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Abstract

For the two-phase incompressible Navier–Stokes equations with surface tension, we derive an appropriate weak formulation incorporating a variational formulation using divergence-free test functions. We prove a consistency result to justify our definition and, under reasonable regularity assumptions, we reconstruct the pressure function from the weak formulation.

1 Introduction

We consider a two-phase flow of two incompressible Newtonian fluids. The isothermal flow in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, and on a finite time interval $[0, T]$ is described in Eulerian coordinates by a velocity field $v\colon \Omega \times [0, T] \to \mathbb{R}^n$ and a scalar pressure function $p\colon \Omega \times [0, T] \to \mathbb{R}$. For each time $t \in [0, T]$, a hypersurface $\Gamma(t)$ separates Ω into two disjoint subsets $\Omega^-(t)$ and $\Omega^+(t)$ of Ω, i.e., we have $\Omega = \Omega^-(t) \cup \Gamma(t) \cup \Omega^+(t)$ and $\Gamma(t) = \partial\Omega^-(t) \cap \Omega$. The regions $\Omega^-(t)$ and $\Omega^+(t)$ are referred to as bulk phases, and correspond to different phases of the fluid. Physically they are characterised by (constant) densities $0 < \beta_1 \leq \beta_2$ and corresponding viscosities $\mu(\beta_i) > 0$, $i = 1, 2$. For convenience, throughout this paper we will require that the interface is compactly contained in the fluid domain, that is, $\Gamma(t) \subset\subset \Omega$. In particular, the interface does not intersect the domain boundary, i.e., $\Gamma(t) \cap \partial\Omega = \emptyset$. This, in turn, means that $\Omega^-(t) \subset\subset \Omega$ and $\Gamma(t) = \partial\Omega^-(t) = \partial\Omega^-(t) \cap \partial\Omega^+(t)$.

Assuming the interface Γ to be sufficiently regular, and the velocity v and the pressure p to be sufficiently smooth functions on $\Omega \setminus \Gamma(t) = \Omega^-(t) \cup \Omega^+(t)$, such that the one-sided limits on $\Gamma(t)$ from $\Omega^\pm(t)$ exist, the flow is described by the following free-boundary problem

$$\beta_1\partial_t v + \beta_1(v \cdot \nabla)v - \mu(\beta_1)\Delta v + \nabla p = 0 \qquad \text{in } \Omega^-(t), \tag{1.1}$$

$$\beta_2\partial_t v + \beta_2(v \cdot \nabla)v - \mu(\beta_2)\Delta v + \nabla p = 0 \qquad \text{in } \Omega^+(t), \tag{1.2}$$

$$\operatorname{div}(v) = 0 \qquad \text{in } \Omega \setminus \Gamma(t), \tag{1.3}$$

$$[v] = 0 \qquad \text{on } \Gamma(t), \tag{1.4}$$

$$V = v \cdot \nu^- \qquad \text{on } \Gamma(t), \tag{1.5}$$

$$[T]\,\nu^- = -2\sigma_{\mathrm{st}}\kappa\nu^- \qquad \text{on } \Gamma(t), \tag{1.6}$$

$$v(\cdot, t) = 0 \qquad \text{on } \partial\Omega, \tag{1.7}$$

$$v(\cdot, 0) = v^{(i)} \qquad \text{in } \Omega \tag{1.8}$$

for every $t \in [0, T]$. The initial phases $\Omega^-(0) = \Omega^{-,(i)} \subset\subset \Omega$, $\Omega^+(0) = \Omega \setminus \overline{\Omega^{-,(i)}}$ and the initial position $\Gamma^{(i)} = \partial(\Omega^{-,(i)})$ of the interface, as well as the initial velocity $v^{(i)}\colon \Omega \to \mathbb{R}^n$, are given.

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The unknowns are the velocity $v(\cdot,t)\colon \Omega\setminus\Gamma(t)\to\mathbb{R}^n$, the pressure $p(\cdot,t)\colon\Omega\setminus\Gamma(t)\to\mathbb{R}$ and the interface (free-boundary) $\Gamma(t)$. Here and in the sequel, $[\,\cdot\,]$ stands for the jump across the interface $\Gamma(t)$ in the direction of the exterior unit-normal field $\nu^-(\cdot,t)$ of $\partial\Omega^-(t)$. For a given quantity f and $x\in\Gamma(t)$, this is, explicitly,

$$[f](x,t)=\lim_{\xi\searrow 0}\Big(f(x+\xi\nu^-(x,t),t)-f(x-\xi\nu^-(x,t),t)\Big).$$

By $V=V(\cdot,t)$ and $\kappa=\kappa(\cdot,t)$, we denote the normal velocity and the mean curvature of $\Gamma(t)$, for fixed t, both taken with respect to $\nu^-(\cdot,t)$. Moreover, in (1.6), $\sigma_{\mathrm{st}}>0$ denotes the surface-tension constant, and the stress tensor $T=T(v,p)$ is defined by

$$T(v(t),p(t))=\begin{cases}2\mu(\beta_1)Dv(t)-p(t)I & \text{in } \Omega^-(t),\\ 2\mu(\beta_2)Dv(t)-p(t)I & \text{in } \Omega^+(t).\end{cases}$$

The partial differential equations (1.1)–(1.3) are the incompressible Navier–Stokes equations. Equations (1.1) and (1.2) model the conservation of linear momentum and the incompressibility condition (1.3) corresponds to conservation of mass in each bulk phase. These partial differential equations in the bulk phases are coupled by the interface conditions (1.4)–(1.6): the velocity field is continuous across the interface $\Gamma(t)$ by (1.4). Due to (1.5), the interface $\Gamma(t)$ is transported purely by the bulk fluid flow. The interface condition (1.6) is (a dynamic version of) the Young–Laplace law relating the jump of the normal stress $[T]\,\nu^-$ to the mean curvature κ. The velocity boundary condition (1.7) is the no-slip condition at the boundary $\partial\Omega$ of the fluid domain Ω. With (1.8), we prescribe initial values $v^{(i)}\colon\Omega\to\mathbb{R}^n$ for the velocity.

The question of (unique) solvability of the free-boundary problem (1.1)–(1.8) and related systems has been studied by many authors: in the framework of Hölder spaces, Denisova and Solonnikov first studied the corresponding two-phase Stokes problem [1]. Later they proved well-posedness of (1.1)–(1.8) for appropriate initial data [2]. Existence results in the context of maximal L^r-regularity (so-called strong solutions), which are even real analytic for positive times, are due to Prüss and Simonett [3] and Köhne, Prüss and Wilke [4], and in a varifold context due to Plotnikov [5, 6] and the first author [7]. In general, the existence of weak solutions to (1.1)–(1.8) is an open problem, cf. [8, Section 2.2].

This paper summarizes the result of [9, Chapter 4] and is organised as follows: In **Section 2** we will introduce our notation and provide some preliminary results. In **Section 3** we will derive a weak notion of solutions which uses divergence-free test functions. This will lead to a weak formulation that does not incorporate the pressure function.

In the remainder of the paper we shall justify our approach and reconstruct a pressure function from the weak formulation: In **Section 4** we shall provide the functional-analytic background and introduce Sobolev spaces on time-dependent domains. In **Section 5**, under reasonable regularity assumptions, we will reconstruct the pressure function from the weak formulation.

2 Notation and Preliminaries

Let $U\subset\mathbb{R}^d$, $d\in\mathbb{N}$, be open. The space of smooth and compactly supported functions in U is denoted by $C_0^\infty(U)$ and $C_{0,\sigma}^\infty(U)$ is the subspace of $C_0^\infty(U)$ of divergence-free functions. Moreover, for $Q\subset\mathbb{R}^d$, we define

$$C_{(0)}^\infty(Q)=\{u\colon Q\to\mathbb{R}\,:\,u=U|_Q\,,\ U\in C_0^\infty(\mathbb{R}^d),\ \operatorname{supp}(u)\subset Q\}.$$

For a Banach space X, its dual is designated by X^*. For a measurable set $M \subset \mathbb{R}^d$ and $r \in [1,\infty]$, $L^r(M)$ and $L^r(M;X)$ denote the standard Lebesgue spaces of scalar and X-valued functions, respectively. If $M=(a,b)$, we simply write $L^r(a,b;X)$. $W^{k,r}(U)$ is the Sobolev space of order $k \in \mathbb{N}$ and integrability exponent r. By $W_0^{k,r}(U)$, we denote the closure of $C_0^\infty(U)$ in $W^{k,r}(U)$, and we set $H^k(U)=W^{k,2}(U)$ and $H_0^k(U)=W_0^{k,2}(U)$. The corresponding dual spaces we abbreviate as $W^{-1,q}(U)=(W_0^{1,q'}(U))^*$, where $q'=\frac{q}{q-1}$, and $H^{-1}(U)=W^{-1,2}(U)$. Furthermore, $L^2_\sigma(U)$ and $H^1_{0,\sigma}(U)$ denote the closure of $C^\infty_{0,\sigma}(U)$ in $L^2(U)$ and $H^1(U)$, respectively. For $r\in[1,\infty)$, we define

$$L^r_\sigma(U)=\overline{C^\infty_{0,\sigma}(U)}^{\|\cdot\|_{L^r(U)^d}} \text{ and } W^{1,r}_{0,\sigma}(U)=\overline{C^\infty_{0,\sigma}(U)}^{\|\cdot\|_{W^{1,r}(U)^d}}.$$

Furthermore, for $r=2$, we use the notation $H^1_{0,\sigma}(U)=W^{1,2}_{0,\sigma}(U)$. The space $W^{1,r}_{0,\sigma}(U)$ has the following useful characterisation; see [10, Lemma II.2.2.3].

Lemma 2.1 (Characterisation of $W^{1,r}_{0,\sigma}(\Omega)$)**.** *For $d\geq 2$ and $r\in(1,\infty)$, let $\Omega\subset\mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Then there holds*

$$W^{1,r}_{0,\sigma}(\Omega)=\Big\{u\in W^{1,r}_0(\Omega)^d\,:\,\operatorname{div}(u)=0\Big\}.$$

It is convenient to introduce the spaces

$$C_0^\infty((0,T);C^\infty_{0,\sigma}(\Omega))=\Big\{\psi\in C_0^\infty(\Omega\times(0,T))^d\,:\,\operatorname{div}(\psi)=0\Big\}$$

and

$$C_0^\infty([0,T);C^\infty_{0,\sigma}(\Omega))=\Big\{\left.\psi\right|_{\Omega\times[0,T)}\,:\,\psi\in C_0^\infty(\Omega\times(-1,T))^d,\ \operatorname{div}(\psi)=0\Big\}.$$

2.1 Functions of Bounded Variation and Sets of Finite Perimeter

For $N\in\mathbb{N}$ and a finite $\mathbb{R}^N$-valued Radon measure μ and a Borel set $E\subset U$, the total-variation measure of E is defined by

$$|\mu|(E)=\sup\sum_{m=1}^\infty|\mu(E_m)|,$$

where the supremum is taken over all pairwise disjoint partitions $(E_m)_{m\in\mathbb{N}}\subset X$ of measurable sets E_m, $m\in\mathbb{N}$, such that $E=\bigcup_{m=1}^\infty E_m$. A function $u\in L^1(U)$ is said to be of bounded variation if its distributional gradient ∇u is a finite $\mathbb{R}^d$-valued Radon measure. The set of all functions of bounded variation is denoted by $BV(U)$, and the set $BV(U,M)$ contains all functions $u\in BV(U)$, such that $u(x)\in M$ for a.e. $x\in U$. A measurable set $E\subset U$ has finite perimeter in U if its characteristic function χ_E belongs to $BV(U)$. By the structure theorem of sets of finite perimeter, there holds $|\nabla\chi_E|(U)=\mathcal{H}^{d-1}(U\cap\partial^*E)$, where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure and ∂^*E is the so-called reduced boundary of E, and, moreover, for all $\psi\in C_0^\infty(U)^d$,

$$\int_E\operatorname{div}(\psi)\,\mathrm{d}x=\int_{\partial^*E}\psi\cdot\nu_E\,\mathrm{d}\mathcal{H}^{d-1}(x),$$

where $\nu_E(x)=-\lim_{\delta\searrow 0}\frac{\nabla\chi_E(B_\delta(x))}{|\nabla\chi_E|(B_\delta(x))}$ is the generalized outer unit normal; cf. e.g. [11, Theorem 3.36]. Note that, if E has C^1-boundary, then $\partial^*E=\partial E$ and ν_E coincides with the usual outer unit normal.

2.2 Hypersurfaces

We briefly recall some facts from differential geometry. For a more complete treatment, cf. for instance [12, Section 2], [13, Section 16.1] and [14]. We call $\Gamma \subset \mathbb{R}^d$, $d \geq 2$, a C^k-hypersurface, $k \in \mathbb{N}$, if, for each $x_0 \in \Gamma$, there exist an open neighbourhood $U \subset \mathbb{R}^d$ of x_0 and a function $u \in C^k(U)$ with

$$U \cap \Gamma = \{x \in U \,:\, u(x) = 0\} \text{ and } \nabla u(x) \neq 0 \text{ for all } x \in U \cap \Gamma.$$

In the case $k = 2$, we briefly call Γ a hypersurface. For a hypersurface Γ, the space $C^1(\Gamma)$ consists of all functions $f \colon \Gamma \to \mathbb{R}$ such that there exist a neighbourhood $U \subset \mathbb{R}^d$ of Γ and a function $g \in C^1(U)$ with $f = g|_U$. The tangent space $T_x\Gamma$ of a hypersurface $\Gamma \subset \mathbb{R}^d$, at a point $x \in \Gamma$, is defined by

$$T_x\Gamma = \{\tau \in \mathbb{R}^d \,:\, \tau \cdot \nabla u(x) = 0\}.$$

A hypersurface Γ is called oriented if there exists a function $\nu \in C^1(\Gamma)^d$ such that, for all $x \in \Gamma$, there holds $|\nu(x)| = 1$ and $\nu(x) \perp T_x\Gamma$, i.e., $\nu(x) \cdot \tau = 0$ for any $\tau \in T_x\Gamma$. The function ν is called unit-normal field (or, briefly, normal). On an oriented hypersurface $\Gamma \subset \mathbb{R}^d$ with unit-normal field ν, for $f \in C^1(\Gamma)$, the tangential gradient $\nabla_\Gamma f \colon \Gamma \to \mathbb{R}^d$ is defined as

$$\nabla_\Gamma f = (\delta_1 f, \ldots, \delta_d f) = \Big(\nabla f - (\nabla f \cdot \nu)\nu\Big)\Big|_\Gamma.$$

For $u \in C^1(\Gamma)^d$, the tangential divergence $\operatorname{div}_\Gamma(u) \colon \Gamma \to \mathbb{R}$ is defined as

$$\operatorname{div}_\Gamma(u) = \sum_{i=1}^{d} \delta_i u_i.$$

Proposition 2.2. *For an oriented hypersurface $\Gamma \subset \mathbb{R}^d$ with unit-normal field ν, define $\mathcal{K} = (\mathcal{K}_{ij})_{i,j=1,\ldots,d} \colon \Gamma \to \mathbb{R}^{d\times d}$ by*

$$\mathcal{K}_{ij}(x) = -\delta_i \nu_j(x) \tag{2.1}$$

for $i, j = 1, \ldots, d$ and $x \in \Gamma$. Then, for every $x \in \Gamma$, the matrix $\mathcal{K}(x)$ is symmetric and $\nu(x)$ is an eigenvector of $\mathcal{K}(x)$ with corresponding eigenvalue 0.

Proof. See [12, Section 2.3] or [14, Theorem 2.10]. □

The foregoing proposition allows one to define the mean curvature of an oriented hypersurface.

Definition 2.3 (Mean curvature)**.** *For an oriented hypersurface $\Gamma \subset \mathbb{R}^d$ with unit-normal field ν, let $x \in \Gamma$ and let $\mathcal{K}$ be defined as in* (2.1)*.*

1 The principal curvatures of Γ in x are the eigenvalues $\kappa_1(x), \ldots, \kappa_{d-1}(x)$ of $\mathcal{K}(x)$ belonging to eigenvectors orthogonal to $\nu(x)$.

