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Boundary coefficient control — A maximal parabolic regularity approach

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ABSTRACT. We investigate a control problem for the heat equation. The goal is to find an optimal heat transfer coefficient in the Robin boundary condition such that a desired temperature distribution at the boundary is adhered. To this end we consider a function space setting in which the heat flux across the boundary is forced to be an L^p function with respect to the surface measure, which in turn implies higher regularity for the time derivative of temperature. We show that the corresponding elliptic operator generates a strongly continuous semigroup of contractions and apply the concept of maximal parabolic regularity. This allows to show the existence of an optimal control and the derivation of necessary and sufficient optimality conditions.

1. INTRODUCTION

The usual function space for the study of linear parabolic optimal control problems is

$$W(0,T) = \{u \in L^2(0,T;X) \,|\, u_t \in L^2(0,T;X^*)\},$$

where X is a Banach space with its dual X^*, see, e.g., [49]. However, for the case of non-Dirichlet boundary control problems this choice is somewhat unsatisfactory. From physical point of view it would be desirable to interpret the controls in terms of measurable functions on the boundary, but this is impossible if the solution is considered only in $W(0,T)$. In other words the control space and the state space are not consistent. Hence, the goal of this paper it to present a new approach, where the boundary flux is forced to be a measurable function.

As a model problem we discuss the optimal control of an active heat sink device, where thermal energy, generated by a heat source $f(t)$ and transported by conduction inside the domain Ω has to be transferred through the boundary Γ. To this end, the boundary heat flux is controlled in such a way that a desired temperature u_d is adhered on Γ. Typically, technical cooling devices only allow for a change of the amount of coolant or the coolant pressure, corresponding to a spatial and temporal change in the heat transfer coefficient q, which will serve as our control variable.

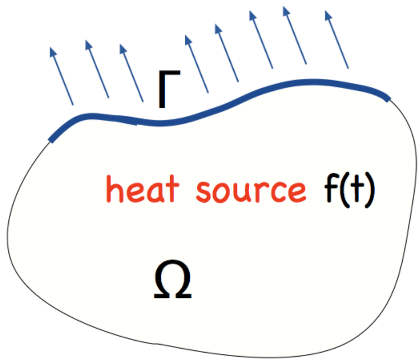

FIGURE 1. Boundary control of a time dependent heat source.

This corresponds to the minimization of the cost functional

$$J(u,q) := \frac{1}{2}\int_0^T \int_\Gamma (u-u_d)^2 dx\, dt + \frac{\alpha}{2}\int_0^T \int_\Gamma q(x,t)^2 d\sigma_\Gamma\, dt \tag{1.1}$$

subject to the parabolic equation

(1.2) $$u' - \nabla \cdot \mu \nabla u = f, \quad u(0) = u_0,$$

combined with mixed Dirichlet/Robin boundary conditions

(1.3) $$u = 0 \quad \text{on} \quad J \times (\partial\Omega \setminus \Gamma)$$

and

(1.4) $$\nu \cdot \mu \nabla u(t) + q(t)u(t) = g(t) \quad \text{on} \quad J \times \Gamma.$$

Here q and g are bounded, measurable functions on the interval $J =]0, T[$, taking their values in the space of functions on Γ which are bounded and measurable with respect to the surface measure.

It is well known that the question how to treat parabolic equations with inhomogeneous Neumann conditions in case of non smooth data of the problem is a delicate one. There are approaches in the literature ([34], [30], see also [35]) where the Banach space under consideration is a negatively indexed Sobolev space of type $H^{-\theta,q}$ or a Sobolev-Morrey space. The serious disadvantage of this, however, is that one does not know in the end that for any time point the divergence of the current is sufficiently regular; one only obtains that it is a distribution. But it would be highly satisfactory to know that the normal flux $\nu \cdot \mu \nabla u(t)$ over any part of the boundary is well defined by Gauss' theorem. The continuity of the normal flux component plays an essential role in connecting and embedding of potential flow systems, as well as in electronic device simulations, where the normal flux on the boundary is often coupled to an electric current which stems from a discrete circuit model [17, 23].

In order to motivate our choice of the Banach space, let us recall the following distributional version of Gauss' theorem

(1.5) $$\int_\Omega \mu \nabla \varphi \cdot \nabla \psi d\mathrm{x} = -\int_\Omega div\big(\mu \nabla \varphi\big) \ \psi d\mathrm{x} + \langle \nu \cdot \mu \nabla \varphi, tr(\psi) \rangle_\partial.$$

Here $\langle \cdot, \cdot \rangle_\partial$ denotes the duality between $H^{\frac{1}{2},2}(\partial\Omega)$ and $H^{-\frac{1}{2},2}(\partial\Omega)$, see [50, Ch. I.1] for further details. This shows the following: if one intends to define the operator $-\nabla \cdot \mu \nabla$ via the form on the left hand side of (1.5) and aims at the inclusion of inhomogeneous Neumann conditions, one must choose the corresponding Banach space in a way, such that it includes distributions, which live on the corresponding boundary part, see [24, Ch. II.2] or [12, Ch. 1.2] for a detailed discussion. In this spirit, we take – following [3], [31], [51], [20] – the Banach space X as $\tilde{L}^p = L^p(\Omega \cup \Gamma; d\mathrm{x} + d\sigma_\Gamma)$, where $d\mathrm{x}$ is the Lebesgue measure on Ω and σ_Γ is the induced boundary measure on the boundary part Γ. This has – in comparison with the concepts in [30, 34] – the advantage, that the distribution $\nu \cdot \mu \nabla$ is forced to be a $L^p(\Gamma; d\sigma_\Gamma)$- function. Consequently, the equation may be tested by indicator functions, and then again Gauss' theorem may be applied. This enables local flux balances, which are crucial for the foundation of Finite Volume methods for the numerical solution of such problems,[5], [21] and [22].
The outline of the paper is as follows. In Section 2 we study properties of elliptic operators on $\tilde{L}^p$. Since the treatment in [20] is restricted to constellations of high regularity for the data and the treatment in [51] is a very abstract setting in terms of Dirichlet forms, we present here another approach which is, firstly, selfcontained and, secondly, avoids any smoothness assumption on the data, in particular, on the domain. This largely extends the applicability of the theory to real-world problems. In doing so, the construction of the operators on the spaces $\tilde{L}^p$ and the investigation of their properties become very transparent. This is based on the assumption that

Ω is a Lipschitz domain and Γ is a suitable part of the boundary ($\Omega \cup \Gamma$ has to be regular in the sense of Gröger [32], see details below). We exploit a C^α-regularity result of Griepentrog and Recke [28] in order to define the corresponding operators A_p as the maximal restrictions from $H_\Gamma^{-1,2}$ to $X := L^p(\Omega \cup \Gamma; d\mathrm{x} + d\sigma_\Gamma)$. Due to a recent result of Cialdea/Maz'ya [11] we are then able to show that the divergence operators are accretive on the above introduced Lebesgue spaces. Let us mention that we require only boundedness/ellipticity of the coefficient function μ – not its symmetry. This leads, via the Lumer-Phillips theorem, to the generator property of a contraction semigroup on every such L^p space.

In Section 3 we apply the results to the parabolic equation. In contrast to the papers [51], [20] we aim here at maximal parabolic $L^r(]0,T]; X)$-regularity, c.f. Theorem 3.2 and Theorem 3.7, in order to get the possibility of the treatment of equations with discontinuous in time boundary conditions. This is achieved by an application of the pioneering result of Lamberton [39, Cor. 1.1]. Choosing the integrability index r sufficiently large, one can show that the solution in fact is Hölder continuous simultaneously in space and time. Using this, a perturbation argument, preserving maximal parabolic regularity, allows to include also inhomogeneous boundary conditions. Afterwards we derive a priori estimates in terms of the data – even uniform for bounded sets of functions q.

In Section 4 we finally study the optimal control problem. The regularity results of the previous section allow to prove the existence of an optimal control and the derivation of necessary and sufficient optimality conditions. Optimality conditions for optimal control problems governed by semilinear partial differential equations have been addressed by numerous contributions in the recent past. Regarding elliptic optimal control problems we mention [7, 8, 9, 10] and the references therein. In particular we refer to [36], where a semilinear elliptic boundary control problem with mixed boundary conditions and non smooth data is considered. Second order optimality conditions for control problems governed by instationary equations have been discussed e.g. in [26] and [44]. In comparison to the very general and abstract setting of the latter contribution the main novelty of this paper is that we can allow for mixed boundary conditions and the control of parameter function, e.g. the heat transfer coefficient function.

2. Elliptic operators on $\tilde{L}^p$

2.1. **Notations, definitions.** Throughout this paper $\mathcal{L}(X;Y)$ denotes the space of bounded linear operators from X to Y, where X and Y are Banach spaces. If $X = Y$, then we abbreviate $\mathcal{L}(X)$.
In the sequel Ω will always be a bounded Lipschitz domain in $\mathbb{R}^d$ and Γ a part of its boundary, which may be empty. If p is from $[1,\infty[$, then $L^p = L^p(\Omega)$ is the space of complex, Lebesgue measurable, p–integrable functions on Ω, and $H^{\theta,p} = H^{\theta,p}(\Omega)$ are the usual spaces of Bessel potentials, see [47, Ch. 4.2.1] or [46, Ch. V.3]. Note that the space $H^{1,q}$ is identical with the Sobolev space $W^{1,q}$. $L^\infty = L^\infty(\Omega)$ is the space of Lebesgue measurable, essentially bounded functions on Ω, and $C^\alpha = C^\alpha(\overline{\Omega})$ the space of up to the boundary α–Hölder continuous functions on Ω.

We assume that $\Omega \cup \Gamma$ is a regular set in the following sense:

Definition 2.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $\Gamma \subset \partial\Omega$ be an open part of its boundary. $\Omega \cup \Gamma$ is a regular set if for every point $\mathrm{x} \in \partial\Omega$ there exist two open sets $\mathcal{U}_\mathrm{x}, \mathcal{V}_\mathrm{x} \subset \mathbb{R}^d$

and a bi–Lipschitz transformation ϕ_{x} from $\mathcal{U}_{\mathrm{x}}$ onto $\mathcal{V}_{\mathrm{x}}$ such that, $\mathrm{x} \in \mathcal{U}_{\mathrm{x}}$, $\phi_{\mathrm{x}}(\mathrm{x}) = 0$, and $\phi_{\mathrm{x}}\big(\mathcal{U}_{\mathrm{x}} \cap (\Omega \cup \Gamma)\big)$ coincides with one of the three model sets

$$\begin{aligned} E_1 &= \{\mathrm{y} \in \mathbb{R}^d : y_1, \dots, y_d \in]-1,1[,\ y_d < 0\}, \\ E_2 &= \{\mathrm{y} \in \mathbb{R}^d : y_1, \dots, y_d \in]-1,1[,\ y_d \le 0\}, \\ E_3 &= \{\mathrm{y} \in E_2 : y_d < 0 \text{ or } y_1 > 0\}. \end{aligned} \tag{2.1}$$

Remark 2.2. The above concept coincides with Gröger's definition of regular sets, cf. [32], which is well adjusted to mixed boundary value problems. We can identify Γ with the Neumann and $\partial\Omega \setminus \Gamma$ with the Dirichlet part of the boundary $\partial\Omega$. An essential point is that one can prove the existence of a continuous extension operator $E : H^{1,q}(\Omega) \to H^{1,q}(\mathbb{R}^d)$ which also continuously extends all L^p spaces, see [25, Thm. 7.25]. Thus one obtains the usual embedding theorems $H^{1,q} \hookrightarrow L^p$.

In two and three space dimensions one can give the following simplifying characterization for a set $\Omega \cup \Gamma$ to be regular in the sense of Gröger, see [36]:

If $\Omega \subseteq \mathbb{R}^2$ is a bounded Lipschitz domain and $\Gamma \subseteq \partial\Omega$ is relatively open, then $\Omega \cup \Gamma$ is regular in the sense of Gröger iff $\partial\Omega \setminus \Gamma$ is the finite union of (non-degenerate) closed arc pieces.

