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To cite this article: Paul Schultz *et al* 2017 *New J. Phys.* **19** 023005

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IOP Publishing New J. Phys. 19 (2017) 023005 https://doi.org/10.1088/1367-2630/aa5a7b

New Journal of Physics
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Deutsche Physikalische Gesellschaft DPG
IOP Institute of Physics

Published in partnership with: Deutsche Physikalische Gesellschaft and the Institute of Physics

CrossMark

OPEN ACCESS

RECEIVED
5 August 2016

REVISED
18 November 2016

ACCEPTED FOR PUBLICATION
19 January 2017

PUBLISHED
2 February 2017

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PAPER

Potentials and limits to basin stability estimation

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Keywords: attractor, basin stability, fractal basin boundaries, riddled basins, intermingled basins

Abstract

Stability assessment methods for dynamical systems have recently been complemented by basin stability and derived measures, i.e. probabilistic statements whether systems remain in a basin of attraction given a distribution of perturbations. Their application requires numerical estimation via Monte Carlo sampling and integration of differential equations. Here, we analyse the applicability of basin stability to systems with basin geometries that are challenging for this numerical method, having fractal basin boundaries and riddled or intermingled basins of attraction. We find that numerical basin stability estimation is still meaningful for fractal boundaries but reaches its limits for riddled basins with holes.

1. Introduction

Going back to the path-breaking ideas of Aleksandr M Lyapunov, dynamical systems are said to be stable if small variations of the initial conditions lead to small reactions of a system, i.e. small perturbations cannot substantially alter the system's time-asymptotic behaviour. This is commonly a statement about the asymptotic behaviour, allowing for large transient deviations if only the system eventually returns to the initial configuration. *Multistable* systems with several attractors add another subtlety to the problem: perturbations may lead to switching from one attractor to another, substantially altering asymptotic behaviour [37]. While infinitesimal perturbations on an attractor have local effects well-studied in the theory of asymptotic stability, finite (including large) perturbations can be critical by causing non-local effects like the transition to another attractor.

A direct method for assessing stability against large perturbations are Lyapunov functions [14, 27, 28], which decrease along trajectories and have local minima on attractors. There are recent approaches to determine them numerically e.g. from radial basis functions [7] or sum of squares decomposition (see e.g. [6] for a comparison of different methods). From an analytic perspective, the method of nonequlibrium potentials [9, 10] determines a special case of Lyapunov functions, namely potentials. This has the additional benefit of yielding transition probabilities between attractors, which is not possible for Lyapunov functions in general. It has further been shown, that nonequilibrium potentials can be constructed for systems with fractal basin boundaries [8, 15]. However, direct methods have in common that Lyapunov functions are typically difficult to find, especially in high dimensions. Furthermore, they return lower bounds on an attractor's basin of attraction, hence it is not possible in general to determine whether a basin shrinks or grows with a parameter change.

Here, we put to test a recent alternative approach to consider non-local perturbations termed *basin stability* $S_{\mathcal{B}}$. The central idea [31, 32] is to use a kind of volume of the basin of attraction to quantify the stability of attractors in multistable systems subject to a given distribution of perturbations. An advantage of basin stability is that it can be efficiently estimated even in high-dimensional systems and has an intuitive interpretation as a probability to return to an attractor, but it relies on the correct identification of the asymptotic behaviour for a Monte Carlo sample of initial conditions. Basin stability and derived concepts have been successfully applied

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recently [39], e.g. for power grids [19, 31, 40, 41], chimera states [29], explosive synchronisation [48] delayed dynamics [25] and resilience measures [34].

The idea of estimating a basin of attraction's area already appears in earlier work, e.g. on the erosion of basin boundaries under parameter variation [38, 43] termed global integrity measure G_τ. It is defined as the fraction from a regular grid of initial conditions that *don't approach* a neighbourhood of an attractor within a given time τ [43]. Hence, the estimate of $S_\mathcal{B}$ ($\hat{S}_\mathcal{B}$) and G_τ are linked by $G_\infty = 1 - \hat{S}_\mathcal{B}$ and our following discussion on numerical estimation uncertainty also applies to the use case of basin integrity studies.

