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Executing large orders in a microscopic market model

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Abstract

In a recent paper, Alfonsi, Schied and Schulz (ASS) propose a simple order book based model for the impact of large orders on stock prices. They use this model to derive optimal strategies for the execution of large orders. We test this model in the context of an agent based microscopic stochastic order book model that was recently proposed by Bovier, Černý and Hryniv. While the ASS model captures some features of real markets, some assumptions in the model contradict our simulation results. In particular, from our simulations the recovery speed of the market after a large order is clearly depended on the order size, whereas the ASS model assumes the speed to be given by a constant. For this reason, we propose a generalisation of the model of ASS that incorporates this dependency, and derive the optimal investment strategies. We show that within our artificial market, correct fitting of this parameter leads to optimal hedging strategies that reduce the trading costs, compared to the ones produced by ASS. Finally, we show that the costs of applying the optimal strategies of the improved ASS model to the artificial market still differ significantly from the model predictions, indicating that even the improved model does not capture all of the relevant details of a real market.

1 Introduction

For a long time, financial mathematics mainly focused on asset pricing, but the scope has been extended in the last years. One of the newly established topics is the theory of optimal trading strategies [4, 16]. Here, the goal is to maximise profit by avoiding costs when executing an order. This is achieved by a clever segmentation of the order volume and a prudential choice of trading times. There have been several models to find these optimal trading strategies (for references see next paragraph); yet, since we do not know if these models capture all relevant features of real markets, we cannot be sure that the strategies work in reality, and tests on real markets would be an expensive experiment. For this reason, *microscopic market models* are an excellent tool for testing theoretical models of optimal trading strategies. Based on assumptions about the market participants' behaviour, these models simulate the trading of financial assets on the level of single traders or orders [11, 17, 8]. The emerging price processes show typical features of real markets [9, 15, 3]. Hence, microscopic models provide artificial, yet reasonable, market environments with full control over all parameters and influences on the price, known dynamics and the opportunity to sample all necessary data as needed without costs.

The particular problem from the theory of optimal trading strategies we focus on in this paper involves large orders. This problem considers a trader who would like to purchase a huge volume of shares up to time T. Since the supply of offered shares for a certain price is limited, the trader will not be able to purchase the whole order for the current price, but he or she will suffer from a price increase. This additional price impact, induced by the trader's own trading, can be lessened if he or she gives the market time to recover; the *best ask price* returns to previous levels. However, the time interval $[0, T]$ is assumed to be too short in order to wait for a full recovery of the market. The *optimal execution* problem asks for the optimal splitting and the optimal trading times to minimise the expected price impact. Clearly, the correct modelling of the market response to the executed orders plays a significant role in determining the optimal strategy. Early models assumed that every purchased block of shares had only a volume dependend permament impact on the price [5], later models introduced additional temporary price impacts that only affected the price of the recently traded block [2, 13]. Yet, it is doubtful if the complex dynamics of a limit order book (*LOB*) can be captured by looking at the best price only. Therefore, recent models attempt to take the dynamics of the whole order book into account. Obizhaeva and Wang introduced a model with an underlying block shaped LOB and calculated the optimal trading strategy in terms of a recursive formula by applying Bellman equations [14]. Alfonsi, Schied and Schulz introduced a generalisation of this model for general order book shapes and gave an explicit solution for the optimal trading strategy with respect to their market model [1]; this model is the one we will test in a microscopic market environment, and we refer to it as the *ASS model*.

The ASS model describes the underlying market by two parameters: The shape of the LOB given by a *shape function* f and a positive constant ρ expressing the *resilience speed* of the order book. There are two versions of the model: In the first one, the consumed volume recovers exponentially fast; in the second version, the best price recovers in this way. The shape of the order book is *static* such that there is a bijection between the impact on the best price and on the volume. Thus, the response of the order book to the execution of a large order depends on the current price impact only, but not on possible executions before.

To test ASS model, we need to select a microscopic market model. The model that serves best as virtual market environment was introduced by Bovier, Černy and Hryniv and is called the *opinion game* [8]. It simulates a family of traders on the level of a *generalised order book*. In contrast to a *classical* order book, a generalised one does not contain placed limit orders but holds the opinions of all market participants about a *fair* price; it also captures traders who are willing to trade for a price close to the best quotes but have not placed (public)[1] orders. These traders offer *hidden liquidity*; they will influence the price impact when an order is executed but do not appear in the order book [10]. Thus, the opinion game provides a more realistic market response to orders than classical order book models.

[1] In some markets, it is possible to place hidden orders [6].

In order to apply the ASS strategies in the opinion game, we have to determine the correct values for f and ρ. There are several problems to find the value for ρ. First, the ASS model does not assume a permanent impact in the assumptions; second, the market recovery is only poorly approximated by an exponential function; third, ρ does not exist as a constant value but depends on the traded volume. While the first two items can be bypassed, the third item strongly conflicts with the assumptions of the ASS model. For this reason we introduce a generalisation of the ASS model that we call the *generalised ASS model* or *gASS model*. The gASS model substitutes ρ by $\bar{\rho}$ that is a function of an order's price impact or volume impact, depending on the model version. Furthermore, we extend the results of the ASS model by proving that there exists a unique, deterministic optimal trading strategy for the gASS model.

We use the parameter values from the last section to calculate the gASS optimal strategies for several parameter sets, apply these strategies to the opinion game, and sample their impact costs. On a general level, the sampled costs show the expected *natural* behaviour; for instance, the costs decrease if the available trading time T or the number of trading opportunities within $[0, T]$ become larger. Furthermore, the simulations reinforce the advantages of the gASS model compared to the ASS model. We show that, although the ASS strategy with respect to the *right* value for ρ coincides with the gASS optimal strategy, a *bad*, yet reasonable, choice of the value for ρ produces significantly higher costs; a justification for the gASS model's benefit. On the other hand, we find that, in comparison to the predicted costs, the sampled costs of the gASS strategies are up to four times higher. This indicates that the (g)ASS model does not capture all relevant details of the opinion game's order book dynamics.

In section 2, we introduce the ASS model and state its optimal trading strategies. In section 3, we present this version of the opinion game that we used to analyse the ASS model. In section 4, we determine f and ρ in the opinion game, which leads to the gASS model. Finally, in section 5, we apply the gASS optimal strategies in the opinion game, and compare the resulting costs for several parameter sets. Furthermore, we show that the gASS strategies perform better than the ASS strategies in the opinion game.

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2 The market model of Alfonsi, Schied and Schulz and its optimal execution strategies

A trader would like to purchase $X_0 > 0$ shares within a time period $[0, T]$, $T > 0$. X_0 is assumed to be large such that the trader's order has an impact on the price and the underlying limit order book. We will refer to this trader as *large trader* in the following. Because we consider a buy order, we first define how the *upper part* of the LOB, which contains the sell limit orders, is modelled. As long as the large trader does not take action, the LOB is described by the *unaffected best ask price* $A^0 := (A^0_t)_{t\geq 0}$ and by a *shape function* $f : \mathbb{R} \to (0, \infty)$. A^0 is a martingale on a given filtered probability space $(\Omega, (\mathscr{F}_t)_{t\geq 0}, \mathscr{F}, P)$ satisfying $A^0_0 = A_0$ for some $A_0 \in \mathbb{R}$; f is a continuous function. The amount of shares available for a price $A^0_t + x$, $x \geq 0$, at time t is then given by $f(x)dx$. Notice that the shape of the order book with respect to the best ask price is static.

Now, assume the large trader acts for the first time and purchases x_0 shares at time t_0; he or she consumes all shares between $A^0_{t_0}$ and $A^0_{t_0} + D^A_{t_0+}$, $D^A_{t_0+}$ being uniquely determined by

$$\int_0^{D^A_{t_0+}} f(x)dx = x_0. \tag{1}$$

$D^A := (D^A)_{t\geq 0}$ is called the *extra spread* caused by the large trader. In general, if we know $D^A_{t_n}$ for a trading time t_n, $D^A_{t_n+}$ is given by

$$\int_{D^A_{t_n}}^{D^A_{t_n+}} f(x)dx = x_n \tag{2}$$

whereby x_n is the amount of shares traded at time t_n. The large trader is inactive between two trading times, t_n and t_{n+1}, and the extra spread recovers. For the exact way of recovery there are two versions considered. To conform to the notation of [1], we first state *version 2*. In this case, D^A_t is defined for $t \in (t_n, t_{n+1}]$ by

$$D^A_t := e^{-\rho(t-t_n)} D^A_{t_n+}. \tag{3}$$

The parameter ρ is a positive constant called the *resilience speed*. To complete the definition, we set $D^A_t := 0$ for $t \leq t_0$. Now, we can introduce the *best ask price* $A := (A_t)_{t\geq 0}$ by

$$A_t := A^0_t + D^A_t. \tag{4}$$

In contrast to A^0, A includes the large trader's impact. In particular, the amount of shares available for a price $A^0_t + x$ at time t is given by

$$\begin{cases} f(x)dx & \text{for } x \geq A_t - A^0_t \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

In other words, every trader in the market experiences the large trader's impact after time t_0.

The *price impact* D^A can also be expressed in terms of the *impact on the volume* $E^A := (E_t^A)_{t\geq 0}$. Because the shape function f is strictly positive, there is a one-to-one relation between E^A and D^A. Given D^A, the process E^A is defined by

$$E_t^A := \int_0^{D_t^A} f(x)dx. \tag{6}$$

We can generally introduce the antiderivative of f,

$$F(x) := \int_0^x f(x)dx, \tag{7}$$

to get the relations

$$E_t^A = F(D_t^A) \quad \text{and} \quad D_t^A = F^{-1}(E_t^A). \tag{8}$$

By (2) and (8), we easily conclude

$$E_{t_n+}^A = E_{t_n}^A + x_n. \tag{9}$$

This motivates to define *version 1*, in which we first define E^A and then derive D^A by relation (8). We set $E_t^A := 0$ for $t \in [0, t_0]$ and

$$E_t^A := e^{-\rho(t-t_n)} E_{t_n+}^A, \ t \in (t_n, t_{n+1}]. \tag{10}$$

The equations (9) and (10) define E^A completely.

Summarising, we have introduced two versions of the ASS model: In version 1, we define the volume impact E^A and assume that it recovers exponentially fast between the large trader's orders. D^A is then derived from E^A by relation (8); in version 2, we first define the price impact D^A, assume an exponentially fast recovery and derive E^A from it.

We cannot exclude a priori that it is reasonable to sell shares and to buy them back later. Thus, we also have to model the impact of (large) sell orders on the LOB. Such orders will be written as orders with negative sign. Let $B^0 = (B_t^0)_{t\geq 0}$ be the *unaffected best bid price* with

$$B_t^0 \leq A_t^0 \text{ for all } t \geq 0 \tag{11}$$

as only constraint for its dynamics. The *lower part* of the LOB is modelled by the shape function f on the negative part of its domain. More precisely, the number of bids for the price $B_t^0 + x$, $x < 0$, is given by $f(x)dx$. As before, we can now introduce the *extra spread* $D^B := (D_t^B)_{t\geq 0}$. Given a sell order $x_n < 0$, a trading time t_n, and $D_{t_n}^B$, $D_{t_n+}^B$ is implicitly defined by

$$\int_{D_{t_n}^B}^{D_{t_n+}^B} f(x)dx = x_n. \tag{12}$$

Note that D^B is non-positive. We equivalently define the *impact on the volume* $E^B := (E_t^B)_{t\geq 0}$ by

$$E_{t_n+}^B := E_{t_n}^B + x_n. \tag{13}$$

E^B is also non-positive, and its connection to D^B is again given by (8). To complete the definitions for sell orders, we set $D_t^B := 0$ and $E_t^B := 0$ for all $t \leq t_0$, and

$$\begin{cases} E_t^B := e^{-\rho(t-t_n)} E_{t_n+}^B & \text{for version 1} \\ \\ D_t^B := e^{-\rho(t-t_n)} D_{t_n+}^B & \text{for version 2} \end{cases} \quad \text{for } t \in (t_n, t_{n+1}], \tag{14}$$

whereby t_n and t_{n+1} are two successive trading times of the large trader.

