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Optimal distributed control of two-dimensional nonlocal Cahn–Hilliard–Navier–Stokes systems with degenerate mobility and singular potential

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Abstract

In this paper, we consider a two-dimensional diffuse interface model for the phase separation of an incompressible and isothermal binary fluid mixture with matched densities. This model consists of the Navier–Stokes equations, nonlinearly coupled with a convective nonlocal Cahn–Hilliard equation. The system rules the evolution of the (volume-averaged) velocity $\boldsymbol{u}$ of the mixture and the (relative) concentration difference φ of the two phases. The aim of this work is to study an optimal control problem for such a system, the control being a time-dependent external force $\boldsymbol{v}$ acting on the fluid. We first prove the existence of an optimal control for a given tracking type cost functional. Then we study the differentiability properties of the control-to-state map $\boldsymbol{v} \mapsto [\boldsymbol{u}, \varphi]$, and we establish first-order necessary optimality conditions. These results generalize the ones obtained by the first and the third authors jointly with E. Rocca in [19]. There the authors assumed a constant mobility and a regular potential with polynomially controlled growth. Here, we analyze the physically more relevant case of a degenerate mobility and a singular (e.g., logarithmic) potential. This is made possible by the existence of a unique strong solution which was recently proved by the authors and C. G. Gal in [14].

1 Introduction

A well-known diffuse interface model for incompressible and isothermal binary fluids is the so-called Cahn–Hilliard–Navier–Stokes system (see, for instance, [1, 24, 25]). It consists of the nonlinear coupling of the Navier–Stokes equations for the volume-averaged velocity $\boldsymbol{u}$ with the convective Cahn–Hilliard equation for the (relative) concentration difference φ of the two fluids. More precisely, assuming matched densities equal to unity, we have to deal with the system of partial differential equations

$$\boldsymbol{u}_t - 2\,\mathrm{div}\,(\nu(\varphi)D\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \mu\nabla\varphi + \boldsymbol{v}, \tag{1.1}$$

$$\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \mathrm{div}(m(\varphi)\nabla\mu), \tag{1.2}$$

$$\mathrm{div}(\boldsymbol{u}) = 0, \tag{1.3}$$

in $Q := \Omega\times(0,T)$, where $\Omega\subset\mathbb{R}^d$, $d = 2, 3$, is a bounded and smooth domain, $T > 0$ is a prescribed final time, and D denotes the symmetric gradient defined by $D\boldsymbol{u} := \big(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}\big)/2$. Here, the viscosity $\nu(\cdot)$ is strictly positive, π stands for the pressure, $\boldsymbol{v}$ is a given external force density, $m(\cdot)$ is the mobility,

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and μ represents the so-called chemical potential. Within the phenomenological framework devised in [7], μ is the functional derivative of the local Ginzburg–Landau type functional

$$\mathcal{G}(\varphi) = \int_\Omega \left(\frac{|\nabla\varphi|^2}{2} + W(\varphi) \right) dx, \tag{1.4}$$

where W is a given double-well potential. Here, and in the following, all of the relevant physical constants have been set equal to unity, for the sake of simplicity.

On the other hand, a physically more rigorous approach shows that μ is the functional derivative of a *nonlocal* functional of the following form (see [5, 21, 22, 23], cf. also [20] for a detailed discussion):

$$\mathcal{F}(\varphi) = -\frac{1}{2}\int_\Omega \int_\Omega K(x-y)\varphi(x)\varphi(y)dxdy + \int_\Omega F(\varphi)dx\,.$$

Here, $K : \mathbb{R}^d \to \mathbb{R}$ is a sufficiently smooth interaction kernel such that $K(x) = K(-x)$, and F is a convex potential (usually of logarithmic type).

In this contribution, we address an optimal control problem for the following nonlocal Cahn–Hilliard–Navier–Stokes system:

$$\boldsymbol{u}_t - 2\,\text{div}\,(\nu(\varphi)D\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \mu\nabla\varphi + \boldsymbol{v}, \tag{1.5}$$

$$\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \text{div}(m(\varphi)\nabla\mu), \tag{1.6}$$

$$\mu = -K*\varphi + F'(\varphi), \tag{1.7}$$

$$\text{div}(\boldsymbol{u}) = 0, \tag{1.8}$$

in Q, subject to the boundary conditions

$$\boldsymbol{u} = \boldsymbol{0}, \qquad m(\varphi)\nabla\mu\cdot\boldsymbol{n} = 0, \tag{1.9}$$

on $\Sigma := \partial\Omega \times (0,T)$, and to the initial conditions

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \varphi(0) = \varphi_0, \tag{1.10}$$

in Ω. Here, $\boldsymbol{n}$ stands for the outward unit normal to the boundary $\partial\Omega$ of Ω, while $\boldsymbol{u}_0$ and φ_0 are given functions.

Problem (1.5)–(1.10) constitutes the *state system* of the control problem to be investigated below. A slightly different version thereof has firstly been analyzed in [18] under rather general assumptions on ν, m, J and F (see also [8, 15, 16, 17] for more restrictive assumptions). More precisely, the mobility degenerates at the pure phases $\varphi = \pm 1$, and F is a bounded (smooth) potential, defined on $(-1,1)$, whose derivatives are unbounded (i.e., a so-called singular potential). In particular, m and F can have the following form:

$$m(s) = 1 - s^2, \qquad F(s) = (1+s)\ln(1+s) + (1-s)\ln(1-s), \qquad s\in(-1,1). \tag{1.11}$$

In [18], the existence of a global weak solution was established for a constant viscosity, but the same argument can easily be extended to nonconstant viscosities as well. Uniqueness and existence of a strong solution are more delicate issues, and restricted to the two-dimensional case. The former was analyzed in

[13], proving a conditional weak-strong uniqueness, i.e., by supposing that a strong solution exists. The existence of a strong solution has been much harder to prove. This was done in the more recent contribution [14] by using a time-discretization scheme combined with a suitable approximation of m and F.

In this paper, we aim to study optimal control problems for the state system (1.5)–(1.10), which, in order to have a well-defined control-to-state operator, postulates the unique solvability of the state system itself. Also, the investigation of the differentiability properties of the control-to-state operator requires that the solution to the state system be sufficiently regular. Both requirements make it necessary to restrict the analysis to the spatially two-dimensional case. For this case, we can exploit the existence of a unique strong solution in order to formulate an optimal distributed control problem which is similar to the one analyzed in [19] under the more restrictive assumptions that m is constant and $F : \mathbb{R} \to \mathbb{R}$ is smooth with a polynomially controlled growth. This is not just a minor generalization, since it requires a considerable technical effort, and, besides, accounts for choices of m and F which are physically more relevant. A similar problem was originally considered in [33] in the spatially three-dimensional case for a convective nonlocal Cahn–Hilliard equation with degenerate mobility and singular potential, where the control was given by the velocity itself. However, the assumptions in [33] were more restrictive than the present ones. Indeed, the authors only considered solutions which are uniformly separated from the pure phases. Here, we do not use this property, so the initial datum can even represent a pure phase. This is possible because the system is conveniently reformulated in a more general form, following the approach devised in [12] for the Cahn–Hilliard equation with degenerate mobility, singular potential and the standard chemical potential (1.4). It is worth observing that for such an equation, and also for the corresponding system (1.1)–(1.3), only the existence of a global weak solution has been proven so far (cf. [6]).

Let us now introduce the control problem we are interested in (see [19]).

(CP) Minimize the tracking type cost functional

$$\mathcal{J}(y, \boldsymbol{v}) := \frac{\beta_1}{2}\|\boldsymbol{u} - \boldsymbol{u}_Q\|^2_{L^2(Q)^2} + \frac{\beta_2}{2}\|\varphi - \varphi_Q\|^2_{L^2(Q)} + \frac{\beta_3}{2}\|\boldsymbol{u}(T) - \boldsymbol{u}_\Omega\|^2_{L^2(\Omega)^2}$$
$$+ \frac{\beta_4}{2}\|\varphi(T) - \varphi_\Omega\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|\boldsymbol{v}\|^2_{L^2(Q)^2}\,, \tag{1.12}$$

where $y := [\boldsymbol{u}, \varphi]$ solves the state system (1.5)–(1.10).

Here, the quantities $\boldsymbol{u}_Q \in L^2(0, T; G_{div})$, $\varphi_Q \in L^2(Q)$, $\boldsymbol{u}_\Omega \in G_{div}$, and $\varphi_\Omega \in L^2(\Omega)$, are given target functions, while β_i, $i = 1, \ldots, 4$, and γ are some fixed nonnegative constants that do not vanish simultaneously. Moreover, G_{div} is the classical Navier–Stokes type space (see, e.g., [34]), that is,

$$G_{div} := \overline{\left\{\boldsymbol{u} \in C_0^\infty(\Omega)^2 : \mathrm{div}(\boldsymbol{u}) = 0\right\}}^{L^2(\Omega)^d}.$$

The control $\boldsymbol{v}$ is supposed to belong to a convenient closed, bounded and convex subset (see below) of the space of controls $L^2(0, T; G_{div})$.

We remind that optimal control problems for the Cahn–Hilliard–Navier–Stokes system with (1.4) have recently been studied for the spatially three-dimensional case, where, however, the time-discretized version case was considered (see [26, 27, 28, 29, 30, 31]). We also refer to the recent contributions [9, 10] for a treatment of the control by the velocity of convective Cahn–Hilliard systems with dynamic boundary conditions in three dimensions.

The plan of this paper is as follows. In Section 2, we introduce notation, the basic assumptions, and the notion of weak solutions to the state system. Then we report the existence theorem mentioned above. Also, we state the existence and uniqueness result on strong solutions to the state system for the case $d = 2$, which is fundamental for the control problem, and the related hypotheses. Section 3 is devoted to establish some global stability estimates that are crucial to analyze the control problem **(CP)**. This is studied in Section 4: first, we prove in a standard way the existence of an optimal control; then we show the Fréchet differentiability of the control-to-state map in suitable Banach spaces (where the stability estimates plays an essential role). Finally, we establish first-order optimality conditions.

Throughout the entire paper, we will repeatedly use Young's inequality

$$a\,b \,\le\, \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for all } a,b \in \mathbb{R} \text{ and } \delta > 0\,, \tag{1.13}$$

and we employ the following notational convention for the use of constants in estimates: the letter C denotes a generic positive constant depending only on the data of the respective problem; the use of subscripts like in $C_{m,K}$ signals that the constant depends in a bounded way on the quantities occurring in the subscript (in this case, m and K), in particular. In any case, the meaning will be clear and no confusion will arise.

2 Notation and known results for the state system

We set $H := L^2(\Omega)$, $V := H^1(\Omega)$, and denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the inner product, respectively, in both H and G_{div}, as well as in $L^2(\Omega)^2$ and $L^2(\Omega)^{2\times 2}$. The notation $\langle\cdot,\cdot\rangle_X$ and $\|\cdot\|_X$ will stand for the duality pairing between a (real) Banach space X and its dual X', and for the norm of X, respectively. The space

$$V_{div} := \overline{\left\{ \boldsymbol{u} \in C_0^\infty(\Omega)^2 : \text{div}(\boldsymbol{u}) = 0 \right\}}^{H^1(\Omega)^d}$$

is endowed with the scalar product

$$(\boldsymbol{u},\boldsymbol{v})_{V_{div}} = (\nabla\boldsymbol{u},\nabla\boldsymbol{v}) = 2(D\boldsymbol{u},D\boldsymbol{v}), \qquad \forall\,\boldsymbol{u},\boldsymbol{v} \in V_{div}.$$

Let us also recall the definition of the Stokes operator $S : D(S) \cap G_{div} \to G_{div}$ in the case of the no-slip boundary condition $(1.9)_1$, i.e., $S = -P\Delta$ with domain $D(S) = H^2(\Omega)^d \cap V_{div}$, where $P : L^2(\Omega)^d \to G_{div}$ is the Leray projector (see [34]). Notice that we have

$$(S\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{u},\boldsymbol{v})_{V_{div}} = (\nabla\boldsymbol{u},\nabla\boldsymbol{v}), \qquad \forall\,\boldsymbol{u} \in D(S), \quad \forall\,\boldsymbol{v} \in V_{div}.$$

We also recall that $S^{-1} : G_{div} \to G_{div}$ is a self-adjoint and compact operator in G_{div}, and the spectral theorem entails the existence of a sequence of eigenvalues λ_j with $0 < \lambda_1 \le \lambda_2 \le \cdots$ and $\lambda_j \to \infty$, and a corresponding family of eigenfunctions $\boldsymbol{w}_j \in D(S)$, which is orthonormal in G_{div} and satisfies $S\boldsymbol{w}_j = \lambda_j \boldsymbol{w}_j$ for all $j \in \mathbb{N}$. We also recall Poincaré's inequality

$$\lambda_1 \|\boldsymbol{u}\|^2 \le \|\nabla\boldsymbol{u}\|^2, \qquad \forall\,\boldsymbol{u} \in V_{div},$$

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and two other inequalities, which are valid in two dimensions of space and will be used repeatedly in the course of our analysis, namely, the following special case of the Gagliardo–Nirenberg inequality (see, e.g., [4]),

$$\|v\|_{L^{2q}(\Omega)} \;\le\; \widehat{C}_2\,\|v\|^{1/q}\,\|v\|_V^{1-1/q}, \qquad \forall\, v\in V, \qquad 2\le q<\infty, \tag{2.1}$$

and Agmon's inequality (see [2])

$$\|v\|_{L^\infty(\Omega)} \;\le\; \widehat{C}_3\,\|v\|^{1/2}\,\|v\|_{H^2(\Omega)}^{1/2}, \qquad \forall\, v\in H^2(\Omega). \tag{2.2}$$

In these inequalities, the positive constant $\widehat{C}_2$ depends on q and on $\Omega\subset\mathbb{R}^2$, while the positive constant $\widehat{C}_3$ depends only on Ω.

The trilinear form b appearing in the weak formulation of the Navier–Stokes equations is defined as usual, namely,

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) := \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}\,dx \qquad \forall\,\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V_{div}\,.$$

The associated bilinear operator $\mathcal{B}$ from $V_{div}\times V_{div}$ into V_{div}' is defined by $\langle \mathcal{B}(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}\rangle := b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$, for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V_{div}$. We set $\mathcal{B}\,\boldsymbol{u} := \mathcal{B}(\boldsymbol{u},\boldsymbol{u})$, for every $\boldsymbol{u}\in V_{div}$. We recall the well-known identity

$$b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) \;=\; -\,b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \qquad \forall\,\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V_{div},$$

and the two-dimensional inequality

$$|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \;\le\; \widehat{C}_1\,\|\boldsymbol{u}\|^{1/2}\,\|\nabla\boldsymbol{u}\|^{1/2}\,\|\nabla\boldsymbol{v}\|\,\|\boldsymbol{w}\|^{1/2}\,\|\nabla\boldsymbol{w}\|^{1/2} \qquad \forall\,\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V_{div},$$

with a constant $\widehat{C}_1>0$ that depends only on Ω.

We now state the assumptions which ensure the existence of a global weak solution. Although weak solutions do not play a role for our control problem, we have decided to include the corresponding existence result for the sake of giving the reader a complete picture of the well-posedness results known for the state system. In particular, we include the result for the case of three dimensions of space, noting that this result is too weak for control purposes. We make the following assumptions:

(V) The viscosity ν is Lipschitz continuous on $[-1,1]$, and there exists some $\nu_1>0$ such that

$$\nu_1 \le \nu(s)\,, \quad \forall s\in[-1,1]\,.$$

(K) $K(\cdot - x)\in W^{1,1}(\Omega)$ for almost every $x\in\Omega$, and it holds that $K(x)=K(-x)$ and

$$\sup_{x\in\Omega}\int_\Omega |K(x-y)|dy<\infty\,, \qquad \sup_{x\in\Omega}\int_\Omega |\nabla K(x-y)|dy<\infty\,.$$

(H1) The mobility satisfies $m\in C^1([-1,1])$, $m\ge 0$, and $m(s)=0$ if and only if $s=-1$ or $s=1$. Moreover, there exists some $\epsilon_0>0$ such that m is nonincreasing in $[1-\epsilon_0,1]$ and nondecreasing in $[-1,-1+\epsilon_0]$.

(H2) $F\in C^2(-1,1)$ and $\lambda := mF''\in C\left([-1,1]\right)$.

(H3) There exists some $\epsilon_0 > 0$ such that F'' is nonincreasing in $[1-\epsilon_0, 1]$ and nondecreasing in $[-1, -1+\epsilon_0]$.

(H4) There exists some $c_0 > 0$ such that

$$F''(s) \geq c_0 \,, \qquad \forall\, s \in (-1,1)\ .$$

(H5) There exists some $\alpha_0 > 0$ such that

$$\lambda(s) \geq \alpha_0 \,, \qquad \forall\, s \in [-1,1]\,.$$

We also recall that if the mobility degenerates, then the notion of weak solution must be formulated in a suitable way (cf. [12], see also [18]).

Definition 1. *Let $\boldsymbol{u}_0 \in G_{div}$ and $\varphi_0 \in L^\infty(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ and $\boldsymbol{v} \in L^2(0,T;V_{div}')$ be given. A couple $[\boldsymbol{u},\varphi]$ is called a weak solution to* (1.5)–(1.10) *on $[0,T]$ if and only if the following conditions hold true:*

- *$\boldsymbol{u}$ and φ satisfy*

$$\begin{aligned}
&\boldsymbol{u} \in L^\infty(0,T;G_{div}) \cap L^2(0,T;V_{div}),\\
&\boldsymbol{u}_t \in L^{4/3}(0,T;V_{div}') \qquad \text{if} \quad d=3,\\
&\boldsymbol{u}_t \in L^2(0,T;V_{div}') \qquad \text{if} \quad d=2,\\
&\varphi \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^\infty(Q),\\
&\varphi_t \in L^2(0,T;V'),\\
&|\varphi(x,t)| \leq 1 \qquad \text{for a.e. } (x,t) \in Q;
\end{aligned}$$

- *for every $\boldsymbol{w} \in V_{div}$, every $\psi \in V$, and almost every $t \in (0,T)$, we have*

$$\langle \boldsymbol{u}_t, \boldsymbol{w}\rangle_{V_{div}} + 2\ (\nu\,(\varphi)\, D\boldsymbol{u}, D\boldsymbol{w}) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}) \ = \ -\big((K*\varphi)\nabla\varphi, \boldsymbol{w}\big) + \langle \boldsymbol{v}, \boldsymbol{w}\rangle_{V_{div}},$$

$$\langle \varphi_t, \psi\rangle_V + \int_\Omega m(\varphi)F''(\varphi)\nabla\varphi\cdot\nabla\psi\, dx - \int_\Omega m(\varphi)(\nabla K*\varphi)\cdot\nabla\psi\, dx \ = \ (\boldsymbol{u}\varphi, \nabla\psi);$$

- *the initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and $\varphi(0) = \varphi_0$ are fulfilled.*

We observe that the regularity properties of the weak solution imply the weak continuity $\boldsymbol{u} \in C_w([0,T];G_{div})$ and $\varphi \in C_w([0,T];H)$. Therefore, the initial conditions are meaningful.

