Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Global–in–time solvability of thermodynamically motivated parabolic systems

Pierre-Étienne Druet

submitted: December 5, 2017

Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: pierre-etienne.druet@wias-berlin.de

No. 2455
Berlin 2017

2010 *Mathematics Subject Classification.* 35K40, 35K51, 35K57, 35K59, 35D30, 35B65.

Key words and phrases. Doubly nonlinear parabolic systems, quasilinear parabolic equations, advection–diffusion–reaction equations, a-priori estimates, generalised solutions, smoothness of solutions.

This research was supported by the Research Center MATHEON through project SE17 funded by the Einstein Center for Mathematics Berlin.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract

In this paper, doubly non linear parabolic systems in divergence form are investigated form the point of view of their global–in–time weak solvability. The non–linearity under the time derivative is given by the gradient of a strictly convex, globally Lipschitz continuous potential, multiplied by a position–dependent weight. This weight admits singular values. The flux under the spatial divergence is also of monotone gradient type, but it is not restricted to polynomial growth. It is assumed that the elliptic operator generates some equi–coercivity on the spatial derivatives of the unknowns. The paper introduces some original techniques to deal with the case of nonlinear purely Neumann boundary conditions. In this respect, it generalises or complements the results by Alt and Luckhaus (1983) and Alt (2012). A field of application of the theory are the multi species diffusion systems driven by entropy.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and $T > 0$. In $Q_T :=]0, T[\times\Omega$ we consider for a vectorial unknown $q :]0, T[\times\Omega \to \mathbb{R}^N$ ($N \in \mathbb{N}$) the following doubly non-linear parabolic system

$$\partial_t R_k(x,\, q) + \operatorname{div} J^k(t,\, x,\, q,\, \nabla q) = f_k(t,\, x,\, q,\, \nabla q) \text{ for } k = 1, \ldots, N\,. \tag{1}$$

The vector fields $R :\ \Omega\times\mathbb{R}^N \to \mathbb{R}^N$, $J :\ Q_T\times\mathbb{R}^N\times\mathbb{R}^{N\times 3} \to \mathbb{R}^{N\times 3}$, and $f :\ Q_T\times\mathbb{R}^N\times\mathbb{R}^{N\times 3} \to \mathbb{R}^N$ are assumed given. We denote $(t, x,\, z,\, D)$ a generic elements of $Q_T\times\mathbb{R}^N\times\mathbb{R}^{N\times 3}$. Accordingly, $R(x,\, z)$, $J(t, x,\, z,\, D)$ etc. stand for the value of the fields in this point.

In connection with (1), we consider the natural lateral flux condition

$$\nu(x)\cdot J^k(t, x,\, q,\, \nabla q) = f_k^\Gamma(t,\, x,\, q) \text{ on } S_T :=]0, T[\times\partial\Omega\,. \tag{2}$$

where ν is the outward pointing unit normal field at $\partial\Omega$, and $f^\Gamma :\ S_T \times \mathbb{R}^N \to \mathbb{R}^N$ is a given vector field.

The system (1), (2) arises for instance in the modelling mass transfer problems involving several chemical species: R, J and f are respectively connected to the mass densities, the diffusion fluxes and the reaction densities. See [DG17] for the derivation of the equations (1) in this case, and for preliminary remarks.

The *parabolicity* of the system (1) is assumed or defined in the following sense: If we linearise the equations in a regular state q^* (say for instance $q^* \in \mathcal{C}^2(\overline{Q_T};\, \mathbb{R}^N)$), the linearisation generates a *uniformly parabolic system* (cp. [LSU68], Ch. 7, Par. 8, Def. 7). In order that this type of structure arises, it is sufficient that:

(a) The vector field $q \mapsto R(x,\, q)$ be *strictly monotone* on $\mathbb{R}^N$ for all $x \in \Omega$;

DOI 10.20347/WIAS.PREPRINT.2455 Berlin 2017

(b) The flux J be given by the gradient of a strictly convex function $D \mapsto \Psi(t, x, z, D)$ in the ∇q variables according to $J(t, x, z, D) = -\Psi_D(t, x, z, D)$.

The structural properties (a), (b) imply the uniform parabolicity, and they allow to prove the local existence of classical solutions to the initial value problem for (1), (2) if the data are smooth. However, formal linearisation in a point $q^* \notin L^\infty(Q_T; \mathbb{R}^N)$ shows that the system (1) under (a), (b) is doubly degenerated if:

- The (positive definite) coefficient matrix $R_z(x, z)$ possesses eigenvalues that tend to zero for $|z| \to \infty$;
- The Hessian $\Psi_{D,D}(t, x, z, D)$ associated with the elliptic operator $-\operatorname{div} \Psi_D$ is singular for $|z| \to \infty$.

This is precisely what happens in the context of the relevant applications.

In this paper we propose a theory of global–in–time weak solutions to (1), (2) based on an energy identity. In order to cope with the natural flux conditions (2), we derive partly original estimates for the integrals $\int_\Omega q(t, x)\, dx$. But note that we allow only partly for degeneracy: the singularity associated with the time derivative is considered, but not the singularity in ellipticity.

There is an essential difference between local–in–time existence results of stronger solutions and the results of this paper: Here, the existence theory essentially relies on the energy identity and the conservation laws for the original system. This is different from the case of the local–in–time existence, where everything relies on the mathematical structure and a certain smoothness of the initial data.

The structural conditions. For the non-linearity R, we assume that there are a given function $\varrho_0 : \Omega \to \mathbb{R}_{0,+} := \{x \in \mathbb{R} \,:\, x \geq 0\}$ and a convex function $\beta \in C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ such that

$$R(x, z) = \varrho_0(x)\, \beta_z(z) \text{ for } (x, z) \in \Omega \times \mathbb{R}^N\,. \tag{3}$$

This particular form will allow to obtain a coercivity estimate in the case of pure flux boundary conditions. It arises naturally in the context of mechanically balanced mass transfer systems: see [DG17] for a derivation of this structure.

We now introduce the structure conditions for the fluxes and the reaction terms (right-hand sides) suggested by a *pointwise entropy principle*: See [BD15], [Guh14] for the original development, and [DG17] for the derivation of closure relations in the particular case of diffusion systems. The fields J and f are derived from convex potentials in order to ensure the non negativity of an entropy production associated with the fluxes J and the production factor f. In order to include the case of chemical reaction in the analysis, we need a particular description of the right-hand side f. We introduce parameters $s,\, s^\Gamma \in \mathbb{N} \cup \{0\}$ – to be interpreted as the number of active chemical reactions in the bulk and on the boundary – and vectors $A^1, \ldots, A^s \in \mathbb{R}^N$ and $A^{\Gamma,1}, \ldots, A^{\Gamma,s^\Gamma} \in \mathbb{R}^N$ so that the right-hand f maps into $\operatorname{span}\{A^1, \ldots, A^s\}$ and so that f^Γ maps into $\operatorname{span}\{A^{\Gamma,1}, \ldots, A^{\Gamma,s^\Gamma}\}$

We denote $(t, x, z, D, D^{\mathrm{R}})$ the generic elements of the compound $Q_T \times \mathbb{R}^N \times \mathbb{R}^{N\times 3} \times \mathbb{R}^s$, and we consider the following conditions:

(c) There is a non-negative potential potential Φ defined on $Q_T \times \mathbb{R}^N \times \mathbb{R}^{N\times 3} \times \mathbb{R}^s$ such that the mapping $(D, D^{\mathrm{R}}) \mapsto \Phi(t, x, z, D, D^{\mathrm{R}})$ is strictly convex for all $(t, x, z) \in Q_T \times \mathbb{R}^N$, and such that it achieves its global minimum in $(D, D^{\mathrm{R}}) = 0$;

(d) There is a non-negative potential potential Φ^{Γ} defined on $S_T \times \mathbb{R}^N \times \mathbb{R}^{s^{\Gamma}}$ such that the mapping $D^{\mathsf{R},\Gamma} \mapsto \Phi^{\Gamma}(t, x, z,\, D^{\mathsf{R},\Gamma})$ is strictly convex for all $(t, x,\, z) \in S_T \times \mathbb{R}^N$ and achieves its global minimum in zero;

and the diffusion fluxes and reaction rates are given by

$$J^i_k = -\Phi_{D^i_k}(t, x,\, q,\, \nabla q,\, A \cdot q) \qquad \text{for } i = 1, \ldots, N \text{ and } k = 1, 2, 3 \tag{4}$$

$$f_i = -\sum_{\alpha=1}^{s} \Phi_{D^{\mathsf{R}}_\alpha}(t, x,\, q,\, \nabla q,\, Aq)\, A^\alpha_i \text{ for } i = 1, \ldots, N \tag{5}$$

$$f^{\Gamma}_i = -\sum_{\alpha=1}^{s^{\Gamma}} \Phi^{\Gamma}_{D^{\mathsf{R},\Gamma}_\alpha}(t, x,\, q,\, A^{\Gamma} q)\, A^{\Gamma,\alpha}_i \text{ for } i = 1, \ldots, N\,. \tag{6}$$

For simplicity, we also introduce abbreviations

$$r_\alpha(t,\, x,\, q,\, \nabla q) := -\Phi_{D^{\mathsf{R}}_\alpha}(t, x,\, q,\, \nabla q,\, Aq) \text{ for } \alpha = 1, \ldots, s \tag{7}$$

$$r^{\Gamma}_\alpha(t,\, x,\, q) := -\Phi^{\Gamma}_{D^{\Gamma,\mathsf{R}}_\alpha}(t, x,\, q,\, A^{\Gamma} q) \quad \text{for } \alpha = 1, \ldots, s^{\Gamma}\,. \tag{8}$$

Remark 1.1 (Notation)**.** + *With $\Phi_{D^i_k}$, Φ_{z_i}, $\Phi_{D^R_\alpha}$ etc. , we denote the partial derivative of Φ with respect to the corresponding scalar variable. With Φ_D, Φ_{D^R} etc. , we denote the gradient of the functions $D \mapsto \Phi(\cdot,\, D)$ on $\mathbb{R}^{N\times 3}$, $D^R \mapsto \Phi(\cdot,\, D^R)$ on $\mathbb{R}^s$ etc.*

+ *For a rectangular matrix $B \in \mathbb{R}^{s_1\times s_2}$, we multiply a vectors $X \in \mathbb{R}^{s_2}$ from the right, and we denote $BX \in \mathbb{R}^{s_1}$ the matrix vector multiplication. For $Y \in \mathbb{R}^{s_1}$, the matrix vector multiplication takes the form $YB = B^T Y \in \mathbb{R}^{s_2}$.*

Remark 1.2 (Simplification)**.** *The condition* (d) *is not the most general. Instead we could assume only that:*

(d') There is a potential Φ^{Γ} defined on $S_T \times \mathbb{R}^N \times \mathbb{R}^{s^{\Gamma}}$, and the mapping $D^{R,\Gamma} \mapsto \Phi^{\Gamma}(t, x, z,\, D^{R,\Gamma})$ is strictly convex for all $(t, x,\, z) \in S_T \times \mathbb{R}^N$.

In this case, we can introduce

$$\tilde{\Phi}^{\Gamma}(t, x, z,\, D^{R,\Gamma}) := \Phi^{\Gamma}(t, x,\, z,\, D^{R,\Gamma}) - \Phi^{\Gamma}_{D^{R,\Gamma}}(t, x,\, z,\, 0) \cdot D^{R,\Gamma}\,,$$
$$r^0(t, x,\, z) := \Phi^{\Gamma}_{D^{R,\Gamma}}(t, x,\, z,\, 0)\,.$$

We obtain that $r^{\Gamma}_\alpha = -\tilde{\Phi}^{\Gamma}_{D^{R,\Gamma}_\alpha}(t, x,\, q,\, A^{\Gamma} q) + r^0_\alpha(t, x,\, q)$, where now the potential $\tilde{\Phi}^{\Gamma}$ achieves its global minimum in zero. The term r^0 can easily be handled, and we will neglect it for reasons of simplicity.

