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On Conjugacy of MASAs and the Outer Automorphism Group of the Cuntz Algebra

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18 August 2013

Abstract

We investigate the structure of the outer automorphism group of the Cuntz algebra and the closely related problem of conjugacy of MASAa in $\mathcal{O}_n$. In particular, we exhibit an uncountable family of MASAs, conjugate to the standard MASA $\mathcal{D}_n$ via Bogolubov automorphisms, that are not inner conjugate to $\mathcal{D}_n$.

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1 Introduction

The main motivation for the present paper comes from the desire to better understand the structure of the outer automorphism group of the Cuntz algebra $\mathcal{O}_n$, [?, ?]. As the Cuntz algebras are among the most intensely investigated operator algebras, it is not surprising that both their single automorphisms and the structure of their automorphism groups attracted a lot of interest. In addition to the obvious intrinsic value of this line of research, we would also like to point out its importance for the current efforts within Elliott's classification program. In this context, for example, the question if $\mathrm{Aut}(\mathcal{O}_2)$ is a universal Polish group is raised in [?].

Our point of departure for the investigations of $\mathrm{Out}(\mathcal{O}_n)$ is the recent progress in understanding of $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$, the group of those automorphisms of $\mathcal{O}_n$ which globally preserve the standard MASA $\mathcal{D}_n$, [?, ?]. This group has the structure of a semi-direct product $\lambda(\mathcal{U}(\mathcal{D}_n)) \rtimes \lambda(\mathcal{S}_n)^{-1}$, where $\lambda(\mathcal{U}(\mathcal{D}_n))$ is a maximal abelian subgroup of $\mathrm{Aut}(\mathcal{O}_n)$ of those automorphisms which fix $\mathcal{D}_n$ point-wise, [?], and $\lambda(\mathcal{S}_n)^{-1}$ is countable, discrete, the so called Weyl group of $\mathcal{O}_n$. Of particular note here is the relation between the image of $\lambda(\mathcal{S}_n)^{-1}$ in $\mathrm{Out}(\mathcal{O}_n)$ and the group of automorphisms of the full two-sided n-shift shown in [?].

The next logical step in the study of $\mathrm{Out}(\mathcal{O}_n)$ would be to learn if every automorphism of $\mathcal{O}_n$ has a representative in $\mathrm{Out}(\mathcal{O}_n)$ coming from $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$, and if not then to classify MASAs of $\mathcal{O}_n$ that are outer but not inner conjugate to $\mathcal{D}_n$ (that is, one is mapped onto the other by an outer automorphism but no such inner automorphism exists). In fact, This question was raised a few years ago by Joachim Cuntz in a conversation with the third named author. In the present paper, we show that Bogolubov automorphisms either globally preserve $\mathcal{D}_n$ or move it to other, not inner conjugate MASAs, see Theorem ?? and Corollary ?? below.

Naturally, any investigations of the structure of an outer automorphism group are significantly helped by classification of single automorphisms up to conjugacy. In the context of the Cuntz algebras, a great deal of progress has been achieved in this direction and we would like to specifically call readers' attention to [?, ?, ?]. By the results of [?] and [?], any two aperiodic automorphisms of $\mathcal{O}_n$ are outer conjugate, and this is in nice analogy with the classification of aperiodic automorphisms of the hyperfinite II_1 factor due to Connes, [?]. Classification of non-aperiodic automorphisms of $\mathcal{O}_n$ is also related to the seminal work of Connes, [?], although in this case the C^*-algebraic setting is much more intricate by comparison with the von Neumann algebraic one. Indeed, non-aperiodic automorphisms of the hyperfinite II_1 factor are completely classified by pairs (k, γ), with k a positive integer (outer period) and γ a k^{th}-root of unity, [?]. It is shown in [?, Proposition 1.6] that each invariant (k, γ) may be realized in $\mathrm{Aut}(\mathcal{O}_k)$ by an automorphim of the form $\lambda_d \lambda_w$ with d a unitary in the canonical MASA $\mathcal{D}_n$ and λ_w a Bogolubov permutation. However, there exist automorphisms of $\mathcal{O}_2$ with $k = 2$ and $\gamma = 1$ which are not outer conjugate to each other, [?]. In the present paper, classification results for single automorphisms of $\mathcal{O}_n$ are used in the proofs of several structural properties of $\mathrm{Out}(\mathcal{O}_n)$ collected in Section 4.

2 Notation and preliminaries

If n is an integer greater than 1, then the Cuntz algebra $\mathcal{O}_n$ is a unital, simple, purely infinite C^*-algebra generated by n isometries $S_1, \ldots, S_n$ satisfying $\sum_{i=1}^n S_i S_i^* = 1$, [?]. We denote by W_n^k the set of k-tuples $\mu = (\mu_1, \ldots, \mu_k)$ with $\mu_m \in \{1, \ldots, n\}$, and by W_n the union $\cup_{k=0}^\infty W_n^k$, where $W_n^0 = \{0\}$. If $\mu \in W_n^k$ then $|\mu| = k$ is the length of μ. If $\mu = (\mu_1, \ldots, \mu_k) \in W_n$ then $S_\mu = S_{\mu_1} \ldots S_{\mu_k}$ ($S_0 = 1$ by convention) is an isometry with range projection $P_\mu = S_\mu S_\mu^*$. Every word in $\{S_i, S_i^* \mid i = 1, \ldots, n\}$ can be uniquely expressed as $S_\mu S_\nu^*$, for $\mu, \nu \in W_n$ [?, Lemma 1.3].

