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OWP 2013 - 03

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Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

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The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

ON THE GEOMETRY OF REGULAR MAPS FROM A QUASI-PROJECTIVE SURFACE TO A CURVE

A.J. PARAMESWARAN AND M. TIBĂR

Abstract. We explore consequences of the triviality of the monodromy group, using the condition of purity of the mixed Hodge structure on the cohomology of the surface X.

1. Introduction

In case of a polynomial function $P : \mathbb{C}^2 \to \mathbb{C}$, Miyanishi and Sugie [MS] proved that the global monodromy group acting on $H^1(F)$ is trivial if and only if the general fibre of P is rational and P is *simple* (cf Definition 2.6). Dimca [Di, Theorem 1, (ii)] showed that in this statement the monodromy group can be replaced by the monodromy at infinity (i.e. around a very large cercle in $\mathbb{C}$). Kaliman [Kal] proved that the number of reducible fibres of a primitive map is at most $\delta - 1$, where $\delta = \#$horizontal components. Dimca also observed that, in addition to the triviality of the monodromy, if all the fibres of the polynomial P are irreducible, then P is linearisable, by using the above sited result of Kaliman and the celebrated theorem of Abhyakar-Moh-Suzuki. One has also studied the monodromy of a polynomial function on $\mathbb{C}^2$ in relation to the monodromy of a plane curve germ, as well as extensions to several variables [Di, ACD, NN1].

We consider here instead of $\mathbb{C}^2$ some other classes of surfaces verifying certain Hodge theoretic conditions in order to draw topological properties of the monodromy. This is also motivated by long standing research on classes of affine surfaces. In several papers [GP], [GPS], [GS1], [GS2] Gurjar, Pradeep and Shastri proved that every $\mathbb{Q}$-homology plane is rational, question which was raised by Miyanishi. This is already a big class of varieties, since it has moduli.

Let X be a nonsingular, quasi-projective, connected surface, and let $f : X \to C$ be a regular map onto a nonsingular affine curve. Let $\bar{f} : \bar{X} \to \bar{C}$ be some resolution of the indeterminacy points at infinity, namely $\bar{X}$ and $\bar{C}$ are nonsingular and such that $D = \bar{X} \setminus X$ is a normal crossing divisor, i.e., $D = \cup_i D_i$, where each D_i a smooth divisor intersecting transversely D_j, $j \neq i$. We shall refer to $\bar{f}$ as "compact resolution" and we shall work throughout the paper with the condition that D is connected.

Date: January 25, 2013.

2000 *Mathematics Subject Classification.* 32S55, 14D06, 14P15, 58K05, 57R45.

Key words and phrases. monodromy, quasi-projective surfaces, mixed Hoge structures, fibrations.

We study here the monodromy of the regular map f in relation to that of $\bar{f}$. The restriction of $\bar{f}$ to some component D_i is either constant and then one says that D_i is *vertical*, or $\bar{f}_| : D_i \to \bar{C}$ is onto, it is namely a (ramified) covering of degree d_i and we say that D_i is a *horizontal* component. We denote by F_p and $\bar{F}_p$ the fibres of f and of $\bar{f}$ over some point $p \in C$ or $p \in \bar{C}$, respectively. Also, F and $\bar{F}$ without lower index will stand for general fibres of f and $\bar{f}$, respectively. One has to take into account that $\bar{F}$ is indeed the closure of the general fibre F, whereas $\bar{F}_p$ is not that of F_p, since it may contain vertical components.

The aim of this paper is to study maps from surfaces with certain Hodge theoretic purity. We mainly investigate the following properties of these maps:

(a) Effect of the triviality of the monodromy group, Propositions 2.7 and 2.8.
(b) Structure of divisor at infinity (Lemma 3.3, Proposition 3.3, Corollary 3.5).
(c) Structure of horizontal components and of fibres, Theorems 3.6, 3.7.
(d) Determination of the monodromy group by a single loop, Theorem 3.9
(e) Relation between reducible fibres and horizontal components, Theorem 4.1.
(f) Trivial monodromy with irreducible fibres, Theorem 5.3.

The paper is organised as follows. In §2 we prove that if the monodromy group $\mathrm{Mon}^1 \bar{f}$ acting on the cohomology group $H^1(\bar{F})$ of the general fibre is trivial then the restriction morphism $H^1(\bar{X}) \to H^1(\bar{F}_p)$ is a surjection for all $p \in \bar{C}$. We moreover show that the kernel of this morphism is precisely $H^1(\bar{C})$, for any choice of $p \in \bar{C}$, whenever the general fibre is connected. We then relate the triviality of the monodromy group of f to that of $\bar{f}$ and to the simplicity of f (Propositions 2.7 and 2.8).

