10 does not depend on the choice of the hermitian elements $h_{\chi} \in A_{-\chi} \otimes A_{\chi}$ in (ii). Thus, it suffices to choose particular hermitian elements satisfying $\mu(h_{\chi}) = \frac{1}{m}$. For example, if $p_{\chi}(\mathbf{1}) = m^{-1}$ is satisfied for all $\chi \in G'$, we can choose

$$h_{\chi} = \sum_{\alpha \equiv \chi} \operatorname{Re}(p(\alpha)) u^{-\alpha} v^{\alpha}, \quad (24)$$

where $p(\alpha)$ are the coefficients of the Laurent polynomial p .

(ii) The same Laurent polynomials q_1, \dots, q_N can be chosen in Theorem 3.10 (ii) and (iii). This is the main advantage of working with the condition (ii) rather than with (i).

3.3 Semi-definite programming

We next devise a constructive method for determining the Laurent polynomials q_j in (10). This method is based on (ii) of Theorem 3.10 and (i) of Remark 3.11.

For a Laurent polynomial $p = \sum_{\alpha} p(\alpha)z^{\alpha}$, let $\mathcal{N} \subseteq \mathbb{Z}^d$ contain $\{\alpha \in \mathbb{Z}^d : p(\alpha) \neq 0\}$. We also define the tautological (column) vector

$$\mathbf{x} = [z^{\alpha} : \alpha \in \mathcal{N}]^T,$$

and the orthogonal projections $E_{\chi} \in \mathbb{R}^{|\mathcal{N}| \times |\mathcal{N}|}$ to be diagonal matrices with diagonal entries given by

$$E_{\chi}(\alpha, \alpha) = \begin{cases} 1, & \alpha \equiv \chi \pmod{M\mathbb{Z}^d}, \\ 0, & \text{otherwise,} \end{cases} \quad \alpha \in \mathcal{N}.$$

Theorem 3.12. *Let*

$$p = \mathbf{p} \cdot \mathbf{x} \in A = \mathbb{C}[T], \quad \mathbf{p} = [p(\alpha) : \alpha \in \mathcal{N}] \in \mathbb{C}^{|\mathcal{N}|}, \quad (25)$$

satisfy $p_{\chi}(\mathbf{1}) = m^{-1}$ for all $\chi \in G'$. The following conditions are equivalent.

(i) *There exist row vectors $\mathbf{q}_j = [q_j(\alpha) : \alpha \in \mathcal{N}] \in \mathbb{C}^{|\mathcal{N}|}$, $1 \leq j \leq N$, satisfying the identities*

$$\mathbf{x}^* E_{\chi} \left(\text{diag}(\text{Re } \mathbf{p}) - \mathbf{p}^* \mathbf{p} - \sum_{j=1}^N \mathbf{q}_j^* \mathbf{q}_j \right) E_{\eta} \mathbf{x} = 0 \quad \text{for all } \chi, \eta \in G'. \quad (26)$$

(ii) *p satisfies the UEP condition (10) with*

$$q_j = \mathbf{q}_j \cdot \mathbf{x} \in \mathbb{C}[T], \quad j = 1, \dots, N,$$

and suitable row vectors $\mathbf{q}_j \in \mathbb{C}^{|\mathcal{N}|}$.

Proof. Define

$$\mathbf{v} = [1 \otimes z^{\alpha} : \alpha \in \mathcal{N}]^T \in (A \otimes A)^{|\mathcal{N}|}.$$

Note that $\mathbf{p}\mathbf{v} = 1 \otimes p$ and the definition of E_{χ} gives $\mathbf{p}E_{\chi}\mathbf{v} = 1 \otimes p_{\chi}$. Therefore, we have

$$\mathbf{v}^* E_{\chi} \mathbf{p}^* \mathbf{p} E_{\eta} \mathbf{v} = p_{\chi}^* \otimes p_{\eta} \quad \text{for all } \chi, \eta \in G',$$

and the analogue for $q_{j,\chi}^* \otimes q_{j,\eta}$. Moreover, we have

$$\mathbf{v}^* E_{\chi} \text{diag}(\text{Re } \mathbf{p}) E_{\eta} \mathbf{v} = \delta_{\chi,\eta} \sum_{\alpha \equiv \chi} \text{Re}(p(\alpha)) u^{-\alpha} v^{\alpha}.$$

Due to $p_{\chi}(\mathbf{1}) = m^{-1}$ and by Remark 3.11 we choose $h_{\chi} = \mathbf{v}^* E_{\chi} \text{diag}(\text{Re } \mathbf{p}) E_{\chi} \mathbf{v}$ as the hermitian elements in Theorem 3.10(ii), and the relation (23) is equivalent to

$$\mathbf{v}^* E_{\chi} \left(\text{diag}(\text{Re } \mathbf{p}) - \mathbf{p}^* \mathbf{p} - \sum_{j=1}^N \mathbf{q}_j^* \mathbf{q}_j \right) E_{\eta} \mathbf{v} \in I \quad \text{for all } \chi, \eta \in G'.$$

Due to $\mu(\mathbf{v}) = \mathbf{x}$, the claim follows from the equivalence of (ii) and (iii) in Theorem 3.10. \square

We suggest the following constructive method based on Theorem 3.12. Given the trigonometric polynomial p and the vector \mathbf{p} in (25), define the matrix

$$R = \text{diag}(\text{Re } \mathbf{p}) - \mathbf{p}^* \mathbf{p} \in \mathbb{C}^{|\mathcal{M}| \times |\mathcal{M}|}. \quad (27)$$

Then the task of constructing tight wavelet frames can be formulated as the following problem of **semi-definite programming**: find a matrix $O \in \mathbb{C}^{|\mathcal{M}| \times |\mathcal{M}|}$ such that

$$S := R + O \quad \text{is positive semi-definite} \quad (28)$$

subject to the constraints

$$\mathbf{x}^* E_\chi O E_\eta \mathbf{x} = 0 \quad \text{for all } \chi, \eta \in G'. \quad (29)$$

If such a matrix O exists, we determine the trigonometric polynomials $q_j = \mathbf{q}_j \mathbf{x} \in \mathbb{C}[T]$ by choosing any decomposition of the form

$$S = \sum_{j=1}^N \mathbf{q}_j^* \mathbf{q}_j$$

with standard methods from linear algebra.