We denote by $\mathcal{F}_n^k$ the C^*-subalgebra of $\mathcal{O}_n$ spanned by all words of the form $S_\mu S_\nu^*$, $\mu, \nu \in W_n^k$, which is isomorphic to the matrix algebra $M_{n^k}(\mathbb{C})$. The norm closure $\mathcal{F}_n$ of $\cup_{k=0}^\infty \mathcal{F}_n^k$ is the UHF-algebra of type n^∞, [?], called the core UHF-subalgebra of $\mathcal{O}_n$, [?]. We denote by τ the unique normalized trace on $\mathcal{F}_n$. Subalgebra $\mathcal{F}_n$ is the fixed-point algebra for the gauge action $\gamma : U(1) \to \mathrm{Aut}(\mathcal{O}_n)$, such that $\gamma_z(S_j) = zS_j$ for $z \in U(1)$ and $j = 1, \ldots, n$. For an integer $m \in \mathbb{Z}$, we denote $\mathcal{O}_n^{(m)} := \{x \in \mathcal{O}_n : \gamma_z(x) = z^m x, \forall z \in U(1)\}$, a spectral subspace for γ. Then $\mathcal{O}_n^{(0)} = \mathcal{F}_n$ and for each positive integer m and each $\alpha \in W_n^m$ we have $\mathcal{O}_n^{(m)} = \mathcal{F}_n S_\alpha$ and $\mathcal{O}_n^{(-m)} = S_\alpha^* \mathcal{F}_n$. Furthermore,

$$E_m(x) = \int_{z \in U(1)} z^{-m} \gamma_z(x) dz$$

is a completely contractive projection from $\mathcal{O}_n$ onto $\mathcal{O}_n^{(m)}$, such that $E_m(xay) = xE_m(a)y$ for all $a \in \mathcal{O}_n$, $x, y \in \mathcal{F}_n$. In particular, $E := E_0$ is the faithful conditional expectation from $\mathcal{O}_n$ onto $\mathcal{F}_n$ given by averaging action γ over $U(1)$ with respect to the Haar measure.

The C^*-subalgebra of $\mathcal{O}_n$ generated by projections P_μ, $\mu \in W_n$, is a MASA (maximal abelian subalgebra) in $\mathcal{O}_n$. We call it the *diagonal* and denote $\mathcal{D}_n$, also writing $\mathcal{D}_n^k$ for $\mathcal{D}_n \cap \mathcal{F}_n^k$. The spectrum of $\mathcal{D}_n$ is naturally identified with X_n — the full one-sided n-shift space (a Cantor set). Occasionally, we will view X_n as metric space equipped with the metric $\mathrm{dist}(x, y) = n^{-k}$, where $k = \min\{m \in \mathbb{N} \mid x_m \neq y_m\}$.

As shown by Cuntz in [?], there exists the following bijective correspondence between unitaries in $\mathcal{O}_n$ (whose collection is denoted $\mathcal{U}(\mathcal{O}_n)$) and unital $*$-endomorphisms of $\mathcal{O}_n$ (whose collection we denote $\mathrm{End}(\mathcal{O}_n)$), determined by

$$\lambda_u(S_i) = uS_i, \quad i = 1, \ldots, n.$$

Composition of endomorphisms corresponds to the 'convolution' multiplication of unitaries: $\lambda_u \circ \lambda_w = \lambda_{\lambda_u(w)u}$. In the case $u, w \in \mathcal{U}(\mathcal{F}_n^1)$ this formula simplifies to $\lambda_u \circ \lambda_w = \lambda_{uw}$ and, in particular, there exists an imbedding $u \mapsto \lambda_u$ of $U(n) \cong \mathcal{U}(\mathcal{F}_n^1)$ into $\mathrm{Aut}(\mathcal{O}_n)$, [?]. If A is either a unital C^*-subalgebra of $\mathcal{O}_n$ or a subset of $\mathcal{U}(\mathcal{O}_n)$, then we denote $\lambda(A) = \{\lambda_u \in \mathrm{End}(\mathcal{O}_n) : u \text{ unitary in } A\}$ and $\lambda(A)^{-1} = \{\lambda_u \in \mathrm{Aut}(\mathcal{O}_n) : u \text{ unitary in } A\}$.

We denote by φ the canonical shift on the Cuntz algebra:

$$\varphi(x) = \sum_{i=1}^n S_i x S_i^*, \quad x \in \mathcal{O}_n.$$

Clearly, $\varphi(\mathcal{F}_n) \subset \mathcal{F}_n$ and $\varphi(\mathcal{D}_n) \subset \mathcal{D}_n$. We denote by $\sigma : X_n \to X_n$ the shift on X_n. Then we have $\varphi(f)(x) = f(\sigma(x))$ for all $f \in C(X_n)$ and $x \in X_n$.