State of the art and brief discussion of the literature. The system (1), (2) arises for instance in the modelling mass transfer problems involving several chemical species, where R, J and f are respectively connected to the mass densities, the diffusion fluxes and the reaction densities. For this type of applications in which the constraints of global mass and momentum conservation have to be satisfied, it is to note that uniform parabolicity is only to expect if the equations (1) correspond to the reduction of a diffusion–reaction system for $N + 1$ species under very strong assumptions on the mechanics of the system. See [DG17] for details and a rigorous derivation. In the context of mass transfer, the vectors $A^1, \ldots, A^s \in \mathbb{R}^N$ are related to the stoichiometric vectors characterising the bulk chemical reactions. The right-hand in the condition (2) has a more complex meaning: It describes not only the chemical reactions with external species occurring at the boundary, but also adsorption, desorption and interaction mechanisms with mass that might be stored at the interface. The structure (2) has been motivated in [DDGG17] following models by [BD15], [Guh14]. The number $s^\Gamma \in \mathbb{N} \cup \{0\}$ is the number of such 'interaction mechanisms' and the vectors $A^{\Gamma,1}, \ldots, A^{\Gamma,s^\Gamma} \in \mathbb{R}^N$ can in some extend be interpreted as stoichiometric vectors.

The system (1) is a *doubly non-linear parabolic system* in divergence form. It cannot be our aim to refer here the entire litterature on parabolic systems of quasilinear type. We only want to put our work into the context of investigations directly related. Let us remark that parabolic systems that are diagonal in the principal part are not addressed in this paper. There, the analysis can rely on the maximum principle, a tool which is yet missing for fully coupled systems.

For the *short–time existence and uniqueness of smooth solutions* in the general case, we refer to the work recent work [Dru17]. Unfortunately, it is not known to the author in which extent the very general estimates in Hölder classes obtained by the Russian school for linear parabolic systems have been applied to non linear problems: We quote the Chapter 7, paragraph 8 of [LSU68] and the book [EZ98]. In the case that the driving potential Φ is quadratic in the variable D (which results into a flux J linear in D), the L^p estimates for strong solutions are available in the context of the theory of maximal parabolic regularity: see [HMPW17] for the short–time well posedness. For the same case of J linear in D, it is possible also in the case of non smooth data to prove the short–time existence of a weak solution with improved regularity: [Ama90], [Ama93] or [HRM16].

For the *global existence of weak solutions*, there is a branching in the theory.

1 **The case of coercive fluxes:** The existence of global weak solutions to doubly non-linear parabolic systems with general monotone non-linearities was proved in [AL83]. This paper treats the case where the elliptic operator $\operatorname{div} J$ generates uniform polynomial control in the gradient variable ∇q with exponent $1 < r < +\infty$. The flux J is assumed of the form $J = \bar{J}(R(q), \nabla q)$. The non-linearity $R = R(q)$ is given by the gradient of a convex function. The right-hand f is allowed to depend only on $R(q)$. The system (1) is considered in connection with mixed-boundary conditions, with a non-empty Dirichlet part $\Gamma_D \subseteq \partial\Omega$. The variables q satisfy

$$q = q^\Gamma \text{ given on }]0, T[\times\Gamma_D, \quad \nu_k(x) \cdot \bar{J}_k(R(q), \nabla q) = 0 \text{ on }]0, T[\times(\partial\Omega \setminus \Gamma_D)\,. \tag{9}$$

On connected domains, these boundary conditions somewhat simplify the task to obtain uniform coercivity estimates, since they now follow directly from the control on the gradient in L^r. In [AL83], the existence is enforced by means of generalising the abstract theory of pseudomonotone operators, and original arguments are developed for the compactness in time. In the more recent paper [Alt12], further generalisations were obtained. Essentially the same remarks remain valid.

In the papers [FK95], global and local existence results for weak solutions are derived in the case of a monotone elliptic part with $r-$growth ($r > 1$). The right-hand f is allowed to depend directly on q, and the optimal growth exponents for the non-linearities are calculated precisely in dependence of each other. Purely Neumann boundary conditions are considered in [FK95], but it is necessary to require the super linear growth of $R(q)$ in q to obtain the coercivity estimates. In the paper [Kac97], the mixed boundary conditions (9) are again considered. The paper [Ben13] also ranges into this line of investigations.

Concerning the uniqueness of weak solutions, the most general results have been obtained in [Ott96] for exactly the setting of [AL83].

2 **Weak solutions in the case of degenerate ellipticity:** All results yet mentioned are based on strong closure relations for the diffusion flux. Mathematically, this means that the elliptic operator generates some coercivity independently on $|q|$ (or the limiting behaviour of $R(q)$ as $|q| \to \infty$).

In some of these cases it is possible to obtain the global existence of weak solutions thanks to the *boundedness by entropy method* (see [J¨15], [CDJ18] or [J¨17] for a general introduction).

This method has been developed in the context of cross diffusion systems, but it in the end relies on the analysis of a doubly non-linear parabolic system of the form (1) with a linear closure relation for the fluxes of the form $J = -M(R(q))\, \nabla q$. In this case, the variables $R(q)$ stand for some volume fractions that have to be positive, and the eigenvalues of the matrix M are allowed to degenerate as $R_i(q)$ tends to zero for some i. This corresponds exactly to the Maxwell–Stefan situation.

Beside the boundedness by entropy method, it is also possible to study the local–in–time well–posedness of the problem (1) in classes of weak solutions that embed into $L^{\infty}(Q;\, \mathbb{R}^N)$. For appropriate initial data and sufficiently short an existence time, the degeneracy in ellipticity does not occur. For an investigation in spaces of maximal parabolic regularity, the reader might consult [HRM16]. Up to now, this approach has been developed for linear closure relations of the type $J = -M(q)\, \nabla q$.

In this paper, we formulate and prove the existence result for the interesting case that the natural flux conditions (2) are considered on the entire lateral surface $]0, T[\times \partial\Omega$. This case is not covered by the analysis in [AL83], [Alt12] where the elliptic part is always assumed to exhibit some coercivity. Moreover, we will consider the case that the mapping $q \mapsto R(x,\, q)$ is bounded, so that the coercivity on q is not a direct consequence of the energy identity. This occurs if the function β occurring in (3) is globally Lipschitz continuous.

Moreover we will consider a general diffusion potential and not restrict to the case of polynomial growth.

There are cases in which global weak solutions exhibit more regularity – also if the system if not diagonal, so that techniques valid for equations would not apply. We will call a *strong solution* to the equations (1), (2) a weak solution so that the weak time derivative of R and second spatial derivatives of q exist, so that the equations are solved also point wise. The main case is that of a closure relation for the fluxes which is thermodynamically consistent in the sense of Wolfgang Dreyer: $J = -\Phi_D(D)$, with a potential depending on the driving force only (see [DG17] for the definition). In this case it is possible to 'square' the operator and to derive higher–order estimates. To find a general approach to the higher regularity is beyond the scope of the paper. We will here restrict to the case of a quadratic growth of Φ.

Structure of the paper. In the next section we formulate our results and the precise structural conditions on which they rely. In the section 3, we recall some basic concepts of convex analysis that we need in order to derive an estimate on the spatial averages of the solutions. The section 4.1 is devoted to the proof of the main *a priori* estimates, that we then use in the section 4 to construct a convergent Galerkin approximation. In the section 5, we prove the higher regularity in the case of thermodynamically consistent quadratic closure relations.

2 Results

Our structural assumptions are mainly the one guaranteeing the uniform parabolicity in smooth points (a) and (b). These are consequences of the thermodynamically motivated assumptions (c), (d). In order to obtain global existence results for weak solutions, we need to formulate additional *growth conditions* on the potentials.

2.1 The growth conditions

Structural conditions for the dual energy potential β. We assume that the non-linearity R possesses the natural structure resulting from the reduction of mechanically balanced diffusion–reaction systems (See [DG17], (3)). In this case, the non-linearity R is associated with a non-linear constraint on the chemical potentials (also called entropy variables) associated with one-homogeneous free energy function.

Given a function $\varrho_0 : \Omega \to \mathbb{R}_{0,+}$, and a strictly convex function $\beta : \mathbb{R}^N \to \mathbb{R}$, we define for $x \in \Omega$ and $z \in \mathbb{R}^N$

$$R_\ell(x,\, z) := \varrho_0(x)\, \beta_{z_\ell}(z) \text{ for } \ell = 1, \ldots, N\,. \tag{10}$$

The assumptions on the function β are the following:

$$\beta \in C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)\,, \tag{11}$$

$$z \mapsto \beta(z) \text{ is closed and strictly convex in } \mathbb{R}^N\,. \tag{12}$$

Due to the convexity of β and the non negativity of ϱ_0, the mapping $z \mapsto R(x,\, z)$ is strictly monotone.

Growth conditions for the bulk potential. We denote Φ the potential driving the diffusion and production mechanisms in the bulk. For this potential, the minimal assumptions are the following:

$$\Phi \in L^\infty(Q_T \times \mathbb{R}^N;\, C^1(\mathbb{R}^{N\times 3} \times \mathbb{R}^s)) \tag{13}$$

$$(D,\, D^{\mathsf{R}}) \mapsto \Phi(x,\, z,\, D,\, D^{\mathsf{R}}) \text{ convex for all } (t,\, x,\, z) \in Q_T \times \mathbb{R}^N \tag{14}$$

Moreover, we require some uniform coercivity conditions that reflect a *strongly thermodynamic closure relation* for the fluxes. We define a convex function

$$\Phi^0(D,\, D^{\mathsf{R}}) := \sup_{(t,x,z)\in Q_T\times\mathbb{R}^N} \Phi(t,x,z,\, D,\, D^{\mathsf{R}})\,. \tag{15}$$

Obviously, $\Phi^0(D,\, D^{\mathsf{R}}) \geq \Phi(t,x,\, z,\, D,\, D^{\mathsf{R}})$. Therefore, Φ^0 is non-negative, and we moreover see that under the condition (c), it achieves a global minimum in zero.

We assume that the function Φ^0 is finite and coercive on $\mathbb{R}^{N\times 3} \times \mathbb{R}^s$, and that there is a constant $0 < c_0$ on $\mathbb{R}^N$ such that

$$c_0\, \Phi^0(D,\, D^{\mathsf{R}}) \leq \Phi(x,\, z,\, D,\, D^{\mathsf{R}}) \quad \text{for all } (t, x,\, z,\, D,\, D^{\mathsf{R}}) \in Q_T \times \mathbb{R}^N \times \mathbb{R}^{N\times 3} \times \mathbb{R}^s\,. \tag{16}$$

Moreover, we assume that there are constants $K_1,\, K_0 > 0$ such that

$$\Phi^0(-D,\, -D^{\mathsf{R}}) \leq K_0\, \Phi^0(D,\, D^{\mathsf{R}}) + K_1\,, \tag{17}$$

for all $(D,\, D^{\mathsf{R}}) \in \mathbb{R}^N \times \mathbb{R}^s$.

We introduce an abbreviation for the fluxes $J(t,x,\, z,\, D) := -\Phi_D(t,x,\, z,\, D,\, Az)$, and the densities $r(t,x,\, z,\, D) := -\Phi_{D^{\mathsf{R}}}(t,x,\, z,\, D,\, Az)$.