In $\mathbb{R}^3$ the following characterization can be proved: If $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain and $\Gamma \subset \partial\Omega$ is relatively open, then $\Omega \cup \Gamma$ is regular in the sense of Gröger iff the following two conditions are satisfied:

i) $\partial\Omega \setminus \Gamma$ is the closure of its interior (within $\partial\Omega$).
ii) for any $\mathrm{x} \in \overline{\Gamma} \cap (\partial\Omega \setminus \Gamma)$ there is an open neighborhood $\mathcal{U} \ni \mathrm{x}$ and a bi-Lipschitz mapping $\kappa : \mathcal{U} \cap \overline{\Gamma} \cap (\partial\Omega \setminus \Gamma) \to]-1,1[$.

Definition 2.3. We define $H^{\theta,q}_\Gamma$ as the closure in $H^{\theta,q}$ of the set

$$C^\infty_\Gamma(\Omega) \stackrel{\text{def}}{=} \Big\{u|_\Omega : u \in C^\infty_0(\mathbb{R}^d),\ \mathrm{supp}(u) \cap (\partial\Omega \setminus \Gamma) = \emptyset\Big\}, \tag{2.2}$$

and $\breve{H}^{-1,p}_\Gamma$ as the space of continuous antilinear forms on $H^{1,p'}_\Gamma$, where $1/p + 1/p' = 1$. We will always denote the (anti-) dual pairing between $H^{1,q}_\Gamma$ and $\breve{H}^{-1,q'}_\Gamma$ by $\langle \cdot, \cdot \rangle$ and note that this pairing extends the scalar product in L^2 (compare [6, Ch. 1] or [42, Ch. 1.4.2]).

As the boundary measure σ on $\partial\Omega$ we take the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}$, restricted to $\partial\Omega$. (Due to the property of Ω of being a Lipschitz domain, the measure σ can be constructed explicitly in terms of the local bi-Lipschitz charts around the boundary points, (compare [19, Ch. 3.3.4 C] and [33]).

2.2. **Prerequisites.** In this section we introduce our notations and collect several important ingredients that are necessary for the establishment of the main results.

Lemma 2.4. i) *σ may be viewed as bounded, positive Radon measure on $\overline{\Omega}$, which additionally satisfies*

$$\sup_{\mathrm{x} \in \mathbb{R}^d} \sup_{r \in]0,1[} \sigma(B(\mathrm{x}, r) \cap \overline{\Omega}) r^{1-d} < \infty, \tag{2.3}$$

where $B(\mathrm{x}, r)$ denotes the ball centered at x with radius r.

ii) *If $\theta \in]\frac{1}{q}, 1]$, then $H^{\theta,q}$ continuously embeds into $L^q(\partial\Omega, d\sigma)$.*

Proof. i) is proved in [33]. Basing on i), ii) is proved in [34, Thm. 3.6]. □

Definition 2.5. We define the measure σ_Γ on Γ as the restriction of σ to Γ. Further, we define, for any $p \in [1, \infty]$, the space $\widetilde{L}^p$ as the usual Lebesgue space $L^p(\Omega \cup \Gamma, d\mathrm{x} + d\sigma_\Gamma)$.

Remark 2.6. $\widetilde{L}^p$ is topologically isomorphic to the direct sum $L^p(\Omega) \oplus L^p(\Gamma; d\sigma_\Gamma)$.

Lemma 2.7. i) *Assume $d = 2$. Then $H^{1,2}_\Gamma$ continuously embeds into $L^r(\partial\Omega; d\sigma) \hookrightarrow L^r(\Gamma; d\sigma_\Gamma)$ for all $r \in [1, \infty[$.*

ii) *Assume $q < d$ and $r \le q\frac{d-1}{d-q}$. Then $H^{1,q}_\Gamma \hookrightarrow H^{1,q}$ continuously embeds into $L^r(\partial\Omega; d\sigma) \hookrightarrow L^r(\Gamma; d\sigma_\Gamma)$.*

iii) *Assume $q < d$ and $r \le q\frac{d-1}{d-q}$. Then $H^{1,q}_\Gamma \hookrightarrow H^{1,q}$ continuously embeds into $\widetilde{L}^r$.*

Proof. i) Sobolev embedding gives $H^{1,2}_\Gamma \hookrightarrow H^{\frac{3}{2r},r}_\Gamma$ for every $r \in]1, \infty[$; thus Lemma 2.4 applies.
ii) The supposition and Sobolev embedding give $H^{1,q} \hookrightarrow H^{\theta,r}$ for some $\theta > \frac{1}{r}$. Hence, one may apply Lemma 2.4.
iii) In view of Remark 2.6 and ii) it remains to show that $H^{1,q}$ continuously embeds into L^r. But, due to Sobolev embedding $H^{1,q}$ continuously embeds into L^s, whenever $s \le \frac{qd}{d-q}$. □

Definition 2.8. Let $\mu = \{\mu_{k,l}\}_{k,l} : \Omega \longrightarrow \mathcal{B}(\mathbb{R}^d; \mathbb{R}^d)$, be a measurable mapping into the set of real $d \times d$ matrices, satisfying the relations

$$\|\mu(\mathrm{x})\|_{\mathcal{L}(\mathbb{R}^d;\mathbb{R}^d)} \le \mu^\bullet \quad \text{and} \quad \sum_{k,l=1}^{d} \mu_{k,l}(\mathrm{x})\, \xi_k\, \xi_l \ge \mu_\bullet \sum_{k=1}^{d} \xi_k^2 \tag{2.4}$$

for all $\mathrm{x} \in \Omega$, all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ and two strictly positive constants $\mu_\bullet$ and $\mu^\bullet$.

Let $\mathfrak{t}$ be the following sesquilinear form on $H^{1,2}_\Gamma \times H^{1,2}_\Gamma$

$$\mathfrak{t}[\psi, \varphi] \overset{\text{def}}{=} \int_\Omega \langle \mu \operatorname{grad} \psi, \operatorname{grad} \varphi \rangle_{\mathbb{C}^d} \, d\mathrm{x}. \tag{2.5}$$

$\mathfrak{t}$ defines an operator $-\nabla \cdot \mu \nabla : H^{1,2}_\Gamma \to \breve{H}^{-1,2}_\Gamma$ by putting $\langle -\nabla \cdot \mu \nabla \psi, \varphi \rangle := \mathfrak{t}[\psi, \varphi]$.

Remark 2.9. It is easy to check that the form $\mathfrak{t}$ is sectorial; precisely: its numerical range is contained in the sector $\mathcal{S} := \{z \in \mathbb{C} : |\Im z| \le \frac{\mu^\bullet}{\mu_\bullet} \Re z\}$.
In the sequel we maintain the notation $-\nabla \cdot \mu \nabla$ for the restriction of the operator $-\nabla \cdot \mu \nabla$ to any of the spaces $\breve{H}^{-1,q}_\Gamma$, if $q > 2$.

The following regularity result for elliptic boundary value problems is one essential ingredient in our subsequent proofs.

Proposition 2.10. *Let $\Omega \cup \Gamma$ be a regular set in the sense of Definition 2.1, and $0 < \mu_\bullet \le \mu^\bullet < \infty$ be the constants from* (2.4). *For every number $q > d$ there exist a constant $\alpha = \alpha(q, \mu_\bullet, \mu^\bullet, \Omega, \Gamma) \in]0, 1[$ such that $dom_{\breve{H}^{-1,q}_\Gamma}(\nabla \cdot \mu \nabla) \hookrightarrow C^\alpha$.*

Proposition 2.10 was proved in [28, 29] within the scale of Sobolev-Campanato spaces. A simpler proof, only in the $\breve{H}_\Gamma^{-1,q}$-scale, is given in [36], but limited to the cases $d = 2, 3, 4$.

Proposition 2.11. *If $q > d$ and $\theta \in]0, 1[$ is sufficiently close to 1, then even the interpolation space $[\breve{H}_\Gamma^{-1,q}, dom_{\breve{H}_\Gamma^{-1,q}}(\nabla \cdot \mu\nabla)]_\theta$ continuously embeds into a suitable Hölder space C^β.*

Proof. We know from Proposition 2.10 the continuous injection $dom_{\breve{H}_\Gamma^{-1,q}}(\nabla \cdot \mu\nabla) \hookrightarrow C^\alpha$. Using the reiteration theorem for complex interpolation (c.f. [47, Ch.1.9.3]), one gets for $\theta \in]\frac{1}{2}, 1[$

$$\begin{aligned}
[\breve{H}_\Gamma^{-1,q}, dom_{\breve{H}_\Gamma^{-1,q}}(\nabla \cdot \mu\nabla)]_\theta &= \big[[\breve{H}_\Gamma^{-1,q}, dom_{\breve{H}_\Gamma^{-1,q}}(\nabla \cdot \mu\nabla)]_{\frac{1}{2}}, dom_{\breve{H}_\Gamma^{-1,q}}(\nabla \cdot \mu\nabla)\big]_{2\theta-1} \\
&\hookrightarrow \big[[\breve{H}_\Gamma^{-1,2}, dom_{\breve{H}_\Gamma^{-1,2}}(\nabla \cdot \mu\nabla)]_{\frac{1}{2}}, dom_{\breve{H}_\Gamma^{-1,q}}(\nabla \cdot \mu\nabla)\big]_{2\theta-1} \\
&= \big[[\breve{H}_\Gamma^{-1,2}, H_\Gamma^{1,2}]_{\frac{1}{2}}, dom_{\breve{H}_\Gamma^{-1,q}}(\nabla \cdot \mu\nabla)\big]_{2\theta-1} \\
&= \big[L^2, dom_{\breve{H}_\Gamma^{-1,q}}(\nabla \cdot \mu\nabla)\big]_{2\theta-1}.
\end{aligned}$$

Forthcoming $\big[L^2, dom_{\breve{H}_\Gamma^{-1,q}}(\nabla \cdot \mu\nabla)\big]_{2\theta-1}$ continuously embeds into $[L^2, C^\alpha]_{2\theta-1}$ by Proposition 2.10, and this last interpolation space is known to embed into a suitable space C^β, if θ is sufficiently close to 1 (see [27, Ch. 7] for details, compare also [48]). ☐

In order to define the maximal restriction of $-\nabla \cdot \mu\nabla$ to $\widetilde{L}^p$ we must first explain in which way $L^p(\Omega \cup \Gamma, d\mathrm{x} + d\sigma_\Gamma)$ embeds into $\breve{H}_\Gamma^{-1,2}$:

Definition 2.12. We define the embedding operator $\mathfrak{E} : L^2(\Omega \cup \Gamma, d\mathrm{x} + d\sigma_\Gamma) \to \breve{H}_\Gamma^{-1,2}$ by

$$\langle \mathfrak{E} f, \psi \rangle := \int_{\Omega\cup\Gamma} f(\mathrm{x})\overline{\psi}(\mathrm{x}) d\mathrm{x} + d\sigma_\Gamma, \quad \psi \in H_\Gamma^{1,2}. \tag{2.6}$$

We recall that this is justified because every $\psi \in H_\Gamma^{1,2}$ is square integrable with respect to $d\mathrm{x} + d\sigma_\Gamma$. Thus, for $p > 2$, $L^p(\Omega \cup \Gamma, d\mathrm{x} + d\sigma_\Gamma)$ is embedded into $\breve{H}_\Gamma^{-1,2}$ via the natural injection $L^p(\Omega \cup \Gamma, d\mathrm{x} + d\sigma_\Gamma) \hookrightarrow L^2(\Omega \cup \Gamma, d\mathrm{x} + d\sigma_\Gamma)$ and $\mathfrak{E}$.

