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1. Introduction

In one fell swoop, with the inauguration of the theory of buildings, Tits [42] introduced a geometric perspective to the study of groups of Lie type. Previously, at the hands Chevalley [14], Steinberg [39] and Ree [30, 31], this class of groups had been given a unified treatment as certain groups of automorphisms of Lie algebras and fixed points of automorphisms of algebraic groups. The utility of buildings was amply demonstrated in [42] where groups with a spherical BN-pair of rank at least 3 are classified. Buildings are important in the study of other classes of groups such as the simple algebraic groups and, with the emergence of twin buildings, Kac-Moody type groups [44]. The various successes of the theory of buildings (see for example [35], [43], [27]) have led to attempts to widen the underlying ideas of buildings to obtain geometric information about other simple groups, with a particular eye upon the sporadic finite simple groups. Early contributions to this endeavor were made by Buekenhout [12], Ronan and Smith [32, 33] and Ronan and Stroth [34].

Here we shall be interested in finite groups. So suppose that G is a finite group, p is a prime number and S a Sylow p-subgroup of G. Set $B = N_G(S)$. A subgroup P of G which properly contains B is called a *p-minimal subgroup* of G (with respect to B) if B is contained in a unique maximal subgroup of P. Put

$$\mathcal{M}(G,B) = \{P \mid B < P \leq G \text{ and } P \text{ is } p\text{-minimal}\}$$

and

$$\mathcal{LL}(G,B) = \{H \mid B < H \leq G\}.$$

So $\mathcal{LL}(G,B)$ is the set of proper overgroups of B in G, and clearly $\mathcal{M}(G,B) \subseteq \mathcal{LL}(G,B)$. Now suppose that G is a group of Lie type whose characteristic is p. Then the associated building of G is the simplicial complex obtained from the poset on $\{H^g \mid g \in G, H \in \mathcal{LL}(G,B)\}$ given by reverse containment. The notion

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of a building may be rephrased in terms of chambers (see [43]). With this reinterpretation $\mathcal{M}(G,B)$ is precisely the set of stabilizers of the panels of the chamber corresponding to B. The subgroups in $\mathcal{M}(G,B)$ in this context are called minimal parabolic subgroups and for each $P \in \mathcal{M}(G,B)$, B is actually a maximal subgroup of P. Indeed, for any $H \in \mathcal{LL}(G,B) \setminus \{G\}$ we also have that $O_p(H) \neq 1$ (see [10]); that is H is a p-local subgroup of G which explains the choice of $\mathcal{LL}$ for local lattice.

Now assume that G is an arbitrary finite group. Attempts to generalize buildings, mentioned above, have used various subsets of $\mathcal{LL}(G,B)$ as a means of passing to a geometric object in the spirit of buildings. Much attention has been focussed upon subsets of $\mathcal{M}(G,B)$. An important notion is that of a minimal parabolic system – a subset $\{P_1,\ldots,P_m\}$ of $\mathcal{M}(G,B)$ is a *minimal parabolic system* for G (of rank m) if $G = \langle P_1,\ldots,P_m\rangle$ and no proper subset of $\{P_1,\ldots,P_m\}$ generates G. The minimal parabolic systems for the sporadic simple groups are collated in Ronan and Stroth[34] for all cases when S is non-cyclic (though they also require $O_p(P_i) \neq 1$ for $i = 1,\ldots,m$). While Lempken, Parker and Rowley in [23] determined all the minimal parabolic systems when G is a symmetric group and $p = 2$. For further work in this direction see Covello [15] and Rowley and Sanita[36]. Unlike the case of Lie type groups of characteristic p, in other groups, such as the sporadic simple groups and the symmetric groups, there is not usually a unique minimal parabolic system.

Lattices of subgroups have long been of interest. For some indication of earlier work see Suzuki [40] and Schmidt [38]. A recent topic of interest was suggested by a theorem of Pálfy and Pudlakand[28] raising the question as to whether each nonempty finite lattice is isomorphic to an overgroup lattice for some subgroup of some finite group.The answer is almost certainly negative – for investigations into this and related questions see Aschbacher [4, 5, 6], Aschbacher and Shareshian [7] and Feit [17]. The set $\mathcal{M}(G,B)$ has some relevance to this type of question as it is the case that for any $H \in \mathcal{LL}(G.B)$ we have that $H = \langle P \mid P \in \mathcal{M}(G,B)\rangle$ (see Lemma 3.2) and therefore the subgroups in $\mathcal{M}(G,B)$ in a certain sense control the lattice of subgroups of G above B.

One of the main purposes of this paper, and its successors, is to describe all of the 2-minimal subgroups for the finite groups of Lie type. A secondary aim is to then probe the minimal parabolic systems. If the characteristic of the Lie type group is also 2, then we just have the panel stabilizers and these subgroups are well understood. Thus we focus our attention upon Lie type groups of odd characteristic. More specifically, here we examine the 2-minimal subgroups of the linear, special linear, unitary and special unitary groups. We begin with the general linear and unitary groups, using the usual notation GL_n^ϵ, $\epsilon = \pm$, to denote these two classes simultaneously. However, before stating our first theorem, we briefly discuss some classes of subgroups, detailed definition being given in later sections. For $G = \mathrm{GL}_n^\epsilon(q)$ where $q = p^a$ is odd, a certain Sylow 2-subgroup S of G was described by Carter and Fong [13] (see also Theorem 5.1). Using this description when $q \equiv \epsilon \pmod 4$ we may view S within H, a subgroup of G which is identified as a wreath product $\mathrm{GL}_1^\epsilon(q) \wr \mathrm{Sym}(n)$ As a consequence the 2-minimal subgroups, called fusers and linkers, appearing in [23] metamorphoses into 2-minimal subgroups of G. Such subgroups we also refer to as fusers and linker, denoting the set of them respectively by $\mathcal{F}$ and $\mathcal{L}$. The base group of H also contributes to our haul of 2-minimal subgroups yielding the set $\mathcal{T}$ of so-called toral 2-minimal subgroups.

Similar 2-minimal subgroups are present when $q \equiv -\epsilon \pmod 4$. A further source of 2-minimal subgroups arises from the parabolic subgroups of G (parabolic being used in the traditional sense) when $\epsilon = +$. These subgroups have non-trivial p-radicals and so are referred to as radical 2-minimal subgroups. We let $\mathcal{R}$ denote the set of all such 2-minimal subgroups of G. When n is odd and $\epsilon = -$ the radical subgroups are replaced by a family of 2-minimal unitary subgroups and these we denote by $\mathcal{U}$. Two additional classes of 2-minimal subgroups of G, denoted by $\mathcal{Q}$ and $\mathcal{S}$, and called quaternion and special linear force their attention upon us. They owe their ancestry to small dimensional linear and unitary groups which in small dimensions can themselves be 2-minimal. So now to our first main result.

Theorem 1.1. *Suppose that $G = \mathrm{GL}_n^\epsilon(q)$ where $n \geq 2$ and $q = p^a$ is odd. Let $S \in \mathrm{Syl}_2(G)$ and set $B = N_G(S)$. Then*

$$\mathcal{M}(G, B) = \mathcal{T} \cup \mathcal{F} \cup \mathcal{L} \cup \mathcal{Q} \cup \mathcal{S} \cup \mathcal{R} \cup \mathcal{U}.$$

As to whether any of the above sets of 2-minimal subgroups are empty depends upon certain specified conditions on ϵ, n and q. For a comprehensive overview of the set $\mathcal{M}(G, B)$ in Theorem 1.1 see Tables 1 and 2. Although there is a deal of complexity in their definition, particularly of the toral 2-minimal subgroups, the overall list of 2-minimal subgroups is pleasingly short. Moreover, aside from the congruences of q mod 8, the 2-minimal subgroups not in $\mathcal{T}$ are defined without reference to the underlying field. A further noteworthy feature is that the groups in $\mathcal{M}(G, B)$ for $G = \mathrm{GL}_n^\epsilon(q)$ are for the most part soluble groups, and these soluble groups have a very restricted structure.

Next we describe the layout of this paper and the main features of the proof of Theorem 1.1. As already mentioned, the wreath product subgroups appearing in [13] demand our attention. Thus in Section 2 we set up notation enabling us to describe explicitly the 2-minimal subgroups of the symmetric groups. In Section 4, for E cyclic of odd order and X a symmetric group we analyze the wreath product $H = E \wr X$– we sometimes call such groups monomial groups. But also observe that, in another guise they are complex reflection groups (denoted $G(m, 1, n)$ in Shephard and Todd's list [37].) The S-module structure of the base group of H is the main focus here resulting in subgroups of the form $U(n_i; s^c; j)$. These subgroups in turn give birth to the toral 2-minimal subgroups. Also, but with less technicalities, the linker and fuser 2-minimal subgroups are introduced in this section.

Section 3 is a repository for general results on p-minimal subgroups (for p an arbitrary prime) which are needed in this paper. A number of these play a critical role in our proofs. For example Lemma 3.9 means that 2-minimal subgroups behave very well with respect to direct products, and hence facilitates certain induction arguments.

The proof of Theorem 1.1 begins in Section 5, where further notation relating to S, B, and the standard vector space of $\mathrm{GL}_n^\epsilon(q)$, and gathers pace in the ensuing sections.

2. Preliminaries

As intimated in Section 1, Section 4 sees us probing the 2-minimal subgroups of monomial groups, that is wreath products $E \wr \mathrm{Sym}(n)$ where E is cyclic of odd order and $\mathrm{Sym}(n)$ is the symmetric group of degree n. Accordingly, we need to assemble appropriate notation relating to $\mathrm{Sym}(n)$ and its 2-minimal subgroups. So let Ω be a set of cardinality $n > 2$ and fix the following notation for the 2-adic decomposition of n:

$$n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r} \text{ where } n_1 > n_2 > \cdots > n_r \geq 0.$$

Set $X = \mathrm{Sym}(\Omega)$, the symmetric group on Ω, and let T be a fixed Sylow 2-subgroup of X. Now T has r orbits on Ω, and we denote these orbits by $\Omega_1, \Omega_2, \dots, \Omega_r$ where $|\Omega_i| = 2^{n_i}$. Putting $I = \{1, \dots, r\}$, we have that

$$T = T_{n_1} \times T_{n_2} \times \cdots \times T_{n_r}$$

where, for $i \in I$, $T_{n_i} \in \mathrm{Syl}_2(\mathrm{Sym}(\Omega_i))$. Observe that T_0 is the trivial group. From [19, Satz 15.3, p. 378] we have that each T_{n_i} is an iterated wreath product of i cyclic groups of order 2 and that $N_X(T) = T$. Thus, we note, for $j, k \geq 0$, $T_{n_j} \wr T_{n_k} = T_{n_{j+k}}$.

We next introduce two types of subgroups of X. Let $i \in I$. Then, for $j \in \{1, \dots, n_i - 1\}$, let $\Sigma_{n_i;n_j}$ be the collection of T-invariant block systems of Ω_i consisting of sets of order 2^k where $k \in \{0, \dots, n_i\} \setminus \{j\}$, and define

$$X(n_i; n_j) = \mathrm{Stab}_{\mathrm{Sym}(\Omega_i)} \Sigma_{n_i;n_j} \times (\prod_{k \in I \setminus \{i\}} T_{n_k}).$$

Put

$$\mathcal{L}(X, T) = \{X(n_i; n_j) \mid i \in I, j \in \{1, \dots, n_i - 1\}\}.$$

For $i, j \in I$, with $i < j$ (so $n_j < n_i$) set $\Lambda_{n_i+n_j} = \Omega_i \cup \Omega_j$. Let Γ_i be the collection of all block systems for T on Ω_i and Γ_j the collection of all block systems of T on Ω_j. We define $\Sigma_{n_i+n_j}$ to be the collection of T-invariant systems of subsets of $\Lambda_{n_i+n_j}$ which are the union of one block system from Γ_i and one from Γ_j with the proviso that the blocks of the two chosen block systems have equal numbers of elements. Then

$$X(n_i + n_j) = \mathrm{Stab}_{\mathrm{Sym}(\Lambda_{n_i+n_j})}(\Sigma_{n_i+n_j}) \times (\prod_{k \in I \setminus \{i,j\}} T_{n_k})$$

and we set

$$\mathcal{F}(X, T) = \{X(n_i + n_j) \mid i, j \in I, i < j\}.$$

The subgroups in $\mathcal{L}(X, T)$ are called *linkers* and those in $\mathcal{F}(X, T)$ *fusers* and, as we see, comprise all the 2-minimal subgroups of X.

Theorem 2.1. *Assume that Ω is a set with $|\Omega| > 2$, $X = \mathrm{Sym}(\Omega)$ and $T \in \mathrm{Syl}_2(X)$. Then $\mathcal{M}(X, T) = \mathcal{L}(X, T) \cup \mathcal{F}(X, T)$.*

Proof. This is proved in [23, Theorem 1.1]. □

In our investigations of monomial groups, or subgroups of $\mathrm{GL}_n^\epsilon(q)$ where subgroups isomorphic to $\mathrm{Sym}(n)$ can be identified the above notational conventions will be employed. So the use of X as a subgroup alerts us to the fact that $X \cong \mathrm{Sym}(n)$ and that (unless indicated otherwise) all the accompanying notation n_i, r, I, $X(n_i; n_j)$, $X(n_i + n_j)$, T and T_{n_i} will be used.

At this point we also note that [3] will be our bible for standard group theoretic notation. We follow the ATLAS [16] conventions in describing the shapes of groups, though, as we have seen, we use $\mathrm{Sym}(n)$ for the symmetric group of degree n, $\mathrm{Alt}(n)$ for the alternating group of degree n and $\mathrm{Mat}(10)$ for the "Mathieu group of degree 10".

For l a positive integer l_2 denotes the largest 2-power which divides l and $\Pi(l)$ the set of all odd prime powers greater than 1 which divide l. So, for example, if $l = 180$, then $l_2 = 2^2$ and $\Pi(l) = \{3, 3^2, 5\}$.

Our next theorem plays an invaluable role in determining the structure of 2-minimal linker subgroups of monomial groups.

Theorem 2.2. *Suppose G is a finite soluble group, Q is a nilpotent normal subgroup of G with K and L subgroups of G. Assume that*

(i) *no G-chief factor of G/Q is G-isomorphic to a G-chief factor of Q; and*
(ii) *K and L are supplements to Q in G with $K \cap Q = L \cap Q$.*

Then K and L are G-conjugate.

Proof. See [29]. □

Lemma 2.3. *Suppose that $X \cong \mathrm{Sym}(n)$ and E is a cyclic group of odd order. Let $H = E \wr X$ and F be the base group of H. Considering F as a $\mathbb{Z}X$-module, we have $\mathrm{H}^1(X, F) = 0$.*

Proof. Let $X_1 \leq X$ be a one-point stabilizer of X. So $X_1 \cong \mathrm{Sym}(n-1)$. Then we can consider E as a trivial $\mathbb{Z}X_1$-module. With this interpretation we have $F = \mathrm{Ind}_{X_1}^X(E)$. Since $|E|$ is odd, we have $\mathrm{H}^1(X_1, E) = 0$. Now the result follows from Shapiro's Lemma [9, Proposition III.6.2]. □

Lemma 2.4. *Let E be a cyclic group of odd order, n a natural number and $X = \mathrm{Sym}(n)$. Let $H = E \wr X$, F be the base group of H and $[F, X]C_F(X) \leq Y \leq F$. Then YX contains exactly $|F/Y|$ conjugacy classes of complements to Y.*

Proof. We view Y and F as $\mathbb{Z}X$-modules. By Lemma 2.3 $\mathrm{H}^1(X, F) = 0$. We have a short exact sequence of X-modules $0 \to Y \to F \to F/Y \to 0$. Hence by [9, III.6.1 (ii)] we have a long exact sequence which starts

$$\begin{aligned} 0 \quad &\to \quad \mathrm{H}^0(X, Y) \to \mathrm{H}^0(X, F) \to \mathrm{H}^0(X, F/Y) \to \mathrm{H}^1(X, Y) \\ &\to \quad \mathrm{H}^1(X, F) \to \dots . \end{aligned}$$

By [9, III.1.8] $\mathrm{H}^0(X, F) \cong \mathrm{H}^0(X, Y) \cong C_F(X) = 0$ and $\mathrm{H}^0(X, F/Y) \cong F/Y$. Hence the map $\mathrm{H}^0(X, F) \to \mathrm{H}^0(X, F/Y)$ is the zero map and as $\mathrm{H}^1(X, F) = 0$, the map $\mathrm{H}^0(X, F/Y) \to \mathrm{H}^1(X, Y)$ is an isomorphism. Hence $|\mathrm{H}^1(X, Y)| = |Y/F|$ and the result now follows from [3, 17.7] or [9, Proposition III.2.3]. □

Finally in this section we have the following elementary lemma.

