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Dispersive stability of infinite dimensional Hamiltonian systems on lattices

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Abstract

We derive dispersive stability results for oscillator chains like the FPU chain or the discrete Klein-Gordon chain. If the nonlinearity is of degree higher than 4, then small localized initial data decay like in the linear case. For this, we provide sharp decay estimates for the linearized problem using oscillatory integrals and avoiding the nonoptimal interpolation between different ℓ^p spaces.

1 Introduction

The phenomenon of dispersive stability is well-studied for partial differential equations. Usually one considers a Hamiltonian system where energy conservation prevents strict spectral stability giving rise to exponential decay. Typically the behavior of small solutions is such that the energy norm is bounded from above and below by constants while the L^∞ norm decays with an algebraic rate of the type $(1+t)^{-\alpha}$. This rate is generated from the fact that initially localized solutions are dispersed by the different group velocities associated with the different wave numbers θ. The fundamental effects derive from the dispersion relation $\hat{\omega} = \omega(\theta)$ of the linear differential operator, where $\hat{\omega}$ is the frequency and $c(\theta) = \nabla_\theta \omega(\theta)$ the group velocity. The dispersion is now related to the fact that c still depends nontrivially on θ, i.e. the second derivative of ω should be nontrivial. We refer to [Seg68, Str74, Ree76, Str78] for results treating the sine-Gordon, the Klein-Gordon, the nonlinear Schrödinger, or the relativistic wave equations. Sometimes the theory is developed under the name scattering theory for small data. In [CW91] a recent improvement was made on the lowest order of nonlinearity for the generalized Korteweg-de Vries equation by a careful combination of sharp estimates for the linear part, obtained via deep harmonic analysis, and careful chain-rule estimates for fractional derivatives of the nonlinearity.

The same dispersive effects are to be expected in discrete systems, which are infinite ODEs on a lattice $\mathbb{Z}^d$. The difference is now that the dispersion relation is now a periodic function in θ, i.e. ω is defined on the torus $\mathbb{T}^d := \mathbb{R}^d/_{(2\pi\mathbb{Z})^d}$. Thus, in contrast to PDEs, where ω is an algebraic function on $\mathbb{R}^d$, the dispersion relation has necessarily a richer degeneracy structure. As a result, the linear decay estimates for periodic lattices need a more careful analysis, and it is the aim of this work to establish a more general approach to this field.

To describe the work done so far and our contributions we start by highlighting the three major equations treated in this field, namely the Fermi-Pasta-Ulam chain (FPU), the Klein-Gordon chain (dKG) and the discrete nonlinear Schrödinger equation (dNLS):

$$\ddot{x}_j = V'(x_{j+1} - x_j) - V'(x_j - x_{j-1}), \qquad j \in \mathbb{Z}; \tag{FPU}$$
$$\ddot{x}_j = x_{j+1} - 2x_j + x_{j-1} + W'(x_j), \qquad j \in \mathbb{Z}; \tag{dKG}$$
$$\mathrm{i}\dot{u}_j = u_{j+1} - 2u_j + u_{j-1} + a|u_j|^{\beta-1}u_j, \qquad j \in \mathbb{Z}. \tag{dNLS}$$

Here the potentials V and W are assumed to be such that $V'(r) = r + \mathcal{O}(|r|^\beta)$ and $W'(x) = x + \mathcal{O}(|x|^\beta)$. In general, $\beta > 1$ is used to measure the order of the nonlinearity.

A very careful study of the linear FPU equation was given in [Fri03], which highlights the synchronization phenomena in compact domains. In [Mie06] general multidimensional linear lattice systems were studied on the shorter hyperbolic scale, where energy transport along the rays dominates but dispersion is not yet seen. Discrete lattice systems as finite-difference approximations of wave equations are analyzed in [Zua05, IZ09], where the proper approximation of dispersion relations is an important point.

Dispersive stability results in the direction of this work are obtained in [SK05, GHM06]. The latter work provides the dispersive stability of FPU under the assumption that the nonlinearity satisfies $\beta > 5$. In this work, we will improve this result to the case $\beta > 4$. In [SK05] dKG and dNLS are studied analytically and numerically; we comment on the result of this paper below.

To describe our result we first restrict to FPU, which will be discussed in full detail in Section 3. There we will also treat a generalized FPU chain which allows for any finite number of interactions. Our main result will be that under a suitable stability and nonresonance condition we have dispersive stability if the nonlinearity is of order $\beta > 4$. In particular we will show that the decay of the solution of the nonlinear problem is the same as that of the linear one. The main point in the analysis is that we obtain an improved estimate for the dispersive decay of the linear semigroup. Writing FPU abstractly in the form

$$\dot{\mathbf{z}}(t) = \mathcal{L}\,\mathbf{z} + \mathcal{K}\mathcal{N}(\mathbf{z})$$

and using the the Banach spaces $\ell^p = \ell^p(\mathbb{Z};\mathbb{R}^2)$ we find, for each $p \in [2,4) \cup (4,\infty]$ a constant C_p such that

$$\|\mathrm{e}^{\mathcal{L}\,t}\mathbf{z}_0\|_{\ell^p} \le \frac{C_p}{(1+t)^{\alpha_p}}\,\|\mathbf{z}_0\|_{\ell^1}, \quad \|\mathrm{e}^{\mathcal{L}\,t}\mathcal{K}\,\mathbf{z}_0\|_{\ell^p} \le \frac{C_p}{(1+t)^{\tilde{\alpha}_p}}\,\|\mathbf{z}_0\|_{\ell^1} \quad \text{for } t > 0, \tag{1.1}$$

where the decay rates are given by

$$\alpha_p = \begin{cases} \dfrac{p-2}{2p} & \text{for } p \in [2,4), \\ \dfrac{p-1}{3p} & \text{for } p \in (4,\infty], \end{cases} \qquad \text{and} \qquad \tilde{\alpha}_p = \frac{p-2}{2p}\,.$$

The operator $\mathcal{K}$ arises from the difference structure of the right-hand side in FPU. The case $p = 4$ is excluded in (1.1), since the first estimate holds only with a logarithmic correction, see (3.9b).

The key observation is that the decay rates for $p \in (2, \infty)$ are strictly better than the ones obtained by interpolating the decay $\alpha_2 = 0$ and $\alpha_\infty = 1/3$, which would lead to $\hat{\alpha}_p = (p-2)/(3p) < \alpha_p$. The main work of Section 3 will be devoted to establish the decay estimates (1.1), which are obtained by analyzing the dispersion relation and estimating the resulting oscillatory integrals. The nonlinear stability result is then obtained using standard arguments, which we have collected in an abstract setting in Section 2. We emphasize that all nonlinear decay estimates are of the form that the nonlinear decay is exactly of the order as the linear decay, which is also found numerically, see Figure 1.1. We also show that our decay rates

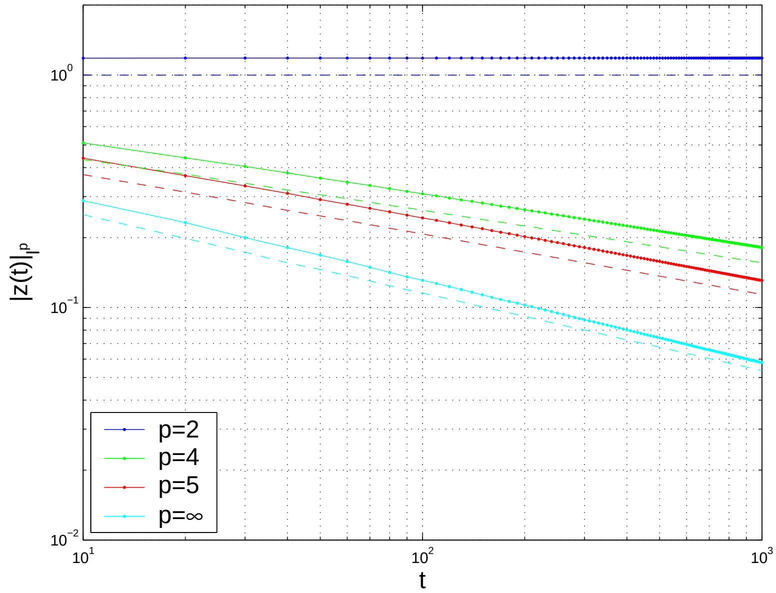


Figure 1.1: Double-logarithmic plot of ℓ^p norms of the solution to the linear FPU (—) and the nonlinear FPU with $V(r) = r + |r|^4$ (- -) as function of t.

are optimal in the sense, that the dispersive decay of the nonlinear system cannot be better than for the linear system.

In Section 4 we will discuss the usage of our method in more general settings such as dKG, dNLS, and a two-dimensional lattice. In particular, we compare our results for dKG with those obtained in [SK05]. There, for $\beta > 5$ dispersive decay in ℓ^p was proved with the rate $\hat{\alpha}_p = (p-2)/(3p)$, while numerically the values 0.226, 0.267, and 0.292 were obtained for $p = 4$, 5, and 6, respectively. We improve the results in a twofold manner: first we reduce the possible order of nonlinearity to the regime $\beta > 4$, and second we establish the better (and sharp) decay rate $\alpha_p = (p-1)/(3p)$, which matches much better with the numerical values.

We conclude with remarking that there is a rich literature on persistent localized solutions in lattices, such as modulated pulses, solitons, and breathers, see e.g. [FW94, FP99, IJ05, GHM06]. From this, it is possible to show that the generalized FPU admits families of solitary waves of KdV type, which for $\beta < 5$ may have arbitrary small energy. However, these solitary waves are the broader the smaller the amplitude is. For the case $\beta < 3$ it follows that dispersive stability cannot hold, see Remark 3.3. It remains open what happens in the case $\beta \in [3, 4]$.

2 General stability result

In this section we present the general method to prove dispersive stability of nonlinear systems, which are based on weak decay estimates of its linearization. The ideas are classical and were established for dispersive stability in PDE theory, see for instance [Seg68] and [Str74]. See also [MSU01] for a survey in the related theory of diffusive stability in parabolic systems. In the context of lattice models the authors of [GHM06] illustrate the ideas in an abstract setting and in [SK05] these arguments are applied to dKG systems and dNLS equations.

To emphasize the general structure we use again an abstract setting in general Banach spaces, which will be specialized to the spaces $\ell^p(\mathbb{Z}^d, \mathbb{R}^n)$ in the following sections. The general aim is to establish conditions that guarantee that the nonlinear system still has the same dispersive decay as the linear one. This will be our first result. In the second result we even go beyond by showing that the different between the solution of the linear systems and the nonlinear systems decays faster than the linear one.

We start with the general system on a Banach space Z given in the form

$$\dot{\mathbf{z}} = \mathcal{L}\,\mathbf{z} + \mathcal{K}\mathcal{N}(\mathbf{z}) \,, \tag{2.1}$$

where $\mathcal{L}$, $\mathcal{K}$ linear and bounded and $\mathcal{N}$ is a nonlinear operator. The operator $\mathcal{L} : Z \to Z$ generates a bounded semi-group $(\mathrm{e}^{\mathcal{L}\,t})_{t\geq 0}$, that is there exists a $C_{\mathcal{L}} > 0$ with $\|\mathrm{e}^{\mathcal{L}\,t}\mathbf{z}\|_Z \leq C_{\mathcal{L}}\|\mathbf{z}\|_Z$ for all $t \geq 0$ and $\mathbf{z} \in Z$. Typically the space Z is chosen such that the solution $\mathbf{z} = \underline{0}$ is a stable solution of (2.1), i.e.

$$\exists\, C_E > 0 \quad \forall\, \text{sln.}\, \mathbf{z}(t)\, \text{with}\, \|\mathbf{z}^0\|_{\ell^2} \leq \varepsilon \quad \forall\, t > 0 : \quad \|\mathbf{z}(t)\|_{\ell^2} \leq C_E \|\mathbf{z}^0\|_{\ell^2}. \tag{2.2}$$

This condition is in particular satisfied if the system is Hamiltonian and the energy functional serves as a Liapunov function. That is, if the energy is bounded from above and below.