We now report the result shown in [18]. In this connection, we point out that there the viscosity ν was assumed to be constant just to avoid technicalities; however, the assertion of the theorem still holds true if ν satisfies only **(V)** (see also [13] for further details).

Theorem 1. *Assume that (V), (K) and (H1)–(H5) are satisfied. Let* $\boldsymbol{u}_0 \in G_{div}$ *and* $\varphi_0 \in L^\infty(\Omega)$ *with* $F(\varphi_0) \in L^1(\Omega)$ *and* $M(\varphi_0) \in L^1(\Omega)$ *be given, where* $M \in C^2(-1,1)$ *solves* $m(s)M''(s) = 1$ *for all* $s \in (-1,1)$ *with* $M(0) = M'(0) = 0$*. Assume also that* $\boldsymbol{v} \in L^2_{loc}([0,\infty); V'_{div})$*. Then, for every* $T > 0$*, the state system* (1.5)–(1.10) *admits a weak solution* $[\boldsymbol{u}, \varphi]$ *on* $[0,T]$ *such that the mean values satisfy* $\overline{\varphi}(t) = \overline{\varphi}_0$ *for all* $t \in [0,T]$*. If* $d = 2$*, then the weak solution* $[\boldsymbol{u}, \varphi]$ *satisfies the energy equation*

$$\frac{1}{2}\frac{d}{dt}\big(\|\boldsymbol{u}\|^2 + \|\varphi\|^2\big) + \int_\Omega m(\varphi)F''(\varphi)|\nabla\varphi|^2\,dx \,+\, 2\|\sqrt{\nu(\varphi)}D\boldsymbol{u}\|^2$$
$$= \int_\Omega m(\varphi)(\nabla K * \varphi)\cdot\nabla\varphi\,dx \,-\, \int_\Omega (K*\varphi)\,\boldsymbol{u}\cdot\nabla\varphi\,dx \,+\, \langle \boldsymbol{v}, \boldsymbol{u}\rangle_{V_{div}} \quad \text{for a.e. } t \in (0,T). \tag{2.3}$$

If $d = 3$*, then* $[\boldsymbol{u}, \varphi]$ *satisfies the energy inequality*

$$\frac{1}{2}\big(\|\boldsymbol{u}(t)\|^2 + \|\varphi(t)\|^2\big) + 2\int_0^t \|\sqrt{\nu(\varphi)}D\boldsymbol{u}\|^2(s)\,ds \,+ \int_0^t\int_\Omega m(\varphi)F''(\varphi)|\nabla\varphi|^2\,dx\,ds$$
$$\le \frac{1}{2}\big(\|\boldsymbol{u}_0\|^2 + \|\varphi_0\|^2\big) + \int_0^t\int_\Omega m(\varphi)(\nabla K*\varphi)\cdot\nabla\varphi\,dx\,ds$$
$$- \int_0^t\int_\Omega (K*\varphi)\boldsymbol{u}\cdot\nabla\varphi\,dx\,ds \,+ \int_0^t \langle \boldsymbol{v}(s), \boldsymbol{u}(s)\rangle_{V_{div}}(s)\,ds\,, \qquad \forall\, t \in (0,T]. \tag{2.4}$$

As noted above, the notion of weak solution does not suffice for purposes of optimal control theory. In order to introduce the notion of strong solution, we need the slightly stronger assumption:

(H2*) $F \in C^3(-1,1)$, and $\lambda := mF'' \in C^1([-1,1])$.

We then set (see [14])

$$B(s) := \int_0^s \lambda(\sigma)d\sigma\,, \qquad \forall s \in [-1,1]\,. \tag{2.5}$$

Moreover, we recall the definition of the notion of admissible kernels (see [3, Definition 1]):

Definition 2. *A kernel* $K \in W^{1,1}_{loc}(\mathbb{R}^d)$ *is called admissible if and only if the following conditions are satisfied:*

(K1) $K \in C^3(\mathbb{R}^d\backslash\{0\})$*;*

(K2) K *is radially symmetric,* $K(x) = \tilde{K}(|x|)$*, and* $\tilde{K}$ *is nonincreasing;*

(K3) $\tilde{K}''(r)$ *and* $\tilde{K}'(r)/r$ *are monotone on* $(0, r_0)$ *for some* $r_0 > 0$*;*

(K4) $|D^3K(x)| \le C_d|x|^{-d-1}$ *for some* $C_* > 0$*.*

For the readers' convenience, we report the following lemma.

Lemma 1. (cf. [3, Lemma 2]) *Let* K *be admissible. Then, for every* $p \in (1,\infty)$*, there exists some* $C_p > 0$ *such that*

$$\|\nabla(\nabla K * \psi)\|_{L^p(\Omega)^{d\times d}} \le C_p\|\psi\|_{L^p(\Omega)} \qquad \forall\, \psi \in L^p(\Omega)\,, \tag{2.6}$$

where $C_p = C_* p$ *for* $p \in [2,\infty)$ *and* $C_p = C_* p/\,(p-1)$ *for* $p \in (1,2)$*, with some constant* $C_* > 0$ *which is independent of* p*.*

After these preliminaries, we assume for the remainder of this paper that $d = 2$, which implies that (2.1) and (2.2) are valid and the embedding $V \subset L^p(\Omega)$ is continuous and compact for $1 \le p < +\infty$. We now report the notion of strong solution introduced in [14]:

Definition 3. *Assume that $\boldsymbol{u}_0 \in V_{div}$, $\varphi_0 \in V \cap C^\beta(\overline{\Omega})$ for some $\beta \in (0,1)$, and $\boldsymbol{v} \in L^2(0,T;G_{div})$ are given. A weak solution $[\boldsymbol{u},\varphi]$ to the state system* (1.5)–(1.10) *on $[0,T]$ corresponding to $[\boldsymbol{u}_0,\varphi_0]$ is called a strong solution if and only if it holds that*

$$\boldsymbol{u} \in L^\infty(0,T;V_{div}) \cap L^2(0,T;H^2(\Omega)^2), \qquad \boldsymbol{u}_t \in L^2(0,T;G_{div}), \tag{2.7}$$

$$\varphi \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \qquad \varphi_t \in L^2(0,T;H), \tag{2.8}$$

$$\boldsymbol{u}_t - 2\mathrm{div}\,(\nu(\varphi)D\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = -(K*\varphi)\nabla\varphi + \boldsymbol{v} \quad \textit{a.e. in } Q, \tag{2.9}$$

$$\varphi_t + \boldsymbol{u}\cdot\nabla\varphi = \Delta B(\varphi) - \mathrm{div}\,\big(m(\varphi)(\nabla K*\varphi)\big) \quad \textit{a.e. in } Q, \tag{2.10}$$

$$\mathrm{div}\,(\boldsymbol{u}) = 0 \quad \textit{a.e. in } Q, \tag{2.11}$$

$$\boldsymbol{u} = \boldsymbol{0}, \quad \big[\nabla B(\varphi) - m(\varphi)(\nabla K*\varphi)\big]\cdot\boldsymbol{n} = 0 \quad \textit{a.e. on } \Sigma, \tag{2.12}$$

for some $\pi \in L^2(0,T;V)$.

We have the following result (see [14, Thm.3.6] and [14, Rem.4.5], cf. also [14, Rem.3.7]).

Theorem 2. *Let the assumptions **(V), (K), (H1), (H2*)–(H5)** hold true, and assume that either $K \in W^{2,1}_{loc}(\mathbb{R}^2)$ or K is admissible. Let $\boldsymbol{u}_0 \in G_{div}$ and $\varphi_0 \in V \cap L^\infty(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ and $M(\varphi_0) \in L^1(\Omega)$ be given, where M is defined as in Theorem 1. Moreover, suppose that $\boldsymbol{v} \in L^2(0,T;G_{div})$. Then, for every $T > 0$, the state system* (1.5)–(1.10) *admits a weak solution $[\boldsymbol{u},\varphi]$ on $[0,T]$. If, in addition, $\boldsymbol{u}_0 \in V_{div}$ and $\varphi_0 \in V \cap C^\beta(\overline{\Omega})$ for some $\beta \in (0,1)$, then the state system* (1.5)–(1.10) *admits a unique strong solution in the sense of Definition 3. Finally, if $\varphi_0 \in H^2(\Omega)$ fulfills the compatibility condition*

$$[\nabla B(\varphi_0) - m(\varphi_0)(\nabla K*\varphi_0)]\cdot\boldsymbol{n} = 0, \quad \textit{a.e. on } \partial\Omega, \tag{2.13}$$

then the strong solution also satisfies

$$\varphi \in L^\infty(0,T;H^2(\Omega)), \qquad \varphi_t \in L^\infty(0,T;H) \cap L^2(0,T;V). \tag{2.14}$$

Moreover, there exists a continuous and nondecreasing function $\mathbb{Q}_1 : [0,\infty) \to [0,+\infty)$, which only depends on F, m, K, ν, Ω, T, $\boldsymbol{u}_0$ and φ_0, such that

$$\begin{aligned}&\|\boldsymbol{u}\|_{L^\infty([0,T];V_{div})\cap L^2(0,T;H^2(\Omega)^2)} + \|\boldsymbol{u}_t\|_{L^2([0,T];G_{div})} + \|\varphi\|_{L^\infty([0,T];H^2(\Omega))}\\ &\quad + \|\varphi_t\|_{L^\infty([0,T];H)\cap L^2(0,T;V)} \le \mathbb{Q}_1\big(\|\boldsymbol{v}\|_{L^2(0,T;G_{div})}\big).\end{aligned} \tag{2.15}$$

3 Stability of the control-to-state mapping

We shall henceforth assume that the initial data $\boldsymbol{u}_0$, φ_0 satisfy the following assumptions:

(H6) $\boldsymbol{u}_0 \in V_{div}$, $\varphi_0 \in H^2(\Omega)$ satisfies (2.13), $F(\varphi_0) \in L^1(\Omega)$, $M(\varphi_0) \in L^1(\Omega)$.

Then we set

$$\begin{aligned}\mathcal{V} &:= L^2\left(0,T;G_{div}\right),\\ \mathcal{H} &:= \left[H^1(0,T;G_{div})\cap C^0([0,T];V_{div})\cap L^2(0,T;H^2(\Omega)^2)\right]\\ &\quad\times\left[C^1([0,T];H)\cap H^1(0,T;V)\cap L^\infty(0,T;H^2(\Omega))\right].\end{aligned} \tag{3.1}$$

On account of Theorem 2, the control-to-state mapping

$$\mathcal{S}:\mathcal{V}\to\mathcal{H},\qquad \boldsymbol{v}\in\mathcal{V}\mapsto\mathcal{S}(\boldsymbol{v}):=[\boldsymbol{u},\varphi]\in\mathcal{H}, \tag{3.2}$$

where $[\boldsymbol{u},\varphi]$ is the (unique) strong solution to (1.5)–(1.10) corresponding to the fixed initial data $\boldsymbol{u}_0$, φ_0 and to the control $\boldsymbol{v}\in\mathcal{V}$, is well defined and locally bounded. We now establish some global stability estimates for the strong solutions to the state system (1.5)–(1.10). In doing this, we can argue formally, since the arguments can be made rigorous within the approximation scheme devised in [14]. The first result is the following.

Lemma 2. *Let the assumptions* ***(V), (K), (H1)***, *and* ***(H2*)–(H6)*** *hold true, and suppose that* $K\in W^{2,1}_{loc}(\mathbb{R}^2)$ *or that* K *is admissible. Assume moreover that controls* $\boldsymbol{v}_i\in\mathcal{V}$, $i=1,2$, *are given and that* $[\boldsymbol{u}_i,\varphi_i]:=\mathcal{S}(\boldsymbol{v}_i)$, $i=1,2$, *are the associated solutions to the state system* (1.5)–(1.10). *Then there exists a continuous function* $\mathbb{Q}_2:[0,\infty)^2\to[0,\infty)$, *which is nondecreasing in both its arguments and depends only on the data* F, m, K, ν_1, Ω, T, $\boldsymbol{u}_0$ *and* φ_0, *such that we have the estimate*

$$\begin{aligned}&\|\boldsymbol{u}_2-\boldsymbol{u}_1\|^2_{C^0([0,t];G_{div})}+\|\boldsymbol{u}_2-\boldsymbol{u}_1\|^2_{L^2(0,t;V_{div})}+\|\varphi_2-\varphi_1\|^2_{C^0([0,t];H)}+\|\varphi_2-\varphi_1\|^2_{L^2(0,t;V)}\\ &\le\mathbb{Q}_2\big(\|\boldsymbol{v}_1\|_{L^2(0,T;G_{div})},\|\boldsymbol{v}_2\|_{L^2(0,T;G_{div})}\big)\,\|\boldsymbol{v}_2-\boldsymbol{v}_1\|^2_{L^2(0,T;V_{div}')}\quad\forall\,t\in(0,T].\end{aligned} \tag{3.3}$$

Proof. In this proof, we omit the explicit dependence on time for the sake of simplicity. Let us test the difference between (2.10), written for each of the two solutions, by $\varphi:=\varphi_2-\varphi_1$ in H. Taking (2.11) into account, we obtain the differential identity

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\varphi\|^2+(\boldsymbol{u}\cdot\nabla\varphi_2,\varphi)+\left(\nabla\left(B\left(\varphi_2\right)-B\left(\varphi_1\right)\right),\nabla\varphi\right)\\ &=\left((m(\varphi_2)-m(\varphi_1))\left(\nabla K*\varphi_2\right)+m(\varphi_1)\nabla K*\varphi,\nabla\varphi\right),\end{aligned} \tag{3.4}$$

where $\boldsymbol{u}=\boldsymbol{u}_1-\boldsymbol{u}_2$. Using **(H5)**, the mean value theorem, the Gagliardo–Nirenberg inequality (2.1), the boundedness of φ_2, the regularity result (2.14), and Young's inequality, we find that the third term on the left-hand side of (3.4) can be estimated as follows (cf. (2.5)):

$$\begin{aligned}&\left(\nabla\left(B\left(\varphi_2\right)-B\left(\varphi_1\right)\right),\nabla\varphi\right)=\left(\lambda(\varphi_2)\nabla\varphi+\left(\lambda(\varphi_2)-\lambda(\varphi_1)\right)\nabla\varphi_1,\nabla\varphi\right)\\ &\ge\alpha_0\|\nabla\varphi\|^2-k_1\|\varphi\|_{L^4(\Omega)}\|\nabla\varphi_1\|_{L^4(\Omega)^2}\|\nabla\varphi\|\\ &\ge\alpha_0\|\nabla\varphi\|^2-C\|\varphi_1\|_{H^2(\Omega)}\left(\|\varphi\|+\|\varphi\|^{1/2}\|\nabla\varphi\|^{1/2}\right)\|\nabla\varphi\|\\ &\ge\frac{\alpha_0}{2}\|\nabla\varphi\|^2-\mathbb{Q}\|\varphi\|^2,\end{aligned} \tag{3.5}$$

where $k_1:=\|\lambda'\|_{C([-1,1])}$. Here, and in the remainder of this proof, $\mathbb{Q}$ stands for a function having similar properties as the function $\mathbb{Q}_2$ in the statement of the theorem.

Concerning the right-hand side of (3.4), we have, setting

$$m_\infty := \max_{\varphi\in[-1,1]} |m(\varphi)| \quad \text{and} \quad m'_\infty := \max_{\varphi\in[-1,1]} |m'(\varphi)|, \tag{3.6}$$

and using the mean value theorem and Young's inequality,

$$\begin{aligned} &|((m(\varphi_2) - m(\varphi_1))\,(\nabla K * \varphi_2) + m(\varphi_1)\nabla K * \varphi, \nabla\varphi)| \\ &\le (m'_\infty + m_\infty)\ \|\nabla K\|_{L^1(\Omega)}\,\|\varphi\|\,\|\nabla\varphi\| \\ &\le \frac{\alpha_0}{4}\,\|\nabla\varphi\|^2 + C\,\|\varphi\|^2. \end{aligned} \tag{3.7}$$

Moreover, invoking (2.14), as well as Hölder's and Young's inequalities, we readily find that

$$|(\boldsymbol{u}\cdot\nabla\varphi_2, \varphi)| \ \le\ \|\boldsymbol{u}\|_{L^4(\Omega)^2}\,\|\nabla\varphi_2\|_{L^4(\Omega)^2}\,\|\varphi\| \ \le\ \frac{\nu_1}{8}\,\|\nabla\boldsymbol{u}\|^2 + \mathbb{Q}\,\|\varphi\|^2. \tag{3.8}$$

Hence, combining (3.4)–(3.8), we obtain that

$$\frac{1}{2}\frac{d}{dt}\,\|\varphi\|^2 + \frac{\alpha_0}{4}\,\|\nabla\varphi\|^2 \ \le\ \mathbb{Q}\,\|\varphi\|^2 + \frac{\nu_1}{8}\,\|\nabla\boldsymbol{u}\|^2 \quad \text{a.e. in } (0,T). \tag{3.9}$$

On the other hand, by testing the difference of (2.9), written for each of the two solutions, by $\boldsymbol{u}$ in G_{div}, and arguing as in the proof of [13, Thm. 7], the following differential inequality can be deduced:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\,\|\boldsymbol{u}\|^2 + \frac{\nu_1}{4}\,\|\nabla\boldsymbol{u}\|^2 \ \le\ \frac{\alpha_0}{8}\,\|\nabla\varphi\|^2 \\ &+ C\left(1 + \|\nabla\boldsymbol{u}_2\|^2\|\boldsymbol{u}_2\|^2_{H^2(\Omega)} + \|\varphi_1\|^2_{L^4(\Omega)} + \|\varphi_2\|^2_{L^4(\Omega)}\right)\|\varphi\|^2 \\ &+ C\,\|\nabla\boldsymbol{u}_1\|^2\|\boldsymbol{u}\|^2 + \frac{1}{\nu_1}\|\boldsymbol{v}\|^2_{V'_{div}} \quad \text{a.e. in } (0,T). \end{aligned} \tag{3.10}$$

Therefore, we get

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2 + \frac{\nu_1}{4}\|\nabla\boldsymbol{u}\|^2 \le \frac{\alpha_0}{8}\,\|\nabla\varphi\|^2 + \Lambda_1\left(\|\varphi\|^2 + \|\boldsymbol{u}\|^2\right) + \frac{1}{\nu_1}\|\boldsymbol{v}\|^2_{V'_{div}} \quad \text{a.e. in } (0,T), \tag{3.11}$$

where $\boldsymbol{v} := \boldsymbol{v}_2 - \boldsymbol{v}_1$ and

$$\Lambda_1 := C\left(1 + \mathbb{Q} + \|\nabla\boldsymbol{u}_2\|^2\|\boldsymbol{u}_2\|^2_{H^2(\Omega)} + \|\varphi_1\|^2_{L^4(\Omega)} + \|\varphi_2\|^2_{L^4(\Omega)}\right) \in L^1(0,T)\,.$$

By adding (3.9) to (3.11), and applying Gronwall's lemma to the resulting differential inequality, we finally obtain the asserted stability estimate (3.3). □

The following higher-order stability estimate for the solution component φ will be crucial for the proof of the Fréchet differentiability of the control-to-state mapping. In order to achieve this, we need to strengthen the hypotheses **(H1)** and **(H2*)** somewhat. More precisely, we postulate the following conditions:

(H1*) The mobility satisfies **(H1)** and also $m \in C^2\left([-1,1]\right)$.