If the semi-definite programming problem does not have a solution, we can increase the set \mathcal{N} and start all over. Note that the identities (29) are equivalent to the following linear constraints on the null-matrices O

$$\sum_{\alpha \equiv \chi, \beta \equiv \eta} O_{\alpha, \beta} z^{\beta - \alpha} = 0 \quad \text{for all } \chi, \eta \in G',$$

or, equivalently,

$$\sum_{\alpha \equiv \chi} O_{\alpha, \alpha + \tau} = 0 \quad \text{for all } \tau \in \{\beta - \alpha : \alpha, \beta \in \mathcal{N}\}.$$

Example 3.13. To illustrate the concept of null-matrices, we consider first a very prominent one-dimensional example of a Daubechies wavelet. Let

$$p = \mathbf{p} \cdot \mathbf{x}, \quad \mathbf{p} = \frac{1}{8} \begin{bmatrix} 1 + \sqrt{3} & 3 + \sqrt{3} & 3 - \sqrt{3} & 1 - \sqrt{3} \end{bmatrix},$$

and $\mathbf{x} = [1, z, z^2, z^3]^T$. In this case $M = m = 2$, $G \simeq \{0, \pi\}$, $G' \simeq \{0, 1\}$ and the orthogonal projections $E_\chi \in \mathbb{R}^{4 \times 4}$, $\chi \in G'$, are given by

$$E_0 = \text{diag}[1, 0, 1, 0] \quad \text{and} \quad E_1 = \text{diag}[0, 1, 0, 1].$$

By (27), we have

$$R = \frac{1}{64} \begin{bmatrix} 4 + 6\sqrt{3} & -6 - 4\sqrt{3} & -2\sqrt{3} & 2 \\ -6 - 4\sqrt{3} & 12 + 2\sqrt{3} & -6 & 2\sqrt{3} \\ -2\sqrt{3} & -6 & 12 - 2\sqrt{3} & -6 + 4\sqrt{3} \\ 2 & 2\sqrt{3} & -6 + 4\sqrt{3} & 4 - 6\sqrt{3} \end{bmatrix},$$

which is not positive semi-definite. Define

$$O = \frac{1}{64} \begin{bmatrix} -8\sqrt{3} & 8\sqrt{3} & 0 & 0 \\ 8\sqrt{3} & -8\sqrt{3} & 0 & 0 \\ 0 & 0 & 8\sqrt{3} & -8\sqrt{3} \\ 0 & 0 & -8\sqrt{3} & 8\sqrt{3} \end{bmatrix}$$

satisfying (29). Then $S = R+O$ is positive semi-definite, of rank one, and yields the well-known Daubechies wavelet, see [15] defined by

$$q_1 = \frac{1}{8} [1 - \sqrt{3} \quad -3 + \sqrt{3} \quad 3 + \sqrt{3} \quad -1 - \sqrt{3}] \cdot \mathbf{x}.$$

Another two-dimensional example of one possible choice of an appropriate null-matrix satisfying (29) is given in Example 4.13.

Remark 3.14. Another, very similar, way of working with null-matrices was pursued already in [6].

4 Existence and constructions of tight wavelet frames

In this section we use results from algebraic geometry and Theorem 3.8 to resolve the problem of existence of tight wavelet frames. Theorem 3.8 allows us to reduce the problem of existence of q_j in (10) to the problem of existence of an sos decomposition of a single nonnegative polynomial

$$f = 1 - \sum_{\sigma \in G} p^{\sigma*} p^{\sigma} \in \mathbb{R}[T].$$

In subsection 4.1, for dimension $d = 2$, we show that the polynomials $q_1, \dots, q_N \in \mathbb{C}[T]$ as in Theorem 3.1 always exist. This result is based on recent progress in real algebraic geometry. We also include an example of a three-dimensional trigonometric polynomial p , satisfying the sub-QMF condition (16), but for which trigonometric polynomials q_1, \dots, q_N as in Theorem 3.1 do not exist. In subsection 4.2, we give sufficient conditions for the existence of the q_j 's in the multidimensional case and give several explicit constructions of tight wavelet frames in section 4.3.

4.1 Existence of tight wavelet frames

In this section we show that in the two-dimensional case ($d = 2$) the question of existence of a wavelet tight frame can be positively answered using the results from [34]. Thus, Theorem 4.1 answers a long standing open question about the existence of tight wavelet frames as in Theorem 3.1. The result of Theorem 4.2 states that in the dimension $d \geq 3$ for a given trigonometric polynomial p satisfying $p(\mathbf{1}) = 1$ and the sub-QMF condition (16) one cannot always determine trigonometric polynomials q_j as in Theorem 3.1.