In §3, we begin by showing some topological consequences of purity of the cohomology (Lemma 3.3). Then we continue to analyse the horizontal and vertical componets of a map from such surface onto an affine curve.

In §4, we bound the reducible fibres with horizontal componets and the second betti number (Theorem 4.1).

In §5 we prove that if the monodromy is trivial and all the fibres are reduced irrducible then it is locally trivial (Theorem 5.3).

The authors express their gratitude to the Mathematische Institut Oberwolfach for hosting them in the RiP program. They thank the School of Mathematics at TIFR Mumbai, and the Paul Painlevé Lab. at Université Lille 1 for supporting their visits during which this project was accomplished.

2. Open and projective monodromies

It is well known that there is a finite set of points $A \subset \bar{C}$ such that, for any $c \in \bar{C} \setminus A$ the restriction:

$$\bar{f}_| : \bar{X} \cap \bar{f}^{-1}(\bar{C} \setminus A) \to \bar{C} \setminus A$$

is a proper stratified submersion hence a locally trivial fibration. In particular the map $f_| : X \cap f^{-1}(C \setminus A) \to C \setminus A$ is a locally trivial fibration.

Let $\bar{F}$ and F denote the general fibres of $\bar{f}_|$ and $f_|$ respectively. The points $\bar{F} \setminus F$ are precisely the intersections of $\bar{F}$ with the horizontal curves. For any point $p \in \bar{F} \setminus F$ one defines a generator of the group $H^1(F) \simeq H_1(F)$ as the equivalence class of the oriented boundary of a small enough disk at the point. It is well-known that there is a single relation among all these $\delta := |\bar{F} \setminus F|$ generators.

We consider homology and cohomology with coefficients in $\mathbb{C}$. The monodromy groups $\mathrm{Mon}^k \bar{f}$ and $\mathrm{Mon}^k f$, for $k = 0, 1, 2$, are by definition the monodromy representations of the fundamental groups, $\pi_1(\bar{C} \setminus A) \to \mathrm{Aut}(H^k(\bar{F}))$ and $\pi_1(\bar{C} \setminus A) \to \mathrm{Aut}(H^k(F))$, respectively. We are more specifically interested in the first cohomology group and its global invariant cycles, denoted by $H^1(\bar{F})^\pi$ and $H^1(F)^\pi$, respectively.

Definition 2.1. We say that f is primitive if its general fibre F is connected.

Note that f is primitive if and only if some compact resolution $\bar{f}$ of f is primitive. Actually this does not depend of the choice of compact resolution (as defined in §1).

By the Stein factorisation, there exists a projective curve $\bar{C}'$ and regular maps $\bar{f}' : \bar{X} \to \bar{C}'$ and $\mu : \bar{C}' \to \bar{C}$ such that $\bar{f} = \mu \circ \bar{f}'$ and that $\bar{f}'$ is primitive.

Proposition 2.2. *Let $p \in \bar{C}$ and let $\bar{f}$ be primitive. Then the following sequence is exact:*

$$0 \to H^1(\bar{C}) \to H^1(\bar{X}) \to H^1(\bar{F}_p). \tag{1}$$

Moreover, $\mathrm{Mon}^1 \bar{f} = 1$ *if and only if the last map is a surjection and* $H^1(\bar{F}_p) \simeq H^1(\bar{F}_q)$ *for all* $q \in \bar{C}$.

For the proof, we need two lemmas.

Lemma 2.3. *Let $f : E \to B$ be a primitive projective submersion, where B is non-complete. Consider the Leray spectral sequence with $E_2^{pq} = H^p(B, R^q f_*(\mathbb{C}_E))$ abutting to $H^{p+q}(E)$. If B and F are complex curves, then we have an exact sequence:*

$$0 \to H^1(B) \to H^1(E) \to H^1(F)$$

Proof. The Leray spectral sequence degenerates in E_2 by [Vo, Theorem 4.15]. The only possibly nontrivial differential in the spectral sequence is $H^0(B, R^1 f_*(\mathbb{C}_E)) \to H^2(B, H^0(F))$. Since $H^2(B) = 0$ for a noncomplete curve, this differential must be 0. The quotient of $H^1(E)$ by $H^1(B, R^0 f_*(\mathbb{C}_E))$ is $H^0(B, R^1 f_*(\mathbb{C}_E))$, which is precisely the invariant cycles. Note that $H^1(B, R^0 f_*(\mathbb{C}_E))$ is isomorphic to $H^1(B)$ since $R^0 f_*(\mathbb{C}_E) = \mathbb{C}_B$ by the primitivity. On the other hand, the invariant cycle theorem of Deligne states that the invariant cycles are precisely the image of $H^1(E) \to H^1(F)$. Thus our sequence is exact. □