(H2)** $F \in C^4(-1,1)$ and $\lambda := mF'' \in C^2\left([-1,1]\right)$.

Moreover, we need the following lemma to handle some boundary terms.

Lemma 3. *Let $\phi, \psi \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$. Then $\phi\psi \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$, and we have*

$$\|\phi\psi\|_{H^{1/2}(\partial\Omega)} \le \|\phi\|_{L^\infty(\partial\Omega)}\|\psi\|_{H^{1/2}(\partial\Omega)} + \|\psi\|_{L^\infty(\partial\Omega)}\|\phi\|_{H^{1/2}(\partial\Omega)}.$$

Proof. The proof is an immediate consequence of the definition of the space $H^{1/2}(\partial\Omega)$ with seminorm given by

$$|\phi|^2_{H^{1/2}(\partial\Omega)} = \int_{\partial\Omega}\int_{\partial\Omega} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^2} d\Gamma(x)d\Gamma(y), \tag{3.12}$$

where $d\Gamma(\cdot)$ is the surface measure on $\partial\Omega$ (see, e.g., [11, Chapter IX, Section 18]). □

We have the following stability result.

Lemma 4. *Let the assumptions **(V)**, **(K)**, **(H1*)**, **(H2**)**, **(H3)**-**(H6)** hold true, and suppose that $K \in W^{2,1}_{loc}(\mathbb{R}^2)$ or that K is admissible. Then there exists a continuous function $\mathbb{Q}_3 : [0,\infty)^2 \to [0,\infty)$, which is nondecreasing in both its arguments and depends only on the data F, m, K, ν_1, Ω, T, $\boldsymbol{u}_0$ and φ_0, such that we have for every $t \in (0,T]$ the estimate*

$$\begin{aligned}&\|\boldsymbol{u}_2-\boldsymbol{u}_1\|^2_{L^\infty(0,t;G_{div})} + \|\boldsymbol{u}_2-\boldsymbol{u}_1\|^2_{L^2(0,t;V_{div})} + \|\varphi_2-\varphi_1\|^2_{L^\infty(0,t;V)} + \|\varphi_2-\varphi_1\|^2_{L^2(0,t;H^2(\Omega))}\\ &+\|\varphi_2-\varphi_1\|^2_{H^1(0,t;H)} \le \mathbb{Q}_3\big(\|\boldsymbol{v}_1\|_{L^2(0,T;G_{div})}, \|\boldsymbol{v}_2\|_{L^2(0,T;G_{div})}\big)\,\|\boldsymbol{v}_2-\boldsymbol{v}_1\|^2_{L^2(0,T;V'_{div})}.\end{aligned} \tag{3.13}$$

Proof. In the following, the explicit dependence on time is omitted for simplicity. Let us take the difference between (2.10) written for each of the two solutions, and test the resulting equation by $(B(\varphi_2)-B(\varphi_1))_t$ in H. As in the proof of the previous lemma, we set $\boldsymbol{u} := \boldsymbol{u}_2-\boldsymbol{u}_1$ and $\varphi := \varphi_2-\varphi_1$. On account of (2.5), we obtain almost everywhere in $(0,T)$ the identity

$$\begin{aligned}&\frac{1}{2}\frac{d\Psi}{dt} + \big(\lambda(\varphi_1)\varphi_t, \varphi_t\big) = -\big(\,(\lambda(\varphi_2)-\lambda(\varphi_1))\,\varphi_{2,t}, \varphi_t\big)\\
&-\big(\boldsymbol{u}\cdot\nabla\varphi_2, (\lambda(\varphi_2)-\lambda(\varphi_1))\,\varphi_{2,t}\big) - \big(\boldsymbol{u}\cdot\nabla\varphi_2, \lambda(\varphi_1)\varphi_t\big)\\
&-\big(\boldsymbol{u}_1\cdot\nabla\varphi, (\lambda(\varphi_2)-\lambda(\varphi_1))\,\varphi_{2,t}\big) - \big(\boldsymbol{u}_1\cdot\nabla\varphi, \lambda(\varphi_1)\varphi_t\big)\\
&-\big(\,(m'(\varphi_2)-m'(\varphi_1))\,\varphi_{2,t}\,(\nabla K*\varphi_2), \nabla(B(\varphi_2)-B(\varphi_1))\,\big)\\
&-\big(m'(\varphi_1)\varphi_t\,(\nabla K*\varphi_2), \nabla(B(\varphi_2)-B(\varphi_1))\,\big)\\
&-\big(\,(m(\varphi_2)-m(\varphi_1))\,(\nabla K*\varphi_{2,t}), \nabla(B(\varphi_2)-B(\varphi_1))\,\big)\\
&-\big(m'(\varphi_1)\varphi_{1,t}\,(\nabla K*\varphi), \nabla(B(\varphi_2)-B(\varphi_1))\,\big)\\
&-\big(m(\varphi_1)\,(\nabla K*\varphi_t), \nabla(B(\varphi_2)-B(\varphi_1))\,\big) = \sum_{j=1}^{10} I_j^{(1)},\end{aligned} \tag{3.14}$$

where the quantities $I_j^{(1)}$, $1 \le j \le 10$, have obvious meaning and the functional Ψ is defined by

$$\begin{aligned}\Psi :=\ &\|\nabla(B(\varphi_2)-B(\varphi_1))\|^2 - 2\big(\,(m(\varphi_2)-m(\varphi_1))\,(\nabla K*\varphi_2), \nabla(B(\varphi_2)-B(\varphi_1))\,\big)\\ &-2\big(m(\varphi_1)\,(\nabla K*\varphi), \nabla(B(\varphi_2)-B(\varphi_1))\,\big).\end{aligned} \tag{3.15}$$

We now estimate individually all of the terms on the right-hand side of (3.14). To this end, we note that the mean value theorem yields that

$$|\lambda(\varphi_2)-\lambda(\varphi_1)| + \max_{0\le k\le 1} |m^{(k)}(\varphi_2)-m^{(k)}(\varphi_1)| \le C_0\,|\varphi| \quad \text{a.e. in } Q,$$

with some global constant C_0. Moreover, we recall the continuity of the embedding $V \subset L^4(\Omega)$, the Gagliardo–Nirenberg inequality (2.1), and the regularity properties stated in Theorem 2. Using Hölder's and Young's inequalities, we obtain, for every $\epsilon > 0$ and $\epsilon' > 0$ (which will be specified later), the following chain of estimates:

$$I_1^{(1)} \le C_0\,\|\varphi\|_{L^4(\Omega)}\,\|\varphi_{2,t}\|_{L^4(\Omega)}\,\|\varphi_t\| \le \epsilon\,\|\varphi_t\|^2 + C_\epsilon\,\|\varphi_{2,t}\|_V^2\,\|\varphi\|_V^2\,, \tag{3.16}$$

$$\begin{aligned} I_2^{(1)} &\le C_0\,\|\boldsymbol{u}\|_{L^4(\Omega)^2}\,\|\nabla\varphi_2\|_{L^4(\Omega)^2}\,\|\varphi\|_{L^4(\Omega)}\,\|\varphi_{2,t}\|_{L^4(\Omega)} \\ &\le \epsilon'\,\|\nabla\boldsymbol{u}\|^2 + C_{\epsilon'}\,\|\varphi_{2,t}\|_V^2\,\|\varphi\|_V^2\,, \end{aligned} \tag{3.17}$$

$$\begin{aligned} I_3^{(1)} &\le C\,\|\boldsymbol{u}\|_{L^4(\Omega)^2}\,\|\nabla\varphi_2\|_{L^4(\Omega)^2}\,\|\varphi_t\| \le \mathbb{Q}\,\|\boldsymbol{u}\|^{1/2}\,\|\nabla\boldsymbol{u}\|^{1/2}\,\|\varphi_t\| \\ &\le \epsilon\,\|\varphi_t\|^2 + \epsilon'\,\|\nabla\boldsymbol{u}\|^2 + \mathbb{Q}\,\|\boldsymbol{u}\|^2\,, \end{aligned} \tag{3.18}$$

$$I_4^{(1)} \le C_0\,\|\boldsymbol{u}_1\|_{L^\infty(\Omega)^2}\,\|\nabla\varphi\|\,\|\varphi\|_{L^4(\Omega)}\,\|\varphi_{2,t}\|_{L^4(\Omega)} \le C\,\|\boldsymbol{u}_1\|_{H^2(\Omega)^2}\,\|\varphi_{2,t}\|_V\,\|\varphi\|_V^2\,, \tag{3.19}$$

$$I_5^{(1)} \le C\,\|\boldsymbol{u}_1\|_{L^\infty(\Omega)^2}\,\|\nabla\varphi\|\,\|\varphi_t\| \le \epsilon\,\|\varphi_t\|^2 + C_\epsilon\,\|\boldsymbol{u}_1\|_{H^2(\Omega)^2}^2\,\|\varphi\|_V^2\,, \tag{3.20}$$

$$\begin{aligned} I_6^{(1)} &\le \|m'(\varphi_2)-m'(\varphi_1)\|_{L^4(\Omega)}\,\|\varphi_{2,t}\|_{L^4(\Omega)}\,\|\nabla K*\varphi_2\|_{L^\infty(\Omega)^2}\,\|\nabla\left(B(\varphi_2)-B(\varphi_1)\right)\| \\ &\le C\,\|\varphi_{2,t}\|_V\,\|\varphi\|_V\,\|\nabla\left(B(\varphi_2)-B(\varphi_1)\right)\|\,, \end{aligned} \tag{3.21}$$

$$\begin{aligned} I_7^{(1)} + I_{10}^{(1)} &\le C\,\|\varphi_t\|\,\|\nabla\left(B(\varphi_2)-B(\varphi_1)\right)\| \\ &\le \epsilon\,\|\varphi_t\|^2 + C_\epsilon\,\|\nabla\left(B(\varphi_2)-B(\varphi_1)\right)\|^2\,, \end{aligned} \tag{3.22}$$

$$\begin{aligned} I_8^{(1)} &\le \|m(\varphi_2)-m(\varphi_1)\|_{L^4(\Omega)}\,\|\nabla K*\varphi_{2,t}\|_{L^4(\Omega)^2}\,\|\nabla\left(B(\varphi_2)-B(\varphi_1)\right)\| \\ &\le C\,\|\varphi_{2,t}\|_V\,\|\varphi\|_V\,\|\nabla\left(B(\varphi_2)-B(\varphi_1)\right)\|\,, \end{aligned} \tag{3.23}$$

$$\begin{aligned} I_9^{(1)} &\le C\,\|\varphi_{1,t}\|_{L^4(\Omega)}\,\|\nabla K*\varphi\|_{L^4(\Omega)^2}\,\|\nabla\left(B(\varphi_2)-B(\varphi_1)\right)\| \\ &\le C\,\|\varphi_{1,t}\|_V\,\|\varphi\|_V\,\|\nabla\left(B(\varphi_2)-B(\varphi_1)\right)\|\,. \end{aligned} \tag{3.24}$$

Here, and in the following, $\mathbb{Q}$ stands for a function having similar properties as the function $\mathbb{Q}_3$ from the statement of the theorem. Inserting the estimates (3.16)–(3.24) in (3.14), and choosing $\epsilon > 0$ small enough, we obtain that almost everywhere in $(0,T)$ it holds

$$\frac{d\Psi}{dt} + \alpha_0\,\|\varphi_t\|^2 \le 4\,\epsilon'\,\|\nabla\boldsymbol{u}\|^2 + \Lambda_2\left(\|\varphi\|_V^2 + \|\nabla\left(B(\varphi_2)-B(\varphi_1)\right)\|^2\right) + \mathbb{Q}\,\|\boldsymbol{u}\|^2, \tag{3.25}$$

where

$$\Lambda_2 := C\left(1 + \|\boldsymbol{u}_1\|_{H^2(\Omega)^2}^2 + \|\varphi_{1,t}\|_V^2 + \|\varphi_{2,t}\|_V^2\right) \in L^1(0,T). \tag{3.26}$$

We now aim to control the $L^2(\Omega)$ norm of $\nabla\left(B(\varphi_2)-B(\varphi_1)\right)$ by the $H^1(\Omega)$ norm of φ (from above and below). Now observe that

$$\nabla\left(B(\varphi_2)-B(\varphi_1)\right) = \left(\lambda(\varphi_2)-\lambda(\varphi_1)\right)\nabla\varphi_2 + \lambda(\varphi_1)\nabla\varphi\,.$$

Hence, we deduce that

$$\begin{aligned}\|\nabla\left(B(\varphi_2)-B(\varphi_1)\right)\|^2 &\geq \alpha_0^2\,\|\nabla\varphi\|^2-2\,\|\lambda\|_{C^0([-1,1])}\,\|\lambda(\varphi_2)-\lambda(\varphi_1)\|_{L^4(\Omega)}\,\|\nabla\varphi_2\|_{L^4(\Omega)^2}\,\|\nabla\varphi\| \\ &\geq \alpha_0^2\,\|\nabla\varphi\|^2-2\,C\,C_0\,\|\varphi\|_{L^4(\Omega)}\,\|\nabla\varphi_2\|_{L^4(\Omega)^2}\,\|\nabla\varphi\| \\ &\geq \alpha_0^2\,\|\nabla\varphi\|^2-\mathbb{Q}\left(\|\varphi\|+\|\varphi\|^{1/2}\,\|\nabla\varphi\|^{1/2}\right)\|\nabla\varphi\| \\ &\geq \frac{1}{2}\,\alpha_0^2\,\|\nabla\varphi\|^2-\mathbb{Q}\,\|\varphi\|^2\,. \end{aligned} \tag{3.27}$$

On the other hand, it is immediately seen that we also have

$$\|\nabla\left(B(\varphi_2)-B(\varphi_1)\right)\|^2 \,\leq\, C\,\|\varphi\|_V^2\,. \tag{3.28}$$

Thanks to (3.27), (3.28), and to the definition (3.15), we then easily find that

$$\frac{\alpha_0^2}{4}\,\|\nabla\varphi\|^2-\mathbb{Q}\,\|\varphi\|^2 \,\leq\, \Psi \,\leq\, C\,\|\varphi\|_V^2. \tag{3.29}$$

Adding (3.11) and (3.25), choosing ϵ' small enough, and employing the bound (3.29), we are thus led to the differential inequality (cf. also (2.15))

$$\begin{aligned}&\frac{d}{dt}\Big(\Psi+\frac{1}{2}\|\boldsymbol{u}\|^2\Big)+\frac{\nu_1}{8}\|\nabla\boldsymbol{u}\|^2+\alpha_0\,\|\varphi_t\|^2 \\ &\leq \Lambda_2\left(\Psi+\frac{1}{2}\|\boldsymbol{u}\|^2\right)+(\Lambda_2+\mathbb{Q})\|\varphi\|^2+\frac{1}{\nu_1}\|\boldsymbol{v}\|_{V_{div}'}^2,\end{aligned}$$

where $\boldsymbol{v}:=\boldsymbol{v}_2-\boldsymbol{v}_1$. Hence, Gronwall's lemma, (3.3), and (3.29) yield the stability estimate (cf. also (2.15))

$$\|\boldsymbol{u}\|^2_{L^\infty(0,t;G_{div})}+\|\boldsymbol{u}\|^2_{L^2(0,t;V_{div})}+\|\varphi\|^2_{L^\infty(0,t;V)}+\|\varphi_t\|^2_{L^2(0,t;H)} \,\leq\, \mathbb{Q}\,\|\boldsymbol{v}\|^2_{L^2(0,T;V_{div}')}\,. \tag{3.30}$$

We now aim to control the $L^2(0,t;H^2(\Omega))$ norm of φ in terms of the $L^2(0,t;H)$ norm of φ_t. This will be achieved in three steps.

Step 1. Control of $\|\Delta\big(B\left(\varphi_2\right)-B\left(\varphi_1\right)\big)\|_{L^2(0,t;H)}$ *in terms of* $\|\varphi_t\|_{L^2(0,t;H)}$.