In numerical simulations, it can be difficult to correctly identify the asymptotic behaviour and determine the attractors. The basin of attraction can practically be defined as the set of all initial conditions whose trajectories enter and stay in some trapping region [35]. Problems may arise if transients are long and chaotic or trajectories stay close to basin boundaries for long, so that numerical errors can move the simulated trajectory across a boundary into a wrong basin and make the simulation converge to a wrong attractor. Principally, three aspects contribute to the overall estimation error: the standard error due to sampling initial conditions, approximation errors in function evaluations or integration of differential equations, and rounding errors due to limited precision. While sampling and approximation errors are controlled by increasing the sample size and the order of approximating polynomials as well as by decreasing step size, rounding errors are typically hard to reduce, which becomes a problem if they are of the same order of magnitude as the other error types. Stochastic systems are additionally subject to the contribution of noise, with intricate effects on the evaluation of trajectories, especially in fractal phase space geometries [22, 44].

Our study thus focuses on the critical case of systems where rounding errors cannot be neglected and may even dominate the overall error due to an intricate state space geometry highly sensitive to numerical imprecision. We put basin stability estimation here to the test by applying it to systems with fractal basin boundaries and riddled or intermingled basins of attraction.

2. Methods

2.1. Basin stability

Consider a system of ordinary differential equations

$$\dot{x} = F(x, t) \tag{1}$$

that has more than one attractor in its state space X. Here, we define an *attractor* as a minimal compact invariant set $A \subseteq X$ whose basin of attraction has positive Lebesgue measure [33]. The basin of attraction of A is the set $B(A) \subseteq X$ of all states from which the system converges to A. Note that in this definition by Milnor, we do not require A to posess an attracting neighbourhood, i.e. to be asymptotically stable. In this sense, also unstable attractors [46] that are separated from their basins of attraction fall into this definition.

Assume the system moves on an attractor A, yet at $t = 0$ a random and not necessarily small perturbation pushes the system to a state $x(0)$ outside A. Assume that $x(0)$ is drawn from a probability distribution with measure μ on X that encodes our knowledge about the frequency of relevant perturbations. E.g., μ may be a uniform distribution on some bounded region $R \supset A$. Note that μ is generally not invariant under a change to the coordinate system.

Will the system converge back to A after the perturbation? To address this, the recent concept of *basin stability* [31, 32] computes the probability measure of B,

$$\begin{aligned} S_\mathcal{B}(A) &:= \mu(B(A)) \\ &= \int_R \mathbf{1}_{B(A)}\, \mathrm{d}\mu \ \in [0, 1], \end{aligned} \tag{2}$$

i.e., the probability that the system will return to A. The indicator function $\mathbf{1}_{B(A)}(x)$ yields 1 if $x \in B(A)$ and 0 otherwise. We use $S_\mathcal{B}(A)$ to quantify just *how* stable the attractor A is against non-infinitesimal perturbations.

The estimation of volume integrals such as equation (2) in high dimensions is a well-known problem, and we assume this is done by simple Monte Carlo sampling [5, 47]. If for each initial state $x(0)$, one can numerically integrate the system $x(t)$ with sufficient precision to decide to which attractor it converges (or whether it diverges), the $S_\mathcal{B}$ estimation procedure is thus:

(i) Draw a sample of $N > 0$ independent initial states from the distribution μ.

(ii) For each, numerically integrate the system until it is clear whether and where it converges.

(iii) Count the number M of times the system has converged to A.

(iv) Use the estimate $\hat{S}_\mathcal{B} = \frac{M}{N}$.

Since this is an N-times repeated Bernoulli experiment with success probability $S_{\mathcal{B}}$, the absolute standard error of the estimate $\hat{S}_{\mathcal{B}}$ due to sampling is $\sqrt{S_{\mathcal{B}}(A)(1-S_{\mathcal{B}}(A))/N}$, independently of the system's dimension. Thus, the procedure can be applied to high-dimensional systems without necessarily increasing the sample size N, although it may take longer to assess convergence. This is of course since we are not interested in the basin of attraction's *geometry* but only in its *volume* w.r.t. the measure μ.

