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The full Keller–Segel model is well-posed on fairly general domains

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Abstract

In this paper we prove the well-posedness of the full Keller-Segel system, a quasilinear strongly coupled reaction-crossdiffusion system, in the spirit that it always admits a unique local-in-time solution in an adequate function space, provided that the initial values are suitably regular. Apparently, there exists no comparable existence result for the full Keller-Segel system up to now. The proof is carried out for general source terms and is based on recent nontrivial elliptic and parabolic regularity results which hold true even on fairly general spatial domains, combined with an abstract solution theorem for nonlocal quasilinear equations by Amann.

Nous considérons le système de Keller et Segel dans son intégralité, un système quasilinéaire à réaction-diffusion fortement couplé. Le résultat principal montre que ce système est bien posé, c'est-à-dire il admet une solution unique existant localement en temps à valeurs dans un espace fonctionnel approprié, pourvu que les valeurs initiales sont réguliers. Apparemment, il n'existe pas encore des résultats comparables. Pour la démonstration, nous utilisons des résultats récents de régularité elliptique et parabolique applicable à des domaines assez généraux, combiné avec un théorème abstrait d'Amann concernant les équations quasilinéaires non locales.

1. Introduction

This paper establishes the local-in-time existence of solutions in a suitable functional-analytic sense to the so-called original *full Keller-Segel model* which is a coupled system of four nonlinear parabolic partial differential equations over a finite time horizon $J =]0, T[$ in a bounded domain Ω in space dimensions $d \in \{2, 3\}$, and reads as follows:

$$u' - \operatorname{div}(\kappa(u,v)\nabla u) = \operatorname{div}(\sigma(u,v)\nabla v) \qquad \text{in } J \times \Omega, \tag{1.1}$$

$$v' - k_v \Delta v = -r_1 v p + r_{-1} w + u f(v) \qquad \text{in } J \times \Omega, \tag{1.2}$$

$$p' - k_p \Delta p = -r_1 v p + (r_{-1} + r_2) w + u g(v,p) \qquad \text{in } J \times \Omega, \tag{1.3}$$

$$w' - k_w \Delta w = r_1 v p - (r_{-1} + r_2) w \qquad \text{in } J \times \Omega, \tag{1.4}$$

combined with homogeneous Neumann conditions

$$\nu \cdot \kappa(u,v)\nabla u = \nu \cdot k_v \nabla v = \nu \cdot k_p \nabla p = \nu \cdot k_w \nabla w = 0 \qquad \text{on } J \times \partial\Omega, \tag{1.5}$$

where ν denotes the outer unit normal to the boundary $\partial\Omega$, and suitable initial values

$$\big(u(0,\cdot), v(0,\cdot), p(0,\cdot), w(0,\cdot)\big) = (u_0, v_0, p_0, w_0) \qquad \text{in } \Omega. \tag{1.6}$$

This model describes the aggregation phase during the life cycle of cellular slime molds like the *Dictyostelium discoideum* and has first being introduced by Keller and Segel in their 1970ies paper "*Initiation of slime mold aggregation viewed as an instability*" [55]. We briefly describe the underlying biological processes. Looking at its life cycle one observes that a myxamoebae population of the Dictyostelium grows by cell division as long as there are enough food resources. When these are depleted, the myxamoebae propagate over the entire domain available to them. Then, after a while, the phase that is covered by the given model is initiated by one cell that starts to exude cyclic Adenosine Monophosphate (cAMP) which attracts the other myxamoebae. As a consequence the other myxamoebae are stimulated to move in direction of the so-called founder cell and commence to release cAMP. This leads to the aggregation of the myxamoebae that also start to differentiate within the myxamoebae aggregates resp. within the aggreagtion centers. The aggregation phase ends with the formation of a pseudoplasmoid in which every myxamoebae maintains its individual integrity. However, Keller and Segel did not model the formation of the pseudoplamoid; thus, this phase of the life cycle of the Dictyostelium is *not* covered in the original equations. This pseudoplasmoid is attracted by light and, therefore, it moves towards light sources. Finally a fruiting body is formed and after some time spores are diffused from which the life cycle begins again. For more details on the life cycle of the Dictyostelium we refer to [15], for example.

In the given model $u(t, x)$ denotes the myxamoebae density of the cellular slime molds at time t in point x, where $v(t, x)$ describes a chemoattractant concentration (like cAMP). The given model for aggregation of a cellular slime population is based on four basic processes that can be observed during the aggregation phase:

a) The chemoattractant is produced per amoeba at a positive rate $f(v)$.
b) The chemoattractant is degraded by an extracellular enzyme, where the concentration of the is enzyme at time t in point x is denoted by $p(t, x)$. This enzyme is produced by the myxamoebae at a positive rate $g(v, p)$ per amoeba.
c) Following Michaelis-Menten the chemoattractant and the enzyme react to form a complex $\mathcal{E}$ of concentration w which dissociates into a free enzyme plus the degraded product:

$$v + p \quad \underset{r_{-1}}{\overset{r_1}{\rightleftarrows}} \quad \mathcal{E} \quad \overset{r_2}{\longrightarrow} \quad p + \text{ degraded product},$$

where r_{-1}, r_1 and r_2 are positive constants representing the reaction rates.
d) The chemoattractant, the enzyme and the complex diffuse according to Fick's law.

As a tribute to the experimental setting and the conservation of the myxamobae density the equations are equipped with homogeneous Neumann boundary data.

Since the influence of chemical substances in the environment on the movement of motile species (in general called chemotaxis) can lead to strictly oriented or to partially oriented and partially tumbling movement of the species, the first equation contains both a pure diffusion term $\operatorname{div}(\kappa(u, v)\nabla u)$ with $\kappa(u, v) \geq 0$ for $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$ and a convection term $\operatorname{div}(\sigma(u, v)\nabla v)$ that describes the movement with respect to the chemical concentration. For a movement towards a higher concentration of the chemical substance, termed *positive* chemotaxis, one assumes $\sigma(u, v) < 0$ for $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$ while for the movement towards regions of lower chemical concentration, called *negative* chemotactical movement, the opposite inequality $\sigma(u, v) > 0$ for $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$ has to hold. For the detailed derivation of the given model we refer to [47, 55].

Chemotaxis is known to be an important device for cellular communication. In development or in living tissues the communication by chemical signals prearanges how cells collocate and organize themselves. Biologists studying chemotaxis often concentrate their experiments on the movement, the self-organisation and pattern formations of the cellular slime mold Dictyostelium discoideum. One reason for the great interest in this cellular slime mold is caused by the fact that "*development in Dictyostelium discoideum results only in two terminal cell types, but processes of morphogenesis and pattern formation occur as in many higher organisms*" (see [71, p. 354]). Thus biologists hope that studying this cellular slime mold gives more insights in understanding cell differentiation.

However, by to a simplification done by Keller and Segel themselves in [55] this original model of four strongly coupled parabolic equations was reduced to a model which is given by a system of only *two* strongly coupled parabolic equations. This was done by assuming that the complex is in a steady state with regard to the chemical reaction and that the total concentration of the free and the bounded enzyme is a constant, assumptions that are well-known for the Michaelis-Menten equations in enzyme kinetics. This reduction was justified by the paradigm "*it is useful for the sake of clarity to employ the simplest reasonable model*" (see [55, p. 403]). The corresponding model was then given by the following parabolic equations:

$$\left.\begin{aligned} u_t - \operatorname{div}\left(\kappa(u,v)\nabla u\right) &= \operatorname{div}\left(\sigma(u,v)\nabla v\right) && \text{in } J\times\Omega,\\ v_t - k_c\Delta v &= -k(v)v + uf(v) && \text{in } J\times\Omega,\\ \nu\cdot\kappa(u,v)\nabla u &= \nu\cdot k_c\nabla v = 0 && \text{on } J\times\partial\Omega,\\ \big(u(0,\cdot),v(0,\cdot)\big) &= (u_0,v_0) && \text{in } \Omega. \end{aligned}\right\} \tag{1.7}$$

This model is nowadays often referred to as the classical chemotaxis model or as the Keller-Segel model in chemotaxis. As in the full model, $\kappa(u,v)$ denotes the density dependent diffusion coefficient and $\sigma(u,v)$ is the chemotactic sensitivity, where now $k(v)v$ and $uf(v)$ describe degradation and production of the chemical signal. For $\kappa(u,v)=1$, $\sigma(u,v)=-\chi\cdot u$ or $-\chi\frac{u}{v}$ with a constant $\chi>0$ and $k(\cdot)$ and $f(\cdot)$ positive constants, this two equation model has been extensively studied during the last twenty years, see for instance [42, 43, 47, 48, 51] and the references therein. In particular the so-called Childress-Percus conjecture for (1.7) has attracted many scientists.

For the κ,σ,k and f as just stated Childress and Percus [17] suggested that in $d=2$ for suitable initial data the solution of (1.7) can blow up in either finite or infinite time, i.e., that there exists a time $T_{\max}$ with $0<T_{\max}\le\infty$ such that

$$\limsup_{t\nearrow T_{\max}} ||u(t,x)||_{L^\infty(\Omega)} = \infty \quad \text{or} \quad \limsup_{t\nearrow T_{\max}} ||v(t,x)||_{L^\infty(\Omega)} = \infty,$$

while they excluded this possibility for spatial dimension $d=1$. For $d\ge 3$ they suggested that under a perturbation of sufficiently high symmetry the solution of (1.7) has to blow up in either finite or infinite time. For $d=1$ the conjecture has been shown to be true by Osaki and Yagi [72]. For $d=2$ there is a huge number of papers that have studied this conjecture. To quote some we only mention [13, 32, 39, 44–46, 69], while the case $d=3$ has been considered for example in [16, 19, 50, 52, 86]. Childress' and Percus' conjecture was originally formulated for system (1.7). However, it has also been studied for the situation where the first equation of (1.7) is parabolic and the second is given by an elliptic equation by several authors, cf. [47] and the references therein.

From the biological point of view, the blow-up behaviour of the solution can be interpreted as the starting point of cell differentiation and therefore the blow-up time $T_{\max}<\infty$ would correspond to the stopping time where the aggregation phase in the life cylce of the Dictyostelium ends and the cell differentiation and formation of the pseudoplamoid starts.

Besides the mathematical interesting question whether the solution can blow up in finite or in infinite time one can also observe interesting pattern formations during the aggregation phase and development of the Dictyostelium such as traveling waves like motion and spiral waves for the chemoattractant (see for instance [48, 49, 84]). Although there have been some attempts to prove the existence of traveling wave solutions for the simplified model (1.7), one can find in general different equations to describe those pattern formations in literature. Hence, it might be worthwhile to remember the original four-equation-system instead if one tries to describe these pattern formations during the aggregation of some particular species. Possibly, the reduction to two equations that was done in [55] was too restrictive to cover all observable patterns and phenomena during the aggregation of mobile species like the Dictyostelium discoideum. For example, one can find an attempt to describe the aggreagtion of the Dictyostelium discoideum along the experimentally observable cAMP spiral waves in [83] where the authors look at a coupled three-equations model that contains a version of the simplified Keller-Segel model equipped with an ODE that should cover the recovery process of the myxamoebae after binding the extracellular cAMP. Thus it might be worth to look at the original model to see whether it can also generate these complex pattern formations.

As far as we know there are no results available for the full four equation model. In particular, the question of blow-up has not been studied for the full four equations model up to now. Of course, there are several local existence and well-posedness results known for parabolic-parabolic and parabolic-elliptic versions of the simplified two equation model (1.7) as for instance the results in [2, 13, 14, 42, 68, 79, 80, 89]. Furthermore, existence results for solutions for the simplified two-equation model with additional population growth are also known, cf. [54, 73, 81, 85, 88]. However, all these results consider the equation either on a smooth domain with boundary of class C^2, on convex domains with smooth boundaries or on the whole space $\mathbb{R}^d$. Furthermore, the inital data have to satisfy certain comparability conditions in some cases. The only result which we are aware of concerning nonsmooth objects is the local existence result in [32] where the authors allow a domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$ that is piecewise of class C^2. Therefore, the results stated in the present paper are completely new and much more general than those known so far. It will turn out that the analysis presented below immediately transfers to the more simple model (1.7).

Our analysis of the system (1.1)–(1.4) fundamentally bases on the fact that it is only *one* equation where the second derivative of another quantity appears. So we solve the equations (1.2)–(1.4) for (v, p, w), where u enters parametrically as a given function. It turns out that the dependence of (v, p, w) on u in this spirit is well-behaved in a suitable sense. This allows to insert (v, p, w) in their dependence of u into (1.1). Thus, one ends up with *one* "scalar" quasilinear parabolic equation whose dependence on u is *nonlocal in time*, since the functions v, p, w, as solutions to evolution equations themselves, depend on the *whole* function u on $]0, t[$ instead of just the value $u(t)$. Such an equation, however, can be solved by a pioneering theorem of Amann which covers such general settings, cf. [3, Thm. 2.1] or Theorem 3.18 below. Still, it is a formidable task to verify the assumptions of the theorem, since the equation under consideration is still quasilinear and nonlocal in nature.

Thereby it is not obvious a priori in which function spaces the problem should be considered, but some hints are given: since *homogeneous* Neumann conditions are prescribed, cf. (1.5), Lebesgue spaces $L^p(\Omega)$ are favorable, see Remark 3.3 below. Fortunately, there are various recent elliptic ([10, 26, 37]) and parabolic ([34, 41]) regularity results available, which are even valid in the case of non-smooth domains. The indeed crucial problem is the adequate choice of p. However, there is a fairly general class of domains Ω for which

$$-\nabla \cdot \mu \nabla + 1 \colon W^{1,q}(\Omega) \to \big(W^{1,q'}(\Omega)\big)' =: W^{-1,q}_{\bullet}(\Omega)$$

is a topological isomorphism for some $q > d$, where μ is a uniformly continuous, strictly positive function on Ω, cf. [26], [37]. Combining this isomorphism property with very recent and powerful results on the square root of elliptic operators as in [10, Thm. 5.1], see also Proposition 3.9 below, provides very precise embedding results for the domains of fractional powers of the elliptic operators on Lebesgue spaces. On the other hand, one can show that the domains of the operators $-\nabla \cdot \phi\mu\nabla$, when considered on $L^{q/2}(\Omega)$, are independent of ϕ, whenever ϕ is a strictly positive function from $W^{1,q}(\Omega)$, cf. e.g. [63]. Combining these results is crucial in the task of establishing *constant* domains for the operators entering in the quasilinear equation (1.1). This is a, or even *the*, central point in the theorem of Amann mentioned above, for which we indeed choose a Lebesgue space $L^p(\Omega)$ with $p = \frac{q}{2}$. Note that for the Keller-Segel model (1.1)– (1.6) one in fact only needs to consider $\mu \equiv 1$. We have included the general case for μ for the sake of transparency and to point out that the technique used is not necessarily restricted to the Laplacian, cf. our statements in Chapter 5 at the end of the paper.