However, for proving dispersive stability we need to choose different spaces and do not rely on (2.2). We consider a scale of Banach spaces $Z_0 \subset Z \subset X$ and a space $Z_{\mathcal{N}} \subset Z$ where the embeddings are assumed to be continuous. The space X is used for the estimation of the solutions, Z_0 is taken for the initial conditions, and

$Z_{\mathcal{N}}$ measures the nonlinearity. We assume that positive constants $C_1, C_2, C_3, \alpha, \gamma$, and $\beta > 1$ exist such that the following estimates hold for all $\mathbf{z}$ and all $t \geq 0$:

$$\|\mathrm{e}^{\mathcal{L}\,t}\mathbf{z}\|_X \leq \frac{C_1}{(1+t)^\alpha}\|\mathbf{z}\|_{Z_0}, \tag{2.3a}$$

$$\|\mathrm{e}^{\mathcal{L}\,t}\,\mathcal{K}\,\mathbf{z}\|_X \leq \frac{C_2}{(1+t)^\gamma}\|\mathbf{z}\|_{Z_{\mathcal{N}}}, \tag{2.3b}$$

$$\|\,\mathcal{N}(\mathbf{z})\|_{Z_{\mathcal{N}}} \leq C_3\|\mathbf{z}\|_X^\beta. \tag{2.3c}$$

The following result is the first simple decay estimate, which we state for reasons of clarity. It is in fact a special case of the more involved result given below. Hence we do not provide an independent proof.

Theorem 2.1:
Let the conditions (2.3) hold with $\min\{\gamma, \alpha\beta, \alpha\beta{+}\gamma{-}1\} \geq \alpha$ and $\gamma \neq 1 \neq \beta\alpha$. Then, there exist positive constants C and ε such that for each $\mathbf{z}_0 \in Z_0$ with $\|\mathbf{z}_0\|_{Z_0} \leq \varepsilon$ the unique solution $\mathbf{z}$ of (2.1) with $\mathbf{z}(0) = \mathbf{z}_0$ satisfies

$$\|\mathbf{z}(t)\|_X \leq \frac{C}{(1+t)^\alpha}\|\mathbf{z}(0)\|_{Z_0} \quad \textit{for } t \geq 0.$$

This and the following result rely on the following lemma that is used to estimate the convolution integral occurring in the variation-of-constants formula. The lower bound in the following result is only given to indicate that the provided exponent γ is optimal.

Lemma 2.2:
For constants $\alpha_1, \alpha_2 \in [0,1) \cup (1,\infty)$ there exists a constant $C > 0$ such that

$$\frac{t}{C(1+t)^{\gamma+1}} \leq \int_0^t \frac{1}{(1+t-s)^{\alpha_1}}\,\frac{1}{(1+s)^{\alpha_2}}\,\mathrm{d}s \leq \frac{C}{(1+t)^\gamma} \quad \textit{for all } t > 0, \tag{2.4}$$

where $\gamma = \min\{\alpha_1, \alpha_2, \alpha_1{+}\alpha_2{-}1\}$.

Proof. To obtain the estimate we split the integral into the two domains $[0, t/2]$ and $[t/2, t]$. In the first interval we estimate $(1+t)/2 \leq 1+t-s \leq 1+t$ and obtain

$$\frac{1}{(1+t)^{\alpha_1}}M_2(t/2) \leq \int_0^{t/2} \frac{1}{(1+t-s)^{\alpha_1}}\,\frac{1}{(1+s)^{\alpha_2}}\,\mathrm{d}s \leq \frac{2^{\alpha_1}}{(1+t)^{\alpha_1}}M_2(t/2)$$

where $M_2(r) = \int_0^r (1+s)^{-\alpha_2}\,\mathrm{d}s$. Evaluating the integral M_2 explicitly, we find a decay estimate with exponent $\gamma_2 = \min\{\alpha_1, \alpha_1{+}\alpha_2{-}1\}$. Treating the interval $[t/2, t]$ similarly, the assertion follows by taking $\gamma = \min\{\gamma_1, \gamma_2\}$. $\square$

The following result gives a refinement of the above result. It is based on an additional Banach space V which satisfies $Z_0 \subset V \subset X$ with continuous embeddings. It will play the role of an intermediate space in which we have already some information, namely

$$\exists\, C, \beta_1, \beta_2 > 0 \text{ with } \beta_1 + \beta_2 > 1 \; \forall \mathbf{z} \in Z : \quad \|\mathcal{N}(\mathbf{z})\|_{Z_\mathcal{N}} \le C \|\mathbf{z}\|_V^{\beta_1} \|\mathbf{z}\|_X^{\beta_2}. \tag{2.5}$$

Such estimates occur naturally by interpolation, see (3.5).

Theorem 2.3:
Let the system (2.1) satisfy (2.3) and (2.5). Assume further that there exist positive δ, C_V, ν such that for all $\mathbf{z}_0 \in Z_0$ with $\|\mathbf{z}_0\|_{Z_0} \le \delta$ the unique solution $\mathbf{z}$ of (2.1) satisfies the estimate

$$\|\mathbf{z}(t)\|_V \le \frac{C_V}{(1+t)^\nu} \|\mathbf{z}(0)\|_{Z_0} \quad \textit{for all } t \ge 0. \tag{2.6}$$

Let $\rho = \min\{\gamma, \beta_1\nu + \beta_2\alpha, \gamma + \beta_1\nu + \beta_2\alpha - 1\}$ and assume $\rho \ge \alpha$, $\beta_1\nu + \beta_2\alpha \ne 1 \ne \gamma$, then there exist positive ε and C_X such that for $\|\mathbf{z}(0)\|_{Z_0} \le \varepsilon$ the solutions satisfy

$$\left.\begin{aligned} \|\mathbf{z}(t)\|_X &\le \frac{C_X}{(1+t)^\alpha} \|\mathbf{z}(0)\|_{Z_0} \quad \textit{and} \\ \|\mathbf{z}(t) - \mathrm{e}^{\mathcal{L}t}\mathbf{z}(0)\|_X &\le \frac{C_X}{(1+t)^\rho} \|\mathbf{z}(0)\|_{Z_0}^{\beta_1+\beta_2} \end{aligned}\right\} \quad \textit{for all } t \ge 0. \tag{2.7}$$

Proof. We give the proof in such a way that the case $\beta_1 = 0$ is included, which provides the proof of Theorem 2.1. Then, (2.5) reduces to (2.3c).

We use the variations-of-constants formula

$$\mathbf{z}(t) = \mathrm{e}^{\mathcal{L}t}\mathbf{z}(0) + \int_0^t \mathrm{e}^{\mathcal{L}(t-s)}\, \mathcal{K}\mathcal{N}(\mathbf{z}(s))\, \mathrm{d}s$$

and estimate the solution in the space X. Using the assumptions we obtain

$$\|\mathbf{z}(t)\|_X \le \frac{C_1}{(1+t)^\alpha} \|\mathbf{z}(0)\|_{Z_0} + \int_0^t \frac{C_2}{(1+t-s)^\gamma} \frac{C(C_V\|\mathbf{z}(0)\|_{Z_0})^{\beta_1}}{(1+s)^{\nu\beta_1}} \|\mathbf{z}(s)\|_X^{\beta_2}\, \mathrm{d}s.$$

Assuming $\zeta = \|\mathbf{z}(0)\|_{Z_0} \le \delta$ and introducing $R(t) = \max_{s\in[0,t]}(1+s)^\alpha \|\mathbf{z}(s)\|_X$ and $\mu = \beta_1\nu + \beta_2\alpha$ we find the estimate

$$R(t) \le C_1\zeta + (1+t)^\alpha \int_0^t \frac{1}{(1+t-s)^\gamma} \frac{1}{(1+s)^\mu}\, \mathrm{d}s\, C_* \zeta^{\beta_1} R(t)^{\beta_2}.$$

Employing Lemma 2.2 we have derived the estimate $R(t) \le C_1\zeta + C^*\zeta^{\beta_1}R(t)^{\beta_2}$. It is now easy to find $\varepsilon > 0$ such that for $\zeta \le \varepsilon$ we have $R(t) \le 2C_1\zeta$, which is the first inequality in (2.7).

Reconsidering the variations-of-constants formula once again gives

$$\|\mathbf{z}(t) - \mathrm{e}^{\mathcal{L}t}\mathbf{z}(0)\|_X \le \int_0^t \frac{1}{(1+t-s)^\gamma} \frac{1}{(1+s)^\mu} \,\mathrm{d}s\, C_* \zeta^{\beta_1} R(t)^{\beta_2},$$

and the second estimate in (2.7) follows by employing Lemma 2.2 and the previous estimate for $R(t)$. □

3 Dispersive decay for generalized FPU systems

We now apply the general result presented in section 2 to Hamiltonian systems on a one-dimensional lattice, also called oscillator chain. Here, we only discuss a generalization of the celebrated Fermi-Past-Ulam chain in detail, while in section 4 we outline how to treat discrete Klein-Gordon systems and nonlinear Schrödinger equations.

3.1 The generalized FPU system

We consider an infinite number of equal particles with unit mass and interacting with a finite number K of neighbors via potentials $V_1, \ldots, V_K$. According to Newton's law the equations of motion are

$$\ddot{x}_j = \sum_{1\le k\le K} \left(V_k'(x_{j+k} - x_j) - V_k'(x_j - x_{j-k}) \right) \,, \qquad j \in \mathbb{Z}. \tag{3.1}$$

Here $x_j \in \mathbb{R}$ denotes the displacements. We write $\mathbf{x} := (x_j)_{j\in\mathbb{Z}}$. For the time being we only assume that $V_k'(r) = a_k r + V_{\mathrm{nl},k}'(r)$, $V_{\mathrm{nl},k}'(r) = \mathcal{O}(|r|^\beta)_{|r|\to 0}$ with $\beta > 1$. System (3.1) is Hamiltonian, i.e. $(\dot{\mathbf{x}}, \dot{\mathbf{p}})^T = \mathcal{J}_{\mathrm{can}}\, \mathrm{d}\,\mathcal{H}_{\mathrm{x}}(\dot{\mathbf{x}}, \dot{\mathbf{p}})$ with momentum $\mathbf{p} := \dot{\mathbf{x}}$, $\mathcal{J}_{\mathrm{can}}$ the Poisson tensor corresponding to the canonical symplectic structure defined by $\langle (\mathbf{x}, \mathbf{p}), \mathcal{J}_{\mathrm{can}}(\tilde{\mathbf{x}}, \tilde{\mathbf{p}}) \rangle_{\ell^2\oplus\ell^2} = \langle \mathbf{x}, \tilde{\mathbf{p}} \rangle_{\ell^2} - \langle \tilde{\mathbf{x}}, \mathbf{p} \rangle_{\ell^2}$ and Hamiltonian

$$\mathcal{H}_{\mathrm{x}}(\mathbf{x}, \mathbf{p}) = \sum_{j\in\mathbb{Z}} \left(\frac{1}{2} p_j^2 + \sum_{1\le k\le K} V_k(x_{j+k} - x_j) \right).$$

The dispersive decay is driven by the linearized system

$$\ddot{x}_j = \sum_{1\le k\le K} a_k \big(x_{j+k} - 2x_j + x_{j-k} \big).$$

The dispersion relation is obtained by looking for plane waves in the form $x_j(t) = \mathrm{e}^{\mathrm{i}(\theta j + \hat{\omega} t)}$. We find the relation

$$\hat{\omega}^2 = \Lambda(\theta) := \sum_{1\le k\le K} a_k 2 \big(1 - \cos(k\theta) \big). \tag{3.2}$$

Obviously, we have $\Lambda(0) = 0$, which is a consequence of Galilean invariance. By periodicity, it suffices to take $\theta \in [-\pi, \pi]$ and by reflection symmetry we may take $\theta \in [0, \pi]$ only. Throughout, we make the following *stability condition*

$$\Lambda(\theta) > 0 \quad \text{for all } \theta \in (0, \pi], \tag{3.3}$$

which certainly holds if all a_k are positive, however more general cases are possible.