We write (2.10) for both solutions and take the difference of the equations. We then get the identity

$$\begin{aligned}\Delta\big(B\left(\varphi_2\right)-B\left(\varphi_1\right)\big) \;=\;& \varphi_t+\boldsymbol{u}\cdot\nabla\varphi_2+\boldsymbol{u}_1\cdot\nabla\varphi+\big(m(\varphi_2)-m(\varphi_1)\big)\,\mathrm{div}(\nabla K*\varphi_2) \\ &+\Big(\left(m'(\varphi_2)-m'(\varphi_1)\right)\nabla\varphi_2+m'(\varphi_1)\nabla\varphi\Big)\cdot(\nabla K*\varphi_2) \\ &+m(\varphi_1)\,\mathrm{div}(\nabla K*\varphi)+m'(\varphi_1)\nabla\varphi_1\cdot(\nabla K*\varphi)\,. \end{aligned}\tag{3.31}$$

It is easy to see that the $L^2(\Omega)$ norms of the fourth to last terms on the right-hand side of (3.31) can, on account of Lemma 1 and of the bound $(2.14)_1$ for φ_1,φ_2, be estimated by $C\,\|\varphi\|_V$. By virtue of Poincaré's inequality, we therefore get that

$$\begin{aligned}\|\Delta\big(B\left(\varphi_2\right)-B\left(\varphi_1\right)\big)\| &\leq \|\varphi_t\|+\|\boldsymbol{u}\|_{L^4(\Omega)^2}\|\nabla\varphi_2\|_{L^4(\Omega)^2}+\|\boldsymbol{u}_1\|_{L^4(\Omega)^2}\|\nabla\varphi\|_{L^4(\Omega)^2}+C\|\varphi\|_V \\ &\leq \|\varphi_t\|+C\|\nabla\boldsymbol{u}\|+C\|\nabla\varphi\|^{1/2}\|\varphi\|^{1/2}_{H^2(\Omega)}+C\|\varphi\|_V\end{aligned}$$

$$\leq \|\varphi_t\| + C\|\nabla \boldsymbol{u}\| + \delta\|\varphi\|_{H^2(\Omega)} + C_\delta\|\varphi\|_V, \tag{3.32}$$

for every $\delta > 0$ (to be fixed later).

Step 2. Control of $\|B(\varphi_2) - B(\varphi_1)\|_{L^2(0,t;H^2(\Omega))}$ *in terms of* $\|\Delta\big(B(\varphi_2) - B(\varphi_1)\big)\|_{L^2(0,t;H)}$.
We need to estimate the trace of the normal derivative of $B(\varphi_2) - B(\varphi_1)$ in $H^{1/2}(\partial\Omega)$. For this purpose, we write (2.12)$_2$ for each solution and then take the difference. From the resulting equation, we get that

$$\frac{\partial}{\partial \boldsymbol{n}}\big(B(\varphi_2) - B(\varphi_1)\big) = (m(\varphi_2) - m(\varphi_1))(\nabla K * \varphi_2)\cdot \boldsymbol{n} + m(\varphi_1)(\nabla K * \varphi)\cdot \boldsymbol{n} \quad \text{a.e. on } \Sigma\,.$$

By applying Lemma 3, we then obtain the estimate

$$\begin{aligned}\Big\|\frac{\partial}{\partial \boldsymbol{n}}\big(B(\varphi_2) - B(\varphi_1)\big)\Big\|_{H^{1/2}(\partial\Omega)} &\leq \|m(\varphi_2) - m(\varphi_1)\|_{L^\infty(\partial\Omega)}\,\|(\nabla K * \varphi_2)\cdot \boldsymbol{n}\|_{H^{1/2}(\partial\Omega)}\\ &+ \|(\nabla K * \varphi_2)\cdot \boldsymbol{n}\|_{L^\infty(\partial\Omega)}\,\|m(\varphi_2) - m(\varphi_1)\|_{H^{1/2}(\partial\Omega)}\\ &+ \|m(\varphi_1)\|_{L^\infty(\partial\Omega)}\,\|(\nabla K * \varphi)\cdot \boldsymbol{n}\|_{H^{1/2}(\partial\Omega)} + \|(\nabla K * \varphi)\cdot \boldsymbol{n}\|_{L^\infty(\partial\Omega)}\,\|m(\varphi_1)\|_{H^{1/2}(\partial\Omega)}\\ &=: \sum_{j=1}^{4} I_j^{(2)}\,,\end{aligned} \tag{3.33}$$

with obvious meaning of $I_j^{(2)}$, $1 \leq j \leq 4$. We now proceed to estimate the four terms on the right-hand side individually. To this end, we employ Lemma 1, Agmon's inequality (2.2), and the classical trace theorem, where C_{tr} denotes the constant of the continuous embedding $H^1(\Omega) \subset H^{1/2}(\partial\Omega)$. We also utilize the fact that if $\psi \in H^1(\Omega)$ and $|\psi| \leq \zeta$ almost everywhere in Ω for some positive constant ζ (with Ω smooth enough), then the trace $\gamma_0\psi := \psi|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$ of ψ on the boundary $\partial\Omega$ satisfies $|\gamma_0\psi| \leq \zeta$ a.e. on $\partial\Omega$, and, moreover, if $g \in C^1(\mathbb{R})$, then $g(\psi) \in H^1(\Omega)$ and $\gamma_0 g(\psi) = g(\gamma_0\psi)$. With these tools at hand, we deduce, for every $\delta > 0$ (to be fixed later), the chain of estimates

$$\begin{aligned}I_1^{(2)} &\leq m'_\infty\,\|\varphi\|_{L^\infty(\Omega)}\,\|K * \varphi_2\|_{H^2(\Omega)} \leq C_{m,K,\Omega}\,\|\varphi\|^{1/2}\,\|\varphi\|^{1/2}_{H^2(\Omega)}\\ &\leq \delta\,\|\varphi\|_{H^2(\Omega)} + C_{\delta,m,K,\Omega}\,\|\varphi\|\,,\end{aligned} \tag{3.34}$$

$$I_2^{(2)} \leq \|(\nabla K * \varphi_2)\cdot \boldsymbol{n}\|_{L^\infty(\partial\Omega)}\,C_{tr}\,\|m(\varphi_2) - m(\varphi_1)\|_V \leq C_{m,K,\Omega}\,\|\varphi\|_V\,, \tag{3.35}$$

$$I_3^{(2)} \leq m_\infty\,\|K * \varphi\|_{H^2(\Omega)} \leq C_{m,K,\Omega}\,\|\varphi\|, \tag{3.36}$$

$$\begin{aligned}I_4^{(2)} &\leq \|\nabla K * \varphi\|_{L^\infty(\Omega)^2}\,C_{tr}\,\|m(\varphi_1)\|_V \leq C_{m,K,\Omega}\,\|\varphi\|_{L^\infty(\Omega)}\\ &\leq \delta\,\|\varphi\|_{H^2(\Omega)} + C_{\delta,m,K,\Omega}\,\|\varphi\|\,.\end{aligned} \tag{3.37}$$

Inserting the estimates (3.34)–(3.37) in (3.33), and invoking (3.28), we deduce that

$$\|B(\varphi_2) - B(\varphi_1)\|_{H^2(\Omega)} \leq C\,\|\Delta(B(\varphi_2) - B(\varphi_1))\| + C\,\delta\,\|\varphi\|_{H^2(\Omega)} + C_\delta\,\|\varphi\|_V \quad \text{a.e. in } (0,T). \tag{3.38}$$

Step 3. Control of $\|\varphi\|_{L^2(0,t;H^2(\Omega))}$ *in terms of* $\|B(\varphi_2) - B(\varphi_1)\|_{L^2(0,t;H^2(\Omega))}$.

We write the identity (cf. (2.5)) $\partial_j\varphi = \lambda^{-1}\partial_j B(\varphi)$, $j = 1, 2$, for the two solutions and take the difference. For the second spatial derivatives $\partial^2_{ij}\varphi$, we get

$$\partial^2_{ij}\varphi = \frac{1}{\lambda(\varphi_1)}\partial^2_{ij}\big(B(\varphi_2) - B(\varphi_1)\big) + \Big(\frac{1}{\lambda(\varphi_2)} - \frac{1}{\lambda(\varphi_1)}\Big)\partial^2_{ij}B(\varphi_2)$$

$$- \Big(\frac{1}{\lambda^2(\varphi_2)} - \frac{1}{\lambda^2(\varphi_1)} \Big) \partial_i \lambda(\varphi_2) \partial_j B(\varphi_2) - \frac{1}{\lambda^2(\varphi_1)} \big(\partial_i \lambda(\varphi_2) - \partial_i \lambda(\varphi_1) \big) \partial_j B(\varphi_2)$$
$$- \frac{1}{\lambda^2(\varphi_1)} \partial_i \lambda(\varphi_1) \big(\partial_j B(\varphi_2) - \partial_j B(\varphi_1) \big) - \frac{1}{\lambda(\varphi_1)} \big(m(\varphi_2) - m(\varphi_1) \big) \partial_i \varphi_2 \,. \tag{3.39}$$

Let us denote by $I_j^{(3)}$, $j = 1, \ldots, 6$, the $L^2(\Omega)$ norms of the six terms on the right-hand side of the above identity. Now observe that (2.14) implies that $\partial_i \varphi_1, \partial_i \varphi_2 \in L^\infty(0,T; L^p(\Omega))$ for $i = 1,2$ and all $p \in [1, +\infty)$. We can therefore infer from $(\mathbf{H2}^{**})$, (2.1), and Young's inequality the estimate

$$\begin{aligned} \|\partial_i \lambda(\varphi_2) - \partial_i \lambda(\varphi_1)\|_{L^4(\Omega)} &\le C\|\varphi\|_V + C\|\nabla \varphi\|_{L^4(\Omega)^2} \\ &\le C\|\varphi\|_V + C\|\nabla\varphi\|^{1/2} \|\varphi\|_{H^2(\Omega)}^{1/2} \\ &\le \eta \|\varphi\|_{H^2(\Omega)} + C_\eta \|\varphi\|_V, \end{aligned} \tag{3.40}$$

for any $\eta > 0$ (to be chosen later). The terms $I_k^{(3)}$ can be estimated in the following way:

$$I_1^{(3)} \;\le\; \frac{1}{\alpha_0} \, \|\partial_{ij}^2 \big(B(\varphi_2) - B(\varphi_1) \big) \| \,, \tag{3.41}$$

$$\begin{aligned} I_2^{(3)} \;&\le\; C \, \|\varphi\|_{L^\infty(\Omega)} \, \|\partial_{ij} B(\varphi_2)\| \;\le\; C \, \|\varphi\|^{1/2} \, \|\varphi\|_{H^2(\Omega)}^{1/2} \\ &\le\; \eta \, \|\varphi\|_{H^2(\Omega)} + C_\eta \, \|\varphi\| \,, \end{aligned} \tag{3.42}$$

$$\begin{aligned} I_3^{(3)} \;&\le\; C \, \|\varphi\|_{L^\infty(\Omega)} \, \|\partial_i \lambda(\varphi_2)\|_{L^4(\Omega)} \, \|\partial_j B(\varphi_2)\|_{L^4(\Omega)} \\ &\le C \, \|\varphi\|^{1/2} \, \|\varphi\|_{H^2(\Omega)}^{1/2} \;\le\; \eta \, \|\varphi\|_{H^2(\Omega)} + C_\eta \|\varphi\| \,, \end{aligned} \tag{3.43}$$

$$\begin{aligned} I_4^{(3)} \;&\le\; C \, \|\partial_i \lambda(\varphi_2) - \partial_i \lambda(\varphi_1)\|_{L^4(\Omega)} \, \|\partial_j B(\varphi_2)\|_{L^4(\Omega)} \\ &\le\; C \, \eta \, \|\varphi\|_{H^2(\Omega)} + C_\eta \, \|\varphi\|_V \,, \end{aligned} \tag{3.44}$$

$$\begin{aligned} I_5^{(3)} \;&\le\; C \, \|\partial_i \lambda(\varphi_1)\|_{L^4(\Omega)} \, \|\partial_j B(\varphi_2) - \partial_j B(\varphi_1)\|_{L^4(\Omega)} \\ &\le\; C \, \|B(\varphi_2) - B(\varphi_1)\|_{H^2(\Omega)} \,, \end{aligned} \tag{3.45}$$

$$I_6^{(3)} \;\le\; C \, \|\varphi\|_V \,. \tag{3.46}$$

Here, we have used Agmon's inequality (2.2), as well as the fact that $B(\varphi_2) \in L^\infty(0,T; H^2(\Omega))$ and $\lambda(\varphi_j) \in L^\infty(0,T; W^{1,p}(\Omega))$, for all $1 \le p < +\infty$, $j = 1,2$. By means of the estimates (3.41)–(3.46), and taking $\eta > 0$ small enough, we deduce from (3.39) that

$$\|\varphi\|_{H^2(\Omega)} \;\le\; C \, \|B(\cdot, \varphi_2) - B(\cdot, \varphi_1)\|_{H^2(\Omega)} + C \, \|\varphi\| \,. \tag{3.47}$$

Now, combining the estimates (3.32), (3.38), (3.47) obtained in the three steps above, and fixing $\delta > 0$ small enough, we finally deduce the desired control

$$\|\varphi\|_{H^2(\Omega)} \;\le\; C \, \|\varphi_t\| + C \, \|\nabla \boldsymbol{u}\| + C \, \|\varphi\|_V \,. \tag{3.48}$$

The stability estimate (3.13) now immediately follows from (3.30) and (3.48). This concludes the proof of the lemma. □

4 Optimal control

We now study the optimal control problem **(CP)**. Throughout this section, we assume that the cost functional $\mathcal{J}$ is given by (1.12). Moreover, we assume that the set of admissible controls $\mathcal{V}_{ad}$ is defined by

$$\mathcal{V}_{ad} := \big\{ \boldsymbol{v} \in L^2(0,T;G_{div}) : \;\; v_{a,i}(x,t) \le v_i(x,t) \le v_{b,i}(x,t), \;\text{ a.e. } (x,t) \in Q, \;\; i = 1,2 \big\}, \tag{4.1}$$

with given functions $\boldsymbol{v}_a, \boldsymbol{v}_b \in L^2(0,T;G_{div}) \cap L^\infty(Q)^2$. Notice that the stability estimate provided by Lemma 4 yields that the control-to-state map $\mathcal{S}$ introduced above (cf. (3.1), (3.2)) is locally Lipschitz continuous from $\mathcal{V}$ into the space

$$\mathcal{W} := \big[C^0([0,T];G_{div}) \cap L^2(0,T;V_{div}) \big] \times \big[H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;H^2(\Omega)) \big]. \tag{4.2}$$

We also point out that problem **(CP)** can be formulated in the form

$$\min_{\boldsymbol{v} \in \mathcal{V}_{ad}} f(\boldsymbol{v}),$$

for the reduced cost functional defined by $f(\boldsymbol{v}) := \mathcal{J}\big(\mathcal{S}(\boldsymbol{v}), \boldsymbol{v}\big)$, for every $\boldsymbol{v} \in \mathcal{V}$.

Let us first prove that an optimal control exists.

Theorem 3. *Let the assumptions of Lemma 4 hold true. Then the optimal control problem* **(CP)** *on* $\mathcal{V}_{ad}$ *admits a solution.*

Proof. In the first part of the proof, we can argue as in [19, Proof of Theorem 2]. We pick a minimizing sequence $\{\boldsymbol{v}_n\} \subset \mathcal{V}_{ad}$ for **(CP)**, and since $\mathcal{V}_{ad}$ is bounded in $\mathcal{V}$, we may assume without loss of generality that $\boldsymbol{v}_n \to \overline{\boldsymbol{v}}$ weakly in $L^2(0,T;G_{div})$ for some $\overline{\boldsymbol{v}} \in \mathcal{V}$. Since $\mathcal{V}_{ad}$ is convex and closed in $\mathcal{V}$, and thus weakly sequentially closed, we have that $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$.

