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Alternative iterative methods for nonexpansive mappings, rates of convergence and applications

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Abstract

Alternative iterative methods for a nonexpansive mapping in a Banach space are proposed and proved to be convergent to a common solution to a fixed point problem and a variational inequality. We give rates of asymptotic regularity for such iterations using proof-theoretic techniques. Some applications of the convergence results are presented.

Keywords: nonexpansive mapping, iterative algorithm, fixed point, viscosity approximation, uniformly smooth Banach space, rates of asymptotic regularity, proof mining, variational inequality problem, accretive operator.

Mathematics Subject Classification: 47H06, 47H09, 47H10, 47J20, 03F60.

1

1   Introduction

Many problems arising in different areas of mathematics such as optimization, variational analysis and game theory, can be formulated as the fixed point problem:

$$\text{find } x \in X \text{ such that } x = Tx, \tag{1.1}$$

where T is a nonexpansive mapping defined on a metric space X, i.e., T satisfies the property $d(Tx, Ty) \leq d(x, y)$, for all $x, y \in X$.

For instance, let $A : C \to H$ be a nonlinear operator where $C \subset H$ is a closed convex subset of a Hilbert space. The variational inequality problem associated to A, VIP(A, C), is formulated as finding a point $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \quad \forall x \in C. \tag{1.2}$$

It is well-known that the VIP(A, C) is equivalent to the problem of finding the fixed point

$$x^* = P_C(x^* - \lambda Ax^*), \tag{1.3}$$

where $\lambda > 0$ and P_C is the metric projection onto C, which is a nonexpansive mapping in this case. Besides, if $f : C \to \mathbb{R}$ is a differentiable convex function and we denote by A the gradient operator of f, then (1.2) is the optimality condition for the minimization problem

$$\min_{x \in C} f(x). \tag{1.4}$$

Bearing in mind that the iterative methods for approximating a fixed point of a nonexpansive mapping can be applied to find a solution to a variational inequality, zeros of an accretive operator and a minimizer of a convex function, in the recent years the study of the convergence of those methods has received a great deal of attention. Basically two types of iterative algorithms have been investigated: Mann algorithm and Halpern algorithm.

In the following, let X be a real Banach space, $C \subset X$ a closed convex subset and $T : C \to C$ a nonexpansive mapping with fixed point set $F = \{x \in C : x = Tx\} \neq \emptyset$.

Mann algorithm generates a sequence according to the following recursive manner:

$$x_{n+1} = (1 - t_n)x_n + t_n Tx_n, \quad n \geq 0, \tag{1.5}$$

where the initial guess $x_0 \in C$ and $\{t_n\}$ is a sequence in $(0, 1)$.

Halpern algorithm generates a sequence via the recursive formula:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0 \tag{1.6}$$

where $x_0, u \in C$ are arbitrary and the sequence $\{\alpha_n\} \subset (0, 1)$.

Whenever a fixed point of the mapping T exists, while Halpern algorithm strongly converges, we just get weak convergence for Mann algorithm, as was established in [11] thanks to a counterexample. The references [29, 21, 15, 35, 10, 48] can be consulted for convergence results of Mann algorithm. Some modifications have been proposed in [32, 16] to get strong convergence. As for Halpern algorithm, see [14, 25, 44, 38, 47, 6, 42, 27] and references therein for studies dedicated to its convergence.

Another iterative approach to solving the problem (1.1) which may have multiple solutions, is to replace it by a family of perturbed problems admitting a unique solution, and then to get a particular original solution as the limit of these perturbed solutions as the perturbation vanishes. For example, Browder [2, 3] proved that if the underlying space H is Hilbert, then, given $u \in H$ and $t \in (0, 1)$, the approximating curve $\{x_t\}$ defined by

$$x_t = tu + (1-t)Tx_t \tag{1.7}$$

strongly converges, as $t \to 0$, to the fixed point of T closest to u from F. Browder's result has been generalized and extended to a more general class of Banach spaces [36, 40, 33]. Combettes and Hirstoaga [8] introduced a new type of approximating curve for fixed point problems in the setting of a Hilbert space. This curve defined by the implicit formula

$$x_t = T(tu + (1-t)x_t), \tag{1.8}$$

was proved to converge to the best approximation to u from F. In [50] Xu studied the behavior of $\{x_t\}$ defined by (1.8) in the setting of a Banach space X and discretized this regularization method studying the strong convergence of the explicit algorithm

$$x_{n+1} = T(\alpha_n u + (1-\alpha_n)x_n), \tag{1.9}$$

where $\{\alpha_n\} \subset (0, 1)$. Moreover, he proved that the convergence point is the image of u under the unique sunny nonexpansive retraction Q from X to F (see, for instance, [34, 36]).

On the other hand, Moudafi in [31] introduced the viscosity approximation method for nonexpansive mappings, which generalizes Browder's (1.7) and Halpern (1.6) iterations, by using a contraction Φ instead of an arbitrary point u. The convergence of the implicit and explicit algorithms has been the subject of many papers because under suitable conditions these iterations strongly converge to the unique solution $q \in F$ to the variational inequality

$$\langle (I - \Phi)q, J(x - q)\rangle \geq 0 \quad \forall x \in F, \tag{1.10}$$

where J is a duality mapping, i.e., q is the unique fixed point of the contraction $Q \circ \Phi$. This fact allows us to apply this method to convex optimization, linear programming and monotone inclusions. See [49, 41, 43] and references therein for convergence results regarding viscosity approximation methods.

In this paper, we analyze the behavior of a new approximating curve in the setting of Banach spaces, which constitutes a hybrid method of the ones presented by Combettes and Hirstoaga (1.8) and Moudafi. This curve is defined by

$$x_t = T(t\Phi(x_t) + (1-t)x_t), \tag{1.11}$$

for some contraction Φ, that is, for any $t \in (0, 1)$ x_t is the unique fixed point of the contraction $T_t = T(t\Phi + (1-t)I)$. The discretized iteration

$$x_{n+1} = T(\alpha_n \Phi(x_n) + (1-\alpha_n)x_n), \tag{1.12}$$

is also considered and studied under suitable conditions on the sequence $\{\alpha_n\} \subset (0, 1)$. From this explicit algorithm we obtain the so-called hybrid steepest descent method

$$x_{n+1} = Tx_n - \alpha_n g(Tx_n). \tag{1.13}$$

This iterative method was suggested by Yamada [45] as an extension of viscosity approximation methods for solving the variational inequality VIP(g, F) (1.2) in the case when g is strongly monotone and Lipschitz continuous, and F is the fixed point set of a mapping T which belongs to a subclass of the quasi-nonexpansive mappings (also see [46, 28]). We will get the convergence of the algorithm (1.13) for a nonexpansive mapping T, just requiring $I - \mu g$ to be a contraction for some $\mu > 0$, which it is satisfied in the particular case when g is strongly monotone and Lipschitz continuous.