An essential feature of the considered model is its Galilean invariance, i.e for all $\xi, c \in \mathbb{R}$ the transformation $(\mathbf{x}, \mathbf{p}) \mapsto (x_j + \xi + ct, x_j + c)_{j\in\mathbb{Z}}$ leaves (3.1) invariant. Therefore it is convenient to use distances $\mathbf{r} := (\partial_+ - \mathbf{1})\mathbf{x} = (x_{j+1} - x_j)_{j\in\mathbb{Z}}$ as new variables instead of the displacements. Introducing $\mathbf{z} := (\mathbf{r}, \mathbf{p})^T$ the Hamiltonian turns into

$$\mathcal{H}_\mathrm{r}(\mathbf{z}) = \frac{1}{2}\langle \mathbf{z}, \mathcal{A}_\mathrm{r}\, \mathbf{z}\rangle_{\ell^2} + \mathcal{V}_\mathrm{nl}(\mathbf{z})$$

with

$$\langle \mathbf{z}, \mathcal{A}_\mathrm{r}\, \mathbf{z}\rangle_{\ell^2} = \sum_{j\in\mathbb{Z}} \left(p_j^2 + \sum_{1\le k\le K} a_k \Big| \sum_{0\le l\le k} r_{j+l} \Big|^2 \right)$$

and

$$\mathcal{V}_\mathrm{nl}(\mathbf{z}) = \sum_{j\in\mathbb{Z}} \sum_{1\le k\le K} V_{\mathrm{nl},k}\Big(\sum_{0\le l\le k} r_{j+l} \Big).$$

The transformed Hamiltonian system (3.1) reads as

$$\dot{\mathbf{z}} = \mathcal{J}_\mathrm{r}\, \mathrm{d}\mathcal{H}_\mathrm{r}(\mathbf{z}) = \mathcal{L}\, \mathbf{z} + \mathcal{J}_\mathrm{r}\, \mathcal{N}(\mathbf{z}) \tag{3.4a}$$

where $\mathcal{L} = \mathcal{J}_\mathrm{r}\, \mathcal{A}_\mathrm{r}$ with $\mathcal{J}_\mathrm{r}$, $\mathcal{A}_\mathrm{r}$, and $\mathcal{N}$ given by

$$\mathcal{J}_\mathrm{r} := \begin{pmatrix} \mathbf{0} & \partial_+ - \mathbf{1} \\ \mathbf{1} - \partial_- & \mathbf{0} \end{pmatrix}, \quad \mathcal{A}_\mathrm{r} := \begin{pmatrix} \sum_{|l|<K} \sum_{|l|<k\le K} (k - |l|) a_k \partial_l & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \tag{3.4b}$$

$$\mathcal{N}(\mathbf{z}) := \mathrm{d}\mathcal{V}_\mathrm{nl}(\mathbf{z}) = \begin{pmatrix} \mathbf{0} \\ \Big(\sum_{1\le k\le K} \sum_{0\le m<k} V'_{k,\mathrm{nl}}\big(\sum_{|l+m|\le k} r_{j+l} \big) \Big)_{j\in\mathbb{Z}} \end{pmatrix}, \tag{3.4c}$$

where $(\partial_l \mathbf{z})_j = z_{j+l}$ and $\partial_\pm = \partial_{\pm 1}$. Clearly, $\mathcal{L}\, \mathbf{z} = \mathcal{J}_\mathrm{r}\, \mathcal{A}_\mathrm{r}\, \mathbf{z}$ gives the linear forces and $\mathcal{J}_\mathrm{r} \mathcal{N}(\mathbf{z})$ the nonlinear interaction forces. Here the operator $\mathcal{J}_\mathrm{r}$ refers to the push-forward of the Poisson tensor $\mathcal{J}_\mathrm{can}$, i.e. $\mathcal{J}_\mathrm{r} = \mathcal{T}\, \mathcal{J}_\mathrm{can}\, \mathcal{T}^*$ where $\mathcal{T}$ is the linear map defined by $(\mathbf{r}, \mathbf{p})^T = \mathcal{T}(\mathbf{x}, \mathbf{p})^T$. Note that now $\mathcal{J}_\mathrm{r}$ is a non-canonical Poisson structure.

3.2 Nonlinear dispersive stability

To study the nonlinear system we use the Banach spaces

$$\ell^p(\mathbb{Z}^d; \mathbb{R}^m) \quad \text{with norm } \|\mathbf{z}\|_{\ell^p} := \Big(\sum_{J\in\mathbb{Z}^d} |z_J|^p \Big)^{1/p}$$

where $p \in [1, \infty]$. We frequently write ℓ^p to denote $\ell^p(\mathbb{Z}^d; \mathbb{R}^m)$, if the lattice $\mathbb{Z}^d$ and the space $\mathbb{R}^m$ are either irrelevant or clear from the context.

For $1 \le p_1 < p_2 \le \infty$ we have the continuous embedding $\ell^{p_1} \subset \ell^{p_2}$ with $\|\mathbf{z}\|_{\ell^{p_2}} \le \|\mathbf{z}\|_{\ell^{p_1}}$. An essential tool is the interpolation estimate

$$\|\mathbf{z}\|_{\ell^{p_\vartheta}} \le \|\mathbf{z}\|_{\ell^{p_0}}^{1-\vartheta} \|\mathbf{z}\|_{\ell^{p_1}}^{\vartheta}, \quad \text{where } \frac{1}{p_\vartheta} = \frac{1-\vartheta}{p_0} + \frac{\vartheta}{p_1}, \tag{3.5}$$

and $p_0, p_1 \in [1, \infty]$ and $\vartheta \in [0, 1]$. This is an easy consequence of Hölder's inequality and plays a crucial role in many estimates concerning dispersive decay. Moreover, we use Young's inequality for convolutions $a * b$ with $(a * b)_J = \sum_{I \in \mathbb{Z}^d} a_{J-I} b_I$. For $r, p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ we have

$$\|a * b\|_{\ell^r} \le \|a\|_{\ell^p} \|b\|_{\ell^q} \quad \text{for all } a \in \ell^p,\ b \in \ell^q. \tag{3.6}$$

To apply the general result of section 2 we first provide the a priori estimate (2.2). The theory in section 3.3 shows that (3.3) is equivalent to the existence of a positive constant C such that

$$\frac{1}{C} \|\mathbf{z}\|_{\ell^2}^2 \le \langle \mathbf{z}, \mathcal{A}_{\mathrm{r}}\, \mathbf{z} \rangle_{\ell^2} \le C \|\mathbf{z}\|_{\ell^2}^2 \quad \text{for all } \mathbf{z} \in \ell^2(\mathbb{Z}; \mathbb{R}^2).$$

Using this it is easy to obtain the classical *energy stability* in $\ell^2(\mathbb{Z}; \mathbb{R}^2)$: there are $C_2 > 0$ and $\varepsilon_0 > 0$ such that for all $\mathbf{z}_0 \in \ell^2$ with $\|\mathbf{z}_0\|_{\ell^2} \le \varepsilon_0$ the solution $\mathbf{z}$ of (3.4) with $\mathbf{z}(0) = \mathbf{z}_0$ exists globally in time and satisfies

$$\|\mathbf{z}(t)\|_{\ell^2} \le C \|\mathbf{z}(0)\|_{\ell^2} \quad \text{for all } t \in \mathbb{R}. \tag{3.7}$$

To state the linear decay result we define the relevant branch $\hat{\omega} = \omega(\theta)$ of the dispersion relation via

$$\omega(\theta) := \sqrt{\Lambda(\theta)} \ge 0.$$

With a slight abuse of notation we simply call ω the *dispersion relation*. Under the stability assumptions (3.3) we have $\omega \in \mathrm{C}^\infty([0, \pi])$ and we are able to define the *set of critical wave numbers* as

$$\Theta_{\mathrm{cr}} := \{\theta \in [0, \pi] \mid \omega''(\theta) = 0\}.$$

Since K in (3.2) is finite, Θ_{cr} is discrete and contains $\theta = 0$. Thus, we have $\Theta_{\mathrm{cr}} = \{\theta_0, \ldots, \theta_M\}$ with $\theta_0 = 0 < \theta_1 < ... < \theta_M \le \pi$ for some $M \in \mathbb{N}$.

The following linear decay results will be proved in section 3.3.

Theorem 3.1:
Consider the group $(\mathrm{e}^{\mathcal{L}\,t})_{t \in \mathbb{R}}$ for $\mathcal{L} = \mathcal{J}_{\mathrm{r}} \mathcal{A}_{\mathrm{r}}$ defined in (3.4b). Assume that the dispersion relation ω satisfies (3.3) and the non-degeneracy condition

$$\omega'(0) > 0 \quad \textit{and} \quad \forall\, \theta \in \Theta_{\mathrm{cr}} : \ \omega'''(\theta) \ne 0. \tag{3.8}$$

Then, for $p \in [2,4) \cup (4,\infty]$ there exists C_p such that, for all $t \geq 0$, we have

$$\|\mathrm{e}^{\mathcal{L}t}\|_{\ell^1,\ell^p} \leq \frac{C_p}{(1+t)^{\alpha_p}}, \quad \text{where} \quad \alpha_p = \begin{cases} \dfrac{p-2}{2p} & \text{for } p \in [2,4), \\ \dfrac{p-1}{3p} & \text{for } p \in (4,\infty]. \end{cases} \tag{3.9a}$$

In the case $p = 4$ there exists $C_4 > 0$ such that

$$\|\mathrm{e}^{\mathcal{L}t}\|_{\ell^1,\ell^4} \leq C_4 \left(\frac{\log(2+t)}{1+t}\right)^{1/4} \quad \text{for all } t > 0. \tag{3.9b}$$

If furthermore $\Theta_{\mathrm{cr}} = \{0\}$, then for $p \in [2,\infty]$ there exists $\tilde{C}_p$ such that

$$\|\mathrm{e}^{\mathcal{L}t}\mathcal{J}_{\mathrm{r}}\|_{\ell^1,\ell^p} \leq \frac{\tilde{C}_p}{(1+t)^{\tilde{\alpha}_p}} \quad \text{for all } t > 0, \quad \text{where } \tilde{\alpha}_p = \frac{p-2}{2p}. \tag{3.10}$$

The philosophy of the decay estimate is that oscillations with wave numbers θ travel along rays $j = c(\theta)t$, where the group velocity is given by $c(\theta) = \omega'(\theta)$. The decay along these rays is like $t^{-1/2}$ if $\omega''(\theta) \neq 0$ and like $t^{-1/3}$ if $\theta \in \Theta_{\mathrm{cr}}$. In Figure 3.1 we plot the dispersion relations ω and the associated solution $r_j(t)$ to display the influence of the critical wave numbers $\theta_j \in \Theta_{\mathrm{cr}}$. Thus, the decay like $t^{-1/3}$ in ℓ^∞ is easily obtained. However, for $\theta \approx \theta_n \in \Theta_{\mathrm{cr}}$ there is a cross-over between the two different decay rates, which needs to be estimated carefully to obtain the decay rate α_p for $p \in (2,\infty)$.