Lemma 2.5. *Suppose that H is a normal subgroup of a group G. Let $S \in \mathrm{Syl}_p(G)$ and $R = S \cap H$. If $N_G(S) = N_G(R)$, then S is the unique Sylow p-subgroup of G which contains R.*

Proof. Using the Frattini Argument we have $|G : N_G(S)| = |N_G(R)H : N_G(R)| = |H : N_H(R)|$. Hence the map $T \mapsto T \cap H$ is a bijection between $\mathrm{Syl}_p(G)$ and $\mathrm{Syl}_p(H)$. □

3. p-minimal subgroups

In this section p is a prime, G is a group, S a Sylow p-subgroup of G and $B = N_G(S)$. We recall that a subgroup P of G containing B is called *p-minimal* so long as $P \neq B$ and B is contained in a unique maximal subgroup of P. We denote the set of p-minimal subgroups of G containing B by $\mathcal{M}(G, B)$.

Lemma 3.1. *If H and K are G-conjugate subgroups of G which contain B, then $H = K$.*

Proof. Let $g \in G$ be such that $H^g = K$. Then both S and $S^{g^{-1}}$ are Sylow p-subgroups of H. By Sylow's Theorem there exist $h \in H$ such that $g^{-1}h = b \in B$. So $g = hb^{-1} \in H$ which means that $K = H^g = H$. □

Lemma 3.2. *Either G is p-closed or $G = \langle \mathcal{M}(G,B) \rangle = \langle O^{p'}(Y) \mid Y \in \mathcal{M}(G,B) \rangle B$.*

Proof. Assume that G is a minimal counterexample to the statement that $G = \langle \mathcal{M}(G,B) \rangle$ and that G is not p-closed. Then $\mathcal{M}(G,B)$ is not empty and $G > \langle \mathcal{M}(G,B) \rangle$. Suppose that U is a maximal subgroup of G containing B. If $U = B$, then $G \in \mathcal{M}(G,B)$, and we have a contradiction. So, by the minimality of G, $U = \langle \mathcal{M}(U,B) \rangle$. Since $\mathcal{M}(U,B) \subseteq \mathcal{M}(G,B)$, we have $U \leq \langle \mathcal{M}(G,B) \rangle < G$. Hence $U = \langle \mathcal{M}(G,B) \rangle$ is the unique maximal subgroup of G containing B. Thus $G \in \mathcal{M}(G,B)$ and again we have a contradiction. For the second equality, we just note that B normalizes $\langle O^{p'}(Y) \mid Y \in \mathcal{M}(G,B) \rangle$ and therefore $\langle O^{p'}(Y) \mid Y \in \mathcal{M}(G,B) \rangle B = \langle O^{p'}(Y)B \mid Y \in \mathcal{M}(G,B) \rangle = \langle \mathcal{M}(G,B) \rangle = G$. □

Definition 3.3. *For H a group and X a group which admits an action of H, we say that X is H-*minimal *provided X has a unique maximal H-invariant subgroup.*

Lemma 3.4. *Suppose that $P = BK \in \mathcal{M}(G,B)$ for some normal subgroup K of order coprime to p. Then $P = B[K,S]$ and $[K,S]$ is B-minimal. If additionally, $[K,S]$ is nilpotent, then it is an r-group for some prime r.*

Proof. Set $L = [K,S]$. Then $K = C_K(S)L$ and so $P = BL$. Assume that L_1 and L_2 are maximal B-invariant subgroups of L. Then BL_1 and BL_2 are both subgroups of P. If, say, $P = BL_1$, then we have

$$L \leq P \cap K = BL_1 \cap K = L_1(B \cap K)$$

which implies that

$$L = [L,S] = [L_1(B \cap K), S] = [L_1 C_K(S), S] \leq L_1.$$

Therefore BL_1 and similarly BL_2 are both proper subgroups of P containing B. Hence BL_1 and BL_2 are both contained in the unique maximal subgroup of P containing B. Thus $B\langle L_1, L_2 \rangle$ is a proper subgroup of P. Hence by maximality $L_1 = L_2$ and so L is B-minimal.

Finally, assuming that L is nilpotent, as L is B-minimal, we conclude that it must be an r-group for some prime r. □

Lemma 3.5. *Suppose that K is a normal subgroup of G and $P \in \mathcal{M}(G,B)$. If $PK \neq BK$, then $PK/K \in \mathcal{M}(G/K, BK/K)$.*

Proof. First observe that $P > B(P \cap K)$ and that PK/K does not normalize SK/K by the Fratinni Argument. Hence $B(P \cap K)$ is contained in the unique maximal subgroup U of P which contains B. Then $U/(P \cap K)$ is the unique maximal subgroup of $P/(P \cap K)$ which contains $B(P \cap K)/(P \cap K)$. Hence BK/K is contained in a unique maximal subgroup of PK/K and so $PK/K \in \mathcal{M}(G/K, BK/K)$. □

Lemma 3.6. *Suppose that K is a normal subgroup of G and G/K is p-minimal. Then there exists $P \in \mathcal{M}(G,B)$ such that $G = PK$.*

Proof. By the Frattini Argument $BK/K = N_G(SK/K)$. Therefore Lemmas 3.2 and 3.6 give the result. □

In the next lemmas, we note that $\mathcal{M}(B,B)$ is the empty set.

Lemma 3.7. *Suppose that K is a normal subgroup of G and $P \in \mathcal{M}(G,B)$. Then either*

(i) $P \in \mathcal{M}(BK,B)$*; or*
(ii) $PK/K \in \mathcal{M}(G/K,BK/K)$ *and* $P \in \mathcal{M}(N_G(S\cap K),B)$.

Proof. Assume that $P \notin \mathcal{M}(BK,B)$. Then $PK/K \in \mathcal{M}(G/K,BK/K)$ by Lemma 3.5. Since $S\cap K \in \mathrm{Syl}_p(P\cap K)$ and $P\cap K$ is a normal subgroup of P, we have $P = N_P(S\cap K)(P\cap K)$ by the Frattini Argument. Thus, because $N_P(S\cap K) \geq B$ and P is p-minimal, we now have $P = N_P(S\cap K)$. Hence $P \in \mathcal{M}(N_G(S\cap K),B)$. □

Lemma 3.8. *Suppose that K is a normal subgroup of G and $G = BKC_G(K)$. Assume that $N_K(S\cap K) = B\cap K$ and $P \in \mathcal{M}(G,B)$. Then $P \in \mathcal{M}(BK,B) \cup \mathcal{M}(BC_G(K),B)$.*

Proof. Since $B\cap K = N_K(S\cap K)$ and $G = BKC_G(K)$, we infer that $N_G(S\cap K) \leq BC_G(K)$. From Lemma 3.7 we have $P \in \mathcal{M}(BK,B)$ or $P \in \mathcal{M}(N_G(S\cap K),B)$. Hence $P \in \mathcal{M}(BK,M) \cup \mathcal{M}(BC_G(K),B)$. □

Lemma 3.9. *Suppose $G = LK$ where L and K are normal subgroups of G with $L\cap K = 1$ and let $P \in \mathcal{M}(G,B)$. Assume that neither K nor L are p-closed. Then either $P\cap K \in \mathcal{M}(K,B\cap K)$ or $P\cap L \in \mathcal{M}(L,B\cap L)$.*

Proof. We have $G = KL = BKC_G(K)$ and, as $S = (S\cap K)(S\cap L)$, $N_K(S\cap K) \leq B$ and so $B\cap K = N_K(S\cap L)$. Furthermore,since $C_G(K) = Z(K)L$ and $Z(K) \leq B$, we have $BC_G(K) = BL$. Hence, using Lemma 3.8, $P \in \mathcal{M}(BK,B) \cup \mathcal{M}(BL,B)$.

If, say, $P \in \mathcal{M}(BL,B)$ and U is the unique maximal subgroup of P containing B, then $U\cap L$ is the unique maximal subgroup of $P\cap L$ containing $B\cap L$. Thus $P\cap L \in \mathcal{M}(L,B\cap L)$. Similarly, if $P \in \mathcal{M}(BK,B)$, we get $P\cap K \in \mathcal{M}(K,B\cap K)$, so proving the lemma. □

Lemma 3.10. *Assume that $H \leq G$ and $G = HZ(G)$. The map $P \mapsto P\cap H$ is a bijection between $\mathcal{M}(G,B)$ and $\mathcal{M}(H,N_H(S\cap H))$.*

Proof. We have that $H/Z(H) \cong G/Z(G)$ and therefore, as $Z(G) \leq N_G(S) = B$, there is a one to one correspondence between the p-minimal subgroups of G and those of H. □

Lemma 3.11. *Suppose that K is a normal subgroup of G and $R = S\cap K$. Assume that $P \in \mathcal{M}(K,N_K(R))$ and PB is a group. If $B\cap K = N_K(R)$, then $PB \in \mathcal{M}(G,B)$.*

Proof. First we observe that

$$B\cap P = B\cap P\cap K = P\cap N_K(R) = N_K(R).$$

Also $PB\cap K = P(B\cap K) = PN_K(R) = P$ and so P is normal in PB. Now suppose that M is a subgroup of PB containing B. Then $M = B(M\cap P)$ and $M\cap P < P$. Since $M\cap P \geq B\cap P = N_K(R)$, we have that $M\cap P \leq U$ where U is the unique maximal subgroup of P containing $N_K(R)$. Since B normalizes both $N_K(R)$ and P and U is the unique maximal subgroup of P containing $N_K(R)$, we get that B normalizes U and $M \leq UB < PB$. Thus UB is the unique maximal subgroup of PB containing B. Hence $PB \in \mathcal{M}(G,B)$ □

Lemma 3.12. *Suppose that $P \in \mathcal{M}(G,B)$, K is a normal subgroup of P which contains $O_p(P)$ and $O_p(P/K) \neq 1$. Then $P = BK$.*

Proof. Suppose that $BK \neq P$. Then K is contained in the unique maximal subgroup U of P containing B. Let W be a Sylow p-subgroup of the preimage of $O_p(P/K) > 1$ which is contained in S. Then $P = N_P(W)WK = N_P(W)K$ by the Frattini Argument. Since $K \leq U$ and B normalizes W, we have $P = N_P(W)$. But then $W \leq O_p(P) \leq K$ and this is a contradiction as $W \not\leq K$. $\square$

Lemma 3.13. *Suppose that $P \in \mathcal{M}(G, B)$, K is a normal subgroup of G. Then either*

(i) *$PK = BK$; or*
(ii) *$PK/K \in \mathcal{M}(G/K, BK/K)$ and one of the following holds:*
  (a) *$P \cap K \leq B$; or*
  (b) *$O_p(P)K/K = O_p(PK/K)$.*

Proof. Assume that (i) does not hold. Then $PK/K \in \mathcal{M}(G/K, BK/K)$ by Lemma 3.5. Set $M = (P \cap K)O_p(P)$. Since $O_p(P) \leq B$, we may suppose that $M \not\leq B$, else (ii)(a) holds. Since $BM \neq P$ and $O_p(P) \leq M$ with M normal in P, Lemma 3.12 implies that $O_p(P/M) = 1$. Since $P/M \cong PK/MK$ by the correspondence theorem, we have $O_p(PK/MK) = 1$. Thus

$$O_p(PK/K) \leq MK/K = O_p(P)K/K = O_p(PK/K)$$

and we have option (ii)(b). $\square$

Definition 3.14. *Let G be a group and $P \in \mathcal{M}(G, B)$. Then P is a* tame *p-minimal subgroup of G provided that for all automorphisms α of G, $P^\alpha \in \mathcal{M}(G, B)$ implies $P^\alpha = P$. We say that G is tame provided all the members of $\mathcal{M}(G, B)$ are tame.*

The next lemma highlights our interest and is the key property of tame p-minimal subgroups of G.

Lemma 3.15. *Suppose that K is a normal subgroup of G, $G = BK$, $R = S \cap K$ and $N_K(R) = B \cap K$. If K is tame, then the map $P \mapsto P \cap K$ is a bijection between $\mathcal{M}(G, B)$ and $\mathcal{M}(K, N_K(R))$.*

Proof. Let $P \in \mathcal{M}(G, B)$. Then $P \cap K \geq B \cap K = N_K(R)$. We claim that $P \cap K$ is not p-closed. For if it were, we get $P \cap K = B \cap K$, whence

$$P = P \cap G = P \cap BK = B(P \cap K) = B,$$

which is a contradiction. Hence, by Lemma 3.2, $P \cap K = \langle Q \mid Q \in \mathcal{M}(P \cap K, N_K(R))\rangle$. Since K is tame, B normalizes each $Q \in \mathcal{M}(P \cap K, N_K(R))$ and hence $BQ \leq P$. Since $P \in \mathcal{M}(G, B)$ and $P = B(P \cap K)$, we get $P \cap K \in \mathcal{M}(K, N_K(R))$. Thus the map $P \mapsto P \cap K$ is a well defined injective map from $\mathcal{M}(G, B)$ to $\mathcal{M}(K, N_K(R))$. Similarly, for $Q \in \mathcal{M}(K, N_K(R))$, we have $QB \leq G$ is a group and so Lemma 3.11 implies that $QB \in \mathcal{M}(G, B)$ and $Q = Q(B \cap K)$. $\square$

We now make some remarks concerning central products and projection maps. Suppose that $K_1, \dots, K_n$ are groups. Then a *central product* of $K_1, \dots, K_n$ is the image of $K_1 \times \dots \times K_n$ by a homomorphism with a central kernel. If $X = K_1 \dots K_n$ is a central product by a homomorphism θ, then the projection of X to K_1 is the composition of the standard projection of $\bar{X} = K_1 \times \dots \times K_n$ to K_1 considered as a homomorphism from $\bar{X}$ to $\bar{X}$ with θ.

Lemma 3.16. *Suppose that G is a group and K is a normal subgroup of G such that $G = KS$. Assume that $K = K_1K_2 \dots K_n$ is a central product and S acts transitively on the set $\{K_1, \dots, K_n\}$ by conjugation. Let π_1 be the projection map from K to K_1. If $Y \in \mathcal{M}(K_1, N_{K_1}(S \cap K_1))$ is tame, then $\pi_1(\langle Y^B \rangle) = Y$.*

Proof. Let $g \in B = S(B \cap K)$. Then $Y^g \le K_1^g = K_j$ for some j and $Y^g \ge N_{K_j}(S \cap K_j)$. If $j \ne 1$, then $\pi_1(Y^g) \le N_{K_1}(S \cap K_1) \le Y$. If $Y^g \le K_1$, then g normalizes K_1 and, as Y is tame, $Y^g = Y$. Hence, as π_1 is a homomorphism from K to K_1, $\pi_1(\langle Y^B \rangle) = Y$. □

The next lemma is fundamentally important when we consider p-minimal subgroups of wreath products.

Lemma 3.17. *Suppose that G is a group and K is a normal subgroup of G such that $G = KS$. Assume, additionally, that $K = K_1K_2 \dots K_n$ is a central product and S acts transitively on the set $\{K_1, \dots, K_n\}$ by conjugation. Let π_1 be the projection map from K to K_1 and assume that*

(a) *$\pi_1(N_K(S)) = N_{K_1}(S \cap K_1)$; and*
(b) *K_1 is tame.*

Then we have the following.

(i) *Let $P \in \mathcal{M}(G, B)$, and set $L = \pi_1(P \cap K)$. Then either $P \in \mathcal{M}(N_G(S \cap K), B)$ or $P = \langle O^{p'}(L)^B \rangle B$ and $L \in \mathcal{M}(K_1, N_{K_1}(S \cap K_1))$.*
(ii) *If $L \in \mathcal{M}(K_1, N_{K_1}(S \cap K_1))$ and $P = \langle O^{p'}(L)^B \rangle B$, then $P \in \mathcal{M}(G, B)$ and $\pi_1(P \cap K) = L$.*

In particular, there is a bijection between the sets

$$\mathcal{M}(G, B) \setminus \mathcal{M}(N_G(S \cap K), B) \text{ and } \mathcal{M}(K_1, N_{K_1}(S \cap K_1)).$$

Proof. Suppose first that $P \in \mathcal{M}(G, B)$ and set $P_0 = P \cap K$. Then $P_0 \ge B \cap K = N_K(S)$. Hence, by assumption (a), $\pi_1(P_0) \ge N_{K_1}(S \cap K_1)$. Set $R_1 = \langle (S \cap K_1)^{\pi_1(P_0)} \rangle$ and $R = \langle R_1^B \rangle (S \cap K)$. Then, as K is a central product of $K_1, \dots, K_n$, $R_1 = \langle (S \cap K_1)^{P_0} \rangle$ is normal in P_0 and so $R \le P_0$. Since R is normal in P_0, the Frattini Argument delivers $P_0 = RN_{P_0}(S \cap K)$ and so $P = P_0S = RN_{P_0}(S \cap K)S$.