Growth conditions for the boundary potential. We denote Φ^Γ the potential driving the production mechanisms on S_T:

$$\Phi^\Gamma \in L^\infty(S_T \times \mathbb{R}^N;\, C^1(\mathbb{R}^{s^\Gamma})) \tag{18}$$

$$D^{\Gamma,\mathsf{R}} \mapsto \Phi^\Gamma(t, x,\, z,\, D^{\Gamma,\mathsf{R}}) \text{ convex for all } (t,\, x,\, z) \in S_T \times \mathbb{R}^N \tag{19}$$

The convex function

$$\Phi^{\Gamma,0}(D^{\Gamma,\mathsf{R}}) := \sup_{(t,x,z)\in S_T\times\mathbb{R}^N} \Phi^\Gamma(t, x, z,\, D^{\Gamma,\mathsf{R}})\,. \tag{20}$$

is assumed finite and coercive on $\mathbb{R}^{s^\Gamma}$. Moreover, it is assumed that there is a constant $0 < c_0^\Gamma$ on $\mathbb{R}^N$ such that

$$c_0^\Gamma\, \Phi^{\Gamma,0}(D^{\Gamma,\mathsf{R}}) \leq \Phi^\Gamma(x,\, z,\, D^{\Gamma,\mathsf{R}}) \text{ for all } (t, x,\, z,\, D^{\Gamma,\mathsf{R}}) \in S_T \times \mathbb{R}^N \times \mathbb{R}^{s^\Gamma}\,. \tag{21}$$

We assume that there is $K_1^\Gamma,\, K_0^\Gamma > 0$ such that

$$\Phi^{\Gamma,0}(-D^{\Gamma,\mathsf{R}}) \leq K_0^\Gamma\, \Phi^{\Gamma,0}(D^{\Gamma,\mathsf{R}}) + K_1^\Gamma \text{ for all } D^{\Gamma,\mathsf{R}} \in \mathbb{R}^{s^\Gamma}\,. \tag{22}$$

We denote $r^\Gamma(t, x,\, z) := -\Phi^\Gamma_{D^{\Gamma,\mathsf{R}}}(t, x,\, z,\, A^\Gamma z)$.

2.2 Definition of the weak solution

Energy identity. The function β is strictly convex by assumption. We denote H its convex conjugate in $\mathbb{R}^N$. For all $R \in \mathrm{dom}\ H$, one has

$$H(R) := \sup_{z\in\mathbb{R}^N} \{z \cdot R - \beta(z)\}\,.$$

Due to convex duality $H(\beta_z(z)) + \beta(z) = \beta_z(z) \cdot z$, yielding

$$R_\ell(x,\, z)\, z_\ell = \varrho_0(x)\, (H\big(\beta_z(z)\big) + \beta(z))\,. \tag{23}$$

If we multiply in (1) with q_k, sum up over $k = 1, \ldots, N$ and then integrate over Ω, we easily deduce by means of (2) the identity

$$\begin{aligned}\int_\Omega \partial_t R(x,\, q) \cdot q + \int_\Omega \{\Phi_D(t, x,\, q,\, \nabla q,\, Aq) \,:\, \nabla q + \Phi_{D^{\mathsf{R}}}(t, x,\, q,\, \nabla q,\, Aq) \cdot Aq\} \\ + \int_{\partial\Omega} \Phi^\Gamma_{D^{\mathsf{R}},\Gamma}(t, x,\, q,\, A^\Gamma q) \cdot A^\Gamma q = 0\,.\end{aligned} \tag{24}$$

We note that

$$\begin{aligned}\partial_t R(x,\, q) \cdot q &= \partial_t(R(x,\, q) \cdot q - \varrho_0(x)\, \beta(q)) \\ &= \varrho_0(x)\, \partial_t H(\beta_z(q))\,.\end{aligned}$$

Therefore, integration of (24) over the interval $[0, t']$ yields

$$\begin{aligned}&\int_\Omega \varrho_0(x)\, H(\beta_z(q(t', x))) \\ &\quad + \int_0^{t'} \int_\Omega \{\Phi_D(t, x,\, q,\, \nabla q,\, Aq) \,:\, \nabla q + \Phi_{D^{\mathsf{R}}}(t, x,\, q,\, \nabla q,\, Aq) \cdot Aq\} \\ &\quad + \int_0^{t'} \int_{\partial\Omega} \Phi^\Gamma_{D^{\mathsf{R}},\Gamma}(t, x,\, q,\, A^\Gamma q) \cdot A^\Gamma q = \int_\Omega \varrho_0(x)\, H(\beta_z(q^0(x)))\,.\end{aligned} \tag{25}$$

Making use of convex duality again

$$\begin{aligned}
&\Phi_D(t,x,\, q,\, \nabla q,\, Aq) \,:\, \nabla q + \Phi_{D^{\mathrm{R}}}(t,x,\, q,\, \nabla q,\, Aq)\cdot Aq\\
&\quad = \Phi(t,x,\, q)(\nabla q,\, Aq) + (\Phi(t,x,\, q))^*(\nabla\Phi(x,\, q,\, \nabla q,\, Aq))\\
&\quad = \Phi(t,x,\, q,\, \nabla q,\, Aq) + (\Phi(t,x,\, q))^*(-J,\, -r)\,.
\end{aligned}$$

Similarly

$$\Phi^{\Gamma}_{D^{\mathrm{R},\Gamma}}(t,x,\, q,\, A^{\Gamma}q)\cdot A^{\Gamma}q = \Phi^{\Gamma}(t,x,\, q,\, A^{\Gamma}q) + (\Phi^{\Gamma}(t,x,q))^*(-r^{\Gamma})\,.$$

We call the identity

$$\begin{aligned}
&\int_{\Omega}\varrho_0(x)\, H(\beta_z(q(t',x))) + \int_{Q_{t'}}\{\Phi(x,\, q,\, \nabla q,\, Aq) + (\Phi(t,x,\, q))^*(-J,\, -r)\}\\
&\quad + \int_{S_{t'}}\{\Phi^{\Gamma}(t,x,\, q,\, A^{\Gamma}q) + (\Phi^{\Gamma}(t,x,q))^*(-r^{\Gamma})\} = \int_{\Omega}\varrho_0(x)\, H(\beta_z(q^0(x)))\,.
\end{aligned}\tag{26}$$

the energy identity (at t'). For $(t,x,z)\in Q_T\times\mathbb{R}^N$, the function $(\Phi(t,x,\, z))^*$ is the convex conjugate in $\mathbb{R}^{N\times 3}\times\mathbb{R}^s$ of the function $(D,\, D^{\mathrm{R}})\mapsto\Phi(t,x,\, z,\, (D,\, D^{\mathrm{R}}))$.

Definition of a weak solution. A distributional solution of the problem (1), (2) is defined as an element q of a vectorial Orlicz *class*. Define

$$\begin{aligned}
&[q]_{\widetilde{W}^{1,0}_{\Phi^0,\,\Phi^{\Gamma,0}}(Q;\mathbb{R}^N)}\\
&\quad := \int_{Q_T}\{\Phi^0(\nabla q,\, Aq) + (\Phi^0)^*(-J(t,x,\, q,\, \nabla q),\, -r(t,x,\, q,\, \nabla q))\}\\
&\qquad + \int_{S_T}\{\Phi^{0,\Gamma}(A^{\Gamma}q) + (\Phi^{\Gamma,0})^*(-r^{\Gamma}(t,x,\, q))\}
\end{aligned}\tag{27}$$

$$\widetilde{W}^{1,0}_{\Phi^0,\,\Phi^{\Gamma,0}}(Q;\,\mathbb{R}^N) := \left\{q\in W^{1,0}_1(Q;\,\mathbb{R}^N)\,:\,[q]_{\widetilde{W}^{1,0}_{\Phi^0,\,\Phi^{\Gamma,0}}(Q;\mathbb{R}^N)} < +\infty\right\}\,.\tag{28}$$

If q satisfies the identity (26) for almost all times, if $[q]_{\widetilde{W}^{1,0}_{\Phi^0,\,\Phi^{\Gamma,0}}(Q;\mathbb{R}^N)} < +\infty$ and if

$$\begin{aligned}
&-\int_Q R(x,\, q)\cdot\phi_t - \int_Q J(t,x,\, q,\, \nabla q)\cdot\nabla\phi - \int_Q r(t,x,\, q,\, \nabla q)\cdot A\phi\\
&-\int_S r^{\Gamma}(t,x,\, q)\cdot A^{\Gamma}\phi = \int_{\Omega}R(x,\, q^0)\cdot\phi(0)\quad\forall\,\phi\in C^1_c([0,T[;\, C^1(\overline{\Omega};\,\mathbb{R}^N))\,,
\end{aligned}\tag{29}$$

then we call q a weak solution for (1), (2). Since the potential Φ^0 does not satisfy the symmetry condition $\Phi(X)=\Phi(-X)$ for its domain of definition, the Orlicz class generated by Φ_0 is not a vectorial Orlicz *space*. We consider $\widetilde{W}^{1,0}_{\Phi^0,\,\Phi^{\Gamma,0}}$ as a subset (a subclass) of the parabolic Sobolev space

$$W^{1,0}_1(Q;\,\mathbb{R}^N) = \{q\in L^1(Q;\,\mathbb{R}^N)\,:\,\nabla u\in L^1(Q;\,\mathbb{R}^{N\times 3})\}$$

2.3 Existence of weak solutions.

Our main theorem is the following:

Theorem 2.1. *Let Ω be a bounded domain of class $\mathcal{C}^{0,1}$. Assume that R satisfies* (11), (12). *Assume that the diffusion potential Φ satisfies the conditions* (13), (14), (16) *and* (17) *and that the boundary potential Φ^{Γ} correspondingly satisfies* (18), (19), (21) *and* (22)

Let $q^0 : \Omega \to \mathbb{R}^N$ and $\varrho_0 : \Omega \to \mathbb{R}_{0,+}$ be measurable, bounded and such that $\int_\Omega \varrho_0(x)\, dx > 0$. Assume that the integral $\int_\Omega \varrho_0(x)\, H(\beta_z(q^0(x)))$ is finite and that the point $R_0 := \fint_\Omega \varrho_0(x)\, \beta_z(q^0(x))$ belongs to the image of β_z.

Assume that one of the following conditions is valid:

1. *$\varrho_0(x) > 0$ for almost all $x \in \Omega$;*
2. *The function Φ can be expressed via*

$$\Phi(t, x, z, D, D^{R}) = \Psi(t, x, R(x, z), D, D^{R}) \text{ for all } (t, x, z, D, D^{R}),$$

where $\Psi(t, x)$ is of class $C(\mathbb{R}^N;\, C^1(\mathbb{R}^{N\times 3} \times \mathbb{R}^s))$ for almost all $(t, x) \in Q_T$.

Then the problem (1), (2) *possesses a weak solution q of class $\widetilde{W}^{1,0}_{\Phi^0,\Phi^{\Gamma,0}}(Q;\, \mathbb{R}^N)$.*

2.4 Existence of strong solutions.

For a non-linear closure relation which is thermodynamically consistent in the sense of Wolfgang Dreyer (Definition of this concept in [DG17]), it is possible to introduce global strong solutions. In this case, the potential Φ is a function of the driving forces only ($\Phi = \Phi(D,\, D^{\mathsf{R}})$. In other words $\Phi = \Phi^0$). A strong solution is here defined as a weak solution such that the weak time derivative $\partial_t R(x,\, q)$ and the weak second spatial derivatives $q_{x,x}$ exist almost everywhere in Q_T. Moreover, the gradient q_x exists in the sense of traces on the lateral surface S_T for such solutions. Then, the equations (1), (2) are valid pointwise almost everywhere in Q_T resp. on S_T with respect to the standard measures.