2 The mean curvature $\kappa \colon \Gamma \to \mathbb{R}$ is the trace of $\mathcal{K}$, i.e.,

$$\kappa = \sum_{i=1}^{d} \mathcal{K}_{ii} = \sum_{i=1}^{d-1} \kappa_i.$$

Note that, in view of the above definitions, there holds $\kappa = -\operatorname{div}_\Gamma(\nu)$. Moreover, for $f \in C^1(\Gamma)$ and $i = 1, \dots, d$, there holds the integration-by-parts formula

$$\int_\Gamma \delta_i f \,\mathrm{d}\mathcal{H}^{d-1}(x) = -\int_\Gamma f\kappa\nu_i \,\mathrm{d}\mathcal{H}^{d-1}(x); \tag{2.2}$$

see [13, Lemma 16.1].

For the treatment of time-dependent interfaces, we need the notion of evolving hypersurface, and have to define its normal velocity.

Definition 2.4 (Evolving hypersurfaces). *Let $I \subset \mathbb{R}$ be an interval. For a family $(\Gamma(t))_{t\in I} \subset \mathbb{R}^d$ of oriented hypersurfaces, define*

$$\Gamma = \bigcup_{t\in I} \Big(\Gamma(t) \times \{t\}\Big). \tag{2.3}$$

1 $(\Gamma(t))_{t\in I}$ is called a $C^{2,1}$-family of evolving oriented hypersurfaces, or, briefly, a family of evolving hypersurfaces, if Γ is a C^1-hypersurface in $\mathbb{R}^{d+1}$ and there exists a function $\nu \in C^1(\Gamma)^d$ such that $\Gamma(t)$ is oriented by $\nu(\cdot, t)$ for every $t \in I$.

2 The normal velocity $V \in C^0(\Gamma)$ of $(\Gamma(t))_{t\in I}$ at a point $(x_0, t_0) \in \Gamma$ is given by

$$V(x_0, t_0) = \eta'(t_0) \cdot \nu(x_0, t_0),$$

where $\eta \in C^1(I_0)^d$, for some subinterval $I_0 \subset I$ with $t_0 \in I_0$, such that $\eta(t_0) = x_0$ and $\eta(t) \in \Gamma(t)$ for all $t \in I_0$.

Remark 2.5. *The definition of the normal velocity V does not depend on the choice of the function η. Moreover, for any $t \in I$, there holds $V(\cdot, t) \in C^1(\Gamma(t))$; see [14, Theorem 5.5].*

Finally, we provide some transport identities for integrals, which allow one to calculate time derivatives of integrals over time-dependent domains and hypersurfaces.

Theorem 2.6 (Transport theorem). *For some interval $I \subset \mathbb{R}$, let $(\Gamma(t))_{t\in I}$ be a family of evolving hypersurfaces in the sense of Definition 2.4. In addition, for every $t \in I$, assume that $\Gamma(t) = \partial\Omega(t)$ for some open, bounded set $\Omega(t) \subset \mathbb{R}^d$. Denote by $\nu = \nu(t)$ the unit-normal field of $\Gamma(t)$ pointing outward to $\Omega(t)$, by $\kappa = \kappa(t)$ the mean curvature of $\Gamma(t)$ and by $V = V(t)$ the normal velocity of $(\Gamma(t))_{t\in I}$, respectively, with respect to $\nu(t)$.*

1 If $U \subset \mathbb{R}^{d+1}$ is an open set such that

$$\bigcup_{t\in I} \Big(\overline{\Omega(t)} \times \{t\}\Big) \subset U,$$

then, for every $f \in C^1(U)$, there holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} f \,\mathrm{d}x = \int_{\Omega(t)} \partial_t f \,\mathrm{d}x + \int_{\Gamma(t)} fV \,\mathrm{d}\mathcal{H}^{d-1}(x).$$

2 Let Γ be as in (2.3). If $f \in C^1(\Gamma)$, then, for and $t \in I$, there holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} f \,\mathrm{d}\mathcal{H}^{d-1}(x) \\ = \int_{\Gamma(t)} \partial_t f \,\mathrm{d}\mathcal{H}^{d-1}(x) - \int_{\Gamma(t)} f\kappa V \,\mathrm{d}\mathcal{H}^{d-1}(x) + \int_{\Gamma(t)} (\nabla f \cdot \nu) V \,\mathrm{d}\mathcal{H}^{d-1}(x).$$

In particular,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^{d-1}(\Gamma(t)) = -\int_{\Gamma(t)} \kappa V \,\mathrm{d}\mathcal{H}^{d-1}(x).$$

Proof. See [12, Appendix] or [14, Theorems 6.1 and 6.4]. □

3 The Notion of Weak Solutions

The free-boundary problem (1.1)–(1.8) incorporates two disjoint subregions $\Omega^-(t)$ and $\Omega^+(t)$ of the domain Ω, where the fluid is of constant density β_1 and β_2, respectively. This means that the associated density function is given by

$$\rho(t) = \beta_1\chi_{\Omega^-(t)} + \beta_2\chi_{\Omega^+(t)} \text{ in } \Omega. \tag{3.1}$$

Note that $\rho(t) = (\beta_1 - \beta_2)\chi_{\Omega^-(t)} + \beta_2$ in $\Omega \setminus \Gamma(t)$. Moreover, the nature of $\rho(t)$ is encoded in the characteristic function

$$\chi(t) = \chi_{\Omega^-(t)} = \frac{\rho(t) - \beta_2}{\beta_1 - \beta_2}, \tag{3.2}$$

and vice versa. In many situations, it is convenient to use that (1.1) and (1.2) are equivalent to

$$\rho\partial_t v + \rho(v\cdot\nabla)v - 2\mu(\rho)\operatorname{div}(Dv) + \nabla p = 0 \text{ in } \Omega\setminus\Gamma(t). \tag{3.3}$$

To motivate a weak formulation, we consider sufficiently smooth solution triplets (v, p, Γ) of (1.1)–(1.8); see Assumptions 3.1 below. More precisely, for the pair (ρ, v), we derive a variational formulation for (3.3) incorporating divergence-free test functions, an energy equality and a weak formulation of the pure transport of the interface (1.5) in terms of a transport equation for χ.

Assumptions 3.1 (Existence of smooth solutions)**.** *Let the following conditions be satisfied.*

1 ***Regularity of initial interface.*** $\Gamma^{(i)}$ *is a* C^2*-hypersurface, inducing a disjoint partition* $\Omega = \Omega^{-,(i)} \cup \Gamma^{(i)} \cup \Omega^{+,(i)}$, *such that*

$$\Gamma^{(i)} = \partial\Omega^{-,(i)} \subset\subset \Omega \text{ and } \Omega^{+,(i)} = \Omega \setminus \overline{\Omega^{-,(i)}} = \Omega\setminus(\Omega^{-,(i)}\cup\Gamma^{(i)}).$$

Define the initial associated density function $\rho^{(i)} \colon \Omega \to \mathbb{R}$ *by*

$$\rho^{(i)}(x) = \beta_1\chi_{\Omega^{-,(i)}}(x) + \beta_2\chi_{\Omega^{+,(i)}}(x) \text{ for } x\in\Omega$$

and define $\chi^{(i)}\colon \Omega\to\mathbb{R}$ *by*

$$\chi^{(i)}(x) = \chi_{\Omega^{-,(i)}}(x) \text{ for } x\in\Omega. \tag{3.4}$$

2 ***Regularity of initial velocity.*** $v^{(i)}\colon\Omega\to\mathbb{R}^n$ *belongs to* $C^0(\overline{\Omega})^n$*. Additionally, the restrictions* $v^{(i)}\big|_{\Omega^{\pm,(i)}}$ *to* $\Omega^{\pm,(i)}$ *satisfy*

$$v^{(i)}\big|_{\Omega^{\pm,(i)}} \in C^1(\overline{\Omega_0^{\pm}})^n \text{ and } \operatorname{div}(v^{(i)})\big|_{\Omega^{\pm,(i)}} = 0.$$

3 ***Existence of smooth solutions.*** (v, p, Γ) *is a solution triplet satisfying equations* (1.1)–(1.8) *with the following regularity properties.*

3.1 ***Regularity of velocity and pressure.*** *There exist open sets* $U^-, U^+ \subset \mathbb{R}^{n+1}$ *with*

$$\bigcup_{t\in[0,T]} \left(\overline{\Omega^\pm(t)} \times \{t\}\right) \subset U^\pm$$

as well as functions $v^\pm \in C^2(U^\pm)^n$ *and* $p^\pm \in C^1(U^\pm)$ *such that*

$$v = v^\pm \text{ and } p = p^\pm \text{ on } \bigcup_{t\in[0,T]} \left(\overline{\Omega^\pm(t)} \times \{t\}\right).$$

3.2 ***Regularity of interface.*** $(\Gamma(t))_{t\in[0,T]}$ *is a family of evolving hypersurfaces in the sense of Definition 2.4 such that* $\Omega^-(t) \cup \Gamma(t) \cup \Omega^+(t)$ *is a pairwise disjoint partition of* Ω *and* $\Gamma(t) = \partial\Omega^-(t) \subset\subset \Omega$ *for all* $t \in [0,T]$*. Additionally, for* $\Gamma = \bigcup_{t\in[0,T]} \left(\Gamma(t) \times \{t\}\right)$*, let* $\nu^- \in C^0(\Gamma)^n$ *be such that* $\nu^-(\cdot, t)$ *is the unit-normal field pointing outward to* $\Omega^-(t)$ *for all* $t \in [0,T]$*.*

3.1 Variational Formulation

In the spirit of the theory of the incompressible Navier–Stokes equations; see for example [15, 10], we will use divergence-free test functions in the weak formulation. This choice leads to a weak formulation lacking the pressure function. In order to justify this approach, one has to reconstruct the pressure from the weak formulation.

For the treatment of time derivatives in (1.1) and (1.2) and for later use, we provide the following consequences of the transport theorem (Theorem 2.6).

Lemma 3.2 (Transport identities)**.** *Suppose that Assumptions 3.1 are valid. Then, for every* $t \in (0,T)$ *and every* $\psi \in C^\infty_{(0)}(\Omega \times [0,T))^n$*, the following statements hold true.*

1 $\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho v \cdot \psi \,\mathrm{d}x = \int_{\Omega\setminus\Gamma(t)} \rho \partial_t (v \cdot \psi) \,\mathrm{d}x + (\beta_1 - \beta_2) \int_{\Gamma(t)} V (v \cdot \psi) \,\mathrm{d}\mathcal{H}^{n-1}(x).$

2 $\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho \left|v\right|^2 \mathrm{d}x = \int_{\Omega\setminus\Gamma(t)} \rho \partial_t \left|v\right|^2 \mathrm{d}x + (\beta_1 - \beta_2) \int_{\Gamma(t)} V \left|v\right|^2 \mathrm{d}\mathcal{H}^{n-1}(x).$

Proof. In view of (1.4), we simply write $v(t) = v^+(t) = v^-(t)$ on $\Gamma(t)$, where

$$v^+(x,t) = \lim_{\xi\searrow 0} v(x + \xi\nu^-(x,t), t) \text{ and } v^-(x,t) = \lim_{\xi\searrow 0} v(x - \xi\nu^-(x,t), t).$$

Let $\psi \in C^\infty_{(0)}(\Omega \times [0,T))^n$. To prove the first statement, we apply Theorem 2.6 to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^-(t)} v \cdot \psi \,\mathrm{d}x = \int_{\Omega^-(t)} \partial_t (v \cdot \psi) \,\mathrm{d}x + \int_{\Gamma(t)} V (v \cdot \psi) \,\mathrm{d}\mathcal{H}^{n-1}(x)$$

and, likewise,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^+(t)} v \cdot \psi \,\mathrm{d}x = \int_{\Omega^+(t)} \partial_t (v \cdot \psi) \,\mathrm{d}x - \int_{\Gamma(t)} V (v \cdot \psi) \,\mathrm{d}\mathcal{H}^{n-1}(x).$$

Recalling the definition of ρ from (3.1), we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho v\cdot\psi\,\mathrm{d}x = \int_{\Omega\setminus\Gamma(t)} \rho\partial_t(v\cdot\psi)\,\mathrm{d}x + (\beta_1-\beta_2)\int_{\Gamma(t)} V(v\cdot\psi)\,\mathrm{d}\mathcal{H}^{n-1}(x).$$

The second claim now follows analogously, with v taking the role of ψ. □

Proposition 3.3 (Weak differentiability of v)**.** *Let $t\in(0,T)$. If Assumptions 3.1 are satisfied, then $v(t)$ is weakly differentiable in Ω.*

Proof. Let $t\in(0,T)$. In view of Assumptions 3.1, there holds

$$v(t)|_{\Omega^-(t)}\in C^2(\overline{\Omega^-(t)})^n \text{ and } v(t)|_{\Omega^+(t)}\in C^2(\overline{\Omega^+(t)})^n.$$

For any $i=1,\dots,n$ and any $\psi\in C_0^\infty(\Omega)^n$, integration by parts yields

$$\begin{aligned}\int_\Omega v_i(t)\operatorname{div}(\psi)\,\mathrm{d}x &= \int_{\Omega^-(t)} v_i(t)\operatorname{div}(\psi)\,\mathrm{d}x + \int_{\Omega^+(t)} v_i(t)\operatorname{div}(\psi)\,\mathrm{d}x\\ &= -\int_{\Omega\setminus\Gamma(t)} \nabla v_i(t)\cdot\psi\,\mathrm{d}x - \int_{\Gamma(t)}[v_i(t)]\psi\cdot\nu^-\,\mathrm{d}\mathcal{H}^{n-1}(x).\end{aligned}$$

Since, by (1.4), there holds $[v_i(t)]=0$, the claim follows. □

The following weak concept of mean curvature will be useful for obtaining a variational formulation of (1.6).

Lemma 3.4 (Weak-mean-curvature functional)**.** *Let $t\in(0,T)$ and suppose that Assumptions 3.1 are satisfied. For every $\psi\in C^1(\Omega)^n$ with $\operatorname{div}(\psi)=0$ in Ω, there holds*

$$\int_{\Gamma(t)} \kappa(t)\nu^-(t)\cdot\psi\,\mathrm{d}\mathcal{H}^{n-1}(x) = \int_{\Gamma(t)} \nu^-(t)\otimes\nu^-(t):\nabla\psi\,\mathrm{d}\mathcal{H}^{n-1}(x). \tag{3.5}$$

Proof. Let $t\in(0,T)$ and $\psi\in C^1(\Omega)^n$ with $\operatorname{div}(\psi)=0$ be arbitrary. We apply the integration-by-parts formula (2.2) to $f=\psi_i$ and sum over $i=1,\dots,n$. Denoting $\kappa=\kappa(t)$, $\nu^-=\nu^-(t)$ and $\Gamma=\Gamma(t)$, as ψ is divergence free, this implies

$$\int_\Gamma \kappa\nu^-\cdot\psi\,\mathrm{d}\mathcal{H}^{n-1}(x) = -\int_\Gamma \operatorname{div}_\Gamma(\psi)\,\mathrm{d}\mathcal{H}^{n-1}(x) = \int_\Gamma \nu^-\otimes\nu^-:\nabla\psi\,\mathrm{d}\mathcal{H}^{n-1}(x).$$

□

Note that the right-hand side of (3.5) is well-defined if Γ is merely the reduced or the essential boundary of a set of finite perimeter. Then one has to interpret ν^- as generalised inner (or outer) normal to Γ.