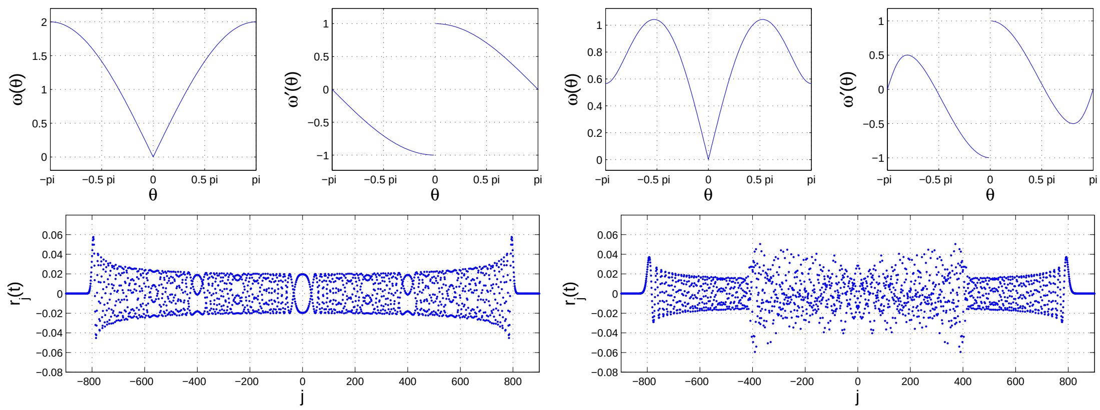


Figure 3.1: Dispersion relations and time evolutions. Left: classical FPU ($K = 1$: $a_1 = -1$). Right: generalized FPU ($K = 2$: $a_1 = 0.08$, $a_2 = 0.23$) with two wave fronts. The upper figures show $\omega(\theta)$ and $\omega'(\theta)$, respectively, and the lower figure shows $r_j(t)$ for $t = 800$ and initial condition $(r_j(0), \dot{x}_j(0)) = (\delta_{j,0}, 0)$.

Because the operator $\mathcal{J}_{\mathrm{r}}$ is related to the difference operators $\partial_1 - 1$ and $\partial_{-1} - 1$, it reduces the amplitudes of very long waves. Thus, in $\mathrm{e}^{\mathcal{L}t}\mathcal{J}_{\mathrm{r}}$ the bad decay associated with $\theta_0 = 0 \in \Theta_{\rm cr}$ is reduced but not for any other $\theta_n \in \Theta_{\rm cr}$. Hence, the last statement needs the requirement $\Theta_{\rm cr} = \{0\}$. In this connection it is interesting to mention that in case $\Theta_{\rm cr} = \{0\}$ the solutions of $\dot{\mathbf{z}} = \mathcal{L}\,\mathbf{z}$ globally decay like $t^{-1/2}$ if we restrict the initial conditions to a suitable subspace. Indeed, if we choose $\mathbf{z}^0 \in \mathcal{J}_{\rm r}\,\ell^1$ this follows from (3.10) and the fact that the operators $\mathcal{J}_{\rm r}$ and $\mathrm{e}^{\mathcal{L}\,t}$ commute.

The following decay result is a direct combination of the abstract results of section 2 and the above linear decay estimates.

Theorem 3.2:
Consider the generalized FPU system satisfying the linearized stability condition (3.3) and the non-degeneracy condition (3.8). Assume that each potential V_k satisfies $V_k(r) = a_k r + \mathcal{O}(|r|^\beta)$ for $\beta > 4$. Then, for each $p \in [2,4) \cup (4,\infty]$ there exist C_p and $\varepsilon > 0$ such that all solutions $\mathbf{z}$ of (3.4) with $\|\mathbf{z}(0)\|_{\ell^1} \le \varepsilon$ satisfy the estimate

$$\|\mathbf{z}(t)\|_{\ell^p} \le \frac{C_p}{(1+t)^{\alpha_p}} \|\mathbf{z}(0)\|_{\ell^1} \quad \textit{for all } t \ge 0, \tag{3.11}$$

where the decay rate α_p is given in (3.9). If additionally $\Theta_{\rm cr} = \{0\}$, then

$$\|\mathbf{z}(t) - \mathrm{e}^{\mathcal{L}\,t}\mathbf{z}(0)\|_{\ell^p} \le \frac{\tilde{C}_p}{(1+t)^{\tilde{\alpha}_p}} \|\mathbf{z}(0)\|_{\ell^1}^\beta \quad \textit{for all } t \ge 0, \tag{3.12}$$

where the decay rate $\tilde{\alpha}_p$ is given in (3.10).

We have omitted the case $p = 4$ to avoid a clumsy presentation. For $p = 4$ one can easily obtain algebraic decay for any $\alpha < 1/4$ by interpolation or a decay as in (3.9b), after generalizing the results in Section 2 to include logarithmic terms.

Proof. In a first step we apply Theorem 2.1 with $Z_0 = Z_\mathcal{N} = \ell^1 \subset X = \ell^{p_1}$ with $4 > p_1 > 2\beta/(\beta-2)$, where we used $\beta > 4$. Because of $\beta > p_1$ we have $\|\mathcal{N}(\mathbf{z})\|_{\ell^1} \le C\|\mathbf{z}\|_{\ell^{p_1}}^\beta$. We estimate $\mathcal{K} = \mathcal{J}_{\rm r}$ by a constant and use $\alpha = \gamma = \alpha_{p_1}$. The choice of p_1 gives $\alpha < 1 < \alpha\beta$ and $\min\{\gamma, \alpha\beta, \alpha\beta+\gamma-1\} = \alpha$, which allows us to apply the theorem. We obtain positive C_{p_1} and ε_0 such that (3.11) holds for $p = p_1$. Since the result holds for $p = 2$ by the nonlinear stability estimate (3.7), the interpolation (3.5) shows that the result holds for $p \in [2, p_1]$. Since p_1 can be chosen as close to $p = 4$ as we like, estimate (3.11) is established for all $p \in [2,4)$.

Next we consider $p \in (4,\beta]$ and see that Theorem 2.3 is applicable with $\nu = \beta_1 = 0$, $\alpha = \alpha_p < \gamma = \tilde{\alpha}_p$, and $Z_0 = Z_\mathcal{N} = \ell^1 \subset X = \ell^p$. Thus, (3.11) and (3.12) hold for $p \in (4,\beta]$.

Finally, we treat the case $p = \infty$ by choosing $p_2 \in (2,4)$ with $p_2 \ge 12-2\beta < 4$. Using $\|\mathcal{N}(\mathbf{z})\|_{\ell^1} \le C\|\mathbf{z}\|_{\ell^{p_2}}^{p_2}\|\mathbf{z}\|_{\ell^\infty}^{\beta-p_2}$ we are able to employ Theorem 2.3 with $Z_0 = Z_\mathcal{N} = \ell^1 \subset V = \ell^{p_2} \subset X = \ell^\infty$, and $\nu = \alpha_{p_2} < \alpha = 1/3 \le \gamma$, where $\gamma = 1/3$ in the general case and $\gamma = 1/2$ if the additional condition $\Theta_{\rm cr} = \{0\}$ holds, see Theorem

3.1. Using $\beta_1 = p_2$ and $\beta_2 = \beta - p_2$ we find $\nu\beta_1 + \alpha\beta_2 > 1$. Hence $\rho = \gamma \geq \alpha$ and the desired estimate (3.11) follows for $p = \infty$. Again, the remaining range $p \in [\beta, \infty]$ follows from interpolation.

If the additional condition $\Theta_{\text{cr}} = \{0\}$ holds, we can apply the last assertion in Theorem 2.3 and obtain (3.12). $\square$

So far, we have only derived estimates for $Z_{\mathcal{N}} = \ell^1$. It is however straight forward to obtain results for $Z_{\mathcal{N}} = \ell^q$ for $q \in (1, 2)$, however the decay rates will be lower and one may need higher order of nonlinearity β. To see this, we simply note that the application of the operator $\mathrm{e}^{\mathcal{L}t}$ is in fact a convolution with a matrix-valued Green's function $\mathbf{G}(t) \in \ell^1(\mathbb{Z}; \mathbb{R}^{2\times 2})$, cf. (3.20). Hence, using Young's inequality (3.6) the operator norm $\|\mathrm{e}^{\mathcal{L}t}\|_{\ell^q,\ell^p}$ can be estimated by $\|\mathbf{G}\|_{\ell^s}$ where $1 + \frac{1}{p} = \frac{1}{s} + \frac{1}{q}$. In fact, the estimates stated above and proved below are obtained by estimating the ℓ^p norm of $\mathbf{G}(t)$.

We emphasize that for $q = 1$ the formula $\|\mathrm{e}^{\mathcal{L}t}\|_{\ell^1,\ell^p} = \|\mathbf{G}(t)\|_{\ell^p}$ holds, since the upper bound follows from Young's inequality and the lower bound is obtained by using the initial condition $\mathbf{z} = (\delta_j)_{j\in\mathbb{Z}}$. Our estimates for $\mathbf{G}_j(t)$ will be sharp enough to establish also lower bounds $\|\mathbf{G}(t)\|_{\ell^p} \geq c/(1+t)^{\alpha_p}$, thus that we cannot hope for better estimates for the linear terms. In fact, using that the decay rates $\alpha_2 = 0$ and $\alpha_\infty = 1/3$ are optimal, it suffices to show that the decay rate α_4 cannot be better than $1/4$ (up to the logarithmic term). Then, for no $p \in (2, \infty)$ the decay rate can be better than α_p, because an interpolation would lead to a better decay rate for $p = 4$. Below we will show that estimate (3.9b) is indeed optimal.

Figure 3.2 displays numerically estimated decay rates, the exact curve α_p, and the curve $\hat{\alpha}_p = (p-2)/(3p)$, which is obtained by interpolation between $p = 2$ and $p = \infty$ and hence is not optimal. The numerical curves agrees well with α_p away from $p = 4$. This effect may be due to the logarithmic correction which spoils the convergence.

In the following remark we argue that the above dispersive decay cannot hold for $\beta < 3$, because of existence of solitary waves with arbitrary small ℓ^1 norm.

Remark 3.3 (Solitary waves)**:**
From [FW94, FP99] the existence of solitary waves for generalized FPU systems can be deduced under additional global conditions on the interaction potentials V_k. Such waves satisfy $z_j(t) = Z(j - ct)$ for a fixed profile $Z : \mathbb{R} \to \mathbb{R}^2$ and a given wave speed c. In particular, [FW94] provides for the case $1 < \beta < 5$ the existence of solitary waves with arbitrarily small energy, i.e. $\|\mathbf{z}^\delta_{\text{soli}}\|_{\ell^2} = \delta \in (0, \delta_0)$. Our stability result implies that for $\beta > 4$ these solution cannot be small in ℓ^1.