Theorem 2.2. *Assumptions of Theorem 2.1. Moreover, assume that Ω is a bounded domain of class $\mathcal{C}^{1,1}$. Assume that the function β belongs to $C^2(\mathbb{R}^N)$. Assume that $q^0 \in W^{1,1}(\Omega;\, \mathbb{R}^N)$ is such that $\int_\Omega \Phi(\nabla q^0,\, A q^0) + \int_\Gamma \Phi^\Gamma(A^\Gamma q^0) < +\infty$. Assume that the potentials $\Phi,\ \Phi^\Gamma$ are twice continuously differentiable over their domain and that Φ satisfies the reinforced assumptions*

1. *There is $\lambda_0 > 0$ such that*

$$D^2\Phi(X) \begin{pmatrix} D \\ D^{\mathsf{R}} \end{pmatrix} \cdot \begin{pmatrix} D \\ D^{\mathsf{R}} \end{pmatrix} \geq \lambda_0\, |D|^2 \text{ for all } X,\ (D,\, D^{\mathsf{R}}) \in \mathbb{R}^{N\times 3} \times \mathbb{R}^s\,.$$

2. *There is a constant $\lambda_1 \in \mathbb{R}_+$ such that*

$$|\Phi_{D,D^{\mathsf{R}}}|_\infty + |\Phi_{D,D}|_\infty \leq \lambda_1\,.$$

3. *There is $\gamma > 1$ such that for all $J \in \mathbb{R}^{N\times 3}$ and $r \in \mathbb{R}^s$*

$$\Phi^*(-J,\, -r) \geq c_0\, |r|^\gamma - c_1\,.$$

Then the problem (1),(2) *possesses a strong solution such that $\partial_t R(x,\, q)$ belongs to $L^2(Q;\, \mathbb{R}^N))$ and the second derivatives q_{x_k, x_ℓ} $(k,\, \ell = 1, 2, 3)$ belong to $L^p(Q;\, \mathbb{R}^N)$ with $p = \min\{2,\, \gamma\}$.*

3 Some properties of the dual energy function

Assume that β satisfies (11), (12). For $R \in \mathbb{R}^N$ we define

$$H(R) = \sup_{z\in\mathbb{R}^N} \{R \cdot z - \beta(z)\} .$$

Then $z \mapsto H(z)$ is a closed, proper convex function (Theorem 12.2 of [Roc70]). The domain of this function, denoted $\operatorname{dom} H$, is defined as the subset in $\mathbb{R}^N$ where it is finite. Due to [Roc70], Th. 13.4, the dimension of $\operatorname{dom} H$ is N. Applying the Theorem 26.3 of [Roc70], we see that $z \mapsto H(z)$ is an essentially smooth convex function of class $C^1(\operatorname{ri}\operatorname{dom} H)$. (Note that the relative interior $\operatorname{ri}\operatorname{dom} H$ is nothing else but its interior.) In this context, *essentially smooth* means that $|H_R(R_n) \cdot b| \to +\infty$ for every sequence $\{R_n\}$ that approaches a (relative) boundary point R of $\operatorname{dom} H$ and for every direction b pointing into the convex set $\operatorname{dom} H$ (see the Definition on page 251 in [Roc70]). In fact we can be more precise and state that for any such sequence $\{R_n\}$ and directions b

$$\liminf_{n\to\infty} b \cdot H_R(R_n) = +\infty . \tag{30}$$

Lemma 3.1. *The set* $\operatorname{dom} H$ *is bounded and convex. We have the identities:*

$$\begin{aligned} \operatorname{ri}\operatorname{dom} H &= \{R \in \mathbb{R}^N \,:\, R = \beta_z(z),\, z \in \mathbb{R}^N\} \\ \partial \operatorname{dom} H &= \{R \in \mathbb{R}^N \,:\, R = \lim_{n\to\infty} \beta_z(z^n),\, |z^n| \to +\infty\} \end{aligned}$$

Proof. Since $z \mapsto \beta(z)$ is strictly convex on $\mathbb{R}^N$, the function $z \mapsto R\cdot z - \beta(z)$ is strictly concave on $\mathbb{R}^N$. Thus, a necessary and sufficient condition to have a global maximum in $\mathbb{R}^N$ is that $R = \beta_z(z^0)$ for a $z_0 \in \mathbb{R}^N$. Thus,

$$\text{Image } \beta_z := \{R \in \mathbb{R}^N \,:\, R = \beta_z(z_0),\, z_0 \in \mathbb{R}^N\} \subseteq \operatorname{dom} H .$$

For $R = \beta_z(z_0) \in \text{Image } \beta_z$ arbitrary

$$H(R) = \max_{z\in\mathbb{R}^N}\{R \cdot z - \beta(z)\} = \beta_z(z_0) \cdot z_0 - \beta(z_0) .$$

Since $z \mapsto \beta(z)$ is of class $C^1(\mathbb{R}^N)$, it follows that $\text{Image } \beta_z \subseteq \operatorname{ri}\operatorname{dom} H$.

Consider next $R \in \mathbb{R}^N \setminus \overline{\text{Image } \beta_z}$. We remark obviously that

$$\overline{\text{Image } \beta_z} = \{R \in \mathbb{R}^N \,:\, R = \lim_{n\to\infty} \beta_z(z^n),\, |z^n| \to +\infty\} .$$

Thus, $R \in \mathbb{R}^N \setminus \overline{\text{Image } \beta_z}$ means that there is $\epsilon > 0$ such that for all z, we have $|R - \beta_z(z)| \geq \epsilon$. For obvious reasons, we can solve for $n \in \mathbb{N}$, the equation $\xi_n + \beta_z(n\,\xi_n) = R$, and see that $|\xi_n| \geq \epsilon$. Thus

$$\begin{aligned} H(R) &\geq (R - \beta_z(n\,\xi_n)) \cdot n\,\xi_n - \beta(0) \\ &= n\,|R - \beta_z(n\,\xi_n)|^2 - \beta(0) \geq n\,\epsilon^2 - \beta(0) . \end{aligned}$$

Thus, $H(R) = +\infty$, and this establishes that $\operatorname{dom} H \subseteq \overline{\text{Image } \beta_z}$. With the help of the inclusions

$$\text{Image } \beta_z \subseteq \operatorname{ri}\operatorname{dom} H, \quad \operatorname{ri}\operatorname{dom} H \subseteq \overline{\text{Image } \beta_z} ,$$

it clearly follows that

$$\partial \operatorname{dom} H = \{R \in \mathbb{R}^N \,:\, R = \lim_{n\to\infty} \beta_z(z^n),\, |z^n| \to +\infty\}$$

□

Lemma 3.2. *Define a subset of* $\mathbb{R}^{N+1}$ *via* $\mathcal{C}^0 := \{\xi \in \mathbb{R}^N \, : \, \xi \cdot (R,\, 1) \leq 0$ *for all* $R \in \operatorname{dom} H\}$. *Then, there is a constant* $c > 0$ *such that* $\operatorname{dist}(R, \partial \operatorname{dom} H) \leq c \, \sup_{\xi \in \mathcal{C}^0, \, |\xi|=1} (R, 1) \cdot \xi$.

Proof. Theorem 13.1 of [Roc70] shows that $R \in \partial \operatorname{dom} H$ if and only if there is $q \in \mathbb{R}^N$, $q \neq 0$ such that $R \cdot q = \delta(q)$, where δ is the support function of $\operatorname{dom} H$. Thus $R \in \partial \operatorname{dom} H$ if and only if there is $\xi \in \{(X',\, X_{N+1}) \in \mathbb{R}^N \times \mathbb{R} \, : \, X_{N+1} = -\delta(X')\}$ such that $(R,\, 1) \cdot \xi = 0$.

On the other hand, $\xi \in \mathbb{R}^{N+1}$ belongs to $\partial \mathcal{C}^0$ if and only if $(R,\, 1) \cdot \xi = 0$ for one $R \in \overline{\operatorname{dom} H}$. □

4 Existence of weak solutions in the case of non smooth data

In this section we prove the Theorem 2.1. In the weak setting, we require some equi-coercivity for the diffusion potential Φ (see (16)) in the D variable.

The system is a *doubly non-linear parabolic system* with no degeneracy in ellipticity. The most general results on weak solvability are to find in [AL83], [Alt12] for the case that the elliptic operator satisfies some coercivity on a complete Sobolev norm.

4.1 The a priori estimates

The 'energy' estimate. Recall the energy identity (26). Thus, since we assume (16)

$$\begin{aligned} &\int_\Omega \varrho_0(x)\, (H(\beta_z(q^0(t))) - H_0) + \int_0^T \int_\Omega \{c_0 \, \Phi^0(\nabla q,\, Aq) + (\Phi^0)^*(-J,\, -r)\} \\ &+ \int_{S_T} \{c_0 \, \Phi^{\Gamma,0}(A^\Gamma q) + (\Phi^{\Gamma,0})^*(-r^\Gamma)\} \leq \int_\Omega \varrho_0 \, (H(\beta_z(q^0)) + H_0) \, . \end{aligned} \tag{31}$$

Here, $H_0 := \inf_{\mathbb{R}^N} H$.

The mean–value estimate. Since we employ flux boundary conditions on $\partial\Omega$, the control on the mean values $\int_\Omega q(t, x)\, dx$ does not follow from the control on the spatial gradient. We exploit here a method a little less general, but far more simple than the ideas first developed in [DDGG17] to prove the following Lemma.

Lemma 4.1. *Let* $q \in \widetilde{W}^{1,0}_{\Phi^0, \Phi^{\Gamma,0}}(Q_T;\, \mathbb{R}^N)$. *Let* $\varrho_0 \, : \, \Omega \to \mathbb{R}_+$ *be measurable, bounded and such that* $\int_\Omega \varrho_0(x)\, dx > 0$. *Assume that* $C_0 := \int_\Omega \varrho_0(x)\, H(\beta_z(q^0(x)))\, dx < +\infty$, *and that the point* $R_0 := \fint_\Omega \varrho_0(x)\, \beta_z(q^0(x))\, dx$ *belongs to the interior of* $\operatorname{dom} H$. *In this case, we denote* $M_0 := \operatorname{dist}(R^0,\, \partial \operatorname{dom} H) > 0$.

If q *satisfies the equations* (1), (2)*, there is a constant* $C(\Omega,\, M_0,\, C_0)$ *and a superlinear, non decreasing function* $\psi \in C([0, +\infty[)$ *such that*

$$\begin{aligned} &\int_0^T \psi(\|q(t)\|_{L^1(\Omega;\, \mathbb{R}^N)})\, dt \\ &\qquad \leq C \left(T + \|\nabla q\|_{L^{2,1}(Q;\, \mathbb{R}^{N\times 3})} + \int_Q \Phi^0(\nabla q,\, Aq) + \int_S \Phi^{\Gamma,0}(A^\Gamma q) \right) . \end{aligned}$$

Proof. We integrate the equations (1) over Q_t, which results for $t \in]0,T[$ arbitrary in

$$\fint_\Omega R_\ell(x,\, q(t)) = \fint_\Omega R(x,\, q^0) \\ + \sum_{\alpha=1}^{s} \int_0^t \bar{r}_\alpha(u)\, du\, A^\alpha_\ell + c_\Omega \sum_{\alpha=1}^{s^\Gamma} \int_0^t \bar{r}^\Gamma_\alpha(u)\, du\, A^{\Gamma,\alpha}_\ell \,. \tag{32}$$

Here we introduced $\bar{r}_\alpha(t) = \fint_\Omega \bar{r}_\alpha$, $\bar{r}^\Gamma_\alpha(t) := \fint_\Gamma r^\Gamma_\alpha(t)$, and $c_\Omega := \frac{|\Gamma|}{|\Omega|}$.