Asymptotic regularity is a very important concept in metric fixed point theory. It was already implicit in [21, 39, 9], but it was formally introduced by Browder and Petryshyn in [4]. In our setting, the mapping T is called asymptotically regular if for all $x \in C$

$$\lim_{n \to \infty} \|x_n - Tx_n\| = 0.$$

Effective rates of asymptotic regularity for both Mann and Halpern iterations have been obtained (see [17, 18, 19, 22, 23]) by applying methods of proof mining. By "proof mining" we mean the logical analysis, using proof-theoretic tools, of mathematical proofs with the aim of extracting relevant information hidden in the proofs. This new information can be both of quantitative nature, such as algorithms and effective bounds, as well as of qualitative nature, such as uniformities in the bounds or weakening the premises. Thus, even if one is not particularly interested in the numerical details of the bounds themselves, in many cases such explicit bounds immediately show the independence of the quantity in question from certain input data. A comprehensive reference for proof mining is Kohlenbach's book [20]. One of the aims of this paper is to give effective rates of asymptotic regularity for the algorithm (1.12) of nonexpansive mappings in the framework of normed spaces.

The organization of the paper is as follows. In section 2 we introduce some preliminary results and present a technical lemma with regards to the behavior of the sequence defined by the algorithm (1.12), which will be useful for the proof of the convergence of the sequence and the evaluation of the rate of asymptotically regularity in following sections. Section 3 contains the main results about the strong convergence of both implicit (1.11) and explicit (1.12) algorithms to the unique solution to the variational inequality (1.10) in the setting of uniformly smooth Banach spaces, and also in the framework of reflexive Banach spaces with weakly continuous normalized duality mapping in the case of the implicit iteration. Section 4 is devoted to the rate of asymptotic regularity for the iterations (1.12) of nonexpansive mappings in normed spaces. Finally, in section 5 we give examples of how to apply the main results of section 3 to find a solution to a variational inequality or a zero of an accretive operator.

2   Preliminaries

Let X be a real Banach space with norm $\| \cdot \|$ and dual space X^*. For any $x \in X$ and $x^* \in X^*$ we denote $x^*(x) = \langle x, x^* \rangle$. Given a nonempty closed convex subset $C \subset X$, $\Phi : C \to C$ will be a a ρ-contraction and $T : C \to C$ a nonexpansive self-mapping with nonempty fixed point set $F := \{x \in C : Tx = x\}$.

We include some brief knowledge about geometry of Banach spaces which can be found

in more details in [7]. The *normalized duality mapping* $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}. \tag{2.1}$$

It is known that

$$J(x) = \partial(\|x\|),$$

where $\partial(\|x\|)$ is the subdifferential of $\|\cdot\|$ at x in the sense of convex analysis. Thus, for any $x, y \in X$, we have the subdifferential inequality

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad j(x + y) \in J(x + y). \tag{2.2}$$

A Banach space X is said to be *smooth* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.3}$$

exists for each $x, y \in S_X$, where S_X is the unit sphere of X, i.e., $S_X = \{v \in X : \|v\| = 1\}$. When this is the case, the norm of X is said to be *Gâteaux differentiable*. If for each $y \in X$ the limit (2.3) is uniformly attained for $x \in X$, we say that the norm of X is *uniformly Gâteaux differentiable*, and we say that X is *uniformly smooth* if the limit (2.3) is attained uniformly for any $x, y \in S_X$.

It is known that a Banach space X is smooth if and only if the duality mapping J is single-valued, and that X is uniformly smooth if and only if the duality mapping J is single-valued and norm-to-norm uniformly continuous on bounded sets of X. Moreover, if X has a uniformly Gâteaux differentiable norm then J is norm-to-weak* uniformly continuous on bounded sets of X.

Following Browder [3] we say that the duality mapping J is *weakly sequentially continuous* (or simply *weakly continuous*) if J is single-valued and weak-to-weak* sequentially continuous; i.e., if $x_n \rightharpoonup x$ in X, then $J(x_n) \rightharpoonup^* J(x)$ in X^*. A Banach space with weakly continuous duality mapping is known (see [24]) to satisfy *Opial's property* (i.e., whenever $x_n \rightharpoonup x$ and $y \ne x$, we have $\overline{\lim}\|x_n - x\| < \overline{\lim}\|x_n - y\|$), and this fact implies (see [12]) that X satisfies the *Demiclosedness principle*: if C is a closed convex subset of X and T is a nonexpansive self-mapping, then $x_n \rightharpoonup x$ and $(I - T)x_n \to y$ imply that $(I - T)x = y$.

Consider a subset $D \subset C$ and a mapping $Q : C \to D$. We say that Q is a *retraction* provided $Qx = x$ for any $x \in D$. The retraction Q is said to be *sunny* if it satisfies the property: $Q(x + t(x - Qx)) = Qx$ whenever $x + t(x - Qx) \in C$, where $x \in C$ and $t \ge 0$.

Lemma 2.1. *[5, 34, 13] Let X be a smooth Banach space and $D \subset C$ be nonempty closed convex subsets of X. Given a retraction $Q : C \to D$, the following three statements are equivalent:*

(a) Q is sunny and nonexpansive.

(b) $\|Qx - Qy\|^2 \le \langle x - y, J(Qx - Qy)\rangle$ for all $x, y \in C$.

(c) $\langle x - Qx, J(y - Qx)\rangle \le 0$ for all $x \in C$ and $y \in D$.

Consequently, there is at most one sunny nonexpansive retraction from C onto D.

In some circumstances, we can construct the sunny nonexpansive retraction. For the nonexpansive mapping T with fixed point set F, an arbitrary $u \in C$ and $t \in (0,1)$, let z_t be the unique fixed point of the contraction $z \mapsto tu + (1-t)Tz$ for $z \in C$; that is, z_t is the unique solution in C to the fixed point equation:

$$z_t = tu + (1-t)Tz_t. \tag{2.4}$$

It is natural to study the behavior of the net $\{z_t\}$ as $t \to 0^+$. However, it is unclear if the strong $\lim_{t \to 0^+} z_t$ always exists in a general Banach space. The answer is yes in some classes of smooth Banach spaces and then the limit defines the sunny nonexpansive retraction from C onto F. Those Banach spaces where the net $\{z_t\}$ strongly converges are said to have *Reich's property* since Reich was the first to show that all uniformly smooth Banach spaces have this property.

Theorem 2.2. *[36, 33] Let X be either a uniformly smooth Banach space or a reflexive Banach space with a weakly continuous duality mapping, C be a nonempty closed convex subset of X, and $T : C \to C$ be a nonexpansive mapping with $F \neq \emptyset$. Then the net $\{z_t\}$ strongly converges as $t \to 0^+$ to a fixed point of T; moreover, the limit*

$$Q(u) := \lim_{t \to 0^+} z_t \tag{2.5}$$

defines the unique sunny nonexpansive retraction from C onto F.

In [37] Reich proved the following two lemmas which will be needed for the convergence results in section 3.

Lemma 2.3. *Let $\{x_n\}$ be a bounded sequence contained in a separable subset D of a Banach space X. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that*

$$\lim_k \|x_{n_k} - y\|$$

exists for all $y \in D$.

Lemma 2.4. *Let D be a closed convex subset a real Banach space X with a uniformly Gâteaux differentiable norm, and let $\{x_n\}$ be a sequence in D such that*

$$f(y) := \lim_n \|x_n - y\|$$

exists for all $y \in D$. If f attains its minimum over D at u, then

$$\limsup_n \langle y - u, j(x_n - u) \rangle \leq 0$$

for all $y \in D$.

The following lemma collects some properties of the iteration (1.12), useful both for proving the convergence of the iteration and for computing the rate of asymptotic regularity.

Lemma 2.5. *Let X be a normed space and $\{x_n\}$ be the sequence defined by the explicit algorithm* (1.12).