Now that all orders are well defined, we introduce the *cost* of a large order x_{t_n} at some trading time t_n by

$$\pi_{t_n}(x_{t_n}) := \begin{cases} \int_{D_{t_n}^A}^{D_{t_n+}^A} (A_{t_n}^0 + x) f(x) dx & \text{for a buy market order } x_{t_n} \geq 0 \\ \\ \int_{D_{t_n}^B}^{D_{t_n+}^B} (B_{t_n}^0 + x) f(x) dx & \text{for a sell market order } x_{t_n} < 0 \end{cases} . \tag{15}$$

We assume that the large trader needs to purchase the X_0 shares in $N+1$ steps at equidistant points in time $0 =: t_0 < \cdots < t_N := T$. His or her *admissible strategies* are sequences $\xi = (\xi_0, \ldots, \xi_N)$ of random variables such that

- $\sum_{n=0}^N \xi_N = X_0$,
- ξ_n is $\mathscr{F}_{t_n}$-measurable for all n, and
- all ξ_n are bounded from below.

We denote the set of all admissible strategies by $\hat{\Xi}$. The goal is to find an admissible strategy ξ^* that minimises the *average cost* $\mathscr{C}(\xi)$ given by the sum of the single trades' costs:

$$\mathscr{C}(\xi) := \mathbb{E}\left(\sum_{n=0}^N \pi_{t_n}(\xi_n)\right). \tag{16}$$

Under the technical assumption that

$$\lim_{x\to\infty} F(X) = \infty \quad \text{and} \quad \lim_{x\to-\infty} F(x) = -\infty, \tag{17}$$

Alfonsi, Schied and Schulz give the unique optimal strategies for both versions explicitly. For the sake of convenience, we set $\tau := T/(N+1) = t_{n+1} - t_n$.

Theorem 2.1 (Optimal strategy for version 1, theorem 4.1 in [1])**.** *Suppose that the function*

$$h_1(x) := F^{-1}(x) - e^{-\rho\tau} F^{-1}(e^{-\rho\tau} x) \tag{18}$$

is one-to-one. Then there exists a unique optimal strategy $\xi^{(1)} = (\xi_0^{(1)}, \dots, \xi_N^{(1)})$*. The initial market order* $\xi_0^{(1)}$ *is the unique solution of the equation*

$$F^{-1}\left(X_0 - N\xi_0^{(1)}(1-e^{-\rho\tau})\right) = \frac{h_1(\xi_0^{(1)})}{1-e^{\rho\tau}}, \tag{19}$$

the intermediate orders are given by

$$\xi_1^{(1)} = \cdots = \xi_{N-1}^{(1)} = \xi_0^{(1)}(1-e^{-\rho\tau}), \tag{20}$$

and the final order is determined by

$$\xi_N^{(1)} = X_0 - \sum_{n=0}^{N-1} \xi_n^{(1)}. \tag{21}$$

In particular, the optimal strategy is deterministic. Moreover, it consists only of nontrivial buy orders, that is $\xi_n > 0$ *for all* n*.*

Theorem 2.2 (Optimal strategy for version 2, theorem 5.1 in [1])**.** *Suppose that the function*

$$h_2(x) := x\frac{f(x) - e^{-2\rho\tau}f(e^{-\rho\tau}x)}{f(x) - e^{-\rho\tau}f(e^{-\rho\tau}x)} \tag{22}$$

is one-to-one and that the shape function satisfies

$$\lim_{|x|\to\infty} x^2 \inf_{y\in[e^{-\rho\tau}x,x]} f(y) = \infty. \tag{23}$$

Then there exists a unique optimal strategy $\xi^{(2)} = (\xi_0^{(2)}, \dots, \xi_N^{(2)})$*. The initial market order* $\xi_0^{(2)}$ *is the unique solution of the equation*

$$F^{-1}\left(X_0 - N[\xi_0^{(2)} - F(e^{-\rho\tau}F^{-1}(\xi_0^{(2)}))]\right) = h(F^{-1}(\xi_0^{(2)})), \tag{24}$$

the intermediate orders are given by

$$\xi_1^{(2)} = \cdots = \xi_{N-1}^{(2)} = \xi_0^{(2)} - F(e^{-\rho\tau}F^{-1}(\xi_0^{(2)})), \tag{25}$$

and the final order is determined by

$$\xi_N^{(2)} = X_0 - \sum_{n=0}^{N-1} \xi_n^{(2)}. \tag{26}$$

In particular, the optimal strategy is deterministic. Moreover, it consists only of nontrivial buy orders, that is $\xi_n > 0$ *for all* n*.*

One can easily check that the orders $\xi_1^{(\cdot)}, \dots, \xi_{N-1}^{(\cdot)}$ have exactly the volume that has recovered since the last trade. In this sense, the theorems just give the right balance between the first and the last order. This balance is found by solving the particular equations, (19) and (24), given in both theorems.

3 The opinon game

Next, we focus on the opinion game. In section 3.1, we recapitulate the original model as introduced by Bovier, Černý and Hryniv in [8]. We have already discussed in the introduction why the underlying generalised order book of this model provides even more information about the market behaviour than a *classical* order book. Yet, the opinion game has no explicit notion of orders and, consequently, also large orders and their executions are not defined. However, we argue in section 3.2 that the generalised order book contains an implicit notion of orders. Furthermore, we state the algorithm that we use to simulate the execution of large orders and show on a qualitative level that this extension leads to a realistic response of the opinion game to large orders.

3.1 The model

We consider a fixed number of traders $N \in \mathbb{N}$ and a fixed number of tradable shares $M < N$. Every trader is described by the pair (p_i, n_i), whereby p_i is the opinion of trader i about the *right* logarithmic price; the opinion is personal and subjective. For numerical reasons, $p_i \in \mathbb{Z}$. The number of shares that trader i posses is given by n_i. In the most general setting, n_i can take values form 0 to M; however, we just divide traders in *buyers* and *sellers* by setting $n_i \in \{0, 1\}$. We define the *best bid price* by

$$p^b := \max_{i:n_i=0} p_i, \tag{27}$$

and the *best ask price* by

$$p^a := \min_{i:n_i=1} p_i; \tag{28}$$

the *price* p is given by

$$p := \frac{p^b + p^a}{2}. \tag{29}$$

The market is said to be in a *stable state* if $p^b < p^a$; no buyer is then willing to pay the lowest asked price and vice versa.

For numerical reasons, the dynamics are defined in discrete time. Every round consists of three steps:

1. **A trader is chosen**
   We define

$$g(i,t) := \begin{cases} (1 + p^b(t) - p_i(t))^{-\gamma} & \text{if trader } i \text{ is buyer} \\ (1 + p_i(t) - p^a(t))^{-\gamma} & \text{if trader } i \text{ is seller} \end{cases}, \tag{30}$$

   and set

$$P(\text{trader } i \text{ is chosen at time t}) := \frac{g(i,t)}{\sum_{j=1}^{N} g(j,t)}. \tag{31}$$

The parameter $\gamma > 0$ can be chosen arbitrarily. Observe that the defined measure prefers traders close to the price. The larger γ is the greater is this preference. Here, we assume that a trader close to the current price reacts faster to price fluctuations than a long time investor with an opinion being completely different from the current price.

2. **The trader's change of opinion**
If a trader is chosen, he or she changes her opinion to $p'_i(t+1) := p_i(t) + d(t)$. The random variable $d(t)$ takes values in $\{-l, -l+1, \dots, l-1, l\}$, $l \in \mathbb{N}$, and is independently sampled for all t. The measure of $d(t)$ is given by

$$P(d(t) = m) = \begin{cases} \frac{1}{2l+1}\left((\delta_{ext}(t)\delta_.)^m \wedge 1\right) & \text{for } m \neq 0 \\ \\ 1 - \sum_{m=1}^{l} P(d(t) = \pm m) & \text{else} \end{cases}, \text{ whereby} \quad (32)$$

$$\delta_. := \begin{cases} \delta_B & \text{if trader } i \text{ is a buyer} \\ \delta_S & \text{if trader } i \text{ is a seller} \end{cases}. \quad (33)$$

We assume that $\delta_S < 1 < \delta_B$ to implement the idea that all traders have a tendency to move into the direction of the price. The $\delta_{ext}(t)$ introduces a drift that changes randomly in time and acts on all traders in the same way, modelling news, rumors and events influencing the price. This drift process is of paramount importance for the stylized facts, statistical features of the price process on large time scales; however, as we want to concentrate on the large orders' impact, which happens on shorter time scales, we assume $\delta_{ext} \equiv 1$ in the remainder of this article.

3. **Trading (if necessary)**
If the market with the changed opinion is stable again, that is

$$p^b((p_1(t), \dots, p'_i(t), \dots, p_N(t))) < p^a((p_1(t), \dots, p'_i(t), \dots, p_N(t))), \quad (34)$$

we set $p_i(t+1) := p'_i(t+1)$, else a trade happens. Let us assume that trader i is a buyer, the other case is symmetric. We uniformly choose a trading partner j with $p_j(t) = p^a(t)$ and set $n_i(t+1) = 0$ and $n_j(t+1) = 1$. After the trade, both traders move away from the best price:

$$p_i(t+1) := p^a(t+1) + g \quad \text{and} \quad p_j(t+1) := p^a(t+1) - g \quad (35)$$

whereby g can be a fixed or random number in $\mathbb{N}$. This last step is justified by the idea that the traders want to make profit and are only willing to trade for a better price than they have paid.

3.2 An extension for large orders

For the existence of orders in the opinion game, let us consider a buyer and a seller with matching opinions such that a trade happens. In order book driven markets, trades can only come about if both traders have placed some kind of orders. From this point of view, the opinion game has an *implicit* notion of orders, at least when trades are happening. This observation motivates a change of our point of view on the opinion game: In the remainder of this article, we rather think about (maybe hidden or unplaced) buy or sell orders instead of traders with opinion. For the sake of convenience, we omit the word *generalised* in the following when we talk about the order book of the opinion game.

To test the ASS model, we have to introduce large orders to the opinion game. Assume we would like to purchase X stocks at time t. Then, we do not apply the standard dynamics explained above at time t; instead, we use the following algorithm:

set $p_k^{(1)} := p_k(t)$ for all $k \in \{1, \dots, N\}$

set $n_k^{(1)} := n_k(t)$ for all $k \in \{1, \dots, N\}$

let $p^a(1)$ be the best ask price of the configuration $\left(p_k^{(1)}, n_k^{(1)}\right)_{k\in\{1,\dots,N\}}$

from $x := 1$ to X do {

find i s.th. $p_i^{(x)} \leq p_j^{(x)}$ for all $j \in \{1, \dots, N\}$
$p_i^{(x+1)} := p^a(x)$
choose uniformly trading partner j s.th. $i \neq j$ and $p_j^{(x)} = p^a(x)$
$n_i^{(x+1)} := 1$ and $n_j^{(x+1)} := 0$
$p_j^{(x+1)} := p^a(x) - g$
$p_i^{(x+1)} := p^a(x) + \hat{g}(x)$

set $p_k^{(x+1)} := p_k^{(x+1)}$ for all $k \in \{1, \dots, N\}\backslash\{i, j\}$
set $n_k^{(x+1)} := n_k^{(x+1)}$ for all $k \in \{1, \dots, N\}\backslash\{i, j\}$
let $p^a(x{+}1)$ be the best ask price of the configuration $\left(p_i^{(x+1)}, n_i^{(x+1)}\right)_{i\in\{1,\dots,N\}}$

}

set $p_k(t+1) := p_k^{(X+1)}$ for all $k \in \{1, \dots, N\}\backslash\{i, j\}$

set $n_k(t+1) := n_k^{(X+1)}$ for all $k \in \{1, \dots, N\}\backslash\{i, j\}$

The value g is the same random or deterministic value as in the original dynamics. The random variables $\hat{g}(x)$ are independently distributed with measure

$$P(\hat{g}(x) = k) = \frac{1}{M} \sum_{n=1}^{N} \mathbb{1}_{\{p_n^{(x)} - p^a(x) = k\}} \text{ for } k \in \mathbb{N}_0. \tag{36}$$

In other words, we execute a large buy order of volume X by taking the lowest X orders one by one and putting them directly to the ask price such that a trade is enforced. The number of market participants is constant in the opinion game, thus taking orders from the tail is an obvious method to simulate a large order that is placed *out of the blue*. After each single trade, we adjust the order prices; the price of the (new) buy order is decreased by g, the price of the sell order is increased by $\hat{g}$. The density function of $\hat{g}$ is given by the order book's current shape.