Moreover, $\mathcal{S}$ is a locally bounded mapping from $\mathcal{V}$ into $\mathcal{H}$. Hence, setting $[\boldsymbol{u}_n, \varphi_n] := \mathcal{S}(\boldsymbol{v}_n)$, $n \in \mathbb{N}$, we may without loss of generality assume that, with appropriate limit points $[\overline{\boldsymbol{u}}, \overline{\varphi}]$,

$$\boldsymbol{u}_n \to \overline{\boldsymbol{u}} \quad \text{weakly* in } L^\infty(0,T;V_{div}), \text{ weakly in } H^1(0,T;G_{div}) \cap L^2(0,T;H^2(\Omega)^2), \tag{4.3}$$

$$\varphi_n \to \overline{\varphi} \quad \text{weakly* in } L^\infty(0,T;H^2(\Omega)) \cap W^{1,\infty}(0,T;H), \text{ weakly in } H^1(0,T;V). \tag{4.4}$$

In particular, it follows from the compactness of the embedding $H^1(0,T;V) \cap L^\infty(0,T;H^2(\Omega)) \subset C^0([0,T];H^r(\Omega))$ for $0 \le r < 2$, given by the Aubin-Lions lemma (cf. [32]), that $\varphi_n \to \overline{\varphi}$ strongly in $C^0(\overline{Q})$. Hence, we have $\nu(\varphi_n) \to \nu(\overline{\varphi})$ strongly in $C^0(\overline{Q})$. Moreover, we also have, by compact embedding, that $\boldsymbol{u}_n \to \overline{\boldsymbol{u}}$ strongly in $L^2(0,T;G_{div})$. By employing these weak and strong convergence properties, we can now pass to the limit in the weak formulation of the state system (1.5)–(1.10) (cf. Definition 1) to see that $[\overline{\boldsymbol{u}}, \overline{\varphi}]$ satisfies the weak formulation corresponding to $\overline{\boldsymbol{v}}$. Notice that, instead of passing to the limit in the weak formulation of the nonlocal Cahn–Hilliard equation (1.6) given in Definition 1, we can alternatively pass to the limit in the weak formulation of (2.10), which reads

$$\langle \varphi_t, \psi \rangle_V + (\nabla B(\cdot,\varphi), \nabla\psi) - (m(\varphi)\,(\nabla K * \varphi)\,, \nabla\psi) = (\boldsymbol{u}\varphi, \nabla\psi) \tag{4.5}$$

for every $\psi \in V$ and almost any $t \in (0,,T)$. Hence, we have $[\overline{\boldsymbol{u}}, \overline{\varphi}] = \mathcal{S}(\overline{\boldsymbol{v}})$, that is, the pair $([\overline{\boldsymbol{u}}, \overline{\varphi}], \overline{\boldsymbol{v}})$ is admissible for **(CP)**. Finally, thanks to the weak sequential lower semicontinuity of $\mathcal{J}$ and to the weak convergences (4.3), (4.4), we infer that the state $[\overline{\boldsymbol{u}}, \overline{\varphi}] = \mathcal{S}(\overline{\boldsymbol{v}})$ is a solution to **(CP)**. $\square$

The linearized system. Assume that the assumptions of Lemma 4 are fulfilled. We fix a control $\overline{\boldsymbol{v}} \in \mathcal{V}$ and let $[\overline{\boldsymbol{u}}, \overline{\varphi}] := \mathcal{S}(\overline{\boldsymbol{v}}) \in \mathcal{H}$ be the associated unique strong solution to the state system (1.5)–(1.10) according to Theorem 2. Let an arbitrary $\boldsymbol{h} \in \mathcal{V}$ be given. In order to prove Fréchet differentiability of the control-to-state operator at $\overline{\boldsymbol{v}}$, we first consider the following system, which is obtained by linearizing the state system (1.5)–(1.10) at $[\overline{\boldsymbol{u}}, \overline{\varphi}]$:

$$\begin{aligned}
&\boldsymbol{\xi}_t - 2\,\mathrm{div}\big(\nu(\overline{\varphi})D\boldsymbol{\xi}\big) - 2\,\mathrm{div}\big(\nu'(\overline{\varphi})\,\eta\,D\overline{\boldsymbol{u}}\big) + (\overline{\boldsymbol{u}}\cdot\nabla)\,\boldsymbol{\xi} + (\boldsymbol{\xi}\cdot\nabla)\,\overline{\boldsymbol{u}} + \nabla\pi^* \\
&\quad = \eta\,(\nabla K * \overline{\varphi}) + \overline{\varphi}\,(\nabla K * \eta) + \boldsymbol{h} \quad \text{in } Q, \qquad (4.6)\\
&\eta_t + \overline{\boldsymbol{u}}\cdot\nabla\eta = -\boldsymbol{\xi}\cdot\nabla\overline{\varphi} + \mathrm{div}\big(\lambda(\overline{\varphi})\nabla\eta\big) - \mathrm{div}\big(m'(\overline{\varphi})\eta\nabla\,(K*\overline{\varphi})\,\big) \\
&\quad - \mathrm{div}\big(m(\overline{\varphi})\,(\nabla K * \eta)\,\big) + \mathrm{div}\big(\eta\lambda'(\overline{\varphi})\nabla\overline{\varphi}\big) \quad \text{in } Q, \qquad (4.7)\\
&\mathrm{div}(\boldsymbol{\xi}) = 0, \quad \text{in } Q, \qquad (4.8)\\
&\boldsymbol{\xi} = \boldsymbol{0} \quad \text{on } \Sigma, \qquad (4.9)\\
&\big[\lambda(\overline{\varphi})\nabla\eta - m'(\overline{\varphi})\eta\nabla\big(K*\overline{\varphi}\big) - m(\overline{\varphi})\big(\nabla K * \eta\big) + \eta\lambda'(\overline{\varphi})\nabla\overline{\varphi}\big]\cdot\boldsymbol{n} = 0 \quad \text{on } \Sigma, \qquad (4.10)\\
&\boldsymbol{\xi}(0) = \boldsymbol{0}, \qquad \eta(0) = 0 \quad \text{in } \Omega. \qquad (4.11)
\end{aligned}$$

We first prove that system (4.6)–(4.11) has a unique weak solution.

Proposition 1. *Let the assumptions of Lemma 4 be satisfied. Then problem* (4.6)–(4.11) *has for every* $\boldsymbol{h} \in \mathcal{V}$ *a unique weak solution* $[\boldsymbol{\xi}, \eta]$ *such that*

$$\begin{aligned}
&\boldsymbol{\xi} \in H^1(0,T;V_{div}') \cap C^0([0,T];G_{div}) \cap L^2(0,T;V_{div}),\\
&\eta \in H^1(0,T;V') \cap C^0([0,T];H) \cap L^2(0,T;V). \qquad (4.12)
\end{aligned}$$

Moreover, there is some constant $K_1^* > 0$*, which depends only on the data of the state system, such that, for every* $t \in (0,T]$*,*

$$\|\boldsymbol{\xi}\|_{H^1(0,t;V_{div}')\cap C^0([0,t];G_{div})\cap L^2(0,t;V_{div})} + \|\eta\|_{H^1(0,t;V')\cap C^0([0,t];H)\cap L^2(0,t;V)} \le K_1^*\,\|\boldsymbol{h}\|_{\mathcal{V}}\,. \qquad (4.13)$$

Proof. We make use of a Faedo–Galerkin approximating scheme. Following the lines of [8], we introduce the family $\{\boldsymbol{w}_j\}_{j\in\mathbb{N}}$ of the eigenfunctions to the Stokes operator S as a Galerkin basis in V_{div} and the family $\{\psi_j\}_{j\in\mathbb{N}}$ of the eigenfunctions to the Neumann operator $A := -\Delta + I$ as a Galerkin basis in V. Both these eigenfunction families $\{\boldsymbol{w}_j\}_{j\in\mathbb{N}}$ and $\{\psi_j\}_{j\in\mathbb{N}}$ are assumed to be suitably ordered and normalized. Moreover, recall that, since $\boldsymbol{w}_j \in D(S)$, we have $\mathrm{div}(\boldsymbol{w}_j) = 0$.

Then we look for two functions of the form

$$\boldsymbol{\xi}_n(t) := \sum_{j=1}^{n} a_j^{(n)}(t)\boldsymbol{w}_j\,, \qquad \eta_n(t) := \sum_{j=1}^{n} b_j^{(n)}(t)\psi_j\,,$$

that solve the following approximating problem:

$$\begin{aligned} &\langle \partial_t \boldsymbol{\xi}_n(t), \boldsymbol{w}_i\rangle_{V_{div}} \,+\, 2\left(\nu(\overline{\varphi}(t))\, D\boldsymbol{\xi}_n(t), D\boldsymbol{w}_i\right) \,+\, 2\left(\nu'(\overline{\varphi}(t))\,\eta_n(t)\, D\overline{\boldsymbol{u}}(t), D\boldsymbol{w}_i\right)\\ &+\, b(\overline{\boldsymbol{u}}(t), \boldsymbol{\xi}_n(t), \boldsymbol{w}_i) \,+\, b(\boldsymbol{\xi}_n(t), \overline{\boldsymbol{u}}(t), \boldsymbol{w}_i)\\ &=\, \left(\eta_n(t)(\nabla K * \overline{\varphi})(t), \boldsymbol{w}_i\right) \,+\, \left(\overline{\varphi}(t)(\nabla K * \eta_n)(t), \boldsymbol{w}_i\right) \,+\, (\boldsymbol{h}(t), \boldsymbol{w}_i)\,, \end{aligned} \tag{4.14}$$

$$\begin{aligned} &\langle \partial_t \eta_n(t), \psi_i\rangle_V \,=\, -\left(\lambda(\overline{\varphi}(t))\nabla\eta_n(t), \nabla\psi_i\right) \,+\, \left(m'(\overline{\varphi}(t))\eta_n(t)\nabla\left(K * \overline{\varphi}\right)(t), \nabla\psi_i\right)\\ &+\, \left(m(\overline{\varphi}(t))\left(\nabla K * \eta_n\right)(t), \nabla\psi_i\right) \,-\, \left(\eta_n(t)\lambda'(\overline{\varphi}(t))\nabla\overline{\varphi}(t), \nabla\psi_i\right) + (\overline{\boldsymbol{u}}(t)\,\eta_n(t), \nabla\psi_i)\\ &+\, (\boldsymbol{\xi}_n(t)\,\overline{\varphi}(t), \nabla\psi_i), \end{aligned} \tag{4.15}$$

$$\boldsymbol{\xi}_n(0) = \mathbf{0}, \quad \eta_n(0) = 0, \tag{4.16}$$

for $i = 1, \ldots, n$, and for almost every $t \in (0,T)$. It is immediately seen that the above system can be reduced to a Cauchy problem for a system of $2n$ linear ordinary differential equations in the $2n$ unknowns $a_i^{(n)}$, $b_i^{(n)}$, in which, owing to the regularity properties of $[\overline{\boldsymbol{u}}, \overline{\varphi}]$, all of the coefficient functions belong to $L^2(0,T)$. Thanks to Carathéodory's theorem, we can conclude that this problem enjoys a unique solution $\boldsymbol{a}^{(n)} := (a_1^{(n)}, \cdots, a_n^{(n)})^{\mathbf{t}}$, $\boldsymbol{b}^{(n)} := (b_1^{(n)}, \cdots, b_n^{(n)})^{\mathbf{t}}$, such that $\boldsymbol{a}^{(n)}, \boldsymbol{b}^{(n)} \in H^1(0,T;\mathbb{R}^n)$, which then specifies $[\boldsymbol{\xi}_n, \eta_n]$.

We now derive a priori estimates for $\boldsymbol{\xi}_n$ and η_n that are uniform in $n \in \mathbb{N}$. To begin with, let us multiply (4.14) by $a_i^{(n)}$, (4.15) by $b_i^{(n)}$, sum over $i = 1, \cdots, n$, and add the resulting identities. Omitting the argument t for the sake of a shorter exposition, we then obtain, almost everywhere in $(0,T)$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\boldsymbol{\xi}_n\|^2 + \|\eta_n\|^2\right) \,+\, 2\left(\nu(\overline{\varphi})\, D\boldsymbol{\xi}_n, D\boldsymbol{\xi}_n\right) \,+\, \left(\lambda(\overline{\varphi})\nabla\eta_n, \nabla\eta_n\right) \,=\, -b(\boldsymbol{\xi}_n, \overline{\boldsymbol{u}}, \boldsymbol{\xi}_n)\\ &-\, 2\left(\nu'(\overline{\varphi})\,\eta_n\, D\overline{\boldsymbol{u}}, D\boldsymbol{\xi}_n\right) \,+\, \left(\eta_n(\nabla K * \overline{\varphi}), \boldsymbol{\xi}_n\right) \,+\, \left(\overline{\varphi}\,(\nabla K * \eta_n), \boldsymbol{\xi}_n\right)\\ &+\, (\boldsymbol{h}, \boldsymbol{\xi}_n) \,+\, \left(m'(\overline{\varphi})\eta_n\nabla\left(K * \overline{\varphi}\right), \nabla\eta_n\right) \,+\, \left(m(\overline{\varphi})\left(\nabla K * \eta_n\right), \nabla\eta_n\right)\\ &-\, \left(\eta_n\lambda'(\overline{\varphi})\nabla\overline{\varphi}, \nabla\eta_n\right) \,+\, (\boldsymbol{\xi}_n\,\overline{\varphi}, \nabla\eta_n). \end{aligned} \tag{4.17}$$

Let us now estimate the terms on the right-hand side of this equation individually. In the remainder of this proof, we use the following abbreviating notation: the letter C will stand for positive constants that depend only on the global data of the state system, on $\overline{\boldsymbol{v}}$, and on $[\overline{\boldsymbol{u}}, \overline{\varphi}]$, but not on $n \in \mathbb{N}$; moreover, by C_σ we denote constants that in addition depend on the quantities indicated by the index σ, but not on $n \in \mathbb{N}$. Both C and C_σ may change within formulas and even within lines.

The first two terms on the right-hand side can be estimated exactly as in [19, Proof of Proposition 1], namely,

$$|b(\boldsymbol{\xi}_n, \overline{\boldsymbol{u}}, \boldsymbol{\xi}_n)| \,\le\, \epsilon\,\|\nabla\boldsymbol{\xi}_n\|^2 + C_\epsilon\,\|\overline{\boldsymbol{u}}\|^2_{H^2(\Omega)^2}\,\|\boldsymbol{\xi}_n\|^2, \tag{4.18}$$

$$\left|2\left(\nu'(\overline{\varphi})\,\eta_n\, D\overline{\boldsymbol{u}}, D\boldsymbol{\xi}_n\right)\right| \,\le\, \epsilon\,\|\nabla\boldsymbol{\xi}_n\|^2 \,+\, \epsilon'\,\|\nabla\eta_n\|^2 \,+\, C_{\epsilon,\epsilon'}\,\|\overline{\boldsymbol{u}}\|^2_{H^2(\Omega)^2}\,\|\eta_n\|^2. \tag{4.19}$$

Concerning the other terms, we have, using Hölder's inequality, Young's inequality, and the global bounds (2.15) as main tools, the following series of estimates:

$$\left|\left(\eta_n(\nabla K * \overline{\varphi}), \boldsymbol{\xi}_n\right)\right| \,\le\, C_K(\|\eta_n\|^2 + \|\boldsymbol{\xi}_n\|^2)\,, \tag{4.20}$$

$$\left|\left(\overline{\varphi}\,(\nabla K * \eta_n), \boldsymbol{\xi}_n\right)\right| \,\le\, C_K(\|\eta_n\|^2 + \|\boldsymbol{\xi}_n\|^2)\,, \tag{4.21}$$

$$
\begin{aligned}
|(\boldsymbol{h}, \boldsymbol{\xi}_n)| &\leq \frac{1}{2}(\|\boldsymbol{h}\|^2 + \|\boldsymbol{\xi}_n\|^2)\,, && (4.22)\\
\big|\big(m'(\overline{\varphi})\eta_n \nabla\left(K * \overline{\varphi}\right), \nabla\eta_n\big)\big| &\leq \|m'(\overline{\varphi})\|_{L^\infty(\Omega)}\|\eta_n\|_{L^4(\Omega)}\|\nabla\left(K*\overline{\varphi}\right)\|_{L^4(\Omega)}\|\nabla\eta_n\| \\
&\leq \epsilon'\|\nabla\eta_n\|^2 + C_{m,\epsilon'}\|\eta_n\|^2_{L^4(\Omega)}\|\nabla\left(K*\overline{\varphi}\right)\|^2_{L^4(\Omega)} \\
&\leq \epsilon'\|\nabla\eta_n\|^2 + C_{m,K,\epsilon'}\|\overline{\varphi}\|^2_{H^2(\Omega)}\big(\|\eta_n\|^2 + \|\eta_n\|\|\nabla\eta_n\|\big) \\
&\leq 2\epsilon'\|\nabla\eta_n\|^2 + C_{m,K,\epsilon'}\|\eta_n\|^2\,, && (4.23)\\
\big|\big(m(\overline{\varphi})\left(\nabla K * \eta_n\right), \nabla\eta_n\big)\big| &\leq \|m(\overline{\varphi})\|_{L^\infty(\Omega)}\|\nabla K * \eta_n\|\|\nabla\eta_n\| \\
&\leq \epsilon'\|\nabla\eta_n\|^2 + C_{m,K,\epsilon'}\|\eta_n\|^2\,, \\
\big|\big(\eta_n\lambda'(\overline{\varphi})\nabla\overline{\varphi}, \nabla\eta_n\big)\big| &\leq \|\eta_n\|_{L^4(\Omega)}\|\lambda'(\overline{\varphi})\|_{L^\infty(\Omega)}\|\nabla\overline{\varphi}\|_{L^4(\Omega)}\|\nabla\eta_n\| && (4.24)\\
&\leq \epsilon'\|\nabla\eta_n\|^2 + C_{\lambda,\epsilon'}\|\eta_n\|^2\,, \\
|(\boldsymbol{\xi}_n\,\overline{\varphi}, \nabla\eta_n)| &\leq \epsilon'\|\nabla\eta_n\|^2 + C_{\epsilon'}\|\boldsymbol{\xi}_n\|^2\,. && (4.25)
\end{aligned}
$$

Hence, inserting the estimates (4.18)-(4.25) in (4.17) and choosing $\epsilon > 0$ and $\epsilon' > 0$ small enough (i.e., $\epsilon \leq \nu_1/4$ and $\epsilon' \leq \alpha_0/12$), we obtain the estimate

$$
\begin{aligned}
&\frac{d}{dt}\big(\|\boldsymbol{\xi}_n\|^2 + \|\eta_n\|^2\big) + \nu_1\,\|\nabla\boldsymbol{\xi}_n\|^2 + \alpha_0\,\|\nabla\eta_n\|^2 \\
&\leq C\left(1 + \|\overline{\boldsymbol{u}}\|^2_{H^2(\Omega)^2}\right)\big(\|\boldsymbol{\xi}_n\|^2 + \|\eta_n\|^2\big) + \|\boldsymbol{h}\|^2. && (4.26)
\end{aligned}
$$

Observe now that, owing to (2.15), the mapping $t \mapsto \|\overline{\boldsymbol{u}}(t)\|^2_{H^2(\Omega)^2}$ belongs to $L^1(0,T)$. Therefore, Gronwall's lemma yields, for every $t \in (0,T]$,

$$
\|\boldsymbol{\xi}_n\|_{L^\infty(0,t;G_{div})\cap L^2(0,t;V_{div})} \leq C\,\|\boldsymbol{h}\|_{\mathcal{V}}, \quad \|\eta_n\|_{L^\infty(0,t;H)\cap L^2(0,t;V)} \leq C\,\|\boldsymbol{h}\|_{\mathcal{V}}\,, \tag{4.27}
$$

for all $n \in \mathbb{N}$.

Moreover, by comparison in (4.14) and (4.15), we can easily deduce also the estimates for the time derivatives $\partial_t\boldsymbol{\xi}_n$ and $\partial_t\eta_n$. Indeed, we have, for every $t \in (0,T)$,

$$
\|\partial_t\boldsymbol{\xi}_n\|_{L^2(0,t;V'_{div})} \leq C\,\|\boldsymbol{h}\|_{\mathcal{V}}, \quad \|\partial_t\eta_n\|_{L^2(0,t;V')} \leq C\,\|\boldsymbol{h}\|_{\mathcal{V}}, \quad \text{for all } n \in \mathbb{N}. \tag{4.28}
$$

From (4.27), (4.28), we deduce the existence of a subsequence, which is again indexed by n, such that, with two functions $\boldsymbol{\xi}$, η satisfying (4.12), we have

$$
\begin{aligned}
\boldsymbol{\xi}_n &\to \boldsymbol{\xi} \quad \text{weakly}^* \text{ in } H^1(0,T;V'_{div}) \cap L^\infty(0,T;G_{div}) \cap L^2(0,T;V_{div})\,, \\
\eta_n &\to \eta \quad \text{weakly}^* \text{ in } H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V)\,.
\end{aligned}
$$

Then, by means of standard arguments, we can pass to the limit as $n \to \infty$ in (4.14)–(4.16) and prove that $\boldsymbol{\xi}$, η satisfy the weak formulation of the problem (4.6)–(4.11). Notice that we actually have the regularity (4.12), since the space $H^1(0,T;V'_{div}) \cap L^2(0,T;V_{div})$ is continuously embedded in $C^0([0,T];G_{div})$; similarly we obtain that $\eta \in C^0([0,T];H)$. Moreover, from (4.27), (4.28) and the weak and weak* sequential semicontinuity of norms it follows that (4.13) is satisfied.