Theorem 4.1. *Let $d = 2$, $p \in \mathbb{C}[T]$ satisfy $p(\mathbf{1}) = 1$ and $\sum_{\sigma \in G} p^{\sigma*} p^\sigma \leq 1$ on $\mathbb{T}^2 = T(\mathbb{R})$. Then there exist $N \in \mathbb{N}$ and trigonometric polynomials $q_1, \dots, q_N \in \mathbb{C}[T]$ satisfying*

$$\delta_{\sigma, \tau} = p^{\sigma*} p^\tau + \sum_{j=1}^N q_j^{\sigma*} q_j^\tau, \quad \sigma, \tau \in G. \quad (30)$$

Proof. The torus T is a non-singular affine algebraic surface over \mathbb{R} , and $T(\mathbb{R})$ is compact. The polynomial f in (17) is in $\mathbb{R}[T]$ and is nonnegative on $T(\mathbb{R})$ by assumption. By Corollary 3.4 of [34], there exist $h_1, \dots, h_r \in \mathbb{C}[T]$ satisfying $f = \sum_{j=1}^r h_j^* h_j$. According to Lemma 3.7 part (b), the polynomials h_j can be taken to be G -invariant. Thus, by Theorem 3.8, there exist polynomials q_1, \dots, q_N satisfying (30). \square

The question may arise, if there exists a trigonometric polynomial p that satisfies $p(\mathbf{1}) = 1$ and the sub-QMF condition $\sum_{\sigma \in G} p^{\sigma*} p^\sigma \leq 1$ on \mathbb{T}^d , but for which there exists no UEP tight frame as in Theorem 3.1. Or, due to Corollary 3.5, if we can find such a p , for which the nonnegative trigonometric polynomial $1 - p^* p$ is not a sum of hermitian squares of trigonometric polynomials?

Theorem 4.2. *There exists $p \in \mathbb{C}[T]$ satisfying $p(\mathbf{1}) = 1$ and the sub-QMF condition on \mathbb{T}^3 , such that $1 - p^* p$ is not a sum of hermitian squares in $\mathbb{R}[T]$.*

The proof is constructive. The following example defines a family of trigonometric polynomials with the properties stated in Theorem 4.2. We make use of the following local-global result from algebraic geometry: if the Taylor expansion of $f \in \mathbb{R}[T]$ at one of its roots has, in local coordinates, a homogeneous part of lowest degree which is not sos of real algebraic polynomials, then f is not sos in $\mathbb{R}[T]$.

Example 4.3. Denote $z_j = e^{-i\omega_j}$, $j = 1, 2, 3$. We let

$$p(z) = \left(1 - c \cdot m(z)\right) a(z), \quad z \in T, \quad 0 < c \leq \frac{1}{3},$$

where

$$m(z) = y_1^4 y_2^2 + y_1^2 y_2^4 + y_3^6 - 3y_1^2 y_2^2 y_3^2 \in \mathbb{R}[T], \quad y_j = \sin \omega_j.$$

In the local coordinates (y_1, y_2, y_3) at $z = \mathbf{1}$, m is the well-known Motzkin polynomial in $\mathbb{R}[y_1, y_2, y_3]$; i.e. m is not sos in $\mathbb{R}[y_1, y_2, y_3]$. Moreover, $a \in \mathbb{R}[T]$ is chosen such that

$$D^\alpha a(\mathbf{1}) = \delta_{0, \alpha}, \quad D^\alpha a(\sigma) = 0, \quad 0 \leq |\alpha| < 8, \quad \sigma \in G \setminus \{\mathbf{1}\}, \quad (31)$$

and $\sum_{\sigma \in G} a^{\sigma*} a^\sigma \leq 1$. Such a can be, for example, any scaling symbol of a 3-D orthonormal wavelet with 8 vanishing moments; in particular, the tensor product Daubechies symbol $a(z) = m_8(z_1) m_8(z_2) m_8(z_3)$ with m_8 in [15] satisfies conditions (31) and $\sum_{\sigma \in G} a^{\sigma*} a^\sigma = 1$. The properties of m and a imply that

1. p satisfies the sub-QMF condition on \mathbb{T}^3 , since m is G -invariant and $0 \leq 1 - c \cdot m \leq 1$ on \mathbb{T}^3 ,
2. p satisfies sum rules of order at least 6,
3. the Taylor expansion of $1 - p^*p$ at $z = \mathbf{1}$, in local coordinates (y_1, y_2, y_3) , has $2 \cdot c \cdot m$ as its homogeneous part of lowest degree.

Therefore, $1 - p^*p$ is not sos of trigonometric polynomials in $\mathbb{R}[T]$. By Corollary 3.5, the corresponding nonnegative trigonometric polynomial f in (17) has no sos decomposition.

4.2 Sufficient conditions for existence of tight wavelet frames

In the general multivariate case $d \geq 2$, in Theorem 4.5, we provide a sufficient condition for the existence of a sums of squares decomposition of f in (17). This condition is based on the properties of the Hessian of $f \in \mathbb{R}[T]$

$$\text{Hess}(f) = (D^\mu f)_{\mu \in \mathbb{N}_0^d, |\mu|=2},$$

where f is a trigonometric polynomial in $\omega \in \mathbb{R}^d$ and D^μ denotes the $|\mu|$ -th partial derivative with respect to $\omega \in \mathbb{R}^d$.

Theorem 4.4. *Let V be a non-singular affine \mathbb{R} -variety for which $V(\mathbb{R})$ is compact, and let $f \in \mathbb{R}[V]$ with $f \geq 0$ on $V(\mathbb{R})$. For every $\xi \in V(\mathbb{R})$ with $f(\xi) = 0$, assume that the Hessian of f at ξ is strictly positive definite. Then f is a sum of squares in $\mathbb{R}[V]$.*

Proof. The hypotheses imply that f has only finitely many zeros in $V(\mathbb{R})$. Therefore the claim follows from [35, Corollary 2.17, Example 3.18]. \square

Theorem 4.4 implies the following result.