Let us emphasize that the analysis of the system (1.1)–(1.4) may be adopted to both the simplified model (1.7) with virtually no changes, and also to the situation where the equations (1.2)–(1.4) for v, p, and w are *elliptic* only. In fact, the general setup and the way to proceed for the latter case would only change very little from the considerations below: one would solve the then elliptic equations for each $t \in J$ in dependence on $u(t)$ alone (topologically with regard to a certain interpolation space), then insert $(v(t), p(t), w(t))$ into the quasilinear equation (1.1) and finally solve this with the theorem of Prüss [77]. See [64] for a display of this technique where the elliptic equation is also quasilinear.

The outline of the paper is as follows: in the next chapter we will establish notations, general assumptions and definitions. In Ch. 3, we collect preliminary results, partly already established in other papers. In particular, the concept of maximal parabolic regularity is introduced – being fundamental for all what follows. The investigation of the model is carried out in Ch. 4, beginning with a precise formulation in Ch. 4.1. The main result, local in time existence and uniquenes for the Keller-Segel system, is formulated in Theorem 4.3. Its proof follows in Ch. 4.2. The paper finishes with concluding comments and remarks in Ch. 5.

2. Notations, general assumptions and definitions

The underlying spatial set Ω is always supposed to be a bounded Lipschitz domain in $\mathbb{R}^d$ for $d = 2$ or $d = 3$ in the sense of [35, Def. 1.2.1.2] or [62, Ch. 1.1.9]. The reader should carefully notice that this is different from a *strong Lipschitz domain*, which is more restrictive and in fact identical with a *uniform cone domain*, see again [35, Def. 1.2.1.1] or [62, Ch. 1.1.9].

Concerning function spaces, $W^{1,q}(\Omega)$ stands for the usual Sobolev space on Ω as a complex vector space (we will switch to real ones later). Accordingly, $W_{\bullet}^{-1,q}(\Omega)$ denotes the *anti*-dual of $W^{1,q'}(\Omega)$. Moreover, for $\theta \in]0,1[$ and $q \in]1,\infty[$, $H^{\theta,q}(\Omega)$ is the symbol for the space of Bessel potentials on Ω, cf. [82, Ch. 4.2.1]. The space of uniformly continuous functions on Ω is denoted by $C(\overline{\Omega})$. For an open set $\Lambda \subset \mathbb{R}^N$, where $N \in \{1,2,3\}$, and a Banach space X, we write $C^\alpha(\Lambda; X)$ for the usual X-valued Hölder spaces of order $\alpha \in]0,1[$, cf. [6, Ch. II.1.1.]. We will mostly encounter these in the incarnations $\Lambda = \Omega$ and $X = \mathbb{R}$ or Λ an interval in $\mathbb{R}$ and X a function space. Since we frequently work with triplets of functions, let $\mathbb{L}^p(\Omega)$ and $\mathbb{W}^{1,q}(\Omega)$ denote the spaces $(L^p(\Omega))^3$ and $(W^{1,q}(\Omega))^3$, respectively. The domain Ω under consideration will not change throughout this work, hence we usually omit the reference to Ω when working with the function spaces.

For two Banach spaces X and Y we denote the space of linear, bounded operators from X into Y by $\mathcal{L}(X;Y)$. If $X = Y$, then we abbreviate $\mathcal{L}(X)$. The norm in a Banach space X will be always indicated by $\|\cdot\|_X$. If a Banach space Y is contained in another Banach space X and the canonic injection of Y into X is continuous, then we say that Y is *embedded* into X and write $Y \hookrightarrow X$. Let Y embed into X. Then $\mathcal{E}(Y;X)$ denotes the *embedding constant*, i.e., the norm of the embedding map. Moreover, in the same situation, if B is the restriction of an operator $A\colon X \supseteq \mathrm{dom}(A) \to X$ to the space Y, then $\mathrm{dom}_Y(B)$ indicates the domain of this operator B in Y.

Finally, we use $J =]0,T[$ for $0 < T < \infty$, and the letter c denotes a generic constant, not always of the same value.

2.1. Assumptions and definitions

In order to allow for concise notation in the later stages of this work, we generalize the nonlinear growth, production and degradation terms on the right hand sides of (1.2)–(1.4) to general functions R_2, R_3, R_4, including a function R_1 for (1.1) which is not present in the above model but poses no problem to include analytically. Note that the differential operator for v in (1.1) will be treated specially. For the R_i and for the coefficient functions κ and σ, we make the following assumptions.

Assumption 2.1. i) The functions $\kappa, \sigma : \mathbb{R}^2 \to \mathbb{R}$ are supposed to be twice continuously differentiable throughout this paper. Moreover, κ takes only positive values.

ii) For $i = 1, \dots, 4$, each function R_i is defined on $\mathbb{R}^4$ and maps into $\mathbb{R}$, and is also assumed to be twice continuously differentiable.

We point out that we have to pose another assumption of completely different nature than the above ones concerning the regularity of the domain Ω, cf. Assumption 3.5 below. This assumption is only posed below to put it in the appropriate context.

Remark 2.2. In the sequel, the functions κ, σ are always readily identified with the induced superposition operators, acting from $C(\overline{\Omega}) \times C(\overline{\Omega})$ into $C(\overline{\Omega})$. The same is, *mutatis mutandis*, done for the functions R_1, R_2, R_3, R_4.

3. Preliminaries: Some operator theoretic results

In this chapter we declare suitable Banach spaces on which the Keller-Segel system will be considered and in which the analysis is carried out, and the corresponding differential operators. The initial point is the (classical) insight that, also for parabolic equations which include homogeneous Neumann conditions, Lebesgue spaces are the adequate function space choice to consider the equations in, cf. [60, Ch. 3.3]. Unfortunately, in view of the nonlinearities, the Hilbert space L^2 is *not* appropriate in general, cf. our comments in Chapter 5 below. It will become clear that L^p-spaces with suitably chosen $p < 2$ are the adequate ones. Thus, it is the aim of the following considerations to provide a consistent definition of the second order divergence operators on such L^p spaces and to show that these operators indeed possess suitable functional analytic properties, in particular, maximal parabolic regularity.

Definition 3.1. Assume that μ is a real-valued, measurable, bounded function on Ω. We define (as usual), for $q \in]1, \infty[$, the continuous operator

$$-\nabla \cdot \mu\nabla \colon W^{1,q} \to W_\bullet^{-1,q}$$

by

$$\langle -\nabla \cdot \mu\nabla v, w\rangle := \int_\Omega \mu\nabla v \cdot \nabla\overline{w}\,\mathrm{dx} \quad \text{for} \quad v \in W^{1,q}, w \in W^{1,q'}, \tag{3.1}$$

here $\langle\cdot,\cdot\rangle$ denoting the anti-dual pairing between $W_\bullet^{-1,q}$ and $W^{1,q'}$, which is in turn an extension of the L^2 scalar product.

Taking $\mu \equiv 1$ in Definition 3.1, one, of course, recovers the (negative) weak Laplacian, $-\nabla \cdot \nabla = -\Delta$.

Remark 3.2. In this context, it is not quite common to admit functions μ which take positive *and* negative values. Nevertheless, this is unavoidable by the properties of the function σ originating from the model, cf. the introduction, see also [32].

3.1. The restriction of $-\nabla \cdot \mu\nabla$ to L^p spaces

Let us in this section consider $-\nabla \cdot \mu\nabla$ as an operator mapping $W^{1,2}$ to $W_\bullet^{-1,2}$ and let $p \in]\frac{2d}{d+2}, 2[$. For these p we have the embedding $L^p \hookrightarrow W_\bullet^{-1,2}$ via taking the adjoint of the embedding $W^{1,2} \hookrightarrow L^{p'}$, and define the *restriction* $A_p(\mu)$ of $-\nabla \cdot \mu\nabla$ to the space L^p as follows: $\psi \in W^{1,2}$ belongs to $\mathrm{dom}_{L^p}(A_p(\mu))$ iff the (anti-) linear form

$$W^{1,2} \ni \varphi \mapsto \int_\Omega \mu\nabla\psi \cdot \nabla\overline{\varphi}\,\mathrm{dx} = \langle -\nabla \cdot \mu\nabla\psi, \varphi\rangle \tag{3.2}$$

is continuous if $W^{1,2}$ is only equipped with the weaker $L^{p'}$ topology, i.e., if there exists a constant $c = c(\psi)$ such that

$$|\langle -\nabla \cdot \mu\nabla\psi, \varphi\rangle| \le c(\psi)\|\varphi\|_{L^{p'}} \quad \text{for all } \varphi \in W^{1,2}.$$

In this case, the functional (3.2) may be extended by continuity from the dense subspace $W^{1,2}$ to whole $L^{p'}$ under preservation of its norm. We denote the representative of this functional on $L^{p'}$ by $\Psi \in L^p$ and define $A_p(\mu)\psi := \Psi$. Then $A_p(\mu)\psi$ satisfies

$$\int_\Omega \big(A_p(\mu)\psi\big)\,\overline{\varphi}\,\mathrm{dx} = \int_\Omega \mu\nabla\psi \cdot \nabla\overline{\varphi}\,\mathrm{dx} = \langle -\nabla \cdot \mu\nabla\psi, \varphi\rangle \quad \text{for all } \varphi \in W^{1,2}, \tag{3.3}$$

which is considered as the *constitutive relation* between $-\nabla \cdot \mu\nabla\psi$ and $A_p(\mu)\psi$. In fact, (3.3) precisely means that $-\nabla \cdot \mu\nabla\psi \in W_\bullet^{-1,2}$ is the image of $A_p(\mu)\psi \in L^p$ under

the embedding $L^p \hookrightarrow W_\bullet^{-1,2}$. Moreover, it is clear that the L^p-norm of $A_p(\mu)\psi$ is nothing else but the norm of the antilinear form (3.2) – $W^{1,2}$ equipped with the $L^{p'}$-norm.
Since the notation $A_p(\mu)$ already indicates the space on which the operator is assumed to act, we write e.g. $\mathrm{dom}(A_p(\mu))$ instead of $\mathrm{dom}_{L^p}(A_p(\mu))$ if there is no need for greater care. Note that the often used technique to construct the "strong" differential operators on the L^p scale by restricting $A_2(\mu)$ to L^p for $p > 2$ and taking adjoints of these resulting operators to define the corresponding operator in L^p for $p < 2$ (or forming the closure of $A_2(\mu)$ there) gives the same operators as the procedure above.

We will mostly consider the case of strictly positive μ and derive results for the corresponding operators; only in Lemma 3.23 properties of the operators $A_p(\mu)$ with possibly non-positive values for μ are pointed out which are fundamental for the treatment of the divergence operator in the right hand side of (1.1). Hence, let us now assume for the rest of this subchapter that μ is bounded from below by a positive constant.

We collect some properties of $A_2(\mu)$. It is a non-negative, selfadjoint operator on L^2, classically called the operator induced by the form (3.1) on $W^{1,2}$, cf. [75, Ch. 1.2.3] or the classical text [56, Ch. VI.2]. Firstly, the following basic properties are of note:

Remark 3.3. i) A particular case of the operator $A_2(\mu)$ with μ bounded from below by a positive constant is of course the case $\mu \equiv 1$, for which one recovers the negative Laplacian $A_2(1) = -\Delta$ on L^2.

ii) It is well-known that the property $\psi \in \mathrm{dom}(A_2(\mu))$ implies a (generalized) homogeneous Neumann condition $\nu \cdot \mu\nabla\psi = \nu \cdot \nabla\psi = 0$ on $\partial\Omega$, cf. [18, Ch. 1.2] or [31, Ch. II.2], ν being the outer normal at the boundary. This fact reflects the homogeneous Neumann boundary conditions (1.5) on the functional analytic level.

Moreover, both $\nabla \cdot \mu\nabla$ and $-A_2(\mu)$ generate analytic semigroups on $W_\bullet^{-1,2}$ and L^2, respectively, and these semigroups are *consistent*, cf. [75, Ch. 1.4.2]. The semigroup $\{\exp(-tA_2(\mu))\}_{t\geq 0}$ is a contractive one on L^2. Even more, by [75, Cor. 4.10], it induces a contraction semigroup on L^∞ and thus does so also on L^p for $p \in]2,\infty[$ by interpolation, and then on all spaces L^p for $p \in [1,2[$ by duality, cf. [75, Cor. 2.16]. We denote the corresponding negative generators as operators on L^p by $B_p(\mu)$, for which $B_2(\mu) = A_2(\mu)$ by definition. Concerning the abstract definition of the semigroups $\{\exp(-tB_p(\mu))\}_{t\geq 0}$ for $p \in [1,2[$, the Lipschitz property of Ω assures that the heat kernel of the semigroup $\{\exp(-tA_2(\mu))\}_{t\geq 0}$ admits upper Gaussian estimates, see [8, 29] or [75, Ch. 6]. Thus, the operators $\{\exp(-tB_p(\mu))\}_{t\geq 0}$ for this range of p are obtained as the continuous extension of the operator $\{\exp(-tA_2(\mu))\}_{t\geq 0}$ to L^p, cf. [75, Prop. 7.1] or [8, Prop. 1.4]. Generally, the semigroups $\{\exp(-tB_p)\}_{t\geq 0}$ are analytic for $p \in [1,\infty[$ as of [75, Cor. 7.5]), while the one for $p = \infty$ is even not strongly continuous.

The natural question to ask at this point is whether $B_p(\mu)$ and $A_p(\mu)$ coincide for $p \neq 2$. This is indeed a crucial question, since various results in the common literature state properties of the *semigroup* generated by A_2 considered on the L^p spaces, whereas other results concern the *operators* A_p directly. The cited proofs for Theorem 3.17 and Proposition 3.9 below are particular examples of this polarity. In this sense, it is necessary to make sure that the generators $B_p(\mu)$ and the operators obtained by restriction $A_p(\mu)$ are actually the same objects. In the subsequent theorem we show that in fact $A_p(\mu)$ and $B_p(\mu)$ coincide in the range of p's which is of interest here.