(1) For all $n \geq 0$,

$$\|\Phi(x_n) - x_n\| \leq (1 + \rho)\|x_n - x_0\| + \|\Phi(x_0) - x_0\|, \tag{2.6}$$

$$\|x_n - Tx_n\| \leq \|x_{n+1} - x_n\| + \alpha_n\|\Phi(x_n) - x_n\|. \tag{2.7}$$

(2) For all $n \geq 1$,

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1 - (1 - \rho)\alpha_n)\|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| \cdot \|\Phi(x_{n-1}) - x_{n-1}\|. \end{aligned} \tag{2.8}$$

(3) If T has fixed points, then $\{x_n\}$ is bounded for every $x_0 \in C$.

Proof.

(1) Let $n \geq 0$.

$$\begin{aligned} \|\Phi(x_n) - x_n\| &\leq \|\Phi(x_n) - \Phi(x_0)\| + \|\Phi(x_0) - x_0\| + \|x_0 - x_n\| \\ &\leq (1 + \rho)\|x_n - x_0\| + \|\Phi(x_0) - x_0\|. \\ \|x_n - Tx_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - Tx_n\| \\ &= \|x_{n+1} - x_n\| + \|T(\alpha_n\Phi(x_n) + (1 - \alpha_n)x_n) - Tx_n\| \\ &\leq \|x_{n+1} - x_n\| + \|\alpha_n\Phi(x_n) + (1 - \alpha_n)x_n - x_n\| \\ &= \|x_{n+1} - x_n\| + \alpha_n\|\Phi(x_n) - x_n\|. \end{aligned}$$

(2) Let $n \geq 1$.

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|T(\alpha_n\Phi(x_n) + (1 - \alpha_n)x_n) \\ &\quad - T(\alpha_{n-1}\Phi(x_{n-1}) + (1 - \alpha_{n-1})x_{n-1})\| \\ &\leq \|\alpha_n\Phi(x_n) + (1 - \alpha_n)x_n - \alpha_{n-1}\Phi(x_{n-1}) \\ &\quad - (1 - \alpha_{n-1})x_{n-1}\| \\ &= \|\alpha_n(\Phi(x_n) - \Phi(x_{n-1})) + (1 - \alpha_n)(x_n - x_{n-1}) \\ &\quad + (\alpha_n - \alpha_{n-1})(\Phi(x_{n-1}) - x_{n-1})\| \\ &\leq \alpha_n\rho\|x_n - x_{n-1}\| + (1 - \alpha_n)\|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| \cdot \|\Phi(x_{n-1}) - x_{n-1}\| \\ &= (1 - (1 - \rho)\alpha_n)\|x_n - x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| \cdot \|\Phi(x_{n-1}) - x_{n-1}\|. \end{aligned}$$

(3) Let p be a fixed point of T.

$$\begin{aligned} \|x_{n+1} - p\| &= \|T(\alpha_n\Phi(x_n) + (1 - \alpha_n)x_n) - Tp\| \\ &\leq \|\alpha_n\Phi(x_n) + (1 - \alpha_n)x_n - p\| \\ &= \|\alpha_n(\Phi(x_n) - \Phi(p)) + (1 - \alpha_n)(x_n - p) + \alpha_n(\Phi(p) - p)\| \\ &\leq \alpha_n\rho\|x_n - p\| + (1 - \alpha_n)\|x_n - p\| + \alpha_n\|\Phi(p) - p\| \\ &= (1 - (1 - \rho)\alpha_n)\|x_n - p\| + (1 - \rho)\alpha_n\frac{\|\Phi(p) - p\|}{1 - \rho} \\ &\leq \max\left\{\|x_n - p\|, \frac{\|\Phi(p) - p\|}{1 - \rho}\right\}. \end{aligned}$$

By induction, we obtain that for all $n \geq 0$,

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\Phi(p) - p\|}{1 - \rho}\right\},$$

thus $\{x_n\}$ is bounded.

$\square$

Let us recall some terminology that is used for expressing the quantitative results in Section 4. We denote by $\mathbb{Z}_+$ the set of nonnegative integers. Let $k \in \mathbb{Z}_+$ and $\{a_n\}_{n \geq k}$ be a sequence of nonnegative real numbers. If $\{a_n\}$ is convergent, then a function $\omega : (0, \infty) \to \{k, k+1, \ldots\}$ is called a *Cauchy modulus* of $\{a_n\}$ if for all $\varepsilon > 0$,

$$|a_{\omega(\varepsilon)+n} - a_{\omega(\varepsilon)}| < \varepsilon, \quad \forall n \in \mathbb{Z}_+. \tag{2.9}$$

If $\lim\limits_{n \to \infty} a_n = a$, then a function $\omega : (0, \infty) \to \{k, k+1, \ldots\}$ is called a *rate of convergence* of $\{a_n\}$ if for any $\varepsilon > 0$

$$|a_n - a| < \varepsilon, \quad \forall n \geq \omega(\varepsilon). \tag{2.10}$$

If the series $\sum\limits_{n=k}^{\infty} a_n$ is divergent, then a function $\omega : \mathbb{Z}_+ \to \{k, k+1, \ldots\}$ is called a *rate of divergence* of the series if $\sum\limits_{i=k}^{\omega(n)} a_i \geq n$ for all $n \in \mathbb{Z}_+$. If the series $\sum\limits_{n=k}^{\infty} a_n$ converges, then by a *Cauchy modulus* of the series we mean a Cauchy modulus of the sequence of partial sums $\{s_n\}_{n \geq k}$, $s_n = \sum\limits_{i=k}^{n} a_i$.

Lemma 2.6. *[47] Assume $\{a_n\}$ is a sequence of nonnegative real number such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n b_n + \epsilon_n, \qquad n \geq 0,$$

where $\{\gamma_n\}$ and $\{\epsilon_n\}$ are sequences in $(0, 1)$ and $\{b_n\}$ is a sequence in $\mathbb{R}$ such that $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$, $\sum\limits_{n=1}^{\infty} \epsilon_n < \infty$ and either $\limsup_n b_n \leq 0$ or $\sum\limits_{n=1}^{\infty} \gamma_n |b_n| < \infty$.
Then $\lim\limits_{n \to \infty} a_n = 0$

The following lemma is a quantitative version of [26, Lemma 2] and has been proved in [23].

Lemma 2.7. *[23, Lemma 9]*
Let $\{\lambda_n\}_{n \geq 1}$ be a sequence in $(0, 1)$ and $\{a_n\}_{n \geq 1}, \{b_n\}_{n \geq 1}$ be sequences of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \lambda_{n+1})a_n + b_n \quad \text{for all } n \in \mathbb{N}.$$

Assume that $\sum_{n=1}^{\infty} \lambda_n$ *is divergent,* $\sum_{n=1}^{\infty} b_n$ *is convergent and let* $\delta : \mathbb{Z}_+ \to \mathbb{N}$ *be a rate of divergence of* $\sum_{n=1}^{\infty} \lambda_n$ *and* $\gamma : (0, \infty) \to \mathbb{N}$ *be a Cauchy modulus of* $\sum_{n=1}^{\infty} b_n$.