Since RB and $N_{P_0}(S \cap K)B$ are both subgroups of P containing B and $P \in \mathcal{M}(G, B)$, either $P = RB$ or $P = N_{P_0}(S \cap K)B \le N_G(S \cap K)$. In the latter case we have $P \in \mathcal{M}(N_G(S \cap K), B)$, so we now show that in the former case we have $L = \pi_1(P_0) \in \mathcal{M}(K_1, N_{K_1}(S \cap K_1))$. Note that $L \ge R_1N_{K_1}(S \cap K_1)$ and so, as R_1 is normal in L, the Frattini Argument implies $L = R_1N_{K_1}(S \cap K_1)$. Let $Y \in \mathcal{M}(L, N_L(S \cap K_1))$ with $Y \ne L$ and set $Q = \langle (S \cap K_1)^Y \rangle = O^{p'}(Y)$. Note that, as $Y \le L = \pi_1(P_0)$ and $S \cap K_1 \le P_0$, we have $Q \le P_0$. Because $Y \in \mathcal{M}(K_1, N_{K_1}(S \cap K_1))$ is tame, Lemma 3.16 implies that $\pi_1(\langle Y^B \rangle) = Y \le L$. It follows that $\langle Q^B \rangle N_K(S) < P_0$. In particular, $\langle Q^B \rangle B$ is contained in the unique maximal subgroup of P. Hence, if $L \notin \mathcal{M}(L, N_L(S \cap K_1))$, $\langle O^{p'}(Y) \mid Y \in \mathcal{M}(L, N_L(S \cap K_1)) \rangle B < P$, but this contradicts $L = \langle O^{p'}(Y) \mid Y \in \mathcal{M}(L, N_L(S \cap K_1)) \rangle \pi_1(N_K(S))$ and $\langle O^{p'}(L), B \rangle = P$. Hence (i) holds.

Now assume that $P = RB$ where $R = \langle O^{p'}(L)^B \rangle$ and $L \in \mathcal{M}(K_1, N_{K_1}(S \cap K_1))$. We have that $P_0 = P \cap K = RN_K(S)$ and as K_1 is tame Lemma 3.16 gives $\pi_1(P_0) = L$. Let U be the unique maximal subgroup of L which contains $N_L(S \cap K_1)$. Assume that $Y \in \mathcal{M}(P, B)$. Then by (i) either $Y = N_Y(S \cap K)$ or $Y = \langle O^{p'}(\pi_1(Y \cap K))^B \rangle B$. In the former case $\pi_1(Y \cap K) = N_{K_1}(S \cap K) \le U$. So suppose the second possibility arises. Then $\pi_1(Y\cap K)\le \pi_1(P_0)=L$. If we have equality, then $O^{p'}(\pi_1(Y\cap K))=O^{p'}(L)$ and so $Y=P$ which means that $P\in\mathcal{M}(G,B)$. So we should assume, using (a), that $\pi_1(Y\cap K)\le U$. Then, for all $Y\in\mathcal{M}(P,B)$, we have $\pi_1(Y\cap K)\le U$. However, $P=\langle Y\mid Y\in\mathcal{M}(P,B)\rangle=\langle S(Y\cap K)\mid Y\in\mathcal{M}(P,B)\rangle$ and $P_0=(S\cap K)\langle Y\cap K\mid Y\in\mathcal{M}(P,B)\rangle=\langle Y\cap K\mid Y\in\mathcal{M}(P,B)\rangle$. Since π_1 is a homomorphism we now have that $\pi_1(P_0)\le U<L=\pi_1(P_0)$ which is absurd. $\square$

We finish this section with a technical lemma. Note that in its statement we are assuming $p=2$.

Lemma 3.18. *Suppose that H is a normal subgroup of G, $R=S\cap H\in\mathrm{Syl}_2(H)$, $P\le H$ and $P\ge N_H(R)$. Assume in addition that*

(i) *$J=J_1\times J_2$ is a normal subgroup of G and G permute $\{J_1,J_2\}$ transitively by conjugation;*
(ii) *$R\cap J=N_J(R\cap J)$;*
(iii) *$S=C_S(J_1)R$; and*
(iv) *$P=N_H(R)(P\cap J)$.*

Then S normalizes P.

Proof. Set $Y=C_S(J_1)$ and $Q_1=\langle (R\cap J_1)^{P\cap J}\rangle$. Then $Q_1\le P\cap J_1$ and is normalized by $\langle P\cap J, N_{N_H(R)}(J_1), Y\rangle$.

Since $N_H(R)$ normalizes $P\cap J$ and $P\cap J\le N_G(Q_1)$, the subgroup $Q=\langle Q_1^{N_H(R)}\rangle$ is normalized by $N_H(R)(P\cap J)$ which by (v) is equal to P. Note that (v) together with (ii) also implies that

$$\begin{aligned} N_P(R\cap J) &= N_P(R\cap J)\cap N_H(R)(P\cap J)=N_H(R)N_{P\cap J}(R\cap J)\\ &= N_H(R)(R\cap J)=N_H(R).\end{aligned}$$

Since $R\cap J_1\le Q_1$ and, by (i), $R\cap J=(R\cap J_1)(R\cap J_2)\le Q\le J\cap P$, we have that $R\cap J\in\mathrm{Syl}_2(Q)$. Thus the Frattini Argument gives

$$P=N_P(R\cap J)Q=N_H(R)Q.$$

Furthermore, we have Y normalizes Q_1 and $N_H(R)$ (as S normalizes $N_H(R)$) and so Y normalizes Q. As $S=YR$ by (iii), we now have S normalizes Q. Hence S normalizes $P=N_H(R)Q$. $\square$

4. 2-minimal subgroups in monomial groups

Recall that T_m is a Sylow 2-subgroup of $\mathrm{Sym}(2^m)$ as described in Section 2. Also the definition of H-minimal groups is given in Definition 3.3.

Lemma 4.1. *Let s be an odd prime and b and m be positive integers. Suppose that $U=\langle u_1,\dots,u_{2^m}\rangle$ is a homocyclic group of rank 2^m and exponent s^b. Let $T=T_m\in\mathrm{Syl}_2(\mathrm{Sym}(2^m))$ permute the set $\{u_1,\dots,u_{2^m}\}$ of generators of U naturally and thereby realize T as a subgroup of $\mathrm{Aut}(U)$. For $0\le j\le m$, define*

$$U_j=U_j(s^b)=\langle (\sum_{i=1}^{2^{m-j}} u_i-u_{2^{m-j}+i})^T\rangle$$

where, by convention, all elements u_k with $k>2^m$ are ignored. Then

(i) *$U_0=C_U(T)$ is cyclic of order s^b and, for $1\le j\le m$, U_j is homocyclic of rank 2^{j-1} and exponent s^b;*

(ii) $U = \oplus_{j=0}^{m} U_j$;

(iii) *the centralizer in T of U_j is the base group of T when T is viewed as the wreath product $T_j \wr T_{m-j}$; and*

(iv) *the set $\{U_j(s^c) = s^{b-c}U_j \mid 0 \leq j \leq m, 1 \leq c \leq b\}$ comprises all the T-minimal subgroups of U.*

Proof. We prove the result by induction on m noting that it is easy to check for $m = 1$. So we now assume that $m > 1$. Let

$$R = \langle (1,2), (3,4), \ldots, (2^m - 1, 2^m) \rangle$$

be the base group of T. Then $[U, R] = \langle u_1 - u_2, \ldots, u_{2^{m-1}} - u_{2^m} \rangle$ and $C_U(R) = \langle u_1 + u_2, \ldots, u_{2^{m-1}} + u_{2^m} \rangle$. Thus $U_m = [U, R]$ and $U = C_U(R) \oplus U_m$. Furthermore $C_U(R)$ is an abelian group of exponent s^b and rank 2^{m-1} which admits T/R as a group of automorphisms permuting its generating set exactly as a Sylow 2-subgroup of $\mathrm{Sym}(2^{m-1})$ does. By induction we obtain $C_U(R) = \oplus_{j=0}^{m-1} U_j$. Thus $U = \oplus_{j=0}^{m} U_j$. Since any minimal T-invariant subgroup of U is contained in either $C_U(R)$ or $[U, R] = U_m$, it remains, again by induction, to show that U_m is a minimal T-invariant subgroup of U of exponent s^b. Suppose that $0 \neq W < U_m$ and that W is T-invariant and of exponent s^b. Then W is homocyclic and $[W, (1,2)] \leq \langle u_1 - u_2 \rangle$. If $[W, (1,2)] \leq s\langle u_1 - u_2 \rangle$, then, as T acts transitively on the given generators of R, we have $[W, R] \leq sU_m$. But then W/sW is centralized by R and consequently $W \leq C_U(R) \cap [U, R] = 0$, which against our assumption. Therefore $[W, (1,2)] = \langle u_1 - u_2 \rangle$ and the action T delivers $W = U_m$. If W has exponent s^c with $c < b$, then $W \leq sU$ and the final statement now follows by an induction on b. $\square$

To clear the air, notationally speaking, we consider the following example.

Example 4.2. *Suppose that $2^m = 16$ and $s^b = 9$. Then the non-zero T_4-minimal subgroups of U are as follows:*

(i) $U_0 = U_0(3^2) = \langle u_1 + \cdots + u_{16} \rangle$ *of rank 1 and order* 9 *and* $3U_0 = U_0(3^1)$ *of order* 3*;*

(ii) $U_1 = U_1(3^2) = \langle u_1 + \cdots + u_8 - (u_9 + \cdots + u_{16}) \rangle$ *of rank 1 and order* 9 *and* $3U_1 = U_1(3^1)$ *of order* 3*;*

(iii) $U_2 = U_2(3^2) = \langle u_1 + \cdots + u_4 - (u_5 + \cdots + u_8), u_9 + \cdots + u_{12} - (u_{13} + \cdots + u_{16}) \rangle$ *of rank 2 and of order* 9^2 *and* $3U_2 = U_2(3^1)$ *of order* 3^2*;*

(iv) $U_3 = U_3(3^2) = \langle u_1 + u_2 - (u_3 + u_4), \ldots, u_{13} + u_{14} - (u_{15} + u_{16}) \rangle$ *of rank 4 and order* 9^4 *and* $3U_3 = U_3(3^1)$ *order* 3^4*; and*

(v) $U_4 = U_4(3^2) = \langle u_1 - u_2, \ldots, u_{15} - u_{16} \rangle$ *of rank 8 and order* 9^8 *and* $3U_4 = U_4(3^1)$ *of order* 3^8*.*

Our next lemma is similar to the preceding one.

Let $j \in \{1, \ldots, m-1\}$. The subgroup of $\mathrm{Sym}(2^m)$ denoted by $X_{2^m}(1; j)$ (note that $r = 1$ and $n_1 = m$ here) in Section 2 has shape $T_{j-1} \wr \mathrm{Sym}(4) \wr T_{m-j-1}$. Set $Y_{m,j} = X_{2^m}(1; j)$. Let $F_{m,j}$ be the base group of $Y_{m,j}$ where we think of $Y_{m,j}$ as the wreath product

$$X_{2^{j+1}}(1; j) \wr T_{m-j-1} = Y_{j+1,j} \wr T_{m-j-1}.$$

So $F_{m,j}$ is a direct product of 2^{m-j-1} copies of $Y_{j+1,j} (= T_{j-1} \wr \mathrm{Sym}(4))$. The set-up just described will be assumed in Lemmas 4.3, 4.4 and 4.5.

Lemma 4.3. *Let s be an odd prime and b and $m \geq 2$ be positive integers. Suppose that $U = \langle u_1, \dots, u_{2^m}\rangle$ is a homocyclic group of rank 2^m and exponent s^b. Let group $Y_{m,j}$ permute the set $\{u_1, \dots, u_{2^m}\}$ of generators of U naturally and thereby realizes $Y_{m,j}$ as a subgroup of* $\mathrm{Aut}(U)$.

For $0 \leq j \leq m$, set

$$U_j = U_j(s^b) = \langle (\sum_{i=1}^{2^{m-j}} u_i - u_{2^{m-j}+i})^{T_m}\rangle.$$

Then the following hold.

(i) *For $1 \leq j \leq m-1$, $U = C_U(F_{m,j}) \oplus [U, F_{m,j}]$ is a $Y_{m,j}$-invariant decomposition of U.*

(ii) *$C_U(F_{m,j}) = \bigoplus_{k=0}^{m-j-1} U_k$ and $[U, F_{m,j}] = W \oplus \bigoplus_{k=m-j+2}^{m} U_k$ where $W = U_{m-j} \oplus U_{m-j+1}$ are decompositions of $C_U(F_{m,j})$ and $[U, F_{m,j}]$ into $Y_{m,j}$-minimal subgroups of exponent s^b.*

Proof. We prove the result by induction on j. Assume that $j = 1$. So $F_{m,1}$ is a direct product of groups isomorphic to Sym(4). Then $C_U(F_{m,1}) = \langle (u_1+u_2+u_3+u_4)^{Y_{m,1}}\rangle$ which has rank 2^{m-2} and $[U, F_{m,1}] = \langle \{u_1-u_2, u_2-u_3, u_1-u_4\}^{Y_{m,1}}\rangle$ which has rank $2^{m-2}+2^{m-1}$. Thus, as $2^{m-2}+2^{m-2}+2^{m-1} = 2^m$ and $C_U(F_{m,1}) \cap [U, F_{m,1}] = 0$, $U = C_U(F_{m,1}) \oplus [U, F_{m,1}]$ and this is a $Y_{m,1}$-invariant decomposition. We may identify $C_U(F_{m,1})$ with the natural permutation module for $Y_{j,1}/F_{j,1} \cong T_{m-2}$ and thus by applying Lemma 4.1 and making the appropriate identifications we have $C_U(F_{m,1}) = \bigoplus_{k=0}^{m-2} U_k$. Applying Lemma 4.1 again this time for T_m, we see that $[U, F_{m,1}] = U_{m-1} \oplus U_m$ and as U_m is not $Y_{j,1}$-invariant we deduce that $W = [U, F_{m,1}]$ is a minimal $Y_{m,1}$-invariant subgroup of exponent s^b. This proves the lemma for $j = 1$.

Now assume that $j > 1$ and let $S_0 = \langle (1,2)^{Y_{m,j}}\rangle$. Then S_0 is elementary abelian of order $2^{2^{m-1}}$ and $Y_{m,j}/S_0 \cong Y_{m-1,j-1}$. Since U has odd order, we have $U = C_U(S_0) \oplus [U, S_0]$ is a $Y_{m,j}$-invariant decomposition of U and we observe that $[U, S_0] = U_m$ is irreducible as a $Y_{m,j}$-module as its restriction to T_m is already irreducible by Lemma 4.1. So $U = C_U(S_0) \oplus U_m$. Since $C_U(S_0) = \langle (u_1 - u_2)^{Y_{m,j}}\rangle$ we may identify $C_U(S_0)$ with the natural $Y_{m,j}/S_0 \cong Y_{m-1,j-1}$-module. By induction we then have $C_U(S_0) = C_{C_U(S_0)}(F_{m-1,j-1}) \oplus [C_U(S_0), F_{m-1,j-1}]$ and we can write $C_{C_U(S_0)}(F_{m-1,j-1}) = \bigoplus_{k=0}^{m-j-1} U_k$ and $[C_U(S_0), F_{m-1,j-1}] = W \oplus \bigoplus_{k=m-j+2}^{m} U_k$. Thus we have decomposed U as a direct sum of irreducible modules as described in the lemma. We complete the lemma by noting that $C_{C_U(S_0)}(F_{m-1,j-1}) = C_U(F_{m,j})$ and that $[U, F_{m,j}] = [C_U(S_0), F_{m-1,j-1}] + U_m$. □

We further embellish Example 4.2 to illustrate the phenomena in Lemma 4.3.