Theorem 4.5. *Let $p \in \mathbb{C}[T]$ satisfy $p(\mathbf{1}) = 1$ and $f = 1 - \sum_{\sigma \in G} p^{\sigma*} p^\sigma \geq 0$ on $T(\mathbb{R}) = \mathbb{T}^d$. If the Hessian of f is positive definite at every zero of f in \mathbb{T}^d , then there exist $N \in \mathbb{N}$ and polynomials $q_1, \dots, q_N \in \mathbb{C}[T]$ satisfying (10).*

Proof. By Theorem 4.4, f is a sum of squares in $\mathbb{R}[T]$. The claim follows then by Theorem 4.1. \square

Due to $p(\mathbf{1}) = 1$, $z = \mathbf{1}$ is obviously a zero of f . We show next how to express the Hessian of f at $\mathbf{1}$ in terms of the gradient $\nabla p(\mathbf{1})$ and the Hessian of p at $\mathbf{1}$, if p additionally satisfies the so-called sum rules of order 2, or, equivalently, satisfies the zero conditions of order 2. We say that $p \in \mathbb{C}[T]$ satisfies zero conditions of order k , if

$$D^\mu p(e^{-i\sigma}) = 0, \quad \mu \in \mathbb{N}_0^d, \quad |\mu| < k, \quad \sigma \in G \setminus \{0\},$$

see [25, 26] for details. The assumption that p satisfies sum rules of order 2 together with $p(\mathbf{1}) = 1$ are necessary for the continuity of the corresponding refinable function ϕ .

Lemma 4.6. *Let $p \in \mathbb{C}[T]$ with real coefficients satisfy the sum rules of order 2 and $p(\mathbf{1}) = 1$. Then the Hessian of $f = 1 - \sum_{\sigma \in G} p^{\sigma*} p^\sigma$ at $\mathbf{1}$ is equal to*

$$-2 \text{Hess}(p)(\mathbf{1}) - 2 \nabla p(\mathbf{1})^* \nabla p(\mathbf{1}).$$

Proof. We expand the trigonometric polynomial p in a neighborhood of $\mathbf{1}$ and get

$$p(e^{-i\omega}) = 1 + \nabla p(\mathbf{1})\omega + \frac{1}{2}\omega^T \text{Hess}(p)(\mathbf{1})\omega + \mathcal{O}(|\omega|^3).$$

Note that, since the coefficients of p are real, the row vector $v = \nabla p(\mathbf{1})$ is purely imaginary and $\text{Hess}(p)(\mathbf{1})$ is real and symmetric. The sum rules of order 2 are equivalent to

$$p^\sigma(\mathbf{1}) = 0, \quad \nabla p^\sigma(\mathbf{1}) = 0 \quad \text{for all } \sigma \in G \setminus \{0\}.$$

Thus, we have $p^\sigma(e^{-i\omega}) = \mathcal{O}(|\omega|^2)$ for all $\sigma \in G \setminus \{0\}$. Simple computation yields

$$\begin{aligned} |p(e^{-i\omega})|^2 &= 1 + (v + \bar{v})\omega + \omega^T (\text{Hess}(p)(\mathbf{1}) + v^* v)\omega + \mathcal{O}(|\omega|^3) \\ &= 1 + \omega^T (\text{Hess}(p)(\mathbf{1}) + v^* v)\omega + \mathcal{O}(|\omega|^3). \end{aligned}$$

Thus, the claim follows. □

Remark 4.7. Note that $\text{Hess}(f)$ is a zero matrix, if p is a symbol of interpolatory subdivision scheme, i.e.,

$$p = m^{-1} + m^{-1} \sum_{\chi \in G' \setminus \{0\}} p_\chi,$$

and p satisfies zero conditions of order at least 3. This property of $\text{Hess}(f)$ follows directly from the equivalent formulation of zero conditions of order k , see [2]. The examples of p with such properties are for example the butterfly scheme in Example 4.13 and the three-dimensional interpolatory scheme in Example 4.11.

Remark 4.8. The sufficient condition of Theorem 4.4 can be generalized to cases when the order of vanishing of f is larger than two. Namely, let V and f as in 4.4, and let $\xi \in V(\mathbb{R})$ be a zero of f . Fix a system x_1, \dots, x_n of local (analytic) coordinates on V centered at ξ . Let $2d > 0$ be the order of vanishing of f at ξ , and let $F_\xi(x_1, \dots, x_n)$ be the homogeneous part of degree $2d$ in the Taylor expansion of f at ξ . Let us say that f is *strongly sos at ξ* if there exists a linear basis g_1, \dots, g_N of the space of homogeneous polynomials of degree d in x_1, \dots, x_n such that $F_\xi = g_1^2 + \dots + g_N^2$. (Equivalently, if F_ξ lies in the interior of the sums of squares cone in degree $2d$.) If $2d = 2$, this condition is equivalent to the Hessian of f at ξ being positive definite.

Then the following holds: If f is strongly sos at each of its zeros in $V(\mathbb{R})$, f is a sum of squares in $\mathbb{R}[V]$. For a proof we refer to [37]. As a result, we get a corresponding generalization of Theorem 4.5: If f as in 4.5 is strongly sos at each of its zeros in \mathbb{T}^d , then the conclusion of 4.5 holds.

For simplicity of presentation, we start by applying the result of Theorem 4.5 to the 2-dimensional polynomial f derived from the symbol of the three-directional piecewise linear box spline. This example also motivates the statements of Remark 4.10.