Then $\lim_{n \to \infty} a_n = 0$ *and moreover for all* $\varepsilon \in (0, 2)$

$$a_n < \varepsilon, \quad \forall n \geq h(\gamma, \delta, D, \varepsilon), \tag{2.11}$$

where $D > 0$ *is an upper bound on* $\{a_n\}$ *and*

$$h(\gamma, \delta, D, \varepsilon) = \delta\left(\gamma\left(\frac{\varepsilon}{2}\right) + 1 + \left\lceil \ln\left(\frac{2D}{\varepsilon}\right)\right\rceil\right).$$

3 Convergence of the algorithms

In this section we prove the convergence of the implicit (1.11), explicit (1.12) and hybrid steepest descent (1.13) algorithms in the setting of Banach spaces, which generalize previous results by Combettes and Hirstoaga [8], Xu [50], Yamada [45], and Xu and Kim [51].

Theorem 3.1. *Let* X *be either a reflexive space with weakly continuous normalized duality mapping* J *or a uniformly smooth Banach space,* C *a nonempty closed convex subset of* X, $T : C \to C$ *a nonexpansive mapping with fixed point set* $F \neq \emptyset$ *and* $\Phi : C \to C$ *a* ρ-contraction. *Then the approximating curve* $\{x_t\} \subset C$ *defined by*

$$x_t = T(t\Phi(x_t) + (1 - t)x_t) \tag{3.1}$$

strongly converges, as $t \to 0$, *to the unique solution* $q \in F$ *to the inequality*

$$\langle (\Phi - I)q, J(x - q) \rangle \leq 0, \quad \forall x \in F. \tag{3.2}$$

Proof. We observe that we may assume that C is separable. To see this, consider the set K defined as

$$K_0 := \{q\},$$
$$K_{n+1} := co(K_n \cup T(K_n) \cup \Phi(K_n)),$$
$$K := \overline{\bigcup_n K_n}.$$

Then $K \subset C$ is nonempty, convex, closed, and separable. Moreover K maps into itself under T, Φ and, therefore, $T_t = T(t\Phi + (1 - t)I)$. Then $\{x_t\} \subset K$ and we may replace C with K.

We will prove that $\{x_t\}$ converges, as $t \to 0$, to the point $q \in F$ which is the unique solution to the inequality (3.2).

The sequence $\{x_t\}$ is bounded. Indeed, given $p \in F$,

$$\begin{aligned} \|x_t - p\| &= \|T(t\Phi(x_t) + (1 - t)x_t) - Tp\| \\ &\leq \|t(\Phi(x_t) - \Phi(p)) + (1 - t)(x_t - p) + t(\Phi(p) - p)\| \\ &\leq (t\rho + (1 - t))\|x_t - p\| + t\|\Phi(p) - p\|. \end{aligned}$$

Then, for any $t \in (0,1)$,

$$\|x_t - p\| \leq \frac{1}{1 - \rho}\|\Phi(p) - p\|.$$

Take an arbitrary sequence $\{t_n\} \subset (0,1)$ such that $t_n \to 0$, as $n \to 0$, and denote $x_n = x_{t_n}$ for any $n \geq 0$. Let $\Gamma := \limsup_{n\to\infty}\langle\Phi(q) - q, J(x_n - q)\rangle$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty}\langle\Phi(q) - q, J(x_{n_k} - q)\rangle = \Gamma.$$

Since $\{x_{n_k}\}$ is bounded, by Lemma 2.3, there exists a subsequence, which also will be denoted by $\{x_{n_k}\}$ for the sake of simplicity, satisfying that

$$f(x) := \lim_k \|x_{n_k} - x\|$$

exists for all $x \in C$.

We define the set

$$A := \{z \in C : f(z) = \min_{x\in C} f(x)\}$$

and note that A is a nonempty bounded, closed and convex set since f is a continuous convex function and $\lim_{\|x\|\to\infty} f(x) = \infty$. Moreover,

$$\|x_{n_k} - Tz\| \leq t_{n_k}\|\Phi(x_{n_k}) - x_{n_k}\| + \|x_{n_k} - z\|,$$

for any $z \in C$. This is enough to prove that T maps A into itself.

Since A is a nonempty bounded, closed and convex subset of either a reflexive space with weakly continuous normalized duality mapping or a uniformly smooth Banach space, it has the fixed point property for nonexpansive mappings (see [12]), that is $F \cap A \neq \emptyset$.

If X is reflexive with weakly continuous normalized duality mapping, we can assume that $\{x_{n_k}\}$ has been chosen to be weakly convergent to a point $\tilde{q}$. Since X satisfies Opial's property, we have $A = \{\tilde{q}\}$. Then, since $\tilde{q} \in F$, we obtain that

$$\Gamma = \langle\Phi(q) - q, J(\tilde{q} - q)\rangle \leq 0.$$

If X is uniformly smooth, let $\tilde{q} \in F \cap A$. Then $\tilde{q}$ minimize f over C and, since the norm is uniformly Gâteaux differentiable, by Lemma 2.4,

$$\limsup_k\langle x - \tilde{q}, J(x_{n_k} - \tilde{q})\rangle \leq 0 \qquad (3.3)$$

holds, for all $x \in C$, and in particular for $x = \Phi(\tilde{q})$.

We shall show that $\{x_{n_k}\}$ strongly converges to $\tilde{q}$. Set $\delta_k := \langle\Phi(\tilde{q}) - \tilde{q}, J(t_{n_k}(\Phi(x_{n_k}) - x_{n_k}) + (x_{n_k} - \tilde{q})) - J(x_{n_k} - \tilde{q})\rangle$. Since J is norm-to-weak* uniformly continuous, we see that $\lim_k \delta_k = 0$. Moreover,

$$\begin{aligned}
\|x_{n_k} - \tilde{q}\|^2 &\leq \|t_{n_k}(\Phi(x_{n_k}) - \Phi(\tilde{q})) + (1 - t_{n_k})(x_{n_k} - \tilde{q}) \\
&\quad + t_{n_k}(\Phi(\tilde{q}) - \tilde{q})\|^2 \\
&\leq \|t_{n_k}(\Phi(x_{n_k}) - \Phi(\tilde{q})) + (1 - t_{n_k})(x_{n_k} - \tilde{q})\|^2 \\
&\quad + 2t_{n_k}\delta_k \\
&\quad + 2t_{n_k}\langle\Phi(\tilde{q}) - \tilde{q}, J(x_{n_k} - \tilde{q})\rangle \\
&\leq (1 - (1 - \rho)t_{n_k})\|x_{n_k} - \tilde{q}\|^2 \\
&\quad + 2t_{n_k}\delta_k \\
&\quad + 2t_{n_k}\langle\Phi(\tilde{q}) - \tilde{q}, J(x_{n_k} - \tilde{q})\rangle.
\end{aligned} \qquad (3.4)$$

From (3.4) and by (3.3), we obtain

$$\lim_k \|x_{n_k} - \tilde{q}\|^2 \leq \limsup_k \frac{2}{1-\rho}\left(\delta_k + 2\langle \Phi(\tilde{q}) - \tilde{q}, J(x_{n_k} - \tilde{q})\rangle\right)$$
$$\leq 0.$$

That is $\lim_k x_{n_k} = \tilde{q}$. Since $\tilde{q}$ is a fixed point of T, we also have

$$\Gamma = \lim_k \langle \Phi(q) - q, J(x_{n_k} - q)\rangle = \langle \Phi(q) - q, J(\tilde{q} - q)\rangle \leq 0.$$

By applying (3.4) to $\{x_n\}$ and q, since $\Gamma \leq 0$ in both cases, we obtain

$$\lim_n x_n = q$$

as required. □

Remark 3.2. It is easily seen that the conclusion of Theorem 3.1 remains true if the uniform smoothness assumption of X is replaced with the following two conditions:

(a) X has a uniformly Gâteaux differentiable norm.