Finally, in order to prove that the solution $\boldsymbol{\xi}, \eta$ is unique, we can test the difference between (4.6), (4.7), written for two solutions $[\boldsymbol{\xi}_1, \eta_1]$ and $[\boldsymbol{\xi}_2, \eta_2]$, by $\boldsymbol{\xi} := \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2$ and by $\eta := \eta_1 - \eta_2$, respectively. Since the problem is linear, the argument is straightforward, and the details can be left to the reader. □

Differentiability of the control-to-state operator. In this subsection, we prove the following result which is crucial to establish optimality conditions.

Theorem 4. *Let the assumptions of Lemma 4 hold true. Then the control-to-state operator $\mathcal{S} : \mathcal{V} \to \mathcal{H}$ is Fréchet differentiable on $\mathcal{V}$ when viewed as a mapping between the spaces $\mathcal{V}$ and $\mathcal{Z}$, where*

$$\mathcal{Z} := \big[C^0([0,T]; G_{div}) \cap L^2(0,T; V_{div})\big] \times \big[C^0([0,T]; H) \cap L^2(0,T; V)\big].$$

Moreover, for any $\overline{\boldsymbol{v}} \in \mathcal{V}$, the Fréchet derivative $\mathcal{S}'(\overline{\boldsymbol{v}}) \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$ is given by $\mathcal{S}'(\overline{\boldsymbol{v}})\boldsymbol{h} = [\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}]$, for all $\boldsymbol{h} \in \mathcal{V}$, where $[\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}]$ is the unique weak solution to the linearized system (4.6)–(4.11) *at $[\overline{\boldsymbol{u}}, \overline{\varphi}] = \mathcal{S}(\overline{\boldsymbol{v}})$ that corresponds to $\boldsymbol{h} \in \mathcal{V}$.*

Proof. Let $\overline{\boldsymbol{v}} \in \mathcal{V}$ be fixed and $[\overline{\boldsymbol{u}}, \overline{\varphi}] = \mathcal{S}(\overline{\boldsymbol{v}})$. Recalling (4.13), we first note that the linear mapping $\boldsymbol{h} \mapsto [\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}]$ belongs to $\mathcal{L}(\mathcal{V}, \mathcal{Z})$, in particular. Moreover, let $\Lambda > 0$ be fixed. In the following, we consider perturbations $\boldsymbol{h} \in \mathcal{V}$ such that $\|\boldsymbol{h}\|_{\mathcal{V}} \le \Lambda$. For any such perturbation $\boldsymbol{h}$, we put

$$[\boldsymbol{u}^{\boldsymbol{h}}, \varphi^{\boldsymbol{h}}] := \mathcal{S}(\overline{\boldsymbol{v}} + \boldsymbol{h}), \qquad \boldsymbol{p}^{\boldsymbol{h}} := \boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}} - \boldsymbol{\xi}^{\boldsymbol{h}}, \qquad q^{\boldsymbol{h}} := \varphi^{\boldsymbol{h}} - \overline{\varphi} - \eta^{\boldsymbol{h}}.$$

Notice that we have the regularity

$$\begin{aligned} &\boldsymbol{p}^{\boldsymbol{h}} \in H^1(0,T; V_{div}') \cap C^0([0,T]; G_{div}) \cap L^2(0,T; V_{div}), \\ &q^{\boldsymbol{h}} \in H^1(0,T; V') \cap C^0([0,T]; H) \cap L^2(0,T; V)\,. \end{aligned} \tag{4.29}$$

By virtue of (2.15) and of (3.13), there is a constant $C_1^* > 0$, which may depend on the data of the problem and on Λ, such that we have: for every $\boldsymbol{h} \in \mathcal{V}$ with $\|\boldsymbol{h}\|_{\mathcal{V}} \le \Lambda$ it holds

$$\big\|[\boldsymbol{u}^{\boldsymbol{h}}, \varphi^{\boldsymbol{h}}]\big\|_{\mathcal{H}} \le C_1^*, \quad \|\varphi^{\boldsymbol{h}}\|_{C^0(\overline{Q})} \le C_1^*, \tag{4.30}$$

$$\|\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}}\|^2_{C^0([0,t]; G_{div}) \cap L^2(0,t; V_{div})} + \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|^2_{H^1(0,t;H) \cap C^0([0,t];V) \cap L^2(0,t;H^2(\Omega))} \le C_1^* \, \|\boldsymbol{h}\|^2_{\mathcal{V}}, \tag{4.31}$$

for every $t \in (0,T]$.

After some straightforward algebraic manipulations, we can see that $\boldsymbol{p}^{\boldsymbol{h}}, q^{\boldsymbol{h}}$ (which, for simplicity, shall henceforth be denoted by $\boldsymbol{p}, q$) is a solution to the weak analogue of the following problem:

$$\begin{aligned} &\boldsymbol{p}_t - 2\,\mathrm{div}\big(\nu(\overline{\varphi})D\boldsymbol{p}\big) - 2\,\mathrm{div}\big((\nu(\varphi^{\boldsymbol{h}}) - \nu(\overline{\varphi}))D(\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\big) - 2\,\mathrm{div}\big((\nu(\varphi^{\boldsymbol{h}}) - \nu(\overline{\varphi}) - \nu'(\overline{\varphi})\eta)D\overline{\boldsymbol{u}}\big) \\ &\quad + \big((\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}}) \cdot \nabla\big)(\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}}) + (\boldsymbol{p} \cdot \nabla)\overline{\boldsymbol{u}} + (\overline{\boldsymbol{u}} \cdot \nabla)\boldsymbol{p} + \nabla\widetilde{\pi}^{\boldsymbol{h}} \\ &= -\big(K * (\varphi^{\boldsymbol{h}} - \overline{\varphi})\big)\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) - (-K * q)\nabla\overline{\varphi} - (K * \overline{\varphi})\nabla q \quad \text{ in } Q, \\ &q_t + (\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}}) \cdot \nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi}) + \boldsymbol{p} \cdot \nabla\overline{\varphi} + \overline{\boldsymbol{u}} \cdot \nabla q = \mathrm{div}\big(\lambda(\overline{\varphi})\nabla q\big) \\ &\quad + \mathrm{div}\big((\lambda(\varphi^{\boldsymbol{h}}) - \lambda(\overline{\varphi}))\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi})\big) + \mathrm{div}\big((\lambda(\varphi^{\boldsymbol{h}}) - \lambda(\overline{\varphi}) - (\lambda'(\overline{\varphi}))\eta)\nabla\overline{\varphi}\big) \\ &\quad - \mathrm{div}\big((m(\varphi^{\boldsymbol{h}}) - m(\overline{\varphi}))\nabla K * (\varphi^{\boldsymbol{h}} - \overline{\varphi})\big) \end{aligned} \tag{4.32}$$

$$
\begin{aligned}
&\quad - \operatorname{div}\big((m(\varphi^{\boldsymbol h}) - m(\overline{\varphi}) - m'(\overline{\varphi})\eta)\nabla K * \overline{\varphi}\big) - \operatorname{div}\big(m(\overline{\varphi})\nabla K * q\big) \quad \text{in } Q, && (4.33)\\
&\operatorname{div}(\boldsymbol p) = 0 \quad \text{in } Q, && (4.34)\\
&\boldsymbol p = \boldsymbol 0 \quad \text{on } \Sigma, && (4.35)\\
&\big[\lambda(\overline{\varphi})\nabla q + \big(\lambda(\varphi^{\boldsymbol h}) - \lambda(\overline{\varphi})\big)\nabla(\varphi^{\boldsymbol h} - \overline{\varphi}) + \big(\lambda(\varphi^{\boldsymbol h}) - \lambda(\overline{\varphi}) - (\lambda'(\overline{\varphi}))\eta\big)\nabla\overline{\varphi}\\
&\quad - (m(\varphi^{\boldsymbol h}) - m(\overline{\varphi}))\nabla K * (\varphi^{\boldsymbol h} - \overline{\varphi}) - (m(\varphi^{\boldsymbol h}) - m(\overline{\varphi}) - m'(\overline{\varphi})\eta)\nabla K * \overline{\varphi}\\
&\quad - m(\overline{\varphi})\nabla K * q\big] \cdot \boldsymbol n = 0 \quad \text{on } \Sigma, && (4.36)\\
&\boldsymbol p(0) = \boldsymbol 0, \qquad q(0) = 0 \quad \text{in } \Omega. && (4.37)
\end{aligned}
$$

That is, $\boldsymbol p$ and q solve the following variational problem (where we avoid to write the argument t of the involved functions):

$$
\begin{aligned}
&\langle \boldsymbol p_t, \boldsymbol w\rangle_{V_{div}} + 2\big(\nu(\overline{\varphi})D\boldsymbol p, D\boldsymbol w\big) + 2\big((\nu(\varphi^{\boldsymbol h}) - \nu(\overline{\varphi}))D(\boldsymbol u^{\boldsymbol h} - \overline{\boldsymbol u}), D\boldsymbol w\big)\\
&\quad + 2\big((\nu(\varphi^{\boldsymbol h}) - \nu(\overline{\varphi}) - \nu'(\overline{\varphi})\eta^{\boldsymbol h})D\overline{\boldsymbol u}, D\boldsymbol w\big) + b(\boldsymbol p, \overline{\boldsymbol u}, \boldsymbol w) + b(\overline{\boldsymbol u}, \boldsymbol p, \boldsymbol w)\\
&\quad + b(\boldsymbol u^{\boldsymbol h} - \overline{\boldsymbol u}, \boldsymbol u^{\boldsymbol h} - \overline{\boldsymbol u}, \boldsymbol w)\\
&= -\big(\big(K * (\varphi^{\boldsymbol h} - \overline{\varphi})\big)\nabla(\varphi^{\boldsymbol h} - \overline{\varphi}), \boldsymbol w\big) - \big((K * q)\nabla\overline{\varphi}, \boldsymbol w\big) - \big((K * \overline{\varphi})\nabla q, \boldsymbol w\big), && (4.38)\\
&\langle q_t, \psi\rangle_V + \big((\boldsymbol u^{\boldsymbol h} - \overline{\boldsymbol u}) \cdot \nabla(\varphi^{\boldsymbol h} - \overline{\varphi}), \psi\big) + \big(\boldsymbol p \cdot \nabla\overline{\varphi}, \psi\big) + \big(\overline{\boldsymbol u} \cdot \nabla q, \psi\big)\\
&= -\big(\lambda(\overline{\varphi})\nabla q, \nabla\psi\big) - \big((\lambda(\varphi^{\boldsymbol h}) - \lambda(\overline{\varphi}))\nabla(\varphi^{\boldsymbol h} - \overline{\varphi}), \nabla\psi\big) - \big((\lambda(\varphi^{\boldsymbol h}) - \lambda(\overline{\varphi}) - (\lambda'(\overline{\varphi}))\eta)\nabla\overline{\varphi}, \nabla\psi\big)\\
&\quad + \big((m(\varphi^{\boldsymbol h}) - m(\overline{\varphi}))\nabla K * (\varphi^{\boldsymbol h} - \overline{\varphi}), \nabla\psi\big) + \big((m(\varphi^{\boldsymbol h}) - m(\overline{\varphi}) - m'(\overline{\varphi})\eta)\nabla K * \overline{\varphi}, \nabla\psi\big)\\
&\quad + \big(m(\overline{\varphi})\nabla K * q, \nabla\psi\big), && (4.39)
\end{aligned}
$$

for every $\boldsymbol w \in V_{div}$, every $\psi \in V$ and almost every $t \in (0, T)$.

We now choose $\boldsymbol w = \boldsymbol p(t) \in V_{div}$ and $\psi = q(t) \in V$ as test functions in equations (4.38) and (4.39), respectively. This gives the identities (omitting the explicit dependence on t)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol p\|^2 + 2\int_\Omega \nu(\overline{\varphi})D\boldsymbol p : D\boldsymbol p\,dx + 2\int_\Omega ((\nu(\varphi^{\boldsymbol h}) - \nu(\overline{\varphi}))\,D(\boldsymbol u^{\boldsymbol h} - \overline{\boldsymbol u}) : D\boldsymbol p\,dx\\
&\quad + 2\int_\Omega \nu'(\overline{\varphi})\,q\,D\overline{\boldsymbol u} : D\boldsymbol p\,dx + \int_\Omega \nu''(\sigma_1^{\boldsymbol h})\,(\varphi^{\boldsymbol h} - \overline{\varphi})^2\,D\overline{\boldsymbol u} : D\boldsymbol p\,dx + \int_\Omega (\boldsymbol p \cdot \nabla)\overline{\boldsymbol u} \cdot \boldsymbol p\,dx\\
&\quad + \int_\Omega \big((\boldsymbol u^{\boldsymbol h} - \overline{\boldsymbol u}) \cdot \nabla\big)(\boldsymbol u^{\boldsymbol h} - \overline{\boldsymbol u}) \cdot \boldsymbol p\,dx = -\int_\Omega \big(K * (\varphi^{\boldsymbol h} - \overline{\varphi})\big)\nabla(\varphi^{\boldsymbol h} - \overline{\varphi}) \cdot \boldsymbol p\,dx\\
&\quad - \int_\Omega (K * q)\nabla\overline{\varphi} \cdot \boldsymbol p\,dx - \int_\Omega (K * \overline{\varphi})\nabla q \cdot \boldsymbol p\,dx\,, && (4.40)\\
&\frac{1}{2}\frac{d}{dt}\|q\|^2 + \int_\Omega \big((\boldsymbol u^{\boldsymbol h} - \overline{\boldsymbol u}) \cdot \nabla(\varphi^{\boldsymbol h} - \overline{\varphi})\big)\,q\,dx + \int_\Omega (\boldsymbol p \cdot \nabla\overline{\varphi})\,q\,dx\\
&= -\int_\Omega \lambda(\overline{\varphi})|\nabla q|^2\,dx - \int_\Omega (\lambda(\varphi^{\boldsymbol h}) - \lambda(\overline{\varphi}))\nabla(\varphi^{\boldsymbol h} - \overline{\varphi}) \cdot \nabla q\,dx\\
&\quad - \int_\Omega (\lambda(\varphi^{\boldsymbol h}) - \lambda(\overline{\varphi}) - (\lambda'(\overline{\varphi}))\eta)\nabla\overline{\varphi} \cdot \nabla q\,dx\\
&\quad + \int_\Omega (m(\varphi^{\boldsymbol h}) - m(\overline{\varphi}))\,\big(\nabla K * (\varphi^{\boldsymbol h} - \overline{\varphi})\big) \cdot \nabla q\,dx
\end{aligned}
$$

$$+\int_\Omega (m(\varphi^{\boldsymbol{h}}) - m(\overline{\varphi}) - m'(\overline{\varphi})\eta)\,(\nabla K * \overline{\varphi}) \cdot \nabla q \,dx + \int_\Omega m(\overline{\varphi})\,(\nabla K * q) \cdot \nabla q \,dx. \tag{4.41}$$

In (4.40), we have used Taylor's expansion

$$\nu(\varphi^{\boldsymbol{h}}) = \nu(\overline{\varphi}) + \nu'(\overline{\varphi})(\varphi^{\boldsymbol{h}} - \overline{\varphi}) + \frac{1}{2}\nu''(\sigma_1^{\boldsymbol{h}})(\varphi^{\boldsymbol{h}} - \overline{\varphi})^2, \tag{4.42}$$

where

$$\sigma_1^{\boldsymbol{h}} = \theta_1^{\boldsymbol{h}}\varphi^{\boldsymbol{h}} + (1-\theta_1^{\boldsymbol{h}})\overline{\varphi}, \quad \theta_1^{\boldsymbol{h}} = \theta_1^{\boldsymbol{h}}(x,t) \in (0,1).$$

Moreover, in the integration by parts on the right-hand side of (4.41), we employed the boundary condition (4.36), which can be written for $\varphi^{\boldsymbol{h}}$ and for $\overline{\varphi}$, and (4.10).

We now estimate all of the terms in (4.40) and in (4.41). Concerning the ones in (4.40), these can be estimated exactly as in [19]. Hence, we just report these estimates omitting the details. We denote by C positive constants that may depend on the data of the system, but not on the choice of $\boldsymbol{h} \in \mathcal{V}$ with $\|\boldsymbol{h}\|_{\mathcal{V}} \leq \Lambda$, while C_σ denotes a positive constant that also depends on the quantity indicated by σ.