Example 4.9. The three-directional piecewise linear box spline is defined by its associated trigonometric polynomial

$$p(e^{-i\omega}) = e^{-i(\omega_1+\omega_2)} \cos\left(\frac{\omega_1}{2}\right) \cos\left(\frac{\omega_2}{2}\right) \cos\left(\frac{\omega_1+\omega_2}{2}\right), \quad \omega \in \mathbb{R}^2.$$

Note that

$$\cos\left(\frac{\omega_1}{2}\right) \cos\left(\frac{\omega_2}{2}\right) \cos\left(\frac{\omega_1+\omega_2}{2}\right) = 1 - \frac{1}{8}\omega^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \omega + \mathcal{O}(|\omega|^4).$$

Therefore, as the trigonometric polynomial p satisfies sum rules of order 2, we get

$$f(e^{-i\omega}) = \frac{1}{8}\omega^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \omega + \mathcal{O}(|\omega|^4).$$

Thus, the Hessian of f at $\mathbf{1}$ is positive definite.

To determine the other zeroes of f , by Lemma 3.7 part (a), we can use either one of the representations

$$\begin{aligned} f(e^{-i\omega}) &= 1 - \sum_{\sigma \in \{0, \pi\}^2} \prod_{\theta \in \{0, 1\}^2 \setminus \{0\}} \cos^2\left(\frac{(\omega + \sigma) \cdot \theta}{2}\right) \\ &= \frac{1}{4} \sum_{\chi \in \{0, 1\}^2} (1 - \cos^2(\omega \cdot \chi)). \end{aligned}$$

It follows that the zeros of f are the points $\omega \in \pi\mathbb{Z}^2$ and, by periodicity of f with period π in both coordinate directions, we get that

$$\text{Hess}(f)(e^{-i\omega}) = \text{Hess}(f)(\mathbf{1}), \quad \omega \in \pi\mathbb{Z}^2,$$

is positive definite at all zeros of f .

Remark 4.10. (i) The result of [5, Theorem 2.4] implies the existence of tight frames for multivariate box-splines. According to the notation in [17, p. 127], the corresponding trigonometric polynomial is given by

$$p(e^{-i\omega}) = \prod_{j=1}^n \frac{1 + e^{-i\omega \cdot \xi^{(j)}}}{2}, \quad \omega \in \mathbb{R}^d,$$

where $\Xi = (\xi^{(1)}, \dots, \xi^{(n)}) \in \mathbb{Z}^{d \times n}$ is unimodular and has rank d . (Unimodularity means that all $d \times d$ -submatrices have determinant 0, 1, or -1 .) Moreover, Ξ has the property that leaving out any column $\xi^{(j)}$ does not reduce its rank. (This property guarantees continuity of the box-spline and that the corresponding polynomial p satisfies at least sum rules of order 2.) Then one can show that

$$f = 1 - \sum_{\sigma \in G} p^{\sigma*} p^{\sigma} \geq 0 \quad \text{on } \mathbb{T}^d,$$

the zeros of f are at $\omega \in \pi\mathbb{Z}^d$ and the Hessian of f at these zeros is positive definite. This yields an alternative proof for [5, Theorem 2.4] in the case of box splines.

(ii) If the summands $m^{-2} - p_{\chi}^* p_{\chi}$ are nonnegative on \mathbb{T}^d , then it can be easier to determine the zeros of f by determining the common zeros of all of these summands.

Example 4.11. There was an attempt to define an interpolatory scheme for 3D-subdivision with dilation matrix $2I_3$ in [13]. There are several inconsistencies in this paper and we give a correct description of the trigonometric polynomial p , the so-called subdivision mask. Note that the scheme we present is an extension of the 2-D butterfly scheme to 3-D data in the following sense: if the data are constant along one of the coordinate directions (or along the main diagonal in \mathbb{R}^3), then the subdivision procedure keeps this property and is identical with the 2-D butterfly scheme.

We describe the trigonometric polynomial p associated with this 3-D scheme by defining its isotypical components. The isotypical components, in terms of $z_k = e^{-i\omega_k}$, $k = 1, 2$, are given by

$$p_{0,0,0}(z_1, z_2, z_3) = 1/8,$$

$$\begin{aligned} p_{1,0,0}(z_1, z_2, z_3) &= \frac{1}{8} \cos \omega_1 + \frac{\lambda}{4} \left(\cos(\omega_1 + 2\omega_2) + \cos(\omega_1 + 2\omega_3) \right. \\ &\quad \left. + \cos(\omega_1 + 2\omega_2 + 2\omega_3) \right) - \frac{\lambda}{4} \left(\cos(\omega_1 - 2\omega_2) + \cos(\omega_1 - 2\omega_3) \right. \\ &\quad \left. + \cos(3\omega_1 + 2\omega_2 + 2\omega_3) \right), \end{aligned}$$

$$\begin{aligned} p_{0,1,0}(z_1, z_2, z_3) &= p_{1,0,0}(z_2, z_1, z_3), \quad p_{0,0,1}(z_1, z_2, z_3) = p_{1,0,0}(z_3, z_1, z_2), \\ p_{1,1,1}(z_1, z_2, z_3) &= p_{1,0,0}(z_1 z_2 z_3, z_1^{-1}, z_2^{-1}), \end{aligned}$$

$$\begin{aligned} p_{1,1,0}(z_1, z_2, z_3) &= \left(\frac{1}{8} - \lambda \right) \cos(\omega_1 + \omega_2) + \lambda \left(\cos(\omega_1 - \omega_2) + \cos(\omega_1 + \omega_2 + 2\omega_3) \right) \\ &\quad - \frac{\lambda}{4} \left(\cos(\omega_1 - \omega_2 + 2\omega_3) + \cos(\omega_1 - \omega_2 - 2\omega_3) + \right. \\ &\quad \left. \cos(3\omega_1 + \omega_2 + 2\omega_3) + \cos(\omega_1 + 3\omega_2 + 2\omega_3) \right), \end{aligned}$$

$$p_{1,0,1}(z_1, z_2, z_3) = p_{1,1,0}(z_1, z_3, z_2), \quad p_{0,1,1}(z_1, z_2, z_3) = p_{1,0,0}(z_2, z_3, z_1),$$

where λ is the so-called tension parameter.