(b) X has Reich's property.

Corollary 3.3. *Let H be a Hilbert space, $C \subset H$ a nonempty closed convex subset, $T : C \to C$ a nonexpansive mapping with fixed point set $F \neq \emptyset$ and $\Phi : C \to C$ a ρ-contraction. Then the approximating curve $\{x_t\} \subset C$ defined by (3.1) strongly converges, as $t \to 0$, to the unique solution $q \in F$ to the inequality*

$$\langle (\Phi - I)q, x - q\rangle \leq 0, \ \ \forall x \in F. \tag{3.5}$$

Theorem 3.4. *Let X be a uniformly smooth Banach space, C a nonempty closed convex subset of X, $T : C \to C$ a nonexpansive mapping with $F \neq \emptyset$, $\Phi : C \to C$ a ρ-contraction and $\{\alpha_n\}$ a sequence in $(0,1)$ satisfying*

(H1) $\lim_{n\to\infty} \alpha_n = 0$

(H2) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(H3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1.$

Then, for any $x_0 \in C$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = T(\alpha_n \Phi(x_n) + (1 - \alpha_n)x_n) \tag{3.6}$$

strongly converges to the unique solution $q \in F$ to the inequality

$$\langle (\Phi - I)q, J(x - q)\rangle \leq 0, \ \ \forall x \in F.$$

Proof. Since T has fixed points, by Lemma 2.5 (3) we have that $\{x_n\}$ is bounded, and therefore so are $\{T(x_n)\}$ and $\{\Phi(x_n)\}$. The fact that $\{x_n\}$ is asymptotically regular is a consequence of Lemmas 2.5 and 2.6. Indeed, by hypothesis we have that $\sum_{n=1}^{\infty}(1-\rho)\alpha_{n-1} = \infty$ and either $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$ or

$$\limsup_{n\to\infty}\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = \lim_{n\to\infty}\left|1 - \frac{\alpha_{n-1}}{\alpha_n}\right| = 0. \tag{3.7}$$

Then inequality (2.8)

$$\|x_{n+1} - x_n\| \le (1 - (1-\rho)\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \cdot \|\Phi(x_{n-1}) - x_{n-1}\|$$

allows us to use Lemma 2.6 to deduce that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.8}$$

Using inequality (2.7) and the hypothesis (H1) we get

$$\lim_{n\to\infty}\|x_n - Tx_n\| \le \lim_{n\to\infty}\left(\|x_{n+1} - x_n\| + \alpha_n\|\Phi(x_n) - x_n\|\right) = 0. \tag{3.9}$$

We will see now that

$$\limsup_n \langle\Phi(q) - q, J(x_n - q)\rangle \le 0. \tag{3.10}$$

Let $\{\beta_k\}$ be a null sequence in $(0, 1)$ and define $\{y_k\}$ by

$$y_k := T(\beta_k\Phi(y_k) + (1 - \beta)y_k).$$

We have proved in Theorem 3.1 that y_k strongly converges to q. For any $n, k \ge 0$ define

$$\delta_{n,k} := \|x_n - Tx_n\|^2 + 2\|x_n - Tx_n\|\|y_k - Tx_n\|$$

and

$$\epsilon_k := \sup_n\{\|\Phi(y_k) - x_n\|\|J(\beta_k(\Phi(y_k) - x_n) + (1 - \beta_k)(y_k - x_n)) - J(y_k - x_n)\|\}.$$

For any fixed $k \in \mathbb{N}$, by (3.9), $\lim_n \delta_{n,k} = 0$. Moreover $\lim_k \epsilon_k = 0$ because of the uniform continuity of J over bounded sets. Using inequality (2.2) we obtain

$$\begin{aligned}
\|y_k - x_n\|^2 &\le (\|Tx_n - x_n\| + \|y_k - Tx_n\|)^2 \\
&\le \delta_{n,k} + \|(1 - \beta_k)(y_k - x_n) + \beta_k(\Phi(y_k) - x_n)\|^2 \\
&\le \delta_{n,k} + (1 - \beta_k)^2\|y_k - x_n\|^2 + 2\beta_k\langle\Phi(y_k) - x_n, J(y_k - x_n)\rangle \\
&\quad + 2\beta_k\epsilon_k \\
&= \delta_{n,k} + ((1 - \beta_k)^2 + 2\beta_k)\|y_k - x_n\|^2 + 2\beta_k\langle\Phi(y_k) - y_k, J(y_k - x_n)\rangle \\
&\quad + 2\beta_k\epsilon_k.
\end{aligned}$$

Then we deduce that

$$\limsup_{n\to\infty}\langle\Phi(y_k)-y_k, J(x_n-y_k)\rangle \le \frac{\beta_k}{2}\limsup_{n\to\infty}\|y_k-x_n\|^2+\epsilon_k. \tag{3.11}$$

On the other hand

$$\begin{aligned}\langle\Phi(q)-q, J(x_n-q)\rangle &= \langle\Phi(q)-q, J(x_n-q)-J(x_n-y_k)\rangle\\ &+\langle(\Phi(q)-q)-(\Phi(y_k)-y_k), J(x_n-y_k)\rangle\\ &+\langle\Phi(y_k)-y_k, J(x_n-y_k)\rangle.\end{aligned} \tag{3.12}$$

Note that

$$\lim_k(\sup_n\{\langle\Phi(q)-q, J(x_n-q)-J(x_n-y_k)\rangle\})=0 \tag{3.13}$$

because J is norm to norm uniform continuous on bounded sets. By using (3.11),(3.13) and passing first to $\limsup_n$ and then to $\lim_k$, from 3.12 we obtain

$$\limsup_n\langle\Phi(q)-q, J(x_n-q)\rangle\le 0.$$

Finally we prove that $\{x_n\}$ strongly converges to q. Set

$$\eta_n := \|J(\alpha_n(\Phi(x_n)-x_n)+(x_n-q))-J(x_n-q)\|.$$

Then $\eta_n\to 0$, as $n\to 0$. We compute

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \|\alpha_n(\Phi(x_n)-q)+(1-\alpha_n)(x_n-q)+\alpha_n(\Phi(q)-q)\|^2\\ &\le \|\alpha_n(\Phi(x_n)-q)+(1-\alpha_n)(x_n-q)\|^2\\ &+2\alpha_n\langle\Phi(q)-q, J(x_n-q)\rangle\\ &+2\alpha_n\eta_n\|\Phi(q)-q\|\\ &\le (1-(1-\rho)\alpha_n)\|x_n-q\|^2+2\alpha_n\langle\Phi(q)-q, J(x_n-q)\rangle\\ &+2\alpha_n\eta_n\|\Phi(q)-q\|\end{aligned}$$

and the result follows from (3.10) and Lemma 2.6.