Example 4.4. *We again take $2^m = 2^4$ and $s^b = 9$. See Example 4.2 for an explicit description of U_0, U_1, U_2, U_3 and U_4. Then*

$$X_{2^4}(1;1) = Y_{4,1} = \mathrm{Sym}(4) \wr T_2$$

with $C_U(F_{4,1}) = U_0 \oplus U_1 \oplus U_2$ and $[U, F_{4,1}] = U_3 \oplus U_4$. Further the $Y_{4,1}$-minimal subgroups of U are U_0, $3U_0$, U_1, $3U_1$, U_2 and $3U_2$, which are (all centralized by the base group of $F_{4,1}$) together with $U_3 \oplus U_4$ and $3(U_3 \oplus U_4)$ which both admit $F_{4,1}$ faithfully. For $X_{2^4}(1;2) = Y_{4,2} = T_1 \wr \mathrm{Sym}(4) \wr T_1$, $C_U(F_{4,2}) = U_0 \oplus U_1$ and

$[U, F_{4,2}] = W \oplus U_4$ with $W = U_2 \oplus U_3$ (so the $Y_{4,2}$-minimal subgroups of U are U_0, $3U_0$, U_1, $3U_1$, $U_2 \oplus U_3$ and $3(U_2 \oplus U_3)$, U_4 and $3U_4$).

Similarly for $X_{2^4}(1;3) = Y_{4,3} = T_2 \wr \mathrm{Sym}(4)$, we get the $Y_{4,3}$ minimal subgroups are U_0 and $3U_0$, $(U_1 \oplus U_2)$, $3(U_1 \oplus U_2)$, U_3, $3U_3$, U_4 and $3U_4$.

Lemma 4.5. *Suppose that $P = Y_{m,j}$, and set $C = O_{2,2'}(P)/O_2(P)$. Then $P/O_{2,2'}(P) \cong T_{m-j}$, C is a composition factor of P and as a T_{m-j}-module over $\mathrm{GF}(3)$, C is isomorphic to $U_{m-j}(3^1)$.*

Proof. We have $P/O_{2,2'}(P) \cong T_1 \wr T_{m-j-1}$ and so $P/O_{2,2'}(P) \cong T_{m-j}$. Since $P/O_2(P) \cong \mathrm{Sym}(3) \wr T_{m-j-1}$, we see that the composition factor C is a faithful T_{m-j}-module. Furthermore we may view T_{m-j} acting on the set of 3-cycles in $\mathrm{Sym}(3) \wr T_{m-j-1}$ which is a set of size 2^{m-j} and we see that the stabilizer of a point in this action has index 2^{m-j} and corresponds to the centralizer of a 3-cycle. From the universal property of permutation modules it follows that the chief factor C is isomorphic to a quotient of the GF(3) permutation module of T_{m-j}. Since C is faithful it follows from Lemma 4.1 that C is isomorphic to $U_{m-j}(3^1)$, as claimed. $\square$

Lemma 4.6. *Let s be an odd prime and b, m and n be positive integers. Suppose that $W = \langle w_{i,j} \mid 1 \leq i \leq 2^m, 1 \leq j \leq n\rangle$ is a homocyclic group of rank $2^m n$ and exponent s^b. Assume that $T = T_m \in \mathrm{Syl}_2(\mathrm{Sym}(2^m))$, set $H = T \wr \mathrm{Sym}(n)$ and let H permute the set $\{w_{i,j} \mid 1 \leq i \leq 2^m, 1 \leq j \leq n\}$ of generators of W naturally and thereby realize H as a subgroup of $\mathrm{Aut}(W)$. For $0 \leq j \leq m$, define*

$$W_j = \langle (\sum_{i=1}^{2^{m-j}} w_{i,1} - w_{2^{m-j}+i,1})^H \rangle$$

where, by convention, all elements w_k with $k > 2^m$ are ignored. Then

- (i) *$W_0 = C_W(T^H)$ has order s^{bn} is the natural permutation module for $H/\langle T^H\rangle \cong \mathrm{Sym}(n)$, and for $1 \leq j \leq m$, W_j is a homocyclic group of rank $2^{j-1}n$ and exponent s^b;*
- (ii) *$W = \oplus_{j=0}^m W_j$;*
- (iii) *the centralizer in H of W_j is the base group of H when H is viewed as the wreath product $T_j \wr (T_{m-j} \wr \mathrm{Sym}(n))$; and*
- (iv) *for $1 \leq j \leq m$, the homocyclic subgroups W_j comprise the minimal H-invariant subgroups of $W_1 \oplus \cdots \oplus W_m$ of exponent s^b.*

Proof. Let F denote the base group of H. We have $F \cong T \times \cdots \times T$ with exactly n factors. For $0 \leq j \leq m$ and $1 \leq k \leq n$, set $W_{j,k} = \langle (\sum_{i=1}^{2^{m-j}} w_{i,k} - w_{2^{m-j}+i,k})^F \rangle$ and note that as a module for the kth direct factor of F, $W_{j,k}$ is isomorphic to $U_j = U_j(s^b)$ as defined in Lemma 4.1. Furthermore, $W_j = \bigoplus_{k=1}^n W_{j,k}$. This together with Lemma 4.1 (i) provides the exponent and rank of the homocyclic groups W_j.

Since F centralizes W_0, W_0 is naturally isomorphic to the permutation module for $H/F \cong \mathrm{Sym}(n)$. This completes the proof of (i).

Part (ii) is transparent from the definition of the subgroups W_j, $0 \leq j \leq m$.

Suppose now that $j > 0$ and let W^* be a non-zero H-invariant subgroup of $W_1 \oplus \cdots \oplus W_m$ of exponent s^b. Since $j \neq 0$ we have $C_F(W_{j,k}) \neq C_F(W_{l,k})$ for $j \neq l$ and as a consequence the homocylic subgroups $W_{j,k}$ are pairwise non-isomorphic as F-modules. Therefore the set $\{W_{j,k} \mid 1 \leq k \leq n\}$ is the set of minimal F-invariant submodules of W_j. In particular, as W^* is F-invariant there exists an l such that $W_{j,l} \leq W^*$. But then W^* contains W_j and we have that $\{W_j \mid 1 \leq j \leq m\}$ is the set of all minimal H-invariant subgroups of exponent s^b contained in $W_1 \oplus \cdots \oplus W_m$. $\square$

As promised in the introduction we now give explicit descriptions of the toral, linking and fuser 2-minimal subgroups. We begin with the toral ones. We take $H = E \wr \mathrm{Sym}(n)$ where E is a finite cyclic group of odd order, F is the base group of H and X is a complement to F in H containing a fixed Sylow 2-subgroup T of H. We have $F = \langle e_1, \dots, e_n \rangle$ where X permutes the generators of F naturally. As usual, we write $n = 2^{n_1} + \cdots + 2^{n_r}$ and accordingly decompose T as $T_{n_1} \times \cdots \times T_{n_r}$ (see Section 2). Corresponding to this decomposition of n, there is an associated decomposition of F namely $F = F_1 \times \cdots \times F_r$ where the generators of F_i, say, are $e_{2^{i-1}+1}, \dots, e_{2^i}$. For $i \in I$, we set $Z_{n_i} = C_{F_i}(T_{n_i})$ and then we have $N_H(T) = \prod_{i \in I} Z_{n_i} T_{n_i}$. Set $\Pi = \Pi(|E|)$. So Π is the set of all prime powers greater than one dividing $|E|$ and hence of $|F_i|$ for each $i \in I$. Each F_i is a direct product of Sylow s-subgroups S_i for primes $s \in \Pi$. These Sylow s-subgroups are homocylic and admit T_{n_i} naturally as in Lemma 4.1. Every $N_H(T)$-minimal subgroup of F is contained in some S_i for appropriate choices of $i \in I$ and prime $s \in \Pi$. Using Lemma 4.1 we see that each such $N_H(T)$-minimal subgroup is of the form $U_j(s^c)$ for some $1 \leq j \leq n_i$ and $s^c \in \Pi$. We now denote these $N_H(T)$-minimal subgroups by $U(i; s^c; j)$. Define $T(n_i; s^c; j) = U(n_i; s^c; j) N_H(T)$. Notice that $U(n_i; s^c; 0) \leq Z_{n_i}$ for each $s^c \in \Pi$. Furthermore, $T(n_i; s^c; j)$ is a 2-minimal subgroup of H by Lemma 3.4.

For $i \in I$ and for $j \in \{1, \dots, n_i - 1\}$ we set

$$P(n_i; n_j) = X_n(n_i; n_j) C_F(T).$$

And for $i, j \in I$ with $i < j$, set

$$P(n_i + n_j) = X_n(n_i + n_j) \langle C_F(T)^{X_n(n_i+n_j)} \rangle.$$

So $P(n_i; n_j)$ and $P(n_i + n_j)$ are subgroups of H which contain $N_H(T)$.

Definition 4.7. *Suppose that E is a cyclic group of odd order and $H = E \wr X$ where $X \cong \mathrm{Sym}(n)$. We employ the notation already developed for H.*

(i) $\mathcal{T}(H, N_H(T)) = \{T(n_i; s^c; j) \mid i \in I, s^c \in \Pi$ *and* $1 \leq j \leq n_i\}$;
(ii) $\mathcal{L}(H, N_H(T)) = \{P(n_i; n_j) \mid i \in I, j \in \{1, \dots, n_i - 1\}\}$;
(iii) $\mathcal{F}(H, N_H(T)) = \{P(n_i + n_j) \mid i, j \in I, i < j\}$.

For future use we observe the following lemma.

Lemma 4.8. (i) $|\mathcal{T}(H, N_H(T))| = |\Pi| \sum_{i \in I} n_i$.
(ii) $|\mathcal{L}(H, N_H(T))| = (\sum_{i \in I} n_i) - r$
(iii) $|\mathcal{F}(H, N_H(T))| = \binom{r}{2}$.

$\square$

The subgroups in Definition 4.7 (i), (ii) and (iii) are, respectively, the 2-minimal *toral, linkers* and *fusers* of H. We have already observed that the $T(n_i; s^c; j)$ are 2-minimal subgroups and it is transparent that the linkers are also 2-minimal subgroups of H. The structure of the subgroups in $\mathcal{F}(H, N_H(T))$ is the subject of our next lemma.

Lemma 4.9. *Suppose that $P = P(n_i + n_j) \in \mathcal{F}(H, N_H(T))$. Then $P \in \mathcal{M}(H, N_H(T))$. Additionally, we have the following.*

(i) $X_n(n_i+n_j)/O_2(X_n(n_i+n_j)) \cong \mathrm{Sym}(2^{n_i-n_j}+1)$ *and in its action on* $\{e_k \mid k \in \Omega_i \cup \Omega_j\}$ *has* 2^{n_j} *orbits each of which is natural for* $\mathrm{Sym}(2^{n_i-n_j}+1)$ *and* $\{e_k \mid k \in \Omega_j\}$ *a maximal block of imprimitivity.*
(ii) $P \cap F = \langle (\prod_{k\in\Omega_j} e_k)^{X_n(n_i+n_j)}\rangle$ *is homocyclic of order* $|E|^{2^{n_i-n_j}+1}$.
(iii) $P/O_2(P) \cong E \wr \mathrm{Sym}(2^{n_1-n_2}+1)$.

Proof. Recall that $P(n_i+n_j) = X_n(n_i+n_j)\langle C_F(T)^{X_n(n_i+n_j)}\rangle$. Set $X^* = X_n(n_i+n_j)$. Then $P = X^*\langle C_F(T)^{X^*}\rangle$. By Lemma 3.6 there exists a 2-minimal subgroup R of P containing $N_H(T)$ such that $RF = PF$. Then

$$R \geq \langle C_F(T)^R\rangle = \langle C_F(T)^P\rangle = P \cap F,$$

whence $P = R$.

From the description of $X_n(n_i+n_j)$ given in Section 2 we have $X^*/O_2(X^*) \cong \mathrm{Sym}(2^{n_i-n_j}+1)$ and in its action on $\{e_k \mid k \in \Omega_i \cup \Omega_j\}$ has 2^{n_j} orbits each of which is natural for $\mathrm{Sym}(2^{n_i-n_j}+1)$ and $\{e_k \mid k \in \Omega_j\}$ a maximal block of imprimitivity. This is the statement in (i).Parts (ii) and (iii) are easy consequences of (i). □

Lemma 4.10. *If* $P \in \mathcal{M}(H, N_H(T))$*, then one of the following holds:*
(i) $P \in \mathcal{M}(TF, N_H(T))$;
(ii) $P \in \mathcal{L}(H, N_H(T)) \cup \mathcal{F}(H, N_H(T))$.

Proof. If $P \leq TF$, then P does indeed belong to $\mathcal{M}(TF, N_H(T))$, so we may as well assume that $P \not\leq TF$. Then $PF/F \in \mathcal{M}(H/F, N_H(T)F/F)$ by Lemma 3.5. Let $X^* \in \mathcal{M}(X,T)$ be such that $X^*F = PF$. Then, as F is abelian, $P\cap F$ is normalized by X^*. Assume that $X^* \in \mathcal{L}(X,T)$. Then, by Theorem 2.2 and Lemma 4.5, P and $X^*(P\cap F)$ are conjugate in PF. Because both $B = TC_F(T)$ and $C_F(T) \leq P \cap F$, we have $P = X^*(P \cap F)$. Since P is 2-minimal, we get that $P = X^*C_F(T) \in \mathcal{L}(H, N_H(T))$. So (ii) holds in this case.

Suppose now that $X^* \in \mathcal{F}(X,T)$ and let $R = O_2(X^*)$ (we may have $R = 1$). Set $J = \langle C_F(T)^{X^*}\rangle$. Then, by Lemma 4.6 (i), $J = C_F(R)$. Since $P \cap F$ is normal in X^*F, $P \cap F \geq J$. Because $R \leq P$, we have that $(P \cap F)R = P \cap FR$ is normalized by P. Therefore $P = N_P(R)(P \cap F)$. Because $P \in \mathcal{M}(H, N_H(T))$ and $P \not\leq BF$, we get $N_P(R) = P$. Since $P \leq X^*F$ and $N_{X^*F}(R) = X^*J$, we now have $P \leq X^*J$ and by comparing the orders of these group we get $P = X^*J \in \mathcal{F}(H, N_H(T))$. This completes the proof of the lemma. □

Lemma 4.11. *If* $P \in \mathcal{M}(TF, N_H(T))$*, then* $P \in \mathcal{T}(H, N_H(T))$.

Proof. Since F is abelian and of odd order, we may apply Lemma 3.4 to see that $P = N_H(T)L$ where $L = [P \cap F, T]$ is a $N_H(T)$-minimal s-group for some prime s. It follows that $P \leq RN_H(T)$ where R is a Sylow s-subgroup of F. Since $N_H(T)\cap F$ centralizes R, we have $R = \langle x_1, \dots, x_n\rangle$ admits $T \in \mathrm{Syl}_2(\mathrm{Sym}(n))$ permuting the generators naturally. Therefore R can be decomposed as a product $R_{n_1} \dots R_{n_r}$ of T-invariant subgroups with R_{n_i} of rank 2^{n_i} which may then each be regarded as T_{n_i}-invariant homocylic subgroups. Since L is T-minimal, we infer that $L \leq R_{n_i}$ for some $i \in I$. By Lemma 4.1 we now have $P = U(n_i; s^c; j)N_H(T) = T(n_i; s^c; j) \in \mathcal{T}(H, N_H(T))$ for some c, as claimed. □

Theorem 4.12. *Suppose that* $H = E\wr\mathrm{Sym}(n)$ *where* $n \geq 2$ *and* E *is a cyclic group of od order. Then*

$$\mathcal{M}(H, N_H(T)) = \mathcal{T}(H, N_H(T)) \cup \mathcal{F}(H, N_H(T)) \cup \mathcal{L}(H, N_H(T)).$$

Proof. Combining Lemmas 4.10 and 4.11 we have

$$\mathcal{M}(H, N_H(T)) \subseteq \mathcal{T}(H, N_H(T)) \cup \mathcal{F}(H, N_H(T)) \cup \mathcal{L}(H, N_H(T)).$$

Since the members of the righthand side of this containment are 2-minimal subgroups of H, we have the result. $\square$

We close this section by presenting a modest example of the 2-minimal subgroups of $H = E \wr X$ where E has order $3^2 5$ and $X \cong \mathrm{Sym}(12)$.

Example 4.13. *We have $n = 2^3 + 2^2$ so $n_1 = 3$, $n_2 = 2$ and $I = \{1, 2\}$. Also $\Pi = \Pi(|E|) = \{3, 3^2, 5\}$. Structurally, we have $T = T_3 \times T_2$ with*

$$N_H(T) = Z_3 Z_2 T = Z_3 T_3 \times Z_2 T_2$$

and $C_F(T) = Z_2 Z_3$ homocylic of rank 2 and order $3^4 5^2$.