In [FP99] the case $\beta = 2$ is investigated, and it is shown that $c = \omega'(0) + \mathcal{O}(\varepsilon^2)$. The constructions there can be generalized to our case to provide small-energy solitary waves of associated with the generalized KdV limit. Moreover, in [SW00] it was shown that solutions of the form $r^\varepsilon_j(t) = \varepsilon^{2/(\beta-1)} R(\varepsilon^3 t, \varepsilon(j + \omega'(0)t)) + \text{h.o.t.}$

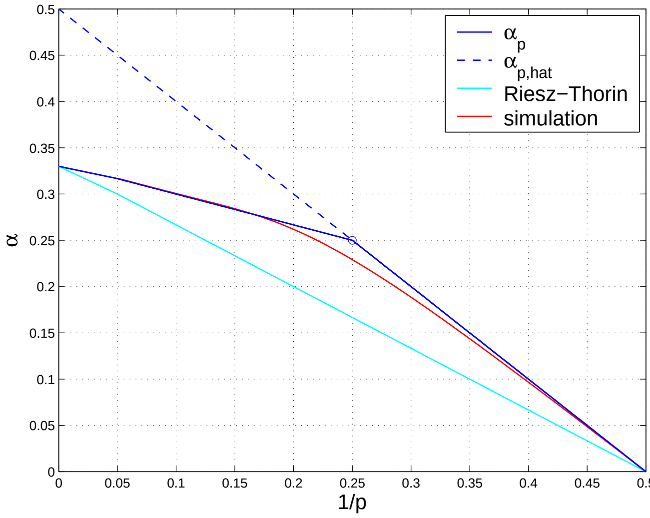


Figure 3.2: Exact decay rate α_p, interpolation rate $\hat{\alpha}_p$, ℓ^2-ℓ^∞ interpolation rate and numerically estimated rates as functions of $1/p$.

exist, where $R : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies the generalized KdV equation

$$\partial_\tau R + b_1 \partial_\eta^3 R + b_2 \partial_\eta V'(R) = 0$$

where $V'(r) = r + \mathcal{O}(|r|^\beta)$. This equation possesses solitary wave solutions with exponentially decaying tales. In terms of the generalized FPU system these solutions satisfy $\|\mathbf{z}^\varepsilon_{\text{soli}}(t)\|_{\ell^1} \sim \varepsilon^{(3-\beta)/(\beta-1)}$, which shows that for $1 < \beta < 3$ there are solitary waves that are arbitrarily small in ℓ^1. We conclude that the above dispersive decay result cannot hold for $\beta < 3$, while the case $\beta \in [3,4]$ remains open.

3.3 ℓ^p-estimates for the linearized system

We consider the linearization of (3.4) in $\mathbf{z} = \mathbf{0}$, i.e. the case $\mathcal{N}(\mathbf{z}) \equiv 0$. To solve the system explicitly we use Fourier transform $\mathcal{F} : \ell^2(\mathbb{Z}, \mathbb{R}^2) \to L^2(\mathcal{S}^1, \mathbb{R}^2)$ defined by $\hat{z}(\theta) = \sum_{j \in \mathbb{Z}} z_j \mathrm{e}^{-\mathrm{i}j\theta}$. Then $\dot{\mathbf{z}} = \mathcal{J}_{\mathrm{r}} \mathcal{A}_{\mathrm{r}} \mathbf{z}$ turns into

$$\begin{pmatrix} \dot{\hat{\mathbf{r}}} \\ \dot{\hat{\mathbf{p}}} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\theta} - 1 \\ 1 - \mathrm{e}^{-\mathrm{i}\theta} & 0 \end{pmatrix} \begin{pmatrix} \omega_{\mathrm{r}}^2(\theta) & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\mathbf{p}} \end{pmatrix} , \tag{3.13}$$

where

$$\begin{aligned} \omega_{\mathrm{r}}^2(\theta) &= \sum_{|l| \le K-1} \sum_{|l| < k \le K} (k - |l|) a_k \mathrm{e}^{\mathrm{i}l\cdot\theta} \\ &= \sum_{0<k\le K} k a_k + 2 \sum_{0<l\le K-1} \left(\sum_{l<k\le K} (k-l) a_k \right) \cos(l \cdot \theta). \end{aligned} \tag{3.14}$$

Since (3.13) implies $\ddot{\hat{\mathbf{r}}} = 2(\cos\theta - 1)\omega_r(\theta)^2\hat{\mathbf{r}}$ and since the previous subsection gives $\ddot{\hat{\mathbf{r}}} = -\Lambda(\theta)\hat{\mathbf{r}}$ we conclude

$$\omega(\theta) = 2\left|\sin\frac{\theta}{2}\right|\omega_{\mathrm{r}}(\theta). \tag{3.15}$$

Using the linear stability condition (3.3) we obtain

$$\exists\, c_{\mathrm{r}} > 0\ \forall\, \theta \in \mathcal{S}^1: \qquad \omega_{\mathrm{r}}(\theta) = \omega_{\mathrm{r}}(-\theta) \geq c_{\mathrm{r}}. \tag{3.16}$$

The fundamental matrix of the linear system (3.13) is

$$\hat{\mathbf{G}}_{\mathrm{r}}(\theta, t) = \begin{pmatrix} \cos(\omega(\theta)t) & \frac{\mathrm{e}^{\mathrm{i}\theta}-1}{\omega(\theta)}\sin(\omega(\theta)t) \\ \frac{-\omega(\theta)}{\mathrm{e}^{-\mathrm{i}\theta}-1}\sin(\omega(\theta)t) & \cos(\omega(\theta)t) \end{pmatrix}. \tag{3.17}$$

The Green's function of our original problem is given by $\mathbf{G}(t) = \mathcal{F}^{-1}\hat{\mathbf{G}}_{\mathrm{r}}(\theta, t)$, that is $G_j(t) = \frac{1}{2\pi}\int_{\mathcal{S}^1}\hat{G}_{\mathrm{r}}(\theta, t)\mathrm{e}^{\mathrm{i}j\theta}\,\mathrm{d}\theta$ for $j \in \mathbb{Z}$. Thus the long time behavior of solutions to the linearized system is determined by oscillatory integrals. For instance, for the classical FPU, i.e. for $\omega_{\mathrm{r}}(\theta) \equiv 1$, the components of G_j turn into Bessel functions, cf. [Fri03]. Below we apply tools from asymptotic analysis to obtain upper bounds on the solutions of the linearized system. To do so it turns out that an alternative representation of the above Green's function is more convenient. Using the symmetry of ω_{r} we find that

$$G_j(t) = \frac{1}{2\pi}\int_0^\pi \begin{pmatrix} h(\theta, t, \frac{j}{t}) & \frac{1}{\omega_{\mathrm{r}}(\theta)}h\big(\theta, t, \frac{j+1/2}{t}\big) \\ \omega_{\mathrm{r}}(\theta)h\big(\theta, t, \frac{j-1/2}{t}\big) & h(\theta, t, \frac{j}{t}) \end{pmatrix}\mathrm{d}\theta$$
$$\text{with } h(\theta, t, c) = \cos\big(t(\omega(\theta)+\theta c)\big) + \cos\big(t(\omega(\theta)-\theta c)\big). \tag{3.18}$$

The new variable $c \in \mathbb{R}$ roughly characterizes the rays $j = ct$ and is used to remind us to the group velocity $c(\theta) = \omega'(\theta)$.

Thus, we obtained the following representation formula for the solution of the linearized problem.

Lemma 3.4 (Explicit solution)**:**
Given some initial conditions $\mathbf{z}^0 = (\mathbf{r}^0, \mathbf{p}^0)^T \in \ell^2(\mathbb{Z}, \mathbb{R}^2)$, the unique solution of $\dot{\mathbf{z}} = \mathcal{L}\,\mathbf{z}$ with $\mathcal{L} = \mathcal{J}_{\mathrm{r}}\mathcal{L}_{\mathrm{r}}$ defined in (3.4b) is determined by

$$\mathbf{z}(t) = \mathrm{e}^{\mathcal{L}t}\mathbf{z}^0 \tag{3.19}$$

where $(\mathrm{e}^{\mathcal{L}t})_{t\in\mathbb{R}}$ is a differentiable group of bounded operators on $\ell^2(\mathbb{Z}, \mathbb{R}^2)$ defined by

$$\big(\mathrm{e}^{\mathcal{L}t}\mathbf{z}\big)_j = \sum_{k\in\mathbb{Z}} G_k(t)\cdot z_{j-k} \qquad \text{for } j \in \mathbb{Z} \tag{3.20}$$

with $G_j(t)$ defined in (3.18).

The asymptotic behavior of (3.18) is determined by terms of the form

$$g(t,c) = \int_0^\pi \psi(\theta) \mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta \quad \text{with } \phi(\theta,c) = \pm(\omega(\theta) - c\theta) \tag{3.21}$$

with ω defined in (3.15) and $\psi(\theta)$ standing for 1, $1/\omega_{\mathrm{r}}(\theta)$ or $\omega_{\mathrm{r}}(\theta)$. In any case ψ is smooth on $[0,\pi]$.

The main result from asymptotic analysis we will use below is van der Corput's lemma, see e.g. [Ste93]. It states that if ϕ is smooth and $\left|\phi^{(k)}(\theta)\right| \geq \lambda > 0$ for $\theta \in (a,b)$ where either $k \geq 2$, or $k=1$ and ϕ' is monotonic, then

$$\left|\int_a^b \mathrm{e}^{\mathrm{i}t\phi(\theta)}\,\mathrm{d}\theta\right| \;\leq\; C_k\,(\lambda t)^{-\frac{1}{k}} \quad \text{with} \quad C_k = (5\cdot 2^{k-1} - 2). \tag{3.22}$$

Note that C_k does neither depend on a and b nor on ϕ explicitly. Writing $F(\theta) = \int_a^\theta \mathrm{e}^{\mathrm{i}t\phi(\xi)}\,\mathrm{d}\xi$ and applying integration by parts to $\int_a^b \psi(\theta)F'(\theta)\mathrm{d}\theta$ we obtain

$$\left|\int_a^b \mathrm{e}^{\mathrm{i}t\phi(z)}\psi(z)\,\mathrm{d}z\right| \;\leq\; C_k\,(\lambda t)^{-\frac{1}{k}}\left(|\psi(b)| + \int_a^b |\psi'(\theta)|\,\mathrm{d}\theta\right). \tag{3.23}$$

In the following lemmas we provide the decay estimates on $g(t,c)$ required to prove the sharp ℓ^p decay rate of the linear group $\mathrm{e}^{\mathcal{L}t}$. We use the notation

$$C_\psi := \max_{\theta\in[0,\pi]} |\psi(\theta)| + \int_0^\pi |\psi'(\theta)|\,\mathrm{d}\theta.$$

Since van der Corput's Lemma only demands assumptions on $\left|\phi^{(k)}(\theta)\right|$ the following considerations are indeed independent of the sign of ϕ in (3.21).

The first lemma provides a global upper bound on $g(t,c)$. Using the classical method of stationary phase, cf. [Won89] it is straight forward to check that the result is sharp.

Lemma 3.5 (Global bound)**:**
Consider the oscillatory integral (3.21) with dispersion relation ω satisfying (3.8) and $\psi \in W^{1,1}([0,\pi])$. Then there exists a constant $C_\omega > 0$ depending only on ω such that

$$\forall\, t \geq 0,\ c \in \mathbb{R}: \quad |g(t,c)| \leq \frac{C_\omega C_\psi}{(1+t)^{1/3}}. \tag{3.24}$$

Proof. Due to $\phi(\theta,c) = \omega''(\theta)$ the following considerations are uniform with respect to the group velocity c.