Define the convex cone $\mathcal{C}$ generated in $\mathbb{R}^{N+1}$ by the domain of H

$$\mathcal{C} := \{\lambda\,(R,\, 1) \in \mathbb{R}^{N+1} \,:\, R \in \operatorname{dom} H,\, \lambda \geq 0\}\,.$$

For $(t,x) \in Q$, we further define $\rho(t,x) := \varrho_0(x)\,(\beta_z(q(t,x)),\, 1)$. Then, ρ is obviously a mapping from Q into $\mathcal{C}$. For $t \in [0,T]$, we define

$$\bar{\rho}(t) = \fint_\Omega \rho(t,x)\, dx \in \mathcal{C}\,,$$

where the latter inclusion follows from the convexity of $\mathcal{C}$. Further, we define the subgraph of $-\beta$ via

$$\operatorname{sub}(-\beta) := \{(X',\, X_{N+1}) \in \mathbb{R}^N \times \mathbb{R} \,:\, X_{N+1} \leq -\beta(X')\}\,.$$

which is obviously a convex set. For $(t,x) \in Q$ we define $\mu(t,x) := (q(t,x),\, -\beta(q(t,x)))$. Then, μ is a mapping into the hyper surface

$$\partial \operatorname{sub}(-\beta) := \{(X',\, X_{N+1}) \in \mathbb{R}^N \times \mathbb{R} \,:\, X_{N+1} = -\beta(X')\}\,.$$

Let $d\nu$ be the positive measure $\varrho_0(x)\, dx$. The convexity of the set $\operatorname{sub}(-\beta)$ again guaranties that

$$\bar{\mu}^\nu(t) := \fint_\Omega \mu(t,x)\, d\nu(x)$$

is a mapping for $[0,T]$ into $\operatorname{sub}(-\beta)$. Now, it is simple to compute that

$$\rho(t,x) \cdot \mu(t,x) = \varrho_0(x)\,(\beta_z(q(t,x)) \cdot q(t,x) - \beta(q(t,x))) = \varrho_0(x)\, H(\beta_z(q(t,x)))\,.$$

Thus $\fint_\Omega \rho(t,x) \cdot \mu(t,x) = \fint_\Omega \varrho_0(x)\, H(\beta_z(q(t,x)))$.

Owing to the Poincaré inequality, there is a constant c_p depending on the measure ν and on Ω such that for all $t \in]0,T[$

$$\int_\Omega |\mu(t,x) - \bar{\mu}^\nu(t)|\, dx \leq c_p \int_\Omega |\nabla \mu(t,x)|\, dx\,.$$

By the definition of μ

$$\int_\Omega |\mu(t,x) - \bar{\mu}^\nu(t)|\, dx \leq c_p\,(1 + \|\beta_z\|_{L^\infty(\mathbb{R}^N)})\, \|\nabla q(t)\|_{L^1(\Omega)} =: d_0(t)\,.$$

Expressing now $\fint_\Omega \rho(t,x) \cdot \mu(t,x) = \bar{\rho}(t) \cdot \bar{\mu}^\nu(t) + \fint_\Omega \rho(t,x) \cdot (\mu(t,x) - \bar{\mu}^\nu(t))$, we obtain the bound

$$\left|\bar{\rho}(t) \cdot \bar{\mu}^\nu(t) - \fint_\Omega \varrho_0(x)\, H(\beta_z(q(t,x)))\right| \leq \|\rho\|_{L^\infty(Q)}\, d_0(t) \\ \leq \|\varrho_0\|_{L^\infty(\Omega)}\,(1 + \|\beta_z\|_{L^\infty(\mathbb{R}^N)})\, d_0(t)\,. \tag{33}$$

Let e^{N+1} be the last standard basis vector in $\mathbb{R}^{N+1}$. Next, we show that the point

$$\Lambda(t) := \bar{\mu}^{\nu}(t) - \frac{1}{\fint_{\Omega} \varrho_0(x)\, dx} \fint_{\Omega} \varrho_0(x)\, H(\beta_z(q(t,x)))\, e^{N+1}$$

belongs for all $t \in]0,T[$ to the polar cone $\mathcal{C}^0$ of $\mathcal{C}$ defined via

$$\mathcal{C}^0 := \{Y \in \mathbb{R}^{N+1} \,:\, Y \cdot X \leq 0 \text{ for all } X \in \mathcal{C}\}\,.$$

Indeed, we must show that for all X of the form $X = \lambda\,(R,\,1)$ with $\lambda \geq 0$ and $R \in \operatorname{dom} H$ that

$$\begin{aligned} 0 &\geq \Lambda(t) \cdot X \\ &= \lambda \left(\bar{q}^{\nu}(t) \cdot_N R - \fint_{\Omega} \beta(q(t,x))\, d\nu(x) - \frac{1}{\fint_{\Omega} \varrho_0(x)\, dx} \fint_{\Omega} \varrho_0(x)\, H(\beta_z(q(t,x)))\right)\,. \end{aligned}$$

Here, $\bar{q}^{\nu}(t) = \fint_{\Omega} q(t,x)\, d\nu(x)$. Obviously

$$\begin{aligned} &\fint_{\Omega} \beta(q(t,x))\, d\nu(x) + \frac{1}{\fint_{\Omega} \varrho_0(x)\, dx} \fint_{\Omega} \varrho_0(x)\, H(\beta_z(q(t,x)))) \\ &= \fint_{\Omega} \{\beta(q(t,x)) + H(\beta_z(q(t,x)))\}\, d\nu(x) = \fint_{\Omega} \sup_{R \in \operatorname{dom} H} q(t,x) \cdot R\, d\nu(x)\,. \end{aligned}$$

Thus, $\Lambda(t) \in \mathcal{C}^0$ for all t. Thanks to the inequality (33), and the fact that $\bar{\rho}(t) \in \mathcal{C}$, it follows that

$$-\bar{\rho}(t) \cdot \Lambda(t) = \left|\bar{\rho}(t) \cdot \bar{\mu}^{\nu}(t) - \fint_{\Omega} \varrho_0(x)\, H(\beta_z(q(t,x)))\right| \leq c\, d_0(t)\,.$$

For such $t \in [0,T]$ such that $|\bar{\mu}^{\nu}(t)| > 0$, we now divide this inequality by $|\bar{\mu}^{\nu}(t)|$, and introduce $\eta(t) := \Lambda(t)/|\bar{\mu}^{\nu}(t)|$. Note that $\eta(t) \in \mathcal{C}^0$ by the definition of a cone. Moreover, since we know from the energy inequality that $\fint_{\Omega} \varrho_0(x)\, H(\beta_z(q(t,x))) \leq C_0$, we can show that

$$\begin{aligned} |\eta(t)| &\geq 1 - \frac{\left|\fint_{\Omega} \varrho_0(x)\, H(\beta_z(q(t,x)))\right|}{|\bar{\mu}^{\nu}(t)|} \\ &\geq \frac{1}{2} \text{ for all } t \,:\, |\bar{\mu}^{\nu}(t)| > 2C_0\,. \end{aligned}$$

We now have obtained

$$-\bar{\rho}(t) \cdot \eta(t) \leq c\, \frac{d_0(t)}{|\bar{\mu}^{\nu}(t)|} \text{ for all } t \,:\, |\bar{\mu}^{\nu}(t)| > 2C_0\,. \tag{34}$$

We next introduce $\bar{\rho}^0 := \fint_{\Omega} \varrho_0(x)\, (\beta_z(q^0(x)),\, 1)\, dx$. On the footing of the identity (32), we verify the identity

$$\bar{\rho}(t) = \bar{\rho}^0 + \sum_{\alpha=1}^{s} \int_0^t \bar{r}_{\alpha}\, (A^{\alpha}, 0) + \sum_{\alpha=1}^{s^{\Gamma}} \int_0^t \bar{r}_{\alpha}^{\Gamma}\, (A^{\Gamma,\alpha}, 0)\,.$$

Thus, inserting the latter into (34)

$$\begin{aligned} -\bar{\rho}^0 \cdot \eta(t) &\leq c\, \frac{d_0(t)}{|\bar{\mu}^{\nu}(t)|} + \left[\int_0^t \bar{r}_{\alpha}\, (A^{\alpha}, 0) + c_{\Omega}\, \bar{r}_{\alpha}^{\Gamma}(t)\, (A^{\Gamma,\alpha}, 0)\right] \cdot \eta(t) \\ &= c\, \frac{d_0(t)}{|\bar{\mu}^{\nu}(t)|} + \frac{1}{|\bar{\mu}^{\nu}(t)|} \left[\int_0^t \bar{r}_{\alpha}\, A^{\alpha} + c_{\Omega} \int_0^t \bar{r}_{\alpha}^{\Gamma}\, A^{\Gamma,\alpha}\right] \cdot \bar{q}^{\nu}(t)\,. \end{aligned}$$

Due to the *a priori* bounds, $|\int_0^t \bar{r}_\alpha| + |\int_0^t \bar{r}^\Gamma_\alpha| \leq C_0$, and moreover to the bound $|A\bar{q}^\nu(t)| \leq \frac{\|\varrho_0\|_{L^\infty(\Omega)}}{\fint_\Omega \varrho_0(x)\,dx} |A\bar{q}(t)|$, it follows that

$$-\bar{\rho}^0 \cdot \eta(t) \leq \tilde{c}\, \tfrac{1}{|\bar{\mu}^\nu(t)|}\, (d_0(t) + |A\bar{q}(t)| + |A^\Gamma\bar{q}(t)|)\,.$$

Thus, for all t such that $|\bar{\mu}(t)| \geq 2C_0$

$$\inf_{\xi\in\mathcal{C}^0,\,|\xi|\geq\frac{1}{2}} |\bar{\rho}^0\cdot\xi| \leq \tilde{c}\,\frac{d_0(t) + |A\bar{q}(t)| + |A^\Gamma\bar{q}(t)|}{|\bar{\mu}^\nu(t)|}\,.$$

The characterisation of Lemma 3.2 yields

$$\mathrm{dist}(\bar{\rho}^0,\, \partial\mathcal{C}) \leq c\,\frac{d_0(t) + |A\bar{q}(t)| + |A^\Gamma\bar{q}(t)|}{|\bar{\mu}^\nu(t)|}\,.$$

and finally

$$\mathrm{dist}\left(\fint_\Omega R(x,\, q^0(x)),\, \partial\,\mathrm{dom}\, H\right) \leq c\,\frac{d_0(t) + |A\bar{q}(t)| + |A^\Gamma\bar{q}(t)|}{|\bar{\mu}^\nu(t)|}\,.$$

We now see that the set where $|\bar{\mu}^\nu(t)| > M_0^{-1}\,c\,(d_0(t) + |A\bar{q}(t)| + |A^\Gamma\bar{q}(t)|)$ and $|\bar{\mu}^\nu(t)| > 2C_0$ has measure zero. Thus

$$|\bar{\mu}^\nu(t)| \leq \max\{2C_0,\, M_0^{-1}\,c\,(d_0(t) + |A\bar{q}(t)| + |A^\Gamma\bar{q}(t)|)\}\,.$$

We know from the energy estimate that d_0 is bounded in $L^2(0,T)$. Moreover, defining

$$\phi(s) := \inf_{|D|+|D^{\mathsf{R}}|\geq s} \Phi^0(D,\, D^{\mathsf{R}}),\ \phi^\Gamma(s) := \inf_{|D^{\Gamma,\mathsf{R}}|\geq s} \Phi^{\Gamma,0}(D^{\mathsf{R}})$$

we easily show that $\int_0^T \{\phi(\|Aq(s)\|_{L^1(\Omega)}) + \phi^\Gamma(\|A^\Gamma q(s)\|_{L^1(\Gamma)})\}\,ds \leq C_0$ for q satisfying the energy identity. Thus, for

$$\psi(s) := \max\{2C_0,\, \min\{s^2,\, \phi(s),\, \phi^\Gamma(s)\}\}\,,$$

we see that $\int_0^T \psi(|\bar{\mu}^\nu(t)|)\,dt$ is bounded by the claimed quantity. Thus, since

$$\|q(t)\|_{L^1(\Omega;\,\mathbb{R}^N)} \leq c_p\,(\|\nabla q(t)\|_{L^1(\Omega;\,\mathbb{R}^{N\times 3})} + |\bar{q}^\nu(t)|)\,,$$

The claim follows. □

4.2 The existence procedure

The section is devoted to the proof of Theorem 2.1.

For $n\in\mathbb{N}$, we choose $\psi^1,\, \psi^2, \ldots, \psi^n \in W^{1,\infty}(\Omega;\, \mathbb{R}^N)$ such that $\bigcup_{n=1}^\infty$ span $\{\psi^1,\, \psi^2, \ldots, \psi^n\}$ is a dense subset of $W^{1,2}(\Omega;\, \mathbb{R}^N)$.