The 2-minimal linkers of H are the groups $P(n_i; n_j) = X_{12}(n_i; n_j) Z_2 Z_3$ where $i \in I$, $j \in \{1, \ldots, n_i - 1\}$. Thus we have

$P(1, 1) = Z_2 Z_3 \times (\mathrm{Sym}(4) \wr 2 \times T_2)$;
$P(1, 2) = Z_2 Z_3 \times (2 \wr \mathrm{Sym}(4) \times T_2)$;
$P(2, 1) = Z_2 Z_3 \times (T_3 \times \mathrm{Sym}(4))$;

There is a single 2-minimal fuser and this, by Lemma 4.9, has shape

$$P(1 + 2) \sim (45) \times (45 \times T_2) \wr \mathrm{Sym}(3),$$

where 45 stands for the cyclic group of order 45.

The toral 2-minimal subgroups of H are $T(n_i; s^c; j)$ where $i \in I$, $s^c \in \Pi$ and $1 \le j \le n_i$. Thus we have

$T(3; 3^1; 1) \sim 3.T_3 Z_3 \times T_2 Z_2$ $\quad T(3; 3^2; 1) \sim 9.T_3 Z_3 \times T_2 Z_2$
$T(3; 5^1; 1) \sim 5.T_3 Z_3 \times T_2 Z_2$ $\quad T(3; 3^1; 2) \sim 3^2.T_3 Z_3 \times T_2 Z_2$
$T(3; 3^2; 2) \sim 9^2.T_3 Z_3 \times T_2 Z_2$ $\quad T(3; 5^1; 2) \sim 5^2.T_3 Z_3 \times T_2 Z_2$
$T(3; 3^1; 3) \sim 3^4.T_3 Z_3 \times T_2 Z_2$ $\quad T(3; 3^2; 3) \sim 9^4.T_3 Z_3 \times T_2 Z_2$
$T(3; 5^1; 3) \sim 5^4.T_3 Z_3 \times T_2 Z_2$ $\quad T(2; 3^1; 1) \sim T_3 Z_3 \times 3.T_2 Z_2$
$T(2; 3^2; 1) \sim T_3 Z_3 \times 9.T_2 Z_2$ $\quad T(2; 5^1; 1) \sim T_3 Z_3 \times 5.T_2 Z_2$
$T(2; 3^1; 2) \sim T_3 Z_3 \times 3^2.T_2 Z_2$ $\quad T(2; 3^2; 2) \sim T_3 Z_3 \times 9^2.T_2 Z_2$
$T(2; 5^1; 2) \sim T_3 Z_3 \times 5^2.T_2 Z_2$.

We note that for each 2-minimal subgroup of H we can give explicit generators. Note that $3^2 5$ is the odd part of $l - 1$ where l is the 3rd strobogrammatic prime.

5. Subgroups of the linear and unitary groups

The purpose of this section is to present some lemmas illustrating structural properties of $\mathrm{GL}_n^\epsilon(q) = \mathrm{GL}^\epsilon(V)$, where $\epsilon = \pm 1$ and q is odd.

We let V be an n-dimensional vector space over $\mathrm{GF}(q)$ or $\mathrm{GF}(q^2)$ and in the latter case we assume that V supports a non-degenerate unitary form. For ease of expression we will refer to orthogonal decompositions of V in both cases – so in effect we are supposing that V supports a trivial form when it is defined over $\mathrm{GF}(q)$.

We let $S_1 \in \mathrm{Syl}_2(\mathrm{GL}_2(q))$ and for $2^m > 2$ a 2-power we set $S_m = S_1 \wr T_{m-1}$. Let Z_m be the centre of $\mathrm{GL}_{2^m}^\epsilon(q)$ then $B_m = S_m Z_m$ is the normalizer of a Sylow 2-subgroup of $\mathrm{GL}_{2^m}^\epsilon(q)$ by [13, Lemma 1]. Finally we let $B_0 = \mathrm{GL}_1^\epsilon(q)$ and S_0 be a Sylow 2-subgroup of B_0. Notice that S_0 is cyclic of order $(q - \epsilon)_2$ and that

$$S_1 \cong (q - \epsilon)_2 \wr T_1$$

when $q \equiv \epsilon \pmod 4$ and otherwise

$$S_1 \cong \langle x, y \mid y^2 = x^{(q^2-1)_2} = 1, x^y = x^{\epsilon q}\rangle,$$

which is a semidihedral group of order $2(q^2-1)_2$.

Theorem 5.1. *Suppose that $G = \mathrm{GL}_n^\epsilon(q)$ and $n = 2^{n_1} + \cdots + 2^{n_r}$ with $n_1 > \cdots > n_r \geq 0$. Let $S = S_{n_1} \times \cdots \times S_{n_r}$ and $B = B_{n_1} \times \cdots \times B_{n_r}$. Then $S \in \mathrm{Syl}_2(G)$ and $B = N_G(S)$.*

Proof. See Theorems 1 and 4 of [13]. □

The decomposition of B leads to a corresponding decomposition of V. Namely, $V = V_{n_1} \oplus \cdots \oplus V_{n_r}$ where $V_{n_i} = [V, B_{n_i}]$, $i \in I$.

If $q \equiv \epsilon \pmod 4$, we let A_0 be a Sylow 2-subgroup of $\mathrm{GL}_1^\epsilon(q)$. Suppose that $q \equiv -\epsilon \pmod 4$. Then A_1 is defined to be the maximal cyclic subgroup of S_1. Thus we have $|A_0| = (q-\epsilon)_2$ and $|A_1| = (q^2-1)_2 = 2(q+\epsilon)_2$ and both groups have order at least 4.

If $q \equiv \epsilon \pmod 4$, then A denotes the base group of $A_0 \wr \mathrm{Sym}(n)$ while if $q \equiv -\epsilon \pmod 4$, we use A to denote the base group of $A_1 \wr \mathrm{Sym}(\lfloor n/2 \rfloor)$.

In the next lemma we encounter the group J_2^ϵ which is defined only when $q \equiv -\epsilon \pmod 4$ and is then the normalizer of A_1 in $\mathrm{GL}_2^\epsilon(q)$. We have that

$$J_2^\epsilon \cong \langle x, s \mid x^{q^2-1} = s^2 = 1, x^s = x^{\epsilon q}\rangle.$$

Thus J_2^ϵ contains a cyclic subgroup C of order q^2-1 and index 2, $|[J_2^\epsilon, J_2^\epsilon]| = q+\epsilon$ and $|Z(J_2^\epsilon)| = q - \epsilon$.

Lemma 5.2. *For $G = \mathrm{GL}_n^\epsilon(q)$, the following hold.*

(i) *If $q \equiv -\epsilon \pmod 4$, then $N_G(A) \cong J_2^\epsilon \wr \mathrm{Sym}(\lfloor \frac{n}{2} \rfloor)$ if n is even and $J_2^\epsilon \wr \mathrm{Sym}(\lfloor \frac{n}{2} \rfloor) \times \mathrm{GL}_1^\epsilon(q)$ if n is odd.*

(ii) *If $q \equiv \epsilon \pmod 4$, then $N_G(A) = \mathrm{GL}_1^\epsilon(q) \wr \mathrm{Sym}(n)$.*

Proof. We consider case (i) first and write $A = A_1 \times \cdots \times A_{\lfloor n/2 \rfloor}$ and set $W_k = [V, A_k]$. Then $\dim W_k = 2$ and we have an orthogonal decomposition

$$[V, A] = W_1 \oplus \cdots \oplus W_{\lfloor n/2 \rfloor}.$$

These 2-dimensional spaces are permuted naturally by $\mathrm{Sym}(\lfloor n/2 \rfloor)$. Since the A_i are the maximal subgroups of A with 2-dimensional commutators, we infer that $N_G(A)$ is as described.

If $q \equiv \epsilon \pmod 4$, then a similar argument shows that $N_G(A) = \mathrm{GL}_1^\epsilon(q) \wr \mathrm{Sym}(n)$. □

Lemma 5.3. *Suppose that $G = \mathrm{GL}_n^\epsilon(q)$ and $g \in G$. If $A^g \leq S$, then $A^g = A$. In particular, if R is a 2-group containing A, then $N_G(R) \leq N_G(A)$.*

Proof. We prove this explicitly for the case $q = -\epsilon \pmod 4$, the case $q \equiv \epsilon \pmod 4$ being easier. Again we let $A = A_1 \times \cdots \times A_{\lfloor n/2 \rfloor}$. For $1 \leq k \leq \lfloor n/2 \rfloor$, set $W_k = [V, A_k]$. Then $\dim W_k = 2$ and again we have an orthogonal decomposition

$$[V, A] = W_1 \oplus \cdots \oplus W_{\lfloor n/2 \rfloor}.$$

These 2-dimensional spaces are permuted naturally by $T \in \mathrm{Syl}_2(\mathrm{Sym}(\lfloor \frac{n}{2} \rfloor))$. Suppose that $A^g \leq S$ and $A^g \neq A$. Then $Y = A_1^g \not\leq A$. If Y centralizes A, then either n is even and $Y \leq A$ or n is odd and $Y \leq AA_0$ where $A_0 \in \mathrm{Syl}_2(\mathrm{GL}_1^\epsilon(q))$ from decomposition of $N_G(A)$ as $J_2^\epsilon \wr \mathrm{Sym}(\lfloor \frac{n}{2} \rfloor) \times \mathrm{GL}_1^\epsilon(q)$. In particular, if y is a generator of Y, then $y^2 \in A$ is non-trivial. But then $[V, Y]$ has dimension at least 3, which is impossible. Since Y is cyclic of order at least 8, and every element of order 8 in the base group C of $N_G(A)$ is contained in $C_G(A)$ (as the Sylow 2-subgroups of C are a direct product of semidihedral groups with a possible direct factor of order 2), we now have that Y permutes the spaces W_k non-trivially. As $\dim[V, Y] = 2$, we deduce that YC/C has order 2 and is generated by a transposition of $\{W_1, \dots, W_{\lfloor n/2 \rfloor}\}$. Let y be an element of order at least 4 in Y which is not contained in C. Then $y^2 \in C$ and $[V, y^2] \le [V, y]$. Since $y^2 \in C$ in non-trivial, $[V, y^2] \cap W_j \ne 0$ for some $1 \le j \le \lfloor n/2 \rfloor$ whereas, for all $1 \le i \le \lfloor n/2 \rfloor$, $[V, y] \cap W_i = 0$. Thus no such y exists and the lemma is proved in this case.

□

Lemma 5.4. *Let $W = \langle c, d, t \mid c^{q^2-1} = d^{q^2-1} = t^2 = 1, c^t = d \rangle$. Then $W \cong C \wr \mathrm{Sym}(2)$ where C is cyclic of order q^2-1 and the assignment $c \mapsto x$, $d \mapsto x^{\epsilon q}$ and $t \mapsto s$ determines a homomorphism from W onto J_2^ϵ with kernel $\langle (cd)^{q-\epsilon}, (cd^{-1})^{q+\epsilon} \rangle$.*

Proof. This is easy to verify. □

Lemma 5.5. *Let C be cyclic of order q^2-1 and $m \ge 1$. Let $F = C \wr (T_1 \wr \mathrm{Sym}(m))$ with $W = C \wr T_1$ having the presentation given in Lemma 5.4. Then there is a surjective homomorphism from $C \wr (T_1 \wr \mathrm{Sym}(m))$ to $J_2^\epsilon \wr \mathrm{Sym}(m)$ with kernel the normal closure in F of $\langle (cd)^{q-\epsilon}, (cd^{-1})^{q+\epsilon} \rangle$.*

Proof. This follows from Lemma 5.4 as generally, if $H/K \cong L$, then $(H \wr M)/(K \wr M) \cong L \wr M$. □

We continue the notation developed in the statement of Lemma 5.5. We intend to make explicit the generators of the images of the $N_F(T)$-minimal subgroups contained in the base group of F. Recall that these subgroups are parameterized by triples $s^e \in \Pi(C)$, $i \in I$ and $1 \le j \le n_i$ giving us subgroups which we denoted by $U(n_i; s^e; j)$. Let $c_1, d_1 \dots c_m, d_m$ be the generating elements (see Lemma 5.4) from the canonical factors of the base group of F permuted transitively by F and having order s^e and satisfying $\{\{d_i, c_i\} \mid 1 \le i \le m\}$ is a system of imprimitivity. Let $i \in I$ and set $w = (2^{n_1} + \dots + 2^{n_{i-1}})/2$. Then, when $j < n_i$, $U(n_i; s^e; j)$ is generated by

$$\langle (\prod_{k=w+1}^{w+2^{n_i-j-1}} c_k d_k c^{-1}_{2^{n_i-j}+k} d^{-1}_{2^{n_i-j}+k})^{T_{n_i}} \rangle.$$

Now we take $x_1, \dots, x_m$ to be the generators of the cyclic subgroups of order $2^{2(q+\epsilon)}$ in the factors of the base group of $J_2^\epsilon \wr \mathrm{Sym}(m)$. Then the image of $U(n_i; s^e; j)$, which we denote by $\overline{U(n_i; s^e; j)}$, is equal to

$$\langle (\prod_{k=w+1}^{w+2^{n_i-j-1}} x_k^{\epsilon q+1} x^{-\epsilon q-1}_{2^{n_i-j}+k})^{T_{n_i}} \rangle.$$

When $j = n_i$, $U(n_i; s^e; j)$ is generated by

$$\langle (c_k d_k^{-1})^{T_{n_i}} \rangle.$$

which maps to

$$\langle (\prod_{k=w+1}^{w+2^{n_i-j-1}} x_k^{-(q+\epsilon)})^{T_{n_i-1}} \rangle.$$

Lemma 5.6. *Assume that $m \geq 2$, $n = a_1 + \cdots + a_m$ with $a_i \geq 2$ for all $1 \leq i \leq m$, $C = \mathrm{GL}^{\epsilon}_{a_1}(q) \times \cdots \times \mathrm{GL}^{\epsilon}_{a_m}(q)$ is a subgroup of $\mathrm{GL}^{\epsilon}_n(q)$ and $C_0 = C \cap \mathrm{SL}^{\epsilon}_n(q)$. Let $S \in \mathrm{Syl}_2(C_0)$. Then S is contained in a unique Sylow 2-subgroup of C.*

Proof. Let R be a Sylow 2-subgroup of C containing S. For $1 \leq i \leq m$, we let K_i be the i^{th} component of C. Thus $K_i \cong \mathrm{GL}^{\epsilon}_{a_i}(q)$. Set $R_i = R \cap K_i$ and $D_i = Z(N_{K_i}(S))$. So $N_C(R) = RD_1 \dots D_m$ by Theorem 5.1. Plainly $D_1 \dots D_m$ centralizes R and hence $D_1 \dots D_m \leq N_C(S)$. Thus $D_i \leq \pi_i(N_C(S))$ and, since $m \geq 2$, $\pi_i(S) = R_i$. It follows that $D_i R_i \leq \pi_i(N_C(S)) \leq N_{K_i}(R_i) = R_i D_i$. Hence $N_C(S) \leq R_1 D_1 \dots R_m D_m = N_C(R) \leq N_C(S)$. Hence $N_C(R) = N_C(S)$ and the lemma follows from Lemma 2.5. $\square$

The next two theorems, which rely upon the simple group classification, are important in telling us where to look for 2-minimal subgroups.