Remark 2.13.
i) $\mathfrak{E}$ is a continuous injection, because the scalar products with all $H_\Gamma^{1,2}$-functions determine an element $f \in L^2(\Omega \cup \Gamma, d\mathrm{x} + d\sigma_\Gamma)$ uniquely.
ii) It is essential to observe that the embedding $\mathfrak{E}$ extends the usual embedding of $L^2(\Omega) \hookrightarrow \breve{H}_\Gamma^{-1,2}$ in the following manner: identifying any element $f \in L^2(\Omega)$ with its extension to $\Omega \cup \Gamma$ by 0, one has for all $\psi \in H_\Gamma^{1,2}$

$$\langle \mathfrak{E} f, \psi \rangle = \int_{\Omega\cup\Gamma} f(\mathrm{x})\overline{\psi}(\mathrm{x}) d\mathrm{x} + d\sigma_\Gamma = \int_\Omega f(\mathrm{x})\overline{\psi}(\mathrm{x}) d\mathrm{x}.$$

Lemma 2.14. *If $p \geq d$, then $\widetilde{L}^p$ continuously embeds into a space $\breve{H}_\Gamma^{-1,d+\epsilon}$ with $\epsilon = \epsilon(d) > 0$.*

Proof. Obviously, it suffices to show the claim for $p = d$. One has the continuous embedding $\widetilde{L}^d \hookrightarrow \breve{H}_\Gamma^{-1,q}$ for some $q > d$ if $H_\Gamma^{1,s}$ continuously embeds into $\widetilde{L}^{\frac{d}{d-1}}$ for some $s \in [1, \frac{d}{d-1}[$. Putting $q := \frac{d}{d-1}$ one obtains $q\frac{d-1}{d-q} = \frac{d-1}{d-2}$, what is larger than $\frac{d}{d-1}$. Hence, taking $s \in [1, \frac{d}{d-1}[$ sufficiently close to $\frac{d}{d-1}$ it remains $\frac{d}{d-1} < s\frac{d-1}{d-s}$, and one may apply ii) of Lemma 2.7. ☐

Lemma 2.15.
i) *There is a $\hat{p} = \hat{p}(d) > 2$, such that $(-\nabla \cdot \mu\nabla + 1)^{-1}$ continuously maps $\widetilde{L}^2$ into $\widetilde{L}^{\hat{p}}$.*

ii) *There is an $\alpha > 0$, such that for every $p \geq d$, $(-\nabla\cdot\mu\nabla+1)^{-1} \in \mathcal{L}(\widetilde{L}^p; C^\alpha) \hookrightarrow \mathcal{L}(\widetilde{L}^p)$.*

iii) *Defining $r = r(\theta) = \left(\frac{1-\theta}{2} + \frac{\theta}{d}\right)^{-1}$ and $s = s(\theta) = \frac{\hat{p}}{1-\theta}$ one has $(-\nabla \cdot \mu\nabla + 1)^{-1} \in \mathcal{L}(\widetilde{L}^r; \widetilde{L}^s)$. The function $\theta \mapsto s(\theta) - r(\theta)$ is positive and strictly increasing on $[0, 1]$.*

Proof. i) In the case $d = 2$ the assertion follows directly from Lemma 2.14 and Proposition 2.10. Assume now $d \geq 3$. Exploiting Lemma 2.7 and duality, one gets $\widetilde{L}^2 \hookrightarrow \breve{H}_\Gamma^{-1,2}$. This implies the continuity of

$$(-\nabla \cdot \mu\nabla + 1)^{-1} : \widetilde{L}^2 \mapsto H_\Gamma^{1,2} \hookrightarrow \widetilde{L}^{\hat{p}}, \tag{2.7}$$

with a $\hat{p} = \hat{p}(d) > 2$, see Lemma 2.7.
ii) is implied by the embedding $\widetilde{L}^p \hookrightarrow \breve{H}_\Gamma^{-1,d+\epsilon}$ $(\epsilon > 0)$ and Proposition 2.10.
iii) The first assertion follows from i) and ii) by Riesz-Thorin. The second is a straightforward calculation. ☐

2.3. **The operators A_p.** We will start with the definition of the operator A_p. Furthermore, we elaborate important properties of the operator.

Definition 2.16. For $p \geq 2$, we define A_p as the maximal restriction of $-\nabla \cdot \mu\nabla$ to $\widetilde{L}^p$.

Lemma 2.17. *Assume $p \in [2, \infty[$.*

i) *The domain of A_p is contained in $\widetilde{L}^p$. Hence, A_p can be understood as an operator on $\widetilde{L}^p$, which is, additionally, a closed one.*
ii) *If $p \geq d$ and $\theta \in]0, 1[$ is sufficiently close to 1, then there is a $\beta > 0$ such that even $[\widetilde{L}^p, dom_{\widetilde{L}^p}(A_p)]_\theta \hookrightarrow C^\beta$.*

Proof. i) It suffices to prove $\left(-\nabla \cdot \mu\nabla + 1\right)^{-1} \in \mathcal{L}(\widetilde{L}^p)$, which is implied by Lemma 2.15.
ii) follows from Proposition 2.11 by means of the continuous embedding $\widetilde{L}^p \hookrightarrow \widetilde{L}^d \hookrightarrow \breve{H}_\Gamma^{-1,d+\epsilon}$, c.f. Lemma 2.14. ☐

Remark 2.18. i) In the spirit of Lemma 2.17 we will only write $dom(A_p)$ instead of $dom_{\widetilde{L}^p}(A_p)$ in the sequel.
ii) A_p+1 is a fortiori a surjection onto $\widetilde{L}^p$ because it is the maximal restriction of $-\nabla\cdot\mu\nabla+1$ to $\widetilde{L}^p$, and $-\nabla \cdot \mu\nabla + 1$ is a surjection onto $\breve{H}_\Gamma^{-1,2}$ by Lax-Milgram.
iii) Lemma 2.17 shows that an element $\psi \in H_\Gamma^{1,2}$ belongs to $dom(A_p)$ iff the antilinear form $H_\Gamma^{1,2} \ni \varphi \mapsto \mathfrak{t}[\psi, \varphi]$ is continuous, when $H_\Gamma^{1,2}$ is equipped only with the $\widetilde{L}^{p'}$ norm.

Let us conclude some basic properties of A_2:

Lemma 2.19. i) *A_2 generates an analytic semigroup of contractions.*
ii) *$A_2^* = \widehat{A_2}$, where $\widehat{A_2}$ is the operator which corresponds to the adjoint coefficient function μ^*.*
iii) *Let α be the Hölder exponent from Proposition 2.10.*
There is a positive integer j such that the mapping

$$\left(A_2 + 1\right)^{-j} : \widetilde{L}^2 \longrightarrow C^\alpha \hookrightarrow \widetilde{L}^\infty \tag{2.8}$$

is well defined and continuous. If $d = 2$, then $j = 1$ works.

iv) *Further, each semigroup operator* e^{-tA_2}*,* $t > 0$*, maps* $\widetilde{L}^2$ *continuously into* $C^\alpha \hookrightarrow \widetilde{L}^\infty$*.*

Proof. i) The form $\mathfrak{t}$ is sectorial (compare Remark 2.9); hence one has the estimate $\|(A_2 - z)^{-1}\|_{\mathcal{L}(\tilde{L}^2)} \le \frac{1}{dist(\mathcal{S},z)}$, since $A_2 + 1$ is a surjection, c.f. [38, Ch. V.3.1]. In particular, one has $\|(A_2 + t)^{-1}\|_{\mathcal{L}(\tilde{L}^2)} \le \frac{1}{t}$ for $t > 0$ and may apply the Hille-Yosida theorem in order to obtain the contraction property of the semigroup.
ii) To the coefficient function μ^* there corresponds the adjoint form $\mathfrak{t}^*$ and to this the adjoint operator A_2^*, see [38, Ch. VI.2.1].
iii) The assertion follows from Lemma 2.17 (ii): when applying $(A_2+1)^{-1}$ several times, starting in $\widetilde{L}^2$, the integrability index in the target spaces improves in every step more as in the previous one. Thus, one ends up after finitely many steps in $\widetilde{L}^d$. Then applying $(A_2 + 1)^{-1}$ a last time, Lemma 2.17 gives the assertion.
iv) Let j as above. There is

$$\left\|e^{-tA_2}\right\|_{\mathcal{L}(\widetilde{L}^2;C^\alpha)} \le \left\|(A_2 + 1)^{-j}\right\|_{\mathcal{L}(\widetilde{L}^2;C^\alpha)} \left\|(A_2 + 1)^j e^{-tA_2}\right\|_{\mathcal{L}(\widetilde{L}^2;\widetilde{L}^2)}.$$

The first factor on the right hand side is finite according to the foregoing assertion. The second one is finite due to the fact that A_2 – as a maximal sectorial operator on a Hilbert space – admits a bounded holomorphic calculus, see [13, Ch. 2.10]. □

The next result contains the essential step towards maximal parabolic regularity of A_p on $\widetilde{L}^p$.

Theorem 2.20. *For every* $p \in [2, \infty[$*,* A_p *generates a strongly continuous semigroup of contractions on* $\widetilde{L}^p$*.*

The proof will follow from several subsequent lemmas.

Lemma 2.21. *Let* $\psi \in H^{1,2}_\Gamma$ *be bounded and* $r > 0$*. Then there is sequence* $\{\psi_n\}_n$ *from* $C_0^\infty(\mathbb{R}^d)$ *such that* $supp(\psi_n) \cap (\partial\Omega \setminus \Gamma) = \emptyset$ *and the sequence* $\{(\psi_n|_\Omega, |\psi_n|^r \psi_n|_\Omega)\}_n$ *converges in* $H^{1,2} \times H^{1,2}$ *to* $(\psi, |\psi|^r\psi)$*. In particular,* $|\psi|^r\psi \in H^{1,2}_\Gamma$*.*

Proof. Due to the definition of $H^{1,2}_\Gamma$, there is a sequence $\{\hat{\psi}_n\}_n$ of $C_0^\infty(\mathbb{R}^d)$-functions with $supp(\hat{\psi}_n) \cap (\partial\Omega \setminus \Gamma) = \emptyset$ and $\hat{\psi}_n|_\Omega \to \psi$ in $H^{1,2}$ for $n \to \infty$. Let c be a bound for $|\psi|$. Let us put $\hat{\varphi}_n := \max(-c, \min(c, \Re\hat{\psi}_n))$ and $\hat{\phi}_n := \max(-c, \min(c, \Im\psi_n))$. By a classical result of [41] we have:

$$\hat{\varphi}_n|_\Omega = \max(-c, \min(c, \Re\hat{\psi}_n|_\Omega)) \to \max(-c, \min(c, \Re\psi)) = \Re\psi \quad \text{in } H^{1,2} \tag{2.9}$$

and

$$\hat{\phi}_n|_\Omega = \max(-c, \min(c, \Im\hat{\psi}_n|_\Omega)) \to \max(-c, \min(c, \Im\psi)) = \Im\psi \quad \text{in } H^{1,2}. \tag{2.10}$$

Obviously, $\hat{\varphi}_n, \hat{\phi}_n$ are no more C^∞ functions, but uniformly bounded, and their supports do not intersect $\partial\Omega \setminus \Gamma$. By a usual mollifier argument, we obtain functions $\varphi_n, \phi_n \in C^\infty(\mathbb{R}^d)$ with the properties $|\varphi_n| \le c, |\phi_n| \le c$, $supp(\varphi_n) \cap (\partial\Omega \setminus \Gamma) = supp(\phi_n) \cap (\partial\Omega \setminus \Gamma) = \emptyset$ and

$$\|\hat{\varphi}_n - \varphi_n\|_{H^{1,2}(\mathbb{R}^d)} + \|\hat{\phi}_n - \phi_n\|_{H^{1,2}(\mathbb{R}^d)} \le \frac{1}{n}. \tag{2.11}$$

Finally, one estimates

$$\||\varphi_n + i\phi_n|^r(\varphi_n + i\phi_n)|_\Omega - |\psi|^r\psi\|_{H^{1,2}} \le$$

$$\||\varphi_n + i\phi_n|^r(\varphi_n + i\phi_n) - |\hat{\varphi}_n + i\hat{\phi}_n|^r(\hat{\varphi}_n + i\hat{\phi}_n)\|_{H^{1,2}(\mathbb{R}^d)} + \tag{2.12}$$

$$\||\hat{\varphi}_n + i\hat{\phi}_n|^r(\hat{\varphi}_n + i\hat{\phi}_n)|_\Omega - |\psi|^r\psi\|_{H^{1,2}} \tag{2.13}$$

and observes that both, (2.12) and (2.13), tend to 0, due to (2.9), (2.10), (2.11) and the uniform boundedness of the functions $\hat{\varphi}_n, \hat{\phi}_n, \varphi_n, \phi_n$. ☐

Lemma 2.22. *If $p \in [d, \infty[$, then the operator $-A_p$ is dissipative, cf.* [43, 1.4 Def. 4.1].