Lemma 3.5 (Weak form of linear-momentum balance)**.** *Let Assumptions 3.1 hold true. For every $\psi\in C_0^\infty([0,T);C_{0,\sigma}^\infty(\Omega))$, there holds*

$$\begin{aligned}&\int_0^T\int_\Omega \rho v\cdot\partial_t\psi + \rho v\otimes v:\nabla\psi - 2\mu(\rho)Dv:D\psi\,\mathrm{d}x\,\mathrm{d}t\\ &= -\int_\Omega \rho^{(i)}v^{(i)}\cdot\psi(0)\,\mathrm{d}x - 2\sigma_{\mathrm{st}}\int_0^T\int_{\Gamma(t)} \nu^-\otimes\nu^-:\nabla\psi\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t.\end{aligned} \tag{3.6}$$

Proof. Multiplying (3.3) by $\psi \in C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega))$ and integrating with respect to space and time leads to

$$\int_0^T \int_{\Omega\setminus\Gamma(t)} \Big(\rho\partial_t v + \rho(v\cdot\nabla)v - 2\mu(\rho)\operatorname{div}(Dv) + \nabla p\Big)\cdot\psi \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.7}$$

Applying the first statement of Lemma 3.2 to deal with the time derivative leads to

$$\begin{aligned} &\int_0^T \int_{\Omega\setminus\Gamma(t)} \rho\partial_t v\cdot\psi \,\mathrm{d}x\,\mathrm{d}t + (\beta_1-\beta_2)\int_0^T\int_{\Gamma(t)} V(v\cdot\psi)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t \\ = &-\int_0^T\int_\Omega \rho v\cdot\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho v\cdot\psi\,\mathrm{d}x\right)\mathrm{d}t \\ = &-\int_0^T\int_\Omega \rho v\cdot\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho^{(i)}v^{(i)}\cdot\psi(0)\,\mathrm{d}x. \end{aligned} \tag{3.8}$$

To each of the remaining terms in (3.7), we shall apply the integration-by-parts formula on the spatial domains $\Omega^-(t)$ and $\Omega^+(t)$: By (1.3) and (1.4), we infer

$$\begin{aligned} &\int_{\Omega\setminus\Gamma(t)} \rho((v\cdot\nabla)v)\cdot\psi\,\mathrm{d}x \\ &= \beta_1\int_{\Omega^-(t)}\operatorname{div}(v\otimes v)\cdot\psi\,\mathrm{d}x + \beta_2\int_{\Omega^+(t)}\operatorname{div}(v\otimes v)\cdot\psi\,\mathrm{d}x \\ &= -\int_\Omega \rho v\otimes v : \nabla\psi\,\mathrm{d}x + (\beta_1-\beta_2)\int_{\Gamma(t)}(v\cdot\nu^-)v\cdot\psi\,\mathrm{d}\mathcal{H}^{n-1}(x). \end{aligned} \tag{3.9}$$

Using Proposition 3.3 and $Dv : \nabla\psi = Dv : D\psi$, we analogously obtain that

$$\begin{aligned} &\int_{\Omega\setminus\Gamma(t)} \mu(\rho)\operatorname{div}(Dv)\cdot\psi\,\mathrm{d}x \\ &= -\int_\Omega \mu(\rho)Dv : D\psi\,\mathrm{d}x - \int_{\Gamma(t)}[\mu(\rho)Dv]\nu^-\cdot\psi\,\mathrm{d}\mathcal{H}^{n-1}(x). \end{aligned} \tag{3.10}$$

In view of $\operatorname{div}(\psi) = 0$, we have

$$\int_{\Omega\setminus\Gamma(t)} \nabla p\cdot\psi\,\mathrm{d}x = -\int_{\Gamma(t)}[p]\nu^-\cdot\psi\,\mathrm{d}\mathcal{H}^{n-1}(x). \tag{3.11}$$

Now combining (3.7)–(3.11) leads to

$$\begin{aligned} &\int_0^T\int_\Omega \rho v\cdot\partial_t\psi + \rho v\otimes v:\nabla\psi - 2\mu(\rho)Dv : D\psi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \rho^{(i)}v^{(i)}\cdot\psi(0)\,\mathrm{d}x \\ = &-(\beta_1-\beta_2)\int_0^T\int_{\Gamma(t)}(V - v\cdot\nu^-)v\cdot\psi\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t \\ &+\int_0^T\int_{\Gamma(t)}\Big[2\mu(\rho)Dv\nu^- - p\nu^-\Big]\cdot\psi\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t \\ = &-2\sigma_{\mathrm{st}}\int_0^T\int_{\Gamma(t)}\kappa\nu^-\cdot\psi\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t, \end{aligned}$$

where the last identity follows by (1.5) and (1.6). Finally, Lemma 3.4 yields (3.6). □

3.2 Energy Equality

In an analogous manner to Lemma 3.5, we may derive the following energy identity.

Lemma 3.6 (Energy equality and a priori bounds)**.** *Let Assumptions 3.1 hold true. For all* $\tau_1, \tau_2 \in [0,T]$ *such that* $\tau_1 \leq \tau_2$*, the following energy equality is satisfied.*

$$\begin{aligned} &2\sigma_{\mathrm{st}}\mathcal{H}^{n-1}(\Gamma(\tau_2)) + \tfrac{1}{2}\int_\Omega \rho(\tau_2)\left|v(\tau_2)\right|^2 \,\mathrm{d}x + 2\int_{\tau_1}^{\tau_2}\int_\Omega \mu(\rho)\left|Dv\right|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &= 2\sigma_{\mathrm{st}}\mathcal{H}^{n-1}(\Gamma(\tau_1)) + \tfrac{1}{2}\int_\Omega \rho(\tau_1)\left|v(\tau_1)\right|^2 \,\mathrm{d}x. \end{aligned} \tag{3.12}$$

Moreover, if the initial energy

$$E^{(i)} = 2\sigma_{\mathrm{st}}\mathcal{H}^{n-1}(\Gamma^{(i)}) + \tfrac{1}{2}\int_\Omega \rho^{(i)}\left|v^{(i)}\right|^2 \,\mathrm{d}x \tag{3.13}$$

is finite, then there holds

$$v \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_0(\Omega)^n) \text{ and } \rho \in L^\infty(0,T;BV(\Omega,\{\beta_1,\beta_2\})).$$

Proof. Let $\tau_1, \tau_2 \in [0,T]$ be such that $\tau_1 \leq \tau_2$. We multiply (3.3) by v and integrate with respect to space and time. This leads to

$$\int_{\tau_1}^{\tau_2}\int_{\Omega\setminus\Gamma(t)} \Big(\rho\partial_t v + \rho(v\cdot\nabla)v - 2\mu(\rho)\operatorname{div}(Dv) + \nabla p\Big)\cdot v \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.14}$$

We shall evaluate the integral expression successively. For the treatment of the time derivative, we apply Lemma 3.2 to obtain

$$\begin{aligned} 2\int_{\tau_1}^{\tau_2}\int_{\Omega\setminus\Gamma(t)} \rho\partial_t v\cdot v\,\mathrm{d}x\,\mathrm{d}t &= \int_{\tau_1}^{\tau_2}\int_{\Omega\setminus\Gamma(t)} \rho\partial_t\left|v\right|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_\Omega \rho(\tau_2)\left|v(\tau_2)\right|^2 \,\mathrm{d}x - \int_\Omega \rho(\tau_1)\left|v(\tau_1)\right|^2 \,\mathrm{d}x \\ &\quad - (\beta_1-\beta_2)\int_{\tau_1}^{\tau_2}\int_{\Gamma(t)} V\left|v\right|^2 \,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t. \end{aligned} \tag{3.15}$$

For the treatment of the remaining terms in (3.14), we shall repeatedly integrate by parts with respect to the spatial variable for fixed $t \in (\tau_1,\tau_2)$: for the computation of the second term in (3.14), we use that, in view of (1.3), there holds $((v\cdot\nabla)v)\cdot v = \operatorname{div}(v\otimes v)\cdot v = \frac{1}{2}\operatorname{div}(\left|v\right|^2 v)$ in $\Omega\setminus\Gamma(t)$, which implies

$$2\int_{\Omega\setminus\Gamma(t)} \rho((v\cdot\nabla)v)\cdot v\,\mathrm{d}x = (\beta_1-\beta_2)\int_{\Gamma(t)} \left|v\right|^2 v\cdot\nu^- \,\mathrm{d}\mathcal{H}^{n-1}(x). \tag{3.16}$$

Proceeding as in (3.10) and using $Dv : \nabla v = \left|Dv\right|^2$ leads to

$$\begin{aligned} &\int_{\Omega\setminus\Gamma(t)} \mu(\rho)\operatorname{div}(Dv)\cdot v\,\mathrm{d}x \\ &= -\int_\Omega \mu(\rho)\left|Dv\right|^2 \,\mathrm{d}x - \int_{\Gamma(t)} [\mu(\rho)Dv]\nu^-\cdot v\,\mathrm{d}\mathcal{H}^{n-1}(x). \end{aligned} \tag{3.17}$$

To treat the pressure term in (3.14), we may again use (1.3). Using calculations as in (3.11), we infer that

$$\int_{\Omega\setminus\Gamma(t)} \nabla p\cdot v\,\mathrm{d}x = -\int_{\Gamma(t)} [p]\nu^-\cdot v\,\mathrm{d}\mathcal{H}^{n-1}(x). \tag{3.18}$$

Now we may combine (3.14)–(3.18). Altogether, by (1.5) and (1.6), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \rho(\tau_2)\,|v(\tau_2)|^2\;\mathrm{d}x - \frac{1}{2}\int_\Omega \rho(\tau_1)\,|v(\tau_1)|^2\;\mathrm{d}x + 2\int_{\tau_1}^{\tau_2}\int_\Omega \mu(\rho)\,|Dv|^2\;\mathrm{d}x\,\mathrm{d}t\\
&= \frac{1}{2}(\beta_1-\beta_2)\int_{\tau_1}^{\tau_2}\int_{\Gamma(t)} |v|^2\,(V - v\cdot\nu^-)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t\\
&\quad - \int_{\tau_1}^{\tau_2}\int_{\Gamma(t)} [2\mu(\rho)Dv\nu^- - p\nu^-]\cdot v\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t\\
&= 2\sigma_{\mathrm{st}}\int_{\tau_1}^{\tau_2}\int_{\Gamma(t)} \kappa v\cdot\nu^-\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t\\
&= 2\sigma_{\mathrm{st}}\int_{\tau_1}^{\tau_2}\int_{\Gamma(t)} \kappa V\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t.
\end{aligned}$$

Now (3.12) follows by observing that, in view of Theorem 2.6, there holds

$$\begin{aligned}
\int_{\tau_1}^{\tau_2}\int_{\Gamma(t)} \kappa V\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t &= -\int_{\tau_1}^{\tau_2} \tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^{n-1}(\Gamma(t))\,\mathrm{d}t\\
&= \mathcal{H}^{n-1}(\Gamma(\tau_1)) - \mathcal{H}^{n-1}(\Gamma(\tau_2)).
\end{aligned}$$

Suppose that the initial energy $E^{(i)}$, defined by (3.13), is finite and let $t\in[0,T]$. Recall from (3.1) that there holds $\rho(t)\in\{\beta_1,\beta_2\}$ a.e. in Ω. Making in (3.12) the choice $\tau_1=0$ and $\tau_2=t$ implies

$$\tfrac{1}{2}\beta_1\int_\Omega |v(t)|^2\;\mathrm{d}x \le E^{(i)}.$$

Hence $v\in L^\infty(0,T;L^2(\Omega)^n)$. Similarly, we obtain that

$$\min\{\mu(\beta_1),\mu(\beta_2)\}\int_0^T\int_\Omega |Dv|^2\;\mathrm{d}x\,\mathrm{d}t \le \int_0^T\int_\Omega \mu(\rho)\,|Dv|^2\;\mathrm{d}x\,\mathrm{d}t \le \tfrac{1}{2}E^{(i)}.$$

Due to the boundary condition (1.7), and using Korn's inequality [16, Theorem 1.33], we infer that $v\in L^2(0,T;H_0^1(\Omega)^n)$.

In the remainder of the proof we fix $t\in(0,T)$. In view of (1.3) and Lemma 2.1, v belongs to $L^\infty(0,T;L^2_\sigma(\Omega))$. To explore the regularity of ρ, we recall that, in view of (3.1), for every $t\in(0,T)$, there holds $\rho(t)\in\{\beta_1,\beta_2\}$ a.e. in Ω and, in particular, ρ belongs to $L^\infty(0,T;L^\infty(\Omega))$. Additionally, for any $\psi\in C_0^\infty(\Omega)^n$, we have

$$\begin{aligned}
\langle\nabla\rho(t),\psi\rangle_{\mathcal{D}(\Omega)^n} &= -\int_\Omega \rho(t)\,\mathrm{div}(\psi)\,\mathrm{d}x\\
&= -(\beta_1-\beta_2)\int_{\Omega^-(t)} \mathrm{div}(\psi)\,\mathrm{d}x\\
&= -(\beta_1-\beta_2)\int_{\Gamma(t)} \psi\cdot\nu^-(t)\,\mathrm{d}\mathcal{H}^{n-1}(x).
\end{aligned}$$

Consequently, $\nabla\rho(t)$ is a finite Radon measure and there holds

$$\begin{aligned}
\|\nabla\rho(t)\|_{\mathcal{M}(\Omega)} &= \sup\left\{\int_\Omega \rho(t)\,\mathrm{div}(\psi)\,\mathrm{d}x \;:\; \psi\in C_0^1(\Omega)^n,\ \|\psi\|_\infty\le 1\right\}\\
&= (\beta_2-\beta_1)\sup\left\{\int_{\Omega^-(t)} \mathrm{div}(\psi)\,\mathrm{d}x \;:\; \psi\in C_0^1(\Omega)^n,\ \|\psi\|_\infty\le 1\right\}\\
&= (\beta_2-\beta_1)\mathcal{H}^{n-1}(\partial\Omega^-(t)\cap\Omega).
\end{aligned}$$

Due to Assumptions 3.1, $\Omega^-(t)$ has a Lipschitz boundary and $\Omega^-(t) \subset\subset \Omega$. Then, we get

$$\|\nabla\rho(t)\|_{\mathcal{M}(\Omega)} = (\beta_2 - \beta_1)\mathcal{H}^{n-1}(\Gamma(t)). \tag{3.19}$$

Finally, from the energy equality (3.12), it follows that $\|\nabla\rho(t)\|_{\mathcal{M}(\Omega)}$ is uniformly bounded in t. Altogether, we have proven that $\rho \in L^\infty(0,T;BV(\Omega,\{\beta_1,\beta_2\}))$. □

3.3 Transport Equation

The interface condition (1.5) can be expressed by the following transport equation for χ in distributional form, cf. [7, Section 2.5].

Lemma 3.7 (Transport equation)**.** *Let Assumptions 3.1 hold true. Then, for all* $\varphi \in C^\infty_{(0)}(\overline{\Omega} \times [0,T))$*, there holds*

$$\int_0^T \int_\Omega \chi(\partial_t\varphi + v\cdot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \chi^{(i)}(x)\varphi(0)\,\mathrm{d}x = 0. \tag{3.20}$$

Proof. Let $\varphi \in C^\infty_{(0)}(\overline{\Omega} \times [0,T))$. Applying Theorem 2.6 to φ and integrating with respect to time yields

$$\int_0^T \int_{\Gamma(t)} V\varphi\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t = -\int_0^T \int_{\Omega^-(t)} \partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega^-(0)} \varphi(0)\,\mathrm{d}x. \tag{3.21}$$

Since $\operatorname{div}(v(t)) = 0$ in $\Omega^-(t)$ by (1.3), we conclude that

$$\begin{aligned}\int_0^T \int_{\Gamma(t)} v\cdot\nu^-\varphi\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t &= \int_0^T \int_{\Omega^-(t)} \operatorname{div}(v\varphi)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\Omega^-(t)} v\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Recalling (1.5), we use that $V = v\cdot\nu^-$ on $\Gamma(t)$ to obtain

$$\int_0^T \int_{\Omega^-(t)} \partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega^-(0)} \varphi(0)\,\mathrm{d}x + \int_0^T \int_{\Omega^-(t)} v\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = 0.$$

As $\chi(t)$ and $\chi^{(i)}$ are the characteristic functions of $\Omega^-(t)$ and $\Omega^{-,(i)} = \Omega^-(0)$, respectively, see (3.2) and (3.4), the identity (3.20) follows. This finishes the proof. □

The previous result motivates the following definition.