The polynomial p also satisfies

$$p(z_1, z_2, z_3) = \frac{1}{8} (1 + z_1)(1 + z_2)(1 + z_3)(1 + z_1 z_2 z_3) q(z_1, z_2, z_3), \quad q(\mathbf{1}) = 1,$$

which implies sum rules of order 2.

- (a) For $\lambda = 0$, we have $q(z_1, z_2, z_3) = 1/(z_1 z_2 z_3)$. Hence, p is the scaling symbol of the trivariate box spline with the direction set $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$ and whose support center is shifted to the origin.
- (b) For $0 \leq \lambda < 1/16$, the corresponding subdivision scheme converges and has a continuous limit function. The only zeros of the associated nonnegative trigonometric polynomial f are at $\pi\mathbb{Z}^3$, and the Hessian of f at these zeros is given by

$$\text{Hess}(f)(\mathbf{1}) = \text{Hess}(f)(e^{-i\omega}) = \begin{pmatrix} 1 - 16\lambda & \frac{1}{2} - 8\lambda & \frac{1}{2} - 8\lambda \\ \frac{1}{2} - 8\lambda & 1 - 16\lambda & \frac{1}{2} - 8\lambda \\ \frac{1}{2} - 8\lambda & \frac{1}{2} - 8\lambda & 1 - 16\lambda \end{pmatrix}$$

for all $\omega \in \pi\mathbb{Z}^3$. The existence of the sos decomposition of f is guaranteed by Theorem 4.5 and one possible decomposition of f is computed as follows.

- (b₁) Denote $u := \cos(\omega_1 + \omega_2)$, $v := \cos(\omega_1 + \omega_3)$, $w := \cos(\omega_2 + \omega_3)$, and $\tilde{u} := \sin(\omega_1 + \omega_2)$, $\tilde{v} := \sin(\omega_1 + \omega_3)$, $\tilde{w} := \sin(\omega_2 + \omega_3)$. Simple computations yield

$$p_{1,1,0} = \frac{1}{8} - (1 - u)\left(\frac{1}{8} - \lambda v^2 - \lambda w^2\right) - \lambda(v - w)^2,$$

and

$$\begin{aligned} \frac{1}{64} - |p_{1,1,0}|^2 &= \lambda^2(v^2 - w^2)^2 + \left(\left(\frac{1}{16} - \lambda v^2\right) \right. \\ &\quad \left. + \left(\frac{1}{16} - \lambda w^2\right)\right) \left(\frac{1}{8}\tilde{u}^2 + \lambda(v - uw)^2 + \lambda(w - uv)^2\right). \end{aligned}$$

Therefore, $\frac{1}{64} - |p_{1,1,0}|^2$ has an sos decomposition with 7 summands h_j , and each h_j has only one nonzero isotypical component.

- (b₂) The isotypical component $p_{1,0,0}$ is not bounded by 1/8; consider, for example, $p_{1,0,0}(e^{-i\omega})$ at the point $\omega = \left(-\frac{\pi}{6}, -\frac{2\pi}{3}, -\frac{2\pi}{3}\right)$. Yet we obtain, by simple computations,

$$p_{1,0,0} = \frac{1}{8} \cos \omega_1 + \frac{\lambda}{2} A \sin \omega_1, \quad A := \sin 2(\omega_1 + \omega_2 + \omega_3) - \sin 2\omega_2 - \sin 2\omega_3,$$

and

$$\frac{1}{16} - |p_{1,0,0}|^2 - |p_{0,1,0}|^2 - |p_{0,0,1}|^2 - |p_{1,1,1}|^2 = E_{1,0,0} + E_{0,1,0} + E_{0,0,1} + E_{1,1,1},$$

where

$$E_{1,0,0} = \frac{3\lambda}{16} \sin^4 \omega_1 + \frac{\lambda}{64} (2 \sin \omega_1 - A \cos \omega_1)^2 + \frac{1 - 16\lambda}{64} \sin^2 \omega_1 (1 + \lambda A^2);$$

the other $E_{i,j,k}$ are given by the same coordinate transformations as $p_{i,j,k}$. Hence, for $\frac{1}{16} - |p_{1,0,0}|^2 - |p_{0,1,0}|^2 - |p_{0,0,1}|^2 - |p_{1,1,1}|^2$, we obtain an sos decomposition with 12 summands g_j , each of which has only one nonzero isotypical component.

Thus, for the trivariate interpolatory subdivision scheme with tension parameter $0 \leq \lambda < 1/16$, by Theorem 3.8, we have explicitly constructed a tight frame with 41 generators q_j as in Theorem 3.1.

- c) For $\lambda = 1/16$, the sum rules of order 4 are satisfied. In this particular case, the scheme is C^1 and the Hessian of f at $\mathbf{1}$ is the zero-matrix, thus the result of Theorem 4.5 is not applicable. Nevertheless, the sos decomposition of $1 - \sum p^{\sigma*} p^\sigma$ in b), with further simplifications for $\lambda = 1/16$, gives a tight frame with 31 generators for the trivariate interpolatory subdivision scheme.