Proof. According to Lemma 2.15, every $\psi \in dom(A_p) \subset H^{1,2}_\Gamma$ is a bounded function on Ω. In view of Lemma 2.21 we thus have $(\psi, |\psi|^{p-2}\psi) \in dom\, \mathfrak{t}$. This implies by the definition of A_p,

$$\int_{\Omega\cup\Gamma} A_p\psi\ |\psi|^{p-2}\overline{\psi}\ dx + d\sigma_\Gamma = \mathfrak{t}[\psi, |\psi|^{p-2}\psi].$$

Hence, one has only to show that

$$-\Re\mathfrak{t}[\psi, |\psi|^{p-2}\psi] \leq 0 \tag{2.14}$$

for every $\psi \in dom(A_p)$ (what is called in [11] the L^p-dissipativity of the form $\mathfrak{t}$). We show (2.14) first for functions $\psi \in C^\infty_\Gamma(\Omega)$ (c.f. (2.2)), thereby proceeding as in the proof of Lemma 1 from [11]: putting $\varphi := |\psi|^{\frac{p-2}{2}}\psi$ one obtains

$$\begin{aligned}
\Re\ \mathfrak{t}[\psi, \psi|\psi|^{p-2}] &= \Re\int_\Omega \langle \mu\nabla\psi, \nabla(|\psi|^{p-2}\psi)\rangle_{\mathbb{C}^d} d\mathrm{x}\\
&= \Re\int_\Omega \langle \mu\nabla\big(|\varphi|^{\frac{2-p}{p}}\varphi\big), \nabla\big(|\varphi|^{\frac{p-2}{p}}\varphi\big)\rangle_{\mathbb{C}^d} d\mathrm{x}\\
&= \Re\Bigg(\int_\Omega \langle \mu\nabla\varphi, \nabla\varphi\rangle_{\mathbb{C}^d} d\mathrm{x} - \left(1 - \frac{2}{p}\right)\int_\Omega \langle (\mu - \mu^*)\nabla|\varphi|, \frac{\overline{\varphi}}{|\varphi|}\nabla\varphi\rangle_{\mathbb{C}^d} d\mathrm{x}\\
&\quad - \left(1 - \frac{2}{p}\right)^2 \int_\Omega \langle \mu\nabla|\varphi|, \nabla|\varphi|\rangle_{\mathbb{C}^d} d\mathrm{x}\Bigg)
\end{aligned} \tag{2.15}$$

Again following [11], we put $\Phi := \Re\big(\frac{\overline{\varphi}}{|\varphi|}\nabla\varphi\big)$ and $\Psi := \Im\big(\frac{\overline{\varphi}}{|\varphi|}\nabla\varphi\big)$. Recalling that μ was a real coefficient matrix, one calculates

$$\begin{aligned}
\Re\int_\Omega \langle \mu\nabla\varphi, \nabla\varphi\rangle_{\mathbb{C}^d} d\mathrm{x} &= \Re\int_\Omega \langle \mu\frac{\overline{\varphi}}{|\varphi|}\nabla\varphi, \frac{\overline{\varphi}}{|\varphi|}\nabla\varphi\rangle_{\mathbb{C}^d} d\mathrm{x}\\
&= \Re\int_\Omega \langle \mu(\Phi + i\Psi), \Phi + i\Psi\rangle_{\mathbb{C}^d} d\mathrm{x}\\
&= \int_\Omega \langle \mu\Phi, \Phi\rangle_{\mathbb{C}^d} dx + \int_\Omega \langle \mu\Psi, \Psi\rangle_{\mathbb{C}^d} d\mathrm{x}
\end{aligned} \tag{2.16}$$

$$\Re\int_\Omega \langle (\mu - \mu^*)\nabla|\varphi|, \frac{\overline{\varphi}}{|\varphi|}\nabla\varphi\rangle_{\mathbb{C}^d} d\mathrm{x} = \int_\Omega \langle \Im(\mu - \mu^*)\Phi, \Psi\rangle_{\mathbb{C}^d} d\mathrm{x} = 0, \tag{2.17}$$

$$\Re\int_\Omega \langle \mu\nabla|\varphi|, \nabla|\varphi|\rangle_{\mathbb{C}^d} d\mathrm{x} = \int_\Omega \langle \mu\Phi, \Phi\rangle_{\mathbb{C}^d} d\mathrm{x}. \tag{2.18}$$

Inserting this into (2.15), one obtains (2.14) for all $\psi \in C_\Gamma^\infty(\Omega)$. Let now ψ be any element from $dom(A_p)$. Since ψ is bounded, there is a sequence $\{\psi_n\}_n$ from $C_0^\infty(\mathbb{R}^d)$ such that the sequence $\{(\psi_n|_\Omega, |\psi_n|^r\psi_n|_\Omega)\}_n \subset C_\Gamma^\infty(\Omega)$ converges in $H^{1,2} \times H^{1,2}$ to $(\psi, |\psi|^r\psi)$, c.f. Lemma 2.21. Thus, (2.14) extends from $C_\Gamma^\infty(\Omega)$ to $dom(A_p)$ by the continuity of $\mathfrak{t}$. □

Lemma 2.23. *Assume* $p \in [d, \infty[$.

i) $dom(A_p)$ *is dense in* $\widetilde{L}^p$.
ii) $-A_p$ *is the generator of a strongly continuous semigroup of contractions.*

Proof. i) The density follows from the dissipativity of $-A_p$, proven in Lemma 2.22, the surjectivity of $A_p + 1$ and a well known theorem, heavily resting on the reflexivity of $\widetilde{L}^p$ (cf. Pazy [43, 1.4 Th. 4.6]).
ii) follows from i), Lemma 2.22 and the Lumer–Phillips theorem [43, 1.4 Th. 4.3]. □

Lemma 2.24. *If* p *is any number from* $[2, \infty[$ *then* $\mathrm{dom}(A_p)$ *is dense in* $\widetilde{L}^p$.

Proof. The assertion is already proved for $p \geq d$. Let now p be from $[2, d[$; then $\mathrm{dom}(A_d) \subset \mathrm{dom}(A_p)$. Hence, as $\mathrm{dom}(A_d)$ is dense in $\widetilde{L}^d$, and $\widetilde{L}^d$ is dense in $\widetilde{L}^p$, $\mathrm{dom}(A_p)$ must be dense in $\widetilde{L}^p$. □

In order to prove Theorem 2.20, it remains to show that A_p also generates a continuous semigroup of contraction on $\widetilde{L}^p$, if $p \in [2, d[$. This is proved for $p = 2$ in Lemma 2.19 i), what, in particular, finishes the case $d = 2$. If $d > 2$ and $p \in]2, d[$ one obtains by Riesz-Thorin

$$\|e^{-tA_p}\|_{\mathcal{L}(\widetilde{L}^p)} \leq \|e^{-tA_2}\|^{1-\theta}_{\mathcal{L}(\widetilde{L}^2)} \|e^{-tA_d}\|^{\theta}_{\mathcal{L}(\widetilde{L}^d)} \leq 1$$

with $\theta = \frac{d}{p}\frac{p-2}{d-2}$. Further, for $\psi \in \widetilde{L}^d$, $\lim_{t\mapsto 0} \|e^{-tA_p}\psi - \psi\|_{\widetilde{L}^p} \leq c \lim_{t\mapsto 0} \|e^{-tA_d}\psi - \psi\|_{\widetilde{L}^d} = 0$. But, by the density of $\widetilde{L}^d$ in $\widetilde{L}^p$, the equality $\lim_{t\mapsto 0} \|e^{-tA_p}\psi - \psi\|_{\widetilde{L}^p} = 0$ extends to all $\psi \in \widetilde{L}^p$, due to the property $\|e^{-tA_p}\|_{\mathcal{L}(\widetilde{L}^p)} \leq 1$.

Theorem 2.20 justifies the following definition, supplementing Definition 2.16:

Definition 2.25. For $p \in [1, 2[$, A_p is the adjoint of $\widehat{A_{p'}}$, where $\widehat{A_{p'}}$ is again the operator which corresponds to the adjoint coefficient function μ^* on the space $\tilde{L}^{p'}$ with $p' = p/(p-1)$.

Remark 2.26. Due to Lemma 2.19, the restriction of A_p to $\widetilde{L}^2$ equals A_2, if $p \in [1, 2[$. In other words: A_p is an extension of A_2 and an extension of A_q, if $q > 2$.

With the help of classical duality results one easily reproduces the statements on A_p for the case $p \in]1, 2[$:

Theorem 2.27. *Suppose* $p \in]1, 2[$.

i) A_p *is closed and densely defined.*
ii) A_p *generates a strongly continuous contraction semigroup on* $\widetilde{L}^p$.

Proof. i) See [38, III.§5 Th. 5.29].
ii) In view of

$$(A_p + \rho)^{-1} = \Big(\big(\widehat{A_{\frac{p}{p-1}}} + \rho\big)^{-1} \Big)^* \tag{2.19}$$

(see [38, III.§5 Th. 5.30]) the assertion follows from i) and the Hille-Yosida theorem. ☐

Lemma 2.28. *The semigroup e^{-tA_2}, $t > 0$ induces semigroups of contractions on $\widetilde{L}^\infty$ and $\widetilde{L}^1$.*

Proof. From Lemma 2.17 i) we know that $e^{-tA_2} \in \mathcal{L}(\widetilde{L}^\infty, \widetilde{L}^\infty)$; and $\{e^{-tA_2}\}_{t>0}$ obviously forms a semigroup on $\widetilde{L}^\infty$. It remains to show the contractivity of e^{-tA_2} on $\widetilde{L}^\infty$. Indeed, due to the contractivity of e^{-tA_2} on $\widetilde{L}^p$ for all $p \in [2, \infty[$, there is for all $\psi \in \widetilde{L}^\infty$

$$\big\| e^{-tA_2}\psi \big\|_{\widetilde{L}^\infty} \xleftarrow{\infty \leftarrow p} \big\| e^{-tA_2}\psi \big\|_{\widetilde{L}^p} \le \|\psi\|_{\widetilde{L}^p} \xrightarrow{p \to \infty} \|\psi\|_{\widetilde{L}^\infty}.$$

N.B. If $\varphi \in \widetilde{L}^\infty$, then $\|\varphi\|_{\widetilde{L}^\infty} = \lim_{p\to\infty} \|\varphi\|_{\widetilde{L}^p}$. Let us turn to the $\widetilde{L}^1$ case: by Lebesgue dominance and the contractivity of e^{-tA_p} on $\widetilde{L}^p$ $(p \neq 1)$ one has for every $\psi \in \widetilde{L}^2$

$$\big\| e^{-tA_2}\psi \big\|_{\widetilde{L}^1} \xleftarrow{1 \leftarrow p} \big\| e^{-tA_2}\psi \big\|_{\widetilde{L}^p} \le \|\psi\|_{\widetilde{L}^p} \xrightarrow{p \to 1} \|\psi\|_{\widetilde{L}^1}.$$

Thus, e^{-tA_2} is a contraction operator on $\widetilde{L}^2$, if this space is equipped with the $\widetilde{L}^1$ norm. Obviously, it extends to a contraction operator on $\widetilde{L}^1$. ☐

3. The parabolic equation

In this section we will draw conclusions for parabolic equations. Throughout this section J always denotes a bounded interval $]0, T[$ and X a Banach space. Let us first define two further function spaces: by $W^{1,s}(J; X)$ we denote the subspace of functions from $L^s(J; X)$ which have a distributional derivative, also belonging to $L^s(J; X)$ (see [2, Ch III.1]). By $W_0^{1,s}(J; X)$ we mean the subspace of $W^{1,s}(J; X)$ whose elements take the value $0 \in X$ in the point $0 \in \bar{J}$.