For all $u \in \mathcal{U}(\mathcal{O}_n)$ we have $\mathrm{Ad}(u) = \lambda_{u\varphi(u^*)}$. If $u \in \mathcal{U}(\mathcal{O}_n)$ then for each positive integer k we denote

$$u_k = u\varphi(u)\cdots\varphi^{k-1}(u). \tag{1}$$

Here $\varphi^0 = \mathrm{id}$, and we agree that u_k^* stands for $(u_k)^*$. If α and β are multi-indices of length k and m, respectively, then $\lambda_u(S_\alpha S_\beta^*) = u_k S_\alpha S_\beta^* u_m^*$. This is established through a repeated application of the identity $S_i x = \varphi(x) S_i$, valid for all $i = 1, \ldots, n$ and $x \in \mathcal{O}_n$.

We often consider elements of $\mathcal{O}_n$ of the form $w = \sum_{(\alpha,\beta)\in\mathcal{J}} c_{\alpha,\beta} S_\alpha S_\beta^*$, where $\mathcal{J}$ is a finite collection of pairs (α,β) of words $\alpha,\beta \in W_n$ and $c_{\alpha,\beta} \in \mathbb{C}$. In particular, we consider the group $\mathcal{S}_n$ of those unitaries in $\mathcal{O}_n$ which can be written as finite sums of words, i.e. in the form $w = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^*$. Each $w \in \mathcal{S}_n$ normalizes $\mathcal{D}_n$ and hence $\lambda_w(\mathcal{D}_n) \subseteq \mathcal{D}_n$, [?]. We denote $\mathcal{P}_n := \mathcal{S}_n \cap \mathcal{U}(\mathcal{F}_n)$ and $\mathcal{P}_n^k := \mathcal{S}_n \cap \mathcal{U}(\mathcal{F}_n^k)$.

For algebras $A \subseteq B$ we denote by $\mathcal{N}_B(A) = \{u \in \mathcal{U}(B) : uAu^* = A\}$ the normalizer of A in B and by $A' \cap B = \{b \in B : (\forall a \in A)\ ab = ba\}$ the relative commutant of A in B. We also denote by $\mathrm{Aut}(B, A)$ the collection of all those automorphisms α of B such that $\alpha(A) = A$, and by $\mathrm{Aut}_A(B)$ those automorphisms of B which fix A point-wise. Likewise, we denote by $\mathrm{End}_A(B)$ the collection of those endomorphisms of B which fix pointwise subalgebra A.

3 Conjugacy of MASAs

If A_1 and A_2 are two MASAs in B then we say they are *conjugate* if there exists an $\alpha \in \mathrm{Aut}(B)$ such that $\alpha(A_1) = A_2$. We say A_1 and A_2 are *inner conjugate* if there exists a $u \in \mathcal{U}(B)$ such that $uA_1u^* = A_2$.

Proposition 3.1 *Let $z \in \mathcal{U}(\mathcal{O}_n)$ be such that $\lambda_z \in \mathrm{Aut}(\mathcal{O}_n)$. Then there exists a $u \in \mathcal{U}(\mathcal{O}_n)$ such that*

$$\lambda_z(\mathcal{D}_n) = \mathrm{Ad}(u)(\mathcal{D}_n) \tag{2}$$

if and only if there exist $u \in \mathcal{U}(\mathcal{O}_n)$ and $w \in \mathcal{S}_n$ such that

$$\lambda_z(d) = \mathrm{Ad}(u)\lambda_w(d), \quad \forall d \in \mathcal{D}_n. \tag{3}$$

Proof. Suppose (??) holds. Then $\mathrm{Ad}(u^*)\lambda_z \in \mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$ and thus there exist $v \in \mathcal{U}(\mathcal{D}_n)$ and $w \in \mathcal{S}_n$ such that $\mathrm{Ad}(u^*)\lambda_z = \lambda_v\lambda_w$, [?]. Since $\lambda_v|_{\mathcal{D}_n} = \mathrm{id}$ and $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$, for each $d \in \mathcal{D}_n$ we have $\mathrm{Ad}(u^*)\lambda_z(d) = \lambda_v\lambda_w(d) = \lambda_w(d)$, and identity (??) holds.

Conversely, suppose that (??) holds. Then $\lambda_z^{-1}\mathrm{Ad}(u)\lambda_w \in \mathrm{End}_{\mathcal{D}_n}(\mathcal{O}_n)$, and thus there exists a $v \in \mathcal{U}(\mathcal{D}_n)$ such that $\lambda_z^{-1}\mathrm{Ad}(u)\lambda_w = \lambda_v$, [?]. Since $\lambda(\mathcal{D}_n) \subseteq \mathrm{Aut}(\mathcal{O}_n)$, [?], λ_w is an automorphism of $\mathcal{O}_n$. Since $\lambda_w(\mathcal{D}_n) \subseteq \mathcal{D}_n$ and $\mathcal{D}_n$ is a MASA in $\mathcal{O}_n$ we may conclude that $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$, and identity (??) follows. □

Before proving our main result, Theorem **??** below, we need some preparation.