We look for approximate solutions $q^n \in C^1([0,T];\ \text{span}\{\psi^1,\, \psi^2, \ldots, \psi^n\})$ of the form

$$q^n(t,x) = \sum_{k=1}^n a_k^{(n)}(t)\,\psi^k(x)\,.$$

We impose as an additional condition on the chosen approximation that $\psi \equiv e^i \in \text{span}\{\psi^1, \ldots, \psi^n\}$ for all considered $n \in \mathbb{N}$ and $i = 1, \ldots, N$.

The approximate problem is a system of n non-linear equations given by

$$\int_\Omega (R_z(x,\, q^n) + \tfrac{1}{n}\,\mathsf{Id})\, \partial_t q^n \cdot \psi^k - \int_\Omega J(t, x,\, q^n,\, \nabla q^n) \cdot \nabla \psi^k \qquad (35)$$
$$- \int_\Omega f(t, x,\, q^n,\, \nabla q^n) \cdot A\psi^k - \int_{\partial\Omega} f^\Gamma(t, x,\, q^n) \cdot A^\Gamma \psi^k = 0$$
$$q^n(0) = \sum_{k=1}^n a_k^0\, \psi_k \qquad (36)$$

Here, the numbers $a_1^0,\, a_2^0, \ldots$ are chosen such that $\sum_{k=1}^n a_k^0\, \psi_k \to q^0$ strongly in $L^1(\Omega;\, \mathbb{R}^N)$.

The equations (36) form a non-linear system of n ordinary differential equations, that we re express in the form

$$A^n(a^{(n)})\, \partial_t a^{(n)} = F(a^{(n)}), \qquad a^{(n)}(0) = (a_1^0, \ldots, a_n^0) \qquad (37)$$
$$A^n_{k,\ell}(a^{(n)}) := \int_\Omega \sum_{i,j=1}^N (R_{i,z_j}(x,\, q^n) + \tfrac{1}{n}\,\delta_{i,j})\, \psi_j^\ell\, \psi_i^k$$
$$F_k(a^{(n)}) := \int_\Omega \{J(t, x,\, q^n,\, \nabla q^n) \cdot \nabla \psi^k + f(t, x,\, q^n,\, \nabla q^n) \cdot A\psi^k\} \qquad (38)$$
$$\int_{\partial\Omega} f^\Gamma(t, x,\, q^n) \cdot A^\Gamma \psi^k\,.$$

Note that the matrix A^n is strictly positive and invertible. Therefore we directly obtain the global solvability of the approximate system.

For $t \in [0,\, T[$, we can multiply the equations (36) with $a_k^{(n)}(t)$, sum over $k = 1, \ldots, n$, and we obtain the identity

$$\int_\Omega [\partial_t R(x,\, q^n) \cdot q^n + \tfrac{1}{2n}\, |q^n|^2] - \int_\Omega J(t, x,\, q^n,\, \nabla q^n) \cdot \nabla q^n$$
$$+ \int_\Omega f(x,\, q^n,\, \nabla q^n) \cdot Aq^n + \int_{\partial\Omega} f^\Gamma(t, x,\, q^n) \cdot A^\Gamma q_n = 0$$

Thus, for all $t \in [0,\, T[$ (cp. the Paragraph on the energy estimate)

$$\int_\Omega [\varrho_0\, H(\beta_z(q^n)) + \tfrac{1}{2n}\, |q^n(t)|^2] - \int_0^t \int_\Omega \{J(t, x,\, q^n,\, \nabla q^n) \cdot \nabla q^n$$
$$+ \int_0^t \int_\Omega f(t, x,\, q^n,\, \nabla q^n) \cdot Aq^n + \int_{S_t} f^\Gamma(t, x,\, q^n) \cdot A^\Gamma q_n$$
$$= \int_\Omega [\varrho_0\, (H(\beta_z(q^{0,n})) + \tfrac{1}{2n}\, |q^n(0)|^2]\,. \qquad (39)$$

We obtain that

$$\frac{1}{\sqrt{n}}\, \|q^n\|_{L^{\infty,2}(Q_T)} \leq C_0, \qquad [q^n]_{\widetilde{W}^{1,0}_{\Phi^0,\, \Phi^0,\Gamma}(Q;\, \mathbb{R}^N)} \leq C_0\,.$$

For $i = 1, \ldots, N$, the choice $\psi \equiv e^i \in \text{span}\{\psi^1, \ldots, \psi^n\}$ as testfunction in (36) yields

$$\partial_t \int_\Omega (R_i(x,\, q^n) + \tfrac{1}{n}\, q_i^n) = \sum_{\alpha=1}^{s} \int_\Omega r_{n,\alpha}(t)\, A_i^\alpha + \sum_{\alpha=1}^{s^\Gamma} \int_{\partial\Omega} r_{n,\alpha}^\Gamma(t)\, A_i^{\Gamma,\alpha} \text{ for } i = 1, \ldots, N\,.$$

For $i = 1, \ldots, N$, it therefore follows for all $t \in]0, T[$ that

$$\begin{aligned}\int_\Omega R_i(x,\, q^n(t)) - \int_\Omega (R_i(x,\, q^{0,n}) + \tfrac{1}{n}\,(q_i^{0,n} - q_i^n(t)))\\ = \sum_{\alpha=1}^{s} \left(\int_{Q_t} r_{n,\alpha}\right) A_i^\alpha + \sum_{\alpha=1}^{s^\Gamma} \left(\int_{S_t} r_{n,\alpha}^\Gamma\right) A_i^{\Gamma,\alpha}\,.\end{aligned}$$

We define $R^{0,n}(t) := \fint_\Omega (R_i(x,\, q^{0,n}) + \frac{1}{n}\,(q_i^{0,n} - q_i^n(t)))$.

Next we make use of the bound

$$\tfrac{1}{n}\, \|q^n(t)\|_{L^1(\Omega;\,\mathbb{R}^N)} \leq (\tfrac{1}{n}\, \|q^n(t)\|^2_{L^2(\Omega;\,\mathbb{R}^N)})^{1/2}\, \tfrac{1}{\sqrt{n}}\, \lambda_3(\Omega)^{1/2} \leq C_0\, n^{-\frac{1}{2}}\,,$$

and we define $R^0 := \fint_\Omega R(x,\, q^0)$, to see that

$$|R^{0,n}(t) - R^0| \leq c \int_\Omega \varrho_0\, |\beta_z(x,\, q^{0,n}) - \beta_z q^0| + C_0\, n^{-\frac{1}{2}} =: c_n\,.$$

By appropriate choice of the sequence $q^{0,n}$, it is possible to assume that $c_n \to 0$. Thus, there is $M_0 > 0$ and $n_0 \in \mathbb{N}$ such that for all $t \in [0, T]$

$$\operatorname{dist}(R^{0,n},\, \partial \operatorname{dom} H) \geq M_0 > 0\,.$$

We apply the Lemma 4.1 on mean–value control, and this yields $\int_0^T \psi(\|q^n(t)\|_{L^1(\Omega;\,\mathbb{R}^N)})\, dt \leq C_0$. Making use of the Sobolev embedding $W^{1,1} \hookrightarrow L^{\frac{3}{2}}$, this implies that q^n satisfies a uniform bound $\|q^n\|_{L_\psi L^{\frac{3}{2}}(Q_T;\,\mathbb{R}^N)}$. We can next extract weakly convergence subsequences:

$$\begin{aligned} q^n &\to q \text{ weakly in } L^1(Q;\, \mathbb{R}^N)\\ \nabla q^n &\to \nabla q \text{ weakly in } L^1(Q;\, \mathbb{R}^{N\times 3})\\ J^n &\to J \text{ weakly in } L^1(Q;\, \mathbb{R}^{N\times 3})\\ R(x,\, q^n) &\to \bar{R} \text{ weakly in } L^2(Q;\, \mathbb{R}^N)\end{aligned}$$

Moreover

$$\begin{aligned} r^n &\to r \text{ weakly in } L^1(Q;\, \mathbb{R}^s)\\ r^{\Gamma,n} &\to r^\Gamma \text{ weakly in } L^1(S;\, \mathbb{R}^{s^\Gamma})\,.\end{aligned}$$

In order to show the compactness of the solution vector we resort to a typical inequality. Let $\nu \in \mathcal{M}^+(\Omega)$ be a positive measure which is absolutely continuous with respect to λ_3. For all $\delta > 0$, there are $C(\delta) > 0$ and $m(\delta) \in \mathbb{N}$ such that

$$\begin{aligned}\|\beta_z(u^1) - \beta_z(u^2)\|_{L^1(\Omega;\,\mathbb{R}^N,\,d\nu)} \leq &\delta \sum_{i=1,2} \|u^1\|_{W^{1,1}(\Omega;\,\mathbb{R}^N)}\\ &+ C(\delta) \sum_{j=1}^{m(\delta)} \left|\int_\Omega \psi^j \cdot (\beta_z(u^1) - \beta_z(u^2))\, d\nu\right|\end{aligned}$$

for all $u^i \in W^{1,1}(\Omega;\, \mathbb{R}^N)$. If we now choose $d\mu = \varrho_0(x)\, dx$, $u^1 = q^n(t)$ and $u^2 = q^{n+p}(t)$, then integration over $[0, T]$ yields

$$\begin{aligned}\|\varrho_0\, (\beta_z(q^n) - \beta_z(q^{n+p}))\|_{L^1(Q;\,\mathbb{R}^N)} &\leq C_0\, \delta \\ &+ C(\delta) \sum_{j=1}^{m(\delta)} \int_0^T \left| \int_\Omega \psi^j \cdot \varrho_0\, (\beta_z(q^n) - \beta_z(q^{n+p})) \right| .\end{aligned}$$

It is readily shown that $R_z(x,\, q^n)$ is a Cauchy sequence in $L^1(Q;\, \mathbb{R}^N)$, which yields the strong convergence. Since R_z is strictly positive definite on the support of ϱ_0, the pointwise convergence of q^n follows at least for a subsequence in almost all (t, x) such that $\varrho_0(x) > 0$. This shows that the weak limits satisfy $\bar{R} = R(x,\, q)$.

Now we can pass to the limit in the differential equations, and we see that for all testfunctions $\phi \in C^1_c(Q;\, \mathbb{R}^N)$

$$- \int_Q R(x,\, q) \cdot \partial_t \phi - \int_Q J \cdot \nabla\phi - \int_S \sum_{\alpha=1}^{s^\Gamma} r^\Gamma_\alpha\, A^{\Gamma,\alpha} \phi = \int_Q \sum_{\alpha=1}^{s} r_\alpha\, A^\alpha \phi\, . \tag{40}$$

Here there is a little bit of technical work done just here after the proof to show for almost all $t \in [0, T]$ the validity of the energy identity

$$\int_\Omega [H(R(x, q)) - H(R^0)] = \int_{Q_t} \{J \cdot \nabla q + r \cdot Aq\} + \int_{S_t} r^\Gamma \cdot A^\Gamma q\, . \tag{41}$$

We recall (39), and it follows for almost all t that

$$\begin{aligned}\limsup_{n\to\infty} \Bigg(- \int_{Q_t} &\{J(t, x,\, q^n,\, \nabla q^n) \cdot \nabla q^n + r(t, x,\, q^n,\, \nabla q^n) \cdot Aq^n\} \\ - \int_{S_t} r^\Gamma(t, x,\, q^n) \cdot A^\Gamma q^n \Bigg) &= - \int_{Q_t} \{J \cdot \nabla q + r \cdot Aq\} - \int_{S_t} r^\Gamma \cdot A^\Gamma q\, .\end{aligned}$$

This is sufficient to show that

$$\limsup_{n\to\infty} - \int_{Q_t} (J(t, x,\, q^n,\, \nabla q^n,\, Aq^n) - J(t, x,\, q_n,\, \nabla q,\, Aq^n)) \cdot \nabla(q^n - q) \leq 0\, .$$

If the set $\{x \in \Omega\, :\, \varrho_0(x) = 0\}$ has not measure zero, we use the additional hypothesis that the dependence $z \mapsto J(\cdot, z)$ is actually a dependence $z \mapsto \Psi_D(\cdot, R(x,\, z))$ to show that

$$\lim_{n\to\infty} J(t, x,\, q_n,\, \nabla q,\, Aq^n) = \lim_{n\to\infty} J(t, x,\, q,\, \nabla q,\, Aq^n)$$

.