Definition 3.1. Let $1 < s < \infty$ and D be a dense subspace of the Banach space X. Assume that $\mathcal{A} : J \ni t \mapsto \mathcal{A}(t) \in \mathcal{L}(D; X)$ is a bounded, strongly (Lebesgue) measurable mapping, where each $\mathcal{A}(t)$ is a closed operator with D as its domain. We say that $\mathcal{A}$ satisfies *maximal parabolic $L^s(J; X)$-regularity*, if for any $f \in L^s(J; X)$ there exists a unique function $u \in W_0^{1,s}(J; X) \cap L^s(J; D)$ satisfying

$$u' + \mathcal{A}(\cdot)u = f, \tag{3.1}$$

where the time derivative is taken in the sense of X-valued distributions on J (see [2, Ch III.1]).

We proceed with some comments concerning maximal parabolic regularity:

i) If $\mathcal{A}$ satisfies maximal parabolic $L^s(J; X)$ regularity, then the mapping $W_0^{1,s}(J; X) \cap L^s(J; D) \ni u \mapsto u' + \mathcal{A}u \in L^s(J; X)$ is a continuous bijection. Hence, the inverse is continuous by the open mapping theorem, and the solution u of (3.1) admits the estimate

$$\|u'\|_{L^s(J;X)} + \|u\|_{L^s(J;D)} \le c\|f\|_{L^s(J;X)}.$$

for some constant c, independent from f.

ii) If $\mathcal{A} \equiv A$ is the constant mapping, then it is well known that the property of maximal parabolic regularity of A is independent of $s \in \,]1, \infty[$ and the specific choice of the interval J (cf. [16]). In this spirit, we then simply say that A satisfies maximal parabolic regularity on X.
iii) If an operator A satisfies maximal parabolic regularity on a Banach space X, then its negative generates an analytic semigroup on X (cf. [16]).
iv) If X is a Hilbert space, then the converse is also true: The negative of every generator of an analytic semigroup on X satisfies maximal parabolic regularity, cf. [15] or [16].
v) Observe that (see [2, Thm. 4.10.2])

$$W^{1,s}(J;X) \cap L^s(J;D) \hookrightarrow C(\overline{J};(X,D)_{1-\frac{1}{s},s}). \tag{3.2}$$

Theorem 3.2. *For any $p \in]1, \infty[$ the operator A_p satisfies maximal parabolic regularity on $\widetilde{L}^p$.*

Proof. For every $p \in [1, \infty]$ the semigroup operators e^{-tA_p} are contractions on $\widetilde{L}^p$, according to Theorem 2.20, Theorem 2.27 and Lemma 2.28. Moreover, the semigroup, generated by A_2 on $\tilde{L}^2$, is analytic, see Lemma 2.19. Thus, the result of Lamberton, cf. [39, Cor. 1.1] gives the assertion. □

Definition 3.3. Let us fix a bounded, measurable $L^p(\Gamma)$-valued function q on J. Then we define for $\psi \in C(\overline{\Omega})$ and any $t \in J$ the operator $B(t;q) : C(\overline{\Omega}) \to L^p(\Gamma)$ by

$$B(t;q)\psi = q(t)\psi|_\Gamma. \tag{3.3}$$

Furthermore, we introduce $\mathcal{B}(\cdot;q) : J \ni t \mapsto B(t;q)$.

Theorem 3.4. *Assume $p \geq d$, $s \in]1, \infty[$ and q as above.*

i) *Then $A_p + \mathcal{B}(\cdot;q)$ also satisfies maximal parabolic $L^s(J;\widetilde{L}^p)$ regularity.*
ii) *The norms $\|\big(\frac{\partial}{\partial t} + A_p + \mathcal{B}(\cdot;q)\big)^{-1}\|_{\mathcal{L}(L^s(J;\tilde{L}^p);L^s(J;dom(A_p))\cap W_0^{1,s}(J;\tilde{L}^p))}$ are uniformly bounded, if q runs through a bounded subset of $L^\infty(J;L^p(\Gamma))$. In particular, for every $f \in L^s(J;\tilde{L}^p)$, the set of solutions for the equations*

$$u' + A_p u + \mathcal{B}(\cdot,q)u = f$$

is bounded in $L^s(J;dom(A_p)) \cap W_0^{1,s}(J;\tilde{L}^p))$, if q runs through a bounded subset of $L^\infty(J;L^p(\Gamma))$.

Proof. i) Due to Lemma 2.17 ii) there is a $\theta \in]0,1[$ which allows the continuous embedding $[\widetilde{L}^p, dom(A_p)]_\theta \hookrightarrow C(\overline{\Omega})$. Applying the interpolation inequality and Young's inequality, this gives for all $\psi \in dom(A_p)$ the estimate

$$\|\psi\|_{C(\overline{\Omega})} \leq c\|\psi\|_{\tilde{L}^p}^{1-\theta}\|\psi\|_{dom(A_p)}^{\theta} \leq \delta\|\psi\|_{dom(A_p)} + c^{\frac{1}{1-\theta}}\left(\frac{1}{\delta}\right)^{\frac{\theta}{1-\theta}}\|\psi\|_{\widetilde{L}^p},$$

for every $\delta > 0$. Let us denote the operator which assigns to the function $f \in L^r(J;\tilde{L}^p)$ the solution u of

$$\frac{\partial u}{\partial t} + A_p u + \lambda u = f, \quad u(0) = 0$$

by $\left(\frac{\partial}{\partial t}+A_p+\lambda\right)^{-1}$ and show

$$\Big\|\Big(\frac{\partial}{\partial t}+A_p+\lambda\Big)^{-1}\Big\|_{\mathcal{L}(L^r(J;\tilde{L}^p))}\le\frac{1}{\lambda},\quad \text{for}\quad \lambda>0. \tag{3.4}$$

As is well known, $\left(\frac{\partial}{\partial t}+A_p+\lambda\right)^{-1}$ acts as

$$\Big(\frac{\partial}{\partial t}+A_p+\lambda\Big)^{-1}f\,(t)=\int_0^t e^{-(t-s)(A_p+\lambda)}f(s)ds.$$

Due to the contractivity of the semigroup operators e^{-tA_p} on $\tilde{L}^p$, one can thus estimate

$$\Big\|\Big(\frac{\partial}{\partial t}+A_p+\lambda\Big)^{-1}f\,(t)\Big\|_{\tilde{L}^p}\le\int_0^t e^{-\lambda(t-s)}\|f(s)\|_{\tilde{L}^p}ds=e^{-\lambda\cdot}\chi_{]0,\infty[}*\|f(\cdot)\|_{\tilde{L}^p}(t).$$

From this, (3.4) follows by an application of Young's inequality, see [45, Ch. IX.4]. Hence, one can estimate for all $\psi\in dom(A_p)$

$$\begin{aligned}\|B(t;q)\psi\|_{\tilde{L}^p}&\le\|q(t)\|_{\tilde{L}^p}\|\psi\|_{C(\overline{\Omega})}\\&\le\|q(t)\|_{\tilde{L}^p}\left(\delta\|\psi\|_{dom(A_p)}+c^{\frac{1}{1-\theta}}\Big(\frac{1}{\delta}\Big)^{\frac{\theta}{1-\theta}}\|\psi\|_{\tilde{L}^p}\right),\quad\delta>0\end{aligned} \tag{3.5}$$

This allows us to estimate for any $f\in L^s(J;\tilde{L}^p)$:

$$\begin{aligned}&\Big\|\mathcal{B}(\cdot,q)\Big(\frac{\partial}{\partial t}+A_p+\lambda\Big)^{-1}f\Big\|_{L^s(J;\tilde{L}^p)}=\Big(\int_J\|q(t)\big(\frac{\partial}{\partial t}+A_p+\lambda\big)^{-1}f(t)\|^s_{\tilde{L}^p}dt\Big)^{1/s}\\
&\le\sup_{t\in J}\|q(t)\|_{L^p(\Gamma)}\Bigg(\int_J\Big(\delta\|A_p\big(\frac{\partial}{\partial t}+A_p+\lambda\big)^{-1}f(t)\|_{\tilde{L}^p}\\
&\quad+c^{\frac{1}{1-\theta}}\Big(\frac{1}{\delta}\Big)^{\frac{\theta}{1-\theta}}\|\big(\frac{\partial}{\partial t}+A_p+\lambda\big)^{-1}f(t)\|_{\tilde{L}^p}\Big)^s dt\Bigg)^{1/s}\\
&\le\sup_{t\in J}\|q(t)\|_{L^p(\Gamma)}\delta\|A_p\big(\frac{\partial}{\partial t}+A_p\big)^{-1}\big(\frac{\partial}{\partial t}+A_p\big)\big(\frac{\partial}{\partial t}+A_p+\lambda\big)^{-1}f\|_{L^s(J;\tilde{L}^p)}\\
&\quad+\sup_{t\in J}\|q(t)\|_{L^p(\Gamma)}c^{\frac{1}{1-\theta}}\Big(\frac{1}{\delta}\Big)^{\frac{\theta}{1-\theta}}\frac{1}{\lambda}\|f\|_{L^s(J;\tilde{L}^p)}\\
&\le\sup_{t\in J}\|q(t)\|_{L^p(\Gamma)}\delta\|\big(\frac{\partial}{\partial t}+A_p\big)^{-1}\|_{\mathcal{L}(L^s(J;\tilde{L}^p);L^s(J;dom(A_p)))}2\|f\|_{L^s(J;\tilde{L}^p)}\\
&\quad+\sup_{t\in J}\|q(t)\|_{L^p(\Gamma)}c^{\frac{1}{1-\theta}}\Big(\frac{1}{\delta}\Big)^{\frac{\theta}{1-\theta}}\frac{1}{\lambda}\|f\|_{L^s(J;\tilde{L}^p)}\end{aligned}$$

Now we choose

$$\delta:=\frac{1}{8\sup_{t\in J}\|q(t)\|_{L^p(\Gamma)}\|\big(\frac{\partial}{\partial t}+A_p\big)^{-1}\|_{\mathcal{L}(L^s(J;\tilde{L}^p);L^s(J;dom(A_p)))}}$$

and afterwards

$$\lambda:=4\sup_{t\in J}\|q(t)\|_{L^p(\Gamma)}c^{\frac{1}{1-\theta}}\Big(\frac{1}{\delta}\Big)^{\frac{\theta}{1-\theta}}.$$

Thus, we get

$$\|\mathcal{B}(\cdot,q)\big(\frac{\partial}{\partial t}+A_p+\lambda\big)^{-1}\|_{\mathcal{L}(L^s(J;\tilde{L}^p))} \le \frac{1}{2}. \tag{3.6}$$