Lemma 3.2 *If $x \in \mathcal{O}_n$, $x \geq 0$, and $x\mathcal{D}_n = \mathcal{D}_n x$ then $x \in \mathcal{D}_n$.*

Proof. We may assume that $0 \le x \le 1$. Let Φ be the faithful conditional expectation from $\mathcal{O}_n$ onto $\mathcal{D}_n$. Since $0 \le x^2 \le x$, we have $0 \le \Phi(x^2) \le \Phi(x)$. Let $d \in \mathcal{D}_n$ be such that $d\Phi(x) = 0$. Then $0 \le d\Phi(x^2)d^* \le d\Phi(x)d^* = 0$ and hence $\Phi(dx^2d^*) = d\Phi(x^2)d^* = 0$. Consequently $dx = 0$. Now, for an arbitrary $a \in \mathcal{D}_n$ let $b \in \mathcal{D}_n$ be such that $xa = bx$. Then $(a-b)\Phi(x) = 0$ and thus $(a-b)x = 0$. This shows that x is in the commutant of $\mathcal{D}_n$ and therefore $x \in \mathcal{D}_n$. □

Remark 3.3 The conclusion of Lemma **??** remains valid if $\mathcal{D}_n \subseteq \mathcal{O}_n$ are replaced by any C^*-algebras $D \subseteq A$ such that D is a MASA in A and there exists a faithful conditional expectation from A onto D. However, it may fail if x is merely self-adjoint but not positive.

Lemma 3.4 *Let $z \in \mathcal{U}(\mathcal{F}_n^1) \setminus \mathcal{N}_{\mathcal{F}_n^1}(\mathcal{D}_n^1)$ and $x \in \mathcal{F}_n$. If $\lambda_z(\mathcal{D}_n)x = x\mathcal{D}_n$ then $x = 0$.*

Proof. Since $z \in \mathcal{U}(\mathcal{F}_n^1) \setminus \mathcal{N}_{\mathcal{F}_n^1}(\mathcal{D}_n^1)$, there exists a minimal projection $p \in \mathcal{D}_n^1$ and a $\delta > 0$ such that for all $m \in \mathbb{N}$, all projections $e \in \mathcal{D}_n^m$ and all $h \in \mathcal{D}_n$ we have

$$||e\lambda_z(\varphi^m(p)) - h|| \ge \delta. \tag{4}$$

Indeed, since $z \notin \mathcal{N}_{\mathcal{F}_n^1}(\mathcal{D}_n^1)$, there exists an $i \in W_n^1$ such that $\lambda_z(P_i) \notin \mathcal{D}_n$. By the Hahn-Banach theorem there exists a functional ω of norm 1 on $\mathcal{O}_n$ such that $\omega(\lambda_z(P_i)) = \delta > 0$ and $\omega|_{\mathcal{D}_n} = 0$. Now, take $m \in \mathbb{N}$, a projection $e \in \mathcal{D}_n^m$ and an $h \in \mathcal{D}_n$, and let $\alpha \in W_n^m$ be such that $eP_\alpha = P_\alpha$. Then

$$\begin{aligned} ||e\lambda_z(\varphi^m(P_i)) - h|| &= ||e\varphi^m(\lambda_z(P_i)) - h|| \\ &\ge ||P_\alpha\varphi^m(\lambda_z(P_i)) - P_\alpha h|| \\ &= ||S_\alpha(\lambda_z(P_i) - S_\alpha^* h S_\alpha)S_\alpha^*|| \\ &= ||\lambda_z(P_i) - S_\alpha^* h S_\alpha|| \\ &\ge |\omega(\lambda_z(P_i) - S_\alpha^* h S_\alpha)| \\ &= \delta. \end{aligned}$$

Now suppose there is an $x \neq 0$ in $\mathcal{F}_n$ such that $\lambda_z(\mathcal{D}_n)x = x\mathcal{D}_n$. We may assume $||x|| = 1$. Since $x^*\lambda_z(\mathcal{D}_n) = \mathcal{D}_n x^*$ as well, we have $x^*x\mathcal{D}_n = \mathcal{D}_n x^*x$ and thus $x^*x \in \mathcal{D}_n$ by Lemma **??**.

Take a small $\epsilon > 0$. For some $l \in \mathbb{N}$, there exists a $y_0 \in \mathcal{F}_n^l$ such that $||x - y_0|| < \epsilon$. Then $||x^*x - y_0^*y_0|| < \epsilon(2+\epsilon)$. For some $k \ge l$ there exists a $d \in \mathcal{D}_n^k$ such that $||x^*x - d|| < \epsilon$ and $d \ge 0$. Then $||y_0^*y_0 - d|| < \epsilon(3+\epsilon)$ and hence $|||y_0| - \sqrt{d}|| < \sqrt{\epsilon(3+\epsilon)}$ due to operator monotonicity of the square root function. Indeed, since $y_0^*y_0 \le d + \epsilon(3+\epsilon)$ we have $|y_0| = \sqrt{y_0^*y_0} \le \sqrt{d+\epsilon(3+\epsilon)} \le \sqrt{d} + \sqrt{\epsilon(3+\epsilon)}$ and likewise $\sqrt{d} \le |y_0| + \sqrt{\epsilon(3+\epsilon)}$. Now, write $y_0 = w|y_0|$ with w a unitary in $\mathcal{F}_n^l \subseteq \mathcal{F}_n^k$. Setting $y := w\sqrt{d}$ we have $y \in \mathcal{F}_n^k$, $y^*y = d \in \mathcal{D}_n^k$ and

$$||x - y|| \le ||x - y_0|| + ||w|y_0| - w\sqrt{d}|| < \epsilon + \sqrt{\epsilon(3+\epsilon)} =: \epsilon'.$$