Note that when the relative std. err. of $\hat{S}_{\mathcal{B}}$ is more relevant than the absolute std. err., smaller values of $S_{\mathcal{B}}(A)$ require larger sample sizes, of the order $N \sim 1/S_{\mathcal{B}}(A)$, since for small $S_{\mathcal{B}}(A)$, the rel. std. err. is $\sim 1/\sqrt{NS_{\mathcal{B}}(A)}$. The divergence of the sample size for very small probabilities to be estimated (i.e. rare attractors with a small basin of attraction and SB (A)<<1) is a common problem where (simple) Monte Carlo methods are likely to fail [5]. However, even if $S_{\mathcal{B}}(A)$ is not small, the geometries of the multiple basins of attraction may still make the estimation of $S_{\mathcal{B}}$ difficult for another reason: for some initial conditions $x(0)$ it may be quite difficult to decide where $x(t)$ converges to, since the trajectory may start or come quite close to the boundary between the different basins. Consequently, approximation and rounding errors (rather than sampling errors) in the integration may become relevant and may make the simulated trajectory hop across a basin border, leading to a wrong assessment of where $x(t)$ actually converges to.

2.2. Challenging types of basins

Particularly, a correct convergence assessment becomes difficult if the basins have *fractal* boundaries, influencing the predictability of a system's behaviour in the long run [13, 30, 35]. Imagine we randomly draw initial states from a box through which the boundary between the basins of two attractors runs. Suppose each initial state is specified up to a certain numerical error ε. Then for an initial state that is closer to the boundary than ε, it is uncertain to which of the two attractors the system will converge. Denote by $f(\varepsilon)$ the fraction of initial states for which the outcome is uncertain subject to an initial error ε, i.e. the *uncertainty fraction* [11, 30]. If the boundary is a smooth curve, then these states are all located inside a strip of width 2ε along this curve, and $f(\varepsilon)$ is just proportional to ε. However, if the boundary is fractal, then $f(\varepsilon) \propto \varepsilon^{\alpha}$. If $\alpha < 1$, the system exhibits *final state sensitivity*, i.e., to decrease the uncertainty one needs a substantial improvement in the knowledge of initial conditions. In a way, this power law scaling leads to an *obstruction of predictability* [11] very similar to the sensitive dependence on initial conditions in chaotic systems.

Predicting the long-term behaviour—the essence of estimating $S_{\mathcal{B}}(A)$—of systems with fractal basin boundaries may be hard [30] although generally, for most initial conditions, the final state sensitivity is much smaller than the unpredictability of the actual trajectory.

Another extreme case are attractors whose basins are not open as for most systems [33] but rather have an empty interior. The complement of such a *riddled basin* intersects every disk in a set of positive measure [1, 21, 23, 36]. This means that *all* points in its basin of attraction have pieces of *another* attractor basin arbitrarily closely nearby [36]. Physical systems exhibiting riddled basins are the damped, periodically-driven particle moving in a special potential landscape [45] or coupled time-delayed systems [2, 4, 16]. There are also experimental observations for laser-cooled ions in a Paul trap [42] indicating a riddled phase space structure. It has been shown that riddled basins of attraction can also be induced by the addition of noise to the dynamics [22]. For an extensive discussion on riddled/intermingled basins of attraction and fractal basin boundaries, a review appeared in [24].

To investigate the behaviour of the estimation procedure, we study two quite different exemplary systems, the continuous-time Wada pendulum with fractal basin boundaries and the discrete-time quadratic map on the complex plane with riddled/intermingled basins of attraction.

3. Results

Let us first investigate how fractal basin boundaries impact the accuracy of $\hat{S}_{\mathcal{B}}$ by studying the *Wada pendulum* [3, 12]. Consider a damped, driven pendulum that is subject to a time-dependent forcing:

$$\begin{aligned} \dot{\phi} &= \omega, \\ \dot{\omega} &= X \cos t - \alpha\omega - K \sin\phi. \end{aligned} \tag{3}$$

For $\alpha = 0.1$, $K = 1$ and $X = 7/4$, this system has several attractors [18]. The four dominant of them, all limit cycles with period 2π, are shown in figure 1(a): the black and red attractors correspond to rotations of the pendulum, and the orange and yellow attractors are librations. Their respective basins of attraction at $t = 0$ are shown in figure 1(b). Certain regions in this figure appear sprinkled with dots belonging to the different basins, i.e. the boundary between the basins is not easily discernible and remains so when zooming in (figure 1(c)). It is a fractal, resulting in this case from the so-called Wada property of the basins.