This shows that the series $\sum_{j=0}^{\infty}(-1)^j\Big(\mathcal{B}(\cdot,q)\big(\frac{\partial}{\partial t}+A_p+\lambda\big)^{-1}\Big)^j$ absolutely converges in $\mathcal{L}(L^s(J;\tilde{L}^p))$ and the operator $\big(\frac{\partial}{\partial t}+A_p+\lambda\big)^{-1}\sum_{j=0}^{\infty}(-1)^j\Big(\mathcal{B}(\cdot,q)\big(\frac{\partial}{\partial t}+A_p+\lambda\big)^{-1}\Big)^j$ equals $\big(\frac{\partial}{\partial t}+A_p+\mathcal{B}(\cdot,q)+\lambda\big)^{-1}$. Moreover, (3.6) implies

$$\left\|\left(\frac{\partial}{\partial t}+A_p+\mathcal{B}(\cdot,q)+\lambda\right)^{-1}\right\|_{\mathcal{L}(L^s(J,\tilde{L}^p);L^s(J;dom(A_p))\cap W_0^{1,s}(J;\tilde{L}^p))} \le 2\left\|\left(\frac{\partial}{\partial t}+A_p+\lambda\right)^{-1}\right\|_{\mathcal{L}(L^s(J,\tilde{L}^p);L^s(J;dom(A_p))\cap W_0^{1,s}(J;\tilde{L}^p))}$$

Finally, one notices that $u\in L^s(J;dom(A_p))\cap W_0^{1,s}(J;\tilde{L}^p)$ satisfies

$$\frac{\partial u}{\partial t}+A_pu+\mathcal{B}(\cdot,q)u=f$$

iff $v:=e^{-\lambda\cdot}u$ satisfies

$$\frac{\partial v}{\partial t}+(A_p+\lambda)v+\mathcal{B}(\cdot,q)v=g,\quad \text{with}\quad g=e^{-\lambda\cdot}f.$$

This can be expressed as the equality

$$\Big(\frac{\partial}{\partial t}+A_p+\mathcal{B}(\cdot,q)\Big)^{-1}=e^{-\lambda\cdot}\Big(\frac{\partial}{\partial t}+A_p+\lambda+\mathcal{B}(\cdot,q)\Big)^{-1}e^{\lambda\cdot}.$$

Since the multiplication operators $e^{\lambda\cdot}$, $e^{-\lambda\cdot}$ act boundedly on $L^s(J;\tilde{L}^p)$ and $L^s(J;dom(A_p))\cap W_0^{1,s}(J;\tilde{L}^p)$, respectively, the assertions are proved. □

Remark 3.5. The proof is closely oriented at [4, Prop. 1.3], we only make some things more transparent in order to obtain uniformity in q.

Let us now consider nonzero initial conditions; for doing so we need the following result:

Proposition 3.6. ([40] *Prop. 2.2.2*) *Let A be injective and a generator of an analytic semigroup on a Banach space X with D as its domain. Then*

$$(X,D)_{1-\frac{1}{s},s}=\{\psi\in X : Ae^{-\cdot A}\psi\in L^s(J;X)\}. \tag{3.7}$$

Theorem 3.7. *Assume $p\ge d$, $s\in]1,\infty[$, q as above and $u_0\in(\widetilde{L}^p,dom(A_p))_{1-\frac{1}{s},s}$.*

i) *Then, for every $f\in L^s(J;\widetilde{L}^p)$ the initial value problem*

$$u'+A_pu+\mathcal{B}(\cdot;q)u=f,\quad u(0)=u_0 \tag{3.8}$$

admits a unique solution $u\in W^{1,s}(J;\widetilde{L}^p)\cap L^s(J;dom(A_p))$.

ii) *If q runs through a bounded set in $L^\infty(J;L^p(\Gamma))$ and u_0 runs through a bounded set in $(\widetilde{L}^p,dom(A_p))_{1-\frac{1}{s},s}$, then the associated set of solutions u of* (3.8) *forms a bounded set in $W^{1,s}(J;\widetilde{L}^p)\cap L^s(J;dom(A_p))$.*

Proof. i) $A_p + 1$ generates an analytic semigroup on $\widetilde{L}^p$ and is injective. Let us denote the function $J \ni t \mapsto e^{-t(A_p+1)}u_0$ by w. Due to Proposition 3.6, the function $(A_p+1)w(\cdot) = -w'$ belongs to $L^s(J; \widetilde{L}^p)$. In other words, $w \in L^s(J; dom(A_p))$. Then, according to (3.5), the function $J \ni t \mapsto B(t; q)w(t)$ belongs to $L^s(J; \widetilde{L}^p)$. Making now an ansatz $u := w + v$, one recognizes that u fulfills (3.8) if $v \in W_0^{1,s}(J; \widetilde{L}^p) \cap L^s(J; dom(A_p))$ satisfies

$$v' + A_p v + \mathcal{B}(\cdot; q)v = f - \mathcal{B}(\cdot; q)w + w. \tag{3.9}$$

But (3.9) is uniquely solvable in $W_0^{1,s}(J; \widetilde{L}^p) \cap L^s(J; dom(A_p))$, due to Theorem 3.4.
ii) It is clear that the functions $J \ni t \mapsto B(t; q)w(t)$ form a bounded subset of $L^s(J; \widetilde{L}^p)$. Thus, the solutions v of (3.9) form a bounded subset of $W_0^{1,s}(J; \widetilde{L}^p) \cap L^s(J; dom(A_p))$, due to Theorem 3.4. □

Remark 3.8. Note that the regularity assumption for q and the corresponding characterization of the operator $B(t; q)$ in Definition 3.3 is related to the control problem to be investigated in Section 4. In particular the solution of the initial value problem is forced to be continuous in space. If we assume q to be measurable and essentially bounded in space and time we can define $B(t, q)$ on $\widetilde{L}^p$. A close inspection of the proofs of Theorem 3.4 and 3.7 then shows that these results also hold in the case $p \in (1, d)$.

Theorem 3.9. *Assume $p \geq d$ and let $\theta \in]0, 1[$ be a number, such that*

$$[\widetilde{L}^p, dom(A_p)]_\theta \hookrightarrow C^\beta \tag{3.10}$$

for some $\beta > 0$. If $s > \frac{1}{1-\theta}$ and f belongs to $L^s(J; \widetilde{L}^p)$, then the solution u to (3.8) *even belongs to a space $C^\gamma(J; C^\beta) \hookrightarrow C^\delta(\overline{J \times \Omega})$ with $\delta = \min(\beta, \gamma)$.*
If q runs through a bounded set in $L^\infty(J; L^p(\Gamma))$, then the set of solutions u to (3.8) *forms a bounded set in $C^\delta(\overline{J \times \Omega})$.*

To prove the theorem, we first formulate

Lemma 3.10. *Assume that X, Z are Banach spaces with continuous injection $Z \hookrightarrow X$. Then, for every $\theta \in [0, 1 - \frac{1}{r}[$ there is a $\gamma = \gamma(\theta)$ such that $L^r(J; Z) \cap W^{1,r}(J; X)$ continuously injects into $C^\gamma(J; [X, Z]_\theta)$.*

Proof. An application of Hölder's inequality yields the embedding

$$W^{1,r}(J; X) \hookrightarrow C^\delta(J; X) \quad \text{with} \quad \delta = 1 - \frac{1}{r}. \tag{3.11}$$

Secondly, one has the continuous embedding

$$W^{1,r}(J; X) \cap L^r(J; Z) \hookrightarrow C(\overline{J}; (X, Z)_{1-\frac{1}{r}, r}) \hookrightarrow C(\overline{J}; [X, Z]_\theta) \tag{3.12}$$

(see [2, Thm. 4.10.2]). From (3.11) and (3.12) the claim follows by a straightforward application of the re-iteration theorem for complex interpolation, see [14] for a complete proof. □

Now, we are in a position to prove Theorem 3.9.

Proof. First, Lemma 2.17 tells us that such θ in fact exists. The condition $s > \frac{1}{1-\theta}$ is obviously equivalent to $\theta < 1 - \frac{1}{s}$. Then Theorem 3.7, combined with Lemma 3.10, proves the assertions. □

4. Application to optimal control problems

4.1. **Problem setting.** For convenience we recall the parabolic optimal control problem to be studied.

$$
\text{(P)}\quad \left.\begin{aligned}
\min J(u,q) := \frac{1}{2}\int_0^T\int_\Gamma (u-u_d)d\sigma_\Gamma\,dt + \frac{\alpha}{2}\int_0^T\int_\Gamma q(x,t)^2 d\sigma_\Gamma\,dt\\
u_t - \nabla\cdot\mu\nabla u = f \quad \text{in } Q = \Omega\times(0,T)\\
\nu\cdot\mu\nabla u + qu = g \quad \text{on } \Sigma_R = \Gamma\times(0,T)\\
u = 0 \quad \text{on } \Sigma_D = (\partial\Omega\setminus\Gamma)\times(0,T)\\
u(x,0) = u_0(x) \quad \text{in } \Omega\\
0 < q_a \le q(x,t) \le q_b \quad \text{a.e. on } \Sigma_R.
\end{aligned}\right\}
$$

The set of admissible controls for problem (P) is denoted by

$$Q_{ad} = \{q \in L^\infty(0,T;L^p(\Gamma)) : q_a \le q(x,t) \le q_b \text{ a.e. in } \Sigma_R\}.$$

We summarize the assumptions for the data.

Assumption 4.1.

- $\Omega\subset\mathbb{R}^d$ and $\Gamma\subset\partial\Omega$ such that $\Omega\cup\Gamma$ is regular in the sense of Gröger, cf. Definition 2.1
- $p \ge d$ and $s = \max\{\tilde{s},2\}$ where $\tilde{s}\in[1,\infty[$ has been chosen according to Theorem 3.9
- The bounds q_a, q_b in the control constraints are real numbers with $0 < q_a < q_b$
- $f \in L^s(0,T,L^p(\Omega))$ and $u_d,\ g \in L^s(0,T;L^p(\Gamma))$
- $u_0 \in (\widetilde{L}^p, dom(A_p))_{1-\frac{1}{s},s}$

4.2. **Discussion of the state equation.** We start with the analysis of the state equation regarding solvability and differentiability with respect to the control variable q. To this end, we consider

$$
\begin{aligned}
u_t - \nabla\cdot\mu\nabla u &= f &&\text{in } Q\\
\nu\cdot\mu\nabla u + qu &= g &&\text{on } \Sigma_R\\
u &= 0 &&\text{on } \Sigma_D\\
u(x,0) &= u_0(x) &&\text{in } \Omega.
\end{aligned}\tag{4.1}
$$

Corollary 4.2. *Let Assumption 4.1 be satisfied. Then, for every control $q\in Q_{ad}$ the state equation* (4.1) *admits a unique solution $u \in W^{1,s}(0,T;\widetilde{L}^p)\cap L^s(0,T;dom(A_p))$. Furthermore, there is a $\delta>0$, independent from $q\in Q_{ad}$, such that u belongs to $C^\delta(\bar{Q})$ and the following a priori estimate holds true*

$$
\begin{aligned}
&\|u\|_{W^{1,s}(0,T;\widetilde{L}^p)} + \|u\|_{L^s(0,T;dom(A_p))} + \|u\|_{C^\delta(\bar{Q})}\\
&\qquad\le \hat{c}\left(\|f\|_{L^s(0,T;L^p(\Omega))} + \|g\|_{L^s(0,T;L^p(\Gamma)} + \|u_0\|_{(\widetilde{L}^p,dom(A_p)))_{1-\frac{1}{s},s}}\right).
\end{aligned}\tag{4.2}
$$

Proof. We rewrite the state equation as an operator equation fitting into the framework of the previous section. First, the functions $f\in L^s(0,T,L^p(\Omega))$ and $g\in L^s(0,T;L^p(\Gamma))$ are considered as functions $\tilde{f},\tilde{g}$ in the space $L^s(0,T;\widetilde{L}^p)$ via the respective extensions by zero. According to Definition 2.16, the maximal restriction of the operator $-\nabla\cdot\mu\nabla$ to $\widetilde{L}^p$ is denoted by

A_p. Furthermore, we introduce for any function $q \in L^\infty(0,T;L^p(\Gamma))$ and any $t \in (0,T)$ the operator $B(t;q) : C(\bar{\Omega}) \to L^p(\Gamma)$, $B(t;q)\psi = q(\cdot,t)\psi|_\Gamma$ and $\mathcal{B}(\cdot;q) : J \ni t \mapsto B(t;q)$, analogously to Definition 3.3. Now, the state equation (4.1) can be written as

$$u' + A_p u + \mathcal{B}(\cdot;q)u = \tilde{f} + \tilde{g}, \quad u(0) = u_0. \tag{4.3}$$

In view of Assumption 4.1, the existence of a unique solution $u \in W^{1,s}(0,T;\widetilde{L}^p) \cap L^s(0,T;dom(A_p))$ immediately follows from Theorem 3.7. The a priori estimate is a direct consequence of Theorem 3.9, since Q_{ad} is a bounded set in $L^\infty(0,T;L^p(\Gamma))$. □

Based on this result, we will introduce a control-to-state mapping. We start with the definition of the state space.