Definition 3.8 (Weak solutions of the transport equation)**.** *For prescribed functions* $v \in L^2(0,T;L^2_\sigma(\Omega))$ *and* $\chi^{(i)} \in L^\infty(\Omega)$*,* $\chi \in L^\infty(\Omega \times (0,T))$ *is called a weak solution of the transport equation*

$$\begin{aligned}\partial_t\chi + v\cdot\nabla\chi &= 0 \quad \text{in } \Omega \times (0,T), \\ \chi(0) &= \chi^{(i)} \quad \text{in } \Omega\end{aligned} \tag{3.22}$$

provided that for every $\varphi \in C^\infty_{(0)}(\overline{\Omega} \times [0,T))$*,* (3.20) *holds true.*

3.4 The Weak Formulation

We seek to introduce a weak formulation for (1.1)–(1.8). To this end, we restrict the class of weak solutions to pairs (ρ, v) satisfying the energy inequality (3.12). For well-prepared initial data $(\rho^{(i)}, v^{(i)})$, this suggests the regularity classes $\rho \in L^\infty(0,T; BV(\Omega, \{\beta_1, \beta_2\}))$ and $v \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_0(\Omega)^n)$. For a.e. $t \in (0,T)$, there exist a measurable set $\Omega^-(t) \subset \Omega$ and an induced characteristic function $\chi(t) \in BV(\Omega, \{0,1\})$ of $\rho(t)$ such that, a.e. in Ω, there holds

$$\chi(t) = \chi_{\Omega^-(t)} = \frac{\rho(t) - \beta_2}{\beta_1 - \beta_2}.$$

Here and subsequently, we refer $\Omega^-(t)$ to as measure-theoretic representative set of $\rho(t)$. This, in turn, leads to the representation

$$\rho(t) = (\beta_1 - \beta_2)\chi_{\Omega^-(t)} + \beta_2 = (\beta_1 - \beta_2)\chi(t) + \beta_2.$$

Notice that this procedure makes the identity $\Omega^-(t) = \{x \in \Omega : \rho(t) = \beta_1\}$ well-defined in a measure-theoretic sense. As $\Omega^-(t)$ is of bounded variation, we may define the interface $\Gamma(t)$ by $\Gamma(t) = \partial^*(\Omega^-(t)) \cap \Omega$, where $\partial^*(\Omega^-(t))$ denotes the reduced boundary of $\Omega^-(t)$. Hence the variational formulation (3.6) remains meaningful if we understand the outer unit normal ν^- in the (measure-theoretic) sense of the generalised outer unit normal given by

$$\nu^-(x,t) = -\lim_{\delta \to 0} \frac{\nabla \chi_{\Omega^-(t)}(B_\delta(x))}{\left|\nabla \chi_{\Omega^-(t)}\right|(B_\delta(x))} \text{ for } x \in \Gamma(t).$$

Additionally, we require χ to solve the corresponding transport equation in the sense of Definition 3.8. and we maintain the assumption that $\Omega^-(t)$ is compactly contained in Ω. Finally, the results of the Lemmas 3.5–3.7 motivate the following weak formulation of (1.1)–(1.8).

Definition 3.9 (Weak formulation). *Let $\left(\rho^{(i)}, v^{(i)}\right) \in BV(\Omega, \{\beta_1, \beta_2\}) \times H^1_{0,\sigma}(\Omega)$ be prescribed initial data, such that the measure-theoretic representative set $\Omega^-(0)$ of $\rho^{(i)}$ is compactly contained in Ω, i.e., $\Omega^-(0) \subset\subset \Omega$, and $\rho^{(i)}$ has the representation*

$$\rho^{(i)} = (\beta_1 - \beta_2)\chi_{\Omega^-(0)} + \beta_2 = (\beta_1 - \beta_2)\chi^{(i)} + \beta_2,$$

where $\chi^{(i)}$ is the induced characteristic function of $\rho^{(i)}$ that is given by

$$\chi^{(i)} = \frac{\rho^{(i)} - \beta_2}{\beta_1 - \beta_2} \in BV(\Omega, \{0,1\}).$$

Then (ρ, v) is called a weak solution of (1.1)–(1.8) *with prescribed initial data $(\rho^{(i)}, v^{(i)})$ if the following conditions are fulfilled.*

1 ***Regularity of associated density.*** *$\rho \in L^\infty(0,T; BV(\Omega, \{\beta_1, \beta_2\}))$, and the measure-theoretic representative set $\Omega^-(t)$ of $\rho(t)$ is compactly contained in Ω; that is, for a.e. $t \in (0,T)$, there holds $\Omega^-(t) \subset\subset \Omega$.*

2 ***Regularity of velocity.*** *$v \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_0(\Omega)^n)$.*

3 ***Weak form of linear-momentum balance.*** *For each* $\psi \in C_0^\infty([0,T);C_{0,\sigma}^\infty(\Omega))$*, there holds*

$$\begin{aligned}&\int_0^T\int_\Omega \rho v\cdot\partial_t\psi + \rho v\otimes v : \nabla\psi - 2\mu(\rho)Dv : D\psi \,\mathrm{d}x\,\mathrm{d}t \\ = &-\int_\Omega \rho^{(i)}v^{(i)}\cdot\psi(0)\,\mathrm{d}x - 2\sigma_{\mathrm{st}}\int_0^T\int_{\Gamma(t)} \nu^-\otimes\nu^- : \nabla\psi\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t,\end{aligned} \tag{3.23}$$

where $\Gamma(t) = \partial^*(\Omega^-(t))$ *is the reduced boundary of* $\Omega^-(t)$*, and* $\nu^-(t)$ *denotes the corresponding generalised outer unit normal.*

4 ***Energy inequality.*** *For a.e.* $\tau_1 \in [0,T)$*, including* $\tau_1 = 0$*, there holds*

$$\begin{aligned}&2\sigma_{\mathrm{st}}\mathcal{H}^{n-1}(\Gamma(\tau_2)) + \tfrac12\int_\Omega \rho(\tau_2)\,|v(\tau_2)|^2\,\mathrm{d}x + 2\int_{\tau_1}^{\tau_2}\int_\Omega \mu(\rho)\,|Dv|^2\,\mathrm{d}x\,\mathrm{d}t \\ \le\ &2\sigma_{\mathrm{st}}\mathcal{H}^{n-1}(\Gamma(\tau_1)) + \tfrac12\int_\Omega \rho(\tau_1)\,|v(\tau_1)|^2\,\mathrm{d}x\end{aligned} \tag{3.24}$$

for all $\tau_2 \in [\tau_1, T)$.

5 ***Transport equation.*** *The induced characteristic function* χ *given by* $\chi = \chi_{\Omega^-(\cdot)}$*, that is,*

$$\chi = \frac{\rho-\beta_2}{\beta_1-\beta_2},$$

is a weak solution of the transport equation (3.22) *with velocity* v *and prescribed initial data* $\chi^{(i)}$ *in the sense of Definition 3.8.*

From now on, we will always consider weak solutions in the sense of the foregoing definition. For convenience, for any weak solution (ρ, v), we will use the notation

$$\Omega^+(t) = \Omega\setminus(\Omega^-(t)\cup\Gamma(t)),$$

where, as in the previous definition, $\Omega^-(t)$ denotes the measure-theoretic representative set of $\rho(t)$ and $\Gamma(t) = \partial^*(\Omega^-(t))$. This means that, via $\Omega = \Omega^-(t)\cup\Gamma(t)\cup\Omega^+(t)$, this notation leads to a pairwise disjoint partition of Ω. Note that if the set $\Omega^-(t)$ is sufficiently smooth, its topological and reduced boundary coincide, i.e., $\Gamma(t) = \partial^*(\Omega^-(t)) = \partial(\Omega^-(t))$. This is consistent with Assumptions 3.1.

Remark 3.10 (Energy inequality)**.** *The energy inequality* (3.24) *restricts the class of weak solutions in Definition 3.9. This approach is in the spirit of the theory of weak solutions for the incompressible Navier–Stokes equations: in this case, for* $n = 2$*, weak solutions are unique, whereas, for* $n = 3$*, it can be shown that weak solutions are unique if one weak solution satisfies an additional regularity assumption, referred to as Serrin's condition, cf. [10, Theorem V.1.5.1].*

4 Lebesgue and Sobolev Spaces on Time-Dependent Domains

We are interested in functions that take values in Lebesgue or Sobolev spaces on time-dependent domains $(\Omega(t))_{t\in[0,T]}$, cf. also [17, 18, 19, 20]. We require the family $(\Omega(t))_{t\in[0,T]}$ to be parametrised in the following way, cf. [20, Assumption 1.1].

Assumptions 4.1 (Time evolution)**.** *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain with boundary $\partial\Omega$ of class C^3. Assume that the time evolution of the family $(\Omega(t))_{t\in[0,T]} \subset \Omega$ is described via a time-dependent C^3-diffeomorphism $\Phi(\cdot;t)\colon \overline{\Omega(0)} \to \overline{\Omega(t)}$, i.e., for every $t \in [0,T]$, there holds*

$$\Omega(t) = \Big\{\Phi(\xi;t) \,:\, \xi \in \Omega(0)\Big\} \text{ and } \overline{\Omega(t)} = \Big\{\Phi(\xi;t) \,:\, \xi \in \overline{\Omega(0)}\Big\}.$$

Denote by $\nu = \nu(\cdot,t)$ the corresponding outer unit normal and by $V = V(\cdot,t)$ the normal velocity of $(\partial\Omega(t))_{t\in[0,T]}$ with respect to ν. For $Q \subset \mathbb{R}^n \times [0,T]$, $C^0_{\mathrm{b}}(Q)$ denotes the set of all bounded, continuous real-valued functions on Q and $C^{3,1}_{\mathrm{b}}(Q)$ is given by

$$\Big\{u \in C^0_{\mathrm{b}}(Q) \,:\, \partial_t^s\partial_x^\alpha u \in C^0_{\mathrm{b}}(Q),\ 1 \le 2s + |\alpha|_* \le 3,\ s \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^n\Big\},$$

where $|\alpha|_* = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

1 ***Regularity of initial domain.*** *The initial domain $\Omega(0) \subset \mathbb{R}^n$ is a bounded domain with C^3-boundary $\partial(\Omega(0))$ and let $Q_0 = \Omega(0) \times (0,T)$.*

2 ***Regularity of** Φ**.*** $\Phi \in C^{3,1}_{\mathrm{b}}(\overline{Q_0})^n$.

3 ***Preservation of volume.*** $\det(\nabla\Phi(\xi;t)) = 1$ *for all* $(\xi,t) \in \overline{Q_0}$.

Corollary 4.2 (Space-time domain)**.** *Let Φ be as in Assumptions 4.1. Then the function $\Lambda\colon \overline{Q_0} \to \mathbb{R}^{n+1}\colon (\xi,t) \mapsto (\Phi(\xi;t),t)$ belongs to $C^{3,1}_{\mathrm{b}}(\overline{Q_0})^{n+1}$. Moreover, Λ is invertible with inverse function $\Lambda^{-1} \in C^{3,1}_{\mathrm{b}}(\overline{\Omega_T})^{n+1}$, where*

$$\Omega_T = \bigcup_{t\in(0,T)} \Big(\Omega(t) \times \{t\}\Big) \subset \mathbb{R}^{n+1}.$$

In particular, $\Phi^{-1} \in C^{3,1}_{\mathrm{b}}(\overline{\Omega_T})^n$ and Ω_T has a Lipschitz boundary.

Proof. As $\Phi \in C^{3,1}_{\mathrm{b}}(\overline{Q_0})^n$ by Assumptions 4.1, it follows that $\Lambda \in C^{3,1}_{\mathrm{b}}(\overline{Q_0})^{n+1}$. Moreover, Λ is invertible and $\Lambda^{-1}(x,t) = (\Phi^{-1}(x;t),t)$. Hence, $\Lambda^{-1} \in C^{3,1}_{\mathrm{b}}(\overline{\Omega_T})^{n+1}$, and thus $\Phi^{-1} \in C^{3,1}_{\mathrm{b}}(\overline{\Omega_T})^n$. Observing that $\partial(Q_0)$ is Lipschitz and that $\Omega_T = \Lambda(Q_0)$ finishes the proof. □

Proposition 4.3 (Normal velocity)**.** *Suppose that Assumptions 4.1 hold true. Then, for every $(x_0,t_0) \in \bigcup_{t\in[0,T]}\Big(\partial\Omega(t) \times \{t\}\Big)$, there holds*

$$V(x_0,t_0) = (\partial_t\Phi)(\Phi^{-1}(x_0;t_0);t_0) \cdot \nu(x_0,t_0).$$

Proof. For $t_0 \in [0,T]$, fix $x_0 \in \partial\Omega(t_0)$. By Assumptions 4.1, restriction to the respective boundaries yields diffeomorphisms $\Phi^{-1}(\cdot;t_0)\colon \partial\Omega(t_0) \to \partial\Omega(0)$ and, for $t \in [0,T]$, $\Phi(\cdot;t)\colon \partial\Omega(0) \to \partial\Omega(t)$. Therefore, $t \mapsto \eta(t) = \Phi(\Phi^{-1}(x_0;t_0);t) \in \partial\Omega(t)$ defines a C^1-mapping $\eta\colon [0,T] \to \mathbb{R}^n$ with $\eta(t_0) = \Phi(\Phi^{-1}(x_0;t_0);t_0) = x_0$. Thus η is an admissible choice in Definition 2.4, which yields

$$V(x_0,t_0) = \eta'(t_0) \cdot \nu(x_0,t_0) = (\partial_t\Phi)(\Phi^{-1}(x_0;t_0);t_0) \cdot \nu(x_0,t_0).$$

Consequently, V has the stated representation in terms of Φ. □

By means of the transformation $\Phi(\cdot;t)\colon \overline{\Omega(0)} \to \overline{\Omega(t)}$, we may transform Lebesgue and Sobolev functions defined on $\Omega(t)$ to functions on $\Omega(0)$. For this purpose, for $t \in [0,T]$, we introduce the transformation $\Phi_*(t)$ defined by

$$(\Phi_*(t)f)(\xi) = (\nabla\Phi)^{-1}(\Phi(\xi;t);t)f(\Phi(\xi;t)) \tag{4.1}$$

for $\xi \in \overline{\Omega(0)}$ and $f\colon \overline{\Omega(t)} \to \mathbb{R}^n$; see [20, equation (10)]. The main properties of the transformation (4.1) are collected in the next lemma; see also [20, Section 3]. In particular, it turns out that $\Phi_*(t)$ defines a divergence-preserving operator.

Lemma 4.4 (Properties of $\Phi_*(t)$)**.** *Suppose that $(\Omega(t))_{t\in[0,T]}$ is as in Assumptions 4.1. Let $k = 0,1,2$, $l = 1,2$, $q \in [1,\infty]$ and $t \in [0,T]$. Then the operator $\Phi_*(t)$ defined by* (4.1) *has the following properties.*

1 *The mapping $\Phi_*(t)\colon W^{k,q}(\Omega(t))^n \to W^{k,q}(\Omega(0))^n$ is an isomorphism. Its inverse operator $\Phi_*^{-1}(t)$ is given by $(\Phi_*^{-1}(t)h)(x) = (\nabla\Phi)(x;t)h(\Phi^{-1}(x;t))$ for $h \in W^{k,q}(\Omega(0))^n$.*

2 *There are constants $C_1, C_2 > 0$, which do not depend on t, such that*

$$C_1\|\Phi_*(t)f\|_{W^{k,q}(\Omega(0))^n} \le \|f\|_{W^{k,q}(\Omega(t))^n} \le C_2\|\Phi_*(t)f\|_{W^{k,q}(\Omega(0))^n}. \tag{4.2}$$

3 *The mapping $\Phi_*(t)\colon W_0^{l,q}(\Omega(t))^n \to W_0^{l,q}(\Omega(0))^n$ is an isomorphism.*

4 *For any $f \in W^{1,q}(\Omega(t))^n$, there holds $\operatorname{div}(f)\circ\Phi(\cdot;t) \in L^q(\Omega(0))$ and $\operatorname{div}(\Phi_*(t)f) = \operatorname{div}(f)\circ\Phi(\cdot;t)$. Moreover, $\Phi_*(t)\colon L^q_\sigma(\Omega(t)) \to L^q_\sigma(\Omega(0))$ is an isomorphism.*

Proof. The proof is straightforward. See [9, Lemma 4.4.6] for details. □

We are interested in functions of the form $t \mapsto f(t) \in L^q(\Omega(t))$ or $t \mapsto f(t) \in W^{k,q}(\Omega(t))$. To define these function spaces, we will always suppose that $(\Omega(t))_{t\in[0,T]}$ satisfies the regularity conditions gathered together in Assumptions 4.1. For functions $f \in L^1_{\mathrm{loc}}(\Omega_T)$, the distributional derivatives $\partial_t^k\partial_x^\alpha f$ with $(k,\alpha) \in \mathbb{N}_0 \times \mathbb{N}_0^n$ are well-defined. This allows us to define the following Bochner-type function spaces.