Theorem 3.4. *Assume that μ is a real, bounded, measurable function on Ω which admits a strictly positive lower bound. Then $A_p(\mu) = B_p(\mu)$ for all $p \in \left]\frac{2d}{d+2}, 2\right]$.*

Proof. Let $t \geq 0$ be arbitrary, but fixed in the following. Recall that the condition on p implies the embeddings $W^{1,2} \hookrightarrow L^{p'}$ and, equivalently, $L^p \hookrightarrow W_\bullet^{-1,2}$. Moreover, it was already stated that $A_2(\mu) = B_2(\mu)$ by definition and that $\{\exp(-tB_p(\mu))\}_{t\geq 0}$ is exactly the continuous extension of the operator $\{\exp(-tA_2(\mu))\}_{t\geq 0}$ to L^p. So, let $\psi \in L^p$ be arbitrary. Then there exists a sequence $(\psi_n) \subset L^2$ such that $\psi_n \to \psi$ and $\exp(-tA_2(\mu))\,\psi_n \to \exp(-tB_p(\mu))\,\psi$, both in L^p. Due to the embedding $L^p \hookrightarrow W^{-1,2}$, we have $\exp(t\,\nabla\cdot\mu\nabla)\,\psi_n \to \exp(t\,\nabla\cdot\mu\nabla)\,\psi$ in $W^{1,2}$ by the analyticity of the semigroup on $W_\bullet^{-1,2}$. On the other hand, also noted above, the semigroups generated by $-A_2(\mu)$ and $\nabla\cdot\mu\nabla$ agree on L^2, hence

$$\exp(t\,\nabla\cdot\mu\nabla)\,\psi_n = \exp(-tA_2(\mu))\,\psi_n \quad\longrightarrow\quad \exp(-tB_p(\mu))\,\psi \quad \text{in } L^p.$$

But this means that $\exp(-tB_p(\mu))\,\psi = \exp(t\,\nabla\cdot\mu\nabla)\,\psi \in L^p$ for every $\psi \in L^p$, i.e.,

$$\exp(-tB_p(\mu)) = \exp(t\,\nabla\cdot\mu\nabla)\,|_{L^p}. \tag{3.4}$$

Since $t \geq 0$ was arbitrary, the resolvent formula via the Laplace transform of the semigroup and (3.4) then imply the coincidence of the resolvents $(B_p(\mu)+1)^{-1}$ and $(-\nabla\cdot\mu\nabla+1)^{-1}|_{L^p} = (A_p(\mu)+1)^{-1}$, cf. [61, Lemma 2.1.6], compare also [7, Prop. 2.4]. From this, $A_p(\mu) = B_p(\mu)$ follows. □

We lastly collect some deeper results about the operators $B_p(\mu)$. It is known that $\exp(-t(B_p(\mu)+1))_{t\geq 0}$ transforms *real* functions into *real* ones and *positive* ones into *positive* ones ([75, Ch. 2.6]). Moreover $B_p(\mu)+1$ admits bounded imaginary powers; in particular, the set of operators $\{(B_p(\mu)+1)^{is} \colon s \in \left]-\epsilon, \epsilon\right[\}$ is bounded in $\mathcal{L}(L^p)$ for every $p \in \left]1, \infty\right[$ and every $\epsilon > 0$, see [21] or [75, Cor. 7.24]. Observing that the fractional powers of $B_p(\mu)+1$ are well defined, due to the contractivity of the semigroups and the Hille-Yosida theorem (cf. [82, Ch. 1.15]), the boundedness of the imaginary powers has quite some interesting implications; the most important for being, at this point, the identity of the domains of fractional powers $(B_p(\mu)+1)^\alpha$ with interpolation spaces, see [82, Ch. 1.15.3] or [6, Ch. 4.6/4.7]. We devote a subchapter to the special fractional powers which we need in the following.

3.2. Fractional powers of the elliptic operators

In this section, we ultimately establish the embedding

$$\operatorname{dom}\big((A_p(\mu)+1)^{\frac{1}{2}+\frac{d}{2q}}\big) \hookrightarrow W^{1,q} \tag{3.5}$$

for suitable $q > d$ and $p \geq \frac{q}{2}$, cf. Theorem 3.11 below. The main tool here, which will be the "anchor" in the derivation of (3.5), is the precise information on the domain of definition of the square root of the operators $-\nabla\cdot\mu+1$, cf. Proposition 3.9, together with the following assumption, which essentially allows to "lift" the obtained regularity to sufficiently high levels:

Assumption 3.5. There is a $q \in \left]d, 4\right[$ such that

$$-\Delta + 1 \colon W^{1,q} \to W_\bullet^{-1,q} \tag{3.6}$$

provides a topological isomorphism, the operator being defined as in Definition 3.1.

Since Assumption 3.5 in fact implicitly determines the class of admissable domains, an (extensive) comment on this should be in order:

Remark 3.6. i) In case of $d = 2$ the assumption is fulfilled for any Lipschitz domain Ω. This is the main result in the classical paper [36], there even established for mixed boundary conditions.

ii) It is exactly this condition which—besides the *a priori* required Lipschitz property—puts a restriction on the geometry of the underlying domain Ω in three spatial dimensions in this paper. For $d = 3$, it is known that Assumption 3.5 holds true in case of *strong* Lipschitz domains Ω, cf. [90]. Moreover, it is also true for Lipschitz domains Ω whose closures form—generally nonconvex—polyhedrons, cf. [37]. Note that this latter class is, by far, *not* contained in the class of strong Lipschitz domains, as the (topologically regularized) double beam shows.

iii) Assumption 3.5 is also fulfilled for domains which are obtained locally as C^1 deformations of the ones mentioned before.

iv) It is well-known that, even for strong Lipschitz domains, the admissable index q exceeds 3 by an arbitrarily small margin only, cf. [90, Introduction], cf. also [53, Thm. A]. In case of C^1-domains Ω, q may be chosen arbitrarily large (cf. [1, Section 15] or [66, p. 156–157]); but if one admits polyhedral domains the isomorphism index q cannot be expected to be larger than 4 in general, since edge and corner singularities appear, cf. [22], [23]. See also [65] and [38, Appendix] for sharp estimates of edge singularities.

v) If ϕ is a uniformly continuous function on Ω with a positive lower bound, then Assumption 3.5 implies that

$$-\nabla \cdot \phi \nabla + 1 : W^{1,q} \to W_{\bullet}^{-1,q} \tag{3.7}$$

is also a topological isomorphism, cf. [26, Ch. 6].

Altogether, this shows that Assumption 3.5 is fulfilled for a fairly rich class of domains which should cover almost all interesting constellations in the applications.

We suppose Assumption 3.5 to be satisfied for the rest of this work. Let us fix a corresponding number q, that is, let us assume that Assumption 3.5 is satisfied for this q from now on.

Remark 3.7. In the introduction it was already noted that the differential operators on L^p with $p = \frac{q}{2}$, where q as in Assumption 3.5, are of critical importance in the following considerations. Due to $q > d$, this implies $p > \frac{d}{2}$. However, $\frac{2d}{d+2} \leq \frac{d}{2}$ if and only if $d \geq 2$. Hence, if p is greater than $\frac{d}{2}$ then it is also greater than $\frac{2d}{d+2}$ in the setting of this paper. By Theorem 3.4, this means that there is *no ambiguity* concerning the differential operators on L^p and we hereby agree to call them $A_p(\mu)$.

Remark 3.8. The domain of the operator $A_p(\mu)$ is always equipped with the usual norm $\|(A_p(\mu) + 1) \cdot \|_{L^p}$, or $\|(A_p(\mu) + 1) \cdot \|_{\mathbb{L}^p}$ when considered on the space L^p or $\mathbb{L}^p$, respectively. This means that $\operatorname{dom} A_p(\mu)$ and $\operatorname{dom}\bigl(A_p(\mu)+1\bigr)$ coincide as Banach spaces and we will use them interchangeably.

The following recent result on the regularity properties of the square root of $-\nabla \cdot \mu\nabla + 1$ is, in coorperation with the isomorphism (3.6), the central instrument for deriving estimates for suitable fractional powers of the differential operators.

Proposition 3.9. *Let μ denote any real, measurable function on Ω which is bounded from below and above by positive constants.*

i) *The isomorphism* $(-\nabla\cdot\mu\nabla+1)^{-\frac{1}{2}} : L^2 \to W^{1,2}$ *continuously extends to an isomorphism from L^p onto $W^{1,p}$ for $p \in]1,2[$. Hence, according to Theorem 3.4, the operator $(A_p(\mu)+1)^{\frac{1}{2}}$ provides a topological isomorphism between the spaces $W^{1,p}$ and L^p, or, in other words:* $\mathrm{dom}(A_p(\mu)+1)^{\frac{1}{2}} = W^{1,p}$ *for all $p \in]\frac{2d}{2+d}, 2[$.*

ii) $(-\nabla\cdot\mu\nabla+1)^{\frac{1}{2}}$ *provides a topological isomorphism between the spaces L^p and $W_\bullet^{-1,p}$, in other words:* $\mathrm{dom}_{W_\bullet^{-1,p}}(-\nabla\cdot\mu\nabla+1)^{\frac{1}{2}} = L^p$ *for all $p \in [2,\infty[$.*

iii) *We have*
$$\mathrm{dom}\big((A_p(\mu)+1)^{\theta/2}\big) = H^{\theta,p} \tag{3.8}$$
for $p \in]\frac{2d}{d+2}, 2]$ and $\theta \in]0,1[\setminus \{\frac{1}{p}\}$.

Proof. i) is the main result in [10], cf. Thm. 5.1 there. ii) follows from i) by duality because $A_2(\mu)$ is selfadjoint on L^2. iii) Since $A_p(\mu)+1$ admits bounded imaginary powers,
$$\mathrm{dom}\big((A_p(\mu)+1)^{\theta/2}\big) = [L^p, \mathrm{dom}(A_p(\mu)+1)^{\frac{1}{2}}]_\theta$$
follows from [82, Ch. 1.15.3]. By i), the latter is equal to $[L^p, W^{1,p}]_\theta$, and this space is exactly $H^{\theta,p}$ as proved in [33, Thm. 3.1]. □

Lemma 3.10. *Let μ denote any real, uniformly continuous function on Ω which is bounded from below by a positive constant. Then, under Assumption 3.5, $\big(-\nabla\cdot\mu\nabla+1\big)^{\frac{1}{2}}$ provides a topological isomorphism between $W^{1,q}$ and L^q.*

Proof. First of all, Remark 3.6 tells us that under the supposition on μ Assumption 3.5 implies the isomorphism (3.7). Having this at hand, the claim follows in a straight forward manner from Proposition 3.9 ii). □

The square root isomorphism on the $W^{1,q}$ space for $q > d$ has the following immediate consequence:

Theorem 3.11. *Let μ denote any real, uniformly continuous function on Ω which is bounded from below by a positive constant. Then, for every p with $2 \ge p \ge \frac{q}{2}$ one has the embedding*
$$\mathrm{dom}\big((A_p(\mu)+1)^{\frac{1}{2}+\frac{d}{2q}}\big) \hookrightarrow W^{1,q}, \tag{3.9}$$
which implies
$$\big(L^p, \mathrm{dom}(A_p(\mu))\big)_{\theta,1} \hookrightarrow W^{1,q} \tag{3.10}$$
for all $\theta \in \big[\frac{1}{2}+\frac{d}{2q}, 1\big[$.

We note that, due to $q > d$, we always have $p > \frac{2d}{2+d}$ in the situation of Theorem 3.11, cf. Remark 3.7.

Proof of Theorem 3.11. It suffices to show the assertion for $p = \frac{q}{2}$. Combining [82, Ch. 1.10.1 and Thm. 1.15.3] gives for $\theta \in \big[\frac{1}{2}+\frac{d}{2q}, 1\big[$

$$\begin{aligned}\big(L^p, \mathrm{dom}(A_p(\mu)+1)\big)_{\theta,1} &\hookrightarrow \big(L^p, \mathrm{dom}(A_p(\mu)+1)\big)_{\frac{1}{2}+\frac{d}{2q},1} \\ &\hookrightarrow \big[L^p, \mathrm{dom}(A_p(\mu)+1)\big]_{\frac{1}{2}+\frac{d}{2q}} = \mathrm{dom}\big((A_p(\mu)+1)^{\frac{1}{2}+\frac{d}{2q}}\big).\end{aligned} \tag{3.11}$$

It was already mentioned in the proof of Proposition 3.9 that $A_p(\mu)+1$ admits bounded imaginary powers which is needed for the equality in (3.11). Hence, modulo identification of $\operatorname{dom} A_p(\mu)$ and $\operatorname{dom}\big(A_p(\mu)+1\big)$, (3.10) follows from (3.9), which we show as follows. Firstly,

$$\big(A_p(\mu)+1\big)^{-\alpha} = \big(-\nabla\cdot\mu\nabla+1\big)^{-\alpha} \quad \text{on } L^p \tag{3.12}$$

for all $\alpha\in[0,1]$. This of course means in particular

$$\big\|(A_p(\mu)+1)^{-(\frac{1}{2}+\frac{d}{2q})}\big\|_{\mathcal{L}(L^p;W^{1,q})} = \big\|(-\nabla\cdot\mu\nabla+1)^{-(\frac{1}{2}+\frac{d}{2q})}\big\|_{\mathcal{L}(L^p;W^{1,q})},$$

and we estimate the latter by

$$\begin{aligned}
&\big\|(-\nabla\cdot\mu\nabla+1)^{-(\frac{1}{2}+\frac{d}{2q})}\big\|_{\mathcal{L}(L^p;W^{1,q})}\\
&\qquad\le \big\|(-\nabla\cdot\mu\nabla+1)^{-\frac{1}{2}}\big\|_{\mathcal{L}(L^q;W^{1,q})}\big\|(-\nabla\cdot\mu\nabla+1)^{-\frac{d}{2q}}\big\|_{\mathcal{L}(L^p;L^q)}\\
&\le \big\|(-\nabla\cdot\mu\nabla+1)^{-\frac{1}{2}}\big\|_{\mathcal{L}(L^q;W^{1,q})}\cdot\mathcal{E}(H^{\frac{d}{q},p};L^q)\big\|(-\nabla\cdot\mu\nabla+1)^{-\frac{d}{2q}}\big\|_{\mathcal{L}(L^p;H^{\frac{d}{q},p})},
\end{aligned}\tag{3.13}$$

where $\mathcal{E}(H^{\frac{d}{q},p};L^q)$ denotes the corresponding embedding constant. The first factor in (3.13) is finite according to Lemma 3.10, and the last is finite due to the equality $\operatorname{dom}_{L^p}\big((-\nabla\cdot\mu\nabla+1)^{\frac{d}{2q}}\big)=\operatorname{dom}\big((A_p(\mu)+1)^{\frac{d}{2q}}\big)=H^{\frac{d}{q},p}$ according to (3.8), cf. also (3.12). Obviously, this altogether gives

$$\operatorname{dom}\big((A_p(\mu)+1)^{\frac{1}{2}+\frac{d}{2q}}\big)\hookrightarrow W^{1,q}$$

which was the claim. □

3.3. Maximal parabolic regularity and consequences for nonlinear problems

Let us now introduce preparatory concepts and results concerning parabolic operators. Throughout the rest of this paper let $T>0$ and set $J=\,]0,T[$. Let us start by introducing the following (standard)

Definition 3.12. If X is a Banach space and $r\in\,]1,\infty[$, then we denote by $L^r(J;X)$ the space of X-valued functions f on J which are Bochner-measurable and for which $\int_J\|f(t)\|_X^r\,\mathrm{d}t$ is finite. We define the Bochner-Sobolev spaces $W^{1,r}(J;X):=\{u\in L^r(J;X)\colon u'\in L^r(J;X)\}$, where u' is to be understood as the time derivative of u in the sense of X-valued distributions (cf. [6, Section III.1]). Moreover, we introduce the subspace of functions with initial value zero $W_0^{1,r}(J;X):=\{\psi\in W^{1,r}(J;X)\colon \psi(0)=0\}$.