$\square$

Corollary 3.5. *Let X be a uniformly smooth Banach space, C a nonempty closed convex subset of X, $T: C\to C$ a nonexpansive mapping with $F\ne\emptyset$ and $g: C\to C$ a mapping such that $I-\mu g$ is a contraction for some $\mu>0$. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying hypotheses (H1)-(H3) in Theorem 3.4. Then the sequence $\{x_n\}$ defined by the iterative scheme*

$$x_{n+1}=Tx_n-\alpha_n g(Tx_n), \tag{3.14}$$

strongly converges to the unique solution $p\in F$ to the inequality problem

$$\langle g(p), x-p\rangle\ge 0 \ \ \forall x\in F. \tag{3.15}$$

Proof. Consider the sequence $\{z_n\}$ defined by $z_n=Tx_n$, for any $n\ge 0$. Then

$$\begin{aligned}z_{n+1} &= T(Tx_n-\alpha_n g(Tx_n))\\ &= T(z_n-\frac{\alpha_n}{\mu}\mu g(z_n))\\ &= T(\alpha'_n(I-\mu g)z_n+(1-\alpha'_n)z_n),\end{aligned}$$

where $\alpha'_n = \frac{\alpha_n}{\mu}$ for all $n \geq 0$, so the sequence $\{\alpha'_n\}$ satisfies hypotheses (H1)-(H3). Since $\Phi := I - \mu g$ is a contraction, Theorem 3.4 implies the strong convergence of $\{z_n\}$ to the unique solution $p \in F$ to the inequality problem

$$\langle (\Phi - I)p, x - p \rangle \geq 0 \ \ \forall x \in F,$$

which is equivalent to (3.15). Therefore, from the iteration scheme (3.14) we deduce that the sequence $\{x_n\}$ strongly converges to p. $\qquad\qquad\square$

4   Rates of asymptotic regularity

In the following, we apply proof mining techniques to get effective rates of asymptotic regularity for the iteration $\{x_n\}$ defined by (1.12). The methods we use in this paper are inspired by those used in [22] to obtain effective rates of asymptotic regularity for Halpern iterations. As in the case of Halpern iterations, the main ingredient turns out to be the quantitative Lemma 2.7.

Theorem 4.1. *Let X be a normed space, $C \subseteq X$ a nonempty convex subset and $T : C \to C$ be nonexpansive. Assume that $\Phi : C \to C$ is a ρ-contraction and that $\{\alpha_n\}_{n \geq 0}$ is a sequence in $(0,1)$ such that $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n$ is divergent and $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n|$ is convergent. Let $x_0 \in C$ and $\{x_n\}_{n \geq 0}$ be defined by (1.12). Assume that $\{x_n\}$ is bounded.*
Then $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$ and moreover for all $\varepsilon \in (0,2)$,

$$\|x_n - Tx_n\| < \varepsilon, \quad \forall n \geq \Psi(\varphi, \beta, \theta, \rho, M, D, \varepsilon),$$

where

1. *$\varphi : (0, \infty) \to \mathbb{Z}_+$ is a rate of convergence of $\{\alpha_n\}$,*

2. *$\beta : (0, \infty) \to \mathbb{Z}_+$ is a Cauchy modulus of $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n|$,*

3. *$\theta : \mathbb{Z}_+ \to \mathbb{Z}_+$ is a rate of divergence of $\sum_{n=0}^{\infty} \alpha_n$,*

4. *$M \geq 0$ is such that $M \geq \|\Phi(x_0) - x_0\|$,*

5. *$D > 0$ satisfies $D \geq \|x_n - x_m\|$ for all $m, n \geq 0$,*

and $\Psi(\varphi, \beta, \theta, \rho, M, D, \varepsilon)$ is defined by

$$\Psi := \max\left\{ 1 + \theta\left(\left\lceil \frac{1}{1 - \rho} \right\rceil \left(\beta\left(\frac{\varepsilon}{4P}\right) + 2 + \left\lceil \ln\left(\frac{4D}{\varepsilon}\right) \right\rceil \right) \right), \ 1 + \varphi\left(\frac{\varepsilon}{2P}\right) \right\},$$

with $P = (1 + \rho)D + M$.

Proof. Applying (2.8) and (2.6), we get that for all $n \geq 1$

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\leq (1 - (1-\rho)\alpha_n)\|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \cdot \|\Phi(x_{n-1}) - x_{n-1}\| \\
&\leq (1 - (1-\rho)\alpha_n)\|x_n - x_{n-1}\| + P \cdot |\alpha_n - \alpha_{n-1}|.
\end{aligned}
$$

Let us denote for $n \geq 1$

$$
a_n := \|x_n - x_{n-1}\|, \quad b_n := P \cdot |\alpha_n - \alpha_{n-1}|, \quad \lambda_n := (1-\rho)\alpha_{n-1}.
$$

Then D is an upper bound on $\{a_n\}$ and

$$
a_{n+1} \leq (1 - \lambda_{n+1})a_n + b_n \quad \text{for all } n \geq 1.
$$

Moreover, $\sum_{n=1}^{\infty} \lambda_n$ is divergent with rate of divergence

$$
\delta : \mathbb{Z}_+ \to \mathbb{Z}_+, \quad \delta(n) = 1 + \theta\left(\left\lceil \frac{1}{1-\rho} \right\rceil \cdot n\right), \tag{4.1}
$$

since for all $n \in \mathbb{Z}_+$,

$$
\begin{aligned}
\sum_{i=1}^{\delta(n)} \lambda_i &= (1-\rho) \sum_{i=0}^{\delta(n)-1} \alpha_i = (1-\rho) \sum_{i=0}^{\theta(\lceil 1/1-\rho \rceil \cdot n)} \alpha_i \geq (1-\rho)\left\lceil \frac{1}{1-\rho} \right\rceil \cdot n \\
&\geq n.
\end{aligned}
$$

Let $t_n := \sum_{i=0}^{n} |\alpha_{i+1} - \alpha_i|$ and $s_n := \sum_{i=1}^{n} b_i = Pt_{n-1}$ and define

$$
\gamma : (0, \infty) \to \mathbb{Z}_+, \quad \gamma(\varepsilon) := 1 + \beta\left(\frac{\varepsilon}{P}\right), \tag{4.2}
$$

Then for all $n \geq 0$,

$$
s_{\gamma(\varepsilon)+n} - s_{\gamma(\varepsilon)} = P\left(t_{\beta(\varepsilon/P)+n} - t_{\beta(\varepsilon/P)}\right) < P \cdot \frac{\varepsilon}{P} = \varepsilon.
$$

Thus, $\sum_{n=1}^{\infty} b_n$ is convergent with Cauchy modulus γ.

It follows that we can apply Lemma 2.7 to get that for all $\varepsilon \in (0, 2)$ and for all $n \geq h_1(\beta, \theta, \rho, M, D, \varepsilon)$

$$
\|x_n - x_{n-1}\| < \frac{\varepsilon}{2}, \tag{4.3}
$$

where

$$
h_1(\beta, \theta, \rho, M, D, \varepsilon) := 1 + \theta\left(\left\lceil \frac{1}{1-\rho} \right\rceil \cdot \left(\beta\left(\frac{\varepsilon}{4P}\right) + 2 + \left\lceil \ln\left(\frac{4D}{\varepsilon}\right) \right\rceil\right)\right)
$$

Using (2.7) and (2.6), we get that for all $n \geq 1$,

$$
\begin{aligned}
\|x_{n-1} - Tx_{n-1}\| &\leq \|x_n - x_{n-1}\| + \alpha_{n-1}\|\Phi(x_{n-1}) - x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + P\alpha_{n-1}.
\end{aligned}
\tag{4.4}
$$

Let $h_2(\varphi, \rho, M, D, \varepsilon) := 1 + \varphi\left(\dfrac{\varepsilon}{2P}\right)$. Since φ is a rate of convergence of $\{\alpha_n\}$ towards 0, it follows that

$$
P\alpha_{n-1} < \frac{\varepsilon}{2} \quad \text{for all } n \geq h_2(\varphi, \rho, M, D, \varepsilon).
\tag{4.5}
$$

As a consequence of (4.3), (4.4) and (4.5), we get that

$$
\|x_{n-1} - Tx_{n-1}\| < \varepsilon
$$

for all $n \geq \max\{h_1(\beta, \theta, \rho, M, D, \varepsilon), h_2(\varphi, \rho, M, D, \varepsilon)\}$, so the conclusion of the theorem follows. $\qquad\square$

If C is bounded, then $\{x_n\}$ is bounded for all $x_0 \in C$. Moreover, we can take $M := D := d_C$ in the above theorem, where $d_C := \sup\{\|x - y\| \mid x, y \in C\}$ is the diameter of C.