Definition 4.5 (Lebesgue and Sobolev spaces on time-dependent domains)**.** *Suppose that $(\Omega(t))_{t\in[0,T]}$ satisfies Assumptions 4.1. Let $s, r \in [1,\infty]$ and $q \in \mathbb{N}_0$.*

1 *The space $L^s(0,T;L^r(\Omega(t)))$ consists of all $f \in L^1_{\mathrm{loc}}(\Omega_T)$ such that $f(t) \in L^r(\Omega(t))$ for a.e. $t \in (0,T)$, and $(t \mapsto \|f(t)\|_{L^r(\Omega(t))}) \in L^s(0,T)$.*

2 *The space $L^s(0,T;W^{q,r}(\Omega(t)))$ consists of all $f \in L^s(0,T;L^r(\Omega(t)))$ such that, for all $\alpha \in \mathbb{N}_0^n$ with $\alpha_1 + \alpha_2 + \cdots + \alpha_n \le q$, there holds $\partial_x^\alpha f \in L^s(0,T;L^r(\Omega(t)))$.*

3 *The space $W^{1,s}(0,T;W^{q,r}(\Omega(t)))$ consists of all $f \in L^s(0,T;W^{q,r}(\Omega(t)))$ such that $\partial_t f \in L^s(0,T;W^{q,r}(\Omega(t)))$.*

4 *The vector-valued versions of the above spaces are given by*

$$L^s(0,T;W^{q,r}(\Omega(t))^n) = L^s(0,T;W^{q,r}(\Omega(t)))^n,$$
$$W^{1,s}(0,T;W^{q,r}(\Omega(t))^n) = W^{1,s}(0,T;W^{q,r}(\Omega(t)))^n.$$

5 *Let $X(t)$ stand for either $W^{q,r}(\Omega(t))$ or $W^{q,r}(\Omega(t))^n$. The space $L^s(0,T;X(t))$ is equipped with the norm*

$$\|f\|_{L^s(0,T;X(t))} = \begin{cases} \left(\int_0^T \|f(t)\|_{X(t)}^s \,\mathrm{d}t\right)^{\frac{1}{s}} & \text{if } s < \infty, \\ \operatorname{ess\,sup}_{t\in(0,T)} \|f(t)\|_{X(t)} & \text{if } s = \infty. \end{cases}$$

The space $W^{1,s}(0,T;X(t))$ is equipped with the norm

$$\|f\|_{W^{1,s}(0,T;X(t))} = \left(\|f\|_{L^s(0,T;X(t))}^2 + \|\partial_t f\|_{L^s(0,T;X(t))}^2\right)^{\frac{1}{2}}.$$

Remark 4.6. *We want to point out that, in the foregoing Definition 4.5 we crucially used the fact that all defined function spaces are subspaces of $L^1(\Omega_T)$.*

We may use $\Phi_*(t)$ to transform functions from the previous definitions to functions taking values in time-independent Lebesgue or Sobolev spaces, i.e., functions belonging to the usual Bochner spaces. To this end, we define $\Phi_* f$ by

$$t \mapsto \Phi_*(t) f(\cdot,t). \tag{4.3}$$

Owing to the time-independent bounds on $\Phi_*(t)$ and its inverse $\Phi_*^{-1}(t)$ from Lemma 4.4, the transformation properties carry over to Φ_*, as we now show. The function spaces introduced in Definition 4.5 are transformed as follows.

Proposition 4.7 (Properties of Φ_*)**.** *Suppose that Assumptions 4.1 hold true. Let $s \in [1,\infty]$, $q \in [1,\infty)$ and $k = 0,1,2$. Denote by $X(\tau)$, $\tau \in [0,T]$, either of the spaces $W^{k,q}(\Omega(\tau))^n$, $W_0^{k,q}(\Omega(\tau))^n$ or $L^q_\sigma(\Omega(\tau))$. Then Φ_*, given by* (4.3)*, is a diffeomorphism between the spaces $L^s(0,T;X(t))$ and $L^s(0,T;X(0))$ as well as between the spaces*

$$W^{1,s}(0,T;L^q(\Omega(t))^n) \cap L^s(0,T;W^{1,q}(\Omega(t))^n)$$

and

$$W^{1,s}(0,T;L^q(\Omega(0))^n) \cap L^s(0,T;W^{1,q}(\Omega(0))^n).$$

Proof. By Lemma 4.4, Φ_* is an isomorphism between spaces of the form $L^s(0,T;X(t))$ and $L^s(0,T;X(0))$. For the proof of the remaining claim, we study the transformation of time derivatives. Let $f \in W^{1,s}(0,T;L^q(\Omega(t))^n) \cap L^s(0,T;W^{1,q}(\Omega(t))^n)$. Hence $\Phi_* f \in L^s(0,T;W^{1,q}(\Omega(0))^n)$. By the definition of $W^{1,s}(0,T;L^q(\Omega(t))^n)$, there holds that $\partial_t f \in L^s(0,T;L^q(\Omega(t))^n)$. To prove that $\partial_t(\Phi_* f)$ belongs to $L^s(0,T;L^q(\Omega(0))^n)$, we use the mapping $\Lambda\colon (\xi,t) \mapsto (\Phi(\xi;t),t)$, which belongs to $C_\mathrm{b}^{3,1}(\Omega(0)\times(0,T))^{n+1}$, by Corollary 4.2, and that we may write

$$\Phi_* g = ((\nabla\Phi)^{-1}\circ\Lambda)(g\circ\Lambda) \tag{4.4}$$

for any $g \in L^s(0,T;L^q(\Omega(0))^n)$. Using the product and the chain rule, we see

$$\begin{aligned} \partial_t(\Phi_* f) &= \partial_t\Big(((\nabla\Phi)^{-1}\circ\Lambda)(f\circ\Lambda)\Big) \\ &= \Big(\partial_t\Big((\nabla\Phi)^{-1}\circ\Lambda\Big)\Big)(f\circ\Lambda) + \Big((\nabla\Phi)^{-1}\circ\Lambda\Big)\partial_t(f\circ\Lambda) \\ &= \Big(\partial_t\Big((\nabla\Phi)^{-1}\circ\Lambda\Big)\Big)(f\circ\Lambda) + \Big((\nabla\Phi)^{-1}\circ\Lambda\Big)\Big((\nabla f\circ\Lambda)\partial_t\Phi + \partial_t f\circ\Lambda\Big) \\ &= \Big(\partial_t\Big((\nabla\Phi)^{-1}\circ\Lambda\Big)\Big)(f\circ\Lambda) + \Big((\nabla\Phi)^{-1}\circ\Lambda\Big)\Big((\sum_{i=1}^n(\partial_i f\circ\Lambda)\partial_t\Phi_i) + \partial_t f\circ\Lambda\Big). \end{aligned}$$

Recalling (4.4), it follows that

$$\partial_t(\Phi_* f) = \partial_t\Big((\nabla\Phi)^{-1} \circ \Lambda\Big)(\nabla\Phi \circ \Lambda)\Phi_* f + \Big(\sum_{i=1}^{n} \Phi_*(\partial_i f)\partial_t \Phi_i\Big) + \Phi_*(\partial_t f).$$

Since Φ and Λ belong to $C^{3,1}_{\mathrm{b}}(\overline{Q_0})^n$ and $C^{3,1}_{\mathrm{b}}(\overline{Q_0})^{n+1}$, respectively, the functions $\partial_t\Big((\nabla\Phi)^{-1}\circ\Lambda\Big)$, $\nabla\Phi \circ \Lambda$ and $\partial_t\Phi$ are continuous and bounded on $Q_0 = \Omega(0) \times (0,T)$. Moreover, $\Phi_* f$, $\Phi_*(\partial_i f)$, $i = 1, \ldots, n$, and $\Phi_*(\partial_t f)$ belong to $L^s(0,T; W^{1,q}(\Omega(0))^n)$. This implies $\partial_t(\Phi_* f) \in L^s(0,T; L^q(\Omega(0))^n)$, and thus $\Phi_* f \in W^{1,s}(0,T; L^q(\Omega(0))^n)$. The remaining claim follows by similar arguments, as in the proof of Lemma 4.4. □

In the spirit of Theorem 2.6, we obtain the following integration-by-parts formula for Sobolev spaces on time-dependent domains.

Lemma 4.8 (Integration by parts)**.** *Suppose that Assumptions 4.1 hold true. For $r \in [1,\infty)$, let $f \in W^{1,r}(\Omega_T)$ and $\varphi \in C_0^\infty(\Omega \times (0,T))$. Then there holds*

$$\int_0^T \int_{\Omega(t)} \partial_t f \varphi \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_{\Omega(t)} f \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{\partial\Omega(t)} V f \varphi \,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t. \tag{4.5}$$

Proof. By Corollary 4.2, the space-time domain Ω_T has a Lipschitz boundary. By density of $C^\infty(\overline{\Omega_T}) \cap W^{1,r}(\Omega_T)$ in

$$W^{1,r}(\Omega_T) = W^{1,r}(0,T; L^r(\Omega(t))) \cap L^r(0,T; W^{1,r}(\Omega(t))),$$

see [21, p. 127, Theorem 3], there exists an approximating sequence $(f_m)_{m\in\mathbb{N}} \subset C^\infty(\overline{\Omega_T}) \cap W^{1,r}(\Omega_T)$ such that $f_m \to f$ in $W^{1,r}(\Omega_T)$ as $m \to \infty$. Using Theorem 2.6, we obtain

$$\begin{aligned} 0 &= \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} f_m \varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\Omega(t)} \partial_t(f_m\varphi) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\partial\Omega(t)} V f_m \varphi \,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t. \end{aligned}$$

Hence

$$\begin{aligned} &\int_0^T \int_{\Omega(t)} \partial_t f_m \varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T \int_{\Omega(t)} f_m \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{\partial\Omega(t)} V f_m \varphi \,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t. \end{aligned}$$

In view of Corollary 4.2 and Proposition 4.3, the normal velocity V is bounded. By standard properties of the trace operator [21, p. 133, Theorem 3], we obtain (4.5) by letting $m \to \infty$ in the final equation. □

5 Consistency of the Weak Formulation

The notion of weak solutions for the sharp-interface model incorporates the variational formulation (3.23), using test functions from the space $C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega))$. Just as in the theory of the incompressible Navier–Stokes equations, this choice removes the pressure function from the weak formulation, cf. [10, Definition V.1.1.1]. Thus it is not clear that the test space $C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega))$ is appropriate. To justify this choice, we will prove that, under additional regularity assumptions given below, it is possible to reconstruct a pressure function from the weak formulation. To this end, we will basically proceed in two steps. Firstly, we will reconstruct an associated pressure function in the whole space-time domain $\Omega \times (0,T)$. Secondly, we shall readjust the associated pressure function separately in the space-time domains

$$\Omega^- = \bigcup_{t\in(0,T)} (\Omega^-(t) \times \{t\}) \text{ and } \Omega^+ = \bigcup_{t\in(0,T)} (\Omega^+(t) \times \{t\})$$

to satisfy the dynamical Young–Laplace law (1.6) in an appropriate trace sense.

Assumptions 5.1. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega$ of class C^3. Let (ρ, v) be a weak solution of the free-boundary problem* (1.1)–(1.8) *in the sense of Definition 3.9 with respect to prescribed initial data $(\rho^{(i)}, v^{(i)}) \in BV(\Omega, \{\beta_1, \beta_2\}) \times H^1_{0,\sigma}(\Omega)$, such that the measure-theoretic representative set $\Omega^{-,(i)} = \Omega^-(0)$ of $\rho^{(i)}$ is compactly contained in Ω and has a C^3-boundary. Moreover, let the following regularity properties hold true.*

1 ***Regularity of interface.*** *For any $t \in [0,T]$, $\Phi^-(\cdot;t)\colon \overline{\Omega^-(0)} \to \overline{\Omega^-(t)}$ is a diffeomorphism as in Assumptions 4.1, such that the time evolution of the measure-theoretic representative set $\Omega^-(t)$ of $\rho(t)$ is described by $\Phi^-(\cdot;t)$, i.e., for every $t \in [0,T]$, there holds*

$$\Omega^-(t) = \left\{\Phi^-(\xi;t) \,:\, \xi \in \Omega^-(0)\right\} \text{ and } \overline{\Omega^-(t)} = \left\{\Phi^-(\xi;t) \,:\, \xi \in \overline{\Omega^-(0)}\right\}.$$

Additionally, for all $t \in [0,T]$, the interface $\Gamma(t) = \partial(\Omega^-(t)) \cap \Omega$ is compactly contained in Ω, that is, $\Gamma(t) = \partial(\Omega^-(t)) \subset\subset \Omega$. Denote by $\nu^- = \nu^-(\cdot,t)$ the unit normal to $\Gamma(t)$ pointing outward to $\Omega^-(t)$ and by $V = V(\cdot,t)$ the normal velocity of $(\Gamma(t))_{t\in[0,T]}$ with respect to ν^-. Similarly, let the time evolution of $\Omega^+(t) = \Omega \setminus (\Omega^-(t) \cup \Gamma(t))$ be described by a diffeomorphism $\Phi^+(\cdot;t)\colon \overline{\Omega^+(0)} \to \overline{\Omega^+(t)}$ satisfying Assumptions 4.1.

2 ***Regularity of velocity.***

$$v|_{\Omega^\pm} \in L^2(0,T; W^{2,2}(\Omega^\pm(t))^n) \cap W^{1,2}(0,T; L^2(\Omega^\pm(t))^n).$$

5.1 The Mean-Curvature Functional for Smooth Interfaces

Due to Assumptions 5.1, the family of interfaces $(\Gamma(t))_{t\in[0,T]}$ has additional regularity properties. This allows us to extend the mean-curvature function to the space-time domain $\Omega \times (0,T)$. For the proof, we study the transformation of the trace spaces $L^2(\Gamma(t)) = L^2(\partial(\Omega^-(t)))$ and $H^{\frac{1}{2}}(\Gamma(t)) = H^{\frac{1}{2}}(\partial(\Omega^-(t)))$.