Theorem 5.7. *Suppose that G is a subgroup of $\mathrm{GL}_n(q)$ containing $\mathrm{SL}_n(q)$ where q is an odd prime power and $n \geq 2$ is an integer. If H is a maximal subgroup of G of odd index then at least one of the following holds.*

(i) *$q_0^c = q$, where c is an odd prime and $H \cong \mathrm{GL}_n(q_0) \circ (q-1)$. (There are $(\frac{q-1}{q_0-1}, n)$-conjugacy classes of these subgroups in $\mathrm{GL}_n(q)$.)*
(ii) *H is a maximal parabolic subgroup of G.*
(iii) *H stabilizes a decomposition of V into spaces of equal dimension.*
(iv) *$n = 4$, $(q-1)_2 = 2$, G has index $2m$ where m is odd, and $G \circ (q-1)$ has two conjugacy classes of subgroup $H \cong \mathrm{GSp}_4(q) \circ (q-1)$.*
(v) *$n = 4$, $(q-1)_2 = 4$, G has index $4m$ or $2m$ where m is odd and $G \circ (q-1)$ has two conjugacy classes of subgroup $H \cong \big(4 \circ 2^{1+4}_{+}.\mathrm{Alt}(6)\big) \circ (q-1)$ or $\big(4 \circ 2^{1+4}_{+}.\mathrm{Sp}_4(2)\big) \circ (q-1)$ respectively.*

Theorem 5.8. *Suppose that G is a subgroup of $\mathrm{GU}_n(q)$ containing $\mathrm{SU}_n(q)$ where q is an odd prime power and $n \geq 2$ is an integer. If H is a maximal subgroup of G of odd index then at least one of the following holds.*

(i) *$q_0^c = q$, where c is an odd prime and $H \cong \mathrm{GU}_n(q_0) \circ (q+1)$. (There are $(\frac{q+1}{q_0+1}, n)$-conjugacy classes of these subgroups in $\mathrm{GU}_n(q)$.)*
(ii) *H stabilizes a decomposition of V into an orthogonal sum of non-degenerate spaces.*
(iii) *$n = 4$, $(q+1)_2 = 2$, G has index $2m$ where m is odd, and $G \circ (q+1)$ has two conjugacy classes of subgroup $H \cong \mathrm{GSp}_4(q) \circ (q+1)$.*
(iv) *$n = 4$, $(q+1)_2 = 4$, G has index $4m$ or $2m$ where m is odd and $G \circ (q+1)$ has two conjugacy classes of subgroup $H \cong \big(4 \circ 2^{1+4}_{+}.\mathrm{Alt}(6)\big) \circ (q+1)$ or $\big(4 \circ 2^{1+4}_{+}.\mathrm{Sp}_4(2)\big) \circ (q+1)$, respectively.*
(v) *$G = \mathrm{SU}_3(5)$ and there are three conjugacy classes of subgroup $H \cong \mathrm{Mat}(10)$.*

Proof of Theorems 5.7 and 5.8. That the given groups contain a Sylow 2-subgroup is readily verified using the orders of the group. We cite either Liebeck and Saxl [24] and Kantor [20] to provide the proof that no other maximal over-groups of a Sylow 2-subgroup exist. Referring to [21, Theorems 4.1.4, 4.1.14, 4.2.9, 4.3.6, 4.6.6] we see that the number of $\mathrm{GL}^{\epsilon}_n(q)$ conjugacy classes, c in their notation, is as indicated in all but the last case of Theorem 5.8 when we refer to the Atlas [16] to see that the number is three. $\square$

We turn our attention for a moment to the 2-minimal subgroups of $\mathrm{GL}^{\epsilon}_n(q)$ in general.

Proposition 5.9. *Suppose that $n \geq 3$, $G = \mathrm{GL}_n(q)$ and $P \in \mathcal{M}(G,B)$. Then either*

(i) *P is contained in a parabolic subgroup of G;*
(ii) *P acts irreducibly on V and there exists $b \geq 1$ such that $n = 2^b m$ and $P \leq \mathrm{GL}_{2^a}(q) \wr \mathrm{Sym}(m)$; or*
(iii) *$q \equiv 3 \pmod 4$, $n = 4$, and $P = \mathrm{GL}_4(p) \circ (q-1)$.*

In particular, if $G \in \mathcal{M}(G,B)$, then $n = 4$ and $q = p \equiv 3 \pmod 4$.

Proof. Suppose that (i) and (ii) do not hold. Then by Lemma 5.7 there exists an odd prime c such that $P \leq \mathrm{GL}_n(q_0) \circ (q-1)$ where $q_0^c = q$. Applying Lemma 5.7 to $\mathrm{GL}_n(q_0)$, we find that $P \leq \mathrm{GL}_4(p^{a_2}) \circ (q-1)$ and then deduce that $P = \mathrm{GL}_4(p^{a_2}) \circ (q-1)$. If n is not a power of 2, then B preserves a non-trivial direct decomposition of V into a sum of two subspaces. Therefore B is contained in at least two maximal subgroups of P. Thus n is a 2-power. Suppose that $p^{a_2} \equiv 1 \mod 4$ and $n > 2$. Then $\mathrm{GL}_n(p^{a_2})$ contains $\mathrm{GL}_2(p^{a_2}) \wr \mathrm{Sym}(n/2)$ and $\mathrm{GL}_1(p^{a_2}) \wr \mathrm{Sym}(n)$ which is also impossible. Thus $p^{a_2} \equiv 3 \pmod 4$. In particular, $a_2 = 1$. If $n > 4$, then $\mathrm{GL}_n(p)$ contains $\mathrm{GL}_2(p) \wr \mathrm{Sym}(n/2)$ and $\mathrm{GL}_4(p) \wr \mathrm{Sym}(n/4)$ and so $\mathrm{GL}_n(p)$ is not 2-minimal in this case. Therefore we have $P = \mathrm{GL}_4(p) \circ (q-1)$ as claimed. □

Proposition 5.10. *Suppose that $n \geq 3$, $G = \mathrm{GU}_n(q)$ and $P \in \mathcal{M}(G,B)$. Then either*

(i) *P preserves an orthogonal decomposition of V;*
(ii) *$q \equiv 1 \pmod 4$, $n = 2^m + 1$ and $P = G$; or*
(iii) *$p^{a_2} \equiv 1 \pmod 4$, $n = 4$ and $P = \mathrm{GU}_4(p^{a_2}) \circ (q+1)$.*

In particular, if $G \in \mathcal{M}(G,B)$, then either $q \equiv 1 \pmod 4$ and $n = 2^m + 1$ or $q = p^{a_2} \equiv 1 \pmod 4$ and $n = 4$.

Proof. It suffices to show that G is not 2-minimal unless $q \equiv 1 \pmod 4$ and $n = 2^m + 1$ or $q = p^{a_2} \equiv 1 \pmod 4$ and $n = 4$. We use Theorem 5.8 liberally. Recall that $n = 2^{n_1} + \cdots + 2^{n_r}$. If $r \geq 3$, then we have that both $\mathrm{GU}_{2^{n_1}}(q) \times \mathrm{GU}_{n-2^{n_1}}(q)$ and $\mathrm{GU}_{n-2^{n_r}}(q) \times \mathrm{GU}_{2^{n_r}}(q)$ are maximal subgroups containing B. Hence we must have $r \leq 2$. If $n_r > 0$, then $\mathrm{GU}_{2^{n_1}}(q) \times \mathrm{GU}_{2^{n_2}}(q)$ and $\mathrm{GU}_2(q) \wr \mathrm{Sym}(n/2)$ both contain B and together generate G. Hence if $r = 2$, we have $n = 2^{n_1} + 1$. If $q \equiv 3 \pmod 4$, then G is generated by $\mathrm{GU}_1(q) \wr \mathrm{Sym}(n)$ and $\mathrm{GU}_{2^n_1}(q) \times \mathrm{GU}_1(q)$ both of which contain B. Thus we must have $q \equiv 1 \pmod 4$. Notice that as the subfield subgroups $\mathrm{GU}_n(q_0)$ where $q_0^c = q$ for some odd prime c do not contain B, we have that G is 2-minimal in this case.

So suppose that $r = 1$. Then $n = 2^{n_1} \geq 4$. Assume that $2^{n_1} \geq 8$. Then both $\mathrm{GU}_2(q) \wr \mathrm{Sym}(n/2)$ and $\mathrm{GU}_4(q) \wr \mathrm{Sym}(n/4)$ contain B and so $n \leq 4$. If $q \equiv 3 \pmod 4$, then we use the subgroups $\mathrm{GU}_1(q) \wr \mathrm{Sym}(n)$ and $\mathrm{GU}_2(q) \wr \mathrm{Sym}(n/2)$. Hence we have $n = 4$ and $q \equiv 1 \pmod 4$. Finally we note that this time B is contained in the normalizer of the subfield subgroups and so if $q \neq p^{a_2}$ we would again have two proper over-groups of B which generate G. Hence $q = p^{a_2}$ and these groups are indeed 2-minimal. □

6. 2-minimal subgroups in linear and unitary groups

The one and only theorem of this section highlights the five subdivisions of our later investigations.

Theorem 6.1. *Suppose that $G = \mathrm{GL}_n^\epsilon(q)$, $S = S_{n_1} \times \cdots \times S_{n_r}$, $B = N_G(S)$ and let A be as in Section 5. Assume that $P \in \mathcal{M}(G, B)$. Then at least one of the following holds.*

- (i) $r = 1$;
- (ii) $P = G = \mathrm{GU}_{2^{n_1}+1}(q)$;
- (iii) $P \in \mathcal{M}(N_G(A), B)$;
- (iv) $\epsilon = +$ *and P is contained in a parabolic subgroup of G; or*
- (v) $P \in \mathcal{M}(\mathrm{GL}^\epsilon(U) \times \mathrm{GL}^\epsilon(W), B)$ *for some non-zero subspaces U and W such that $V = U \oplus W$.*

Proof. Assume that $r > 1$. Thus, if $G = P$, Propositions 5.9 and 5.10 yield alternative (ii). So we may now suppose that $G \neq P$. Employing Theorems 5.9 and 5.10 again shows that either (iv) or (v) holds, or $P \leq H \leq \mathrm{GL}_{2^d}^\epsilon(q) \wr \mathrm{Sym}(n/2^d)$ for some d such that 2^d divides n. So assume that $P \leq H = \mathrm{GL}_{2^d}^\epsilon(q) \wr \mathrm{Sym}(n/2^d)$ where 2^d divides n. Let K be the base group of H. If P is not transitive on the wreathed direct factors of K, then (v) holds. Therefore we may suppose that $PK \neq PB$. Finally, Lemma 3.7 implies that $P = N_P(S \cap K)$. Since $S \cap K$ contains A, we now have that (iii) holds by Lemma 5.3. □

7. 2-minimal radical subgroups

In this section we assume that $G = \mathrm{GL}_n(q)$ and that $\mathrm{SL}_n(q) \leq H \leq G$. We investigate 2-minimal subgroups of H which lie in a parabolic subgroup of G (so we are pursuing case (iv) of Theorem 6.1). Notice that in this case we must have $r > 1$.

Lemma 7.1. *Suppose that $P \in \mathcal{M}(H, B \cap H)$ and that P does not act irreducibly on V. Then either*

- (i) *there exist non-zero subspaces U and W of V such that $V = U \oplus W$ and $P \leq \mathrm{GL}(U) \times \mathrm{GL}(W)$; or*
- (ii) $O_p(P) = O_p(R) \cap P$ *and* $P = O_p(P)(B \cap H)$ *for all maximal parabolic subgroups R of G which contain P.*

Proof. Since P is contained in a parabolic subgroup of G, there exist maximal parabolic subgroups of G containing P. Let R be any such maximal parabolic subgroup. Then $R = N_G(W)$ where W is a non-zero proper subspace of V which is of course P-invariant. Let L be a Levi complement in R chosen so as $B \leq L$. Then there is a complement U to W in V such that $L = \mathrm{GL}(U) \times \mathrm{GL}(W)$. Let $w \in Z(L)$ act fixed-point-freely on $O_p(R)$. Obviously $w \in Z(B)$. Now $P = C_P(w)(O_p(R) \cap P)$ by a Frattini Argument. Since P is 2-minimal, $B \cap H \leq C_P(w)$, and $B \cap H \leq (O_p(R) \cap P)B$, we get that either $P = C_P(w) \leq L$ or $P = (O_p(R) \cap P)(B \cap H) = O_p(P)(B \cap H)$. Hence either (i) or (ii) holds. □

Theorem 7.2. *Suppose that $P \in \mathcal{M}(H, B \cap H)$ and P is contained in a parabolic subgroup of G. Then either*

- (i) *there exist non-zero subspaces U and W of V such that $V = U \oplus W$ and $P \leq \mathrm{GL}(U) \times \mathrm{GL}(W)$; or*

(ii) $n = 2^{n_1} + 2^{n_2}$ *and there exists* $i \in I = \{1, 2\}$ *such that* $V_{n_i} = [V, O_p(P)] = C_V(O_p(P))$ *and* $P = O_p(N_G(V_{n_i}))(B \cap H)$. *In particular,* P *is normalized by* B.

Proof. We suppose that (i) does not hold. Deploying Lemma 7.1 we now have $O_p(P) \leq O_p(R)$ for all maximal parabolic subgroups R of G containing P. Let $V = W_1 > \cdots > W_k > 0$ be a P-invariant flag such that $\bar{W_i} = W_i/W_{i+1}$ is an irreducible P-module. Then $O_p(P)$ centralizes $\bar{W_i}$ and thus $\bar{W_i}$ is an irreducible $(B \cap H)$-module. Thus $\{\bar{W_i} \mid 1 \leq i \leq k\}$ is in natural correspondence with $\{V_{n_i} \mid 1 \leq i \leq r\}$. In particular, $k = r$. Let V_{n_j} correspond to $\bar{W_r}$ and V_{n_i} correspond to W_1/W_2. Set $R_2 = N_G(W_2)$ and $R_r = N_G(W_r)$. Then $P \leq R_2 \cap R_r$ from which we infer that $O_p(P) \leq O_p(R_2) \cap O_p(R_r)$. Set $U_0 = V_{n_i} + V_{n_j}$ and $U_1 = \oplus_{m \notin \{i,j\}} V_{n_m}$ and note that U_0 and U_1 are both $(B \cap H)$-invariant. As $U_1 \leq W_2$ and $[W_2, O_p(R_2)] = 0$, we have that U_1 is P-invariant and that P acts on U_1 just as $(B \cap H)$ does. Similarly we have that $[V_{n_j}, O_p(P)] = 0$. Now $[V_{n_i}, O_p(P)] \leq [V_{n_i}, O_p(R_r)] \leq [V, O_p(R_r)] = W_r = V_{n_j} \leq U_0$. So U_0 is also $O_p(P)$-invariant. Hence U_0 is P-invariant. Now we have that $P \leq \mathrm{GL}(U_0) \times \mathrm{GL}(U_1)$ where $V = U_0 \oplus U_1$ which, as (i) is assumed not to hold, implies that $U_1 = 0$. Hence $r = 2$ and $V = V_{n_1} \oplus V_{n_2}$ and V_{n_1} and V_{n_2} are the only $B \cap H$ invariant subspaces of V. It follows that either $C_V(O_p(P)) = V_{n_1}$ or $C_V(O_p(P)) = V_{n_2}$. So suppose that $C_V(O_p(P)) = V_{n_1}$, for example. Then $P \leq N_G(V_{n_1})$ and $O_p(N_G(V_{n_1}))$ is elementary abelian and admits $B \cap H$ irreducibly. Since $O_p(P) \leq O_p(N_G(V_{n_1}))$ by Lemma 7.1, we now have $O_p(P) = O_p(N_G(V_{n_1}))$. Thus (ii) holds and the theorem is proved. □

Recalling our standard setup of $n = 2^{n_1} + \cdots + 2^{n_r}$ with $I = \{1, \ldots, r\}$, we now define another type of 2-minimal subgroup an example of which has just emerged in Theorem 7.2 (ii).

Definition 7.3. *Let* $\{i, j\}$ *be a* 2*-element subset of* I, $W = V_{n_i} \oplus V_{n_j}$ *and* $M = (\mathrm{GL}(W) \cap H)(B \cap H)$. *Then the* 2*-minimal subgroups of* M *which do not act irreducibly on* W *are determined in Theorem 7.2. The example arising in Theorem 7.2 (ii) with* $C_W(O_p(N_{\mathrm{GL}(W)}(V_{n_i})) = V_{n_j}$ *will be denoted by* $R(n_i \ggg n_j)$. *These* 2*-minimal subgroups will collectively be called* 2*-minimal* radical *subgroups and the set of such subgroups of* H *is denoted by* $\mathcal{R}$.

Note that $|O_p(R(n_i \ggg n_j))| = q^{n_i n_j} = |O_p(R(n_j \ggg n_i))|$.

From Definition 7.3 we see that each two element subset of I gives us two 2-minimal radical subgroups. Thus we have

Lemma 7.4. $|\mathcal{R}| = r(r-1)$.