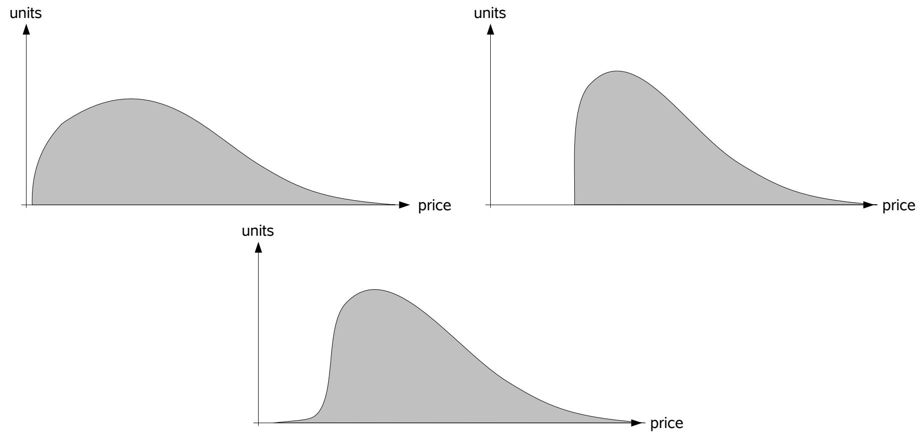

Figure 1: Sketch of the order book shape in the opinion game when a large order is executed. Before the execution, the order book is in equilibrium (upper left figure); directly afterwards, the best ask price is increased, and there is more liquidity close to it (upper right figure). When the LOB recovers from the order, the best ask price decreases, but the best quotes have a low volume only (lower figure); it takes more time until the order book is in equilibrium again.

This choice of $\hat{g}$ leads to a realistic response of the order book to the execution of large orders (see figure 3.2). While the large order is executed, the new sell orders have a great probability to be placed in vicinity to the peak of the order book's seller part; thus the peak grows, and the order book provides more liquidity for prices in this region. Here, we implement the idea that the execution of a large buy order leads to a conspicuous rise in the price that attracts more traders to place sell orders close to the current best ask price; these traders hope that the price increase continues such that their orders are executed. At the same time, these additional offers provide more liquidity that slows down the price increase. If we consider the immediate price impact of the large order as function of the executed volume, the additional liquidity leads to a sublinear function shape. Sublinear behaviour of an order's immediate price impact has also been observed for real world markets in several empirical studies [7, 3]. After the execution of the large order, the price increase stops and some traders realise quickly that orders for higher prices will probably not be executed in the near future; they place new orders for lower prices. However, most traders need more time to acknowledge that their price claims are probably too high. In result, the best ask price decreases, but the order book volume in proximity to the new best quote is low. It takes more time until the LOB is back in equilibrium. This recovery behaviour of the order book is technically implemented by the preference for traders close to the best quotes in (31) when we update opinions. As another feature that is known from real world markets, the best ask price does not return to the value it has had before the execution, but it stabilizes at higher values after the order book has returned to equilibrium. We discuss this *permanent impact* on the best price in section 4.2.

Since the dynamics are symmetric, the algorithm applies to large sell orders in the same way.

4 Determining the parameters

The opinion game provides a variety of parameters to influence the characteristics of the modelled market. For instance, it is possible to change the size of the market or the volatility in the opinion game to simulate *different* markets. Nevertheless, we restrict ourselves in the following to one parameter set, which is stated in subsection 4.1. Although the variation of parameters surely leads to additional insight, our choice already gives a sound understanding of the problems that occur when applying the ASS model. In the same subsection, we also describe the *averaged* shape of the opinion game's order book that will serve as shape function f for the ASS model.

Having set up the opinion game, we try to determine the ASS model's resilience speed, ρ. It turns out that the assumption of a constant ρ is not valid in the opinion game. Thus, we substitute ρ by a function $\bar{\rho}$ that maps both the order's impact and the time elapsed since the last trade to the resilience speed. We describe how we can extract the function values from the sampled data, and argue that it is sufficient to know the impact dependend function $\bar{\rho}(\cdot) := \bar{\rho}(\cdot, \tau)$ only; recall that $\tau = T/N$ was the recovery time between two successive trades. Finally, we introduce the generalised ASS theorems that assume the resilience speed to be a function of the price impact (in version 2, subsection 4.2) or the volume impact (in version 1, subsection 4.3).

4.1 The parameters of the opinion game and the shape of its order book

There is a high degree of freedom in the parameters for the opinion game. Nevertheless, certain parameter sets have been shown to be a reasonable choice. Calibrated

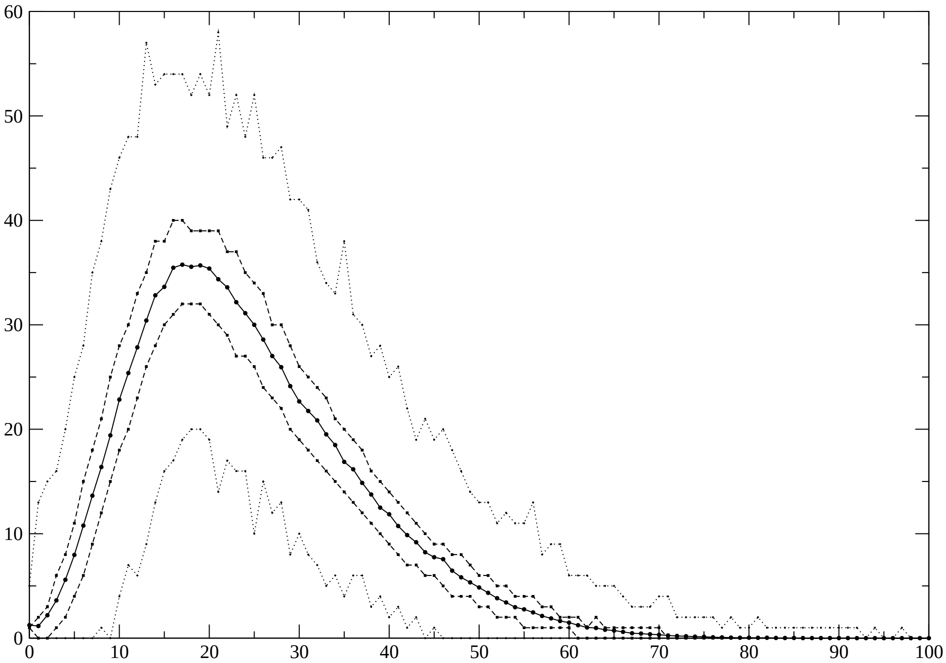

Figure 2: The seller part of the LOB relative to the best ask price. The solid line marks the mean values, the dashed lines illustrate the quartiles. The minimal and maximal values are illustrated by the dotted lines.

with these parameter values, the opinion game results in a realistic price process in terms of stylized facts. However, not all choices can be justified rigorously. For an extensive discussion about the choice of parameter γ, for instance, we refer to [18].

We used the following values in all simulations throughout this article:

$\sharp$ of traders N	2000
$\sharp$ of shares M	1000
speed of adaption γ	1.5
jump range $\{-l, \ldots, l\}$	$\{-4, \ldots, 4\}$
drift of buyers	$e^{0.1}$
drift of sellers	$e^{-0.1}$
jump range g	random variable, uniformly distributed on $\{5, \ldots, 20\}$, sampled idependently every time it is used

All sample runs that we did in the opinion game, either to extract necessary parameters or to test execution strategies, were started independently with a new instance of the random number generator. Furthermore, the recording of data or the execution of large orders was started after $1,000,000$ simulation steps only such that the model had sufficient time to get close to a stable state.

To determine f, we recorded 500 times the opinion game's LOB relative to the best prices. Figure 4.1 shows the resulting upper part of the order book. The lower part is symmetric up to small deviations caused by the object's random nature. Even if the shape is not static as assumed in the ASS model, an *averaged shape* is clearly visible. We use these mean values to define the shape function f for the opinion game. For non-integer values, we interpolate f by assuming that the function is a right-continuous step function. This means that we violate the assumption of the ASS model about f being continuous. Yet, this choice for f has the advantage that the integral of f from 0 to an integer n is equal to the sum of the integer function values from 0 to $n-1$. Furthermore, for all parameter sets that we considered, we were still able to find unique solutions for the optimal trading strategies.

Recall that the price scale in the opinion game is logarithmic, whereas the ASS model assumes a linear scale. However, it is possible to scale the grid of the opinion game with a factor ϵ, and the difference between logarithmic and linear scale is negligible if ϵ is small. To determine the order of ϵ, we consider an order of 200 units of shares, 20% of the market volume in the opinion game; it is mentioned in [1] that the size of large orders can amount up to twenty percent of the daily traded volume. We assume that the shape of the LOB, f, is determined as described above, and the best ask price before our trade is denoted by A^0. Then the relative impact costs are given by

$$\frac{1}{200e^{\epsilon A^0}} \int_0^{D_{0+}} e^{\epsilon(A^0+x)} f(x)dx - 1 \approx \frac{\epsilon}{200} \underbrace{\int_0^{D_{0+}} xf(x)dx}_{\approx 2039.47} \approx 10.20\epsilon. \tag{37}$$

An empirical study at the US stock market show that large orders can cause relative costs up to 3.55% [3]. If we assume that $\epsilon \leq 0.0355/10.2$, then ϵ is of order 10^{-3} at most. Thus, it is reasonable to assume ϵ to be small. However, we are interested in qualitative results; thus, and for the sake of convenince, we will simply assume that the opinion game operates on $\mathbb{Z}$.

4.2 Determining ρ for the ASS model, version 2

In the following, we describe our approach to calibrate ρ for the opinion game. Recall that, in version 2, this parameter determines the recovery speed of the price impact. We first describe how we sampled the necessary data. Afterwards, we focus on the main problems of extracting ρ from those data. Possible solutions are discussed and culminate in this section's main result: The gASS theorem for version 2, which assumes that the resilience speed is a function $\bar{\rho}$ depending on the order's price impact.

We fixed a price impact $D \in \{1, \ldots, 20\}$ and run 2500 simulations for each value of D. Each run consisted of a trading part in which a large sell order was executed at once. The particular order's volume was determined by its price impact: The trading part was finished as soon as the impact was equal to D. In a second experiment's

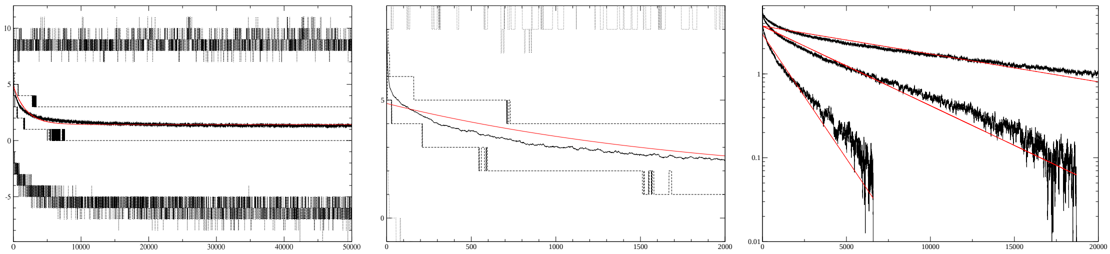

Figure 3: The left and the middle graph show quartiles and extremal values of 2500 samples of $\bar{p}$ for $D = 8$, and the corresponding $\langle \bar{p} \rangle$ and $\hat{p}$ (red). The left graph illustrates the long time behaviour on the domain $t \in [0, 50000]$. Clearly, $\hat{p}$ converges to a level $A_D > 0$. The middle graph displays $t \in [0, 2000]$ showing the poor approximation by $\hat{p}$. The right graph shows $\langle \bar{p} \rangle$ (black) for $D = 16$, $D = 12$ and $D = 8$ (top down) as well as their regression functions $\hat{p}$ (red) on a logarithmic scale and with respect to the new asymptotic levels A_D. If $\bar{\rho}_{num}$ was constant the $\langle \bar{p} \rangle$ should be approximately parallel.

part, we recorded the relaxiation of the price. In particular, the large execution took place at time $\bar{t} := 1,000,000$; we recorded

$$\bar{p}(t) := p^a(t + 1 + \bar{t}) - p^a(\bar{t}) \tag{38}$$

for $t \in \{0, \dots, 50000\}$. The process $(\bar{p}(t))_{t \in \mathbb{N}_0}$ is the discrete counterpart of the ASS model's process D^A.

To avoid problems caused by random fluctuations in $\bar{p}$, we consider the pointwise average of the samples denoted by $\langle \bar{p} \rangle$ and defined by

$$\langle \bar{p} \rangle_t := \frac{1}{2500} \sum_{i=0}^{2500} \bar{p}_t^i \tag{39}$$

for all $t \in \{0, \dots, 50000\}$, $\bar{p}^i$ denoting the ith sample. For a clear distinction, we denote the value for ρ that we extract from $\langle \bar{p} \rangle$ by $\bar{\rho}_{num}$. The ASS model assumes $\langle \bar{p} \rangle$ to be of the form

$$\langle \bar{p} \rangle_t = D e^{-\bar{\rho}_{num} t} \tag{40}$$

with a static value $\bar{\rho}_{num}$; this follows from equation (3). Thus we should be able to determine $\bar{\rho}_{num}$ by

$$\bar{\rho}_{num} = \frac{\ln D - \ln \langle \bar{p} \rangle_t}{t} \tag{41}$$

for an arbitrary t. However, the right hand side of the equation depends on D and t; thus, we would like to consider $\bar{\rho}_{num}(D, t)$ as a function.