Definition 4.3. Let p, s, be chosen according to Assumption 4.1. Then, the state space is defined by

$$Y := W^{1,s}(0,T;\widetilde{L}^p) \cap L^s(0,T;dom(A_p)),$$

where the operator A_p is defined in Definition 2.16.

Definition 4.4. Based on Corollary 4.2, we introduce the control-to-state operator $S : L^\infty(0,T;L^p(\Gamma)) \to Y$ by $u = S(q)$, which assigns to $q \in L^\infty(0,T;L^p(\Gamma))$ the unique solution $u \in Y$ of (4.3).

Remark 4.5. The previous result regarding continuity of the state variable would allow to discuss pointwise state constraints within the optimal control problem. In that case the establishment of first order necessary conditions becomes more delicate, since Lagrange multipliers regarding pointwise state constraints are only measures, see, e.g., [1, 8]. Moreover, the elaboration of second order sufficient optimality conditions is very difficult due to the low regularity of an adjoint state caused by the presence of measures in the right hand side of the respective adjoint equation. In particular, the derivation of sufficient optimality conditions in the case of parabolic boundary control control problems with pointwise control constraints is still an open question even for smooth data and simpler boundary conditions, at least in higher space dimensions. For detailed information concerning second order sufficient optimality conditions for parabolic optimal control problems with state constraints we refer to [44].

In order to establish optimality conditions for problem (P), it is essential to show Fréchet-differentiability of the control-to-state operator S, mapping q to $u = S(q)$.

Theorem 4.6. *Let Assumption 4.1 be satisfied. Then, the control-to-state operator S is twice continuously Fréchet-differentiable from $L^\infty(0,T;L^p(\Gamma))$ to Y. Its derivative $u_h := S'(q)h$ at the point q in direction h is given by the solution of*

$$u_h' + A_p u_h + \mathcal{B}(\cdot;q)u_h = -\mathcal{B}(\cdot;h)u, \quad u_h(0) = 0, \tag{4.4}$$

where $u = S(q) \in Y$ is the solution of the state equation w.r.t q. Furthermore, $u_{h_1h_2} = S''(q)[h_1,h_2]$, $h_i \in L^\infty(0,T;L^p(\Gamma))$, $i = 1,2$ is the solution of

$$u_{h_1h_2}' + A_p u_{h_1h_2} + \mathcal{B}(\cdot;q)u_{h_1h_2} = -(\mathcal{B}(\cdot;h_1)u_{h_2} + \mathcal{B}(\cdot;h_2)u_{h_1}), \quad u_{h_1h_2}(0) = 0, \tag{4.5}$$

with $u_{h_i} = S'(q)h_i$, $i = 1,2$.

Proof. We will utilize the implicit function theorem to prove the Fréchet-differentiability of the solution operator S. Analogously to the proof of Theorem 3.7, we denote the function $t \mapsto e^{-t(A_p+1)}u_0$ by w. Due to $u = S(q)$ and Definition 4.4, $v := u - w$ satisfies

$$v' + A_p v + \mathcal{B}(\cdot; q)v = \tilde{f} + \tilde{g} - \mathcal{B}(\cdot; q)w + w, \quad v(0) = 0.$$

Next, we introduce the mapping $\mathcal{F} : Y \times L^\infty(0, T; L^p(\Gamma)) \to L^s(0, T; \widetilde{L}^p)$ by

$$\mathcal{F}(v, q) := v' + A_p v + \mathcal{B}(\cdot; q)v - \tilde{f} - \tilde{g} + \mathcal{B}(\cdot; q)w - w.$$

One can easily see that according to Corollary 4.2 and the previous ansatz $v := u - w$, for every $q \in L^\infty(0, T; L^p(\Gamma))$ there is a unique $v(q) \in Y$ such that $\mathcal{F}(v(q), q) = 0$. Obviously, the mapping $\mathcal{F}$ is continuously differentiable with respect to v. Its partial derivative $\partial_v \mathcal{F}(v, q)$ with respect to v is the mapping $v_h \mapsto v_h' + A_p v_h + \mathcal{B}(\cdot; q)v_h$, which is, due to Theorem 3.4, a topological isomorphism between $W_0^{1,s}(J; \widetilde{L}^p) \cap L^s(J; dom(A_p))$ and $L^s(J; \widetilde{L}^p)$. Hence, the Implicit function theorem applies, i.e. $v(q)$ and $u = S(q)$, respectively, are continuously differentiable. The particular form of the derivative of v w.r.t. q immediately follows from

$$\partial_q v(q)h = -(\partial_v \mathcal{F}(v, q))^{-1} \partial_q \mathcal{F}(v, q)h = (\partial_v \mathcal{F}(v, q))^{-1}(-\partial_q \mathcal{B}(\cdot; q)h(v + w)).$$

Using again the ansatz $v := u - w$ and $\partial_q \mathcal{B}(\cdot; q)h = \mathcal{B}(\cdot; h)$, see Definition 3.3, the derivative $S'(q)h =: u_h$ of the control-to-state operator $u = S(q)$ at point g in direction h is given by

$$u_h' + A_p u_h + \mathcal{B}(\cdot; q)u_h = -\mathcal{B}(\cdot; h)u, \quad u_h(0) = 0.$$

Applying again the Implicit function theorem to the previous equation using the notations $u_{h_1 h_2} = S''(q)[h_1, h_2]$ and $u_{h_i} = S'(q)h_i,\ i = 1, 2$, respectively, one obtains (4.5).

□

Remark 4.7. It can be proved that the mapping is not only C^2, but C^∞ and even analytic (see [37, Ch. III.3]), since $\mathcal{F}$ in the previous proof is C^∞ and even analytic w.r.t. the first variable.

4.3. **Existence of optimal control and first order necessary conditions.** In this section we will elaborate first order necessary conditions for the optimal control problem (P). Since the state equation is nonlinear, we cannot expect uniqueness of an optimal control and we have to deal with local optimal controls. First, we want to clarify the existence of an optimal solution for problem (P).

Theorem 4.8. *Let Assumption 4.1 be satisfied. Then there exists at least one solution of problem* (P)*.*

Proof. Due to Corollary 4.2, there exists a unique solution

$$u \in W^{1,s}(0, T; \tilde{L}^p) \cap L^s(0, T; dom(A_p)) \hookrightarrow C^\delta(\overline{J \times \Omega})$$

of the state equation (4.1) for every control $q \in Q_{ad}$. Since the set of admissible controls Q_{ad} is bounded in $L^\infty(\Sigma_R)$, the set of solutions u is bounded in $W^{1,s}(0, T; \tilde{L}^p) \cap L^s(0, T; dom(A_p))$ and also bounded in $C^\delta(\bar{Q})$, see Theorem 3.7 and Theorem 3.9. Obviously, there exist a minimizing sequence $\{q_n\}$ converging to

$$l = \inf_{q \in Q_{ad}} J(S(q), q).$$

Since Q_{ad} is bounded and convex, we can extract a subsequence $\{q_{n_k}\}$ which converges weakly in $L^s(J; L^p(\Gamma))$ to a function $\bar{q} \in Q_{ad}$. The sequence $\{u_n = s(q_n)\}$ is bounded in

$W^{1,s}(0,T;\tilde{L}^p) \cap L^s(0,T;dom(A_p))$ and also bounded in $C^\delta(\bar{Q})$. Hence we may extract a further subsequence $\{u_{n_k}\}$ still subscripted by n_k, which weakly converges in $W^{1,s}(0,T;\tilde{L}^p) \cap L^s(0,T;dom(A_p))$ to an element $\bar{u}$ and in $C(\overline{J \times \Omega})$ strongly to $\bar{u}$. It is not hard to see that then the sequence $\{\mathcal{B}(\cdot;q_{n_k})u_{n_k}\}$ converges in $L^s(J;\tilde{L}^p)$ weakly to $\mathcal{B}(\cdot;\bar{q})\bar{u}$. As in the proof of Theorem 3.7, we rewrite equation 3.8 with $u = u_{n_k}$ as (3.9) with $v_{n_k} = u_{n_k} - w$ or, equivalently,

$$v_{n_k} = \Big(\frac{\partial}{\partial t} + A_p\Big)^{-1}\Big(f - \mathcal{B}(\cdot;q_{n_k})u_{n_k} + w\Big). \tag{4.6}$$

It is clear that v_{n_k} converges weakly in $W^{1,s}(0,T;\tilde{L}^p) \cap L^s(0,T;dom(A_p))$ to $\bar{u} - w$ and $f - \mathcal{B}(\cdot;q_l)u_l + w$ to $f - \mathcal{B}(\cdot;\bar{q})\bar{u} + w$, weakly in $L^s(J;\tilde{L}^p)$. But $\Big(\frac{\partial}{\partial t} + A_p\Big)^{-1} : L^s(J;\tilde{L}^p) \to W^{1,s}(0,T;\tilde{L}^p) \cap L^s(0,T;dom(A_p))$ is also continuous, if both spaces are equipped with their weak topologies (see [18, Ch. V.3]). This implies

$$\frac{\partial v}{\partial t} + A_p v = f - \mathcal{B}(\cdot;\bar{q})\bar{u} + w, \quad \text{or, equivalently,} \quad \frac{\partial \bar{u}}{\partial t} + A_p\bar{u} + \mathcal{B}(\cdot;\bar{q})\bar{u} = f.$$

The optimality of $\{\bar{u}, \bar{q}\}$ follows by standard arguments using the lower semicontinuity of the cost functional w.r.t. q and Assumption 4.1, respectively. □

In order to establish first order necessary optimality conditions the present control-to-state mapping S is rather abstract due to the choice of the state space Y, see Definition 4.3. In the following, we will consider the Hilbert space setting $Y = W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^{1,2}_\Gamma)$ as the state space such that $u = S(q)$ is equivalent to the classical weak formulation of the state equation (4.1), i.e.

$$\iint_Q u_t v\,dxdt + \iint_Q \mu \nabla u \nabla v\,dxdt + \iint_{\Sigma_R} quv\,dsdt = \iint_Q fv\,dxdt + \iint_{\Sigma_R} gv\,dsdt \quad \forall v \in L^2(0,T;H^{1,2}_\Gamma) \tag{4.7}$$

$$u(0) = u_0. \tag{4.8}$$

Let us introduce the reduced objective functional $j : L^\infty(0,T;L^p(\Gamma)) \to \mathbb{R}$ by

$$j(q) := J(S(q),q) = \frac{1}{2}\iint_{\Sigma_R} (S(q) - u_d)^2 d\sigma_\Gamma\,dt + \frac{\alpha}{2}\iint_{\Sigma_R} q^2 d\sigma_\Gamma\,dt.$$

By chain rule and Theorem 4.6, the cost functional $J(q)$ is continuously Fréchet differentiable w.r.t q from $L^\infty(0,T;L^p(\Gamma))$ to $\mathbb{R}$, since the solution operator S is of course continuously Fréchet differentiable from $L^\infty(0,T;L^p(\Gamma))$ to Y. The derivative at point $\bar{q}$ in direction h is given by

$$j'(\bar{q})h = \iint_{\Sigma_R} (\bar{u} - u_d)u_h d\sigma_\Gamma dt + \alpha \iint_{\Sigma_R} \bar{q}h d\sigma_\Gamma dt, \tag{4.9}$$

where we have again used the abbreviation $u_h = S'(\bar{q})h$.