Lemma 5.2 (Transformation of trace spaces)**.** *Suppose that Assumptions 5.1 hold true, and let $t \in [0,T]$. Then the pullback operator Φ^-_{-t}, defined by $\Phi^-_{-t}u = u \circ \Phi^-(\cdot,t)$ for $u \in L^2(\Gamma(t))$, induces linear homeomorphisms*

$$\Phi^-_{-t}\colon L^2(\Gamma(t)) \to L^2(\Gamma(0)) \text{ and } \Phi^-_{-t}\colon H^{\frac{1}{2}}(\Gamma(t)) \to H^{\frac{1}{2}}(\Gamma(0)),$$

such that

$$C_1\|u\|_{L^2(\Gamma(t))} \le \|\Phi^-_{-t}u\|_{L^2(\Gamma(0))} \le C_2\|u\|_{L^2(\Gamma(t))} \tag{5.1}$$

for every $u \in L^2(\Gamma(t))$*, and*

$$C_1\|u\|_{H^{\frac{1}{2}}(\Gamma(t))} \leq \|\Phi^-_{-t}u\|_{H^{\frac{1}{2}}(\Gamma(0))} \leq C_2\|u\|_{H^{\frac{1}{2}}(\Gamma(t))}$$

for every $u \in H^{\frac{1}{2}}(\Gamma(t))$ *with constants* $C_1, C_2 > 0$ *independent of* u *and* t*. In particular, there are constants* $C_3, C_4 > 0$ *such that*

$$C_3\mathcal{H}^{n-1}(\Gamma(t))) \leq \mathcal{H}^{n-1}(\Gamma(0))) \leq C_4\mathcal{H}^{n-1}(\Gamma(t))). \tag{5.2}$$

Proof. The estimate (5.2) follows from (5.1) applied to the constant function $u \equiv 1$. The proof of the remaining claims can be found in [18, Section 5.4.1]. □

Lemma 5.3 (Mean-curvature functional)**.** *If Assumptions 5.1 hold true, then there exists a function* $m \in L^\infty(0, T; H^1(\Omega^-(t)))$ *with the following properties.*

1 Let $t \in [0, T]$*. For the trace of* $m(t)$ *on the boundary* $\Gamma(t) = \partial\Omega^-(t)$*, there holds*

$$m(t)|_{\Gamma(t)} = \kappa(t). \tag{5.3}$$

2 The zero extension K *of* ∇m *to* $\Omega \times (0, T)$ *belongs to* $L^\infty(0, T; L^2(\Omega)^n)$ *and, for every* $\psi \in C_0^\infty([0, T); C^\infty_{0,\sigma}(\Omega))$*, there holds*

$$\begin{aligned}\int_0^T\int_\Omega K \cdot \psi \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\Gamma(t)} \kappa\nu^- \cdot \psi \,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t \\ &= \int_0^T\int_{\Gamma(t)} \nu^- \otimes \nu^- : \nabla\psi \,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t.\end{aligned} \tag{5.4}$$

Proof. Let $t \in [0, T]$. We apply the pullback operator $\Phi^-_{-t}\colon H^{\frac{1}{2}}(\Gamma(t)) \to H^{\frac{1}{2}}(\Gamma(0))$, introduced in Lemma 5.2, to the mean-curvature function $\kappa(t) \in C^1(\Gamma(t)) \subset H^{\frac{1}{2}}(\Gamma(t))$. For notational convenience, we suppress the upper index $^-$ and simply write $\Phi = \Phi^-$ and $\Phi_{-t} = \Phi^-_{-t}$ in the remainder of this proof. We define $\tilde{\kappa}(x, t) = \Phi_{-t}\kappa(x, t) = \kappa(\Phi(x; t), t)$ for $x \in \Gamma(0) = \partial(\Omega^-(0))$. Since $\Omega^-(0)$ has a C^3-boundary, there exists a weak solution $\tilde{u} = \tilde{u}(t) \in H^1(\Omega^-(0))$ of

$$\begin{aligned}\Delta\tilde{u}(t) &= 0 \qquad \text{in } \Omega^-(0), \\ \tilde{u}(t) &= \tilde{\kappa}(t) \quad \text{on } \Gamma(0)\end{aligned} \tag{5.5}$$

depending on t, which additionally satisfies the estimate

$$\|\tilde{u}(t)\|_{H^1(\Omega^-(0))} \leq C(\Omega^-(0))\|\tilde{\kappa}(t)\|_{H^{\frac{1}{2}}(\Gamma(0))}$$

for some constant $C(\Omega^-(0)) > 0$, depending on $\Omega^-(0)$, but independent of t; see [15, Theorem III.4.1]. By Assumptions 5.1 and the foregoing Lemma 5.2, we infer that for a suitable constant $C > 0$, independent of t, there holds

$$\|\tilde{u}(t)\|_{H^1(\Omega^-(0))} \leq C(\Omega^-(0))\|\kappa(t)\|_{H^{\frac{1}{2}}(\Gamma(t))} \leq C.$$

Therefore, the function $m\colon \Omega^- \to \mathbb{R}$ defined by $m(x, t) = \tilde{u}(\Phi^{-1}(x; t), t)$ for $x \in \Omega^-(t)$ belongs to $L^\infty(0, T; H^1(\Omega^-(t)))$ and, by construction, satisfies (5.3).

Concerning the second claim, we define $K\colon \Omega\times(0,T)\to\mathbb{R}$, for any $(x,t)\in\Omega\times(0,T)$, by

$$K(x,t)=\begin{cases}\nabla m(x,t) & \text{if } x\in\Omega^-(t),\\ 0 & \text{if } x\in\Omega\setminus\Omega^-(t).\end{cases}$$

Then K belongs to $L^\infty(0,T;L^2(\Omega)^n)$. For every $\psi\in C_0^\infty([0,T);C_{0,\sigma}^\infty(\Omega))$, we then obtain

$$\int_0^T\int_\Omega K(t)\cdot\psi(t)\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_{\Gamma(t)} m(t)\psi(t)\cdot\nu^-(t)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t.$$

Taking into account (5.3), we conclude the first identity in (5.4). Noting that the last equality in (5.4) follows from Lemma 3.4 finishes the proof. □

For our purposes, it is important to note that, if Assumptions 5.1 are satisfied, then Lemma 5.3 allows one to replace (3.23) by

$$\begin{aligned}&\int_0^T\int_\Omega \rho v\cdot\partial_t\psi+\rho v\otimes v:\nabla\psi-2\mu(\rho)Dv:D\psi\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega\rho^{(i)}v^{(i)}\cdot\psi(0)\,\mathrm{d}x\\ =&-2\sigma_{\mathrm{st}}\int_0^T\int_\Omega K\cdot\psi\,\mathrm{d}x\,\mathrm{d}t\end{aligned}\tag{5.6}$$

for all $\psi\in C_0^\infty([0,T);C_{0,\sigma}^\infty(\Omega))$. It is also convenient to introduce $\mathcal{G}\in\mathcal{D}'(\Omega\times(0,T))^n$ given by

$$\begin{aligned}\langle\mathcal{G},\psi\rangle_{\mathcal{D}(\Omega\times(0,T))^n}=&\int_0^T\int_\Omega\rho v\cdot\partial_t\psi+\rho v\otimes v:\nabla\psi-2\mu(\rho)Dv:D\psi\,\mathrm{d}x\,\mathrm{d}t\\ &+2\sigma_{\mathrm{st}}\int_0^T\int_\Omega K\cdot\psi\,\mathrm{d}x\,\mathrm{d}t\end{aligned}\tag{5.7}$$

for $\psi\in C_0^\infty(\Omega\times(0,T))^n$. Note that, for all $\psi\in C_0^\infty((0,T);C_{0,\sigma}^\infty(\Omega))$, there holds

$$\langle\mathcal{G},\psi\rangle_{\mathcal{D}(\Omega\times(0,T))^n}=0.\tag{5.8}$$

5.2 Existence of an Associated Pressure Function

We shall prove the existence of an associated pressure function, that is, a distribution $p\in\mathcal{D}'(\Omega\times(0,T))$ such that

$$\nabla p=-\rho\partial_t v-\operatorname{div}(\rho v\otimes v)-2\mu(\rho)\operatorname{div}(Dv)+2\sigma_{\mathrm{st}}K\ \text{in}\ \mathcal{D}'(\Omega\times(0,T))^n.$$

The theory of the incompressible Navier–Stokes equations provides us with the following key tool.

Theorem 5.4. *Let $r,s\in(1,\infty)$ and let $r'=\frac{r}{r-1}$. If $\mathcal{F}\in L^s(0,T;W^{-1,r}(\Omega)^n)$ satisfies*

$$\int_0^T\langle\mathcal{F}(t),\psi(t)\rangle_{W_0^{1,r'}(\Omega)^n}\,\mathrm{d}t=0\ \textit{for all}\ \psi\in C_0^\infty((0,T);C_{0,\sigma}^\infty(\Omega)),$$

then there exists a unique $p\in L^s(0,T;L^r(\Omega))$ satisfying $\mathcal{F}=\nabla p$ in $\mathcal{D}'(\Omega\times(0,T))^n$; that is,

$$\langle\mathcal{F},\psi\rangle_{\mathcal{D}(\Omega\times(0,T))^n}=\int_0^T\langle\nabla p(t),\psi(t)\rangle_{\mathcal{D}(\Omega)^n}\,\mathrm{d}t=-\int_0^T\int_\Omega p\operatorname{div}(\psi)\,\mathrm{d}x\,\mathrm{d}t$$

for all $\psi\in C_0^\infty(\Omega\times(0,T))^n$, and, for a.e. $t\in(0,T)$, there holds

$$\int_\Omega p(t)\,\mathrm{d}x=0.$$

Proof. See [10, Lemma IV.1.4.1]. □

Although the functional $\mathcal{G}$, defined by (5.7), vanishes on $C_0^\infty((0,T);C_{0,\sigma}^\infty(\Omega))$ due to (5.8), the functional

$$\psi \mapsto \int_0^T \int_\Omega \rho v \cdot \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t$$

does not in general belong to any $L^s(0,T;W^{-1,r}(\Omega))$-space. To circumvent this problem, we improve the properties of this functional by taking into account Assumptions 5.1.

Proposition 5.5. *Suppose that Assumptions 5.1 are satisfied. Then there holds*

$$\begin{aligned}&\int_0^T \int_{\Omega\setminus\Gamma(t)} \rho \partial_t v \cdot \psi \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T \int_\Omega \rho v \cdot \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t - (\beta_1-\beta_2)\int_0^T \int_{\Gamma(t)} V(v\cdot\psi)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t\end{aligned} \tag{5.9}$$

for any $\psi \in C_0^\infty(\Omega\times(0,T))^n$.

Proof. Since, by Assumptions 5.1, v belongs to

$$L^2(0,T;H^1(\Omega^\pm(t))^n) \cap W^{1,2}(0,T;L^2(\Omega^\pm(t))^n),$$

for any $\psi \in C_0^\infty(\Omega\times(0,T))^n$, the integration-by-parts formula (Lemma 4.8) yields

$$\begin{aligned}&\beta_1\int_0^T \int_{\Omega^-(t)} \partial_t v \cdot \psi \,\mathrm{d}x\,\mathrm{d}t + \beta_2\int_0^T \int_{\Omega^+(t)} \partial_t v \cdot \psi \,\mathrm{d}x\,\mathrm{d}t \\ &= -\beta_1\int_0^T \int_{\Omega^-(t)} v\cdot\partial_t\psi \,\mathrm{d}x\,\mathrm{d}t - \beta_1\int_0^T \int_{\Gamma(t)} V(v\cdot\psi)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t \\ &\quad -\beta_2\int_0^T \int_{\Omega^+(t)} v\cdot\partial_t\psi \,\mathrm{d}x\,\mathrm{d}t + \beta_2\int_0^T \int_{\Gamma(t)} V(v\cdot\psi)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t.\end{aligned}$$

Recalling that $\rho = (\beta_1-\beta_2)\chi + \beta_2$ finally yields the claim. □

Remark 5.6 (Time derivatives across the interface)**.** *In* (5.9)*, the domain of integration is* $\Omega\setminus\Gamma(t) = \Omega^-(t)\cup\Omega^+(t)$ *instead of the whole domain* Ω*, despite the fact that* $\Gamma(t)$ *has Lebesgue measure zero. This is because, by Assumptions 5.1, the restrictions of* v *to* $\Omega^\pm$ *belong to some* $W^{1,q}(0,T;L^q(\Omega^\pm(t))^n)$*-space. However, this does not give any information about the behaviour of* $\partial_t v$ *on the interface* $\Gamma(t)$*. In particular, we cannot assume that* $\partial_t v$ *exists in the sense of weak derivatives on* $\Omega\times(0,T)$*.*

We now prove some preparatory results, which incorporate the additional properties from Assumptions 5.1, before we reconstruct the pressure function with the help of Theorem 5.4.

Proposition 5.7. *If Assumptions 5.1 are satisfied, then* v *has the following properties.*

1 $\operatorname{div}(v) = 0$ *in* $\Omega\times(0,T)$.

2 For every $\varphi \in C_0^\infty(\Omega\times(0,T))$*, there holds*

$$\begin{aligned}\int_0^T \int_{\Gamma(t)} V\varphi \,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t &= -\int_0^T \int_\Omega \chi\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\Gamma(t)} (v\cdot\nu^-)\varphi\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t.\end{aligned} \tag{5.10}$$

3 For every $\psi \in C_0^\infty(\Omega \times (0,T))^n$*, there holds*

$$\int_0^T \int_{\Gamma(t)} V(v \cdot \psi) \,\mathrm{d}\mathcal{H}^{n-1}(x) \,\mathrm{d}t = \int_0^T \int_{\Gamma(t)} (v \cdot \psi)(v \cdot \nu^-) \,\mathrm{d}\mathcal{H}^{n-1}(x) \,\mathrm{d}t. \tag{5.11}$$

Proof. 1 By Definition 3.9, $v \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_0(\Omega)^n)$. In particular, this means that $v \in L^2(0,T;H^1_{0,\sigma}(\Omega)^n)$. Finally, Lemma 2.1 implies the first claim.

2 Let $\varphi \in C_0^\infty(\Omega \times (0,T))$. The first equality in (5.10) follows from the first statement of Theorem 2.6. For the proof of the second equality in (5.10), we use that χ is a weak solution of the transport equation (3.22). Thus, by (3.20), we have

$$-\int_0^T \int_\Omega \chi \partial_t \varphi \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_\Omega \chi v \cdot \nabla \varphi \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_{\Omega^-(t)} v \cdot \nabla \varphi \,\mathrm{d}x \,\mathrm{d}t.$$

Using integration by parts and $\operatorname{div}(v) = 0$ in $\Omega \times (0,T)$, it follows

$$\begin{aligned} &-\int_0^T \int_\Omega \chi \partial_t \varphi \,\mathrm{d}x \,\mathrm{d}t \\ =\ &-\int_0^T \int_{\Omega^-(t)} \operatorname{div}(v) \varphi \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_{\Gamma(t)} (v \cdot \nu^-) \varphi \,\mathrm{d}\mathcal{H}^{n-1}(x) \,\mathrm{d}t \\ =\ &\int_0^T \int_{\Gamma(t)} (v \cdot \nu^-) \varphi \,\mathrm{d}\mathcal{H}^{n-1}(x) \,\mathrm{d}t. \end{aligned}$$

This proves (5.10).

3 Due to Assumptions 5.1, there holds

$$v \in W^{1,2}(0,T;L^2(\Omega^-(t))^n) \cap L^2(0,T;H^1(\Omega^-(t)))^n) = H^1(\Omega_T^-)^n.$$

By Corollary 4.2, Ω_T^- has a Lipschitz boundary, and therefore $C^\infty\left(\overline{\Omega_T^-}\right) \cap H^1(\Omega_T^-)$ is dense in $H^1(\Omega_T^-)$; see [21, p. 127, Theorem 3]. This means that there exists an approximating sequence $(v_m)_{m \in \mathbb{N}} \subset C^\infty\left(\overline{\Omega_T^-}\right)^n \cap H^1(\Omega_T^-)^n$ such that $v_m \to v$ in $H^1(\Omega_T^-)^n$ as $m \to \infty$. For any $\psi \in C_0^\infty(\Omega \times (0,T))^n$, for $m \to \infty$, we obtain

$$\chi \partial_t (v_m \cdot \psi) \to \chi \partial_t (v \cdot \psi) \text{ in } L^2(\Omega_T^-) \tag{5.12}$$

and

$$\chi \nabla (v_m \cdot \psi) \to \chi \nabla (v \cdot \psi) \text{ in } L^2(\Omega_T^-)^n \tag{5.13}$$

since $\chi \in L^\infty(\Omega \times (0,T))$ and $v \in L^2(0,T;L^2(\Omega)^n)$. As χ is a weak solution of the transport equation, by (3.20), we infer that

$$\int_0^T \int_\Omega \chi \partial_t (v_m \cdot \psi) \,\mathrm{d}x \,\mathrm{d}t = -\int_0^T \int_\Omega \chi v \cdot \nabla (v_m \cdot \psi) \,\mathrm{d}x \,\mathrm{d}t.$$

Now (5.12) and (5.13) allow us to pass to the limit $m \to \infty$. This yields

$$\begin{aligned} \int_0^T \int_{\Omega^-(t)} \partial_t (v \cdot \psi) \,\mathrm{d}x \,\mathrm{d}t &= \int_0^T \int_\Omega \chi \partial_t (v \cdot \psi) \,\mathrm{d}x \,\mathrm{d}t \\ &= -\int_0^T \int_\Omega \chi v \cdot \nabla (v \cdot \psi) \,\mathrm{d}x \,\mathrm{d}t. \end{aligned}$$

As the integration-by-parts formula (see Lemma 4.8) applies to the left-hand side and as $\operatorname{div}(v) = 0$ in Ω^-, we obtain

$$\begin{aligned}\int_0^T \int_{\Gamma(t)} V(v\cdot\psi)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t &= \int_0^T \int_{\Omega^-(t)} \operatorname{div}((v\cdot\psi)v)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\Gamma(t)} (v\cdot\psi)(v\cdot\nu^-)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t.\end{aligned}$$

This justifies (5.11), which completes the proof.