Denoting by $I_3^{(4)}, \ldots, I_7^{(4)}$ the absolute values of the third to seventh terms on the left-hand side of (4.40), and by $I_1^{(5)}, \ldots, I_3^{(5)}$ the three terms on the right-hand side, we have, with constants $\epsilon > 0$ and $\epsilon' > 0$ that will be fixed later, the following series of estimates:

$$I_3^{(4)} \leq \epsilon\,\|\nabla \boldsymbol{p}\|^2 + C_\epsilon\,\|\nabla(\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\|\left(\|\boldsymbol{u}^{\boldsymbol{h}}\|_{H^2(\Omega)^2} + \|\overline{\boldsymbol{u}}\|_{H^2(\Omega)^2}\right)\|\boldsymbol{h}\|_{\mathcal{V}}^2, \tag{4.43}$$

$$I_4^{(4)} \leq \epsilon\,\|\nabla \boldsymbol{p}\|^2 + \epsilon'\,\|\nabla q\|^2 + C_{\epsilon,\epsilon'}\left(1 + \|\overline{\boldsymbol{u}}\|_{H^2(\Omega)^2}^2\right)\|q\|^2, \tag{4.44}$$

$$I_5^{(4)} \leq \epsilon\,\|\nabla \boldsymbol{p}\|^2 + C_\epsilon\,\|\overline{\boldsymbol{u}}\|_{H^2(\Omega)^2}^2\,\|\boldsymbol{h}\|_{\mathcal{V}}^4, \tag{4.45}$$

$$I_6^{(4)} \leq \epsilon\,\|\nabla \boldsymbol{p}\|^2 + C_\epsilon\,\|\overline{\boldsymbol{u}}\|_{H^2(\Omega)^2}^2\,\|\boldsymbol{p}\|^2, \tag{4.46}$$

$$I_7^{(4)} \leq \epsilon\,\|\nabla \boldsymbol{p}\|^2 + C_\epsilon\,\|\nabla(\boldsymbol{u}^{\boldsymbol{h}} - \overline{\boldsymbol{u}})\|^2\,\|\boldsymbol{h}\|_{\mathcal{V}}^2, \tag{4.47}$$

$$I_1^{(5)} \leq \epsilon\,\|\nabla \boldsymbol{p}\|^2 + C_\epsilon\,\|\boldsymbol{h}\|_{\mathcal{V}}^4, \tag{4.48}$$

$$I_2^{(5)} \leq \epsilon\,\|\nabla \boldsymbol{p}\|^2 + C_\epsilon\,\|q\|^2, \tag{4.49}$$

$$I_3^{(5)} \leq \epsilon'\,\|\nabla q\|^2 + C_{\epsilon'}\,\|\boldsymbol{p}\|^2. \tag{4.50}$$

Let us now consider (4.41). To estimate some of the terms in this equation, we shall employ the following identity, which holds for general functions $G \in C^2([-1,1])$:

$$G(\varphi^{\boldsymbol{h}}) - G(\overline{\varphi}) - G'(\overline{\varphi})\eta = G'(\overline{\varphi})q + \frac{1}{2}G''(\sigma^{\boldsymbol{h}})\left(\varphi^{\boldsymbol{h}} - \overline{\varphi}\right)^2, \tag{4.51}$$

with $\sigma^{\boldsymbol{h}} = \theta^{\boldsymbol{h}}\varphi^{\boldsymbol{h}} + (1-\theta^{\boldsymbol{h}})\overline{\varphi},\ \theta^{\boldsymbol{h}} = \theta^{\boldsymbol{h}}(x,t) \in (0,1)$. We denote by $I_1^{(6)}, I_2^{(6)}$ the absolute values of the two terms on the left-hand side, which can be estimated exactly as in [19] (we therefore omit the details), and by $I_1^{(7)}, \ldots, I_6^{(7)}$ the six terms on the right-hand side of (4.41). Using the mean value theorem, (2.1), (4.30), (4.31), Hölder's and Young's inequalities, and the continuity of the embedding $V \subset L^p(\Omega)$ for $1 \leq p < +\infty$ in two dimensions of space, we obtain the following series of estimates:

$$I_1^{(6)} \le \epsilon' \|\nabla q\|^2 + \|q\|^2 + C_{\epsilon'} \|\nabla(\boldsymbol{u^h} - \overline{\boldsymbol{u}})\|^2 \|\boldsymbol{h}\|_{\mathcal{V}}^2 , \tag{4.52}$$

$$I_2^{(6)} \le \epsilon \|\nabla \boldsymbol{p}\|^2 + C_\epsilon \|q\|^2 , \tag{4.53}$$

$$I_1^{(7)} \le -\alpha_0 \|\nabla q\|^2 , \tag{4.54}$$

$$\begin{aligned} I_2^{(7)} &\le \|\lambda(\varphi^{\boldsymbol{h}}) - \lambda(\overline{\varphi})\|_{L^4(\Omega)} \|\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi})\|_{L^4(\Omega)} \|\nabla q\| \\ &\le C \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|_{L^4(\Omega)} \|\nabla(\varphi^{\boldsymbol{h}} - \overline{\varphi})\|_{L^4(\Omega)}\| \|\nabla q\| \\ &\le \epsilon' \|\nabla q\|^2 + C_{\epsilon'} \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|_V^2 \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|_{H^2(\Omega)^2}^2 , \end{aligned} \tag{4.55}$$

$$\begin{aligned} I_3^{(7)} &\le \Big(\|\lambda'(\overline{\varphi}) q\|_{L^4(\Omega)} + \frac{1}{2} \|\lambda''(\sigma_2^{\boldsymbol{h}})(\varphi^{\boldsymbol{h}} - \overline{\varphi})^2\|_{L^4(\Omega)} \Big) \|\nabla\overline{\varphi}\|_{L^4(\Omega)} \|\nabla q\| \\ &\le C\big(\|q\|_{L^4(\Omega)} + \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|_{L^8(\Omega)}^2\big) \|\nabla q\| \\ &\le C\,(\|q\| + \|q\|^{1/2} \|\nabla q\|^{1/2}) \|\nabla q\| + \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|_V^2 \|\nabla q\| \\ &\le \epsilon' \|\nabla q\|^2 + C_{\epsilon'} \|q\|^2 + C_{\epsilon'} \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|_V^4 \le \epsilon' \|\nabla q\|^2 + C_{\epsilon'} \|q\|^2 + C_{\epsilon'} \|\boldsymbol{h}\|_{\mathcal{V}}^4 , \end{aligned} \tag{4.56}$$

$$\begin{aligned} I_4^{(7)} &\le \|m(\varphi^{\boldsymbol{h}}) - m(\overline{\varphi})\|_{L^4(\Omega)} \|\nabla K * (\varphi^{\boldsymbol{h}} - \overline{\varphi})\|_{L^4(\Omega)} \|\nabla q\| \le C \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|_V^2 \|\nabla q\| \\ &\le \epsilon' \|\nabla q\|^2 + C_{\epsilon'} \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|_V^4 \le \epsilon' \|\nabla q\|^2 + C_{\epsilon'} \|\boldsymbol{h}\|_{\mathcal{V}}^4 , \end{aligned} \tag{4.57}$$

$$\begin{aligned} I_5^{(7)} &\le \Big(\|m'(\overline{\varphi})\, q\| + \frac{1}{2} \|m''(\sigma_4^{\boldsymbol{h}})(\varphi^{\boldsymbol{h}} - \overline{\varphi})^2\| \Big) \|\nabla K * \overline{\varphi}\|_{L^\infty(\Omega)} \|\nabla q\| \\ &\le C\big(\|q\| + \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|_{L^4(\Omega)}^2\big) \|\nabla q\| \\ &\le \epsilon' \|\nabla q\|^2 + C_{\epsilon'} \|q\|^2 + C_{\epsilon'} \|\varphi^{\boldsymbol{h}} - \overline{\varphi}\|_V^4 \le \epsilon' \|\nabla q\|^2 + C_{\epsilon'} \|q\|^2 + C_{\epsilon'} \|\boldsymbol{h}\|_{\mathcal{V}}^4 , \end{aligned} \tag{4.58}$$

$$I_6^{(7)} \le \|m(\overline{\varphi})\|_{L^\infty(\Omega)} \|\nabla K * q\| \|\nabla q\| \le \epsilon' \|\nabla q\|^2 + C_{\epsilon'} \|q\|^2 . \tag{4.59}$$

We now insert estimates (4.43)–(4.50) in (4.40) and the estimates (4.52)–(4.59) in (4.41). Adding the resulting inequalities, and taking $\epsilon, \epsilon' > 0$ small enough (in particular, $\epsilon \le \nu_1/16$ and $\epsilon' \le \alpha_0/20$), we find that

$$\frac{d}{dt}\big(\|\boldsymbol{p^h}\|^2 + \|q^{\boldsymbol{h}}\|^2\big) + \nu_1 \|\nabla \boldsymbol{p^h}\|^2 + \alpha_0 \|\nabla q^{\boldsymbol{h}}\|^2 \le \Xi\,(\|\boldsymbol{p^h}\|^2 + \|q^{\boldsymbol{h}}\|^2) + \Xi\,\|\boldsymbol{h}\|_{\mathcal{V}}^4 + \Xi^{\boldsymbol{h}} \|\boldsymbol{h}\|_{\mathcal{V}}^2,$$

where the functions $\Xi, \Xi^{\boldsymbol{h}} \in L^1(0,T)$ are given by

$$\begin{aligned} \Xi(t) &:= C\big(1 + \|\overline{\boldsymbol{u}}(t)\|_{H^2(\Omega)^2}^2\big), \\ \Xi^{\boldsymbol{h}}(t) &:= C\Big(\big(\|\boldsymbol{u^h}(t)\|_{H^2(\Omega)^2} + \|\overline{\boldsymbol{u}}(t)\|_{H^2(\Omega)^2}\big) \|\nabla(\boldsymbol{u^h} - \overline{\boldsymbol{u}})(t)\| + \|\nabla(\boldsymbol{u^h} - \overline{\boldsymbol{u}})(t)\|^2 \\ &\qquad + \|(\varphi^{\boldsymbol{h}} - \overline{\varphi})(t)\|_{H^2(\Omega)}^2 \Big) . \end{aligned}$$

Recalling that $\|\boldsymbol{h}\|_{\mathcal{V}} \le \Lambda$, thanks to (4.30) and (4.31), we get

$$\int_0^T \Xi^{\boldsymbol{h}}(t)\, dt \le C\, \|\boldsymbol{h}\|_{\mathcal{V}}.$$

Taking (4.37) into account, an application of Gronwall's lemma yields the estimate

$$\|\boldsymbol{p^h}\|_{C^0([0,T];G_{div})}^2 + \|\boldsymbol{p^h}\|_{L^2(0,T;V_{div})}^2 + \|q^{\boldsymbol{h}}\|_{C^0([0,T];H)}^2 + \|q^{\boldsymbol{h}}\|_{L^2(0,T;V)}^2 \le C\, \|\boldsymbol{h}\|_{\mathcal{V}}^3.$$

We therefore have

$$\frac{\|\mathcal{S}(\overline{\boldsymbol{v}}+\boldsymbol{h})-\mathcal{S}(\overline{\boldsymbol{v}})-[\boldsymbol{\xi}^{\boldsymbol{h}},\eta^{\boldsymbol{h}}]\|_{\mathcal{Z}}}{\|\boldsymbol{h}\|_{\mathcal{V}}} \;=\; \frac{\|[\boldsymbol{p}^{\boldsymbol{h}},q^{\boldsymbol{h}}]\|_{\mathcal{Z}}}{\|\boldsymbol{h}\|_{\mathcal{V}}} \;\le\; C\,\|\boldsymbol{h}\|_{\mathcal{V}}^{1/2} \to 0,$$

as $\|\boldsymbol{h}\|_{\mathcal{V}} \to 0$. This concludes the proof of the assertion. □

First-order necessary optimality conditions. From Theorem 4, by arguing as in the proof of [19, Corollary 1], we can deduce the following necessary optimality condition:

Corollary 1. *Let the assumptions of Lemma 4 hold true. If $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$ is an optimal control for* **(CP)** *with associated state $[\overline{\boldsymbol{u}},\overline{\varphi}] = \mathcal{S}(\overline{\boldsymbol{v}})$, then the following inequality holds true:*

$$\beta_1 \int_0^T\!\!\int_\Omega (\overline{\boldsymbol{u}}-\boldsymbol{u}_Q)\cdot\boldsymbol{\xi}^{\boldsymbol{h}}\,dx\,dt \;+\; \beta_2\int_0^T\!\!\int_\Omega(\overline{\varphi}-\varphi_Q)\,\eta^{\boldsymbol{h}}\,dx\,dt \;+\; \beta_3\int_\Omega(\overline{\boldsymbol{u}}(T)-\boldsymbol{u}_\Omega)\cdot\boldsymbol{\xi}^{\boldsymbol{h}}(T)\,dx$$
$$+\;\beta_4\int_\Omega(\overline{\varphi}(T)-\varphi_\Omega)\,\eta^{\boldsymbol{h}}(T)\,dx \;+\;\gamma\int_0^T\!\!\int_\Omega \overline{\boldsymbol{v}}\cdot(\boldsymbol{v}-\overline{\boldsymbol{v}})\,dx\,dt \;\ge\; 0 \qquad \forall\,\boldsymbol{v}\in\mathcal{V}_{ad}, \tag{4.60}$$

where $[\boldsymbol{\xi}^{\boldsymbol{h}},\eta^{\boldsymbol{h}}]$ is the unique solution to the linearized system (4.6)–(4.11) *corresponding to $\boldsymbol{h}=\boldsymbol{v}-\overline{\boldsymbol{v}}$.*

The adjoint system and first-order necessary optimality conditions. We now aim to eliminate the variables $[\boldsymbol{\xi}^{\boldsymbol{h}},\eta^{\boldsymbol{h}}]$ from the variational inequality (4.60). To this end, let us introduce the following *adjoint system*:

$$\widetilde{\boldsymbol{p}}_t \;=\; -\;2\,\mathrm{div}\big(\nu(\overline{\varphi})\,D\widetilde{\boldsymbol{p}}\big)-(\overline{\boldsymbol{u}}\cdot\nabla)\,\widetilde{\boldsymbol{p}}+(\widetilde{\boldsymbol{p}}\cdot\nabla^T)\,\overline{\boldsymbol{u}}\;+\;\widetilde{q}\,\nabla\overline{\varphi}-\beta_1(\overline{\boldsymbol{u}}-\boldsymbol{u}_Q), \quad \text{in } Q, \tag{4.61}$$

$$\begin{aligned}\widetilde{q}_t \;=\; &-\mathrm{div}\big(\lambda(\overline{\varphi})\nabla\widetilde{q}\big)\;-\;m'(\overline{\varphi})\nabla(K*\overline{\varphi})\cdot\nabla\widetilde{q}\\ &-\nabla K\dot{*}(m(\overline{\varphi})\nabla\widetilde{q})\;+\;\lambda'(\overline{\varphi})\nabla\overline{\varphi}\cdot\nabla\widetilde{q}\;-\;(\nabla K*\overline{\varphi})\cdot\widetilde{\boldsymbol{p}}\;-\;\nabla K*(\overline{\varphi}\,\widetilde{\boldsymbol{p}})\\ &+\;2\nu'(\overline{\varphi})\,D\overline{\boldsymbol{u}}:D\widetilde{\boldsymbol{p}}\;-\;\overline{\boldsymbol{u}}\cdot\nabla\widetilde{q}\;-\;\beta_2(\overline{\varphi}-\varphi_Q), \quad \text{in } Q,\end{aligned} \tag{4.62}$$

$$\mathrm{div}(\widetilde{\boldsymbol{p}})=0, \quad \text{in } Q, \tag{4.63}$$

$$\widetilde{\boldsymbol{p}}=\boldsymbol{0}, \qquad \frac{\partial\widetilde{q}}{\partial\boldsymbol{n}}=0, \quad \text{on } \Sigma, \tag{4.64}$$

$$\widetilde{\boldsymbol{p}}(T)=\beta_3(\overline{\boldsymbol{u}}(T)-\boldsymbol{u}_\Omega), \quad \widetilde{q}(T)=\beta_4(\overline{\varphi}(T)-\varphi_\Omega), \quad \text{in } \Omega. \tag{4.65}$$

Here, we have set

$$(\nabla K\dot{*}\nabla\widetilde{q})(x) := \int_\Omega \nabla K(x-y)\cdot\nabla\widetilde{q}(y)\,dy \quad \text{for a.e. } x\in\Omega\,.$$

Recalling that $\boldsymbol{u}_\Omega \in G_{div}$ and $\varphi_\Omega \in H$, we expect the solution to (4.61)–(4.65) to have the regularity properties

$$\widetilde{\boldsymbol{p}} \in H^1(0,T;V_{div}')\cap C([0,T];G_{div})\cap L^2(0,T;V_{div}), \tag{4.66}$$
$$\widetilde{q} \in H^1(0,T;V')\cap C([0,T];H)\cap L^2(0,T;V). \tag{4.67}$$

Hence, the pair $[\widetilde{\boldsymbol{p}}, \widetilde{q}\,]$ must be understood as a solution to the weak formulation of the system (4.61)–(4.65). In particular, the following identities must hold:

$$\langle \widetilde{\boldsymbol{p}}_t, \boldsymbol{z}\rangle_{V_{div}} \;=\; 2\left(\nu(\overline{\varphi})D\widetilde{\boldsymbol{p}}, D\boldsymbol{z}\right) \;-\; b(\overline{\boldsymbol{u}},\widetilde{\boldsymbol{p}},\boldsymbol{z}) + b(\boldsymbol{z},\overline{\boldsymbol{u}},\widetilde{\boldsymbol{p}}) + \left(\widetilde{q}\nabla\overline{\varphi},\boldsymbol{z}\right) - \beta_1\big((\overline{\boldsymbol{u}}-\boldsymbol{u}_Q),\boldsymbol{z}\big), \quad (4.68)$$

$$\begin{aligned}\langle \widetilde{q}_t, \chi\rangle_V \;=\; & \left(\lambda(\overline{\varphi})\nabla\widetilde{q},\nabla\chi\right) \;-\; \left(m'(\overline{\varphi})\nabla(K*\overline{\varphi})\cdot\nabla\widetilde{q},\chi\right) \\ & - \left(\nabla K \dot{*} (m(\overline{\varphi})\nabla\widetilde{q}),\chi\right) \;+\; \left(\lambda'(\overline{\varphi})\nabla\overline{\varphi}\cdot\nabla\widetilde{q},\chi\right) \;-\; \left((\nabla K*\overline{\varphi})\cdot\widetilde{\boldsymbol{p}},\chi\right) \\ & - \left(\nabla K*(\overline{\varphi}\,\widetilde{\boldsymbol{p}}),\chi\right) \;+\; 2\left(\nu'(\overline{\varphi})\,D\overline{\boldsymbol{u}} : D\widetilde{\boldsymbol{p}},\chi\right) \\ & - \left(\overline{\boldsymbol{u}}\cdot\nabla\widetilde{q},\chi\right) \;-\; \left(\beta_2(\overline{\varphi}-\varphi_Q),\chi\right), \end{aligned} \quad (4.69)$$

for every $\boldsymbol{z}\in V_{div}$, every $\chi\in V$ and almost every $t\in(0,T)$. We have the following result.

Proposition 2. *Let the assumptions of Lemma 4 hold true. Then the adjoint system* (4.61)–(4.65) *has a unique weak solution* $[\widetilde{\boldsymbol{p}}, \widetilde{q}]$ *satisfying* (4.66)–(4.67).