Let us define a suitable notion of maximal parabolic regularity in the non-autonomous case and point out some basic facts on this:

Definition 3.13. Let X, D be Banach spaces with D densely embedded in X. Let $J\ni t\mapsto\mathcal{A}(t)\in\mathcal{L}(D;X)$ be a bounded and measurable map and suppose that the operator $\mathcal{A}(t)$ is closed in X for all $t\in J$. Let $r\in(1,\infty)$. Then we say that the family $\{\mathcal{A}(t)\}_{t\in J}$ satisfies *(non-autonomous) maximal parabolic $L^r(J;D,X)$-regularity*, if for any $f\in L^r(J;X)$ there is a unique function $u\in L^r(J;D)\cap W_0^{1,r}(J;X)$ which satisfies

$$u'(t)+\mathcal{A}(t)u(t)=f(t) \tag{3.14}$$

for almost all $t \in J$. We write

$$\mathrm{MR}^r(J; D, X) := L^r(J; D) \cap W^{1,r}(J; X)$$

and

$$\mathrm{MR}^r_0(J; D, X) := L^r(J; D) \cap W^{1,r}_0(J; X)$$

for the spaces of maximal parabolic regularity. From the open mapping theorem, we further obtain that there exists a constant c such that

$$\|u\|_{\mathrm{MR}^r_0(J;D,X)} \le c\|f\|_{L^r(J;X)} \tag{3.15}$$

for all $f \in L^r(J; X)$ and u being the associated unique solution of (3.14).

If all operators $\mathcal{A}(t)$ are equal to one (fixed) operator $\mathcal{A}_0$, and there exists an $r \in (1, \infty)$ such that $\{\mathcal{A}(t)\}_{t\in J}$ satisfies maximal parabolic $L^r(J; D, X)$-regularity, then $\{\mathcal{A}(t)\}_{t\in J}$ satisfies maximal parabolic $L^s(I; D, X)$-regularity for all $s \in (1, \infty)$ and all other (finite) intervals I (cf. [27]), and we say that $\mathcal{A}_0$ satisfies *maximal parabolic regularity on X*.

The following embedding result for the spaces of maximal parabolic regularity is essentially used in the sequel.

Lemma 3.14. *Let X, Y be two Banach spaces, with dense embedding $Y \hookrightarrow X$.*

i) *There is an embedding*

$$\mathrm{MR}^s(J; Y, X) \hookrightarrow C\big(\overline{J}; (X, Y)_{1-\frac{1}{s},s}\big). \tag{3.16}$$

ii) *Conversely, if the operator A generates an analytic semigroup on the Banach space X with Y as its domain, and $\psi \in (X, Y)_{1-\frac{1}{s},s}$, then the function $\exp(\cdot A)\,\psi$ belongs to $W^{1,s}(J; X) \cap L^s(J; Y)$ for every bounded interval $J = [0, T[$.*

iii) *There is an embedding*

$$\mathrm{MR}^r(J; Y, X) \hookrightarrow C^\alpha\big(J; (X, Y)_{\zeta,1}\big) \tag{3.17}$$

where $0 < \alpha = 1 - \zeta - \frac{1}{r}$.

Proof. i) is proved in [6, Ch. 4.10], ii) is shown in [61, Ch. 2.2.1 Prop. 2.2.2], and iii) is proved in [5, Ch. 3, Thm. 3], see also [25] for a simple proof. □

Remark 3.15. The first two points of Lemma 3.14 together show that the space $(X, \mathrm{dom}_X(A))_{1-\frac{1}{s},s}$, is the adequate space of initial values in the framework of maximal parabolic regularity.

In the immediate context of maximal parabolic regularity, Y is taken as $\mathrm{dom}_X(A)$ equipped with the graph norm, of course. Moreover, we need the following results.

Theorem 3.16 ([78, Thm. 2.5])**.** *Let the following two suppositions be satisfied:*

(H1) The family of operators $\{\mathcal{A}(t)\}_{t\in\overline{J}}$, acting on a Banach space X has a common dense domain D and the mapping $\overline{J} \ni t \mapsto \mathcal{A}(t) \in \mathcal{L}(D; X)$ is continuous. Moreover, each operator $\mathcal{A}(s)$, $s \in \overline{J}$, generates an analytic semigroup on X.

(H2) For some $r \in]1,\infty[$, every (fixed) $s \in [0,T]$ and all $f \in L^r(J;X)$ there is a unique element $u \in \mathrm{MR}_0^r(J;D;X)$ which satisfies the equation $u' + \mathcal{A}(s)u = f$.

Then $\{\mathcal{A}(t)\}_{t\in\overline{J}}$ satisfies maximal parabolic $L^r(J;D,X)$-regularity.

Theorem 3.17. *Let μ be a real, bounded, measurable function on Ω which admits a positive lower bound. Then, for every $p \in]1,\infty[$, the operators $B_p(\mu)$ admit maximal parabolic regularity on L^p. In particular, $A_p(\mu)$ admits maximal parabolic regularity on L^p for $p \in [\frac{2d}{d+2}, 2]$, due to Theorem 3.4.*

Proof. The theorem can be proved in different ways: in [41, Thm. 5.4] it is shown via Gaussian estimates for the heat kernel, heavily resting on [40], see also [20].

On the other hand, the theorem is proved in [34, Ch. 7], there resting on the contractivity of the induced semigroups on all L^p spaces, $p \in [1,\infty]$, and the pioneering result of Lamberton [58]. The latter allows to prove maximal parabolic regularity on even more general Lebesgue spaces, see [30]. □

Theorem 3.18 ([3, Thm. 2.1])**.** *Let $J =]0,T[$ for some $T \in]0,\infty[$ and $r \in]1,\infty[$, and suppose that X, Y are Banach spaces with dense embedding $Y \hookrightarrow X$. Also assume that*

i) *$\mathcal{A}$ is a map from $\mathrm{MR}^r(J;Y,X)$ into $L^\infty(J,\mathcal{L}(Y;X))$, the latter space being identified with a subset of the* non-autonomous *parabolic operators on X. $\mathcal{A}$ is Lipschitz continuous on bounded subsets.*
ii) *For each $u \in \mathrm{MR}^r(J;Y,X)$ and every $S \in]0,T]$ the non-autonomous operator $\mathcal{A}(u)|_{]0,S[}$ provides a topological isomorphism between $\mathrm{MR}_0^r(0,S;Y,X)$ and $L^r(0,S;X)$.*
iii) *There is an $s > r$, and a mapping $F : \mathrm{MR}^r(J;Y,X) \to L^s(J;X)$, which is Lipschitzian on every bounded subset.*
iv) *Both $\mathrm{MR}^r(J;Y,X) \ni u \mapsto \mathcal{A}(u) \in L^\infty(J;\mathcal{L}(Y;X))$ and $F\colon \mathrm{MR}^r(J;Y,X) \to L^s(J;X)$ are* Volterra maps*, i.e.*

$$u|_{]0,S[} = v|_{]0,S[} \quad \Longrightarrow \quad (A(u),F(u))|_{]0,S[} = (A(v),F(v))|_{]0,S[}$$

for every $S \in]0,T[$.
v) *$u_0 \in (X,Y)_{1-\frac{1}{r},r}$.*

Then there is a (maximal) interval $I_\bullet :=]0,S_\bullet[\subseteq J$ such that the equation

$$u' + \mathcal{A}(u)u = F(u), \quad u(0) = u_0$$

has a solution u on every subinterval $I =]0,S[\subseteq I_\bullet$, which belongs to the maximum regularity space $\mathrm{MR}^r(I;Y,X)$. Moreover, this solution is unique.

Remark 3.19. It is known since long that the Volterra property allows to derive results which are not available in a general context without this property, see e.g. [31, Ch. V]. Nevertheless, we feel that Amann's result is very close to the "optimum" what can be achieved. The reader is advised to consult [4, Thm. 3.1] for comments on the result by its inventor and a (fixable) shortcoming in the proof in [3].

3.4. Transferring to real spaces

Up to now, we have worked in a *complex* setting, but the Keller-Segel system has to be read as a *real* one. Therefore we transfer the results which we need in the sequel to the corresponding real spaces. In order to do this, we denote the real parts of L^p, $W^{1,q}$ by $L^p_{\mathbb{R}}$ and $W^{1,q}_{\mathbb{R}}$, respectively.

Remark 3.20. The necessity to start with complex spaces and to re-evaluate the assertions to also hold in the real case can be explained as follows: Most results up to this chapter 3.4 are complex in their very nature, a particular example being Proposition 3.9. This makes it evident that, at this point, complex spaces are the correct setting. On the other hand, the condition of being twice continuously differentiable for the nonlinear functions is more or less inevitable in the context presented above, cf. Lemma 4.14, Corollary 4.15 and Lemma 4.16. But imposing this condition in a *complex* setting in fact necessitates the *analyticity* of the corresponding functions, which is drastically and more importantly unnecessarily more restrictive. Hence we "do the twist" and switch to real spaces for the actual investigation of the model.

The starting point is the insight that the semigroup operators $\exp(-tA_p(\mu))$ map real functions into real functions (cf. [75, Prop. 2.5]) if the coefficient function μ is real-valued. Hence, the operators $(A_p(\mu)+\lambda)^{-1}\colon L^p \to L^p$ *also* map real functions into real ones if $\lambda \in]0,\infty[$. This makes clear that the operator $A_p(\mu)$ has a meaningful restriction to $L^p_{\mathbb{R}}$, the domain of which also consisting of real functions only. In this sense, the symbol $\mathrm{dom}(A_p(\mu))$ from now on denotes the domain of $A_p(\mu)$ considered on the *real* space $L^p_{\mathbb{R}}$.

Lemma 3.21. *Let μ be a real, uniformly continuous function which is bounded from above and below by positive constants. The assertion of Theorem 3.11 remains true in case of real spaces, i.e. one has for $p \ge \frac{q}{2}$ the embedding*

$$(L^p_{\mathbb{R}}, \mathrm{dom}(A_p(\mu)))_{\frac{1}{2}+\frac{d}{2q},1} \hookrightarrow W^{1,q}_{\mathbb{R}}. \tag{3.18}$$

Proof. Let us first recall, see Remark 3.8, that we have topologized $\mathrm{dom}(A_p(\mu))$ by the norm $\|(A_p(\mu)+1)\cdot\|_{L^p_{\mathbb{R}}}$. Further, by Theorem 3.11, there is a positive constant c such that the following inequality holds true for all $\psi \in \mathrm{dom}(A_p(\mu))$:

$$\|\psi\|_{W^{1,q}} \le c\,\|\psi\|_{L^p}^{1-\theta}\,\|\psi\|_{\mathrm{dom}(A_p(\mu))}^{\theta} = c\,\|\psi\|_{L^p}^{1-\theta}\,\|(A_p(\mu)+1)\psi\|_{L^p}^{\theta}. \tag{3.19}$$

In particular, inequality 3.19 holds for *real* $\psi \in \mathrm{dom}(A_p(\mu))$, and then reads

$$\|\psi\|_{W^{1,q}_{\mathbb{R}}} \le c\,\|\psi\|_{L^p_{\mathbb{R}}}^{1-\theta}\,\|(A_p(\mu)+1)\psi\|_{L^p_{\mathbb{R}}}^{\theta} = c\,\|\psi\|_{L^p_{\mathbb{R}}}^{1-\theta}\|\psi\|_{\mathrm{dom}(A_p(\mu))}^{\theta}. \tag{3.20}$$

But (3.20) is constitutive for the embedding (3.18), cf. [12, Ch. 3.5] or [11, Ch. 5, Prop. 2.10]. □

Theorem 3.22. *Let μ be a real, bounded, measurable function on Ω which admits a positive lower bound. Then, for every $p \in]1,\infty[$, $B_p(\mu)$ admits maximal parabolic $L^p_{\mathbb{R}}$ regularity. In particular, $A_p(\mu)$ admits maximal parabolic $L^p_{\mathbb{R}}$ regularity for $p \in [\frac{2d}{d+2}, 2]$ due to Theorem 3.4.*

Proof. Let $f \in L^p_{\mathbb{R}}$. Then, by maximal parabolic L^p regularity of $B_p(\mu)$, there exists a unique solution $u \in \mathrm{MR}^r_0(J; \mathrm{dom}(B_p(\mu)), L^p)$ such that

$$u' + B_p(\mu)u = f, \quad u(0) = 0.$$

But then this solution is given by the variation of constants formula

$$u(t) = \int_0^t \exp(-(t-s)B_p(\mu))\, f(s)\,\mathrm{d}s$$

and since the semigroup operators transform real functions into real ones, cf. [75, Prop. 2.5], it is clear that the solution in fact belongs to the space $W^{1,r}_0(J; L^p_{\mathbb{R}}) \cap L^r(J; \mathrm{dom}(B_p(\mu)))$, what proves the claim. □

We will need that the domains of the differential operators $A_p(\mu)$ are uniform w.r.t. μ from a certain regularity class, as per the assumptions in Theorem 3.18. In general, this is not to be expected if μ does not have a positive lower bound, cf. Remark 3.2. Still, we need that the differential operator on the right-hand side in (1.1), which is the "culprit" having potentially negative coefficient function values, is compatible with the domain of definition for the function $v(t)$.