Let p be the projection in D_n^1 and $\delta > 0$ be such that identity (**??**) holds. Let $g \in \mathcal{D}_n$ satisfy $\lambda_z(\varphi^k(p))x = xg$. Also, let q be the spectral projection of d corresponding to eigenvalue $||d||$. Then we have

$$\begin{aligned} 0 &= ||\lambda_z(\varphi^k(p))x - xg|| \\ &= ||(\lambda_z(\varphi^k(p))y - yg) + (\lambda_z(\varphi^k(p))(x-y) - (x-y)g)|| \\ &> ||y(\lambda_z(\varphi^k(p)) - g)|| - \epsilon'(1 + ||g||) \\ &\geq \frac{1}{1+\epsilon'}||y^*y(\lambda_z(\varphi^k(p)) - g)|| - \epsilon'(1 + ||g||). \end{aligned}$$

We have $y^*y = d$, $dq = ||d||q$ and $||d|| \geq ||x^*x|| - \epsilon = 1 - \epsilon$. Thus

$$\begin{aligned} \frac{1}{1+\epsilon'}||y^*y(\lambda_z(\varphi^k(p)) - g)|| - \epsilon'(1 + ||g||) &\geq \frac{1-\epsilon}{1+\epsilon'}||q(\lambda_z(\varphi^k(p)) - g)|| - \epsilon'(1 + ||g||) \\ &\geq \frac{1-\epsilon}{1+\epsilon'}\delta - \epsilon'(1 + ||g||) \end{aligned}$$

by (**??**). Since ϵ and ϵ' can be simultaneously arbitrarily small, this is a contradiction which shows that $x = 0$. □

Remark 3.5 MASAs of the form $\lambda_z(\mathcal{D}_n)$, $z \in \mathcal{U}(\mathcal{F}_n^1)$, are called *standard* and used to compute noncommutative topological entropy of certain endomorphisms, [**?**, **?**, **?**]. They are abstractly characterized in [**?**].

For the following lemma, note that given any partial isometry $S_\alpha S_\beta^*$ with $|\alpha|, |\beta| \geq 1$ there exists a $w \in \mathcal{S}_n$ of the form $w = S_\alpha S_\beta^* + \sum_{(\mu,\nu)} S_\mu S_\nu^*$. This is easily verified with help of the pigeon hole principle. Also recall that E denotes the faithful conditional expectation from $\mathcal{O}_n$ onto $\mathcal{F}_n$.

Lemma 3.6 *Let $a \in \mathcal{O}_n$. If $E(av) = 0$ for all $v \in \mathcal{S}_n$ then $a = 0$.*

Proof. For each projection $P_\beta \in \mathcal{D}_n$, $\beta \in W_n$, we have $0 = E(av)P_\beta = E(a(vP_\beta))$. Thus $E(aS_\alpha S_\beta^*) = 0$ for all α, β with $|\alpha|, |\beta| \geq 1$. Since the linear span of such elements $S_\alpha S_\beta^*$ is dense in $\mathcal{O}_n$ and E is faithful, we conclude that $a = 0$. □

Theorem 3.7 *If $z \in \mathcal{U}(\mathcal{F}_n^1) \setminus \mathcal{N}_{\mathcal{F}_n^1}(\mathcal{D}_n^1)$ and $a \in \mathcal{O}_n$ is such that $\lambda_z(\mathcal{D}_n)a = a\mathcal{D}_n$ then $a = 0$. In particular, there is no unitary $u \in \mathcal{O}_n$ such that $\lambda_z(\mathcal{D}_n) = \mathrm{Ad}(u)(\mathcal{D}_n)$.*

Proof. Suppose by way of contradiction that $\lambda_z(\mathcal{D}_n)a = a\mathcal{D}_n$. Then, since unitaries from $\mathcal{S}_n$ normalize $\mathcal{D}_n$, for any $v \in \mathcal{S}_n$ we have $\lambda_z(\mathcal{D}_n)av = av\mathcal{D}_n$. Since $\lambda_z(\mathcal{D}_n) \subseteq \mathcal{F}_n$, this implies $\lambda_z(\mathcal{D}_n)E(av) = E(av)\mathcal{D}_n$. Therefore $E(av) = 0$ by Lemma **??**, and consequently $a = 0$ by Lemma **??**. □

An immediate consequence of Theorem **??** is existence of two MASAs of the Cuntz algebra $\mathcal{O}_n$ which are outer but not inner conjugate. In fact, Theorem **??** implies the following stronger fact.

Corollary 3.8 *There exists an uncountable family of MASAs in $\mathcal{O}_n$ indexed by the cosets $\mathcal{U}(\mathcal{F}_n^1)/\mathcal{N}_{\mathcal{F}_n^1}(\mathcal{D}_n^1)$ such that each of them is outer conjugate to $\mathcal{D}_n$ but no two of them are inner conjugate.*

To the best of our knowledge, Corollary **??** exhibits the very first example of two MASAs in a simple, purely infinite C^*-algebra that are outer but not inner conjugate[1].