□

Example 7.5. *Suppose that* $G = \mathrm{GL}_{26}(q)$. *Then* $26 = 2^4 + 2^3 + 2^1$ *so that* $n_1 = 4$, $n_2 = 3$, $n_3 = 1$ *and* $r = 3$. *By Lemma 7.4 there are* 6 *conjugacy classes of* 2*-minimal radical subgroups of* G. *Matrices representing these* p*-minimal subgroups are depicted in the following schematic where a* * *indicates an appropriate* $M_{x,y}(q)$ *and* B_{n_i} *denotes the Sylow* 2*-normalizer in* $\mathrm{GL}_{2^{n_i}}(q)$. *Also we shall assume that* G *acts on* V *by right matrix multiplication.*

$$R(4 \ggg 3) = \begin{pmatrix} B_4 & * & 0 \\ 0 & B_3 & 0 \\ 0 & 0 & B_1 \end{pmatrix} \qquad R(3 \ggg 4) = \begin{pmatrix} B_4 & 0 & 0 \\ * & B_3 & 0 \\ 0 & 0 & B_1 \end{pmatrix}$$

$$R(4 \ggg 1) = \begin{pmatrix} B_4 & 0 & * \\ 0 & B_3 & 0 \\ 0 & 0 & B_1 \end{pmatrix} \qquad R(1 \ggg 4) = \begin{pmatrix} B_4 & 0 & 0 \\ 0 & B_3 & 0 \\ * & 0 & B_1 \end{pmatrix}$$

$$R(3 \ggg 1) = \begin{pmatrix} B_4 & 0 & 0 \\ 0 & B_3 & * \\ 0 & 0 & B_1 \end{pmatrix} \qquad R(1 \ggg 3) = \begin{pmatrix} B_4 & 0 & 0 \\ 0 & B_3 & 0 \\ 0 & * & B_1 \end{pmatrix}$$

8. Toral, fuser and linker 2-minimal subgroups of linear and unitary groups

In this section, we first describe the 2-minimal subgroups of $G = \mathrm{GL}_n^\epsilon(q)$ which normalize A where A is defined as in Section 5 immediately after Theorem 5.1. Throughout this section we set $H = N_G(A)$.

We first consider the case when $q \equiv \epsilon \pmod 4$. In this case $H \cong (q-\epsilon) \wr \mathrm{Sym}(n)$ where $(q-\epsilon)$ denotes a cyclic group of order $q-\epsilon$. Recall that A is a direct product of n cyclic groups of order $(q-\epsilon)_2$ and so has order at least 4^n. The 2-minimal subgroups of H are in one to one correspondence with the 2-minimal subgroups of $H/A \cong (q-\epsilon)_{2'} \wr \mathrm{Sym}(n)$ (which if $q-\epsilon$ is a power of 2, we understand to be isomorphic to $\mathrm{Sym}(n)$). We extend the notation from Section 4 by taking preimages. Thus we set

$$\mathcal{T}(H,B) = \{T(n_i; s^c; j) \mid i \in I, s^c \in \Pi(q-\epsilon) \text{ and } 1 \le j \le n_i\}.$$

The linkers and fusers for H are defined in a similar fashion by pulling back from $H/O_2(H)$ and we continue to denote these sets by $\mathcal{L}(H,B)$ and $\mathcal{F}(H,B)$. So our first result is

Theorem 8.1. *Suppose that $q \equiv \epsilon \pmod 4$. Then $\mathcal{M}(H,B) = \mathcal{T}(H,B) \cup \mathcal{F}(H,B) \cup \mathcal{L}(H,B)$.*

Proof. Taking into account our modified notation, this is just a restatement of Theorem 4.12. □

The corresponding subsets of 2-minimal subgroups when $q \equiv -\epsilon \pmod 4$ are more technical to define. Recall that in this case $H = N_G(A) \cong J_2^\epsilon \wr \mathrm{Sym}(n/2)$ when n is even and $H = N_G(A) \cong J_2^\epsilon \wr \mathrm{Sym}(\lfloor n/2 \rfloor) \times \mathrm{GL}_1^\epsilon(q)$ when n is odd. When n is odd, the final factor is contained in B and is normal in H and so we can, and will, be suppressed in our considerations. By Lemma 5.5, we have that $N_G(A)$ is a quotient of W where $W = C \wr (T_1 \wr \mathrm{Sym}(\lfloor n/2 \rfloor)$ and C has order q^2-1. By Lemma 3.6 every 2-minimal subgroup of H is an image of a 2-minimal subgroup of W. Hence we read off the 2-minimal subgroups of H from those that we have described in Theorem 4.12 for W considered as a subgroup of $C \wr \mathrm{Sym}(n)$. Let $L = \mathrm{Sym}(\lfloor n/2 \rfloor)$ be a complement to the base group of H containing T.

Using bars to denote images, we have

$$\mathcal{F}(H,B) = \{\overline{P} \mid P \in \mathcal{F}(W)\} = \{\langle B, P^* \rangle \mid P^* \in \mathcal{F}(L,T)\}$$

are the fusers and these all have images great than B.

The linkers become

$$\mathcal{L}(H,B) = \{\overline{P} \mid P \in \mathcal{F}(W), P \neq P(i;1)\} = \{BP^* \mid P^* \in \mathcal{F}(L,T)\}$$

and finally the toral 2-minimal subgroups of H are

$$\begin{aligned}\mathcal{T}(H,B) &= \{T(n_i;s^c;j) \mid i\in I, 1\le j<n_i, s^c\in\Pi(q-\epsilon)\}\\ &\quad\cup\{T(n_i;s^c;n_i)\mid i\in I, s^c\in\Pi(q+\epsilon)\}.\end{aligned}$$

We refer to the discussion in Section 5 for a vibrant description of these toral subgroups.

Theorem 8.2. *Suppose that $q\equiv -\epsilon \pmod 4$. Then*

$$\mathcal{M}(H,B)=\mathcal{T}(H,B)\cup\mathcal{F}(H,B)\cup\mathcal{L}(H,B).$$

Proof. This follows from the foregoing discussion. □

Corollary 8.3. *$N_G(A)$ is tame.*

Proof. This follows from the description of the 2-minimal subgroups of $N_G(A)$ given in Theorems 8.1 and 8.2. □

9. 2-minimal subgroups in dimensions 2 and 4

In this section we determine the 2-minimal subgroups of $\mathrm{GL}_2^\epsilon(q)$ and $\mathrm{GL}_4^\epsilon(q)$. These are the base cases for our inductive proof of Theorem 1.1. We first look at the dimension 2 case. Let V be the natural $\mathrm{GL}_2^\epsilon(q)$-module. Two subgroups of $\mathrm{GL}_2^\epsilon(q)$ play a leading role. The first is the monomial group $\mathrm{GL}_1^\epsilon(q)\wr T_1$ which has order $2(q-\epsilon)^2$ and the second is the group J_2^ϵ which we have already introduced in Section 5. We now give an alternative description of J_2^ϵ. If $\epsilon=+$, $J_2^+=\mathrm{GL}_1(q^2):\langle\alpha\rangle$ where α is the field automorphism of $\mathrm{GF}(q^2)$ which maps every element to its q^{th} power. If $\epsilon=-$, then J_2^- preserves a decomposition of V as a sum of two isotropic subspaces and is isomorphic to $\mathrm{GL}_1(q^2):\langle\beta\rangle$ where β is the automorphism of the multiplicative group of $\mathrm{GF}(q^2)$ which maps every element to the inverse of its q^{th} power. In particular, note that $Z(J_2^\epsilon)$ is cyclic of order $q-\epsilon$.

Lemma 9.1. *Suppose that p is an odd prime, $q=p^a>5$ and $G=\mathrm{GL}_2^\epsilon(q)$. Then the maximal subgroups of G of odd index are as follows.*

(i) $\mathrm{GL}_1^\epsilon(q)\wr T_1$ *when* $q\equiv\epsilon \pmod 4$.
(ii) J_2^ϵ *when* $q\equiv-\epsilon \pmod 4$.
(iii) $\mathrm{GL}_2^\epsilon(p^{a/c})\circ(q-\epsilon)$ *for each odd prime divisor c of a.*
(iv) $\mathrm{Q}_8.\mathrm{Sym}(3)\circ(q-\epsilon)$ *when* $q\equiv 3,5 \pmod 8$ *is a prime.*

Furthermore, in each case there is exactly one conjugacy class of such subgroups.

Proof. This result is deduced from the list of maximal subgroups of $\mathrm{GL}_2(q)$ given in [8, Theorem 3.4]. □

Corollary 9.2. *With $G=\mathrm{GL}_2^\epsilon(q)$, we have $G\in\mathcal{M}(G,B)$ if and only if one of the following holds:*

(i) $a=a_2>1$;
(ii) $a=1$, $q\not\equiv 3,5 \pmod 8$; *or*
(iii) $G=\mathrm{GL}_2^\epsilon(3)$ *or* $\mathrm{GL}_2^\epsilon(5)$.

Proof. If $G=\mathrm{GL}_2^\epsilon(3)$ or $\mathrm{GL}_2^\epsilon(5)$, then it is easily verified that G is 2-minimal. So we may assume that $q>5$.

We first check that if (i) or (ii) hold, then G is 2-minimal. Note first that exactly one of the groups in (i) and (ii) of Lemma 9.1 can contain B. If (i) holds, then, as

$a = a_2$, (iii) of Lemma 9.1 cannot occur, and, as $a_2 > 1$, $q \not\equiv 3, 5 \pmod 8$ and (iv) of Lemma 9.1 cannot occur. Hence G is 2-minimal in this case. If (ii) holds, then once again there is only one conjugacy class of maximal subgroups of odd index in G.

Suppose now that $G \in \mathcal{M}(G, B)$. Then as exactly one of the subgroups listed in (i) and (ii) of Lemma 9.1 contain B, the groups listed in (iii) and (iv) of the same lemma cannot arise in G. Hence either (i) or (ii) holds and the lemma is proved. □

We can now harvest the 2-minimal subgroups for the groups $\mathrm{GL}_2^\epsilon(q)$.

Proposition 9.3. *Assume that $G = \mathrm{GL}_2^\epsilon(q)$ (where $q = p^a$). Then under the given conditions $\mathcal{M}(G, B)$ is as follows.*

(i) $q \equiv \epsilon \pmod 8$ *and*

$$\mathcal{M}(\mathrm{GL}_1^\epsilon(q) \wr T_1, B) \cup \{\mathrm{GL}_2^\epsilon(p^{a_2}) \circ (q - \epsilon)\}.$$

(ii) $q \equiv -\epsilon \pmod 8$ *and*

$$\mathcal{M}(J_2^\epsilon, B) \cup \{\mathrm{GL}_2^\epsilon(p^{a_2}) \circ (q - \epsilon)\}.$$

(iii) $q \equiv 4 - \epsilon \pmod 8$*,* $p \neq 5$*, and*

$$\mathcal{M}(J_2^\epsilon, B) \cup \{\mathrm{Q}_8.\mathrm{Sym}(3) \circ (q - \epsilon)\}.$$

(iv) $q \equiv 4 + \epsilon \pmod 8$*,* $p \neq 5$ *and*

$$\mathcal{M}(\mathrm{GL}_1^\epsilon(q) \wr T_1, B) \cup \{\mathrm{Q}_8.\mathrm{Sym}(3) \circ (q - \epsilon)\}.$$

(v) $q = 5^a$ *with a odd and*

$$\mathcal{M}(\mathrm{GL}_1^\epsilon(q) \wr T_1, B) \cup \{\mathrm{GL}_2^\epsilon(5) \circ (q - \epsilon)\} \cup \{\mathrm{Q}_8.\mathrm{Sym}(3) \circ (q - \epsilon)\}.$$

Proof. Assume that $P \notin \mathcal{M}(\mathrm{GL}_1^\epsilon(q) \wr T_1, B)$ when $q \equiv \epsilon \pmod 4$ and $P \notin \mathcal{M}(J_2^\epsilon, B)$ when $q \equiv -\epsilon \pmod 4$. We prove the result by induction on a. Assume that $a = 1$. If $q = 3$ or $q = 5$, then we observe that the proposition holds. Hence we may take $q > 5$. If $P = G$, then (i) or (ii) holds by Lemma 9.2. If $P < G$, then Lemma 9.1 indicates that $q \equiv 3, 5 \pmod 8$ and that one of (iii) and (iv) holds. Assume now that $a > 1$. Again if $P = G$, we get $a = a_2 > 1$ from Lemma 9.2 and (i) or (ii) holds. For $P < G$ we again apply Lemma 9.1 to get $P \leq \mathrm{GL}_2^\epsilon(q_0)$ where $q_0^c = q$ for some odd prime c. Noting that $q \equiv q_0 \pmod 8$, induction yields the result. □

For completeness we rerecord, from Theorems 8.1 and 8.2, the 2-minimal subgroups of $\mathcal{M}(\mathrm{GL}_1^\epsilon(q), B)$ for $q \equiv \epsilon \pmod 4$ and $\mathcal{M}(J_2^\epsilon, B)$ for $q \equiv -\epsilon \pmod 4$.

Lemma 9.4. (i) *For $q \equiv \epsilon \pmod 4$, $\mathcal{M}(\mathrm{GL}_1^\epsilon(q), B) = \{T(1, s^c, 1) \mid s^c \in \Pi(q - \epsilon)\}$.*

(ii) *For $q \equiv -\epsilon \pmod 4$, $\mathcal{M}(J_2^\epsilon, B) = \{T(1, s^c, 1) \mid s^c \in \Pi(q + \epsilon)\}$.*

□

Corollary 9.5. *$G = \mathrm{GL}_2^\epsilon(q)$ is tame.*

Proof. From Proposition 9.3 and Lemma 9.4 it follows that pairs of distinct members of $\mathcal{M}(G, B)$ are not isomorphic. Hence G is tame. □

In the next theorem we determine the 2-minimal subgroups of $H = \mathrm{GL}_2^\epsilon(q) \wr T_{n-1} \le \mathrm{GL}_{2^n}^\epsilon(q)$ where $B \le H$. These subgroups break into two types as indicated by Lemma 9.3. Thus we introduce the *quaternion* 2-minimal subgroups when $q \equiv 3, 5 \pmod 8$

$$Q(n) = Z_n\left((q-\epsilon)_2 \circ \mathrm{Q}_8.\mathrm{Sym}(3)\right) \wr T_{n-1}$$

and the *special linear* 2-minimal subgroups

$$S(2,n) = Z_n\left(\mathrm{SL}_2^\epsilon(p^{a_2}).(q-\epsilon)_2\right) \wr T_{n-1}$$

for $q \equiv 1, 7 \pmod 8$ or $q = 5^a$ with a odd. With reference to our notation at this point, we note that $\mathrm{SL}_2^\epsilon(p^{a_2}).(q-\epsilon)_2 = O^{2'}(\mathrm{GL}_2^\epsilon(p^{a_2}))$ is the subgroup of $\mathrm{GL}_2^\epsilon(p^{a_2}) \circ (q-\epsilon)$ consisting of elements with determinant in the subgroup of $\mathrm{GF}(q)^*$ when $\epsilon = +$ or $\mathrm{GF}(q^2)^*$ when $\epsilon = -$ of order $(q-\epsilon)_2$.

Theorem 9.6. *Suppose that $H = \mathrm{GL}_2^\epsilon(q) \wr T_{n-1}$ for some natural number n. Then*

$$\mathcal{M}(H,B) = \mathcal{M}(N_H(A),B) \cup \{Q(n), S(2,n)\}.$$

In particular, H is tame.

Proof. Let K be the base group of H and suppose that $P \in \mathcal{M}(H,B)$. Then by the construction of H, $P \le KS$ and S operates transitively on the factors $K_1, \ldots, K_{2^{n-1}}$ of K. Now $S \cap K \in \mathrm{Syl}_2(K)$ and $N_K(S\cap K) = (S \cap K)Z_n$ by Theorem 5.1. It follows that $\pi_1(N_K(S\cap K)) = N_{K_1}(S \cap K)$. Finally, K_1 is tame by Corollary 9.5. Thus the conditions of Lemma 3.17 are satisfied and so we have $P \in \mathcal{M}(N_H(S \cap K), B) = \mathcal{M}(N_H(A), B)$ by Lemma 5.3 or $P = Z_n\langle O^{2'}(L)^{T_{n-1}}\rangle T_{n-1}$ where $L \in \mathcal{M}(K_1, N_{K_1}(S \cap K_1))$. If $L \le N_{K_1}(A_1)$, then we also have $P \in \mathcal{M}(N_H(A), B)$. Proposition 9.3 now delivers the result. □

By Propositions 5.9 and 5.10 we now see why $\mathrm{GL}_4^\epsilon(q)$ is most interesting for us when $q \equiv -\epsilon \pmod 4$.