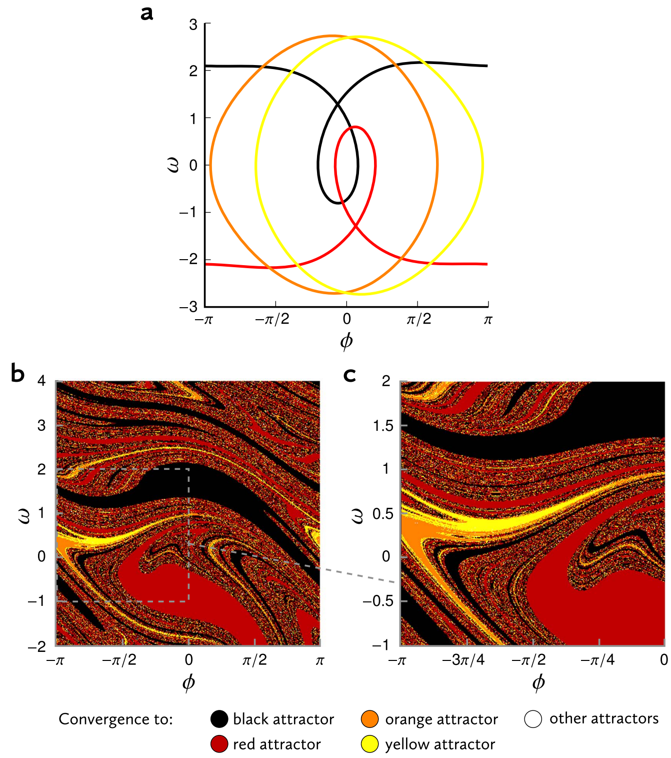

Figure 1. Damped pendulum with fractal basin boundaries. Damped pendulum with fractal basin boundaries. (a) Attractors of the damped pendulum with time-dependent forcing from equation (3). (b) State space of the pendulum at $t = 0$. Black/red/orange/yellow colouring indicates convergence to the black/red/orange/yellow attractor. Convergence to other attractors is indicated by white colouring. (c) Detail of dashed square from (b).

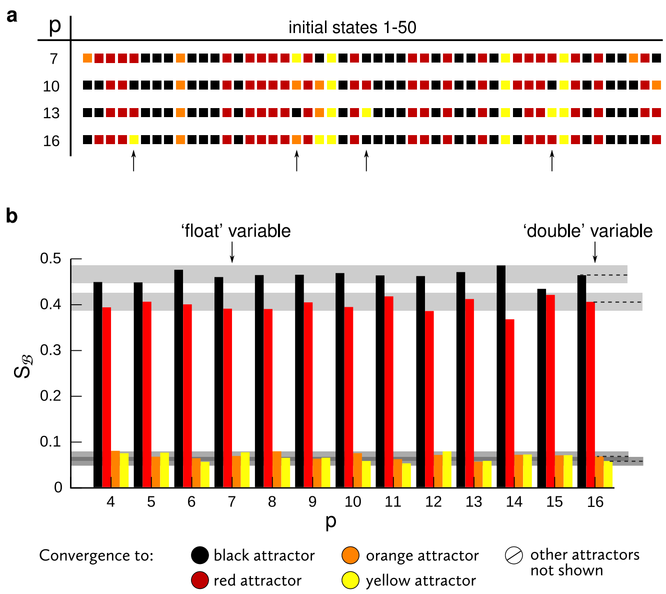

Figure 2. Basin stability in the pendulum with fractal basin boundaries. (a) Numerical integrations for a fixed set of fifty initial states at different values of the numerical precision p. The squares in each column correspond to the same initial state, and their respective colours indicate which state the system converges to from there at given precision p. Black/red/orange/yellow colouring indicates convergence to the black/red/orange/yellow attractor. The arrows highlight a selection of initial conditions for which $\hat{S}_{\mathcal{B}}$ is rather uncertain. (b) Estimated basin stability $\hat{S}_{\mathcal{B}}$ of the four attractors at different levels of p using $N = 1000$. The basin stability of the black/red/orange/yellow attractor is shown by the height of the black/red/orange/yellow bar. The grey shadows indicate the standard error of $\hat{S}_{\mathcal{B}}(p = 16)$.