It will turn out that both the latter *and* the constant domain of definition for the differential operators on the left-hand side in (1.1) is exactly $\mathrm{dom}_{L^p}(\Delta)$. We prove the following lemma which covers all these considerations in its generality, there writing Δ instead of $-A_p(1)$ and already supposing that all occurring spaces are in fact *real* ones.

Lemma 3.23. *Let $p = \frac{q}{2}$ and assume $\mu \in W^{1,q}$.*

i) *The domain of the Laplacian is embedded into the domain of $A_p(\mu)$, that is,*

$$\mathrm{dom}_{L^p}(\Delta) \hookrightarrow \mathrm{dom}(A_p(\mu)).$$

ii) *If μ has, additionally, a positive lower bound, then the reverse embedding*

$$\mathrm{dom}(A_p(\mu)) \hookrightarrow \mathrm{dom}_{L^p}(\Delta)$$

is also true, and $\mathrm{dom}_{L^p}(\Delta)$ and $\mathrm{dom}(A_p(\mu))$ coincide as Banach spaces.

Proof. i) Let $\psi \in \mathrm{dom}_{L^p_{\mathbb{R}}}(\Delta)$ and consider the linear form

$$W^{1,2} \ni \varphi \mapsto \langle -\nabla \cdot \mu \nabla \psi, \varphi \rangle. \tag{3.21}$$

We show that $\psi \in \mathrm{dom}(A_p(\mu))$ by showing that (3.21) is continuous w.r.t. the $L^{p'}$-topology on $W^{1,2}$. Therefore we estimate

$$\begin{aligned} \Big| \int_\Omega \mu \nabla \psi \cdot \nabla \varphi \,\mathrm{dx} \Big| &= \Big| \int_\Omega \nabla \psi \cdot \nabla(\mu\varphi)\,\mathrm{dx} - \int_\Omega \varphi \nabla \psi \cdot \nabla \mu \,\mathrm{dx} \Big| \\ &\le \Big| \int_\Omega \nabla \psi \cdot \nabla(\mu\varphi)\,\mathrm{dx} \Big| + \Big| \int_\Omega \varphi \nabla \psi \cdot \nabla \mu \,\mathrm{dx} \Big| \\ &\le \|\mu\|_{L^\infty} \|\Delta \psi\|_{L^p} \|\varphi\|_{L^{p'}} + \|\nabla \psi\|_{L^q} \|\nabla \mu\|_{L^q} \|\varphi\|_{L^{p'}}. \end{aligned} \tag{3.22}$$

Since $\mathrm{dom}_{L^p}(\Delta)$ was topologized by $\|(-\Delta+1)\cdot\|_{L^p}$, we thus find

$$\sup_{\varphi\in W^{1,2},\|\varphi\|_{L^{p'}}\le 1}\Big|\int_\Omega \mu\nabla\psi\cdot\nabla\varphi\,\mathrm{dx}\Big|$$
$$\le\Big(\|\mu\|_{L^\infty}\|\Delta(-\Delta+1)^{-1}\|_{\mathcal{L}(L^p)}+\mathcal{E}(\mathrm{dom}_{L^p}(\Delta),W^{1,q})\|\nabla\mu\|_{L^q}\Big)\|\psi\|_{\mathrm{dom}_{L^p}(\Delta)}. \quad (3.23)$$

This means that the linear form (3.21) is bounded, such that $\psi\in\mathrm{dom}(A_p(\mu))$. Moreover, $\|A_p(\mu)\psi\|_{L^p}$ is bounded by the right-hand side in (3.23). From here, the embedding $\mathrm{dom}_{L^p}(\Delta)\hookrightarrow\mathrm{dom}(A_p(\mu))$ follows immediately.

ii) One reasons analogously as in the previous case, but exploits instead of (3.22) the equality

$$\int_\Omega\nabla\psi\cdot\nabla\varphi\,\mathrm{dx}=\int_\Omega\mu^{-1}\mu\nabla\psi\cdot\nabla\varphi\,\mathrm{dx}=\int_\Omega\mu\nabla\psi\cdot\nabla(\mu^{-1}\varphi)\,\mathrm{dx}-\int_\Omega\varphi\mu\nabla\psi\nabla(\mu^{-1})\,\mathrm{dx}.$$

This gives $\mathrm{dom}(A_p(\mu))\hookrightarrow\mathrm{dom}_{L^p}(\Delta)$, from which the identity $\mathrm{dom}_{L^p}(\Delta)=\mathrm{dom}(A_p(\mu))$ as Banach spaces follows. □

Lemma 3.23 directly yields the following corollary:

Corollary 3.24. *For $p=\frac{q}{2}$ and $\mu\in W^{1,q}$, the mapping*

$$C\big(\overline{J};W^{1,q}\big)\ni\omega\mapsto-\nabla\cdot\omega\nabla$$

takes its values in the space $C(\overline{J};\mathcal{L}(\mathrm{dom}_{L^p}(\Delta);L^p))$ and is Lipschitzian on bounded subsets.

4. Investigation of the model

4.1. Precise formulation of the problem and main result

In this section, we give a rigorous analysis of (1.1)–(1.4) in the sense of Definition 4.1 below. In fact, most of this section will consist of the proof of the main Theorem 4.3, which we state in the following. An explanation of the strategy for the proof can be found in Section 4.2.

Let us first agree on the following: from now on all appearing function spaces are supposed to be *real* ones, without indicating this explicitely in the sequel.

For all what follows, we moreover fix p as $p=\frac{q}{2}$ with q being the number from Assumption 3.5. We abbreviate $A_p(\mu)$ *for this fixed* p by $A(\mu)$ in the sequel. Fix also, from now on, a number $r>2(1-\frac{d}{q})^{-1}$ and another number $s>r$.

In the following we want to establish a precise notion of the *solution of the Keller-Segel-Model.*

Definition 4.1. Given a subinterval $I=]0,S[$ of J, we call a quadrupel $(u,(v,p,w))\in\mathrm{MR}^r(I;\mathrm{dom}_{L^p}(\Delta),L^p)\times\mathrm{MR}^s(I;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$ a solution of (1.1)–(1.4) on I, if this satisfies

$$\begin{aligned}
u'(t) + A\big(\kappa(u(t), v(t))\big)u(t) &= A\big(\sigma(u(t), v(t))\big)v(t) \\
&\quad + R_1\big(u(t), v(t), p(t), w(t)\big), \qquad (4.1)\\
v'(t) - k_v \Delta v(t) &= R_2\big(u(t), v(t), p(t), w(t)\big), \qquad (4.2)\\
p'(t) - k_p \Delta p(t) &= R_3\big(u(t), v(t), p(t), w(t)\big), \qquad (4.3)\\
w'(t) - k_w \Delta w(t) &= R_4\big(u(t), v(t), p(t), w(t)\big), \qquad (4.4)\\
\big(u(0), v(0), p(0), w(0)\big) &= (u_0, v_0, p_0, w_0) \qquad (4.5)
\end{aligned}$$

for almost all $t \in I$ in $L^p \times \mathbb{L}^p$ for (4.1)–(4.4), where the time derivative is taken in the sense of vector valued distributions and the initial values satisfy

$$(u_0, v_0, p_0, w_0) \in \big(L^p, \mathrm{dom}_{L^p}(\Delta)\big)_{1-\frac{1}{r},r} \times \big((L^p, \mathrm{dom}_{L^p}(\Delta))_{1-\frac{1}{s},s}\big)^3 =: \mathrm{IV}(r, s).$$

Here, the operator $-\Delta$ is to be understood as $A_p(1)$, i.e., the restriction of the weak (negative) Laplacian to L^p.

Remark 4.2. i) In the original model, we had the specific inhomogeneities

$$\begin{aligned}
R_1 &= 0,\\
R_2 &= -r_1 vp + r_{-1} w + u f(v),\\
R_3 &= -r_1 vp + (r_{-1} + r_2) w + u g(v, p),\\
R_4 &= r_1 vp - (r_{-1} + r_2) w,
\end{aligned}$$

cf. (1.1)–(1.4).

ii) For almost all $t \in I$ the functions $u(t, \cdot), v(t, \cdot), p(t, \cdot), w(t, \cdot)$ each lie in the space $\mathrm{dom}_{L^p}(\Delta)$. This tells us that for these t the *homogeneous Neumann condition* $\nu \cdot \nabla u = \nu \cdot \nabla v = \nu \cdot \nabla p = \nu \cdot \nabla w = 0$ is fulfilled in a generalized sense, cf. Remark 3.3.

iii) The regularity of the initial values in $\mathrm{IV}(r, s)$ is exactly the optimal one for the class of solutions as defined in Definition 4.1, cf. Remark 3.15.

iv) Definition 4.1 is in fact faithful to itself in the sense that the functions and mappings indeed map into the correct spaces, see Remark 4.6.

We formulate now the main result of this paper.

Theorem 4.3. *Under Assumption 3.5, Problem* (1.1)–(1.4) *admits exactly one local in time solution in the spirit of Definition 4.1. Moreover, the the components* (v, p, w) *of the solution are uniformly bounded in* L^∞ *over the interval of existence.*

Remark 4.4. Considering the derivation of the model in the introductionary chapter, the question of *positivity* of the solutions (u, v, p, w), provided their initial values were positive in the first place, arises very naturally. It is a folklore result in the theory of reaction-diffusion systems (cf. e.g. [76]) that a system in the form (4.2)–(4.4) is positivity preserving if and only if the inhomogeneities $R_2(\bar{u}, \cdot), R_3(\bar{u}, \cdot), R_4(\bar{u}, \cdot)$ are *quasipositive* for every $\bar{u} \in \mathbb{R}$, that is, if $(\bar{v}, \bar{p}, \bar{w})$ is an arbitrary vector in $\mathbb{R}^3$ with nonnegative entries, then

$$R_2(\bar{u}, 0, \bar{p}, \bar{w}) \geq 0, \quad R_3(\bar{u}, \bar{v}, 0, \bar{w}) \geq 0 \quad \text{and} \quad R_4(\bar{u}, \bar{v}, \bar{p}, 0) \geq 0.$$

Intuitively, the condition prevents the source term for each of the quantitities to be negative whenever the quantity itself is in danger of becoming negative, thus preventing a further decrease of the quantity. The specific inhomogenenites in (1.2)–(1.4), cf. Remark 4.2, indeed satisfy this condition for g satisfying $g(\bar{v}, 0) \geq 0$ for all $\bar{v} \geq 0$ and nonnegative f with nonnegative $\bar{u}$. Hence, (4.2)–(4.4) is positivity preserving for (v, p, w) if u is also a positive function, i.e., (4.1) is *also* positivity preserving. Unfortunately, the latter seems very difficult to show in the very general context of Definition 4.1, even with $R_1 = 0$ (and is generally not true for seemingly easy cases, see [70, Ch. 5]). However, for the specific choices $\kappa(u, v) = 1$ and $\sigma(u, v) = -u$, already mentioned in the introduction as well-researched model choices, positivity of u is shown in [32, Thm. 3.3] *independent* of the sign of v. The proof in [32] only relies on the fact that v is uniformly bounded in time and space, which is the case for our solutions obtained from Theorem 4.3. Hence, for this choice of κ and σ, $R_1 = 0$ and $R_2(\bar{u}, \cdot), R_3(\bar{u}, \cdot), R_4(\bar{u}, \cdot)$ quasipositive for $\bar{u} \geq 0$, system (4.1)–(4.4) is indeed posivitity preserving. This includes in particular system (1.1)–(1.4) for this choice of κ and σ.

We now proceed with the proof of the main result.

4.2. The proof

The actual proof of Theorem 4.3 works as follows. It should be evident to the reader that we plan to use the abstract result of Amann, Theorem 3.18. The general idea is to solve the semilinear equations for (v, p, w), (4.2)–(4.4), in dependence of u, and to show that this dependence satisfies the assumptions in Theorem 3.18. Here, it is clear that the dependence of (v, p, w) on u will be *nonlocal in time*, which indeed makes Theorem 3.18—instead of other well-known abstract quasilinear existence results—necessary.

However, as (4.2)–(4.4) are *nonlinear* equations themselves, it is not *a priori* clear that they in fact admit global solutions on the whole time horizon $]0, T[$, and a local-in-time existence interval $I(u)$ for (v, p, w) depending on u would clearly thwart any attempt to establish the assumptions from Theorem 3.18. Hence, we modify the right-hand sides in (4.2)–(4.4) by introducing a suitable cut-off, which then allows to show global existence, uniqueness, and a well-behaved dependence on u for the solutions $(\hat{v}, \hat{p}, \hat{w})$ of the modified system ((4.10)–(4.12) below); this is Theorem 4.10. After establishing that the involved operators and functions satisfy the assumptions of Theorem 3.18, we then use that very theorem to show existence and uniqueness of a local-in-time solution u to the *modified* system (4.9)–(4.12), including the equation for u, in Theorems 4.13 and 4.9. From there, we finally obtain Theorem 4.3 by showing that the local-in-time solution obtained for the modified system is indeed also the solution to the original system (4.1)–(4.4) at the cost of a possibly still smaller existence interval.

Aside from the dependence of (v, p, w) on u, there is another major obstacle when working to satisfy the assumptions of Theorem 3.18: Innocently looking, assumption i) of said theorem in fact requires, in our notation, that the differential operators, which will be $A_p\big(\kappa(u(t), [v(u)](t))\big)$, have *uniform* domains Y for *all* $u \in \mathrm{MR}^r(J; Y, L^p)$ and for almost every $t \in J$. As already hinted right before Lemma 3.23, we will be able to use $Y = \mathrm{dom}_{L^p}(\Delta)$, provided that the coefficient functions $\kappa(u(t), [v(u)](t))$ are from $W^{1,q}$ for almost every $t \in J$. Luckily, we already laid the fundations for this by Lemma 3.21 and have the maximal regularity embedding (3.16) at hand. Together, they immediately yield the following introductory result which is of importance in all what follows.