It was shown in [**?**] that

$$\{v \in \mathcal{U}(\mathcal{O}_n) \mid \lambda_v \in \mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)\} = \{dw \mid d \in \mathcal{U}(\mathcal{D}_n),\ w \in \mathcal{S}_n \text{ s.t. } \lambda_w \in \mathrm{Aut}(O_n)\}.$$

On the other hand, the set $\{u\varphi(u^*) \mid u \in \mathcal{U}(\mathcal{O}_n)\}$ is dense in $\mathcal{U}(\mathcal{O}_n)$ by [**?**]. Furthermore, $\mathcal{U}_a(\mathcal{O}_n) := \{v \in \mathcal{U}(\mathcal{O}_n) \mid \lambda_v \in \mathrm{Aut}(\mathcal{O}_n)\}$ is a dense G_δ-subset of $\mathcal{U}(\mathcal{O}_n)$ such that $\mathcal{U}(\mathcal{O}_n)\setminus\mathcal{U}_a(\mathcal{O}_n)$ is also dense, [**?**]. In this context, we would like to mention the following corollary.

Corollary 3.9 *The following inclusion is proper:*

$$\{udw\varphi(u^*) \mid u \in \mathcal{U}(\mathcal{O}_n),\ d \in \mathcal{U}(\mathcal{D}_n),\ w \in \mathcal{S}_n \textit{ s.t. } \lambda_w \in \mathrm{Aut}(O_n)\} \subset \mathcal{U}_a(\mathcal{O}_n).$$

We close this section by posing the following question. Suppose that $z \in \mathcal{U}(\mathcal{F}_n)$ is such that $\lambda_z \in \mathrm{Aut}(\mathcal{O}_n)$ and that there exists a $u \in \mathcal{U}(\mathcal{O}_n)$ such that $\lambda_z(\mathcal{D}_n) = \mathrm{Ad}(u)(\mathcal{D}_n)$. Does this imply existence of a $v \in \mathcal{U}(\mathcal{F}_n)$ such that $\lambda_z(\mathcal{D}_n) = \mathrm{Ad}(v)(\mathcal{D}_n)$?

4 The outer automorphism group of $\mathcal{O}_n$

In this section, we collect a few observations about the structure of the outer automorphism group of $\mathcal{O}_n$. We denote by $\pi : \mathrm{Aut}(\mathcal{O}_n) \to \mathrm{Out}(\mathcal{O}_n)$ the canonical surjection.

Proposition 4.1 *If $d \in \mathcal{U}(\mathcal{D}_n)$ and $u \in \mathcal{S}_n$ then $\lambda_d\lambda_u \in \mathrm{Inn}(\mathcal{O}_n)$ if and only if there exist $v \in \mathcal{U}(\mathcal{D}_n)$ and $y \in \mathcal{S}_n$ such that $\lambda_d = \mathrm{Ad}(v)$ and $\lambda_u = \mathrm{Ad}(y)$.*

Proof. If $\lambda_d\lambda_u = \mathrm{Ad}(w)$ for some $w \in \mathcal{U}(\mathcal{O}_n)$ then w normalizes $\mathcal{D}_n$ and thus $w = vy$ for some $v \in \mathcal{U}(\mathcal{D}_n)$ and $y \in \mathcal{S}_n$, [**?**]. Hence $\mathrm{Ad}(v^*)\lambda_d = \mathrm{Ad}(y)\lambda_u^{-1}$ and consequently $\lambda_d = \mathrm{Ad}(v)$ and $\lambda_u = \mathrm{Ad}(y)$, since the intersection of $\lambda(\mathcal{U}(\mathcal{D}_n))$ and $\lambda(\mathcal{S}_n)^{-1}$ is trivial. □

The semi-direct product decomposition in the following Proposition **??** is a special case of [**?**, Theorem 6.5] pertaining a broader class of algebras. We include a short self-contained proof, different from Matsumoto's argument.

Recall that an automorphism is *aperiodic* if its image in the outer automorphism group has infinite order. A Bogolubov automorphism λ_z of $\mathcal{O}_n$ is aperiodic if and only if the corresponding unitary z has infinite order. Thus, in particular, a gauge automorphism γ_t is aperiodic if and only if t is not a root of unity.

[1] We are grateful to Mikael Rørdam for his comments on this point.

Proposition 4.2 *The subgroup* $\pi(\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n))$ *of* $\mathrm{Out}(\mathcal{O}_n)$ *is not normal, and has the structure of a semi-direct product*

$$\pi(\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)) = \pi(\lambda(\mathcal{U}(\mathcal{D}_n))) \rtimes \pi(\lambda(\mathcal{S}_n)^{-1}).$$

Proof. Since all aperiodic automorphisms of $\mathcal{O}_n$ are outer conjugate, [?, ?], a normal subgroup of $\mathrm{Out}(\mathcal{O}_n)$ contains either none or all of them. Clearly, $\lambda(\mathcal{U}(\mathcal{F}_n^1))$ contains aperiodic automorphisms λ_z such that z does not normalize $\mathcal{D}_n^1$. Thus Theorem **??** implies that $\pi(\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n))$ contains some but not all aperiodic automorphisms of $\mathcal{O}_n$. Consequently, it is not a normal subgroup of $\mathrm{Out}(\mathcal{O}_n)$.