Thus, $\nabla q^n \to \nabla q$ pointwise almost everywhere, and strongly in $L^1(Q;\, \mathbb{R}^{N\times 3})$. It follows that $J = J(t, x,\, q, \nabla q)$, $r = r(t, x,\, q, \nabla q)$. Finally, we obtain the strong convergence of q^n on S_T, the identity $r^\Gamma = r^\Gamma(t, x,\, q)$, and the proof of Theorem 2.1 is finished.

The technical statement. We give the proof that (40) implies (41). At first, it is standard to show that for $\phi \in C^1_c(\overline{\Omega};\, \mathbb{R}^N)$, the function $t \mapsto \int_\Omega R(x,\, q(t, x)) \cdot \phi(x)\, dx$ is weakly differentiable, and that

$$\frac{d}{dt} \int_\Omega R(x,\, q(t)) \cdot \phi = \int_\Omega \{J(t) \cdot \nabla\phi + f(t) \cdot \phi\} + \int_{\partial\Omega} f^\Gamma(t) \cdot \phi\, .$$

Therefore, for all $t_1 < t_2$

$$\int_\Omega \frac{R(x,\, q(t_2)) - R(x,\, q(t_1))}{t_2 - t_1} \cdot \phi + \int_\Omega \left\{ \fint_{t_1}^{t_2} -J(t)\, dt \cdot \nabla\phi + \fint_{t_1}^{t_2} -r(t)\, dt \cdot A\phi \right\} + \int_\Gamma \fint_{t_1}^{t_2} -r^\Gamma(t)\, dt \cdot A^\Gamma \phi = 0\,. \tag{42}$$

Owing to the convexity of Φ^0 and $(\Phi^0)^*$

$$\begin{aligned} &\pm \left\{ \fint_{t_1}^{t_2} -J(t)\, dt \cdot \nabla\phi + \fint_{t_1}^{t_2} -r(t)\, dt \cdot A\phi \right\} \\ &\leq (\Phi^0)^* \left(\fint_{t_1}^{t_2} -J(t)\, dt,\, \fint_{t_1}^{t_2} -r(t)\, dt \right) + \Phi^0(\nabla\phi,\, A\phi) + \Phi^0(-\nabla\phi,\, -A\phi) \\ &\leq \fint_{t_1}^{t_2} (\Phi^0)^*(-J(t),\, -r(t))\, dt + \Phi^0(\nabla\phi,\, A\phi) + \Phi^0(-\nabla\phi,\, -A\phi)\,. \end{aligned}$$

Thus, owing to the assumption (17)

$$\begin{aligned} &\int_\Omega \left\{ \fint_{t_1}^{t_2} -J(t)\, dt \cdot \nabla\phi + \fint_{t_1}^{t_2} -r(t)\, dt \cdot A\phi \right\} \\ &\quad \leq \fint_{t_1}^{t_2} \int_\Omega (\Phi^0)^*(-J,\, -r) + (1 + K_0) \int_\Omega \Phi^0(\nabla\phi,\, A\phi) + K_1\, |\Omega|\,. \end{aligned}$$

Similarly, we show that

$$\begin{aligned} \int_{\partial\Omega} \left\{ \fint_{t_1}^{t_2} -r^\Gamma(t)\, dt \cdot A^\Gamma \phi \right\} &\leq \fint_{t_1}^{t_2} \int_{\partial\Omega} (\Phi^{\Gamma,0})^*(-r^\Gamma) \\ &\quad + (1 + K_0^\Gamma) \int_{\partial\Omega} \Phi^{\Gamma,0}(A^\Gamma\phi) + K_1^\Gamma\, |\Gamma|\,. \end{aligned}$$

Owing to standard approximation arguments, the identity (42) therefore extends by density to all $\phi \in W^{1,1}(\Omega;\, \mathbb{R}^N)$ such that $\Phi^0(\nabla\phi,\, A\phi) \in L^1(\Omega)$ and $\Phi^{\Gamma,0}(A^\Gamma\phi) \in L^1(\partial\Omega)$. Accepting first this point, for almost all $t \in]0, T[$, we obtain that

$$\begin{aligned} 0 = &\int_\Omega \frac{R(x,\, q(t_2)) - R(x,\, q(t_1))}{t_2 - t_1} \cdot q(t) \\ &+ \int_\Omega \left\{ \fint_{t_1}^{t_2} -J(t)\, dt \cdot \nabla q(t) + \fint_{t_1}^{t_2} -r(t)\, dt \cdot Aq(t) \right\} \\ &+ \int_\Gamma \left\{ \fint_{t_1}^{t_2} -r^\Gamma(t)\, dt \cdot A^\Gamma q(t) \right\}\,. \end{aligned} \tag{43}$$

For a function $u \in L^1(Q_T)$ and $t \leq T - h$ we extend $u(t)$ by zero to $\mathbb{R}^-$ and we denote $u_{\underline{h}}(t, x) = h^{-1} \int_{t-h}^{h} u(s,\, x)\, ds$ the lower Steklov averaging, and $u_h(t, x) := h^{-1} \int_t^{t+h} u(s,\, x)\, ds$ the upper Steklov averaging.

We next observe that there are $t_{1,h} \leq t_{2,h}$, $t_{2,h} - t_{1,h} \leq 2h$ such that $q_h(t) = q(t_{2,h})$ and $q_h(t-h) = q(t_{1,h})$. We compute

$$\begin{aligned} \partial_t (R(x,\, q_h))_{\underline{h}}(t) &= \frac{R(x,\, q_h(t)) - R(x,\, q(t-h))}{h} \\ &= \frac{R(x,\, q(t_{2,h})) - R(x,\, q(t_{1,h}))}{h}\,. \end{aligned}$$

DOI 10.20347/WIAS.PREPRINT.2455

Thus, making use of (43)

$$
\begin{aligned}
0 = &\int_\Omega \partial_t(R(x,\, q_h))_{\underline{h}} \cdot q(t) \\
&+ \frac{t_{2,h}-t_{1,h}}{h} \int_\Omega \left\{ \fint_{t_{1,h}}^{t_{2,h}} -J(t)\, d_t \cdot \nabla q(t) + \fint_{t_{1,h}}^{t_{2,h}} -r(t)\, dt \cdot Aq(t) \right\} \\
&+ \frac{t_{2,h}-t_{1,h}}{h} \int_\Gamma \{ \fint_{t_{1,h}}^{t_{2,h}} -r^\Gamma(t)\, dt \cdot A^\Gamma q(t) \} \,.
\end{aligned} \tag{44}
$$

Denote $f_h(t) := \int_\Omega \partial_t(R(x,\, q_h))_{\underline{h}} \cdot q(t)$. For all $h < a < b < T$, the standard properties of the Steklov averaging operator yield

$$
\begin{aligned}
\int_a^b f_h(t)\, dt &= \int_a^b \int_\Omega \partial_t(R(x,\, q_h))_{\underline{h}} \cdot q \\
&= \int_a^b \int_\Omega (\partial_t R(x,\, q_h))_{\underline{h}} \cdot q = \int_{a-h}^b \int_\Omega \partial_t R(x,\, q_h) \cdot q_h \\
&= \int_{a-h}^b \int_\Omega \varrho_0(x)\, \partial_t(\beta_z(q_h) \cdot q_h - \beta(q_z)) \\
&= \int_\Omega \varrho_0(x)\, (H(\beta_z(q_h(b))) - H(\beta_z(q_h(a-h)))) \,.
\end{aligned}
$$

Next we consider

$$
\begin{aligned}
g_h(t) :=& \int_\Omega \left\{ \fint_{t_{1,h}}^{t_{2,h}} -J(s)\, d_s \cdot \nabla q(t) + \fint_{t_{1,h}}^{t_{2,h}} -r(s)\, ds \cdot Aq(t) \right\} \\
&+ \int_\Gamma \left\{ \fint_{t_{1,h}}^{t_{2,h}} -r^\Gamma(s)\, ds \cdot A^\Gamma q(t) \right\} \,.
\end{aligned}
$$

The functions g_h satisfy the majoration

$$
\begin{aligned}
\pm g_h(t) \leq & \fint_{t_{1,h}}^{t_{2,h}} \int_\Omega (\Phi^0)^*(-J,\, -r) + (1+K_0) \int_\Omega \Phi^0(\nabla q,\, Aq) + K_1\, |\Omega| \\
&+ \fint_{t_{1,h}}^{t_{2,h}} \int_\Gamma (\Phi^{\Gamma,0})^*(-r^\Gamma) + (1+K_0^\Gamma) \int_\Gamma \Phi^{0,\Gamma}(A^\Gamma q) + K_1^\Gamma\, |\Gamma| \,.
\end{aligned}
$$

For $h \to 0$, the right-hand converges strongly in $L^1(0,T)$. Therefore, the functions g_h converge strongly in $L^1(0,T)$ for $h \to 0$ to their pointwise limit g defined via

$$
g = \int_\Omega \{-J(t) \cdot \nabla q(t) - r(t) \cdot Aq(t)\} + \int_\Gamma \{-r^\Gamma(t) \cdot A^\Gamma q(t)\} \,.
$$

It remains to integrate (44) over $]a,b[\subset]0,T[$. For h tending to zero, we obtain for almost all a, b that

$$
\begin{aligned}
&\int_\Omega \varrho_0(x)\, (H(\beta_z(q(b))) - H(\beta_z(q(a)))) \\
&= \int_a^b \int_\Omega \{-J(t) \cdot \nabla q(t) - r(t) \cdot Aq(t)\} + \int_a^b \int_{\partial\Omega} \{-r^\Gamma(t) \cdot A^\Gamma q(t)\} \,.
\end{aligned}
$$

5 More regularity in the case of strongly thermodynamic closure

The regularity analysis turns out particularly convenient in the case of a thermodynamically consistent closure. We here can obtain higher order estimates. Recall that a thermodynamic consistent closure in the sense of Wolfgang Dreyer means that the fluxes and densities are derived from potentials Φ and Φ^{Γ} that are functions of the driving forces D and D^{R} *only*.