We write $U_\delta(\theta_m) = \{\theta \in [0,\pi] \mid |\theta - \theta_m| < \delta\}$. Due to the non-degeneracy condition (3.8) it is possible to choose $\delta > 0$ such small that $|\omega'''(\theta)| \geq A$ for all

$\theta \in \bigcup_{m=0}^{M} U_\delta(\theta_m)$ for some constant $A > 0$. Since $\omega''(\theta) = 0$ if and only if $\theta \in \Theta_{\rm cr}$ there exists $B > 0$ with $|\omega''(\theta)| \geq B$ for all $\theta \in [0, \pi] \setminus \bigcup_{m=0}^{M} U_\delta(\theta_m)$. Now we write

$$g(t, c) = \int\limits_{\bigcup_{m=0}^{M} U_\delta(\theta_m)} \psi(\theta) \mathrm{e}^{\mathrm{i}t\phi(\theta,c)} \, \mathrm{d}\theta + \int\limits_{[0,\pi]\setminus\bigcup_{m=0}^{M} U_\delta(\theta_m)} \psi(\theta) \mathrm{e}^{\mathrm{i}t\phi(\theta,c)} \, \mathrm{d}\theta$$

and apply (3.23). Thus

$$|g(t, c)| \leq (M + 1) \left(18A^{-1/3} + 8B^{-1/2}\right) C_\psi t^{-1/3}$$

holds for $t > 1$. Using $|g(t, c)| \leq \pi \max_{\theta\in[0,\pi]} |\psi(\theta)|$ in case $0 \leq t \leq 1$ proves the conclusion for $C_\omega = 2 \max\{\pi, (M + 1)(18A^{-1/3} + 8B^{-1/2})\}$. $\square$

The next result provides the decay rate $t^{-1/2}$ along noncritical rays. The importance is to characterize the width of the regions around the critical rays with decay rate $t^{-1/3}$ that has to be excluded. This result provides sharp estimates for cross-over between the two decay rates. Excluding group velocities near the critical ones corresponding to the critical wave numbers, i.e. allowing only for c with $|c| \notin \bigcup_{\theta_n\in\Theta_{\rm cr}} [|\omega'(\theta_n)| - \varepsilon, |\omega'(\theta_n)| + \varepsilon]$ where $\varepsilon > 0$, (3.25) implies a uniform bound $\sim t^{-1/2}$ on $g(t, c)$. In fact, the result shows that the excluded regions may be taken smaller, namely of width growing like $t^{2/3}$. Using a suitable Airy scaling, it can be shown that this width cannot be decreased, see (3.32) for more details.

Lemma 3.6 (Envelope function)**:**
Consider the oscillatory integral (3.21) with dispersion relation ω satisfying (3.8) and $\psi \in W^{1,1}([0, \pi])$. Then, there exists a constant $C_\omega > 0$ depending only on ω such that for all $t > 0$ and all $c \in \mathbb{R} \setminus \{c \,|\, \exists\, \theta \in \Theta_{\rm cr} : ||\omega'(\theta)| - |c|| \leq t^{-2/3}\}$ holds

$$\big|g(t, c)\big| \leq \frac{C_\omega C_\psi}{(1 + t)^{1/2}} \left(1 + \sum_{\theta\in\Theta_{\rm cr}} \frac{1}{|\omega'(\theta)^2 - c^2|^{1/4}}\right). \tag{3.25}$$

Proof. For $0 < t \leq 1$ we use $|g(t, c)| \leq C_\psi \pi$. Below we assume $t > 1$.

To simplify the considerations let us first assume that there is only one critical wave number $\theta_0 = 0$. For θ near 0 the phase function of (3.21) behaves like $\phi(\theta, c) = \pm(c_0 - c)\theta \pm \frac{\omega'''(0)}{6}\theta^3 + \mathcal{O}(\theta^5)$ with $c_0 = \omega'(0)$. Now we write

$$g(t, c) = \int_0^{\tilde\delta} \psi(\theta) \mathrm{e}^{\mathrm{i}t\phi(\theta,c)} \, \mathrm{d}\theta + \int_{\tilde\delta}^{\delta} \psi(\theta) \mathrm{e}^{\mathrm{i}t\phi(\theta,c)} \, \mathrm{d}\theta + \int_\delta^\pi \psi(\theta) \mathrm{e}^{\mathrm{i}t\phi(\theta,c)} \, \mathrm{d}\theta. \tag{3.26}$$

Due to $\omega'''(0) \neq 0$ there exists $0 < \delta < 1$ and constants $\underline{A}, \bar{A} > 0$ such that

$$\forall\, \theta \in (0, \delta) : \qquad |\omega''(\theta)| \geq \underline{A}\theta \quad \text{and} \quad |\omega'(\theta) - c_0| \leq \bar{A}\theta^2. \tag{3.27}$$

Then we have in particular $|\partial_\theta^2\phi(\theta,c)| = |\omega''(\theta)| \geq \underline{A}\tilde{\delta}$ for all $\theta \in (\tilde{\delta},\delta)$. Since we assumed $\Theta_{\mathrm{cr}} = \{0\}$ there also exists $B > 0$ such that we have $|\partial_\theta^2\phi(\theta,c)| \geq B$ for all $\theta \in (\delta,\pi)$. Thus van der Corput's Lemma (3.23) implies

$$\left|\int_{\tilde{\delta}}^{\delta} \psi(\theta)\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta\right| \leq \frac{8C_\psi}{(\underline{A}\tilde{\delta}t)^{1/2}} \quad \text{and} \quad \left|\int_{\delta}^{\pi} \psi(\theta)\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta\right| \leq \frac{8C_\psi}{(Bt)^{1/2}}. \tag{3.28}$$

Here $\delta, \underline{A}, \bar{A}$ and B do not depend on c but only on ω.

If $\tilde{\delta}$ with $0 < \tilde{\delta} \leq \delta$ is so small that $\tilde{\delta}^2 \leq \frac{1}{\bar{A}+1}|c_0 - |c||$, then using (3.27) we obtain $|\partial_\theta\phi(\theta,c)| = |\omega'(\theta) - c| \geq |c_0 - |c|| - \bar{A}\theta^2 \geq \tilde{\delta}^2$ for all $\theta \in (0,\tilde{\delta})$. Hence, again according to (3.23) we obtain

$$\left|\int_{0}^{\tilde{\delta}} \psi(\theta)\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta\right| \leq \frac{3C_\psi}{\tilde{\delta}^2 t}. \tag{3.29}$$

Now we distinguish two cases. If $\delta^2 \leq \frac{1}{\bar{A}+1}|c_0 - |c||$ we choose $\tilde{\delta} := \delta$. Hence the right hand side of (3.29) is independent of c. Substituting this bound together with the second estimate in (3.28) in (3.26) gives

$$|g(t,c)| \leq \frac{8C_\psi}{(Bt)^{1/2}} + \frac{3C_\psi}{\delta^2 t} \leq \frac{C_\psi}{t^{1/2}}\left(\frac{8}{\sqrt{B}} + \frac{3}{\delta^2}\right). \tag{3.30}$$

In case $\frac{1}{\bar{A}+1}|c_0 - |c|| < \delta^2$ we choose $\tilde{\delta}^2 := \frac{1}{\bar{A}+1}|c_0 - |c||$. Then the assumption $|c_0 - |c|| \geq t^{-2/3}$ yields $\tilde{\delta}^{3/2}t^{1/2} \geq (\bar{A}+1)^{-3/4}$. Thus, combining the upper bound (3.29) with the first estimate in (3.28) leads to

$$\left|\int_{0}^{\delta} \psi(\theta)\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta\right| \leq \frac{8C_\psi}{(\underline{A}\tilde{\delta}t)^{1/2}} + \frac{3C_\psi}{\tilde{\delta}^2 t} \leq \frac{C_\psi}{|c_0 - |c||^{1/4}t^{1/2}}\left(\frac{8}{\underline{A}^{1/2}} + 3(\bar{A}+1)^{3/4}\right).$$

Finally, using this together with the second estimate in (3.28) and $|g(t,c)| \leq C_\psi\pi$ for $0 < t \leq 1$ yields

$$|g(t,c)| \leq \frac{C_\omega C_\psi}{(1+t)^{1/2}}\left(1 + \frac{1}{|c_0^2 - c^2|^{1/4}}\right)$$

with $C_\omega > 0$ depending only on $\omega(\theta)$. The last estimate also covers (3.30) if we choose C_ω sufficiently large. This completes the proof for $\Theta_{\mathrm{cr}} = \{0\}$.

To prove the general case assume we have $\Theta_{\mathrm{cr}} = \{\theta_0, \theta_1, \ldots, \theta_M\}$ with $\theta_0 < \theta_1 < \ldots \theta_M$. We decompose the integral defining $g(c,t)$ like

$$g(c,t) = \cdots + \int_{\theta_m}^{\theta_m+\tilde{\delta}_m} \ldots + \int_{\theta_m+\tilde{\delta}_m}^{\theta_m+\delta_m} \ldots + \int_{\theta_m+\delta_m}^{\theta_{m+1}+\delta_{m+1}} \ldots + \ldots$$

with δ_m and δ_{m+1} sufficiently small such that $\omega'''(\theta) \neq 0$ for $\theta \in (\theta_m - \delta_m, \theta_m + \delta_m) \cup (\theta_{m+1} - \delta_{m+1}, \theta_{m+1} + \delta_{m+1})$. Then similar estimates like (3.27) hold and we use the same arguments as above to get the upper bound

$$\Big| \int_{\theta_m}^{\theta_{m+1}-\delta_{m+1}} \psi(\theta) \mathrm{e}^{\mathrm{i}t\phi(\theta,c)} \,\mathrm{d}\theta \Big| \leq \frac{C_{\omega,m} C_\psi}{(1+t)^{1/2}} \left(1 + \frac{1}{|c_m^2 - c^2|^{1/4}} \right).$$

Since Θ_{cr} is finite this implies the statement. □

Now we state a result that provides a global decay rate $t^{-1/2}$ under the additional assumption that only $\theta = 0$ is a critical wave number and that the function ψ satisfies $\psi(0)$. It will be used to estimate $\mathrm{e}^{\mathcal{L}t} \mathcal{J}_{\mathrm{r}}$, where the bad behavior of the fronts, which relate to long waves (i.e. $\theta = 0$) are filtered out by the difference operators $\partial_\pm - 1$ in $\mathcal{J}_{\mathrm{r}}$.

Lemma 3.7:
Consider the oscillatory integral (3.21) with dispersion relation ω satisfying (3.8) and $\psi \in W^{1,1}([0,\pi])$. If additionally $\Theta_{\mathrm{cr}} = \{0\}$ and $\psi(0) = 0$, then there exists a constant $C_\omega > 0$ depending only on $\omega(\theta)$ such that

$$\forall t \geq 0: \quad \big| g(t,c) \big| \leq \frac{C_\omega C_\psi}{(1+t)^{1/2}}. \tag{3.31}$$

The proof relies on an uniform asymptotic expansion of the oscillatory integrals. Since we think that the technical details would dislocate the focus of the paper we forbear to give the full proof but only highlight the main idea. The detail can be found in [Pat09].