For the semi-direct product decomposition it suffices to observe that $\pi(\lambda(\mathcal{U}(\mathcal{D}_n)))$ and $\pi(\lambda(\mathcal{S}_n)^{-1})$ have trivial intersection. Indeed, suppose that $d \in \mathcal{U}(\mathcal{D}_n)$, $w \in \mathcal{S}_n$, $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$ and $u \in \mathcal{U}(\mathcal{O}_n)$ are such that $\lambda_d = \mathrm{Ad}(u)\lambda_w$. Then for each $p \in \mathcal{D}_n$ we have $p = \lambda_d(p) = u\lambda_w(p)u^*$ and thus $u \in \mathcal{N}_{\mathcal{O}_n}(\mathcal{D}_n)$. Hence there exist $d_1 \in \mathcal{U}(\mathcal{D}_n)$ and $w_1 \in \mathcal{S}_n$ such that $u = d_1 w_1$, [?]. But then $\mathrm{Ad}(d_1^*)\lambda_d = \mathrm{Ad}(w_1)\lambda_w$ and equivalently $\lambda_{d_1^* d\varphi(d_1)} = \lambda_{w_1 w \varphi(w_1^*)}$. This yields $d_1^* d\varphi(d_1) = 1 = w_1 w \varphi(w_1^*)$ and thus both $\lambda_d = \mathrm{Ad}(d_1)$ and $\lambda_w = \mathrm{Ad}(w_1^*)$ are inner. □

We have seen in Proposition **??** above that subgroup $\pi(\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n))$ is not normal in $\mathrm{Out}(\mathcal{O}_n)$. It follows from Proposition **??** below that the smallest normal subgroup of $\mathrm{Out}(\mathcal{O}_n)$ containing $\pi(\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n))$ is quite large.

Proposition 4.3 *If G is a normal subgroup of* $\mathrm{Out}(\mathcal{O}_n)$ *containing at least one aperiodic element, then*

$$\pi\big(\mathcal{Z}(\mathrm{Out}(\mathcal{O}_n)) \cup \lambda(\mathcal{U}(\mathcal{F}_n))^{-1} \cup \{\alpha \in \mathrm{Aut}(\mathcal{O}_n) \mid \alpha \text{ aperiodic}\}\big) \subseteq G. \tag{5}$$

*In particular, (**??**) holds with G the commutator subgroup of* $\mathrm{Out}(\mathcal{O}_n)$.

Proof. Since all aperiodic automorphisms of $\mathcal{O}_n$ are outer conjugate to one another, [?] and [?], group G contains classes of all of them. Now, let $\alpha \in \mathrm{Aut}(\mathcal{O}_n)$ and suppose that there exists an aperiodic $\beta \in \mathrm{Aut}(\mathcal{O}_n)$ such that $\alpha\beta$ is aperiodic. Then there exists a $\psi \in \mathrm{Aut}(\mathcal{O}_n)$ such that $\pi(\alpha\beta) = \pi(\psi\beta\psi^{-1})$ and consequently $\pi(\alpha) = \pi(\alpha\beta\beta^{-1}) = \pi(\psi\beta\psi^{-1})\pi(\beta^{-1})$ belongs to G. It is clear that if $\alpha \in \mathrm{Aut}(\mathcal{O}_n)$ and either $\pi(\alpha) \in \mathcal{Z}(\mathrm{Out}(\mathcal{O}_n))$ or $\alpha \in \lambda(\mathcal{U}(\mathcal{F}_n))^{-1}$ then for any aperiodic gauge automorphism γ_t the product $\alpha\gamma_t$ is again aperiodic. This shows the first claim of the proposition. For the remaining one, simply note that if $\theta \in \mathrm{Aut}(\mathcal{O}_n)$ is aperiodic then θ^2 is outer conjugate to θ. Thus $\pi(\theta^2) = \pi(\psi\theta\psi^{-1})$ for some $\psi \in \mathrm{Aut}(\mathcal{O}_n)$, and hence $\pi(\theta) = \pi(\theta^2)\pi(\theta^{-1}) = \pi(\psi\theta\psi^{-1}\theta^{-1})$ belongs to the commutator subgroup $[\mathrm{Out}(\mathcal{O}_n), \mathrm{Out}(\mathcal{O}_n)]$. □

Remark 4.4 Classes of non-trivial gauge automorphisms of $\mathcal{O}_n$ do not belong to the center of $\mathrm{Out}(\mathcal{O}_n)$. Indeed, consider unitary $u = S_{11}S_{121}^* + S_{121}S_{11}^* + P_{122} + P_2$ in $\mathcal{S}_2$, discussed in [?, Theorem 5.2]. Then λ_u is an automorphism of $\mathcal{O}_2$ such that $\lambda_u^2 = \mathrm{id}$. For a $1 \neq t \in U(1)$ we have $\lambda_u\gamma_t\lambda_u^{-1}\gamma_t^{-1} = \lambda_w$, with $w = t\lambda_u(P_{121}) + \bar{t}\lambda_u(P_{11}) + P_{122} + P_2$. Automorphism λ_w of $\mathcal{O}_2$ is outer, for otherwise there existed a $d \in \mathcal{U}(\mathcal{D}_2)$ such that $w = d\varphi(d^*)$. But then w, viewed as a function on X_2, would take value 1 at the infinite word $111\ldots$ (fixed by the shift on X_2). However, this is not the case. A similar argument applies to all n, with a suitably modified $u \in \mathcal{S}_n$ (cf. [?, Theorem 5.2]).