Corollary 4.2. *In the hypotheses of Theorem 4.1, assume moreover that C is bounded. Then $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$ for all $x_0 \in C$ and moreover for all $\varepsilon \in (0, 2)$,*

$$
\|x_n - Tx_n\| < \varepsilon, \quad \forall n \geq \Psi(\varphi, \beta, \theta, \rho, d_C, \varepsilon),
$$

where $\Psi(\varphi, \beta, \theta, \rho, d_C, \varepsilon)$ is defined as in Theorem 4.1 by replacing M and D with d_C.

Thus, for bounded C, we get asymptotic regularity for general $\{\alpha_n\}$ and an explicit rate of asymptotic regularity $\Psi(\varphi, \beta, \theta, \rho, d_C, \varepsilon)$ that depends weakly on C (via its diameter d_C) and on the ρ-contraction Φ (via ρ), while it does not depend on the nonexpansive mapping T, the starting point $x_0 \in C$ of the iteration or other data related with C and X.

The rate of asymptotic regularity can be simplified when the sequence $\{\alpha_n\}$ is decreasing.

Corollary 4.3. *Let $X, C, T, \Phi, \{x_n\}$ be as in the hypotheses of Corollary 4.2. Assume that $\{\alpha_n\}$ is a decreasing sequence in $(0, 1)$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=0}^{\infty} \alpha_n$ is divergent. Then $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$ for all $x_0 \in C$ and furthermore, for all $\varepsilon \in (0, 2)$,*

$$
\|x_n - Tx_n\| < \varepsilon, \quad \forall n \geq \Psi(\varphi, \theta, \rho, d_C, \varepsilon),
$$

where $\varphi : (0, \infty) \to \mathbb{Z}_+$ is a rate of convergence of $\{\alpha_n\}$, $\theta : \mathbb{Z}_+ \to \mathbb{Z}_+$ is a rate of divergence of $\sum\limits_{n=0}^{\infty} \alpha_n$ and $\Psi(\varphi, \theta, \rho, d_C, \varepsilon)$ is defined by

$$
\Psi := \max\left\{1 + \theta\left(\left\lceil\frac{1}{1-\rho}\right\rceil\left(\varphi\left(\frac{\varepsilon}{4P}\right) + 2 + \left\lceil\ln\left(\frac{4d_C}{\varepsilon}\right)\right\rceil\right)\right), \ 1 + \varphi\left(\frac{\varepsilon}{2P}\right)\right\}
$$

with $P = (2 + \rho)d_C$.

Proof. Remark that $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n|$ is convergent with Cauchy modulus φ. $\square$

Finally, we get, as in the case of Halpern iterates, an exponential (in $1/\varepsilon$) rate of asymptotic regularity for $\alpha_n = \dfrac{1}{n+1}$.

Corollary 4.4. *Let $X, C, T, \Phi, \{x_n\}$ be as in the hypotheses of Corollary 4.2. Assume that $\alpha_n = \dfrac{1}{n+1}$ for all $n \geq 0$.*
Then $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$ for all $x_0 \in C$ and furthermore, for all $\varepsilon \in (0, 2)$,

$$\|x_n - Tx_n\| < \varepsilon, \quad \forall n \geq \Theta(\rho, d_C, \varepsilon),$$

where

$$\Theta(\rho, d_C, \varepsilon) = \exp\left(\frac{4}{1-\rho}\left(\frac{16d_C}{\varepsilon} + 2\right)\right)$$

Proof. We can apply Corollary 4.3 with

$$\varphi : (0, \infty) \to \mathbb{Z}_+, \quad \varphi(\varepsilon) = \left\lceil \frac{1}{\varepsilon} \right\rceil - 1.$$

and

$$\theta : \mathbb{Z}_+ \to \mathbb{Z}_+, \quad \theta(n) = 4^n - 1$$

to conclude that for all $\varepsilon \in (0, 2)$,

$$\|x_n - Tx_n\| < \varepsilon, \quad \forall n \geq \Psi(\rho, d_C, \varepsilon),$$

where $P = (2 + \rho)d_C$ and

$$
\begin{aligned}
\Psi(\rho, d_C, \varepsilon) &= \max\left\{ \exp\left(\ln 4 \cdot \left\lceil \frac{1}{1-\rho} \right\rceil \cdot \left(\left\lceil \frac{4P}{\varepsilon} \right\rceil + 1 + \left\lceil \ln\left(\frac{4d_C}{\varepsilon}\right) \right\rceil\right)\right), \left\lceil \frac{2P}{\varepsilon} \right\rceil \right\} \\
&= \exp\left(\ln 4 \cdot \left\lceil \frac{1}{1-\rho} \right\rceil \cdot \left(\left\lceil \frac{4P}{\varepsilon} \right\rceil + 1 + \left\lceil \ln\left(\frac{4d_C}{\varepsilon}\right) \right\rceil\right)\right) \\
&< \exp\left(\frac{4}{1-\rho}\left(\frac{16d_C}{\varepsilon} + 2\right)\right) = \Theta(\rho, d_C, \varepsilon).
\end{aligned}
$$

as $\rho \in (0, 1)$, $\lceil a \rceil < a + 1$ and $1 + \ln a \leq a$ for all $a > 0$. The conclusion follows now immediately. $\square$

5 Applications

As it was pointed out in the introduction, iterative methods for nonexpansive mappings have been applied to solve the problem of finding a solution to the VIP(A, C) (1.2) which, in fact, is equivalent under suitable conditions to finding the minimizer of a certain function. On the other hand, the relation between the set of zeros of an accretive operator and the fixed point

set of its resolvent allows us to use those iterative techniques for nonexpansive mappings to approximate zeros of such operators. We first apply the explicit iterative method for approximating fixed points, presented in section 3, to solve a particular variational inequality problem in the setting of Hilbert spaces. Then we focus on the asymptotic behavior of the resolvent of an accretive operator in the framework of Banach spaces.

5.1 A variational inequality problem

Let H be a Hilbert space, $T : H \to H$ be a nonexpansive mapping with fixed point set $F \neq \emptyset$, and $\Phi : H \to H$ be a contraction. Assume that A is a Lipschitzian self-operator on H which is strongly monotone; that is, there exist a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \eta \|x - y\|^2, \ \forall x, y \in H.$$

It is known that the following variational inequality

$$\langle (A - \gamma\Phi)q, q - x \rangle \leq 0 \ \forall x \in F, \tag{5.1}$$

where $\gamma > 0$, is the optimality condition for the minimization problem

$$\min_{x \in F} f(x) - h(x)$$

where f is a differentiable function with differential $\partial f = A$ and h is a potential function for $\gamma\Phi$ (i.e. $h'(x) = \gamma\Phi(x)$ for $x \in H$). Marino and Xu [30] presented an iterative method to solve the variational inequality (5.1) for a linear bounded operator. We apply the explicit method (3.6), in particular algorithm (3.14), to solve such variational inequality dispensing with the linear condition on the operator A. For that purpose, we need the following lemma.