Lemma 4.5. *Set* $\alpha = \frac{1}{2} - \frac{d}{2q} - \frac{1}{r}$. *By the choice of* r, *we have* $\alpha > 0$.

i) *The space* $\mathrm{MR}^r(J; \mathrm{dom}_{L^p}(\Delta), L^p)$ *embeds into* $C^\alpha(J; W^{1,q})$ *and, hence, compactly into* $C(\overline{J}; C(\overline{\Omega}))$.
ii) *Analogously,* $\mathrm{MR}_0^s(J; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p)$ *embeds into* $C^\alpha(J; \mathbb{W}^{1,q})$ *and, hence, compactly into* $C(\overline{J}; C(\overline{\Omega})^3)$.

Proof. The compactness in both cases follows by the vector-valued Arzelà-Ascoli theorem, cf. [59, Ch. III.3]. For i), the condition on r implies $1 - \frac{1}{r} - \frac{1}{2}\big(1 + \frac{d}{q}\big) > 0$. Thus, the claim follows from Lemma 3.14, cf. (3.17), in conjunction with Lemma 3.21. ii) is proved analogously. □

Remark 4.6. For $u \in \mathrm{MR}^r(I; \mathrm{dom}_{L^p}(\Delta), L^p)$ and $v \in \mathrm{MR}^s(I; \mathrm{dom}_{L^p}(\Delta), L^p)$ with I as in Definition 4.1, Lemma 4.5 in conjunction with Lemma 3.23 and the assumptions on κ and σ (cf. Assumption 3.5) tells us that $\kappa(u(t), v(t))$ and $\sigma(u(t), v(t))$ are each functions from $W^{1,q}$ for every $t \in \overline{I}$. Together with $u(t), v(t) \in \mathrm{dom}_{L^p}(\Delta)$ for almost every $t \in I$, this shows that the expressions $A(\kappa(u(t), v(t)))u(t)$ and $A(\sigma(u(t), v(t)))v(t)$ in (4.1) are indeed well-defined. See also Lemmata 4.16 and 4.17 below.

As laid out in the beginning of this section, we will now modify the abstract system (4.1)–(4.4) in such a way that the terms on the right hand sides of (4.2)–(4.4) become bounded in space and time. This will ultimatively lead to a solution in the spirit of Definition 4.1 on a *smaller* time interval, since the modification becomes "active", only after some time point $T_\bullet > 0$, allowing to re-obtain the correct solution to the unmodified system on $[0, T_\bullet]$.

We consider

$$(v_0, p_0, w_0) \in \big((L^p, \mathrm{dom}_{L^p}(\Delta))_{1-\frac{1}{s},s}\big)^3 \tag{4.6}$$

to be given and fixed from now on.

Definition 4.7. For $\delta > 0$, we put $M := \delta + \max(\|v_0\|_{L^\infty}, \|p_0\|_{L^\infty}, \|w_0\|_{L^\infty})$. Let $\eta \in C^\infty(\mathbb{R})$ be a smooth function which is the identity on the interval $[-M, M]$ and is equal to $-(M+1)$ on the interval $]-\infty, -(M+1)]$ and equal to $M+1$ on the interval $[M+1, \infty[$. Moreover, we put $R_i^\eta := R_i(\cdot, \eta(\cdot), \eta(\cdot), \eta(\cdot))$ for $i = 2, 3, 4$.

Note that, due to Lemma 3.21 and the choice of s, we have the embedding $(L^p, \mathrm{dom}_{L^p}(\Delta))_{1-\frac{1}{s},s} \hookrightarrow C(\overline{\Omega})$, such that the number M in Definition 4.7 is well-defined.

We further split off the initial values for the functions v, p, w. In this spirit, we put $v_{\mathcal{I}}(t) = \exp(t\, k_v \Delta)\, v_0$ as well as $p_{\mathcal{I}}(t) = \exp(t\, k_p \Delta)\, p_0$ and $w_{\mathcal{I}}(t) = \exp(t\, k_w \Delta)\, w_0$, and write

$$v = v_{\mathcal{I}} + \check{v}, \quad p = p_{\mathcal{I}} + \check{p}, \quad w = w_{\mathcal{I}} + \check{w}, \tag{4.7}$$

where $\check{v}, \check{p}$ and $\check{w}$ have the initial value 0, of course.

For convenience, we collect some of the properties for the functions $v_{\mathcal{I}}, p_{\mathcal{I}}$ and $w_{\mathcal{I}}$ which will be of importance later.

Lemma 4.8. *Let the initial values* (v_0, p_0, w_0) *satisfy* (4.6).

i) *One has*

$$v_{\mathcal{I}}' - k_v \Delta v_{\mathcal{I}} = p_{\mathcal{I}}' - k_p \Delta p_{\mathcal{I}} = w_{\mathcal{I}}' - k_w \Delta w_{\mathcal{I}} \equiv 0 \tag{4.8}$$

on any time interval $]0, S[\subset J$.

ii) *The functions* $v_{\mathcal{I}}, p_{\mathcal{I}}$ *and* $w_{\mathcal{I}}$ *are each from* $\mathrm{MR}^s(J; \mathrm{dom}_{L^p}(\Delta), L^p)$*, take their values pointwise on* J *in* $W^{1,q}$*, and are continuous on every time interval* $[0, S[\subset \overline{J}$*.*
iii) *The functions* $v_{\mathcal{I}}, p_{\mathcal{I}}$ *and* $w_{\mathcal{I}}$ *are continuous on every time interval* $[0, S[\subset \overline{J}$ *when considered with values in* $C(\overline{\Omega})$*. Moreover, in this case we have*

$$\|v_{\mathcal{I}}(t)\|_{C(\overline{\Omega})} \leq \|v_{\mathcal{I}}(0)\|_{C(\overline{\Omega})}, \quad \|p_{\mathcal{I}}(t)\|_{C(\overline{\Omega})} \leq \|p_{\mathcal{I}}(0)\|_{C(\overline{\Omega})}, \quad \text{and} \quad \|w_{\mathcal{I}}(t)\|_{C(\overline{\Omega})} \leq \|w_{\mathcal{I}}(0)\|_{C(\overline{\Omega})}$$

for every $s \in S$

Proof. i) is clear. ii) Lemma 3.14 ii) shows that the functions $v_{\mathcal{I}}, p_{\mathcal{I}}, w_{\mathcal{I}}$ are continuous, when considered as $(L^p, \mathrm{dom}_{L^p}(\Delta))_{1-\frac{1}{s},s}$-valued ones. Thus, the assertion follows from Lemma 3.21 and the definition of s. iii) The first assertion follows from ii) by embedding $W^{1,q} \hookrightarrow C(\overline{\Omega})$. Moreover, since the semigroups act as *contractive* ones in L^∞, cf. Ch. 3.1, the evolution of the initial values v_0, p_0, w_0 does not lead to larger L^∞-norms. The latter is identical with the $C(\overline{\Omega})$-norm in our case. □

Having introduced the modified nonlinearities R_i^η and the split-off of the initial values, we combine both into the functions $\widehat{R}_i \colon J \times C(\overline{\Omega})^4 \to L^p$ by

$$\widehat{R}_i(t; \mathfrak{u}, \mathfrak{v}, \mathfrak{p}, \mathfrak{w}) := R_i^\eta(\mathfrak{u}, v_{\mathcal{I}}(t) + \mathfrak{v}, p_{\mathcal{I}}(t) + \mathfrak{p}, w_{\mathcal{I}}(t) + \mathfrak{w})$$

for $i = 2, 3, 4$, and

$$\widehat{R}_1(t; \mathfrak{u}, \mathfrak{v}, \mathfrak{p}, \mathfrak{w}) := R_1(\mathfrak{u}, v_{\mathcal{I}}(t) + \mathfrak{v}, p_{\mathcal{I}}(t) + \mathfrak{p}, w_{\mathcal{I}}(t) + \mathfrak{w}).$$

Then we consider instead of (4.1)–(4.4) the system

$$\begin{aligned} u'(t) + A\big(\kappa(u(t), v_{\mathcal{I}}(t) + v(t))\big)u(t) &= A\big(\sigma(u(t), v_{\mathcal{I}}(t) + v(t))\big(v_{\mathcal{I}}(t) + v(t)\big) \\ &\quad + \widehat{R}_1(t; u(t), v(t), p(t), w(t)), && (4.9) \\ v'(t) - k_v \Delta v(t) &= \widehat{R}_2(t; u(t), v(t), p(t), w(t)), && (4.10) \\ p'(t) - k_p \Delta p(t) &= \widehat{R}_3(t; u(t), v(t), p(t), w(t)), && (4.11) \\ w'(t) - k_w \Delta w(t) &= \widehat{R}_4(t; u(t), v(t), p(t), w(t)), && (4.12) \\ (u(0), v(0), p(0), w(0)) &= (u_0, 0, 0, 0) && (4.13) \end{aligned}$$

as equations in the Banach space $L^p \times \mathbb{L}^p \times \mathrm{IV}(r, s)$, holding for almost every $t \in I$ for the first four components. Note that we have, by abuse of notation, returned to writing v, p and w instead of $\check{v}, \check{p}$ and $\check{w}$ as introduced in (4.7) for better readability. Since we work exclusively with the functions with initial value 0 from here on, this should not give rise to confusion to the reader.

After these preparations we prove the subsequent theorem, from which our main result, Theorem 4.3, then follows (and which is in fact only a slight reformulation of this).

Theorem 4.9. *For given* $(u_0, v_0, p_0, w_0) \in \mathrm{IV}(r, s)$*, the system* (4.9)–(4.13) *admits exactly one local-in-time solution*

$$(u, (v, p, w)) \in \mathrm{MR}^r(I; \mathrm{dom}_{L^p}(\Delta), L^p) \times \mathrm{MR}_0^s(I; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p),$$

with $I =]0, S[\subseteq J$*.*

Let us point out some of the strategy for the proof of Theorem 4.9: Firstly, we will solve the equations (4.10)–(4.12) with $u \in C(\overline{J}; C(\overline{\Omega}))$ fixed by a fixed-point argument. The crucial point is that the dependence of these solution (v, p, w) from u is well-behaved in the space $\mathrm{MR}_0^s(J; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p)$. So implicitly inserting this into (4.9), this equation decouples from the other ones and is tractable by means of Amann's result, Theorem 3.18. Having then u at hand (we prove that the assumptions of Theorem 3.18 are satisfied in Theorem 4.13), one "rediscovers" (v, p, w) by (4.10)–(4.12).

Theorem 4.10. i) *Assume* $u \in C(\overline{J}; C(\overline{\Omega}))$ *to be given. Then the system* (4.10)–(4.12) *has a unique solution* $(v, p, w) \in \mathrm{MR}_0^s(J; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p)$.

ii) *Let* $\mathcal{S} : C(\overline{J}; C(\overline{\Omega})) \to \mathrm{MR}_0^s(J; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p)$ *denote the mapping which assigns to* u *the solution of* (4.10)–(4.12). *Then* $\mathcal{S}$ *is continuously differentiable.*

Proof. i) As announced above, we tackle this problem with a fixed-point argument which is rather standard in nature. We thus describe it only briefly. Let us investigate the right-hand sides in (4.10)–(4.12) for fixed $u \in C(\overline{J}; C(\overline{\Omega}))$ and an arbitrary subinterval $J_0 \subseteq J$. Observing that

$$L_R := \max_i L_{i,R}, \quad \text{where} \quad L_{i,R} := \max_{\substack{|\bar{u}| \le \|u\|_{C(\overline{J}; C(\overline{\Omega}))}, \\ |\bar{v}| \vee |\bar{p}| \vee |\bar{w}| \le M+1}} \left\| \partial_{(2,3,4)} R_i(\bar{u}, \bar{v}, \bar{p}, \bar{w}) \right\|_{\mathbb{R}^3}$$

is finite due to Assumption 2.1 and the definition of η, we easily find that $(v, p, w) \mapsto \widehat{R}(\cdot; u(\cdot), v(\cdot), p(\cdot), w(\cdot))$ is Lipschitz-continuous as a mapping from $L^\varsigma(J_0; \mathbb{L}^p)$ to $L^s(J_0; \mathbb{L}^p)$ for every $\varsigma > s$, where we have collected the functions $\widehat{R}_2, \widehat{R}_3, \widehat{R}_4$ into $\widehat{R}$. Moreover, the Lipschitz constant depends on the time interval J_0 via the term $\ell_{\varsigma,s}(J_0) := |J_0|^{\frac{1}{s} - \frac{1}{\varsigma}}$, with the usual convention $\frac{1}{\infty} = 0$.

This paves the way for a fixed point argument employing Banach's fixed point theorem for the mapping Φ_u, associated to the given $u \in C(\overline{J}; C(\overline{\Omega}))$, which assigns to $(v, p, w) \in L^\varsigma(J_0; \mathbb{L}^p)$ the solution $(\hat{v}, \hat{p}, \hat{w}) \in \mathrm{MR}_0^s(J_0; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p) \hookrightarrow L^\varsigma(J; \mathbb{L}^p)$ of the parabolic equations

$$\hat{v}'(t) - k_v \Delta \hat{v}(t) = \widehat{R}_2(t; u(t), v(t), p(t), w(t)), \tag{4.14}$$

$$\hat{p}'(t) - k_p \Delta \hat{p}(t) = \widehat{R}_3(t; u(t), v(t), p(t), w(t)), \tag{4.15}$$

$$\hat{w}'(t) - k_w \Delta \hat{w}(t) = \widehat{R}_4(t; u(t), v(t), p(t), w(t)). \tag{4.16}$$

This mapping is well-defined, since

$$-\widetilde{\Delta} := \mathrm{diag}(-k_v \Delta, -k_p \Delta, -k_p \Delta)$$

satisfies maximal parabolic regularity on $\mathbb{L}^p$ due do Theorem 3.22, with domain $\mathrm{dom}_{\mathbb{L}^p}(-\widetilde{\Delta}) = \mathrm{dom}_{\mathbb{L}^p}(\Delta)$, and the triplet of functions on the right-hand sides in (4.14)–(4.16) is from $L^s(J_0; \mathbb{L}^p)$. Now let $I_0 = \,]0, S_0[\, \subseteq J$. By the Lipschitz continuity of the right-hand sides $\widehat{R}$ as noted above, with $J_0 = I_0$, and the maximal parabolic regularity estimate (3.15), Φ_u is Lipschitz-continuous and its Lipschitz constant consists of $\ell_{\varsigma,s}(I_0)$ and quantities which are monotonically increasing w.r.t. the interval length $|I_0|$. Thus, it is possible to choose S_0 such that we achieve a certain Lipschitz constant for Φ_u. In fact, choosing S_0 small enough such that Φ_u is a *contraction* gives a fixed point $(v,p,w) = \Phi_u(v,p,w)$ on $L^\varsigma(I_0;\mathbb{L}^p)$ which is, by construction, the unique solution of (4.14)–(4.16) on I_0, and in fact an element of $\mathrm{MR}_0^s(I_0;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$.