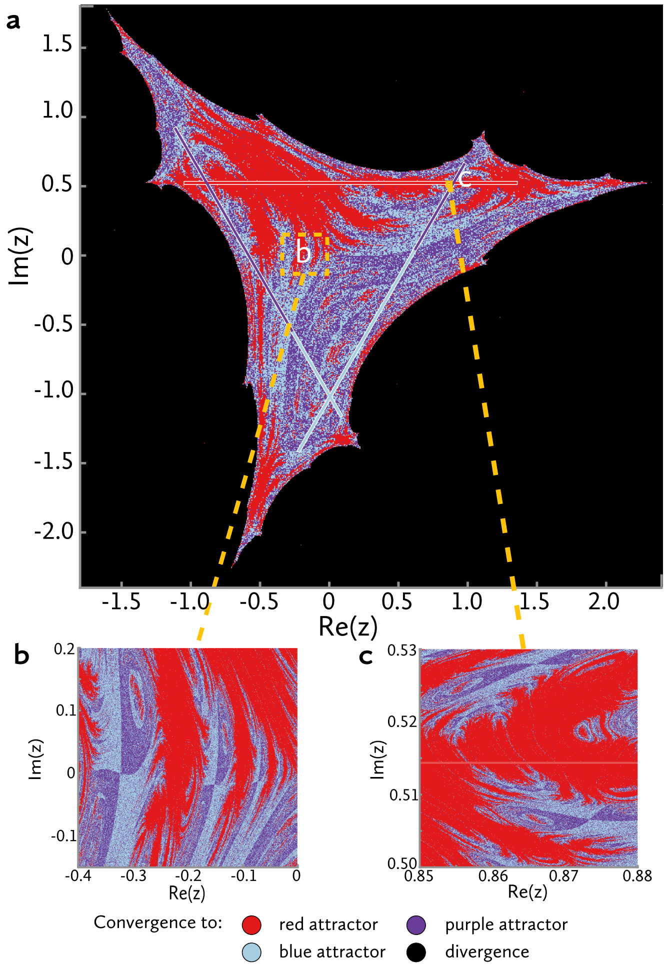

Figure 3. Intermingled basins of the quadratic map. (a) Phase space portrait of the three attractors (red/blue/purple line segments) of the map (equation (4)) with their intermingled basins of attraction coloured alike. The black area corresponds to initial conditions for which the dynamics diverge. Below are zoom-ins of two regions, (b) and (c). The locations of the attractors (line segments, see [1] are highlighted by red/blue/purple bars (not in scale).

Three (or more) subsets of a space are said to have the *Wada property* if any point on the boundary of one subset is also on the boundary of the two others [18, 35]. For the pendulum, the black basin, the red basin and the union of the orange and yellow basins have the Wada property [18, 35]. This means that starting within the rounding error ε of the boundary, a trajectory could in principle converge to *any of the four* attractors.

To verify this empirically, we write $\varepsilon = 10^{-p}$ with p denoting *precision*, and discard all information after the pth significant decimal digit in the floating point variables used in all individual operations of the numerical integration. We use 64 bit double precision to allow for a maximum of $p = 16$, while using untruncated 32 bit single precision would correspond to $p \approx 7$. For different values of p, we integrate a fixed set of 50 initial states $x(0)$, drawn uniformly at random from the rectangle $R = [-\pi, \pi] \times [-2, 4]$. The integration stops when a trajectory is close to an attractor within the given precision.

Figure 2(a), reveals that some initial states, particularly those indicated by arrows, indeed lead to different outcomes for different values of p. To investigate how $\hat{S}_{\mathcal{B}}$ depends on p, we let μ be the uniform distribution on R yielding a sample of $N = 1000$ random initial states which are integrated with different precision p, leading to estimates $\hat{S}_{\mathcal{B}}(p)$. As depicted in figure 2(b), there seems to be no systematic influence of p on $\hat{S}_{\mathcal{B}}(p)$. Indeed, most of the individual values of $\hat{S}_{\mathcal{B}}(p)$ are within one standard error of the most precise value $\hat{S}_{\mathcal{B}}(16)$. This suggests that, in contrast to long-term prediction for *individual* initial states (see figure 2(a)), $\hat{S}_{\mathcal{B}}$ is robust under variation of p.