To see the filter effect of the difference operators $\partial_\pm - 1$ we apply the method of stationary phase, cf. [Won89], to $g(t,c)$ for $c = c_0 := \omega'(0)$ and find that it behaves like $t^{-2/3}$. According to [Hör90, 7.7.18] there is a generalization of the classical method of stationary phase which is uniform in terms of the group velocity c. In fact, for $y \in [-\varepsilon, \varepsilon]$ with $\varepsilon > 0$ sufficiently small and $c_0 := \omega'(0)$ holds

$$\begin{aligned} g(t, c_0 + y) \sim t^{-1/3} \mathrm{Ai}\big(a(y) t^{2/3}\big) \big[u_0(y) + \mathcal{O}(t^{-1}) \big] \\ + t^{-2/3} \mathrm{Ai}'\big(a(y) t^{2/3}\big) \big[u_1(y) + \mathcal{O}(t^{-1}) \big]. \end{aligned} \tag{3.32}$$

Here $\mathrm{Ai}(\cdot)$ refers to the Airy function, and a, u_0 and u_1 are smooth functions with $a(0) = 0$. Making these functions explicit we find that the leading order term cancels. Together with Lemma 3.6 this implies (3.31).

In this connection one should note that there is a smooth cross-over between the different scales. Indeed, employing the asymptotic behavior of Airy's function, cf. [Olv74], we obtain for $y < 0$ the asymptotic behavior $t^{-1/3} \mathrm{Ai}\big(a(y) t^{2/3}\big) \sim C_1 t^{-1/2}$ and $t^{-2/3} \mathrm{Ai}'\big(a(y) t^{2/3}\big) \sim C_2 t^{-1/2}$ as $t \to \infty$. Furthermore, the asymptotic expansion (3.32) implies that the width-scaling of the fronts in Lemma 3.5 is sharp. This holds for $\theta = 0$ as well as for general $\theta \in \Theta_{\rm cr}$, where in (3.32) occurs an additional modulating factor $\mathrm{e}^{i\omega(\theta)t}$, see again [Hör90] and [Pat09] for details.

Up to now we provided the decay rates along critical and noncritical rays but we did not use that the effective propagation speed is finite. The light cone corresponds to $c \in [-c_*, c_*]$ where $c_* := \max_{\theta\in\Theta_{\rm cr}} |\omega'(\theta)|$. Outside of this region the decay is faster than algebraic in terms of t as well as in terms of the velocity $c \in (-\infty, -c_*) \cup (c_*, \infty)$.

Applying partial integration, which is the standard argument to see this, cf. [Ste93], is not straight forward due to the occurring boundary terms. In [Fri03] the exponential decay is proved, for the standard FPU case, using a dilation-analytic argument with respect to Fourier frequency. Using (3.32) and the asymptotic behavior of Ai as $t \to \infty$ it turn out that the decay is even faster. In any case one finds for each $\delta > 0$ a decay constant $\kappa_\delta > 0$ such that

$$\forall\, t \geq 0\ \forall\, c \in (-\infty, -c_*{-}\delta] \cup [c_*{+}\delta, \infty): \qquad |g(t,c)| \leq \mathrm{e}^{-\kappa_\delta(|c|-c_*)t}. \tag{3.33}$$

With the above lemmas we are now prepared to prove the ℓ^p decay rate of $\mathrm{e}^{\mathcal{L}\,t}$.

Proof of Theorem 3.1. According to Lemma 3.4 the group $\mathrm{e}^{\mathcal{L}\,t}$ acts as convolution with the matrix-valued Green's function $\mathbf{G}(t) = (\mathbf{G}^{k,m}(t))_{k,m=1,2}$. Using Young's inequality (3.6) we obtain

$$\|\mathrm{e}^{\mathcal{L}\,t}\mathbf{z}^0\|_{\ell^p} \leq \|\mathbf{G}(t)\|_{\ell^p}\|\mathbf{z}^0\|_{\ell^1}.$$

Thus, it is sufficient to prove the desired decay rates in (3.9) and (3.10), respectively, for the components of $\mathbf{G}(t)$.

We only carry out the details of the proof for $\mathbf{G}^{1,1}(t)$. Let us first consider the case $p \neq 4$. We aim to prove

$$\left\|\mathbf{G}^{1,1}(t)\right\|_{\ell^p} \leq \frac{C_p}{(1+t)^{\alpha_p}} \tag{3.34}$$

which according to (3.18) and by introducing the velocity $c = j/t$ as new variable follows from

$$t\int_{-\infty}^{\infty}\left|\frac{1}{2\pi}\int_0^\pi h(\theta,t,c)\,\mathrm{d}\theta\right|^p \mathrm{d}c = \mathcal{O}(t^{-p\alpha_p}) \quad \text{as } t\to\infty.$$

The left hand side is bounded by terms of the form

$$B_p(t) := t\int_{-\infty}^{\infty}|g(t,c)|^p\,\mathrm{d}c$$

with $g(t,c)$ defined in (3.21), $\phi(\theta,c) = \pm(\omega(\theta) \pm c\theta)$ and $\psi(\theta)$ standing for 1, $1/\omega_{\rm r}(\theta)$ or $\omega_{\rm r}(\theta)$. Without loss of generality we only consider $\phi(\theta,c) = \omega(\theta) - c\theta$ and may assume $t > 1$.

To estimate the contributions of each $\theta \in \Theta_{\mathrm{cr}}$ we choose $\varepsilon > 0$ and consider $c \in [\omega'(\theta){-}\varepsilon, \omega'(\theta){+}\varepsilon]$. Using Lemma 3.5 and Lemma 3.6 we find

$$\begin{aligned} B_p(t) &= t\left(\int_{\omega'(\theta)-\varepsilon}^{\omega'(\theta)-t^{-2/3}} + \int_{\omega'(\theta)-t^{-2/3}}^{\omega'(\theta)+t^{-2/3}} + \int_{\omega'(\theta)+t^{-2/3}}^{\omega'(\theta)+\varepsilon}\right) |g(t,c)|^p \,\mathrm{d}c \\ &\leq \frac{C_\omega C_\psi}{(1+t)^{p/3-1/3}} + \frac{2\tilde{C}_\omega C_\psi}{(1+t)^{p/2-1}}\left(1 + \int_{\omega'(\theta)+t^{-2/3}}^{\omega'(\theta)+\varepsilon} \frac{\mathrm{d}c}{|\omega'(\theta)^2 - c^2|^{p/4}}\right) \\ &\leq \frac{C_\omega C_\psi + 2\tilde{C}_\omega C_\psi C}{(1+t)^{(p-1)/3}} + \frac{2\tilde{C}_\omega C_\psi C}{(1+t)^{(p-2)/2}} \end{aligned} \tag{3.35}$$

with C depending on $\omega'(\theta), \varepsilon$ and p. Taking the leading order term we get the decay rate $p\alpha_p$. Thus, using (3.33) and Lemma 3.6 for $c \notin [\omega'(\theta){-}\varepsilon, \omega'(\theta){+}\varepsilon]$ we obtain

$$B_p(t) \leq 2M \frac{C_\omega C_\psi + 2\tilde{C}_\omega C_\psi C}{(1+t)^{p\alpha_p}} + \mathcal{O}(t^{-(p-2)/2}) + \mathcal{O}(\mathrm{e}^{-\kappa_\varepsilon \varepsilon p t})$$

which implies (3.34). Hence, the case $p \neq 4$ is established.

In the case $p = 4$ the additional factor $\log t$ contributing to the leading order term appears on the right hand side of (3.35). Indeed, we obtain

$$B_4(t) \leq \frac{C_\omega C_\psi + 2\tilde{C}_\omega C_\psi C}{1+t} + \frac{2\tilde{C}_\omega C_\psi C}{1+t}\Big(\log t + \log \varepsilon\Big).$$

This is sufficient to see that $\left\|\mathbf{G}^{1,1}(t)\right\|_{\ell^p} \leq C_p\big((1+t)\log(2+t)\big)^{1/4}$.

For the other components of $\mathbf{G}(t)$ we may use exactly the same arguments. This proves the first statement of Theorem 3.1.

To prove the second statement we proceed like above but we use the global upper bound Lemma 3.7 instead of Lemma 3.5 and Lemma 3.6. Then, the leading order term behaves like $t^{(2-p)/p}$. $\square$

4 Outlook: Further applications

4.1 The discrete Klein-Gordon and nonlinear Schrödinger equation

Here we outline how to apply the tools developed in Sections 2 and 3 to other models in one-dimensional chains, namely the discrete Klein-Gordon (dKG) and the discrete nonlinear Schrödinger equation (dNLS), see Section 1.

For (dKG) we have an on-site potential with $W'(x) = bx + \mathcal{O}(|x|^\beta)$. Like in the FPU case our results are not restricted to nearest neighbor interaction. Indeed, we may allow for any finite range interaction as long as the stability condition (3.8) is satisfied; but for simplicity we restrict ourselves to the simplest case, where the dispersion relation reads

$$\omega(\theta) = \sqrt{2 + b - 2\cos\theta}.$$

The stability condition immediately implies $b \leq 0$. In Figure 4.1 we plot the dispersion relation and the time evolution of a prototypical dKG chain. A major

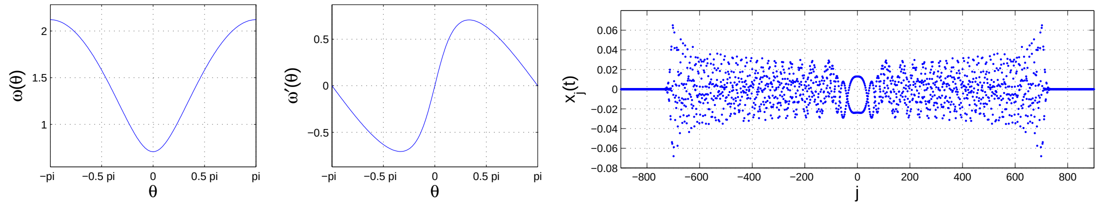


Figure 4.1: Dispersion relation and time evolution for the prototypical dKG chain ($a_1 = -1$, $b = 0.5$): $\omega(\theta)$, $\omega'(\theta)$ and $x_j(t)$ at $t = 800$ to initial condition $(x_j(0), \dot{x}_j(0)) = (\delta_{j,0}, 0)$.

difference to FPU is that the propagation fronts do not correspond to the macroscopic wave number $\theta \approx 0$. Hence, the fronts are not monotone but have an Airy expansion as in (3.32) but multiplied with a factor $e^{i\omega(\theta_*)t}$, where $\omega'(\theta_*) = c_*$ and hence $\omega''(\theta_*) = 0$. Now $\theta = 0$ does not lie in Θ_{cr} because the on-site potential W destroyed the Galilean invariance.

But apart from these two difference the results and the approaches to prove these are the same like in the FPU case. Using the explicit solution of the linearized system, we may prove the analog to Theorem 3.1 with the same decay rates. This relies on the fact that the key ingredients for its proof is the representation of the solutions in terms of oscillatory integral of the form (3.21) and quite general conditions on ω, namely (3.8).

Theorem 4.1:
Consider the discrete Klein-Gordon system (dKG) with $W'(x) = bx + \mathcal{O}(|x|^\beta)$, $b < 0$ and $\beta > 4$. Then, for each $p \in [2, 4) \cup (4, \infty]$ there exist C_p and $\varepsilon > 0$ such that all solutions $\mathbf{z} = (\mathbf{x}, \dot{\mathbf{x}})$ with $\|\mathbf{z}(0)\|_{\ell^1} \leq \varepsilon$ satisfy the estimate

$$\|\mathbf{z}(t)\|_{\ell^p} \leq \frac{C_p}{(1+t)^{\alpha_p}} \|\mathbf{z}(0)\|_{\ell^1} \quad \text{for all } t \geq 0, \tag{4.1}$$

where the decay rate α_p is given in (3.9).

Again the case $p = 4$ can be included by adding a suitable logarithmic term.