□

Next, we explore the regularity of the convective term $(v\cdot\nabla)v$.

Lemma 5.8 (Regularity of convective term)**.** *Let Assumptions 5.1 be satisfied. Then* $(v\cdot\nabla)v$ *belongs to* $L^2(0,T;L^3(\Omega^\pm(t))^n)$.

Proof. Let Φ_* be given by (4.3). In view of Proposition 4.7, there holds that

$$w = \Phi_* v \in L^2(0,T;W^{2,2}(\Omega^\pm(0))^n) \cap W^{1,2}(0,T;L^2(\Omega^\pm(0))^n), \tag{5.14}$$

and it is sufficient to verify that $w\cdot\nabla w_i \in L^2(0,T;L^3(\Omega^\pm(0)))$ for $i = 1,\ldots,n$. To this end, we will use the continuous embedding

$$\begin{aligned}&L^2(0,T;H^1(\Omega^\pm(0))) \cap W^{1,2}(0,T;W^{-1,2}(\Omega^\pm(0))) \\ &\quad\hookrightarrow C^0([0,T];L^2(\Omega^\pm(0)));\end{aligned} \tag{5.15}$$

see [22, Chapter III, Theorem 4.10.2] and [23, Théorème 12.4]. As (5.14) implies that $w, \nabla w_i \in L^2(0,T;H^1(\Omega^\pm(0))^n) \cap W^{1,2}(0,T;W^{-1,2}(\Omega^\pm(0))^n)$, taking into account the embedding (5.15), we conclude that

$$w \in C^0([0,T];H^1(\Omega^\pm(0))^n) \hookrightarrow L^\infty(0,T;L^6(\Omega^\pm(0))^n).$$

Then, by Hölder's inequality, we obtain

$$\|w\cdot\nabla w_i\|_{L^2(0,T;L^3(\Omega^\pm(0)))} \le \|w\|_{L^\infty(0,T;L^6(\Omega^\pm(0))^n)} \|\nabla w_i\|_{L^2(0,T;L^6(\Omega^\pm(0))^n)},$$

which completes the proof. □

Using Proposition 5.5, we improve the regularity of the functional $\mathcal{G}$; see (5.7).

Proposition 5.9. *Suppose that Assumptions 5.1 hold true and let* $\mathcal{G}$ *be as in* (5.7)*. For* $\psi \in C_0^\infty(\Omega\times(0,T))^n$*, define* $\mathcal{G}_{\mathrm{reg}} \in \mathcal{D}'(\Omega\times(0,T))^n$ *by*

$$\begin{aligned}&\langle \mathcal{G}_{\mathrm{reg}}, \psi\rangle_{\mathcal{D}(\Omega\times(0,T))^n} \\ &= -\int_0^T \int_{\Omega\setminus\Gamma(t)} \rho\partial_t v\cdot\psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \rho v\cdot\nabla v\cdot\psi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - 2\int_0^T\int_\Omega \mu(\rho) Dv : D\psi\,\mathrm{d}x\,\mathrm{d}t + 2\sigma_{\mathrm{st}} \int_0^T\int_\Omega K\cdot\psi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{5.16}$$

Then $\mathcal{G}_{\mathrm{reg}}$ *extends to a functional belonging to* $L^2(0,T;H^{-1}(\Omega)^n)$*. Moreover, there holds* $\langle \mathcal{G}, \psi\rangle_{\mathcal{D}(\Omega\times(0,T))^n} = \langle \mathcal{G}_{\mathrm{reg}}, \psi\rangle_{\mathcal{D}(\Omega\times(0,T))^n}$ *for all* $\psi \in C_0^\infty(\Omega\times(0,T))^n$.

Proof. In view of Assumptions 5.1, Lemma 5.8, Definition 3.9 and Lemma 5.3, $\mathcal{G}_{\mathrm{reg}}$ extends to a functional belonging to the class $L^2(0,T;H^{-1}(\Omega)^n)$.

Let $\psi \in C_0^\infty(\Omega \times (0,T))^n$. It suffices to show that

$$\begin{aligned}&\int_0^T\int_\Omega \rho(v\cdot\partial_t\psi + v\otimes v:\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t\\ =\ &-\int_0^T\int_{\Omega\setminus\Gamma(t)} \rho\partial_t v\cdot\psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \rho\,\mathrm{div}(v\otimes v)\cdot\psi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{5.17}$$

To this end, we integrate by parts on $\Omega^\pm(t)$, and use Proposition 5.7, to see that

$$\begin{aligned}&\int_0^T\int_\Omega \rho\,\mathrm{div}\left((v\otimes v)\psi\right)\,\mathrm{d}x\,\mathrm{d}t\\ &= (\beta_1-\beta_2)\int_0^T\int_{\Gamma(t)}(v\cdot\nu^-)(v\cdot\psi)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t\\ &= (\beta_1-\beta_2)\int_0^T\int_{\Gamma(t)}V(v\cdot\psi)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t.\end{aligned}$$

Finally, applying Proposition 5.5 implies (5.17). □

Taking into account the additional smoothness Assumptions 5.1, we can prove the existence of an associated pressure function.

Theorem 5.10 (Reconstruction of associated pressure)**.** *Let Assumptions 5.1 be satisfied. Then there exists some function $p \in L^2(0,T;L^2(\Omega))$ such that its restrictions $p^\pm = p|_{\Omega^\pm}$ to $\Omega^\pm$ belong to $L^2(0,T;H^1(\Omega^\pm(t)))$ and satisfy*

$$\begin{aligned}\nabla p^- &= -\beta_1\partial_t v + \mu(\beta_1)\Delta v - \beta_1(v\cdot\nabla)v - 2\sigma_{\mathrm{st}}K \text{ a.e. in } \Omega^-,\\ \nabla p^+ &= -\beta_2\partial_t v + \mu(\beta_2)\Delta v - \beta_2(v\cdot\nabla)v - 2\sigma_{\mathrm{st}}K \text{ a.e. in } \Omega^+,\end{aligned} \tag{5.18}$$

where $K \in L^2(0,T;L^2(\Omega)^n)$ denotes the extension of the mean-curvature function as in Lemma 5.3.

Proof. In view of Proposition 5.9 and (5.8), for any $\psi \in C_0^\infty((0,T);C_{0,\sigma}^\infty(\Omega))$, there holds $\langle\mathcal{G}_{\mathrm{reg}},\psi\rangle_{\mathcal{D}(\Omega\times(0,T))^n} = \langle\mathcal{G},\psi\rangle_{\mathcal{D}(\Omega\times(0,T))^n} = 0$. Since $\mathcal{G}_{\mathrm{reg}}$ belongs to $L^2(0,T;H^{-1}(\Omega)^n)$, by Theorem 5.4, there exists a function $p \in L^2(0,T;L^2(\Omega))$ such that, for the distributional gradient ∇p, there holds

$$\langle\nabla p,\psi\rangle_{\mathcal{D}(\Omega\times(0,T))^n} = \langle\mathcal{G}_{\mathrm{reg}},\psi\rangle_{\mathcal{D}(\Omega\times(0,T))^n} \text{ for all } \psi\in C_0^\infty(\Omega\times(0,T))^n.$$

For any $\psi \in C_0^\infty(\Omega^-)^n$, since $\rho = \beta_1$ in Ω^- and $v \in L^2(0,T;W^{2,2}(\Omega^-(t))^n)$, this leads to

$$\begin{aligned}&\langle\nabla p,\psi\rangle_{\mathcal{D}(\Omega\times(0,T))^n}\\ =\ &-\int_0^T\int_{\Omega^-(t)}\rho(\partial_t v + \mathrm{div}(v\otimes v))\cdot\psi\,\mathrm{d}x\,\mathrm{d}t\\ &-2\int_0^T\int_{\Omega^-(t)}\mu(\rho)Dv:D\psi\,\mathrm{d}x\,\mathrm{d}t + 2\sigma_{\mathrm{st}}\int_0^T\int_{\Omega^-(t)}K\cdot\psi\,\mathrm{d}x\,\mathrm{d}t\\ =\ &-\int_0^T\int_{\Omega^-(t)}\beta_1(\partial_t v + \mathrm{div}(v\otimes v))\cdot\psi\,\mathrm{d}x\,\mathrm{d}t\\ &+2\int_0^T\int_{\Omega^-(t)}\mu(\beta_1)\,\mathrm{div}(Dv)\cdot\psi\,\mathrm{d}x\,\mathrm{d}t + 2\sigma_{\mathrm{st}}\int_0^T\int_{\Omega^-(t)}K\cdot\psi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

As $\partial_t v \in L^2(0,T;L^2(\Omega^-(t))^n)$, due to Assumptions 5.1, this implies that p^- belongs to $L^2(0,T;H^1(\Omega^-(t)))$. Additionally, since v is divergence free, we conclude the first identity in (5.18). As the statements about p^+ follow analogously, this finishes the proof. $\square$

5.3 The Pressure Jump

The question left to answer is whether there are pressure functions $p^\pm$ such that the Young–Laplace law (1.6) holds true. That is, whether

$$[p]\,\nu^- = (p^+ - p^-)\nu^- = 2\Big(\mu(\beta_2)(Dv)^+ - \mu(\beta_1)(Dv)^-\Big)\nu^- + 2\sigma_{\mathrm{st}}\kappa\nu^- \tag{5.19}$$

is satisfied on the interface $\Gamma(t)$. A first step towards an affirmative answer to this question is to understand the "jump brackets"$[\,\cdot\,]$ in an appropriate sense: in Theorem 5.10 we reconstructed a pressure function p such that, for its restrictions $p^\pm$ to $\Omega^\pm$, there holds

$$p^+ = p|_{\Omega^+} \in L^2(0,T;H^1(\Omega^+(t))) \text{ and } p^- = p|_{\Omega^-} \in L^2(0,T;H^1(\Omega^-(t))).$$

In particular, for a.e. $t \in (0,T)$, the traces $p^\pm(t)|_{\Gamma(t)}$ are well-defined in the Sobolev sense, and there holds

$$p^\pm(t)\Big|_{\Gamma(t)} \in H^{\frac{1}{2}}(\Gamma(t)).$$

Therefore, the statement of Theorem 5.10 suggests that the pressure jump $[p]$ on the space-time interface

$$\Gamma = \bigcup_{t\in(0,T)} \Big(\Gamma(t) \times \{t\}\Big) \tag{5.20}$$

belongs to the space $L^2(0,T;H^{\frac{1}{2}}(\Gamma(t)))$ which is given by either of the equivalent definitions

$$L^2(0,T;H^{\frac{1}{2}}(\Gamma(t))) = \Big\{ u|_\Gamma \,:\, u \in L^2(0,T;H^1(\Omega^-(t)))\Big\}$$

and

$$L^2(0,T;H^{\frac{1}{2}}(\Gamma(t))) = \Big\{ u|_\Gamma \,:\, u \in L^2(0,T;H^1(\Omega^+(t)))\Big\}.$$

Likewise, we introduce the n-dimensional version of the latter space and define

$$L^2(0,T;H^{\frac{1}{2}}(\Gamma(t))^n) = L^2(0,T;H^{\frac{1}{2}}(\Gamma(t)))^n.$$

To give the jump condition (5.19) a meaning, in the remaining part of this chapter, we will interpret the "jump brackets"$[\,\cdot\,]$ in the sense of Sobolev traces without changing the notation. More precisely, for a function $f \in L^2(0,T;L^2(\Omega))$ such that the restrictions $f^\pm = f|_{\Omega^\pm}$ to $\Omega^\pm$ belong to $L^2(0,T;H^1(\Omega^\pm(t)))$, we denote

$$[f(t)] = f^+(t)\Big|_{\partial\Omega^+(t)\cap\Omega} - f^-(t)\Big|_{\partial\Omega^-(t)},$$

where $f^+(t)|_{\partial\Omega^+(t)\cap\Omega}$ and $f^-(t)|_{\partial\Omega^-(t)}$ denote the traces of $f^\pm(t)$ on the interface $\Gamma(t) = \partial\Omega^+(t)\cap\Omega = \partial\Omega^-(t)$ taken with respect to the domains $\Omega^+(t)$ and $\Omega^-(t)$, respectively. Analogously, for a function $f \in L^2(0,T;L^2(\Omega)^n)$ such that the restrictions $f^\pm = f|_{\Omega^\pm}$ to $\Omega^\pm$ belong to $L^2(0,T;H^1(\Omega^\pm(t))^n)$, we denote $[f(t)] = ([f_i(t)])_{i=1,\dots,n}$. To construct a pressure function respecting the Young–Laplace law, we provide the following two technical lemmas.

Lemma 5.11. *Let $D \subset \Omega$ be a bounded subdomain of Ω with Lipschitz boundary ∂D and outer normal ν_D. Then, for $a \in H^{\frac{1}{2}}(\partial D)^n$, there holds $\int_{\partial D} a \cdot \nu_D \,\mathrm{d}\mathcal{H}^{n-1}(x) = 0$ if and only if there exists a function $u \in H^1(D)^n$ such that*

$$\operatorname{div}(u) = 0 \text{ in } D \text{ and } u = a \text{ on } \partial D. \tag{5.21}$$

Proof. Let $u \in H^1(D)^n$ satisfy (5.21). Then there holds

$$\int_{\partial D} a \cdot \nu_D \,\mathrm{d}\mathcal{H}^{n-1}(x) = \int_{\partial D} u \cdot \nu_D \,\mathrm{d}\mathcal{H}^{n-1}(x) = \int_D \operatorname{div}(u)\,\mathrm{d}x = 0.$$

The opposite direction follows by [24, Theorem IV.1.1]. □

The following variant of the fundamental lemma of the calculus of variations allows one to deal with divergence-free test functions.

Lemma 5.12. *Let Assumptions 5.1 hold true. If $b \in L^2(0,T; H^{\frac{1}{2}}(\Gamma(t))^n)$ satisfies*

$$\int_0^T \int_{\Gamma(t)} b(t) \cdot \psi(t) \,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t = 0$$

for all $\psi \in C_0^\infty((0,T); C_{0,\sigma}^\infty(\Omega))$, then the tangential projection $t \mapsto \mathcal{P}_\tau(b(t)) = b(t) - (b(t)\cdot\nu^-(t))\nu^-(t)$ vanishes on $\Gamma(t)$, i.e., for a.e. $t \in (0,T)$, there holds $\mathcal{P}_\tau(b(t)) = 0$ on $\Gamma(t)$. Moreover, the normal projection $t \mapsto \mathcal{P}_{\nu^-}(b(t)) = (b(t)\cdot\nu^-(t))\nu^-(t)$ belongs to $L^2(0,T;H^{\frac{1}{2}}(\Gamma(t))^n)$. Moreover, for a.e. $t \in (0,T)$, there holds

$$\mathcal{P}_{\nu^-}(b(t)) = C(t)\nu^-(t) \tag{5.22}$$

on $\Gamma(t)$, where the function $t \mapsto C(t)$ belongs to $L^2(0,T)$ and is given by

$$C(t) = \tfrac{1}{\mathcal{H}^{n-1}(\Gamma(t))}\left(\int_{\Gamma(t)} b(t)\cdot\nu^-(t)\,\mathrm{d}\mathcal{H}^{n-1}(x)\right). \tag{5.23}$$

Proof. We split the proof of the lemma into several steps.