In the following, we investigate the impact of riddled basins of attraction on $\hat{S}_{\mathcal{B}}$ using a conceptual example [1, 26], i.e. the following quadratic map on the complex plane:

$$F_\lambda(z) = z^2 - (1 + \lambda \mathrm{i})\bar{z}. \tag{4}$$

Following the treatment in [1], we study the map for $\lambda = 1.02871376822$. This map has three different attractors on the complex plane which are shown in figure 3; for simplicity they are referred to as the red/blue/

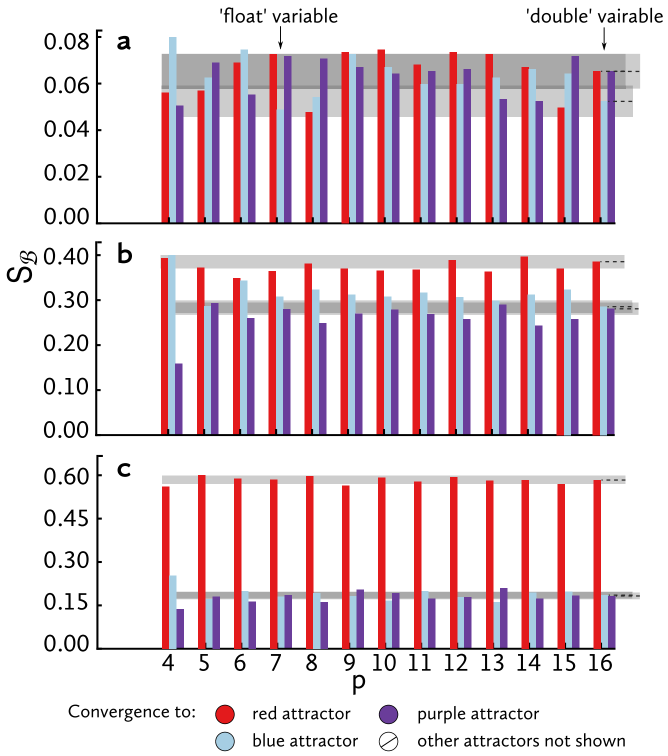

Figure 4. Basin stability estimation for the quadratic map. (a) $\hat{S}_{\mathcal{B}}$ of the the red/blue/purple attractors at different levels of p, using $R = [-1.8,\ 2.4] \times [-2.4,\ 1.8]$. (b) $\hat{S}_{\mathcal{B}}$ with R corresponding to figure 3 inset (1), (c) $\hat{S}_{\mathcal{B}}$ with R corresponding to figure 3 inset (2). The basin stability is shown by the height of the red/blue/purple bar, the grey shadows indicate the standard error of $\hat{S}_{\mathcal{B}}(16)$.

purple attractors with their respective basin of attraction in the following. Interestingly, the three basins of attraction are not just riddled, they are *intermingled*. A basin of attraction is called intermingled if *any* open set which intersects one basin in a set of positive measure also intersects *each* of the other basins in a set of positive measure [17, 20].

The fact that there is a positive probability to end up in a different attractor around each initial condition inside a riddled/intermingled basin of attraction renders these systems effectively non-deterministic [45]. As in the case of Wada boundaries, slight variations of initial conditions or numerical imprecision will affect any forecast of the system's long-term behaviour.

Again, we investigate the effect of limited numerical precision on the significance of $\hat{S}_{\mathcal{B}}$. In figure 4(a) we depict the result of estimating $S_{\mathcal{B}}$ for varying p using $R = [-1.8,\ 2.4] \times [-2.4,\ 1.8]$, i.e. the region pictured in figure 3(a). We observe a large variation of $\hat{S}_{\mathcal{B}}$ of up to 50% compared to the most precise estimation $\hat{S}_{\mathcal{B}}(16)$ and no systematic dependence on p.