Given $\langle \bar{p} \rangle$, let $\hat{p} : [0, \infty) \to \mathbb{R}$ the corresponding regression function of the form

$$\hat{p}_t := A + B e^{\hat{\rho} t}, \tag{42}$$

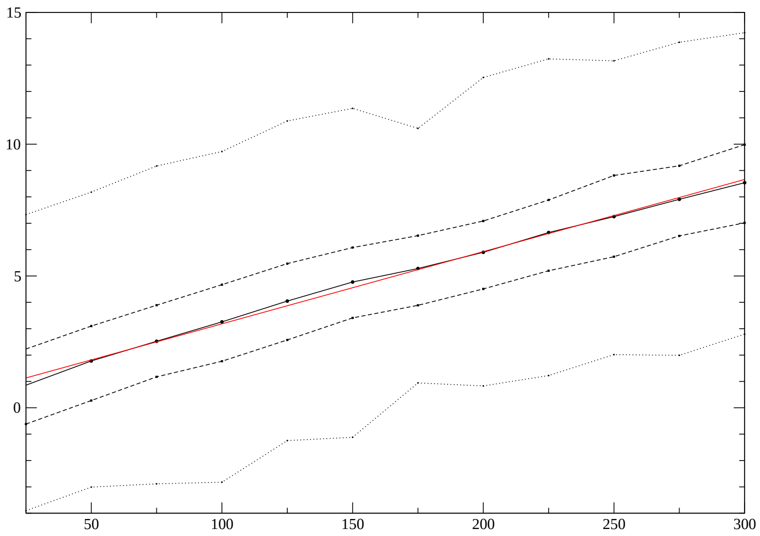

Figure 4: Mean, quartiles and extremal values of 500 samples of the permanent impact in dependence on the purchased volume $V \in \{25, 50, \ldots, 275, 300\}$. For every volume V, we recorded the best ask price before the trade and the averaged best ask price 500,000 steps after the trade. Here, the averaged best ask price is the mean of the best ask price sampled all 100 steps over a time interval of 100,000 steps. The linear regression of the mean is displayed in red.

for $t \in [0, \infty)$. It is determined by a Newton Gauß algorithm with three degrees of freedom: A, B, $\hat{\rho}$. Observe that all three values can depend on D. The form of the regression function is motivated by assumption (40), which also leads to the expectation that $A = 0$ and $B = D$. Figure 4.2 shows the statistical behaviour of $\bar{p}$ for $D = 8$, the corresponding $\langle \bar{p} \rangle$ and $\hat{p}$. Furthermore, we compare $\langle \bar{p} \rangle$ for different D values. The three main problems are visible:

1. The ASS model assumes A_D to be 0; this is not the case.
2. The measured data is only well-approximated by an exponential function for large times. For small t, it is doubtful that the assumption of an exponential decay is the right choice at all.
3. If $\bar{\rho}_{num}$ was constant the $\langle \bar{p} \rangle$ should be approximately parallel on a logarithmic scale; instead, $\bar{\rho}_{num}$ depends D.

These problems occured for all tested values of D. Next, we discuss the problems and their consequences for the determination of $\bar{\rho}_{num}$ one by one.

4.2.1 Existence of a permanent price impact

The reason for this observation is a permament impact on the order book that a large trade causes . After having recovered, the LOB is shifted by $I_{per}(X)$, whereby

$I_{per} : \mathbb{R} \to \mathbb{R}$ is assumed to be increasing and $I_{per}(0) = 0$. Huberman and Stanzl [12] argued on a theoretic level that linearity of I_{per} is equivalent to the absence of arbitrage opportunities. Empirical studies by Almgren et al [3] reinforce the conjecture of a linear permanent impact: The authors state that the permanent impact is well described by the power law $x^{0.891 \pm 0.1}$ with respect to a Gaussian error model; the assumption of linearity can not be rejected by this result. Figure 4.2.1 shows the permanent impact for the opinion game. The mean is well aproximated by a linear function with coefficient 0.02738. Interestingly enough, it can be slightly better approximated by the power law $x^{0.90752}$, a result close to the empirical findings of Almgren et al.

Concerning the problems in determining $\bar{\rho}_{num}$, caused by the positive A_D, we have two possibilities: First, we could ignore the permanent impact such that $\bar{\rho}_{num}$ would be given by (41). This would be an appropriate solution for small t, but it would cause the ASS model to assume that even for large t the LOB is still not close to equilibrium; $\bar{\rho}_{num}$ could become arbitrarily small. Second, we could assume that the whole model has been shifted by A_D such that A_D is the new zero line. In this case, $\bar{\rho}_{num}$ would be given by

$$\bar{\rho}_{num}(D, t) = \frac{\ln D - \ln(\langle \bar{p} \rangle_t - A_D)}{t}, \tag{43}$$

which is fine for large t but grows to infinity as t goes to zero. To avoid this problem, we define

$$\bar{\rho}_{num}(D, t) := \frac{\ln D - \ln(\langle \bar{p} \rangle_t - (1 - e^{-t}) A_D)}{t}. \tag{44}$$

Furthermore, let us point out that there is no special reason to choose $1 - \exp(-t)$. However, at this point, it becomes clear that the complex dynamics within the LOB are poorly described by an added permanent impact function.

4.2.2 $\langle \bar{p} \rangle$ is poorly approximated by an exponential function

Since $\langle \bar{p} \rangle$ should decay exponentially fast, $\bar{\rho}_{num}$ should be a constant. However, the existence of a permanent impact and the consequential definition of $\bar{\rho}_{num}$ in (44) makes the validity of this assumption unlikely here. Yet, even without the permanent impact, the description of $\langle \bar{p} \rangle$ by an exponential function is poor as the upper right graph of figure 4.2 shows. As a result, $\bar{\rho}_{num}$ is time dependend. A time dependend resilience speed seems to be incompatible with theorem 2.2 at first, but a closer look at the theorem's statement reveals that ρ is only needed to determine the order book state *before* the next trade, given the state *after* the current trade. The time between two succeeding trades is given by τ. Thus, we focus on $\bar{\rho}_{num}(\cdot, \tau)$ and use the notation

$$\bar{\rho}_{num}(D) := \bar{\rho}_{num}(D, \tau) \tag{45}$$

assuming that τ, which is given by the input parameters N and T, is fixed. Figure 4.2.2 shows the function $\bar{\rho}_{num}(\cdot, \tau)$ for several values of τ.

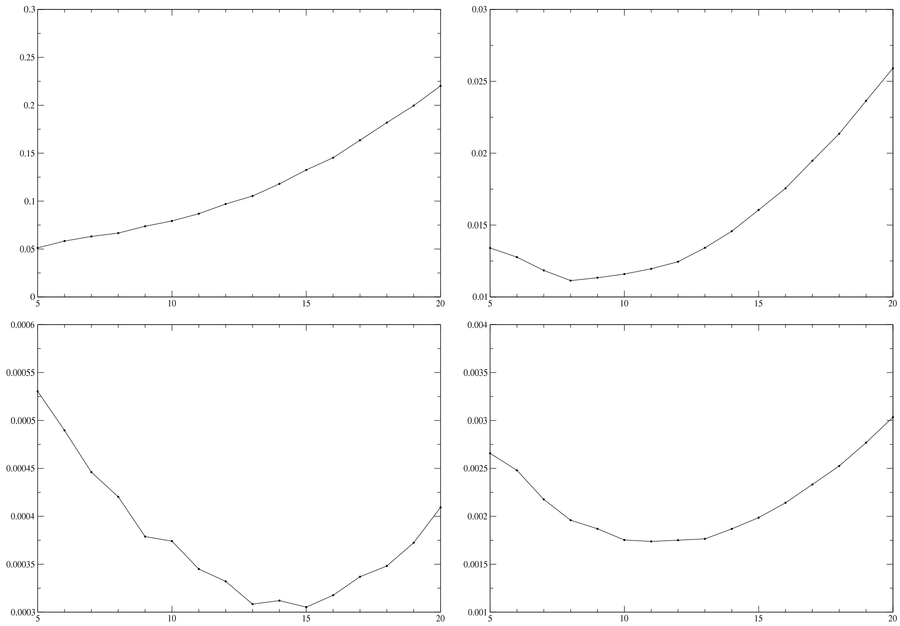

Figure 5: Starting in the upper left corner and proceeding clockwisely, we show the graphs of $\rho_{num}(D, \tau)$ in dependence on D for $\tau = 7, 70, 700, 7000$. Observe that the x-axis only begin in 5 due to the fact that small price impacts cannot be distinguished from the noise contained in the signal.

4.2.3 $\bar{\rho}_{num}$ is a function of D

In contrast to the time dependence, the dependence on the order's price impact requires a generalisation of theorem 2.1. Now, the resilience speed $\bar{\rho} : \mathbb{R} \to (0, \infty)$ is a continously differentiable function of D^A. In particular, the formulas (3) and (14), which describe the price recovery in the ASS model, change to

$$D_t^A := e^{-\bar{\rho}(D_{t_n+}^A)(t-t_n)} D_{t_n+}^A \text{ for } t \in (t_n, t_{n+1}], \tag{46}$$

$$D_t^B := e^{-\bar{\rho}(D_{t_n+}^B)(t-t_n)} D_{t_n+}^B \text{ for } t \in (t_n, t_{n+1}]. \tag{47}$$

We denote this modified model as *version 2* of the *generalised ASS model.*

For the following theorem concerning the optimal trading strategy for the gASS model, we need two technical assumptions:

$$\text{The range of } \bar{\rho} \text{ is a subset of } [k, K],\ 0 < k < K < \infty, \text{ and} \tag{48}$$

$$1 - \tau \bar{\rho}'(x) x > 0 \text{ for all } x \in \mathbb{R}. \tag{49}$$

The first assumption bounds the resilience speed, the second assumption ensures that a larger impact cannot overtake a smaller one in the recovery phase as we will see in lemma A.1.

Theorem 4.1 (Optimal stratey for the generalised ASS model, version 2)**.** *Suppose that $\bar\rho$ fulfils (48) and (49), and that f satisfies*

$$\lim_{|x|\to\infty} x^2 \inf_{y\in[e^{\tau\bar\rho(x)}x,x]} f(y) = \infty. \tag{50}$$

Furthermore, let the function

$$h_2(x) := x\frac{f(x) - e^{-2\tau\bar\rho(x)} f(e^{-\tau\bar\rho(x)}x)(1-\tau\bar\rho'(x)x)}{f(x) - e^{-\tau\bar\rho(x)} f(e^{-\tau\bar\rho(x)}x)(1-\tau\bar\rho'(x)x)} \tag{51}$$

be one-to-one. Then there exists a unique optimal strategy $\xi^{(2)} = (\xi_0^{(2)}, \ldots, \xi_N^{(2)}) \in \hat\Xi$. The initial market order $\xi_0^{(2)}$ is the unique solution of the equation

$$F^{-1}\left(X_0 - N\left[\xi_0^{(2)} - F\left(e^{-\tau\bar\rho(F^{-1}(\xi_0^{(2)}))}F^{-1}(\xi_0^{(2)})\right)\right]\right) = h_2(F^{-1}(\xi_0^{(2)})), \tag{52}$$

the intermediate orders are given by

$$\xi_1^{(2)} = \cdots = \xi_{N-1}^{(2)} = \xi_0^{(2)} - F\left(e^{-\tau\bar\rho(F^{-1}(\xi_0^{(2)}))}F^{-1}(\xi_0^{(2)})\right), \tag{53}$$

and the final order is determined by

$$\xi_N^{(2)} = X_0 - \sum_{n=0}^{N} \xi_n^{(2)}. \tag{54}$$

In particular, the optimal stratey is deterministic. Moreover, it consists only of nontrivial buy orders, that is $\xi_n^{(2)} > 0$ for all n.

Proof. See appendix A.2. □

Observe that the intermediate orders of the optimal strategy, defined in (53), have the same size. Furthermore, they suggest to purchase exactly that volume that has recovered since the last trade. The gASS model has inherited this feature from the ASS model. Yet, this observation means that also the $D^A_{t_n+}$ are equal to each other for all $n \in \{0, \ldots, N-1\}$, and thus, $\bar\rho$ is only evaluated for one value. In other words, although $\bar\rho$ is a function, the optimal strategy *uses* only one value. Of course, if $\bar\rho \equiv \rho$ for some constant ρ in the gASS model both models, the gASS and the ASS, coincide. This is the main advantage of the gASS theorem: It determines the *right* resilience speed from $\bar\rho$; a manual calibration, as in the ASS model, is not needed anymore.