This is in stark contrast with what happens for the weak closure M of $\mathcal{O}_n$ in the GNS representation of the canonical KMS state $\omega = \tau \circ E$, which is the AFD factor of type $III_{1/n}$. Indeed, gauge automorphisms of $\mathcal{O}_n$ extend to $M = \pi_\omega(\mathcal{O}_n)''$, [?], thereby providing the ($2\pi/\log(n)$-periodic) modular automorphisms (w.r.t. the normal extension of ω) which then lie in the center of $\mathrm{Out}(M)$ by the Connes-Radon-Nikodym theorem, [?, Theorem 1.2.8]. It also follows from the above that the automorphism λ_u above does not extend to M (i.e., it is not normal).

Remark 4.5 Classes under inner equivalence of all automorphisms of $\mathcal{O}_2$ known to us at the moment belong to the commutator subgroup of $\mathrm{Out}(\mathcal{O}_2)$. For example, consider the unitary $u \in \mathcal{S}_2$ discussed in Remark **??** above. Then for an aperiodic gauge automorphism γ_t the automorphism $\lambda_u\gamma_t$ is aperiodic as well. Indeed, since λ_u has order 2, it suffices to show that all even powers of $\lambda_u\gamma_t$ are outer. But we have $\lambda_u\gamma_t\lambda_u\gamma_t = \lambda_v$, with $v = t^3\lambda_u(P_{121}) + t\lambda_u(P_{11}) + P_{122} + P_2$, and λ_v is aperiodic for the same reason as given in Remark **??** above. Now, the same argument as in Proposition **??** gives the conclusion that λ_u is a commutator modulo an inner automorphism of $\mathcal{O}_2$.

We close this paper with a few simple albeit potentially useful observations about the outer automorphism group of $\mathcal{O}_2$. First of all, it is worth noting that by combining some of the results from [?] and [?] one easily obtains the following.

Proposition 4.6 *Every element of infinite order in* $\mathrm{Out}(\mathcal{O}_2)$ *is a product of two involutions in the commutator subgroup. In particolar, both* $\mathrm{Out}(\mathcal{O}_2)$ *and* $[\mathrm{Out}(\mathcal{O}_2), \mathrm{Out}(\mathcal{O}_2)]$ *are generated by elements of finite order.*

Proof. We consider the inner equivalence classes of the automorphisms λ_A and λ_F defined in [?, Section 5.3]. As shown therein, the commutator $\lambda_F\lambda_A\lambda_F\lambda_A^{-1}$ has infinite order in $\mathrm{Out}(\mathcal{O}_2)$ and is a product of two involutions. The conclusion follows immediately as every aperiodic automorphism is a conjugate of such commutator. □

We believe that the same result holds true for $\mathrm{Out}(\mathcal{O}_n)$ for all $n \geq 2$, with a suitable modification of the lenghty computations in [?]. As this falls outside the scope of the present work, we leave the task to the interested reader.

Corollary 4.7 *The normal subgroup of* $\mathrm{Out}(\mathcal{O}_2)$ *generated by* $\lambda(\mathcal{S}_2)^{-1}$ *is generated by elements of finite order.*

Proof. There are two possible cases:

1) If $\pi(\lambda(\mathcal{S}_2)^{-1})$ is not contained in the commutator subgroup of $\mathrm{Out}(\mathcal{O}_2)$, then there is an element in $\pi(\lambda(\mathcal{S}_2)^{-1})$ but not in the commutator subgroup, say g, necessarily of finite order. Moreover, gh must have finite order for any h of infinite order in $\pi(\lambda(\mathcal{S}_2)^{-1})$. Accordingly, any h of infinite order in $\pi(\lambda(\mathcal{S}_2)^{-1})$ can be written as $g^{-1}(gh)$, a product of finite order elements.

2) On the other hand, if $\pi(\lambda(\mathcal{S}_2)^{-1})$ is in $[\mathrm{Out}(\mathcal{O}_2), \mathrm{Out}(\mathcal{O}_2)]$ then any element of infinite order in $\pi(\lambda(\mathcal{S}_2)^{-1})$ is a product of two conjugates of involutions in $\pi(\lambda(\mathcal{P}_2)^{-1})$, by Proposition **??**. □

To the best of our knowledge, the following result provides the first non-trivial structural result about the rather mysterious group $\lambda(\mathcal{S}_2)^{-1}$. Recall that a group G is called almost simple if there exists a non-abelian simple group H such that $H \subseteq G \subseteq \mathrm{Aut}(H)$.

Proposition 4.8 *The group* $\lambda(\mathcal{S}_2)^{-1}|_{\mathcal{S}_2}$ *is almost simple.*

Proof. Clearly any automorphism of the form λ_w, with $w \in \mathcal{S}_2$, restricts to an automorphism of $\mathcal{S}_2$ and thus one has inclusions

$$\mathcal{S}_2 \simeq \mathrm{Inn}(\mathcal{S}_2) \subseteq \lambda(\mathcal{S}_2)^{-1}|_{\mathcal{S}_2} \subseteq \mathrm{Aut}(\mathcal{S}_2).$$

The conclusion now follows from [**?**] and simplicity of the Higman-Thompson group $\mathcal{S}_2$. □

Of course, one might wonder whether the kernel of the restriction map $\lambda(\mathcal{S}_2)^{-1} \to \lambda(\mathcal{S}_2)^{-1}|_{\mathcal{S}_2}$ is trivial (cf. [**?**]).

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