In figure 3(c) we zoomed into the neighbourhood of the red attractor, where the share of the corresponding red basin is increasing in proximity of the attractor. In particular, the measure of this basin of attraction, restricted to an ϵ-neighbourhood of the attractor, approaches unit probability for $\epsilon \to 0$ [1]. This apparent behaviour provides an explanation for figure 4(c) where we determined $\hat{S}_{\mathcal{B}}(p)$ for figure 3(c). In contrast to our previous observation, the fluctuations of $\hat{S}_{\mathcal{B}}(p)$ almost stay within one standard error and the estimation appears to be more robust. For reference, figure 4(b) depicts $\hat{S}_{\mathcal{B}}(p)$ for figure 3(b) not containing any (part of) an attractor. On the one hand, the variation of $\hat{S}_{\mathcal{B}}(p)$ exceeds one standard error, up to about 20% compared to $\hat{S}_{\mathcal{B}}(16)$, such that our estimation is more sensitive to numerical imprecision than in figure 4(b); on the other hand the variations are smaller than in our first experiment.

4. Discussion

We applied the Monte Carlo estimation procedure of basin stability in two cases, basins with fractal boundaries and riddled/intermingled basins of attraction. In the fractal boundaries case, we find that while the asymptotic properties of individual trajectories still cannot be determined robustly, the converse is true for the basin stability estimation. It remains an open question for future research, how exactly (in a quantitative sense) the numerical estimation uncertainty might be derived from the actual basin geometry. In the riddled/intermingled case, however, we find that the results can vary drastically with the chosen precision. The effect of rounding errors is comparable or even larger than the standard error of the sampling. Only if the sample region R is chosen in some sense 'close enough' to the actual attractor of interest, the foliated structure of the surrounding basins allows for a meaningful numerical estimation.

While we here study the effect of small numerical errors on the asymptotics in deterministic systems, a somewhat complementary work [44] considers the effect of noise on transient properties, i.e. the escape probability from a constrained region. As this example shows, it is an interesting aspect for further research to combine these approaches and study the additional effect of noise on final state determination.

In our two prototypical examples, the phase space dimension is low, i.e. two and three, while basin stability estimation is especially advantegous in high-dimensional systems compared to complementary methods (e.g. Lyapunov functions). Given that the phase space dimension does not affect the standard error of the estimation process, we have no reason to assume a different behaviour between low and high dimensions. Conversely, we expect that estimation problems inherent to our examples and strategies to cope with them equally apply to high-dimensional systems and are important to be considered in future research.

5. Conclusion

What are practical implications for the application of basin stability? In general, it is sufficient if the rounding error of an estimation is smaller than its sampling error to get a significant result. However, any numerical procedure is subject to a finite numerical precision and we have to assume that in practice it will not be high enough to reach this goal in dynamical systems with intricate basin geometries. If there is no prior knowledge available, a good starting point is to actually visualise the interesting part of the phase space to get a first idea of the appearance of, e.g., fractal sets. If any are detected, it is necessary to use the highest available numerical precision p_h to get $\hat{S}_{\mathcal{B}}(p_h)$, potentially avoiding artefacts respectively insignificant estimations. We suggest to repeat the $S_{\mathcal{B}}$ estimation at a lower numerical precision p_l and take the difference $\hat{e}_p = |\hat{S}_{\mathcal{B}}(p_h) - \hat{S}_{\mathcal{B}}(p_l)|$ as a straight-forward (rough) estimator of the variability of $\hat{S}_{\mathcal{B}}(p)$ with p and, by way of extrapolation, as a rough estimate of the remaining standard error of $\hat{S}_{\mathcal{B}}(p_h)$ as an estimate of $S_{\mathcal{B}}$ due to finite numerical precision. To assess the influence of rounding errors on $\hat{S}_{\mathcal{B}}$ then compare $\hat{e}_p$ with the standard error of $\hat{S}_{\mathcal{B}}(p_h)$ as an estimate of $S_{\mathcal{B}}(p_h)$ due to sampling, which can be estimated as $\hat{s}_p = \sqrt{\hat{S}_{\mathcal{B}}(p_h)(1 - \hat{S}_{\mathcal{B}}(p_h))/N}$. If $\hat{e}_p < \hat{s}_p$, rounding has no significant effect on the estimation quality. For instance, this could be implemented by comparing the results at double and single precision computations.

Acknowledgments

The authors gratefully acknowledge the support of BMBF, CoNDyNet, FK. 03SF0472A.

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