4.3 Determining ρ for the ASS model, version 1

The procedure to determine ρ for version 1 and the occuring problems are similar as in section 4.2. Hence, we do not elaborate on the details again, but describe how we recorded the data and then focus on the gASS model for version 1.

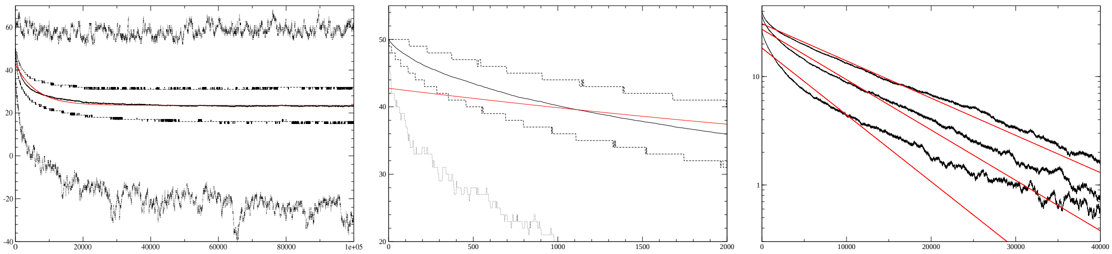

Figure 6: The left and the middle graph show quartiles and extremal values of 2500 samples of v for $D = 50$, and the corresponding $\langle v \rangle$ and $\hat{v}$ (red). The left graph illustrates the long time behaviour on the domain $t \in [0, 100000]$; the middle graph displays $t \in [0, 2000]$. The right graph shows $\langle v \rangle$ (black) for $D = 150$, $D = 100$ and $D = 50$ (top down) as well as their regression functions $\hat{v}$ (red) on a logarithmic scale and with respect to the new asymptotic levels A_E.

Version 1 of the ASS model considers the impact onto the volume. Thus, we fixed a volume impact $E \in \{10, 20, 30, \dots, 200\}$, and run 2500 simulations for each value. Having executed an order of size E at once, we recorded how the consumed volume recovered afterwards. In particular, the large execution took place at time $\bar{t} := 1000000$; we set

$$\bar{v} := \sum_{i=1}^{2000} \mathbb{1}_{\{p_i(\bar{t}+1)=p^a(\bar{t}+1)\}}, \tag{55}$$

which is the volume at the best ask price after the order execution and results from the discrete nature of the model. Then, we recorded $v(t)$ defined by

$$v(t) := E - \left[\sum_{i=1}^{2000} \mathbb{1}_{\{(n_i(\bar{t}+1+t)=1)\wedge(p_i(\bar{t}+1+t)\leq p^a(\bar{t}+1))\}} - \bar{v} \right] \tag{56}$$

for $t \in \{0, 2, 4, \dots, 99998\}$. In comparison to our proceedings for version 2, we chose a longer time interval to look at, because the volume recovers slower than the best price. The process v is the discrete counterpart of the ASS model's process E^A.

We consider the averaged volume impact $\langle v \rangle$, pointwisely defined by

$$\langle v \rangle_t := \frac{1}{2500} \sum_{i=0}^{2500} v^i(t) \tag{57}$$

for $t \in \{0, 2, 4, \dots, 99998\}$; v^i denoting the ith sample. Let $\hat{v} : [0, \infty) \to \mathbb{R}$ be $\langle v \rangle$'s regression function of the form

$$\hat{v}_t = A + Be^{-\hat{\rho}t}, \tag{58}$$

and let $\bar{\rho}_{num}$ denote that value for ρ that we extract from $\langle v \rangle$. Figure 4.3 shows the typical behaviour of v.

As in version 2, we have three main problems:

1. A positive aymptote A_E,
2. a poor approximation by an exponential function, and
3. a volume dependend $\bar{\rho}_{num}$.

The problems (1) and (2) can be treated as in the section before. In particular, we set

$$\bar{\rho}_{num}(E,t) := \frac{\ln E - \ln(\langle v \rangle_t - (1-e^{-t})A_E)}{t} \tag{59}$$

and point out that theorem 2.1 is only interested in

$$\bar{\rho}_{num}(E) := \bar{\rho}_{num}(E,\tau). \tag{60}$$

For problem (2), we generalise the ASS model, version 1, to the gASS model, version 1. Now, the resilience speed $\bar{\rho} : [0,\infty) \to (0,\infty)$ is a *twice* differentiable function of E^A. In particular, the formulas (10) and (14) from the ASS model become

$$E_t^A := e^{-\bar{\rho}(E^A_{t_n+})(t-t_n)} E^A_{t_n+}, \;\; t \in (t_n, t_{n+1}], \tag{61}$$

$$E_t^B := e^{-\bar{\rho}(E^B_{t_n+})(t-t_n)} E^B_{t_n+}, \;\; t \in (t_n, t_{n+1}]. \tag{62}$$

in the gASS model. Then, the following theorem determines the optimal trading strategy in the set of all admissible strategies $\hat{\Xi}$:

Theorem 4.2 (Optimal stratey for the generalised ASS model, version 1)**.** *Suppose that $\bar{\rho}$ fulfils the assumptions (48) and (49), and additionally*

$$e^{-\bar{\rho}(x)\tau}\left(1 - \tau\bar{\rho}'(x)x\right) < 1 \;\; \text{for all } x \in \mathbb{R}. \tag{63}$$

Furthermore, let the function

$$h_1(x) := \frac{F^{-1}(x) - e^{-\bar{\rho}(x)\tau}\left(1-\tau\bar{\rho}'(x)x\right)F^{-1}(e^{-\bar{\rho}(x)\tau}x)}{1 - e^{-\bar{\rho}(x)\tau}\left(1-\tau\bar{\rho}'(x)x\right)} \tag{64}$$

be one-to-one. Then there exists a unique optimal strategy $\xi^{(1)} = (\xi_0^{(1)}, \ldots, \xi_N^{(1)}) \in \hat{\Xi}$. The initial market order $\xi_0^{(1)}$ is the unique solution of the equation

$$F^{-1}\left(X_0 - N\xi_0^{(1)}(1-e^{-\bar{\rho}(\xi_0^{(1)})\tau})\right) = h_1(\xi_0^{(1)}), \tag{65}$$

the intermediate orders are given by

$$\xi_1^{(1)} = \cdots = \xi_{N-1}^{(1)} = \xi_0^{(1)}(1-e^{-\bar{\rho}(\xi_0^{(1)})\tau}), \tag{66}$$

and the final order is determined by

$$\xi_N^{(1)} = X_0 - \sum_{n=0}^{N} \xi_n^{(1)}. \tag{67}$$

In particular, the optimal stratey is deterministic. Moreover, it consists only of nontrivial buy orders, that is $\xi_n > 0$ for all n.

N	T	$\xi_0^{(2)}$	$\xi_1^{(2)}$	$\xi_N^{(2)}$	Predicted	Sampled	Samp/Pred
40	400	8.953	4.735	6.376	701.47	1867.74	266.26%
40	4000	6.128	4.813	6.154	500.24	1573.50	314.55%
40	40000	5.156	4.858	5.401	392.42	1076.89	274.42%
50	400	8.290	3.804	5.293	691.94	1853.37	267.85%
50	4000	5.195	3.875	4.937	462.51	1535.96	332.09%
50	40000	4.255	3.896	4.821	349.26	1014.42	290.44%
80	400	7.546	2.365	5.631	691.65	1832.69	264.97%
80	4000	3.732	2.442	3.371	387.98	1464.03	377.35%
80	40000	2.647	2.461	2.909	231.67	914.17	394.61%

Table 1: The optimal strategies according to the gASS model, version 2, for $X = 200$ and several values for N and T.

Proof. See appendix A.1. □

As in version 2 of the gASS model, $\bar{\rho}$ is only evaluated in one value, and if $\bar{\rho} \equiv \rho$ the best strategies of the gASS and the ASS models coincide.

5 Numerical results

Let us turn to the numerical results of this paper. We use the parameter values determined in the last section to calculate the gASS optimal strategies and apply them in the opinion game. We show first that the resulting costs show an *expected* behaviour on a general level, and that the ASS model with a suboptimal value for ρ suggests a strategy that produces significantly higher costs than the corresponding gASS strategy. Afterwards, we compare the costs sampled in the opinion game to the costs predicted by the gASS model, and find large differences. Again, we first treat version 2 in detail in section 5.1 and, afterwards, version 1 in section 5.2.

5.1 Results for version 2

Here, we refer to the values for f and $\bar{\rho}$, $\bar{\rho}_{num}$, as determined in sections 4.1 and 4.2.

Table 5.1 shows the gASS optimal strategies and their costs for different values of T and N. We consider two kinds of costs. The *predicted costs* are the impact costs that are theoretically predicted by the (g)ASS model. Here, we assume that the market behaves as described in section 2. The *sampled costs* are the average of 500 samples with the given strategy in the opinion game. Observe first that the predicted and sampled costs decrease if the trading time or the number of trading opportunities increase. Of course, this is no special feature of the (g)ASS strategies;

Strategy	$\xi_0^{(2)}$	$\xi_1^{(2)}$	$\xi_N^{(2)}$	Predicted	Sampled
gASS	3.732	2.442	3.371	387.98	1464.03
ASS	21.016	0.971	102.262	979.97	1584.12

Table 2: The optimal strategies and their costs for the ASS model with $\rho = \hat{\rho}$ and the gASS model with $\bar{\rho}$ from section 4.2. $(X, T, N) = (200, 4000, 80)$.

every fixed strategy benefits from a larger τ, which is implied by a greater T, and additional trading opportunities can be used, but do not have to be used. Thus, every reasonable strategy can only perform better with larger T or N. Nevertheless, the costs of the gASS strategies show a *reasonable* behaviour.

Furthermore, the gASS strategies perform better than the ASS strategies: Recall that the ASS model with the right value for ρ results in the same optimal strategy as the gASS model. Moreover, the (g)ASS model assumes an exponential decay of the price impact (see (40)). We have taken this assumption into account by introducing $\langle\bar{\rho}\rangle$'s regression function $\hat{p}$ in (42), which was of the form

$$\hat{p}_t := A + Be^{-\hat{\rho}t}. \tag{68}$$

Table 2 shows the optimal strategies and their costs for $(X, T, N) = (200, 4000, 80)$ with respect to the ASS model with $\rho = \hat{\rho}$ and the gASS model with $\bar{\rho} = \bar{\rho}_{num}$. The example shows that a naive guess of a good ρ can lead to much higher costs: The ASS costs amount 252.58% of the gASS costs in prediction, and still 108.02% in the samples.

The last two paragraphs have shown that the gASS strategies are reasonable and superior to the ASS stratgies. However, returning to table 5.1, we see that the predicted and the sampled costs for the individual parameter sets differ stronly from each other. The last column shows both kinds of costs in relation to each other. Obviously, the sampled costs are multiple times higher. This observation is a strong evidence that the assumptions of the (g)ASS model are insufficient to capture the whole complexity of the order book dynamics in the opinion game. It is doubtful if the (g)ASS model really suggests optimal trading strategies for this artificial market environment. With regard to the opinion game features concerning the order book behaviour that we have discussed in section 3, it is highly unlikely that the (g)ASS strategies minimise the costs in real world markets.

5.2 Results for version 1

Recall that version 1 of the gASS model assumes an exponentially fast recovery of the volume. Table 5.2 shows the optimal strategies for several values of N and T. Again, the predicted and sampled costs decrease with an increasing number of trading opportunities, N, or a longer trading horizon, T. Nevertheless, also version 1 shows large differences of the sampled costs compared to the predicted once. Even

N	T	$\xi_0^{(2)}$	$\xi_1^{(2)}$	$\xi_N^{(2)}$	Predicted	Sampled	Samp/Pred
40	400	16.95	4.407	11.169	1002.02	1846.48	184.28%
40	4000	12.764	4.509	11.371	872.63	1565.66	179.42%
40	40000	7.166	4.747	7.716	591.42	1064.72	180.03%
50	400	14.956	3.558	10.708	959.38	1837.80	191.56%
50	4000	11.613	3.668	8.656	845.09	1532.69	181.36%
50	40000	6.291	3.850	5.052	562.05	1012.16	180.08%
80	400	11.478	2.296	7.118	863.04	1869.49	216.62%
80	4000	9.619	2.332	6.156	794.95	1479.48	186.11%
80	40000	4.155	2.303	13.888	469.40	940.49	200.36%

Table 3: The optimal strategies according to the gASS model, version 1, for $X = 200$ and several values for N and T.

if not as large as for version 2, the sampled costs are approximately twice as high. The slightly better perfomance can be seen as a hint that the assumptions of version 1 are closer to the order book dynamics in the opinion game; yet, there is still a big gap between prediction and samples showing that, also in version 1, important features of the market dynamics are missing in the (g)ASS model.