Proof. We only give a sketch of the proof, which can be carried out arguing as the proof of Proposition 1. In particular, we omit the details of the construction of an approximating Faedo–Galerkin scheme and only derive the basic a priori estimates. To this end, we take $\boldsymbol{z} = \widetilde{\boldsymbol{p}}(t)\in V_{div}$ in (4.68) and $\chi = \widetilde{q}(t)\in H$ in (4.69), and add the resulting equations. Omitting the argument t again, we now estimate all the terms on the right-hand side of the resulting identity. We denote by C positive constants that only depend on the global data and on $[\overline{\boldsymbol{u}},\overline{\varphi}]$, while C_σ stands for positive constants that also depend on the quantity indicated by the index σ. Using the elementary Young's inequality (1.13), the Hölder and Gagliardo–Nirenberg inequalities (cf. (2.1)), Young's inequality for convolution integrals, as well as the assumptions and the global bound (2.15), we obtain (with positive constants ϵ and ϵ' that will be fixed later) the following series of estimates:

$$\Big|\int_\Omega (\widetilde{\boldsymbol{p}}\cdot\nabla^T)\overline{\boldsymbol{u}}\cdot\widetilde{\boldsymbol{p}}\,dx\Big| \;\le\; \|\widetilde{\boldsymbol{p}}\|\,\|\nabla\overline{\boldsymbol{u}}\|_{L^4(\Omega)^{2\times2}}\,\|\widetilde{\boldsymbol{p}}\|_{L^4(\Omega)^2} \;\le\; \epsilon\,\|\nabla\widetilde{\boldsymbol{p}}\|^2 + C_\epsilon\,\|\overline{\boldsymbol{u}}\|^2_{H^2(\Omega)^2}\,\|\widetilde{\boldsymbol{p}}\|^2, \quad (4.70)$$

$$\Big|\int_\Omega \widetilde{q}\,\nabla\overline{\varphi}\cdot\widetilde{\boldsymbol{p}}\,dx\Big| \;\le\; \|\widetilde{q}\|\,\|\nabla\overline{\varphi}\|_{L^4(\Omega)^2}\,\|\widetilde{\boldsymbol{p}}\|_{L^4(\Omega)^2} \;\le\; \epsilon\,\|\nabla\widetilde{\boldsymbol{p}}\|^2 + C_\epsilon\,\|\widetilde{q}\|^2, \quad (4.71)$$

$$\Big|\beta_1\int_\Omega (\overline{\boldsymbol{u}}-\overline{\boldsymbol{u}}_Q)\cdot\widetilde{\boldsymbol{p}}\,dx\Big| \;\le\; \beta_1\,\|\overline{\boldsymbol{u}}-\overline{\boldsymbol{u}}_Q\|\,\|\widetilde{\boldsymbol{p}}\| \;\le\; \|\widetilde{\boldsymbol{p}}\|^2 + \frac{\beta_1^2}{4}\,\|\overline{\boldsymbol{u}}-\overline{\boldsymbol{u}}_Q\|^2, \quad (4.72)$$

$$\begin{aligned}&\Big|\int_\Omega m'(\overline{\varphi})\,\widetilde{q}\,\nabla(K*\overline{\varphi})\cdot\nabla\widetilde{q}\,dx\Big| \;\le\; m'_\infty\,\|\widetilde{q}\|_{L^4(\Omega)}\,\|\nabla(K*\overline{\varphi})\|_{L^4(\Omega)^2}\,\|\nabla\widetilde{q}\| \\ &\le\; C_{m,K}\left(\|\widetilde{q}\| + \|\widetilde{q}\|^{1/2}\,\|\nabla\widetilde{q}\|^{1/2}\right)\|\nabla\widetilde{q}\| \;\le\; \epsilon'\,\|\nabla\widetilde{q}\|^2 + C_{\epsilon',m,K}\,\|\widetilde{q}\|^2\,, \end{aligned} \quad (4.73)$$

$$\Big|\int_\Omega \widetilde{q}\,\nabla K\dot{*}(m(\overline{\varphi})\nabla\widetilde{q})\,dx\Big| \;\le\; C_K\,\|\widetilde{q}\|\,\|m(\overline{\varphi})\nabla\widetilde{q}\| \;\le\; \epsilon'\,\|\nabla\widetilde{q}\|^2 + C_{\epsilon',m,K}\,\|\widetilde{q}\|^2\,, \quad (4.74)$$

$$\Big|\int_\Omega \lambda'(\overline{\varphi})\,\widetilde{q}\,\nabla\overline{\varphi}\cdot\nabla\widetilde{q}\,dx\Big| \;\le\; \lambda'_\infty\,\|\widetilde{q}\|_{L^4(\Omega)}\,\|\nabla\overline{\varphi}\|_{L^4(\Omega)^2}\,\|\nabla\widetilde{q}\| \quad (4.75)$$

$$\le\; C_\lambda\left(\|\widetilde{q}\| + \|\widetilde{q}\|^{1/2}\,\|\nabla\widetilde{q}\|^{1/2}\right)\|\nabla\widetilde{q}\| \;\le\; \epsilon'\,\|\nabla\widetilde{q}\|^2 + C_{\epsilon',\lambda}\,\|\widetilde{q}\|^2\,, \quad (4.76)$$

$$\Big|\int_\Omega (\nabla K*\overline{\varphi})\cdot\widetilde{\boldsymbol{p}}\,\widetilde{q}\,dx\Big| \;\le\; C_K(\|\widetilde{\boldsymbol{p}}\|^2 + \|\widetilde{q}\|^2)\,, \quad (4.77)$$

$$\Big| \int_\Omega \widetilde{q}\, \nabla K * (\overline{\varphi}\, \widetilde{\boldsymbol{p}})\, dx \Big| \,\le\, C_K(\|\widetilde{\boldsymbol{p}}\|^2 + \|\widetilde{q}\|^2)\,, \tag{4.78}$$

$$\Big| 2 \int_\Omega \big(\nu'(\overline{\varphi})\, D\overline{\boldsymbol{u}} : D\widetilde{\boldsymbol{p}}\big)\, \widetilde{q}\, dx \Big| \,\le\, C_\nu\, \|D\overline{\boldsymbol{u}}\|_{L^4(\Omega)^{2\times 2}}\, \|D\widetilde{\boldsymbol{p}}\|\, \|\widetilde{q}\|_{L^4(\Omega)} \tag{4.79}$$

$$\le\, C_\nu\, \|D\overline{\boldsymbol{u}}\|_{L^4(\Omega)^{2\times 2}}\, \|D\widetilde{\boldsymbol{p}}\| \left(\|\widetilde{q}\| + \|\widetilde{q}\|^{1/2}\, \|\nabla\widetilde{q}\|^{1/2}\right) \tag{4.80}$$

$$\le\, \epsilon\, \|\nabla\widetilde{\boldsymbol{p}}\|^2 + \epsilon'\, \|\nabla\widetilde{q}\|^2 + C_{\epsilon,\epsilon',\nu}\, \big(1 + \|\overline{\boldsymbol{u}}\|^2_{H^2(\Omega)^2}\big)\, \|\widetilde{q}\|^2, \tag{4.81}$$

$$\Big| \beta_2 \int_\Omega (\overline{\varphi} - \varphi_Q)\, \widetilde{q}\, dx \Big| \,\le\, \beta_2\, \|\overline{\varphi} - \varphi_Q\|\, \|\widetilde{q}\| \,\le\, \|\widetilde{q}\|^2 + \frac{\beta_2^2}{4}\, \|\overline{\varphi} - \varphi_Q\|^2\,. \tag{4.82}$$

Choosing now $\epsilon > 0$ and $\epsilon' > 0$ small enough (in particular, $3\,\epsilon \le \nu_1/2$ and $5\,\epsilon' \le \alpha_0/2$), we arrive at the following differential inequality:

$$\frac{d}{dt}\left(\|\widetilde{\boldsymbol{p}}\|^2 + \|\widetilde{q}\|^2\right) \,+\, \theta_1 \left(\|\widetilde{\boldsymbol{p}}\|^2 + \|\widetilde{q}\|^2\right) + \theta_2 \,\ge\, \nu_1\, \|\nabla\widetilde{\boldsymbol{p}}\|^2 + \alpha_0\, \|\nabla\widetilde{q}\|^2, \tag{4.83}$$

where the functions $\theta_1, \theta_2 \in L^1(0,T)$ are given by

$$\theta_1(t) := C\left(1 + \|\overline{\boldsymbol{u}}(t)\|^2_{H^2(\Omega)^2}\right), \qquad \theta_2(t) := \beta_1^2\, \|(\overline{\boldsymbol{u}} - \boldsymbol{u}_Q)(t)\|^2 + \,\beta_2^2\, \|(\overline{\varphi} - \varphi_Q)(t)\|^2.$$

By applying the (backward) Gronwall lemma to (4.83), we obtain

$$\begin{aligned} \|\widetilde{\boldsymbol{p}}(t)\|^2 + \|\widetilde{q}(t)\|^2 &\le \Big[\|\widetilde{\boldsymbol{p}}(T)\|^2 + \|\widetilde{q}(T)\|^2 + \int_t^T \theta_2(\tau)d\tau\Big] e^{\int_t^T \theta_1(\tau)d\tau} \\ &\le\, C\,\Big[\|\widetilde{\boldsymbol{p}}(T)\|^2 + \|\widetilde{q}(T)\|^2 + \beta_1^2\, \|\overline{\boldsymbol{u}} - \boldsymbol{u}_Q\|^2_{L^2(0,T;G_{div})} + \beta_2^2\, \|\overline{\varphi} - \varphi_Q\|^2_{L^2(Q)}\Big], \end{aligned}$$

for all $t \in [0,T]$. From this estimate, and by integrating (4.83) over $[t,T]$, we can deduce the estimates for $\widetilde{\boldsymbol{p}}$ and $\widetilde{q}$ in $C^0([0,T];G_{div}) \cap L^2(0,T;V_{div})$ and in $C^0([0,T];H) \cap L^2(0,T;V)$, respectively. A comparison argument in (4.61) and (4.62) entails the estimates for $\widetilde{\boldsymbol{p}}_t$ and $\widetilde{q}_t$ in $L^2(0,T;V'_{div})$ and in $L^2(0,T;V')$, respectively. We therefore can deduce the existence of a weak solution to system (4.61)–(4.65) satisfying (4.66)–(4.67). The proof of uniqueness is rather straightforward, and we may allow ourselves to leave it to the interested reader. □

Using the adjoint system, we can now eliminate $\boldsymbol{\xi}^{\boldsymbol{h}}, \eta^{\boldsymbol{h}}$ from (4.60). Indeed, we have the following result.

Theorem 5. *Let the assumptions of Lemma 4 hold true. If $\overline{\boldsymbol{v}} \in \mathcal{V}_{ad}$ is an optimal control for* **(CP)** *with associated state $[\overline{\boldsymbol{u}}, \overline{\varphi}] = \mathcal{S}(\overline{\boldsymbol{v}})$ and adjoint state $[\widetilde{\boldsymbol{p}}, \widetilde{q}]$, then the following variational inequality holds true:*

$$\gamma \int_0^T\!\!\int_\Omega \overline{\boldsymbol{v}} \cdot (\boldsymbol{v} - \overline{\boldsymbol{v}})\, dx\, dt \,+ \int_0^T\!\!\int_\Omega \widetilde{\boldsymbol{p}} \cdot (\boldsymbol{v} - \overline{\boldsymbol{v}})\, dx\, dt \,\ge\, 0\,, \quad \forall\, \boldsymbol{v} \in \mathcal{V}_{ad}. \tag{4.84}$$

Proof. Note that, thanks to (4.65), we have for the sum (that we denote by $\mathcal{I}$) of the first four terms on the left-hand side of (4.60) the identity

$$\mathcal{I} := \beta_1 \int_0^T\!\!\int_\Omega (\overline{\boldsymbol{u}} - \boldsymbol{u}_Q) \cdot \boldsymbol{\xi}^{\boldsymbol{h}}\, dx\, dt + \beta_2 \int_0^T\!\!\int_\Omega (\overline{\varphi} - \varphi_Q)\eta^{\boldsymbol{h}}\, dx\, dt + \beta_3 \int_\Omega (\overline{\boldsymbol{u}}(T) - \boldsymbol{u}_\Omega) \cdot \boldsymbol{\xi}^{\boldsymbol{h}}(T)\, dx$$

$$+\,\beta_4\int_\Omega(\overline{\varphi}(T)-\varphi_\Omega)\eta^{\boldsymbol{h}}(T)\,dx\;=\;\beta_1\!\int_0^T\!\!\int_\Omega(\overline{\boldsymbol{u}}-\boldsymbol{u}_Q)\cdot\boldsymbol{\xi}^{\boldsymbol{h}}\,dx\,dt\;+\beta_2\int_0^T\!\!\int_\Omega(\overline{\varphi}-\varphi_Q)\eta^{\boldsymbol{h}}\,dx\,dt$$
$$+\int_0^T\big(\langle\widetilde{\boldsymbol{p}}_t(t),\boldsymbol{\xi}^{\boldsymbol{h}}(t)\rangle_{V_{div}}+\langle\boldsymbol{\xi}^{\boldsymbol{h}}_t(t),\widetilde{\boldsymbol{p}}(t)\rangle_{V_{div}}\big)\,dt+\int_0^T\big(\langle\widetilde{q}_t(t),\eta^{\boldsymbol{h}}(t)\rangle_V+\langle\eta^{\boldsymbol{h}}_t(t),\widetilde{q}(t)\rangle_V\big)\,dt\,. \tag{4.85}$$

Recalling the weak formulation of the linearized system (4.6)–(4.11) for $\boldsymbol{h}=\boldsymbol{v}-\overline{\boldsymbol{v}}$, we obtain, omitting the argument t,

$$\begin{aligned}\langle\boldsymbol{\xi}^{\boldsymbol{h}}_t,\widetilde{\boldsymbol{p}}\rangle_{V_{div}}\;=\;&-2\left(\nu(\overline{\varphi})\,D\boldsymbol{\xi}^{\boldsymbol{h}},D\widetilde{\boldsymbol{p}}\right)\;-\;2\left(\nu'(\overline{\varphi})\,\eta^{\boldsymbol{h}}\,D\overline{\boldsymbol{u}},D\widetilde{\boldsymbol{p}}\right)\;-\;b(\overline{\boldsymbol{u}},\boldsymbol{\xi}^{\boldsymbol{h}},\widetilde{\boldsymbol{p}})\\&-\;b(\boldsymbol{\xi}^{\boldsymbol{h}},\overline{\boldsymbol{u}},\widetilde{\boldsymbol{p}})\;+\;\left(\eta^{\boldsymbol{h}}(\nabla K*\overline{\varphi}),\widetilde{\boldsymbol{p}}\right)\;+\;\left(\overline{\varphi}\,(\nabla K*\eta^{\boldsymbol{h}}),\widetilde{\boldsymbol{p}}\right)\;+\;(\boldsymbol{v}-\overline{\boldsymbol{v}},\widetilde{\boldsymbol{p}})\,,\end{aligned} \tag{4.86}$$

$$\begin{aligned}\langle\eta^{\boldsymbol{h}}_t,\widetilde{q}\rangle_V\;=\;&(\overline{\boldsymbol{u}}\,\eta^{\boldsymbol{h}},\nabla\widetilde{q})\;+\;(\xi^{\boldsymbol{h}}\,\overline{\varphi},\nabla\widetilde{q})\;-\;\left(\lambda(\overline{\varphi})\nabla\eta^{\boldsymbol{h}},\nabla\widetilde{q}\right)\\&+\;\left(m'(\overline{\varphi})\,\eta^{\boldsymbol{h}}\,\nabla(K*\overline{\varphi}),\nabla\widetilde{q}\right)\;+\;\left(m(\overline{\varphi})(\nabla K*\eta^{\boldsymbol{h}}),\nabla\widetilde{q}\right)\\&-\;\left(\eta^{\boldsymbol{h}}\lambda'(\overline{\varphi})\,\nabla\overline{\varphi},\nabla\widetilde{q}\right).\end{aligned} \tag{4.87}$$

We now insert these two identies, as well as (4.68) and (4.69), in (4.85). Integrating by parts, using the boundary conditions for the involved quantities and the fact that $\boldsymbol{\xi}^{\boldsymbol{h}}$ and $\widetilde{\boldsymbol{p}}$ are divergence free vector fields, and observing that the symmetry of the kernel K implies the identity

$$\int_\Omega(K*\eta)\,\omega\,dx\;=\;\int_\Omega(K*\omega)\,\eta\,dx\,,\quad\forall\,\eta,\omega\in H,$$

we arrive at the conclusion that

$$\mathcal{I}=\int_0^T\!\!\int_\Omega\widetilde{\boldsymbol{p}}\cdot(\boldsymbol{v}-\overline{\boldsymbol{v}})\,dx\,dt\,.$$

Therefore, (4.84) follows from this identity and (4.60). ☐

Remark 1. System (2.7)–(2.12) subject to (1.10), written for $[\overline{\boldsymbol{u}},\overline{\varphi}]$, the adjoint system (4.61)–(4.65), and the variational inequality (4.84), form together the first-order necessary optimality conditions. Moreover, since $\mathcal{V}_{ad}$ is a nonempty, closed and convex subset of $L^2(Q)^2$, the condition (4.84) is, in the case $\gamma>0$, equivalent to the following condition for the optimal control $\overline{\boldsymbol{v}}\in\mathcal{V}_{ad}$,

$$\overline{\boldsymbol{v}}=\mathbb{P}_{\mathcal{V}_{ad}}\Big(-\frac{\widetilde{\boldsymbol{p}}}{\gamma}\Big),$$

where $\mathbb{P}_{\mathcal{V}_{ad}}$ is the orthogonal projector in $L^2(Q)^2$ onto $\mathcal{V}_{ad}$. From standard arguments it follows from this projection property the pointwise condition

$$\overline{v}_i(x,t)\;=\;\max\,\left\{v_{a,i}(x,t),\,\min\,\left\{-\gamma^{-1}\,\widetilde{p}_i(x,t),\,v_{b,i}(x,t)\right\}\right\},i=1,2,\quad\text{for a.e. }(x,t)\in Q\,.$$

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