Now one may consider Φ_u again, this time on another interval $I_1 =]S_0,S_1[\subset J$, with the modification that now the initial values $(v(S_0),p(S_0),w(S_0))$ have to be split off again as we did above, cf. (4.7). It remains to observe that choosing S_1 such that $|I_1| = |I_0|$ again makes Φ_u a contraction and its fixed point the unique solution of (4.14)–(4.16), this time on I_1, "gluing" the solutions on I_0 and I_1 together, and to iteratively repeat this procedure a *finite* number k times to obtain a unique solution $(v,p,w) \in \mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$ of (4.14)–(4.16) with the correct regularity on the *whole* time interval.

ii) For this we apply the implicit function theorem, considering the mapping

$$\Psi\colon C(\overline{J};C(\overline{\Omega})) \times \mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p) \to L^s(J;\mathbb{L}^p),$$

which is given by

$$\begin{aligned}\Psi(u,v,p,w)(t) = \Big(&v'(t) - k_v\Delta v(t) - \widehat{R}_2(t;u(t),v(t),p(t),w(t)),\\ &p'(t) - k_p\Delta p(t) - \widehat{R}_3(t;u(t),v(t),p(t),w(t)),\\ &w'(t) - k_w\Delta w(t) - \widehat{R}_4(t;u(t),v(t),p(t),w(t))\Big).\end{aligned}$$

Obviously, for given $u \in C(\overline{J};C(\overline{\Omega}))$, the triple (v,p,w) is a solution of (4.10)–(4.12) iff $\Psi(u,v,p,w) = 0$ in $L^s(J;\mathbb{L}^p)$. By the assumptions on R_2, R_3 and R_4, Ψ is continuously differentiable and the partial derivative with respect to the second variable in a given point $\big(\bar u,(\bar v,\bar p,\bar w)\big) \in C(\overline{J};C(\overline{\Omega})) \times \mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$ is the linear mapping which assigns to the triple $(h_2,h_3,h_4) \in \mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$ the expression

$$\begin{aligned}&\left[\left(\partial_{(2,3,4)}\Psi\right)(\bar u,\bar v,\bar p,\bar w)(h_2,h_3,h_4)\right](t)\\ &\quad= \Bigg[\Big(h_2'(t) - k_v\Delta h_2(t) - \sum_{i=2}^4 \partial_i\widehat{R}_2(t;\bar u(t),\bar v(t),\bar p(t),\bar w(t))h_i(t)\Big), \qquad (4.17)\\ &\qquad\Big(h_3'(t) - k_v\Delta h_3(t) - \sum_{i=2}^4 \partial_i\widehat{R}_3(t;\bar u(t),\bar v(t),\bar p(t),\bar w(t))h_i(t)\Big), \qquad (4.18)\\ &\qquad\Big(h_4'(t) - k_v\Delta h_4(t) - \sum_{i=2}^4 \partial_i\widehat{R}_4(t;\bar u(t),\bar v(t),\bar p(t),\bar w(t))h_i(t)\Big)\Bigg], \qquad (4.19)\end{aligned}$$

a function from $L^s(J;\mathbb{L}^p)$. We know already that the *autonomous* operator $-\tilde\Delta$ satisfies maximal parabolic regularity on the space $\mathbb{L}^p$. Moreover, it is clear that the remaining terms in (4.17)–(4.19), considered as time-dependent multipliers on the corresponding L^p-space, form *bounded* operators in $L^s(J;\mathbb{L}^p)$, since the corresponding multipliers are *bounded and continuous* in space and time. Hence, according to a suitable perturbation theorem as in [9, Prop. 1.3], the equation

$$\left(\partial_{(2,3,4)}\Psi\right)(\bar u,\bar v,\bar p,\bar w)(h_2,h_3,h_4) = \mathfrak{f}$$

is uniquely solvable with $(h_2,h_3,h_4) \in \mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$ for every $\mathfrak{f} \in L^s(J;\mathbb{L}^p)$. This means that the partial derivative $\left(\partial_{(2,3,4)}\Psi\right)(\bar u,\bar v,\bar p,\bar w)$ is a topological isomorphism between $\mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$ and $L^s(J;\mathbb{L}^p)$, what makes the implicit function theorem applicable. Considering $(\bar u,\bar v,\bar p,\bar w)=(\bar u,\mathcal{S}(\bar u))$, i.e., $\Psi(\bar u,\bar v,\bar p,\bar w)=0$, we thus obtain that the implicit function defined on a neighborhood of $\bar u$, whose existence is guaranteed by the implicit function theorem, coincides with $\mathcal{S}$ on that neighborhood and is continuously differentiable. Since this is true for *every* $\bar u\in C(\overline{J},C(\overline{\Omega}))$, $\mathcal{S}$ is continuously differentiable on that space. □

Remark 4.11. In addition to the results of Theorem 4.10, the above considerations make it clear that the set of solutions $\{\mathcal{S}(u)\colon u\in\mathfrak{B}\}$ which correspond to a *bounded* subset $\mathfrak{B}$ of $C(\overline{J};C(\overline{\Omega}))$ in turn forms a *bounded* subset in the space $\mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$, and, hence, a precompact one in $C_0(\overline{J};C(\overline{\Omega})^3)$, cf. Lemma 4.5. This can be seen by observing that the real functions R_i, being right hand sides in (4.14)–(4.16), are uniformly bounded in $L^s(J;\mathbb{L}^p)$ in the following way: We set, analogously to the definition of $L_{i,R}$ in the foregoing proof,

$$M_R:=\max_i M_{i,R}<\infty,\quad\text{where}\quad M_{i,R}:=\sup_{\substack{|\bar u|\le M_{\mathfrak{B}},\\ |\bar v|\vee|\bar p|\vee|\bar w|\le M+1}}\bigl|R_i(\bar u,\bar v,\bar p,\bar w)\bigr|,$$

using $M_{\mathfrak{B}}:=\max_{u\in\mathfrak{B}}\|u\|_{C(\overline{J};C(\overline{\Omega}))}$. Then

$$\max_i\bigl\|\widehat{R}_i(\cdot;\bar u(\cdot),\bar v(\cdot),\bar p(\cdot),\bar w(\cdot))\bigr\|_{L^s(J;\mathbb{L}^p)}\le \ell_{s,\infty}(J)|\Omega|^{\frac1p}M_R$$

for *all* $\bar u\in\mathfrak{B}$ and $(v,p,w)\in\mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p))$, which by the maximal parabolic regularity estimate (3.15) shows that $\{\mathcal{S}(u)\colon u\in\mathfrak{B}\}$ forms a bounded set in the space $\mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$.

Out next intention is to show that the mapping $\mathcal{S}$ is Lipschitzian on bounded subsets of $\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)$.

Corollary 4.12. *Let $\mathcal{B}$ be any bounded subset of* $\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)$. *Then the mapping $\mathcal{S}$ is Lipschitzian as a mapping from $\mathcal{B}$ into* $\mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$*, and hence, also into* $C(\overline{J};\mathbb{W}^{1,q})$.

Proof. Without loss of generality we may assume that $\mathcal{B}$ is a—sufficiently large—ball. Any bounded subset $\mathcal{B}$ of $\mathrm{MR}_0^s(J;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$ forms a precompact subset of $C(\overline{J};C(\overline{\Omega}))$, according to Lemma 4.5. Accordingly, its closure $\overline{\mathcal{B}}$ in $C(\overline{J};C(\overline{\Omega}))$ forms a compact set in this space which is convex, too. Now Theorem 4.10 (ii) tells us that the derivative of $\mathcal{S}$ is bounded on $\overline{\mathcal{B}}$. Since this set contains with any two points also the segment between them, an application of the mean value theorem gives the first claim. Finally, the assertion for $C(\overline{J};\mathbb{W}^{1,q})$ is obtained from the previous one via Lemma 4.5. □

Having introduced the solution operator $\mathcal{S}$ for (4.10)–(4.12), we now turn back to Theorem 4.9. Inserting $\mathcal{S}(u)$ with $u\in\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)$ for (v,p,w) in (4.9), one obtains a self-consistent equation for u alone together with the initial value condition $u(0)=u_0$. This equation can be solved via Theorem 3.18, as we will show below. Afterwards, having the solution $\bar u$ at hand, the functions $(\bar v,\bar p,\bar w)$ are determined via Lemma 4.10 or $\mathcal{S}(\bar u)$, from which they satisfy (4.10)–(4.12) automatically by construction. The quality of the whole solution of (4.9)–(4.12) is then $\bar u\in\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)$ and $(\bar v,\bar p,\bar w)\in\mathrm{MR}_0^s(I;\mathrm{dom}_{\mathbb{L}^p}(\Delta),\mathbb{L}^p)$.

We have formulated the next big step—the application of Theorem 3.18—as a theorem on its own. For this, let $\mathcal{S}_1$ denote the v-component of $\mathcal{S}$, $\mathcal{S}_2$ the p-component of $\mathcal{S}$, and $\mathcal{S}_3$ the w-component of $\mathcal{S}$.

Theorem 4.13. *Suppose $(u_0, v_0, p_0, w_0) \in \mathrm{IV}(r, s)$. Then there exists a maximal interval $I_\bullet =]0, S_\bullet[\subseteq J$ such that the equation*

$$\begin{aligned} u'(t) &+ A\big(\kappa(u(t), v_{\mathcal{I}}(t) + \mathcal{S}_1(u)(t))\big)u(t) \\ &= A\big(\sigma(u(t), v_{\mathcal{I}}(t) + \mathcal{S}_1(u)(t))\big)\big(v_{\mathcal{I}}(t) + \mathcal{S}_1(u)(t)\big) + \widehat{R}_1\big(t; u(t), \mathcal{S}(u)(t)\big), \end{aligned} \tag{4.20}$$

has a unique solution $u \in \mathrm{MR}^r(I; \mathrm{dom}_{L^p}(\Delta), L^p)$ with initial value $u(0) = u_0$ on every subinterval $I =]0, S[\subset I_\bullet$.

In order to validate the suppositions in Theorem 3.18, we will formulate some lemmata:

Lemma 4.14. *Let $\xi \colon \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable. Then the superposition operator $C(\overline{\Omega}) \times C(\overline{\Omega}) \ni (\psi, \varphi) \to \xi(\psi(\cdot), \varphi(\cdot))$ induced by ξ is well defined and Lipschitzian on bounded sets when considered as an operator from $W^{1,q} \times W^{1,q}$ into $W^{1,q}$.*

Proof. Let $\mathcal{B}$ be a bounded set in $W^{1,q}$ and assume firstly that $\psi, \varphi \in \mathcal{B} \cap C^\infty(\Omega)$. Taking into account that $\mathcal{B}$ forms a bounded subset of $C(\overline{\Omega})$, a straight forward calculation shows the existence of a constant $c = c(\mathcal{B}, \xi)$ such that

$$\|\xi(\psi_1, \varphi_1) - \xi(\psi_2, \varphi_2)\|_{W^{1,q}} \le c\big(\|\psi_1 - \psi_2\|_{W^{1,q}} + \|\varphi_1 - \varphi_2\|_{W^{1,q}}\big), \tag{4.21}$$

holds for all $\psi, \varphi \in \mathcal{B} \cap C^\infty(\Omega)$. Thus, the superposition operator induced by ξ is defined on a dense subset of $\mathcal{B} \times \mathcal{B} \subset W^{1,q} \times W^{1,q}$ and is uniformly continuous in $W^{1,q}$ w.r.t. the $W^{1,q} \times W^{1,q}$-toplogy. Hence, it can be extended to all of $\mathcal{B} \times \mathcal{B}$, with the same estimate as in (4.21). ☐

We immediately obtain the following extension from the preceding lemma.

Corollary 4.15. *Let $\xi \colon \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable. In the spirit of Lemma 4.14, ξ induces a superposition operator $C(\overline{J}; W^{1,q}) \times C(\overline{J}; W^{1,q}) \to C(\overline{J}; W^{1,q})$ via*

$$C(\overline{J}; W^{1,q}) \times C(\overline{J}; W^{1,q}) \ni (\psi, \varphi) \mapsto \big[t \mapsto \xi(\psi(t), \varphi(t))\big] \in C(\overline{J}; W^{1,q}),$$

and this mapping is also Lipschitzian on bounded sets.

The next lemma covers the differential operators occurring in (4.9).