A Proofs of the theorems 4.1 and 4.2

The structure of the proofs remains the same as in the proofs of the corresponding ASS theorems (see sections 8 to 10 in [1]). Nevertheless, we need to justify the constraints on $\bar{\rho}$; furthermore, the calculations become more complicated by our generalisation.

We start with the introduction of slightly changed dynamics for the gASS model and the reduction of the admissible strategies to deterministic ones. For any admissible strategy ξ, the new dynamics are defined by the processes $D := (D_t)_{t\geq 0}$ and $E := (E_t)_{t\geq 0}$. We set $D_0 = D_{t_0} := 0 =: E_{t_0} = E_0$ and

$$E_{t_n+} := E_{t_n} + \xi_n \text{ and } D_{t_n+} := F^{-1}(F(E_{t_n}) + \xi_n). \tag{69}$$

for the trading times $t_0, \dots, t_N$. The processes' values between two successive trading times $t \in (t_n, t_{n+1})$ are given by

$$\begin{aligned} E_t &:= e^{-\bar{\rho}(E_{t_n+})(t-t_n)} E_{t_n+} \quad \text{for version 1;} \\ D_t &:= e^{-\bar{\rho}(D_{t_n+})(t-t_n)} D_{t_n+} \quad \text{for version 2.} \end{aligned} \tag{70}$$

Given one process, we can recover the other one by the equations (8):

$$E_t = F(D_t) \quad \text{and} \quad D_t = F^{-1}(E_t). \tag{71}$$

Lemma A.1. *Under assumption (49),*

$$E_t^B \leq E_t \leq E_t^A \text{ and } D_t^B \leq D_t \leq D_t^A \tag{72}$$

for all $t \geq 0$. In the special case that all ξ_n are non-negative, we have $D^A = D$ and $E^A = E$.

Proof. To see that $D^A = D$ and $E^A = E$ if ξ consists of buy orders only, observe that the new dynamics match exactly the original ones for such a ξ.

We prove (72) by showing the following inequality:

$$|x+y|e^{-\bar{\rho}(x+y)\tau} > |x|e^{-\bar{\rho}(x)\tau} \tag{73}$$

for all $(x, y) \in \mathbb{R}^2$ such that $|x+y| > |x|$. Observe that we have equality in the equation above if we consider the trivial case that $y = 0$. We define a function $u_x : \mathbb{R} \to \mathbb{R}$ by

$$u_x(y) := (x+y)e^{-\bar{\rho}(x+y)\tau}. \tag{74}$$

Differentiation yields

$$u_x'(y) = e^{-\bar{\rho}(x+y)\tau}(1 - \tau\bar{\rho}'(x)x) \tag{75}$$

The right hand side of this equation is positive by assumption (49), thus u_x is strictly increasing. Since $u_x(0) = xe^{-\bar{\rho}(x)\tau}$, (73) is proven. □

It remains to define the *simplified price of ξ_n under the new dynamics* by

$$\bar{\pi}_{t_n}(\xi_n) := \int_{D_{t_n}}^{D_{t_n+}} (A^0_{t_n} + x)f(x)dx = A^0_{t_n}\xi_n + \int_{D_{t_n}}^{D_{t_n+}} xf(x)dx. \tag{76}$$

Observe that

$$\bar{\pi}_{t_n}(\xi_n) \leq \pi_{t_n}(\xi_n) \tag{77}$$

for all admissible strategies ξ because of lemma A.1. In particular, if ξ consists of buy orders only we have equality.

We show in the next two sections that, the strategies given in the theorems 4.2 and 4.1, $\xi^{(1)}$ and $\xi^{(2)}$, are the unique minimizers of the *price functional*

$$\bar{\mathscr{C}}(\xi) := \mathbb{E}\left[\sum_{n=0}^{N} \bar{\pi}_{t_n}(\xi_n)\right] \tag{78}$$

for the corresponding version of the model. As $\xi^{(1)}$ and $\xi^{(2)}$ consist of buy orders only, (77) and the remark afterwards imply that these strategies are also the minimizers of the original price functional $\mathscr{C}$.

We turn to the reduction of $\hat{\Xi}$ to deterministic strategies. Let us define the *remaining trading volume, X_t,* by

$$X_t := \begin{cases} X_0 - \sum_{t_n < t} \xi_n & \text{for } t \leq T \\ 0 & \text{for } t > T \end{cases}. \tag{79}$$

Observe that $(X_t)_{t\geq 0}$ is a bounded, predictable process for an admissible strategy ξ. We can transform the price of a strategy $\xi \in \hat{\Xi}$ by

$$\sum_{n=0}^{N} \bar{\pi}_{t_n}(\xi_n) = \sum_{n=0}^{N} A^0_{t_n}\xi_n + \sum_{n=0}^{N} \int_{D_{t_n}}^{D_{t_n+}} xf(x)dx, \tag{80}$$

and use definition (79) as well as *integration by parts* to rewrite the first term on the right hand side:

$$\sum_{n=0}^{N} A^0_{t_n}\xi_n = -\sum_{n=0}^{N} A^0_{t_n}\left(X_{t_{n+1}} - X_{t_n}\right) = X_0 A_0 + \sum_{n=0}^{N} X_{t_n}\left(A^0_{t_n} - A^0_{t_{n+1}}\right). \tag{81}$$

But, we know that $(X_t)_{t\geq 0}$ is a bounded, predictable process and A^0 is a martingale, and thus, the expectation of (81) must be $X_0 A_0$. The second term on the right hand side of (80) is deterministic for a given realization of a strategy $\xi(\omega)$. We denote this term by

$$\begin{array}{llll} C^{(i)}(\xi): & \mathbb{R}^{N+1} & \rightarrow & \mathbb{R} \\ & \xi & \mapsto & \sum_{n=0}^{N} \int_{D_{t_n}}^{D_{t_n+}} xf(x)dx \end{array} \tag{82}$$

for version i, $i \in \{1, 2\}$. Now, we can express $\bar{\mathscr{C}}$ by

$$\bar{\mathscr{C}}(\xi) = A_0 X_0 + \mathbb{E}(C^{(i)}(\xi)). \tag{83}$$

We spend the next two section to show $C^{(i)}$ has a unique minimiser in the set

$$\Xi := \left\{ x := (x_0, \ldots, x_N) \in \mathbb{R}^{N+1} : \sum_{n=0}^{N} x_n = X_0 \right\} \tag{84}$$

and this minimiser is determined by the formula given in theorem 4.2 or 4.1, respectively.
For the sake of convenience, we introduce some more notation:

$$\bar{a}_x := \exp(-\tau\bar{\rho}(x)) \text{ for } x \in \mathbb{R}, \tag{85}$$

$$a_n := \begin{cases} \exp(-\tau\bar{\rho}(E_{t_n+})) & \text{in section A.1} \\ \exp(-\tau\bar{\rho}(D_{t_n+})) & \text{in section A.2} \end{cases} \quad \text{for } n \in \{0, \ldots, N\}. \tag{86}$$

Because the range of $\bar{\rho}$ is assumed to be $[k, K]$, $0 < k < K < \infty$, by (48),

$$e^{-\tau K} \leq \bar{a}_x \leq e^{-\tau k} \quad \text{and} \quad e^{-\tau K} \leq a_n \leq e^{-\tau k}. \tag{87}$$

Additionally, we will need these functions:

$$\tilde{F}(x) := \textstyle\int_0^x xf(x)dx \quad \text{and} \quad G(x) := \tilde{F}(F^{-1}(x)). \tag{88}$$

Observe that

$$G'(x) = \tilde{F}'(F^{-1}(x))(F^{-1})'(x) = F^{-1}f(F^{-1}(x))\frac{1}{f(F^{-1}(x))} = F^{-1}(x), \tag{89}$$

and thus, G is twice continuously differentiable, non-negative, convex and has a fixed point in 0.

A.1 The optimal strategy for version 1

In this section, we calculate the unique minimiser of $C^{(1)}$ in Ξ. For any $\xi = (x_0, \ldots, x_N) \in \Xi$, we have

$$\begin{aligned} C^{(1)}(\xi) &= \sum_{n=0}^{N} \int_{D_{t_n}}^{D_{t_n+}} x f(x) dx & (90) \\ &= \sum_{n=0}^{N} \left[\tilde{F}(F^{-1}(E_{t_n+})) - \tilde{F}(F^{-1}(E_{t_n})) \right] \\ &= \sum_{n=0}^{N} \left[G(E_{t_n} + x_n) - G(E_{t_n}) \right] \end{aligned}$$

Lemma A.2. *The function $C^{(1)}$ has at least one local minimum in Ξ.*

Proof. The statement will follow from

$$C^{(1)}(\xi) \to \infty \text{ for } ||\xi||_\infty \to \infty \tag{91}$$

because $C^{(1)}$ is continuous. At first, we use the properties of G to find a lower bound for $G(x) - G(cx)$, $x \in \mathbb{R}$ and $c \in [0, 1]$:

$$\begin{aligned} G(x) - G(cx) &\geq G(cx) + (x - cx)G'(cx) - G(cx) & (92) \\ &= (1 - c)|F^{-1}(cx)||x|. \end{aligned}$$

The inequality (92) also leads to a lower bound for $C^{(1)}$ by using (90):

$$\begin{aligned} C^{(1)}(\xi) &= G(E_{t_N} + x_N) - G(E_{t_0}) + \sum_{n=0}^{N-1} \left[G(E_{t_n} + x_n) - G(E_{t_{n+1}}) \right] \\ &= G\left(\left(\Pi_{n=0}^{N} a_n \right) x_0 + \cdots + a_{N-1} x_{N-1} + x_N \right) - G(0) \\ &\quad + \sum_{n=0}^{N-1} \Big[G\left(\left(\Pi_{m=0}^{n-1} a_m \right) x_0 + \cdots + a_{n_1} x_{n-1} + x_n \right) & (93) \\ &\qquad - G\left(a_n \left[\left(\Pi_{m=0}^{n-1} a_m \right) x_0 + \cdots + a_{n_1} x_{n-1} + x_n \right] \right) \Big] \\ &\geq G\left(\left(\Pi_{n=0}^{N} a_n \right) x_0 + \cdots + a_{N-1} x_{N-1} + x_N \right) - G(0) \\ &\quad + \sum_{n=0}^{N-1} \Big[(1 - a_n) \left| F^{-1}\left(a_n \left[\left(\Pi_{m=0}^{n-1} a_m \right) x_0 + \cdots + a_{n_1} x_{n-1} + x_n \right] \right) \right| \\ &\qquad \left| a_n \left[\left(\Pi_{m=0}^{n-1} a_m \right) x_0 + \cdots + a_{n_1} x_{n-1} + x_n \right] \right| \Big]. \end{aligned}$$

We define a linear mapping $T : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ by

$$T(\xi) := \left(x_0, a_0 x_0 + x_1, \ldots, \left[\Pi_{n=0}^{N-1} a_n \right] x_0 + \cdots + a_{N-1} x_{N-1} + x_N \right), \tag{94}$$

and the smallest a_n by

$$a := \min\{a_n : n \in \{0, \dots, N\}\}. \tag{95}$$

Observe that

$$||T(\xi)||_\infty \geq ||(x_0, ax_0 + x_1, \dots, a^n x_0 + \dots + ax_{N-1} + x_N)||_\infty \to \infty \tag{96}$$

for $||\xi||_\infty \to \infty$ as well as $G(x) \to \infty$ and $|F^{-1}(ax)||x| \to \infty$ for $|x| \to \infty$. The last statement follows because F is unbounded. Finally, we define

$$H(x) := \min\left(G(x), |F^{-1}(ax)||x|\right). \tag{97}$$

Also $H(x) \to \infty$ for $x \to \infty$, and consequently,

$$C^{(1)}(\xi) \geq H(||T(\xi)||_\infty) - G(0) \to \infty. \tag{98}$$

□

One has to determine $\xi_0^{(1)}$ by solving

$$F^{-1}\left(X_0 - N\xi_0^{(1)}(1 - a_0)\right) = h_1(\xi_0^{(1)}) \tag{99}$$

in theorem (4.2). We define the function

$$\hat{h}_1(x) := h_1(x) - F^{-1}\left(X_0 - N(1 - \bar{a}_x)x\right) \tag{100}$$

for which $\xi_0^{(1)}$ is a zero.