Lemma 5.1. *Assume that A is a L-Lipschitzian η-strongly monotone operator, and let Φ be a ρ-contraction. Then, for any $\gamma < \eta/\rho$, $A - \gamma\Phi$ is R-Lipschitzian and δ-strongly monotone with $R = L + \gamma\rho$ and $\delta = \eta - \gamma\rho$. Besides, for any $0 < \mu < 2\delta/R^2$, the mapping $I - \mu(A - \gamma\Phi)$ is a contraction.*

Proof. Since A is L-Lipschitzian and Φ is a ρ-contraction,

$$\|(A - \gamma\Phi)x - (A - \gamma\Phi)y\| \leq \|Ax - Ay\| + \gamma\|\Phi x - \Phi y\| \leq (L + \gamma\rho)\|x - y\|,$$

that is, $A - \gamma\Phi$ is Lipschitzian with constant $R = L + \gamma\rho$. The strong monotonicity of $A - \gamma\Phi$ is consequence of the strong monotonicity of A as it is showed as follow.

$$\begin{aligned}
\langle (A - \gamma\Phi)x - (A - \gamma\Phi)y, x - y \rangle &= \langle Ax - Ay, x - y \rangle - \gamma\langle \Phi x - \Phi y, x - y \rangle \\
&\geq \eta\|x - y\|^2 - \gamma\|\Phi x - \Phi y\|\|x - y\| \\
&\geq (\eta - \gamma\rho)\|x - y\|^2,
\end{aligned}$$

where $\delta = \eta - \gamma\rho > 0$. By applying the R-Lipschitz continuity and δ-strong monotonicity of $B := A - \gamma\Phi$ we obtain

$$\begin{aligned}
\|(I - \mu B)x - (I - \mu B)y\|^2 &= \|x - y\|^2 + \mu^2\|Bx - By\|^2 - \mu\langle x - y, Bx - By \rangle \\
&\leq \|x - y\|^2 + \mu^2 R^2\|x - y\|^2 - 2\mu\delta\|x - y\|^2 \\
&= (1 - \mu(2\delta - \mu R^2))\|x - y\|^2.
\end{aligned}$$

Then, for any $0 < \mu < 2\delta/R^2$, the mapping $I - \mu(A - \gamma\Phi)$ is a contraction with constant $\sqrt{1 - \mu(2\delta - \mu R^2)}$. $\qquad\square$

Theorem 5.2. *Let T be a nonexpansive mapping with fixed point set F, A a L-Lipschitzian η-strongly monotone operator and Φ a ρ-contraction on a Hilbert space. Then, for any $\gamma < \eta/\rho$, the sequence defined by the iterative scheme*

$$x_{n+1} = Tx_n - \alpha_n(A - \gamma\Phi)Tx_n,$$

where $\{\alpha_n\} \subset (0,1)$ satisfies hypotheses (H1)-(H2) in Theorem 3.4, strongly converges to the unique solution to the variational inequality (5.1).

Proof. Note that, for any $\gamma < \eta/\rho$, Lemma 5.1 implies that there exists $\mu > 0$ such that $I - \mu g$ is a contraction, where $g = A - \gamma\Phi$. Then, by Theorem 3.5 we obtain the strong convergence of the sequence $\{x_n\}$ to the unique solution to the variational inequality problem (5.1).

$\qquad\square$

5.2 Zeros of m-accretive operators

Let X be a real Banach space. A set-valued operator $A : X \to 2^X$ with domain $D(A)$ and range $R(A)$ in X is said to be *accretive* if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2, j(x_1 - x_2) \rangle \geq 0,$$

where J is the normalized duality mapping. An accretive operator A is *m-accretive* if $R(I + \lambda A) = X$ for all $\lambda > 0$. Denote the set of zeros of A by

$$Z := A^{-1}(0) = \{z \in D(A) : 0 \in Az\}.$$

Throughout this subsection it is assumed that A is m-accretive and $A^{-1}(0) \neq \emptyset$. Set $C = \overline{D(A)}$ and assume it is convex. It is known that the *resolvent* of A, defined by

$$J_\lambda = (I + \lambda A)^{-1},$$

for $\lambda > 0$, is a single-valued nonexpansive mapping from C into itself (cf. [1]).

If we consider the problem of finding a zero of A, i.e.,

$$\text{find } z \in C \text{ such that } 0 \in Az,$$

it is straightforward to see that the set of zeros $A^{-1}(0)$ coincides with the fixed point set of J_λ, $Fix(J_\lambda)$, for any $\lambda > 0$. Therefore an equivalent problem is to find $z \in Fix(J_\lambda)$.

As a consequence of the convergence of the implicit iterative scheme (3.1), we obtain Reich's result (cf. [36]) for approximating zeros of accretive operators in uniformly smooth Banach spaces. Besides, the following theorem in the setting of reflexive Banach spaces with weakly continuous normalized duality mapping constitutes a new approach.

Theorem 5.3. *Let A be a m-accretive operator in either a uniformly smooth Banach space or a reflexive Banach spaces with weakly continuous normalized duality mapping X. Then, for each $x \in X$, the sequence $\{J_\lambda(x)\}$ strongly converges, as $\lambda \to \infty$, to the unique zero of A, $q \in A^{-1}(0)$, which satisfies the variational inequality*

$$\langle x - q, J(y - q) \rangle \leq 0 \ \ \forall y \in A^{-1}(0). \tag{5.2}$$

Proof. Given $x \in X$ we consider the approximating curve $\{x_t\}$ such that $x_t = J_{1/t}x$, for any $t \in (0,1)$. By definition of the resolvent of A, we obtain the following equivalence:

$$\begin{aligned}
x_t = (I + \frac{1}{t}A)^{-1}x \ &\Leftrightarrow \ x \in x_t + \frac{1}{t}Ax_t \\
&\Leftrightarrow \ t(x - x_t) \in Ax_t \\
&\Leftrightarrow \ x_t + t(x - x_t) \in (I + A)x_t \\
&\Leftrightarrow \ x_t = (I + A)^{-1}(x_t + t(x - x_t)) \\
&\Leftrightarrow \ x_t = T(t\Phi(x_t) + (1-t)x_t),
\end{aligned}$$

where $T = (I + A)^{-1}$ is the nonexpansive resolvent with constant 1, and $\Phi = x$ is a constant mapping which is a contraction. Therefore, Theorem 3.1 implies the strong convergence of $\{x_t\}$, as $t \to 0$, to the unique solution to the inequality (3.2); in other words, $\{J_\lambda x\}$ strongly converges, as $\lambda \to \infty$, to the unique solution $q \in A^{-1}(0)$ to the inequality (5.2). $\qquad \square$

Remark 5.4. If we define the mapping $Q : X \to A^{-1}(0)$ such that, for any $x \in X$,

$$Qx = \lim_{\lambda \to \infty} J_\lambda x,$$

then, since Qx satisfies the inequality (5.2), by Lemma 2.1 we can claim that Q is the unique sunny nonexpansive retraction from X to $A^{-1}(0)$.

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