The additional regularity Applying the same approximation scheme than in the Section 4, we multiply the equations (37) with $\partial_t a^{(n)}$, and it follows that

$$\int_{\Omega}(R_z(x,\, q^n) + \tfrac{1}{n}\,\mathsf{Id})\partial_t q^n \cdot \partial_t q^n \\ + \partial_t \left(\int_{\Omega} \Phi(\nabla q^n,\, Aq^n) + \int_{\Gamma} \Phi^{\Gamma}(A^{\Gamma} q^n)\right) = 0\,.$$

Time integration yields

$$\int_{Q_t} R_z(x,\, q^n)\partial_t q^n \cdot \partial_t q^n + \int_{\Omega} \Phi(\nabla q^n(t),\, Aq^n(t)) + \int_{\Gamma} \Phi^{\Gamma}(A^{\Gamma} q^n(t)) \\ = \int_{\Omega} \Phi(\nabla q^{0,n},\, Aq^{0,n}) + \int_{\Gamma} \Phi^{\Gamma}(A^{\Gamma} q^{0,n})\,.$$

Thus, if $\int_{\Omega} \Phi(\nabla q^0,\, Aq^0) + \int_{\Gamma} \Phi^{\Gamma}(A^{\Gamma} q^0) < +\infty$, we can choose an appropriate sequence $q^{0,n}$ such as to ensure that

$$\int_{Q_t} R_z(x,\, q^n)\partial_t q^n \cdot \partial_t q^n + \int_{\Omega} \Phi(\nabla q^n(t),\, Aq^n(t)) + \int_{\Gamma} \Phi^{\Gamma}(A^{\Gamma} q^n(t)) \leq C_0\,. \tag{45}$$

We note next that for $\ell = 1, \ldots, N$

$$|R_z(x,\, q^n)\partial_t q^n \cdot e^{\ell}| \leq (R_z(x,\, q^n)\partial_t q^n \cdot \partial_t q^n)^{\frac{1}{2}}\,(R_z(x,\, q^n)\, e^{\ell} \cdot e^{\ell})^{\frac{1}{2}} \\ \leq \|\varrho_0\|_{L^{\infty}(\Omega)}^{\frac{1}{2}}\, \|\beta_z\|_{L^{\infty}(\mathbb{R}^N)}^{\frac{1}{2}}\,(R_z(x,\, q^n)\partial_t q^n \cdot \partial_t q^n)^{\frac{1}{2}}\,.$$

Thus

$$\|\partial_t R(x,\, q^n)\|^2_{L^2(Q;\,\mathbb{R}^N)} \leq \|\varrho_0\|_{L^{\infty}(\Omega)}\, \|\beta_z\|_{L^{\infty}(\mathbb{R}^N)} \int_Q R_z(x,\, q^n)\partial_t q^n \cdot \partial_t q^n \leq C_0\,.$$

Thus, the limit $R(x,\, q)$ possesses a weak time–derivative in $L^2(Q;\, \mathbb{R}^N)$, and we can rely on the identity

$$\int_Q J(\nabla q,\, Aq) \cdot \nabla\phi = \int_Q (-f(q,\, \nabla q) + \partial_t R(x,\, q)) \cdot \phi - \int_S f^{\Gamma}(q) \cdot \phi\,,$$

for all $\phi \in C^1(\overline{Q_T};\, \mathbb{R}^N)$. Thus, for almost all $t \in [0, T]$, the vector $u := q(t)$ is a weak solution to the equations

$$-\operatorname{div}(\Phi_D(\nabla u,\, Au)) + \Phi_{D^{\mathsf{R}}}(\nabla u,\, Au)A = g \text{ in } \Omega \tag{46}$$

$$-\Phi_D(\nabla u,\, Au)\,\nu(x) + \Phi^{\Gamma}_{D^{\Gamma,\mathsf{R}}}(A^{\Gamma} u)\, A^{\Gamma} = 0 \text{ on } \partial\Omega\,. \tag{47}$$

Here, $g = \partial_t R(x,\, q)(t) \in L^2(\Omega;\, \mathbb{R}^N)$.

If the boundary of Ω is of class $\mathcal{C}^{1,1}$, the assumptions of Theorem 2.2 and classical results for quasilinear elliptic systems (resumed here below in Lemma 5.1) yield $\|D^2 q\|_{L^2(Q)} \leq C\, \|\partial_t R(q)\|_{L^2(Q)}$ and the proof of Theorem 2.2 is finished.

We at last show how to obtain the higher regularity in the quasilinear elliptic system. For simplicity, we present a formal proof of the *a priori* bound. This bound can rigorously be established by means of local flattening of the boundary $\partial\Omega$ and applying difference quotients.

Lemma 5.1. *Assume that $u \in W^{2,2}(\Omega;\, \mathbb{R}^N)$ satisfies* (46), (47)*. Denote $D^2\Phi$ is the Hessian of Φ as a field defined on $\mathbb{R}^{N\times 3} \times \mathbb{R}^s$, and assume that there is $\lambda_0 > 0$ such that*

$$D^2\Phi(X)\,(D,\, D^R)\cdot(D,\, D^R) \geq \lambda_0\, |D|^2 \text{ for all } X,\, (D,\, D^R) \in \mathbb{R}^{N\times 3}\times\mathbb{R}^s\,.$$

Assume moreover that there is a constant $\lambda_1 \in \mathbb{R}_+$ such that

$$|\Phi_{D,D^R}|_\infty + |\Phi_{D,D}|_\infty \leq \lambda_1\,.$$

Finally, assume that there is $\gamma > 1$ such that for all $J \in \mathbb{R}^{N\times 3}$ and $r \in \mathbb{R}^s$

$$(\Phi^0)^*(-J,\, -r) \geq c_0\, |r|^\gamma - c_1\,.$$

Then $\|D^2 u\|_{L^{\min\{\gamma,2\}}(\Omega)} \leq c\, \Big(1 + \|g\|^2_{L^2(\Omega)} + \int_\Omega (\Phi^0)^*(-J,\, -r)\Big)^{\frac{1}{\min\{\gamma,2\}}}$.

Proof. We start from the weak form of (46)

$$\begin{aligned}\int_\Omega \{\Phi_D(\nabla u,\, Au) \,:\, \nabla\zeta + \Phi_{D^R}(\nabla u,\, Au)\cdot A\zeta\}\qquad&\\ + \int_{\partial\Omega} \Phi^\Gamma_{D^{\Gamma,R}}(A^\Gamma u)\cdot A^\Gamma\zeta = \int_\Omega g\,\zeta\,.&\end{aligned}$$

for all $\zeta \in C^1(\overline{\Omega};\, \mathbb{R}^N)$. We choose testfunctions of the form $\zeta = \eta\, \sum_{\beta=1}^3 \tau_\beta(x)\, \partial_{x_\beta}\psi$. Here, $\eta \in C^1(\overline{\Omega})$ is a cut-off function, $\tau \in C^{0,1}(\Omega;\, \mathbb{R}^3)$ is a vector field assumed tangent at $\partial\Omega$ on the support of η, and $\Psi \in C^1(\overline{\Omega};\, \mathbb{R}^N)$ is arbitrary.

For $i = 1, \ldots, N$, we denote $w^\tau_i = \eta\, \sum_{\beta=1}^3 \tau_\beta(x)\, \partial_{x_\beta} u_i$. After obvious shifting, we see that w^τ satisfies the relations

$$\begin{aligned}\int_\Omega D^2\Phi(\nabla u,\, Au)(\nabla w^\tau,\, Aw^\tau)\cdot(\nabla\psi,\, A\psi) + \int_\Gamma \Phi_{D^{\Gamma,R},D^{\Gamma,R}}(A^\Gamma u)\, A^\Gamma w^\tau\cdot A^\Gamma\psi&\\ = -\int_\Omega g\cdot \operatorname{div}(\eta\,\tau\,\psi) - \int_{\partial\Omega} (\tau\cdot\nabla\nu)\cdot\Phi_D(\nabla u,\, Au)\cdot\psi\,\eta&\end{aligned}$$

Here, $D^2\Phi$ is the Hessian of Φ as a field defined on $\mathbb{R}^{N\times 3}\times\mathbb{R}^s$ with blocks $\Phi_{D,D}$, Φ_{D,D^R}, etc. Obviously

$$\left|\int_\Omega g\cdot\operatorname{div}(\eta\,\tau\,\psi)\right| \leq c\,\|\eta\,\tau\|_{C^1(\overline{\Omega})}\,\|g\|_{L^2}\,\|\psi\|_{W^{1,2}}\,.$$

We express

$$\begin{aligned}\Phi_D(\nabla u,\, Au) &= \Phi_D(\nabla u,\, Au) - \Phi_D(0)\\ &= \int_0^1 \Phi_{D,D}(\theta\,\nabla u,\, \theta\, Au)\, d\theta\, \nabla u + \int_0^1 \Phi_{D,D^R}(\theta\,\nabla u,\, \theta\, Au)\, d\theta\, Au\\ &=: \Phi^\theta_{D,D}\,\nabla u + \Phi^\theta_{D,D^R}\, Au\,.\end{aligned}$$

Therefore, since the vector $\tau \cdot \nabla \nu$ is tangent on $\partial\Omega$ and bounded by the curvatures, it follows that

$$\left|\int_{\partial\Omega} (\tau \cdot \nabla \nu) \cdot \Phi_D(\nabla u,\, Au) \cdot \psi\, \eta\right|$$
$$\leq \|\,\delta\, \nu\|_{L^\infty(\partial\Omega)}\, \lambda_1\, (\|\,\delta\, u\|_{L^2(\partial\Omega)} + \|Au\|_{L^2(\partial\Omega)})\, \|\psi\|_{L^2(\partial\Omega)}\,.$$

Here, δ is the tangential gradient on $\partial\Omega$. We clearly obtain an estimate

$$\lambda_0 \int_\Omega |\nabla w^\tau|^2 \leq c\, (\|g\|_{L^2} + \|u\|_{W^{1,2}} + \|\,\delta\, u\|_{L^2(\partial\Omega)})\, \|w^\tau\|_{W^{1,2}(\Omega)}\,.$$

Since $W^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega)$ compactly, there is for all $\epsilon > 0$ a constant c_ϵ such that $\|f\|_{L^2(\partial\Omega)} \leq \epsilon\, \|\nabla f\|_{L^2(\Omega)} + c_\epsilon\, \|f\|_{L^2(\Omega)}$. We decompose the tangential gradient $\delta\, u$ via $\delta\, u = \sum_{k=1,2} \tau^k \cdot \nabla u\, \tau^k$ where $\tau^1,\, \tau^2$ are chosen orthonormal on $\partial\Omega$. In the end we obtain

$$\sum_{k=1}^2 \int_\Omega |\nabla w^{\tau^k}|^2 \leq c\, \epsilon \sum_{k=1}^2 \int_\Omega |\nabla w^{\tau^k}|^2 + C_\epsilon\, (\|g\|_{L^2}^2 + \|u\|_{W^{1,2}}^2)\,.$$

In order to obtain a complete estimate, we next make use of the strong form (46). Performing the differentiation, it follows that

$$\begin{aligned} &- \sum_{j=1}^N \sum_{k,\ell=1}^3 D^2\Phi_{D_k, D_\ell^j}(\nabla u,\, Au)\, u^j_{x_k, x_\ell} \\ &= g + \Phi_{D, D^{\mathrm{R}}}(\nabla u,\, Au) A \nabla u - \Phi_{D^{\mathrm{R}}}(\nabla u,\, Au) A\,. \end{aligned} \tag{48}$$

We choose locally in Ω on orthonormal system $\{\tau^1,\, \tau^2,\, \nu\}$, so that the vectors τ^k and ν belong to $C^1(\overline{\Omega};\, \mathbb{R}^3)$ and $\{\tau^1,\, \tau^1\}$ span the tangential plane to $\partial\Omega$ in each point of the considered local subregion adjacent to the boundary.

Denoting $M_{i,j} := \sum_{k,\ell=1}^3 \nu_k,\, \nu_\ell\, D^2\Phi_{D^i_k, D^j_\ell}(\nabla u,\, Au)$, we see that M is symmetric and positive definite as an element of $\mathbb{R}^{N\times N}$ and that $\lambda_{\inf}(M) \geq \lambda_0$. We make use of the relation (48) to obtain an estimate of the type

$$|\nu\, u_{x,x}\, \nu| \leq C\, (|g| + |\nabla u| + |w^\tau_x| + |\Phi_{D^{\mathrm{R}}}(\nabla u,\, Au)|)\,.$$

If $(\Phi^0)^*(-J,\, -r) \geq c\, |r|^\gamma$ for $r \in \mathbb{R}^s$, we obtain finally that

$$\|\nu\, u_{x,x}\, \nu\|_{L^{\min\{\gamma,2\}}(\Omega)} \leq C\, \left(\|g\|_{L^2}^2 + \|u\|_{W^{1,2}}^2 + \int_\Omega (\Phi^0)^*(-J,\, -r)\right)^{\frac{1}{\min\{\gamma,2\}}}\,.$$

The claim follows. □

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