Lemma A.3. *Given that the assumptions of theorem 4.2 hold, function $\hat{h}_1$ has at most one zero, which is positive if it exists.*

Proof. For the existence of at most one zero, it is sufficient to show that $\hat{h}_1$ is strictly increasing. The function h_1 has a fixed point in 0, is positive for positive arguments and continuous as well as bijective, thus it must be strictly increasing or, equivalently, its slope must be strictly positive. Consequently, the slope of $\hat{h}_1$ is also positive because

$$\begin{aligned} \hat{h}_1'(x) &= h_1'(x) - \left[F^{-1}\left(X_0 - Nx(1 - \bar{a}_x)\right)\right]' \qquad (101) \\ &= h_1'(x) + N\frac{1 - \bar{a}_x\left(1 - \tau\bar{\rho}'(x)x\right)}{f(F^{-1}(X_0 - Nx(1 - \bar{a}_x)))}, \qquad (102) \end{aligned}$$

and the numerator of the second term is positive by assumption (63). The positivity of the zero (if existing) follows simply from

$$\hat{h}_1(0) = -F^{-1}(X_0) < 0. \tag{103}$$

□

Lemma A.4. *For all $n \in \{0, \dots, N-1\}$, the partial derivatives of $C^{(1)}$ can be expressed by*

$$\frac{\partial}{\partial x_n} C^{(1)}(x) = F^{-1}(E_{t_n+}) + a_n \left(1 - \bar{\rho}'(E_{t_n+}) E_{t_n+}\right) \left[\frac{\partial}{\partial x_{n+1}} C^{(1)}(x) - F^{-1}(E_{t_{n+1}})\right].$$

Proof. We show first that

$$\frac{\partial}{\partial x_n} E_{t_m} = a_n \left(1 - \tau \bar{\rho}'(E_{t_n+}) E_{t_n+}\right) \frac{\partial}{\partial x_{n+1}} E_{t_m} \tag{104}$$

for $n \in \{0, \dots, m-2\}$. Convince yourself that

$$a_n = \bar{a}_{x_n + E_{t_n}} \tag{105}$$

to see that excessive usage of the the chain rule leads to

$$\begin{aligned}
\frac{\partial}{\partial x_n} E_{t_m} &= \frac{\partial}{\partial x_n} \left[a_{m-1} \left(x_{m-1} + a_{m-2} \left(\dots + a_n \left(x_n + E_{t_n}\right) \dots\right)\right)\right] \\
&= \left[\prod_{k=n+1}^{m-1} \left[\frac{d}{dx} \bar{a}_{x_k + x}(x_k + x)\right]_{x = E_{t_k}}\right] \left[\frac{\partial}{\partial x_n} \bar{a}_{x_n + E_{t_n}}(x_n + E_{t_n})\right] \qquad (106) \\
&= \underbrace{\left[\prod_{k=n+1}^{m-1} \left[\frac{d}{dx} \bar{a}_{x_k + x}(x_k + x)\right]_{x = E_{t_k}}\right]}_{= \frac{\partial}{\partial x_{n+1}} E_{t_m}} \left[a_n \left(1 - \tau \bar{\rho}'(E_{t_n+}) E_{t_n+}\right)\right].
\end{aligned}$$

Now, we use (90) and (104) to get

$$\begin{aligned}
& \frac{\partial}{\partial x_n} C^{(1)}(x) \\
= \; & G'(E_{t_n} + x_n) + \sum_{m=n+1}^{N} \frac{\partial}{\partial x_n} \left[G(E_{t_m} + x_m) - G(E_{t_m})\right] \\
= \; & F^{-1}(E_{t_n} + x_n) + \sum_{m=n+1}^{N} \left[\frac{\partial}{\partial x_n} E_{t_m}\right] \left[F^{-1}(E_{t_m} + x_m) - F^{-1}(E_{t_m})\right] \\
= \; & F^{-1}(E_{t_n+}) + a_n \left[1 - \tau \bar{\rho}'(E_{t_n+}) E_{t_n+}\right] \left[F^{-1}(E_{t_{n+1}+}) - F^{-1}(E_{t_{n+1}})\right] \\
& + a_n \left[1 - \tau \bar{\rho}'(E_{t_n+}) E_{t_n+}\right] \sum_{m=n+2}^{N} \left[\frac{\partial}{\partial x_{n+1}} E_{t_m}\right] \left[F^{-1}(E_{t_m+}) - F^{-1}(E_{t_m})\right].
\end{aligned}$$

The same calculation for $\partial / \partial x_{n+1}$ results in

$$\frac{\partial}{\partial x_{n+1}} C^{(1)}(x) = F^{-1}(E_{t_{n+1}}) + \sum_{m=n+2}^{N} \left[\frac{\partial}{\partial x_{n+1}} E_{t_m}\right] \left[F^{-1}(E_{t_{m+1}+}) - F^{-1}(E_{t_{m+1}})\right].$$

Combining both results yields the desired equation. □

Now, we are prepared to prove theorem 4.2.

Lemma A.5. *Strategy $\xi^{(1)}$ is the unique minimiser of function $C^{(1)}$ and all components of $\xi^{(1)}$ are positive.*

Proof. We showed in lemma A.2 that there is an optimal strategy $\xi^* = (x_0^*, \dots, x_N^*) \in \Xi$. Thus, there must be a Lagrange multiplier $\nu \in \mathbb{R}$ such that

$$\frac{\partial}{\partial x_n^*} C^{(1)}(\xi^*) = \nu \quad \text{for } n \in \{0, \dots, N\}. \tag{107}$$

Applying lemma A.4 yields

$$F^{-1}(E_{t_n+}) + a_n \left(1 - \bar{\rho}'(E_{t_n+})E_{t_n+}\right) \left[\nu - F^{-1}(E_{t_{n+1}})\right] = \nu \tag{108}$$

$$\Leftrightarrow \nu = \frac{F^{-1}(E_{t_n+}) - a_n \left(1 - \bar{\rho}'(E_{t_n+})E_{t_n+}\right) F^{-1}(a_n E_{t_n+})}{1 - a_n \left(1 - \bar{\rho}'(E_{t_n+})E_{t_n+}\right)} = h_1(E_{t_n+}) \tag{109}$$

for $n \in \{0, \dots, N-1\}$. The function h_1 is bijective by assumption, and thus,

$$\begin{aligned} x_0^* &= h_1^{-1}(\nu) && (110)\\ x_n^* &= (1-a_0)x_0^* \text{ for } n \in \{1, \dots, N-1\} && (111)\\ x_N^* &= X_0 - x_0^* - (N-1)x_0^*(1-a_0). && (112) \end{aligned}$$

Therefore, the optimal strategy ξ^* is completely defined if we can determine x_0^*. By (90),

$$\begin{aligned} C^{(1)}(x^*) &= G(x_0^*) - G(0) + (N-1)\left[G(a_0 x_0^* + (1-a_0)x_0^*) - G(a_0 x_0^*)\right] \\ &\quad + G(a_0 x_0^* + X_0 - x_0^* - (N-1)(1-a_0)x_0^*) - G(a_0 x_0^*) \\ &= N\left[G(x_0^*) - G(a_0 x_0^*)\right] + G(X_0 - N(1-a_0)x_0^*) - G(0) \qquad (113)\\ &=: C_0^{(1)}(x_0^*). \end{aligned}$$

We know that $C_0^{(1)}$ has a minimum because of lemma A.2. We can find it by differentiation:

$$\begin{aligned} &\frac{d}{dx} C_0^{(1)}(x) \qquad (114)\\ &= N\big[F^{-1}(x) - \bar{a}_x[1 - \tau\bar{\rho}'(x)x]F^{-1}(\bar{a}_x x) \\ &\qquad - \left(1 - \bar{a}_x[1 - \tau\bar{\rho}'(x)x]\right) F^{-1}(X_0 - N(1-\bar{a}_x)x)\big] \\ &= N\left(1 - \bar{a}_x[1 - \tau\bar{\rho}'(x)x]\right) \hat{h}_1(x). \end{aligned}$$

Assumption (63) and lemma A.3 tell us $C^{(1)}$ has exactly one minimum, and this minimum is positive. We have established the uniqueness and representation of the optimal strategy.

It remains to show that all components of x^* are positive. We already know that $x_0^* > 0$. The positivity of x_n^* follows from (111) for all $n \in \{1, \dots, N-1\}$. For the last order x_N^*, observe that (114) vanishes in x_0^*. Furthermore, F^{-1} is strictly increasing, and thus,

$$\begin{aligned} 0 &= F^{-1}(x_0^*) - a_0(1-\tau\bar{\rho}'(x_0^*)x_0^*)F^{-1}(a_0x_0^*) \\ &\quad - [1-a_0(1-\tau\bar{\rho}'(x_0^*)x_0^*)]\,F^{-1}(\underbrace{X_0 - N(1-a_0)x_0^*}_{=x_N^*+a_0x_0^*}) && (115) \\ &> [1-a_0(1-\tau\bar{\rho}'(x_0^*)x_0^*)]\left[F^{-1}(a_0x_0^*) - F^{-1}(a_0x_0^*+x_N^*)\right], && (116) \end{aligned}$$

which, indeed, implies the positivity of x_N^*. □

A.2 The optimal strategy for version 2

In this section, we determine the unique minimiser of $C^{(2)}$ in Ξ. For $\xi = (x_0, \dots, x_N) \in \Xi$, we have

$$\begin{aligned} C^{(2)}(\xi) &= \sum_{n=0}^{N} \int_{D_{t_n}}^{D_{t_n+}} xf(x)dx && (117) \\ &= \sum_{n=0}^{N} \left(G(x_n + F(D_{t_n})) - \tilde{F}(D_{t_n}) \right) \end{aligned}$$

Lemma A.6. *The function $C^{(2)}$ has a local minimum in Ξ.*

Proof. Again, it suffices to show

$$C^{(2)}(\xi) \to \infty \text{ for } ||\xi||_\infty \to \infty. \tag{118}$$

We rearrange (117) and get

$$\begin{aligned} & C^{(2)}(\xi) && (119) \\ = & \sum_{n=0}^{N} \left(\tilde{F}(F^{-1}(x_n + F(D_{t_n}))) - \tilde{F}(D_{t_n}) \right) \\ = & \tilde{F}(a_N F^{-1}(x_N + F(D_{t_N}))) \\ & + \sum_{n=0}^{N} \left(\tilde{F}(F^{-1}(x_n + F(D_{t_n}))) - \tilde{F}(a_n F^{-1}(x_n + F(D_{t_n}))) \right) \\ \geq & \sum_{n=0}^{N} \left(\tilde{F}(F^{-1}(x_n + F(D_{t_n}))) - \tilde{F}(a_n F^{-1}(x_n + F(D_{t_n}))) \right) \end{aligned}$$

A lower bound for the last line of (119) is given by

$$\begin{aligned} \tilde{F}(x) - \tilde{F}(\bar{a}_x x) &= \left| \int_{\bar{a}_x x}^{x} zf(z)dz \right| && (120) \\ &\geq \inf_{y \in [\bar{a}_x x, x]} f(y) \left| \int_{\bar{a}_x x}^{x} zdz \right| \\ &= \frac{1}{2}(1-\bar{a}_x^2)x^2 \inf_{y \in [\bar{a}_x x, x]} f(y) \;\geq\; 0. \end{aligned}$$

Because of the assumptions (50) and (17), we know

$$H(x) := \frac{1}{2}(1-\bar{a}^2_{F^{-1}(x)})(F^{-1}(x))^2 \inf_{y\in[\bar{a}_{F^{-1}(x)}F^{-1}(x),F^{-1}(x)]} f(y). \tag{121}$$

tends to infinity for $|x| \to \infty$. Finally, we introduce the mapping

$$T(x) := (x_0, x_1 + F(D_{t_1}), \ldots, x_N + F(D_{t_N})), \tag{122}$$

for which $C^{(2)}(x) \geq H(||T(x)||_\infty)$ holds. It remains to show that $||T(x)||_\infty \to \infty$ for $|x| \to \infty$. Let us assume there is a sequence x^k such that $||x^k||_\infty \to \infty$ but $T(x^k)$ remains bounded. This implies especially the boundedness of (x^k_0). But then again, $D^k_{t_1} = a^k_0 F^{-1}(x^k_0)$ remains bounded. We can continue the argumentation for all coordinates of $T(x)$ and conclude that (x^k_n) is a bounded sequence for all $n \in \{0,\ldots,N\}$. This contradicts the assumption, and thus the lemma is proven. □