Step 1. By assumption, for a.e. $t \in (0,T)$, the fundamental lemma of the calculus of variations implies

$$\int_{\Gamma(t)} b(t)\cdot\psi \,\mathrm{d}\mathcal{H}^{n-1}(x) = 0$$

for all $\psi \in C_{0,\sigma}^\infty(\Omega)$. Passing on to $H_{0,\sigma}^1(\Omega)$, for any $u \in H_{0,\sigma}^1(\Omega)$, there holds

$$\int_{\Gamma(t)} b(t)\cdot u \,\mathrm{d}\mathcal{H}^{n-1}(x) = 0. \tag{5.24}$$

Let $a \in H^{\frac{1}{2}}(\Gamma(t))^n$ be such that $\int_{\Gamma(t)} a\cdot\nu^-(t)\,\mathrm{d}\mathcal{H}^{n-1}(x) = 0$. Applying Lemma 5.11 on $\Omega^-(t)$ and $\Omega \setminus \overline{\Omega^-(t)}$, respectively, there exist functions $u_1 \in H^1(\Omega^-(t))^n$ and $u_2 \in H^1(\Omega\setminus\overline{\Omega^-(t)})^n$ such that $\operatorname{div}(u_1) = 0$ in $\Omega^-(t)$, $\operatorname{div}(u_2) = 0$ in $\Omega\setminus\overline{\Omega^-(t)}$, $u_1 = u_2 = a$ on $\Gamma(t)$ and $u_2 = 0$ on $\partial\Omega$. As $H_{0,\sigma}^1(\Omega) = \{u \in H_0^1(\Omega)^n : \operatorname{div}(u) = 0\}$, see Lemma 2.1, the composite function

$$u = \begin{cases} u_1 & \text{in } \Omega^-(t), \\ u_2 & \text{in } \Omega\setminus\overline{\Omega^-(t)} \end{cases}$$

is an admissible test function in (5.24), which finally yields

$$\int_{\Gamma(t)} b(t)\cdot a \,\mathrm{d}\mathcal{H}^{n-1}(x) = \int_{\Gamma(t)} b(t)\cdot u \,\mathrm{d}\mathcal{H}^{n-1}(x) = 0. \tag{5.25}$$

Step 2. For a.e. $t\in(0,T)$, there holds $b(t)\in H^{\frac{1}{2}}(\Gamma(t))^n$ and $\nu^-(t)\in C^1(\Gamma(t))^n$, by assumption. Hence, the function $\mathcal{P}_\tau(b(t))$ belongs to $H^{\frac{1}{2}}(\Gamma(t))^n$ and satisfies

$$\mathcal{P}_\tau(b(t))\cdot\nu^-(t) = (b(t)-(b(t)\cdot\nu^-(t))\nu^-(t))\cdot\nu^-(t) = 0.$$

In particular, $\mathcal{P}_\tau(b(t))$ is an admissible choice in (5.25), which implies

$$\int_{\Gamma(t)} |\mathcal{P}_\tau(b(t))|^2 \,\mathrm{d}\mathcal{H}^{n-1}(x) = \int_{\Gamma(t)} b(t)\cdot\mathcal{P}_\tau(b(t)) \,\mathrm{d}\mathcal{H}^{n-1}(x) = 0,$$

since $|\mathcal{P}_\tau(b)|^2 = b\cdot\mathcal{P}_\tau(b)$. This proves the first claim.

Step 3. From $\mathcal{P}_\tau(b)=0$, we infer that $\mathcal{P}_{\nu^-}(b) = b \in L^2(0,T;H^{\frac{1}{2}}(\Gamma(t))^n)$. For the proof of (5.22), we consider $a(t)\in H^{\frac{1}{2}}(\Gamma(t))^n$ given by

$$\begin{aligned} a(t) &= \mathcal{P}_{\nu^-}(b(t)) - \frac{1}{\mathcal{H}^{n-1}(\Gamma(t))}\left(\int_{\Gamma(t)} b(t)\cdot\nu^-(t)\,\mathrm{d}\mathcal{H}^{n-1}(x)\right)\nu^-(t) \\ &= \left(b(t)\cdot\nu^-(t) - \tfrac{1}{\mathcal{H}^{n-1}(\Gamma(t))}\int_{\Gamma(t)} b(t)\cdot\nu^-(t)\,\mathrm{d}\mathcal{H}^{n-1}(x)\right)\nu^-(t). \end{aligned}$$

Since, by the definition of $a(t)$, $\int_{\Gamma(t)} a(t)\cdot\nu^-(t)\,\mathrm{d}\mathcal{H}^{n-1}(x)$ vanishes, $a(t)$ is an admissible function in (5.25). Thus we get

$$\int_{\Gamma(t)} b(t)\cdot a(t)\,\mathrm{d}\mathcal{H}^{n-1}(x) = 0$$

and, by the definition of $a(t)$, we conclude that

$$\int_{\Gamma(t)} (b(t)\cdot\nu^-(t))^2\,\mathrm{d}\mathcal{H}^{n-1}(x) = \tfrac{1}{\mathcal{H}^{n-1}(\Gamma(t))}\left(\int_{\Gamma(t)} b(t)\cdot\nu^-(t)\,\mathrm{d}\mathcal{H}^{n-1}(x)\right)^2$$

and hence, we have

$$\begin{aligned} &\int_{\Gamma(t)} |a(t)|^2 \,\mathrm{d}\mathcal{H}^{n-1}(x) \\ &= \int_{\Gamma(t)} (b(t)\cdot\nu^-(t))^2\,\mathrm{d}\mathcal{H}^{n-1}(x) - \tfrac{1}{\mathcal{H}^{n-1}(\Gamma(t))}\left(\int_{\Gamma(t)} b(t)\cdot\nu^-(t)\,\mathrm{d}\mathcal{H}^{n-1}(x)\right)^2 \\ &= 0. \end{aligned}$$

Hence $a(t)$ vanishes $\mathcal{H}^{n-1}$-a.e. on $\Gamma(t)$.

Step 4. We shall prove that the function $t\mapsto C(t)$, given by (5.23), is measurable. To this end, we use the pullback operator $\Phi^-_{-t}\colon L^2(\Gamma(t))\to L^2(\Gamma(0))$ introduced in Lemma 5.2 and define $B_i(t) = \Phi^-_{-t}\circ b_i(t)\in L^2(\Gamma(0))^n$ for $i=1,2,\dots,n$ and $t\in[0,T]$. Then, in view of Lemma 5.2, the function $B=(B_1,B_2,\dots,B_n)$ belongs to $L^2(0,T;L^2(\Gamma(0))^n)$. In particular, B is Bochner measurable. This means that there exists a sequence $(B_m)_{m\in\mathbb{N}}$ of simple functions

$B_m\colon [0,T] \to L^2(\Gamma(0))^n$ such that, for a.e. $t \in [0,T]$, there holds $B_m(t) \to B(t)$ in $L^2(\Gamma(0))^n$ as $m \to \infty$. For $t \in [0,T]$ and $A \in L^2(\Gamma(0))^n$, define

$$\mathcal{I}(t,A) = \tfrac{1}{\mathcal{H}^{n-1}(\Gamma(t))}\left(\int_{\Gamma(t)} (\Phi_t^- A)\cdot \nu^-(t)\,\mathrm{d}\mathcal{H}^{n-1}(x)\right),$$

where Φ_t^- denotes the inverse of Φ_{-t}^-. Using Lemma 5.2 again, we conclude that, for any $t \in [0,T]$, $\mathcal{I}(t,\cdot)\colon L^2(\Gamma(0))^n \to \mathbb{R}$ is a linear functional that, for any $A \in L^2(\Gamma(0))^n$, satisfies

$$\begin{aligned}\mathcal{H}^{n-1}(\Gamma(t))\,|\mathcal{I}(t,A)| &= \left|\int_{\Gamma(t)} (\Phi_t^- A)\cdot \nu^-(t)\,\mathrm{d}\mathcal{H}^{n-1}(x)\right| \\ &\le \|\Phi_t^- A\|_{L^2(\Gamma(t))^n}\|\nu^-(t)\|_{L^2(\Gamma(t))^n}\end{aligned}$$

and, consequently,

$$|\mathcal{I}(t,A)| = \mathcal{H}^{n-1}(\Gamma(t))^{-\frac{1}{2}}\|\Phi_t^- A\|_{L^2(\Gamma(t))^n} \le D\|A\|_{L^2(\Gamma(0))^n} \tag{5.26}$$

for a constant $D > 0$ independent of $t \in [0,T]$. Hence $\mathcal{I}(t) = \mathcal{I}(t,\cdot)$ defines an element of $\left(L^2(\Gamma(0))^n\right)^*$, where the constant of continuity does not depend on $t \in [0,T]$. Then, $C_m\colon [0,T] \to \mathbb{R}$, defined by $C_m(t) = \mathcal{I}(t, B_m(t))$ for $t \in [0,T]$, is a simple function for every $m \in \mathbb{N}$. Moreover, for a.e. $t \in [0,T]$, we infer that $C_m(t) = \mathcal{I}(t,B_m(t)) \to \mathcal{I}(t,B(t))$ as $m \to \infty$. Since there holds

$$\begin{aligned}\mathcal{I}(t,B(t)) &= \tfrac{1}{\mathcal{H}^{n-1}(\Gamma(t))}\left(\int_{\Gamma(t)} (\Phi_t^- B)\cdot \nu^-(t)\,\mathrm{d}\mathcal{H}^{n-1}(x)\right) \\ &= \tfrac{1}{\mathcal{H}^{n-1}(\Gamma(t))}\left(\int_{\Gamma(t)} b(t)\cdot \nu^-(t)\,\mathrm{d}\mathcal{H}^{n-1}(x)\right) \\ &= C(t),\end{aligned} \tag{5.27}$$

we conclude that $t \mapsto C(t)$ is a measurable function.

Step 5. In view of (5.26) and (5.27), there is some constant $D > 0$ such that

$$\int_0^T |C(t)|^2\,\mathrm{d}t = \int_0^T |\mathcal{I}(t,B(t))|^2\,\mathrm{d}t \le D\int_0^T \|B(t)\|^2_{L^2(\Gamma(0))^n}\,\mathrm{d}t.$$

Hence we have $\|C\|_{L^2(0,T)} \le D\|B\|_{L^2(0,T;L^2(\Gamma(0))^n)}$ and, as $B \in L^2(0,T;L^2(\Gamma(0))^n)$, it follows that $C \in L^2(0,T)$. This finishes the proof. □

The existence statement of Theorem 5.10 and the preparatory Lemma 5.12 now allow us to construct a pressure function satisfying the jump condition of the Young–Laplace law.

Theorem 5.13 (Reconstruction of pressure)**.** *Let Assumptions 5.1 be satisfied. Then there exists a unique function $p \in L^2(0,T;L^2(\Omega))$ with the following properties.*

1 $p|_{\Omega^\pm} \in L^2(0,T;H^1(\Omega^\pm(t)))$.

2 For a.e. $t \in (0,T)$, there holds $\int_\Omega p(t)\,\mathrm{d}x = 0$.

3 $\nabla p = -\beta_1\partial_t v + \mu(\beta_1)\Delta v - \beta_1(v\cdot\nabla)v$ *a.e. in* Ω^-.

4 $\nabla p = -\beta_2\partial_t v + \mu(\beta_2)\Delta v - \beta_2(v\cdot\nabla)v$ *a.e. in* Ω^+.

5 $[p] = 2\,[\mu(\rho)Dv\nu^-]\cdot\nu^- + 2\sigma_{\rm st}\kappa$ *in* $L^2(0,T;H^{\frac{1}{2}}(\Gamma(t)))$.

Proof. The uniqueness of p is a direct consequence of the zero-mean condition. In the remainder we shall construct the desired function p with the help of the functions $p^\pm$ from Theorem 5.10 and the function $K = \nabla m\chi_{\Omega^-}$ from Lemma 5.3: Consider the function

$$\tilde{p} = \begin{cases} p^- - 2\sigma_{\rm st} m & \text{in } \Omega^-, \\ p^+ & \text{in } \Omega^+. \end{cases}$$

Notice that, by Lemma 5.3 and Theorem 5.10, $\tilde{p}|_{\Omega^\pm}$ belongs to $L^2(0,T;H^1(\Omega^\pm(t)))$ and, in the almost-everywhere sense, there holds

$$\begin{aligned}\nabla\tilde{p} = \nabla p^- - 2\sigma_{\rm st}\nabla m &= -\beta_1\partial_t v + \mu(\beta_1)\Delta v - \beta_1(v\cdot\nabla)v + 2\sigma_{\rm st}K - 2\sigma_{\rm st}\nabla m \\ &= -\beta_1\partial_t v + \mu(\beta_1)\Delta v - \beta_1(v\cdot\nabla)v\end{aligned}$$

in Ω^- and, likewise, $\nabla\tilde{p} = \nabla p^+ = -\beta_2\partial_t v + \mu(\beta_2)\Delta v - \beta_2(v\cdot\nabla)v$ a.e. in Ω^+. We remark that these properties remain valid for

$$p = \begin{cases} \tilde{p}|_{\Omega^-} + C^- & \text{in } \Omega^-, \\ \tilde{p}|_{\Omega^+} + C^+ & \text{in } \Omega^+ \end{cases}$$

for arbitrary functions $C^-, C^+ \in L^2(0,T)$. Therefore, it is sufficient to prove that there exists some $C \in L^2(0,T)$ such that, for a.e. $t \in (0,T)$, there holds

$$[\tilde{p}(t)] = 2[\mu(\rho(t))(Dv(t)\nu^-(t))\cdot\nu^-(t)] - 2\sigma_{\rm st}\kappa(t) + C(t) \tag{5.28}$$

on $\Gamma(t)$. This is because the functions C^- and C^+ provide two degrees of freedom: for a.e. $t \in (0,T)$, the first may be used to remove the function C from the previous equation. For example, by making the choice $C^- = C$ and $C^+ = 0$, the function p satisfies the desired jump condition. If p does not have the zero-mean property, the second degree of freedom may be used to subtract its mean value.

Let $\psi \in C_0^\infty((0,T);C_{0,\sigma}^\infty(\Omega))$. By the definition of $\tilde{p}$, there holds

$$[\tilde{p}] = (\tilde{p})^+ - (\tilde{p})^- = p^+ - (p - 2\sigma_{\rm st}m)^- = [p] + 2\sigma_{\rm st}\kappa.$$

This implies

$$\begin{aligned}&\int_0^T\int_{\Gamma(t)} [\tilde{p}]\,\nu^-\cdot\psi\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t \\ =\; &-\int_0^T\int_{\Omega^-(t)}\nabla p^-\cdot\psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\Omega^+(t)}\nabla p^+\cdot\psi\,\mathrm{d}x\,\mathrm{d}t \\ &+ 2\sigma_{\rm st}\int_0^T\int_{\Gamma(t)}\kappa\nu^-\cdot\psi\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t.\end{aligned}$$

In view of Proposition 5.7 and Theorem 5.10, we conclude that

$$\begin{aligned}&\int_0^T\int_{\Gamma(t)}([\tilde{p}] - 2\sigma_{\rm st}\kappa)\nu^-\cdot\psi\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t \\ =\; &\int_0^T\int_{\Omega\setminus\Gamma(t)}(\rho\partial_t v + \rho\,\mathrm{div}(v\otimes v) - 2\mu(\rho)\,\mathrm{div}(Dv) - 2\sigma_{\rm st}K)\cdot\psi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Recalling that $v \in L^2(0,T;H^1(\Omega)^n)$ and applying integration by parts leads to

$$\begin{aligned}
&\int_0^T \int_{\Gamma(t)} ([\tilde{p}] - 2\sigma_{\rm st}\kappa)\nu^- \cdot \psi \,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t \\
=& \int_0^T \int_{\Omega\setminus\Gamma(t)} \rho\partial_t v \cdot \psi \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^T \int_\Omega (\rho \operatorname{div}(v\otimes v) + 2\mu(\rho)Dv : D\psi - 2\sigma_{\rm st} K)\cdot\psi \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^T \int_{\Gamma(t)} 2\,[\mu(\rho)Dv]\,\nu^- \cdot \psi \,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t \\
=& -\langle \mathcal{G}_{\rm reg}, \psi\rangle_{\mathcal{D}(\Omega\times(0,T))^n} + \int_0^T \int_{\Gamma(t)} 2\,[\mu(\rho)Dv]\,\nu^- \cdot \psi \,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t,
\end{aligned}$$

where $\mathcal{G}_{\rm reg}$ is given by (5.16). Since $\langle \mathcal{G}_{\rm reg}, \psi\rangle_{\mathcal{D}(\Omega\times(0,T))^n} = 0$, in view of Proposition 5.9, we conclude that

$$\int_0^T \int_{\Gamma(t)} \left(\left[\tilde{p}\nu^- - 2\mu(\rho)Dv\nu^-\right] - 2\sigma_{\rm st}\kappa\nu^-\right)\cdot\psi \,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}t = 0.$$

Finally, in view of Lemma 5.12, there exists a function $C \in L^2(0,T)$ such that (5.28) is valid. □

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