Lemma 4.16. *Let $\xi \colon \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable.*

i) *The operator*

$$\mathcal{A}(u)(t) := A\big(\xi(u(t), v_{\mathcal{I}}(t) + \mathcal{S}_1(u)(t))\big) \tag{4.22}$$

defines a mapping

$$\mathcal{A} : \mathrm{MR}^r(J; \mathrm{dom}_{L^p}(\Delta), L^p) \to C(\overline{J}; \mathcal{L}(\mathrm{dom}_{L^p}(\Delta); L^p)).$$

Moreover, $\mathcal{A}$ is Lipschitzian on bounded subsets of $\mathrm{MR}^r(J; \mathrm{dom}_{L^p}(\Delta), L^p)$.

ii) *If, additionally, ξ is a strictly positive function, then $\mathcal{A}(u)|_I$ provides a topological isomorphism between $\mathrm{MR}_0^r(I;\mathrm{dom}_{L^p}(\Delta),L^p)$ and $L^r(I;L^p)$ for every subinterval $I=\,]0,S[\,\subseteq J$ and every $u\in\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)$. In particular, $\mathcal{A}$ satisfies assumptions i) and ii) in Theorem 3.18 for the spaces $X=L^p$ and $Y=\mathrm{dom}_{L^p}(\Delta)$ in this case.*

Proof. i) According to Lemma 4.5, both spaces $\mathrm{MR}(J;\mathrm{dom}_{L^p}(\Delta),L^p)$ and $\mathrm{MR}_0^s(\mathrm{dom}_{L^p}(\Delta),L^p)$ each embed continuously into $C(\overline{J};W^{1,q})$. Hence, both u and $\mathcal{S}_1(u)$ are from $C(\overline{J};W^{1,q})$, cf. Theorem 4.10. Due to to Lemma 4.8 and (4.6), this is also true for the function $v_{\mathcal{I}}(\cdot)$. Thanks to Corollary 4.15, then the function $\xi\big(u(\cdot),v_{\mathcal{I}}(\cdot)+\mathcal{S}_1(u)(\cdot)\big)$ is also from $C(\overline{J},W^{1,q})$. This allows to apply Corollary 3.24, which shows that $\mathcal{A}$ as given in (4.22), is well-defined as a mapping into the space $C(\overline{J};\mathcal{L}(\mathrm{dom}_{L^p}(\Delta);L^p))$.

Let us further show the Lipschitz continuity of $\mathcal{A}$ on bounded subsets of the space $\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)$. Combining Corollary 4.12 and Lemma 4.14 shows that the mapping

$$\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)\ni u\mapsto \xi\big(u(\cdot),v_{\mathcal{I}}(\cdot)+\mathcal{S}_1(u)(\cdot)\big)\in C(\overline{J};W^{1,q})$$

is well-defined and Lipschitzian on any bounded subset of $\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)$. Now it remains to apply Corollary 3.24.

ii) Clearly, assumption i) of Theorem 3.18 is already covered by the first assertion in this lemma. Let u be a fixed function from $\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)$. Under the positivity condition on ξ, the functions $\xi\big(u(t),v_{\mathcal{I}}(\cdot)+\mathcal{S}_1(u)(t)\big)\in W^{1,q}$ are measurable and bounded from above and below by positive constants, uniformly for all $t\in\overline{J}$. Thus, the operators $\mathcal{A}(u)(t)$ satisfy maximal parabolic regularity on L^p for each fixed $t\in J$, cf. Theorem 3.17. Moreover, $t\mapsto\mathcal{A}(u)(t)$ belongs to $C(\overline{I};\mathcal{L}(\mathrm{dom}_{L^p}(\Delta);L^p))$ for every subinterval $I=\,]0,S[\,\subseteq J$ by i). But then Theorem 3.16 tells us that the non-autonomous operator $\mathcal{A}(u)$ on every such I satisfies maximal parabolic $L^r(I;\mathrm{dom}_{L^p}(\Delta),L^p)$-regularity. This is exactly assumption ii) in Theorem 3.18. □

Let us now turn to the right-hand side in (4.9).

Lemma 4.17. *Define for $u\in\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)$ the following operators:*

$$F_1(u):=A\big(\sigma(u(\cdot),v_{\mathcal{I}}(\cdot)+\mathcal{S}_1(u)(\cdot))\big)v_{\mathcal{I}}(\cdot),\tag{4.23}$$

$$F_2(u):=A\big(\sigma(u(\cdot),v_{\mathcal{I}}(\cdot)+\mathcal{S}_1(u)(\cdot))\big)\big(\mathcal{S}_1(u)(\cdot)\big),\tag{4.24}$$

$$F_3(u):=\widehat{R}_1\big(\cdot\,;u(\cdot),\mathcal{S}(u)(\cdot)\big).\tag{4.25}$$

Then F_1,F_2 and F_3 are well-defined as mappings from $\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)$ into $L^s(J;L^p)$ and Lipschitzian on bounded sets.

Proof. We first consider F_1 and F_2. Taking $\xi=\sigma$ in Lemma 4.16, we see that the operator function in (4.22) belongs to the space $C(\overline{J};\mathcal{L}(\mathrm{dom}_{L^p}(\Delta);L^p))$ for every $u\in\mathrm{MR}^r(J;\mathrm{dom}_{L^p}(\Delta),L^p)$. Due to the supposition $v_0\in(L^p,\mathrm{dom}_{L^p}(\Delta))_{1-\frac{1}{s},s}$, cf. (4.5) and (4.6), we already know that in fact $v_{\mathcal{I}}\in L^s(J;\mathrm{dom}_{L^p}(\Delta))$, see Lemma 4.8. For F_2, we recall that $\mathcal{S}_1(u)$ belongs to $L^s(J;\mathrm{dom}_{L^p}(\Delta))$, cf. Theorem 4.10. This shows that F_1 and F_2 are well-defined.

Let us prove the Lipschitz properties for F_1 and F_2. For F_1, this directly follows from Lemma 4.16 with $\xi=\sigma$, and the property $v_{\mathcal{I}}\in L^s(J;\mathrm{dom}_{L^p}(\Delta))$. On the other hand,

F_2 is of the form $F_2(u) = \mathcal{A}_\sigma(u)\mathcal{S}_1(u)$, where $\mathcal{A}_\sigma$ is the operator in (4.22) for $\xi = \sigma$, i.e., a product of two functions in u which are Lipschitzian and bounded on bounded sets in $\mathrm{MR}^r(J; \mathrm{dom}_{L^p}(\Delta), L^p)$ with values in the correct spaces, by Lemma 4.16 and Corollary 4.12, see also Remark 4.11. Hence F_2 is also Lipschitzian on bounded sets.

The assertions on F_3 are also satisfied: It remains to collect the continuity of $v_\mathcal{I}, p_\mathcal{I}$ and $w_\mathcal{I}$ due to Lemma 4.8 with the regularity of v_0, p_0 and w_0 as in (4.6), the assumptions on R_1 (cf. Assumption 2.1) and the properties of $\mathcal{S}(\cdot)$ as in Theorem 4.10 combined with Corollary 4.12. □

Lemma 4.18. *Define $\mathcal{A}$ as in* (4.22)*, there setting $\xi := \sigma$. Further, put $F := F_1 + F_2 + F_3$ as given in* (4.23)*–*(4.25)*. Then both $\mathcal{A}$ and F satisfy the Volterra property, cf. Theorem 3.18.*

Proof. We only need to check the supposition for $\mathcal{S}$. Since $\mathcal{S}(u)$ is obtained as the solution of a system of semilinear parabolic *forward* equations into which u enters *pointwise* with respect to the time variable, it is clear that if $u_1, u_2 \in C(\overline{J}; C(\overline{\Omega}))$ with $u_1 = u_2$ on a subinterval $I =]0, S[\subseteq J$, then also $\mathcal{S}(u_1)|_I = \mathcal{S}(u_2)|_I$. But this is exactly the Volterra property. □

Now all suppositions of Theorem 3.18 are proved to be satisfied in order to prove Theorem 4.13.

Proof of Theorem 4.13. Since we presupposed the correct regularity for the initial value $u_0 \in (L^p, \mathrm{dom}_{L^p}(\Delta))_{1-\frac{1}{r},r}$, it remains to collect all the assertions from Lemmata 4.16, 4.17 and 4.18. With these, Theorem 3.18 is applicable and, hence, proves Theorem 4.13. □

With Theorem 4.13 at hand, we are now in turn able to prove the main Theorem 4.3 via Theorem 4.9.

Proof of Theorem 4.9. Let $u \in \mathrm{MR}^r(I; \mathrm{dom}_{L^p}(\Delta), L^p)$ be the local-in-time solution of (4.20) on an interval $I \subset I_\bullet$ as given by Theorem 4.13. Lemma 4.5 shows that u admits the regularity to obtain $(v, p, w) := \mathcal{S}(u)$ via Theorem 4.10. This proves Theorem 4.9 by construction. □

Proof of Theorem 4.3. We use Theorem 4.9. Let

$$(u, (\check{v}, \check{p}, \check{w})) \in \mathrm{MR}^r(I; \mathrm{dom}_{L^p}(\Delta), L^p) \times \mathrm{MR}_0^s(I; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p)$$

be the solutions of (4.9)–(4.13) as given by Theorem 4.9 (we need to return to the accented way of denoting the functions, as introduced in (4.7), now). It suffices to "remove" the cut-off introduced in Definition 4.7 for $(\check{v}, \check{p}, \check{w})$. Let M be the number from Definition 4.7 for given $\delta > 0$. Firstly, from Lemma 4.8, we know that

$$\|v_\mathcal{I}\|_{C(\overline{I}; C(\overline{\Omega}))} \vee \|p_\mathcal{I}\|_{C(\overline{I}; C(\overline{\Omega}))} \vee \|w_\mathcal{I}\|_{C(\overline{I}; C(\overline{\Omega}))} \leq M.$$

On the other hand, since $\check{v}, \check{p}$ and $\check{w}$ are functions from $C_0(\overline{I}; C(\overline{\Omega}))$ by Lemma 4.5, there exists an interval $I_0 =]0, S_0[\; \subseteq I$ such that

$$\|\check{v}\|_{C(\overline{I}_0; C(\overline{\Omega}))} \vee \|\check{p}\|_{C(\overline{I}_0; C(\overline{\Omega}))} \vee \|\check{w}\|_{C(\overline{I}_0; C(\overline{\Omega}))} \leq \frac{\delta}{2}.$$

This means that

$$R_j^\eta\big(u(t), v_{\mathcal{I}}(t) + \check{v}(t), p_{\mathcal{I}}(t) + \check{p}(t), w_{\mathcal{I}}(t) + \check{w}(t)\big)$$
$$= R_j\big(u(t), v_{\mathcal{I}}(t) + \check{v}(t), p_{\mathcal{I}}(t) + \check{p}(t), w_{\mathcal{I}}(t) + \check{w}(t)\big)$$

for every $t \in \overline{I}_0$, hence $(u,(v,p,w))$ with (v,p,w) as in (4.7) are a solution to (4.1)–(4.5) on I_0, cf. (4.8). Moreover, (v,p,w) admits the correct regularity due to $(v_{\mathcal{I}}, p_{\mathcal{I}}, w_{\mathcal{I}}) \in \mathrm{MR}^s(J; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p)$, see Lemma 4.8. □

5. Concluding remarks

In this concluding chapter we want to comment on possible relaxations and possible modifications that can be done to apply our results also to some slightly different situations than those that we have proposed in the present paper.

i) Reduction to simplified models: We want to point out again that the simplified model (1.7) may also be treated by the strategy used above for the full model, with very little changes. The same is true for the case of only elliptic equations for v, p, and w, and for which one would not need to deal with a nonlocal equations. We refer to the paragraph in the introduction and to [64], where such a system was treated.

ii) Regularity of initial data: We suggest that one can reduce the requirements on the initial values considerably, if one is willing and able to work in spaces with temporal weights. The basis of such an approach are the results in [57] where it is shown that maximal parabolic regularity carries over to spaces with temporal weights. The demanding task would be to prove an analogue of Amann's theorem also in this case and, finally, carry out the programm of this paper in that setting. Clearly, this would be an ambitious program and is completely out of scope here.

iii) Boundary conditions in the model: Of course, one can also impose other boundary conditions than homogenous Neumann conditions. For example, one can also find references where no-flux boundary conditions for the equation of the population density and homogeneous Dirichlet conditions for the chemoattractant or homogeneous Dirichelt boundary conditions for both equations of the simplified system (1.7) are considered (see for example [24] and [87]). If other boundary conditions are imposed (cf. for instance [67]) or if the inhomogenities R_i consist of more delicate terms such as ones "living on the boundary" $\partial\Omega$, one can proceed in a quite similar way, basing on Assumption 3.5 in case of pure Dirichlet conditions or mixed boundary conditions. There also exist large classes of domains for which the assumptions is satisfied in these cases, cf. [26]. Then spaces of type $W^{-1,q}$ would be adequate to consider the system in and the prinicipal functional analytical framework would be very similiar. In particular, the needed elliptic and parabolic regularity results are also available here, cf. [10, Ch. 11].

iv) Convex domains: In contrast to the known results so far we did not assume the domain Ω to be convex. However, if the domain Ω *is* convex, then one can prove the result of well-posedness much easier: one is enabled to treat the problem in L^2, basing on the classical result $(-\Delta+1)^{-1}\colon L^2 \to H^2$, cf. [35, Ch. 3.2]. Namely, from this one deduces

$$\big(L^2, \mathrm{dom}_{L^2}(\Delta)\big)_{\theta,1} \hookrightarrow \big[L^2, \mathrm{dom}_{L^2}(\Delta)\big]_\theta \hookrightarrow \big[L^2, H^2\big]_\theta \hookrightarrow W^{1,4},$$

as long as $\theta \geq \frac{1}{2}(1+\frac{d}{4})$, the bound on θ being strictly smaller than 1 for space dimensions $d = 2$ or $d = 3$. Thus, one can prinicpally proceed as in our more general proof, thereby avoiding the highly nontrivial considerations in the non-Hilbert spaces we used.

v) Regularity of solutions: Concerning the equations for (v, p, w), one could chose any other integrability index $p \in]\frac{q}{2}, \infty[$ with respect to the spatial variable. Moreover, it is possible to *bootstrap* the regularity of the solutions by inserting the solutions $(v, p, w) \in \mathrm{MR}_0^s(J; \mathrm{dom}_{\mathbb{L}^p}(\Delta), \mathbb{L}^p) \hookrightarrow C(\overline{J}; C(\overline{\Omega}))$ of (4.10)–(4.12) into the right hand sides, which then each belong to a space $C^\beta(J; C(\overline{\Omega}))$ for some $\beta > 0$. Now exploiting the fact that $-\Delta$ also generates an analytic semigroup on $C(\overline{\Omega})$ (see [74, Rem. 2.6]) and the well known results of [61, Ch. 4], one obtains even more regularity for (v, p, w).

vi) Matrix-valued coefficient functions: Finally we want to point out a technicality concerning our considerations in Ch. 3.1 and 3.2. As already mentioned in the introduction, these considerations may also be generalized to *real matrix-valued* coefficients, since the underlying results are available also in this case, cf. [28] and the references therein, see also [26]. We did not undertake this here because the considered Keller-Segel model is restricted to scalar coefficients and the general way to proceed is clear.

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