Lemma A.7. *Under the assumptions of theorem 4.1, equation (52) has at most one solution, which is positive if existing. Furthermore,* $g(x) := f(x) - \bar{a}_x f(\bar{a}_x x)(1 - \tau\bar{\rho}'(x)x)$ *is positive.*

Proof. We show that both $h_2 \circ F^{-1}$ and

$$\hat{h}_2(x) := -F^{-1}\left(X_0 - N\left[x - F\left(\bar{a}_{F^{-1}(x)}F^{-1}(x)\right)\right]\right) \tag{123}$$

are strictly increasing. In this case, at most one zero can exist, and its positivity is guaranteed by $h_2(F^{-1}(0)) = 0$ and $\hat{h}_2(0) = -F(X_0) < 0$. The function h_2 is strictly increasing because it is continuous, bijective, has a fixed point at zero and

$$\lim_{\epsilon\to 0}\frac{h_2(\epsilon)-h_2(0)}{\epsilon} = \lim_{\epsilon\to 0}\frac{f(\epsilon)-\bar{a}^2_\epsilon f(\bar{a}_\epsilon\epsilon)(1-\tau\bar{\rho}'(\epsilon)\epsilon)}{f(\epsilon)-\bar{a}_\epsilon f(\bar{a}_\epsilon\epsilon)(1-\tau\bar{\rho}'(\epsilon)\epsilon)} \tag{124}$$

$$= \frac{1-\bar{a}^2_0}{1-\bar{a}_0} > 0. \tag{125}$$

Since F^{-1} is also strictly increasing, we have proven the same property for $h_2 \circ F^{-1}$. We differentiate $\hat{h}_2$:

$$\hat{h}'_2(x) = N\left[\frac{f(F^{-1}(x)) - \bar{a}_{F^{-1}(x)}f(\bar{a}_{F^{-1}(x)}F^{-1}(x))(1-\tau\bar{\rho}'(F^{-1}(x))F^{-1}(x))}{f(F^{-1}(x))f\left(F^{-1}\left(X_0 - N\left[x - F(\bar{a}_{F^{-1}(x)}F^{-1}(x))\right]\right)\right)}\right] \tag{126}$$

This expression is strictly positive because the numerator is strictly positive as we show next. We define both

$$\begin{aligned} k(x) &:= f(x) - \bar{a}_x f(\bar{a}_x x)(1-\tau\bar{\rho}'(x)x) \text{ and} \\ k_2(x) &:= f(x) - \bar{a}^2_x f(\bar{a}_x x)(1-\tau\bar{\rho}'(x)x). \end{aligned} \tag{127}$$

The numerater of (126) can be expressed by $k(F^{-1}(x))$, and furthermore, $h_2(x) = xk_2(x)/k(x)$. Both functions k and k_2 are continuous, and due to the properties of h explained in the beginning of the proof, the functions must have the same sign for all $x \in \mathbb{R}$. The function k_2 is greater than k for all $x \in \mathbb{R}$; thus, there can be no change of signs and we have either $k(x) > 0$ and $k_2(x) > 0$ or $k(x) < 0$ and $k_2(x) < 0$ for all x. Because $k(0) = f(0)(1 - \bar{a}_0) > 0$, positivity is proven. □

Lemma A.8. *For all $n \in \{0, \ldots, N-1\}$, the partial derivatives of $C^{(1)}$ can be expressed by*

$$\frac{\partial}{\partial x_n} C^{(2)}(x) \;=\; D_{t_n+} + \frac{a_n f(D_{t_{n+1}})(1-\tau\bar{\rho}'(D_{t_n+})D_{t_n+})}{f(D_{t_n+})} \left[\frac{\partial}{\partial x_{n+1}} C^{(2)}(x) - D_{t_{n+1}}\right].$$

Proof. First, observe

$$\frac{\partial}{\partial x_n} D_{t_m} = \frac{a_n f(D_{t_{n+1}})}{f(D_{t_n+})} \left(1-\tau\bar{\rho}'(D_{t_n+})D_{t_n+}\right) \frac{\partial}{\partial x_{n+1}} D_{t_m} \tag{128}$$

for $n \in \{0, \ldots, m-2\}$. This follows from

$$\begin{aligned} & \frac{\partial}{\partial x_n} D_{t_m} \qquad (129) \\ = \;& \frac{\partial}{\partial x_n} \left[a_{m-1} F^{-1}(x_{m-1} + F(\ldots(a_n F^{-1}(x_n + F(D_{t_n})))\ldots))\right] \\ = \;& \left[\prod_{k=n+1}^{m-1} \left[\frac{d}{dx} \bar{a}_{F^{-1}(x_k+F(x))} F^{-1}(x_k+F(x))\right]_{x=D_{t_k}}\right] \left[\frac{\partial}{\partial x_n} \bar{a}_{F^{-1}(x_n+F(D_{t_n}))} F^{-1}(x_n + F(D_{t_n}))\right] \\ = \;& \left[f(D_{t_{n+1}}) \frac{\partial}{\partial x_{n+1}} D_{t_m}\right] \left[\frac{a_n\left(1-\tau\bar{\rho}'(D_{t_n+})D_{t_n+}\right)}{f(D_{t_n+})}\right]. \end{aligned}$$

We use (128) and (117) for the transformation

$$\begin{aligned} & \frac{\partial}{\partial x_n} C^{(2)}(x) \qquad (130) \\ = \;& F^{-1}(x_n + F(D_{t_n})) + \sum_{m=n+1}^{N} \frac{\partial}{\partial x_n} \left[G(x_m + F(D_{t_m})) - \tilde{F}(D_{t_m})\right] \\ = \;& D_{t_n+} + \sum_{m=n+1}^{N} f(D_{t_m}) \left[\frac{\partial}{\partial x_n} D_{t_m}\right] \left[F^{-1}(x_m + F(D_{t_m})) - D_{t_m}\right] \\ = \;& D_{t_n+} + \frac{a_n f(D_{t_{n+1}})}{f(D_{t_n+})} \left[1-\tau\bar{\rho}'(D_{t_n+})D_{t_n+}\right] \Big(D_{t_{n+1}+} - D_{t_{n+1}} \\ & + \sum_{m=n+2}^{N} f(D_{t_m}) \left[\frac{\partial}{\partial x_{n+1}} D_{t_m}\right] \left[F^{-1}(x_m + F(D_{t_m})) - D_{t_m}\right]\Big). \end{aligned}$$

Now, the same calculation for $\partial C^{(2)}(x)/\partial x_{n+1}$ results in

$$\frac{\partial}{\partial x_n} D_{t_m} = D_{t_{n+1}+} + \sum_{m=n+2}^{N} f(D_{t_m}) \left[\frac{\partial}{\partial x_{n+1}} D_{t_m}\right] \left[F^{-1}(x_m + F(D_{t_m})) - D_{t_m}\right], \tag{131}$$

and combining (130) and (131) yields the desired result. □

Finally, we are prepared to prove theorem 4.1.

Lemma A.9. *Strategy $\xi^{(2)}$ is the unique minimiser of function $C^{(2)}$ and all components of $\xi^{(2)}$ are positive.*

Proof. Lemma A.6 guarantees the existence of at least one optimal strategy $\xi^* \in \Xi$. By standard arguments, there is a Lagrange multiplier $\nu \in \mathbb{R}$ such that

$$\frac{\partial}{\partial x_n^*} C^{(2)}(\xi^*) = \nu \quad \text{for } n \in \{0, \dots, N\}. \tag{132}$$

We use lemma A.8 to get

$$\nu = D_{t_n+} + \frac{a_n f(a_n D_{t_n+})}{f(D_{t_n+})}(1 - \tau\bar{\rho}'(D_{t_n+})D_{t_n+})\left[\nu - a_n D_{t_n+}\right] \tag{133}$$

$$\Leftrightarrow \nu = D_{t_n+}\frac{f(D_{t_n+}) - a_n^2 f(a_n D_{t_n+})(1 - \tau\bar{\rho}'(D_{t_n+})D_{t_n+})}{f(D_{t_n+}) - a_n f(a_n D_{t_n+})(1 - \tau\bar{\rho}'(D_{t_n+})D_{t_n+})} = h_2(D_{t_n+}) \tag{134}$$

for $n \in \{0, \dots, N-1\}$. Function h_2 is one-to-one, and thus,

$$\nu = h_2(F^{-1}(x_n^* + F(D_{t_n}))) \tag{135}$$

implies that $x_n^* + F(D_{t_n})$ does not depend on $n \in \{0, \dots, N-1\}$. Consequently, $D_{t_n+} = F^{-1}(x_n^* + F(D_{t_n}))$ is constant in n such that we can conclude

$$\begin{aligned} x_0^* &= F(h_2^{-1}(\nu)), & (136)\\ x_n^* &= x_0^* - F(D_{t_n}) = x_0^* - F(a_0 F^{-1}(x_0^*)) \text{ for } n \in \{1, \dots, N-1\}, & (137)\\ x_N^* &= X_0 - x_0^* - (N-1)\left[x_0^* - F(a_0 F^{-1}(x_0^*))\right]. & (138) \end{aligned}$$

The value x_0^* determines the optimal solution completely, and thus, it must minimise

$$\begin{aligned} & C_0^{(2)}(x_0) \\ := \; & C^{(2)}\left(x_0, x_0 - F\left(a_0 F^{-1}(x_0)\right), \dots, X_0 - x_0 - (N-1)\left[x_0 - F(a_0 F^{-1}(x_0))\right]\right) \\ \overset{(117)}{=} \; & G(x_0) + \sum_{n=1}^{N-1}\left[G(x_0) - \tilde{F}(a_0 F^{-1}(x_0))\right] \\ & \qquad + G(X_0 - N[x_0 - F(a_0 F^{-1}(x_0))]) - \tilde{F}(a_0 F^{-1}(x_0)) \\ = \; & N[G(x_0) - \tilde{F}(a_0 F^{-1}(x_0))] + G(X_0 - N[x_0 - F(a_0 F^{-1}(x_0))]). \end{aligned}$$

Differentiation results in

$$\begin{aligned} \frac{dC_0^{(2)}(x_0)}{dx_0} &= N\Bigg[D_{0+} - a_0^2 D_{0+}\frac{f(D_{t_1})}{f(D_{0+})}(1 - \tau\bar{\rho}'(D_{0+})D_{0+}) & (139)\\ &\qquad + D_{t_n+}\left(a_0\frac{f(D_{t_1})}{f(D_{0+})}(1 - \tau\bar{\rho}'(D_{0+})D_{0+}) - 1\right)\Bigg] \end{aligned}$$

such that we can calculate the minimiser by

$$\begin{array}{rcl} \frac{d}{dx_0^*}C_0^{(2)}(x_0^*) & = & 0 \\ \Leftrightarrow \quad D_{t_N+} & = & D_{0+}\frac{f(D_{0+})-a_0^2 f(D_{t_1})(1-\tau\bar{\rho}'(D_{0+})D_{0+})}{f(D_{0+})-a_0 f(D_{t_1})(1-\tau\bar{\rho}'(D_{0+})D_{0+})}. \end{array} \tag{140}$$

The left hand side of the last line can be rewritten as

$$\begin{aligned} D_{t_N+} &= F^{-1}(F(D_{t_N}) + x_N^*) \\ &= F^{-1}(F(D_{t_1}) + X_0 - x_0^* - (N-1)(x_0^* - F(D_{t_1}))) \\ &= F^{-1}(X_0 - N(x_0^* - F(D_{t_1})) \end{aligned} \tag{141}$$

and the right hand side is just $h_2(F^{-1}(x_0^*))$. We know by lemma A.7 that equation (140) has at most one zero such that we are finished with the existence, uniqueness and representation of the optimal strategy.

At last, we show that all components of this strategy are positive. We already know $x_0^* > 0$ and thus also $x_n^* > 0$ for all $n \in \{1, \ldots, N-1\}$ by (137). For the positivity of x_N^*, we transform (140) into

$$D_{t_N+} = D_{0+}\left[1 + \frac{a_0 f(a_0 D_{0+}) - a_0^2 f(a_0 D_{0+})}{f(D_{0+}) - a_0 f(a_0 D_{t_{0+}})(1-\tau\bar{\rho}'(D_{0+})D_{0+})}\,(1-\tau\bar{\rho}'(D_{0+})D_{0+})\right]. \tag{142}$$

The fraction on the right hand side is strictly positive by lemma A.7; positivity of x_N^* follows from

$$D_{t_N+} > D_{0+} = \frac{D_{t_N}}{a_0} > D_{t_N}. \tag{143}$$

□

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