4.3 Constructions of tight wavelet frames

Lemma 3.7 part (a) sometimes yields an elegant method for determining the sum of squares decomposition of the polynomial f in (17) and, thus, constructing the trigonometric polynomials q_j in Theorem 3.1. Note that

$$f = 1 - \sum_{\sigma \in G} p^{\sigma*} p^\sigma = 1 - m \sum_{\chi \in G'} p_\chi^* p_\chi = m \sum_{\chi \in G'} \left(\frac{1}{m^2} - p_\chi^* p_\chi \right). \quad (32)$$

So it suffices to find an sos decomposition for each of the polynomials $m^{-2} - p_\chi^* p_\chi$, provided that they are all nonnegative. This nonnegativity assumption is satisfied, for example, for the special case when all coefficients $p(\alpha)$ of p are nonnegative. This is due to the simple fact that for nonnegative $p(\alpha)$ we get

$$p_\chi^* p_\chi \leq |p_\chi(\mathbf{1})|^2 = m^{-2}$$

on \mathbb{T}^d , for all $\chi \in G'$.

The last equality in (32) allows us to simplify the construction of frame generators considerably. In Example 4.12 we apply this method to the three-directional piecewise linear box spline. Example 4.13 illustrates the advantage of the representation in (32) for the butterfly scheme [19], an interpolatory subdivision method with the corresponding mask $p \in \mathbb{C}[T]$ of a larger support, some of whose coefficients are negative. Example 4.14 shows that our method is also applicable for at least one of the interpolatory $\sqrt{3}$ -subdivision schemes studied in [27]. For the three-dimensional example that also demonstrates our constructive approach see Example 4.11 part (b1).

Example 4.12. Consider the three-directional piecewise linear box spline with the symbol

$$p(z_1, z_2) = \frac{1}{8} (1 + z_1)(1 + z_2)(1 + z_1 z_2), \quad z_j = e^{-i\omega_j}.$$

The sos decomposition for the isotypical components yields

$$f = 1 - m \sum_{\chi \in G'} p_\chi^* p_\chi = \frac{1}{4} \sin^2(\omega_1) + \frac{1}{4} \sin^2(\omega_2) + \frac{1}{4} \sin^2(\omega_1 + \omega_2).$$

Thus, in (18) we have a decomposition with $r = 3$. Since each of h_1, h_2, h_3 has only one isotypical component, we get a representation $f = \tilde{h}_1^2 + \tilde{h}_2^2 + \tilde{h}_3^2$ with 3 G -invariant polynomials \tilde{h}_j . By Theorem 3.8 we get 7 frame generators. Note that the method in [29, Example 2.4] yields 6 generators of slightly larger support. The method in [7, Section 4] based on properties of the Kronecker product leads to 7 frame generators whose support is the same as the one of p . One can also employ the technique discussed in [20, Section] and get 7 frame generators.

Another prominent example of a subdivision scheme is the so-called butterfly scheme. This example shows the real advantage of treating the isotypical components of p separately for p with larger support.

Example 4.13. The butterfly scheme describes an interpolatory subdivision scheme that generates a smooth regular surface interpolating a given set of points [19]. The trigonometric

polynomial p associated with the butterfly scheme is given by

$$\begin{aligned} p(z_1, z_2) &= \frac{1}{4} + \frac{1}{8} \left(z_1 + z_2 + z_1 z_2 + z_1^{-1} + z_2^{-1} + z_1^{-1} z_2^{-1} \right) \\ &\quad + \frac{1}{32} \left(z_1^2 z_2 + z_1 z_2^2 + z_1 z_2^{-1} + z_1^{-1} z_2 + z_1^{-2} z_2^{-1} + z_1^{-1} z_2^{-2} \right) \\ &\quad - \frac{1}{64} \left(z_1^3 z_2 + z_1^3 z_2^2 + z_1^2 z_2^3 + z_1 z_2^3 + z_1^2 z_2^{-1} + z_1 z_2^{-2} \right. \\ &\quad \left. + z_1^{-1} z_2^2 + z_1^{-2} z_2 + z_1^{-3} z_2^{-1} + z_1^{-3} z_2^{-2} + z_1^{-2} z_2^{-3} + z_1^{-1} z_2^{-3} \right). \end{aligned}$$

Its first isotypical component is $p_{0,0} = \frac{1}{4}$, which is the case for every interpolatory subdivision scheme. The other isotypical components, in terms of $z_k = e^{-i\omega_k}$, $k = 1, 2$, are given by $p_{1,0}(z_1, z_2) = \frac{1}{4} \cos(\omega_1) + \frac{1}{16} \cos(\omega_1 + 2\omega_2) - \frac{1}{32} \cos(3\omega_1 + 2\omega_2) - \frac{1}{32} \cos(\omega_1 - 2\omega_2)$, i.e.,

$$p_{1,0}(z_1, z_2) = \frac{1}{4} \cos(\omega_1) + \frac{1}{8} \sin^2(\omega_1) \cos(\omega_1 + 2\omega_2),$$

and $p_{0,1}(z_1, z_2) = p_{1,0}(z_2, z_1)$, $p_{1,1}(z_1, z_2) = p_{1,0}(z_1 z_2, z_2^{-1})$. Note that on \mathbb{T}^2

$$|p_\chi| \leq \frac{1}{4} \quad \text{for all } \chi \in G',$$

thus, our method is applicable. Simple computation shows that

$$\begin{aligned} 1 - 16 |p_{1,0}(z_1, z_2)|^2 &= 1 - \cos^2(\omega_1) - \cos(\omega_1) \sin^2(\omega_1) \cos(\omega_1 + 2\omega_2) \\ &\quad - \frac{1}{4} \sin^4(\omega_1) \cos^2(\omega_1 + 2\omega_2). \end{aligned}$$