Lemma 9.7. *Suppose that $G = \mathrm{GL}_4^\epsilon(q)$ and $q \equiv -\epsilon \pmod 4$. Then $\mathcal{M}(G,B) = \mathcal{M}(\mathrm{GL}_2^\epsilon(q) \wr T_1, B) \cup \{\mathrm{GL}_4^\epsilon(p^{a_2}) \circ (q-\epsilon)\}$. In particular, G is tame.*

Proof. The first part follows from Propositions 5.9 and 5.10 and then we see that G is tame by applying Theorem 9.6. □

Finally, for $q \equiv -\epsilon \pmod 4$, we consider groups of the form $H = \mathrm{GL}_4^\epsilon(q) \wr T_{n-2}$ contained in $\mathrm{GL}_{2^n}^\epsilon(q)$ and containing B. Our aim is to determine all the 2-minimal subgroups. Thus we additionally define

$$S(4,n) = Z_n\left(\mathrm{SL}_4^\epsilon(p^{a_2}).(q-\epsilon)_2\right) \wr T_{n-2}$$

for $q \equiv -\epsilon \pmod 4$. Note that if $\epsilon = +$, then $a_2 = 1$. The group $S(4,n)$ is also called a *special linear* 2-minimal subgroup.

Theorem 9.8. *Suppose that $H = \mathrm{GL}_4^\epsilon(q) \wr T_{n-2}$ with $q \equiv -\epsilon \pmod 4$. Then $\mathcal{M}(H,B) = \mathcal{M}(\mathrm{GL}_2^\epsilon(q) \wr T_{n-1}, B) \cup \{S(4,n)\}$. In particular, H is tame.*

Proof. Just as in Theorem 9.6 we get that Lemma 3.17 is applicable. It then follows from Lemma 9.7 that $\mathcal{M}(H,B)$ is precisely as described. □

10. 2-MINIMAL SUBGROUPS OF $\mathrm{GL}^\epsilon_{2^m}(q)$

In this section we assume that $n = 2^m$ and intend to describe in detail the members of $\mathcal{M}(G, B)$. We first examine the basic action of the 2-minimal subgroups of G.

Proposition 10.1. *Suppose that $G = \mathrm{GL}^\epsilon_{2^m}(q)$ with $m > 1$.*

(i) *If $q \equiv \epsilon \pmod 4$, then*

$$\mathcal{M}(G, B) = \mathcal{M}(\mathrm{GL}^\epsilon_1(q) \wr \mathrm{Sym}(2^m), B) \cup \mathcal{M}(\mathrm{GL}^\epsilon_2(q) \wr T_{m-1}, B).$$

(ii) *If $q \equiv -\epsilon \pmod 4$, then*

$$\mathcal{M}(G,B) = \mathcal{M}(J^\epsilon_2 \wr \mathrm{Sym}(2^{m-1}), B) \cup \mathcal{M}(\mathrm{GL}^\epsilon_2(q) \wr T_{m-1}), B) \cup \mathcal{M}(\mathrm{GL}^\epsilon_4(q) \wr T_{m-2}, B).$$

In particular, G is tame.

Proof. Define

$$\mathcal{M}_* = \mathcal{M}(\mathrm{GL}^\epsilon_1(q) \wr \mathrm{Sym}(2^m), B) \cup \mathcal{M}(\mathrm{GL}^\epsilon_2(q) \wr T_{m-1}, B)$$

if $q \equiv \epsilon \pmod 4$ and

$$\mathcal{M}_* = \mathcal{M}(\mathrm{GL}^\epsilon_2(q) \wr \mathrm{Sym}(2^{m-1}), B) \cup \mathcal{M}(\mathrm{GL}^\epsilon_4(q) \wr T_{m-2}, B)$$

if $q \equiv -\epsilon \pmod 4$. Note that by Lemmas 3.7, 5.2 and 5.3 we have

$$\mathcal{M}(\mathrm{GL}^\epsilon_2(q) \wr \mathrm{Sym}(2^{m-1}), B) = \mathcal{M}(J^\epsilon_2 \wr \mathrm{Sym}(2^{m-1}), B) \cup \mathcal{M}(\mathrm{GL}^\epsilon_2(q) \wr T_{m-1}, B).$$

Observe that the members of $\mathcal{M}_*$ are tame in their signified over-groups by Corollary 8.3 and Theorems 9.6 and 9.8.

We may assume that $m > 1$ when $q \equiv \epsilon \pmod 4$ and that $m > 2$ when $q \equiv -\epsilon \pmod 4$. Denote by $\mathcal{M}_j$ the set of 2-minimal subgroups of $\mathrm{GL}^\epsilon_{2^j}(q) \wr \mathrm{Sym}(2^{m-j})$ and note that $\mathcal{M}_1$ is non-empty if and only if $q \equiv \epsilon \pmod 4$. Then using Propositions 5.9 and 5.10, $\mathrm{GL}^\epsilon_{2^m}(q)$ is not 2-minimal so long as $m > 1$ when $q \equiv \epsilon \pmod 4$ and $m > 2$ when $q \equiv -\epsilon \pmod 4$, employing Propositions 5.9 and 5.10 again gives

$$\mathcal{M}(G) = \bigcup_{j=1}^{m-1} \mathcal{M}_j.$$

Suppose that the theorem is false. Then there exist a minimal $j \le m-1$ such $P \in \mathcal{M}_j$ but P is not in $\mathcal{M}_*$. Let $M = \mathrm{GL}^\epsilon_{2^j}(q) \wr \mathrm{Sym}(2^{m-j})$ and C be the base group of M. Lemma 3.7 implies that $P = N_P(S \cap C)$ or $P \in \mathcal{M}(CB, B)$. As $S \cap C$ contains A as described before Lemma 5.3 we can apply Lemma 5.3 when $P = N_P(S \cap C)$ to get $P \le N_G(A)$ and consequently $P \in \mathcal{M}_1$ if $q \equiv \epsilon \pmod 4$ and $P \in \mathcal{M}_2$ if $q \equiv -\epsilon \pmod 4$, which is against the choice of P. Hence $PC = BC = SC$ as $Z(G) \le C$. In particular we have $P \le \mathrm{GL}^\epsilon_{2^j}(q) \wr T_{m-j}$. Thus $j > 1$ if $q \equiv \epsilon \pmod 4$ and $j > 2$ if $j \equiv -\epsilon \pmod 4$. We now intend to apply Lemma 3.17, so write $C = K_1 \times \cdots \times K_{2^{m-j}}$ where $K_l \cong \mathrm{GL}^\epsilon_{2^j}(q)$, $1 \le l \le 2^{m-j}$. Proceeding by induction we may assume $G = \mathrm{GL}^\epsilon_{2^j}(q)$ is tame and $\pi_1(N_C(S)) = \pi_1((S \cap C)Z(G)) = N_{K_1}(S \cap K_1)$, hence Lemma 3.17 and induction shows that there exists P_0 with $P = \langle P_0, B\rangle$ where $P_0 \in \mathcal{M}(\mathrm{GL}^\epsilon_1(q) \wr \mathrm{Sym}(2^j), B) \cup \mathcal{M}(\mathrm{GL}^\epsilon_2(q) \wr T_{j-1}, B)$ when $q \equiv \epsilon \pmod 4$ and $P_0 \in \mathcal{M}(G) = \mathcal{M}(\mathrm{GL}^\epsilon_2(q) \wr \mathrm{Sym}(2^{j-1}), B) \cup \mathcal{M}(\mathrm{GL}^\epsilon_4(q) \wr T_{j-2}, B)$ when $q \equiv -\epsilon \pmod 4$. But then $P \in \mathcal{M}_*$ and we have a contradiction. Consequently $\mathcal{M}(G, B) = M^*$ so proving the proposition.

□

11. Proof of main theorem

We first recollect the 2-minimal toral subgroups

$$\mathcal{T} = \mathcal{T}(G,B) = \{T(n_j; s^c; k) \mid j \in I, s^c \in \Pi(q-\epsilon) \text{ and } 1 \leq k \leq n_j\}$$

when $q \equiv \epsilon \pmod 4$ and

$$\mathcal{T} = \mathcal{T}(G,B) = \{T(n_i; s^c; j), T(n_i; t^d; n_i) \mid i \in I, 1 \leq j < n_i, s^c \in \Pi(q-\epsilon), t^d \in \Pi(q+\epsilon)\}$$

when $q \equiv -\epsilon \pmod 4$.

The 2-minimal linkers and fusers also vary according to the congruence of q so we have

$$\mathcal{F} = \mathcal{F}(G,B) = \mathcal{F}(H,B) = \{\langle B, P\rangle \mid P = P(n_i + n_j) \in \mathcal{F}(\mathrm{Sym}(n), T)\}$$

when $q \equiv \epsilon \pmod 4$ and

$$\mathcal{F} = \mathcal{F}(G,B) = \mathcal{F}(H,B) = \{\langle B, P\rangle \mid P = P(n_i + n_j) \in \mathcal{F}(\mathrm{Sym}(\lfloor n/2\rfloor), T)\}$$

when $q \equiv -\epsilon \pmod 4$. Similarly

$$\mathcal{L} = \mathcal{L}(G,B) = \mathcal{L}(H,B) = \{BP \mid P = P(n_i; n_j) \in \mathcal{L}(\mathrm{Sym}(n), T)\}$$

when $q \equiv \epsilon \pmod 4$ and

$$\mathcal{L} = \mathcal{L}(G,B) = \mathcal{L}(H,B) = \{BP \mid P = P(n_i; n_j) \in \mathcal{L}(\mathrm{Sym}(\lfloor n/2\rfloor), T)\}$$

when $q \equiv -\epsilon \pmod 4$.

The quaternion 2-minimal subgroups $Q(m)$ defined so far only in $\mathrm{GL}^{\epsilon}_{2^m}(q)$ (see just after Corollary 9.5) extend to 2-minimal subgroups

$$Q(n_i) \times \prod_{k \in I\setminus\{i\}} B_{n_k}$$

of $\mathrm{GL}^{\epsilon}_n(q)$. We abuse notation and also denote this 2-minimal subgroup of $\mathrm{GL}^{\epsilon}_n(q)$ by $Q(n_i)$. The set of quaternion 2-minimal subgroups is

$$\mathcal{Q} = \mathcal{Q}(G,B) = \{Q(n_i) \mid i \in I\}.$$

We recollect that this set is non-empty precisely when $q \equiv 3, 5 \pmod 8$. Similarly we have special linear 2-minimal subgroups

$$S(2, n_i) \times \prod_{k \neq i} B_{n_k}$$

for $q \equiv 1, 7 \pmod 8$ or $q = 5^a$ with a odd and

$$S(4, n_i) \times \prod_{k \neq i} B_{n_k}$$

for $q \equiv -\epsilon \pmod 4$ of $\mathrm{GL}^{\epsilon}_n(q)$. We again abuse notation and denote these subgroups by $S(2, n_i)$ and $S(4, n_i)$ respectively. Put

$$\mathcal{S} = \mathcal{S}(G,B) = \{S(2,n_i), S(4,n_i) \mid i \in I\}.$$

When $\epsilon = +$ we have radical 2-minimal subgroups

$$R(n_i \ggg n_j)$$

and the set of radical 2-minimal subgroups is

$$\mathcal{R} = \mathcal{R}(G,B) = \{R(n_i \ggg n_j) \mid \{i,j\} \subseteq I, i \neq j\}.$$

subgroups	conditions	number
$T(n_i; s^c; j)$	$i \in I$, $s^c \in \Pi(q-\epsilon)$, $1 \leq j \leq n_i$	$\lvert\Pi(q-\epsilon)\rvert \sum_{i\in I} n_i$
$P(n_i + n_j)$	$\{i, j\} \subseteq I$, $i \neq j$	$r(r-1)/2$
$P(n_i; n_j)$	$i \in I$, $n_i \geq 2$	$\sum_{i\in I, n_i\geq 2}(n_i - 1)$
$R(n_i \ggg n_j)$	$\{i, j\} \subseteq I$, $i \neq j$, $\epsilon = +$	$r(r-1)$
$Q(n_i)$	$i \in I$, $n_i \geq 2$, $q \equiv 1, 7 \pmod 8$	$\begin{cases} r-1 & n \text{ odd} \\ r & n \text{ even} \end{cases}$
$S(2, n_i)$	$i \in I$, $n_i \geq 2$, $q \equiv 3, 5 \pmod 8$ or 5^a a odd	$\begin{cases} r-1 & n \text{ odd} \\ r & n \text{ even} \end{cases}$

TABLE 1. The 2-minimal subgroup of $\mathrm{GL}_n^\epsilon(q)$, $q \equiv \epsilon \pmod 4$

When $\epsilon = -$, n is odd and $q \equiv 1 \pmod 4$, the counterparts of the 2-minimal radical subgroups are the 2-minimal unitary subgroups

$$U(n_j) = \mathrm{GU}_{2^{n_j}+1}(q) \times \prod_{k \notin \{j,r\}} B_{n_k}$$

where $j \in I \setminus \{r\}$ and the set of these subgroups is

$$\mathcal{U} = \mathcal{U}(G, B) = \{U(n_j) \mid 1 \leq j \leq r-1\}.$$

Theorem 11.1. *For $G = \mathrm{GL}_n^\epsilon(q)$,*

$$\mathcal{M}(G, B) = \mathcal{T} \cup \mathcal{F} \cup \mathcal{L} \cup \mathcal{Q} \cup \mathcal{S} \cup \mathcal{R} \cup \mathcal{U}.$$

Proof. We proceed by induction on n noting that the result is true for $n = 1$ and $n = 2$. Suppose that $P \in \mathcal{M}(G, B)$. Then by Theorem 6.1, either $P = G \in \mathcal{U}(G, B)$ or $r = 1$, $P\mathcal{M}(N_G(A), B)$, $\epsilon = +$ and $P = O_p(P)B$ or $P \in \mathcal{M}(\mathrm{GL}^\epsilon(U) \times \mathrm{GL}^\epsilon(W), B)$ for some non-zero subspaces U and W of V. If $r = 1$, Proposition 10.1 together with Theorems 9.6 and 9.8 show that either $P \in \mathcal{S}(G, B)$, $\mathcal{Q}(G, B)$ or $\mathcal{M}(N_G(A), B)$. If indeed $P \in \mathcal{M}(N_G(A), B)$, Theorems 8.1 and 8.2 indicate that $P \in \mathcal{T}(G, B) \cup \mathcal{F}(G, B) \cup \mathcal{L}(G, B)$. So we may suppose that $P \in \mathcal{M}(\mathrm{GL}^\epsilon(U) \times \mathrm{GL}^\epsilon(W), B)$ for some non-zero subspaces U and W of V. Let $K = \mathrm{GL}^\epsilon(U)$ and $L = \mathrm{GL}^\epsilon(W)$. Then by Lemma 3.9 either $P \cap K \in \mathcal{M}(K, B \cap K)$ or $P \cap L \in \mathcal{M}(L, B \cap L)$. The proof is now completed by using induction. $\square$

subgroups	conditions	number
$T(n_i; s^c; j)$	$i \in I$, $n_i \geq 1$ $s^c \in \Pi(q-\epsilon)$ $1 \leq j \leq n_i - 1$	$\lvert\Pi(q-\epsilon)\rvert \sum_{i \in I}(n_i - 1)$
$T(n_i; s^c; n_i)$	$i \in I$, $s^c \in \Pi(q+\epsilon)$,	$r\lvert\Pi(q+\epsilon)\rvert$
$P(n_i + n_j)$	$\{i, j\} \subseteq I$, $i \neq j$	$r(r-1)/2$
$P(n_i; n_j)$	$i \in I$, $n_i \geq 2$	$\sum_{i \in I, n_i \geq 2}(n_i - 1)$
$R(n_i \ggg n_j)$	$\{i, j\} \subseteq I$, $i \neq j$, $\epsilon = +$	$r(r-1)$
$U(n_i)$	$i \in I \setminus \{r\}$, $n_r = 0$, $\epsilon = -$	$r - 1$
$Q(n_i)$	$i \in I$, $n_i \geq 1$, $q \equiv 1, 7 \pmod 8$	$\begin{cases} r-1 & n \text{ odd} \\ r & n \text{ even} \end{cases}$
$S(2, n_i)$	$i \in I$, $n_i \geq 1$, $q \equiv 3, 5 \pmod 8$ or 5^a a odd	$\begin{cases} r-1 & n \text{ odd} \\ r & n \text{ even} \end{cases}$
$S(4, n_i)$	$i \in I$, $n_i \geq 2$	$\begin{cases} r & n \equiv 0 \pmod 4 \\ r-1 & n \equiv 2, 3 \pmod 4 \\ r-2 & n \equiv 1 \pmod 4 \end{cases}$

TABLE 2. The 2-minimal subgroup of $\mathrm{GL}_n^\epsilon(q)$, $q \equiv -\epsilon \pmod 4$

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