In the following, we will express this derivative in terms of an adjoint state φ. To this end, we introduce the following formal adjoint equation

$$\begin{aligned} -\varphi_t - \nabla\cdot\mu^*\nabla\varphi &= 0 && \text{in } Q \\ \nu\cdot\mu^*\nabla\varphi + \bar{q}\varphi &= \bar{u} - u_d && \text{on } \Sigma_R \\ \varphi &= 0 && \text{on } \Sigma_D \\ \varphi(x,T) &= 0 && \text{in } \Omega, \end{aligned} \tag{4.10}$$

where μ^* is the adjoint coefficient function to μ. Similarly to Definition 2.16, we denote the maximal restriction of $-\nabla\cdot\mu^*\nabla$ to $\widetilde{L}^p$ as the formal adjoint of the differential operator $-\nabla\cdot\mu\nabla$ by A_p^*. Since $\bar{u}_{|\Gamma} - u_d \in L^s(0,T,L^p(\Gamma))$ we can apply Corollary 4.2 and conclude that there exists a unique solution $\varphi \in W^{1,s}(0,T;\widetilde{L}^p) \cap L^s(0,T;dom(A_p^*)) \cap C^\delta(\bar{Q})$ satisfying the a priori estimate

$$\|\varphi\|_{W^{1,s}(0,T;\widetilde{L}^p)} + \|\varphi\|_{L^s(0,T;dom(A_p^*))} + \|\varphi\|_{C^\delta(\bar{Q})} \quad\leq\quad \hat{c}\|\bar{u}_{|\Gamma} - u_d\|_{L^s(0,T;L^p(\Gamma))}. \tag{4.11}$$

In particular, φ satisfies

$$\begin{aligned} -\iint\limits_{Q}\varphi_t v\,dxdt + \iint\limits_{Q}\mu^*\nabla\varphi\nabla v\,dxdt + \iint\limits_{\Sigma_R}\bar{q}\varphi v\,dsdt &= \iint\limits_{\Sigma_R}(\bar{u} - u_d)v\,d\sigma_\Gamma dt \\ \varphi(T) &= 0 \end{aligned} \tag{4.12}$$

for all test functions $v \in L^2(J;H^{1,2}_\Gamma)$.

According to Theorem 4.6, we know that $u_h = S'(\bar{q})h$ with $h \in L^\infty(0,T;L^p(\Gamma))$ solves the initial boundary value problem (4.4), which in weak form reads as

$$\begin{aligned} \iint\limits_{Q}u_{h,t}v\,dxdt + \iint\limits_{Q}\mu\nabla u_h\nabla v\,dxdt + \iint\limits_{\Sigma_R}\bar{q}u_h v\,dsdt &= \\ -\iint\limits_{\Sigma_R}h\bar{u}v\,d\sigma_\Gamma dt &\quad \forall v \in L^2(J;H^{1,2}_\Gamma) \\ u_h(0) &= 0. \end{aligned} \tag{4.13}$$

Now we insert u_h as a test function in (4.12), replace the first term on the right-hand side of (4.9) by the left-hand side of equation (4.12) and use (4.13) with test function φ to obtain

$$j'(q)h = \iint\limits_{\Sigma_R}(-\varphi\bar{u} + \alpha\bar{q})h\,d\sigma_\Gamma dt.$$

Hence, the first order necessary optimality condition reads as follows:

$$\iint\limits_{\Sigma_R}(-\varphi\bar{u} + \alpha\bar{q})(q - \bar{q})\,d\sigma_\Gamma dt \geq 0 \quad \forall q \in Q_{ad},$$

where $\varphi \in W^{1,s}(0,T;\widetilde{L}^p) \cap L^s(0,T;dom(A_p))$ is the unique solution of the adjoint equation (4.10). To summarize, the first order optimality conditions for problem (P) can be formulated as follows.

Theorem 4.9. *Let $\bar{q} \in Q_{ad}$ be a (local) optimal control with associated state $\bar{u} \in W^{1,s}(0,T;\widetilde{L}^p) \cap L^s(0,T;dom(A_p))$ of problem* (P). *Then, there exists an adjoint state $\bar{\varphi} \in W^{1,s}(0,T;\widetilde{L}^p) \cap L^s(0,T;dom(A_p^*))$, such that the following optimality system is satisfied*

$$\bar{u}' + A_p\bar{u} + \mathcal{B}(\cdot;\bar{q})\bar{u} = \tilde{f} + \tilde{g}, \quad \bar{u}(0) = u_0, \tag{4.14}$$

$$-\bar{\varphi}' + A_p^*\bar{\varphi} + \mathcal{B}(\cdot;\bar{q})\bar{\varphi} = \bar{u} - \tilde{u}_d, \qquad \varphi(T) = 0, \tag{4.15}$$

$$\int_{\Sigma_R} (-\bar{\varphi}\bar{u} + \alpha\bar{q})(q - \bar{q})d\sigma_\Gamma dt \geq 0 \quad \forall q \in Q_{ad}. \tag{4.16}$$

4.4. **Second order sufficient optimality conditions.** To avoid too many technicalities, we will concentrate on the establishment of second order sufficient optimality conditions for local solutions in L^∞-sense. The consideration of local solutions w.r.t to weaker norms is more evolved and will be part of a forthcoming work.

By means of Theorem 4.6, the reduced cost functional $j(q)$ is of course also twice differentiable from $L^\infty(\Sigma_R)$ to $\mathbb{R}$ and we obtain for the second derivative of the reduced objective at the local optimal point $\bar{q} \in Q_{ad}$ in directions $h_1,\ h_2 \in L^\infty(\Sigma_R)$

$$j''(\bar{q})[h_1,h_2] = \iint_{\Sigma_R} S'(\bar{q})h_2 S'(\bar{q})h_1 d\sigma_\Gamma dt + \iint_{\Sigma_R} (S(\bar{q}) - u_d)S''(\bar{q})[h_1,h_2]d\sigma_\Gamma dt + \alpha \iint_{\Sigma_R} h_1 h_2 d\sigma_\Gamma dt.$$

As before we utilize the adjoint equation (4.10) and the equation for 2nd derivative $S''(\bar{q})[h_1,h_2]$, see (4.5) to replace the second term on the right-hand , yielding the expression

$$j''(\bar{q})[h_1,h_2] = \iint_{\Sigma_R} u_{h_1}u_{h_2} d\sigma_\Gamma dt + \alpha \iint_{\Sigma_R} h_1 h_2 d\sigma_\Gamma dt - \iint_{\Sigma_R} (u_{h_1}h_2 + u_{h_2}h_1)\bar{\varphi} d\sigma_\Gamma dt, \tag{4.17}$$

with $u_{h_i} = S'(q)h_i,\ i = 1,2$ and $\bar{\varphi}$ is the solution of the adjoint equation (4.10).

The crucial point in the analysis of second order sufficient optimality conditions is the fact that the quadratic form $j''(\bar{q})[h_1,h_2]$ has to depend continuously on $h_i,\ i = 1,2$ in the L^2-norm, i.e. we have to ensure the following continuity estimate

$$|j''(\bar{q})[h_1,h_2]| \leq c\|h_1\|_{L^2(\Sigma_R)}\|h_2\|_{L^2(\Sigma_R)} \tag{4.18}$$

for all $h_i \in L^\infty(\Sigma_R)$. This is motivated by consideration on the so-called two-norm discrepancy, see e.g. [49, Ch. 4.10.2].

The first term in $j''(\bar{q})[h_1,h_2]$ (see (4.17)) can be estimated with respect to the L^2-norms of $h_i,\ i = 1,2$ by applying standard a priori estimates and embeddings, e.g.

$$\|u_{h_i}\|_{L^2(\Sigma_R)} \leq c\|u_{h_i}\|_{L^2(0,T;H^{1,2}_\Gamma(\Omega))} \leq c\|\bar{u}\|_{C(\bar{Q})}\|h_i\|_{L^2(\Sigma_R)}, \tag{4.19}$$

since the control $\bar{q}$ and the directions h_i are considered as functions in $L^\infty(\Sigma_R)$. Moreover, the optimal state $\bar{u}$ is bounded in $C(\bar{Q})$, see (4.2). The third term is the more delicate one. Here we take advantage of the regularity and the respective a priori estimate of the adjoint state, see (4.11), such that we derive the estimate

$$\left| \iint_{\Sigma_R} (u_{h_1} h_2 + u_{h_2} h_1) \bar{\varphi} d\sigma_\Gamma dt \right| \leq c \|\bar{\varphi}\|_{C(\bar{Q})} \|h_1\|_{L^2(\Sigma_R)} \|h_2\|_{L^2(\Sigma_R)}.$$

We note that the previous estimate heavily rests on the continuity of the adjoint state φ and for this reason on the results derived in section 3. The continuity estimate (4.18) allows to estimate the second order remainder term of the reduced cost functional j. Based on

$$j(q) = j(\bar{q}) + j'(\bar{q})(q - \bar{q}) + \frac{1}{2} j''(\bar{q})[q - \bar{q}, q - \bar{q}] + r(\bar{q}, q - \bar{q})$$

one derives the estimate

$$|r(\bar{q}, q - \bar{q})| \leq c \|q - \bar{q}\|_{L^\infty(\Sigma_R)} \|q - \bar{q}\|^2_{L^2(\Sigma_R)} \tag{4.20}$$

applying (4.18). This estimate is an essential part for the establishment of second order sufficient optimality conditions, see [49, Ch. 4.10.2].

In all what follows, we denote by $\bar{q}$ an admissible control of problem (P) with associated state $\bar{u} = S(\bar{q})$. Furthermore, we suppose that the first order necessary optimality conditions given in Corollary 4.9 are fulfilled by $\bar{q}$, the respective state $\bar{u}$ and adjoint state $\bar{\varphi}$. For the statement of second order sufficient optimality conditions we will count on so called strongly active sets. We start with the definition of the τ-critical cone associated to $\bar{q}$:

$$C_\tau(\bar{q}) := \{h \in L^2(\Sigma_R) : \ h := q - \bar{q} \text{ satisfies (4.21)}, \ q \in Q_{ad}\},$$

where

$$h(x,t) \begin{cases} \geq 0, & \text{if } \bar{q}(x,t) = q_a \\ \leq 0, & \text{if } \bar{q}(x,t) = q_b \\ = 0, & \text{if } |-\bar{\varphi}(x,t)\bar{u}(x,t) + \alpha \bar{q}(x,t)| > \tau. \end{cases} \tag{4.21}$$

We are now in the position to formulate second order sufficient optimality conditions for problem (P). With the above results, in particular the regularity results and the remainder estimate (4.20), the proof of the following theorem is completely analogous to the one presented in [49, Theorem 5.17]

Theorem 4.10. *Let $\bar{q}$ be an admissible control of problem* (P) *with associated state $\bar{u} = S(\bar{q})$ satisfying the first order necessary optimality conditions given in Corollary 4.9 with associated adjoint state $\bar{\varphi}$. Further, it is assumed that there are two constants $\tau > 0$ and $\delta > 0$ such that*

$$j''(\bar{q})h^2 \geq \delta \|h\|^2_{L^2(\Sigma_R)}$$

holds for all directions $h \in C_\tau(\bar{q})$. Then there exist a $\tilde{\delta} > 0$ and $\rho > 0$ such that

$$j(q) \geq j(\bar{q}) + \tilde{\delta} \|q - \bar{q}\|^2_{L^2(\Sigma_R)} \tag{4.22}$$

holds for all $q \in Q_{ad}$ with $\|q - \bar{q}\|_{L^\infty(\Sigma_R)} \leq \rho$.

Such kind of sufficient optimality conditions are an indispensable tool basis for carrying out numerical analysis of optimal control problems, e.g. error estimates in numerical discretizations or convergence analysis of the sequential quadratic programming method in order to solve optimal control problems.

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