This theorem improves the result in [SK05] in a twofold manner, namely in terms of β as well as in terms of the decay rate α_p for $p \in (2, \infty)$. In particular, Theorem 4.1 explains the discrepancy the numerical simulation and the theoretical decay rate $\hat{\alpha}_p$ in [SK05]. We see that our decay rate α_p fits the numerics much better.

p	4	5	6
$\hat{\alpha}_p = \frac{p-2}{3p}$	$\frac{1}{6} \approx 0.167$	$\frac{1}{5}0.2$	$\frac{1}{3} \approx 0.133$
numerics in [SK05]	0.226	0.267	0.292
$\alpha = \frac{p-1}{3p}$	$\frac{1}{4} = 0.25$	$\frac{4}{15} \approx 0.267$	$\frac{5}{18} \approx 0.278$

The above theory can be easily transferred to the discrete nonlinear Schrödinger equation (dNLS), where the dispersion relation reads $\omega(\theta) = 2 - 2\cos\theta$. Obviously $\Theta_{\mathrm{cr}} = \{\pi/2\}$ and the non-degeneracy condition $\omega'''(\pi) \neq 0$ holds. In this case the ℓ^2 norm is in fact a first integral, and hence is preserved exactly along solutions. Using this, it is not difficult to show that for $\beta > 4$ we have dispersive stability with the same decay rates as above.

4.2 Applications to systems in 2D

Here we discuss the application of our general theory to a system on a two-dimensional lattice. The crucial point in higher space dimensions are the estimates for the linear group. Here we only present a conjecture for the decay rates; the rigorous proof being ongoing work, cf. [Pat09]. For methods to handle 2D oscillatory integrals we refer to [Won89, BH86, Hör90] and [GWF81], which is based on techniques derived in [Dui74].

We consider the Hamiltonian system

$$\ddot{x}_j = V'(x_{j+e_1} - x_j) - V'(x_j - x_{j-e_1}) + V'(x_{j+e_2} - x_j) - V'(x_j - x_{j-e_2}) \tag{4.2}$$

with $j = (j_1, j_2) \in \mathbb{Z}^2$. Here $e_1 = (1,0)^T$ and $e_2 = (0,1)^T$ are the unit vectors, $\mathbf{x} := (x_j)_{j\in\mathbb{Z}^2}$ with $x_j \in \mathbb{R}$ and $V'(r) = r + \mathcal{O}(|r|^\beta)$ with $\beta > 1$. To avoid difficulties by introducing an analog to the distances $\mathbf{r}$ in one dimension we restrict ourselves to initial conditions $(\mathbf{x}(0), \dot{\mathbf{x}}(0)) = (\mathbf{x}^0, \mathbf{0}) \in \ell^1(\mathbb{Z}^2, \mathbb{R}^2)$.

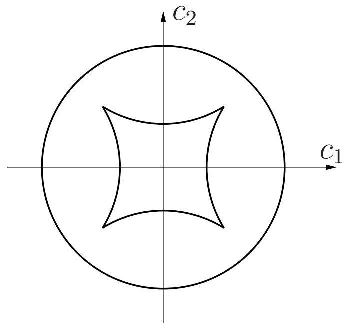
Like in the one-dimensional case it is possible to solve the linearization of (4.2) explicitly and the behavior of the solutions relies on oscillatory integrals of the form

$$g(t,c) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \psi(\theta) \mathrm{e}^{\mathrm{i}t\phi(\theta,c)} \,\mathrm{d}\theta \tag{4.3}$$

with $\phi(\theta, c) = \pm(\omega(\theta) - c\cdot\theta)$, where now $\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$ and $c \in \mathbb{R}^2$. For (4.2) the dispersion relation is given by

$$\omega(\theta) = \sqrt{4 - 2\cos\theta_1 - 2\cos\theta_2}.$$

Although we do not state the formula note that in this case it is possible calculate the critical set $\Theta_{\mathrm{cr}} = \{\theta \in \mathbb{T}^2 \,|\, \det \mathrm{D}^2\omega(\theta) = 0\}$ explicitly. The mapping $\mathrm{D}\omega : \mathbb{T}^2 \to \mathbb{R}^2$ has the range $\{c \in \mathbb{R}^2 \,|\, 0 < |c| < 1\}$ of possible group velocities and maps Θ_{cr} into a closed curve with four vertices, see Figure 4.2, left. The right-hand side of Figure 4.2 displays the time evolution of the first component of the Green's



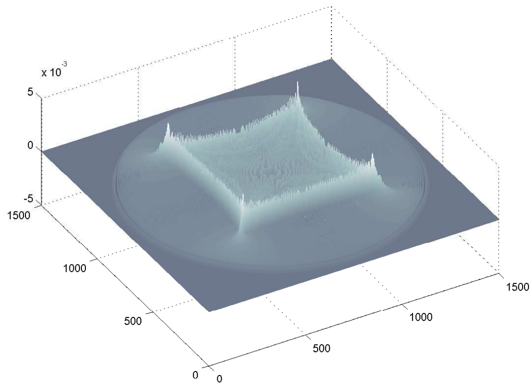


Figure 4.2: Left: The circle is the set of possible group velocities and the curve with four vertices denotes the critical group velocities. Right: Time evolution of the linearization of (4.2) with initial condition $x^0_{j_1 j_j} = \delta_{j_1}\delta_{j_2}$ and $\dot{\mathbf{x}}^0 = \underline{0}$.

function, which clearly shows different regimes at the critical wave numbers. We can roughly distinguish three regions: (i) four vertices, (ii) four edges connecting these vertices and (iii) the remaining region inside the light cone, which is a circle of radius t.

To obtain the decay properties of $\|\mathrm{e}^{\mathcal{L}t}\|_{\ell^1,\ell^p}$, where $\mathcal{L}$ again stands for the linear part of the operator on the right hand side of (4.2), we first determine the local asymptotic behavior of (4.3). Then, assuming a reasonable width of the three different regions we infer the ℓ^p decay rate like in the proof of Theorem 3.1. To do so we apply the localization principle: For $c = \mathrm{D}\omega(\theta)$, the main contribution to $g(t,c)$ is given by

$$I_\theta(t) = \int_{\mathbb{T}^2} h(\delta)\,\mathrm{e}^{\mathrm{i}t\varphi(\delta)}\,\mathrm{d}\delta \quad \text{where } \varphi(\delta) := \phi(\theta+\delta, \mathrm{D}\omega(\theta)). \tag{4.4}$$

Here $h \in C^\infty$, $\operatorname{supp} h \subset U_\varepsilon(0)$ for $\varepsilon > 0$ sufficiently small, and $h(0) = 1$. The function h arises via partition of unity on $\mathbb{T}^2$. The local decay rate $t^{-\alpha(\theta)}$ of $I_\theta(t)$, and hence of $g(t, \mathrm{D}\omega(\theta))$, is determined by the leading-order terms of the Taylor expansion

$$\varphi(\delta) = \varphi(0) + \frac{1}{2}\delta^T \cdot \mathrm{D}^2\omega(\theta)\cdot\delta + \text{h.o.t.}, \quad \mathrm{D}\varphi(0) = 0. \tag{4.5}$$

For $\theta \in \mathbb{T}^2 \setminus \Theta_{\mathrm{cr}}$ a linear coordinate transformation $\delta = A\xi$ leads to $\varphi(A\xi) = \varphi(0) + \xi_1^2 \pm \xi_2^2 + \text{h.o.t.}$ Thus, scaling ξ with $\sqrt{t}$ leads to $|I_\theta(t)| \sim \frac{2\pi}{|\det \mathrm{D}^2\omega(\theta)|^{1/2}}\, t^{-1} + \mathcal{O}(t^{-2})$. This decay rate corresponds to the region inside the light cone, but away from the fronts.

For $\theta \in \Theta_{\mathrm{cr}}$ we have to distinguish two cases. The four vertices correspond to the degenerated points $(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$. If $\theta \in \Theta_{\mathrm{cr}} \setminus \{(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})\}$, then, following the ideas in [GWF81] we find a local coordinate transformation to get $\varphi(A\xi) = \varphi(0) + \xi_1^2 \pm \xi_2^3 + \text{h.o.t.}$ Thus, $|I_\theta(t)| \sim b(\theta)t^{-5/6} + \mathcal{O}(t^{-7/6})$, where $b(\theta)$ is singular in $(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$.

Finally, for $\theta = (\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$ there exists a coordinate transform $\delta = b(\xi)$ such that $\phi(b(\xi)) = \phi(0) - \xi_1^2 - \xi_2^4$. Scaling ξ_1 and ξ_2 with $t^{1/2}$ and $t^{1/4}$, respectively, gives $|I_\theta(t)| \sim b(\theta)t^{-3/4}$, which is also the global ℓ^∞ decay rate.

The decay rate of $\|\mathrm{e}^{\mathcal{L}t}\|_{\ell^1,\ell^p}$ is roughly determined by

$$\left\|g\left(t, \tfrac{\cdot}{t}\right)\right\|^p_{\ell^p(\mathbb{Z}^2,\mathbb{R})} \sim t^2 \int_{|c|\le 1} |g(t,c)|^p \mathrm{d}c = \frac{t^2}{(2\pi)^{2p}} \int_{|c|<1} \left|\int_{\mathbb{T}^2} \mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta\right|^p \mathrm{d}c. \tag{4.6}$$

Using the normal forms given above we estimate the amount of the three regions on the right-hand side and obtain

$$\|g(t, \tfrac{\cdot}{t})\|^p_{\ell^p} \sim t^2\Big(C_{\text{cone}}t^{-p} + C_{\text{curve}}t^{-\beta}t^{-5p/6} + C_{\text{vertex}}t^{-\gamma}t^{-3p/4}\Big),$$

where $t^{-\beta}$ gives area of the regions around the four curves and $t^{-\gamma}$ the area of the regions around the four vertices measured relatively to the disc $|c| < 1$. We conjecture that the correct values are $\beta = 2/3$ and $\gamma = 3/4$.

This conjecture leads to the decay rates $\alpha_p^{\text{2D}} = \min\{\frac{p-2}{p}, \frac{3p-5}{4p}\}$, which is obtained from interpolating the three values $\alpha_2 = 0$, $\alpha_3 = 1/3$, and $\alpha_\infty = 3/4$. It seems reasonably that the case $p = 3$ needs a logarithmic correction. The numerical simulations shown in Figure 4.3 agree quite well with this rate for $p \in [2,3]$, however there are major discrepancies for larger p. In any case, the numerics clearly suggests that the optimal decay rates are better that the ones, which can be obtained by interpolating between $\alpha_2 = 0$ and $\alpha_\infty = 3/4$.

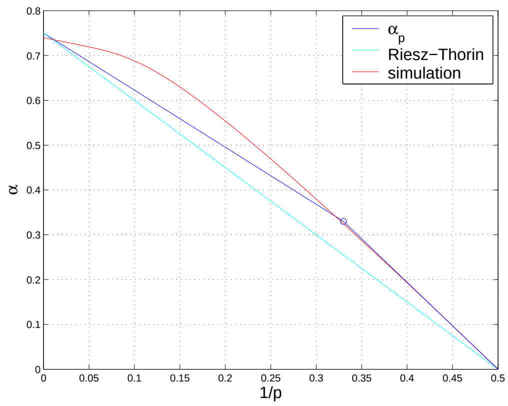


Figure 4.3: Conjectured exact decay rate $\alpha 2\mathrm{D}_p$, interpolation rate and numerically estimates rates as function of $1/p$.

Nevertheless, if α_p^{2D} hat the above form for $p \in [2,3)$, then the nonlinear dispersive decay theory of Theorem 2.1 provides dispersive decay with this rate whenever the nonlinearity is of degree $\beta > 3$.

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