Setting $u_j := \sin(\omega_j)$, $j = 1, 2$, $v := \sin(\omega_1 + \omega_2)$, $v' := \sin(\omega_1 - \omega_2)$, $w := \sin(\omega_1 + 2\omega_2)$ and $w' := \sin(2\omega_1 + \omega_2)$, we get

$$1 - 16 |p_{1,0}(z_1, z_2)|^2 = \frac{1}{4} u_1^2 \left(w^2 + (u_2^2 + v^2)^2 + 2u_2^2 + 2v^2 \right).$$

Therefore,

$$\begin{aligned} f = 1 - \sum_{\sigma \in G} p^{\sigma*} p^\sigma &= \frac{1}{4} \left(u_1^2 u_2^2 + u_1^2 v^2 + u_2^2 v^2 \right) + \frac{1}{16} \left(u_1^2 w^2 + u_2^2 w'^2 + v^2 v'^2 \right) \\ &\quad + \frac{1}{16} \left(u_1^2 (u_2^2 + v^2)^2 + u_2^2 (u_1^2 + v^2)^2 + v^2 (u_1^2 + u_2^2)^2 \right). \end{aligned}$$

This provides a decomposition $f = \sum_{j=1}^9 h_j^* h_j$ into a sum of 9 squares. As in the previous example, each h_j has only one nonzero isotypical component h_{j,χ_j} . Thus, by part (b) of Lemma 3.7 and by Theorem 3.8, there exists a tight frame with 13 generators. Namely, as in the proof of Theorem 3.8, we get

$$\begin{aligned} q_1(z_1, z_2) &= \frac{1}{2} - \frac{1}{2} p(z_1, z_2), \quad q_2(z_1, z_2) = \frac{1}{2} z_1 - 2p(z_1, z_2) p_{(1,0)}^*(z_1, z_2) \\ q_3(z_1, z_2) &= q_2(z_2, z_1), \quad q_4(z_1, z_2) = q_2(z_1 z_2, z_2^{-1}) \\ q_{4+j}(z_1, z_2) &= p(z_1, z_2) \tilde{h}_{j,\chi_j}^*, \quad j = 1, \dots, 9, \end{aligned}$$

where \tilde{h}_{j,x_j} are the lifted isotypical components defined as in Lemma 3.7. Let $\mathcal{N} = \{0, \dots, 7\}^2$, $p = \mathbf{p} \cdot \mathbf{x}$ and $q_j = \mathbf{q}_j \cdot \mathbf{x}$ with $\mathbf{x} = [z^\alpha : \alpha \in \mathcal{N}]^T$. The corresponding null-matrix $O \in \mathbb{R}^{64 \times 64}$ satisfying (29) is given by

$$\mathbf{x}^* \cdot O \cdot \mathbf{x} = \mathbf{x}^* \left[\sum_{j=1}^{13} \mathbf{q}_j^T \mathbf{q}_j - \text{diag}(\mathbf{p}) + \mathbf{p}^T \mathbf{p} \right] \mathbf{x}.$$

Note that other factorizations of the positive semi-definite matrix $\text{diag}(\mathbf{p}) - \mathbf{p}^T \mathbf{p} + O$ of rank 13 lead to other possible tight frames with at least 13 frame generators. An advantage of using semi-definite programming techniques is that it can possibly yield q_j of smaller degree and reduce the rank of $\text{diag}(\mathbf{p}) - \mathbf{p}^T \mathbf{p} + O$.

Using the technique of semi-definite programming the authors in [6] constructed numerically a tight frame for the butterfly scheme with 18 frame generators. The advantage of our construction is that the frame generators are determined analytically. The disadvantage is that their support is approximately twice as large as that of the frame generators in [6].

The next example is one of the family of interpolatory $\sqrt{3}$ -subdivision studied in [27]. The associated dilation matrix is $M = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$ and $m = 3$.

Example 4.14. The symbol of the scheme is given by

$$p(z_1, z_2) = p_{(0,0)}(z_1, z_2) + p_{(1,0)}(z_1, z_2) + p_{(0,1)}(z_1, z_2)$$

with isotypical components $p_{(0,0)} = \frac{1}{3}$,

$$p_{(0,1)}(z_1, z_2) = \frac{4}{27}(z_2 + z_1^{-1} + z_1 z_2^{-1}) - \frac{1}{27}(z_1^{-2} z_2^2 + z_1^2 + z_2^{-2})$$

and $p_{(1,0)}(z_1, z_2) = p_{(0,1)}(z_2, z_1)$. We have by Lemma 3.7 and due to the equality $|p_{(0,1)}(z_1, z_2)|^2 = |p_{(1,0)}(z_1, z_2)|^2$

$$1 - \sum_{\sigma \in G} p^{\sigma*} p^\sigma = 2 \left(\frac{1}{9} - p_{(0,1)}^* p_{(0,1)} \right),$$

thus it suffices to consider only

$$\begin{aligned} \frac{1}{9} &- |p_{(0,1)}(z_1, z_2)|^2 = 3^{-2} - 27^{-2} \left(51 + 16 \cos(\omega_1 + \omega_2) + 16 \cos(2\omega_1 - \omega_2) \right. \\ &+ 16 \cos(\omega_1 - 2\omega_2) + 2 \cos(2\omega_1 + 2\omega_2) + 2 \cos(2\omega_1 - 4\omega_2) \\ &+ \left. 2 \cos(4\omega_1 - 2\omega_2) - 8 \cos(3\omega_1) - 8 \cos(3\omega_2) - 8 \cos(3\omega_1 - 3\omega_2) \right). \end{aligned}$$

Numerical tests show that this polynomial is nonnegative.

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