Lemma 2.4. (cf [Vo, 4.3.3]) *Let Y be a connected nonsingular projective variety and let $j : U \hookrightarrow Y$ denote the inclusion of some Zariski open subset U. Then, for* *any $k \geq 0$, the weight k filtration of the mixed Hodge structure on $H^k(U)$ is precisely the image of the morphism induced by the restriction $j^* : H^k(Y) \to H^k(U)$.* □

Proof of Proposition 2.2. We first prove the second statement. If $\bar{F}$ denotes the general fibre of $\bar{f}$ then, by Deligne's invariant cycle theorem [De1], the global invariant cycles coincide with the image of $H^1(\bar{X}) \to H^1(\bar{F})$. Then $\mathrm{Mon}^1 \bar{f} = 1$ if and only if this map is surjective. For proving the surjectivity for a special fibre $\bar{F}_p$ we use the following construction. Let U be a tubular neighbourhood of $\bar{F}_p$ which is invariant under $\bar{f}$. Let $\bar{F} \subset U$ be a general fibre contained in U. Then $H^1(U) \cong H^1(\bar{F}_p)$ is an isomorphism and $H^1(\bar{X}) \to H^1(\bar{F})$ factors through $H^1(U)$. Hence $H^1(U) \cong H^1(\bar{F}_p) \to H^1(\bar{F})$ is surjective. Since $b_1(\bar{F}) \geq b_1(\bar{F}_p)$, this must be an isomorphism and the proof of the surjection in (1) is done. For the sake of completeness, let us provide here a brief proof of the inequality $b_1(\bar{F}) \geq b_1(\bar{F}_p)$ for any special fibre $\bar{F}_p$ of $\bar{f}$. We consider the pair $(X_{\Delta^*}, \bar{F})$ where Δ is a small disk centered at the image point $\bar{f}(\bar{F}_p)$ and Δ^* is the punctured disk. We have $H_*(X_{\Delta^*}, \bar{F}) \simeq \tilde{H}_*(S^1 \vee S(\bar{F}))$ where $S(\bar{F})$ denotes the suspension over the general fibre $\bar{F}$. It follows that $H_1(X_{\Delta^*}, \bar{F}) \simeq \mathbb{C}$. The exact sequence of the pair yields:

$$
\begin{array}{ccccc}
H_1(\bar{F}) & \stackrel{i}{\to} & H_1(X_{\Delta^*}) & \stackrel{k}{\to} & \mathbb{C} \to 0 \\
j \downarrow & & \downarrow \pi & & \\
H_1(\bar{F}_p) & \simeq & H_1(X_\Delta) & &
\end{array}
\tag{2}
$$

where j is induced by the restriction to $\bar{F}$ of the topological contraction of a tube to the special fibre $\bar{F}_p$. The surjectivity of the natural map π holds already at the level of fundamental groups since X_{Δ^*} is the complement of a proper complex analytic subset. Since there exists a lift by k of $[S^1]$ which is sent to 0 by π, it follows that the restriction $\pi_{| \ker k}$ is a surjection too. By the exactness, $\pi \circ i$ is surjective and therefore j is surjective too. Hence $b_1(\bar{F}) \geq b_1(\bar{F}_p)$.

To prove the exactness of the sequence (1), let us show that the kernel of $\gamma : H^1(\bar{X}) \to H^1(\bar{F})$ is $H^1(\bar{C})$. First we observe that $\ker \gamma$ has weight 1 since $H^1(\bar{X})$ has pure Hodge structure. Let $B \subset \bar{C}$ be the open subset over which $\bar{f}$ is submersive. Let $\bar{X}_B := \bar{f}^{-1}(B)$. We have the following commutative diagram, where the botom sequence is exact by Lemma 2.3:

$$
\begin{array}{ccccccc}
0 & \to & H^1(\bar{B}) & \to & H^1(\bar{X}) & \stackrel{\gamma}{\to} & H^1(\bar{F}) \\
 & & \downarrow & & \downarrow & & \downarrow = \\
0 & \to & H^1(B) & \to & H^1(\bar{X}_B) & \to & H^1(\bar{F}) \quad \to 0
\end{array}
\tag{3}
$$

The restriction maps $H^1(\bar{B}) \to H^1(B)$ and $H^1(\bar{X}) \to H^1(\bar{X}_B)$ are injective since the inclusion of a Zariski open subset in a smooth variety induces a surjection of their fundamental groups. This implies that $H^1(\bar{B}) \to H^1(\bar{X})$ is injective and that we have the equality $\ker \gamma = H^1(\bar{X}) \cap H^1(B)$, where the intersection is taken in $H^1(\bar{X}_B)$, i.e. $H^1(B)$ and $H^1(\bar{X})$ are identified with their isomorphic images in $H^1(\bar{X}_B)$. Since by Lemma 2.4 the weight 1 part of $H^1(B)$ is precisely the image of $H^1(\bar{B})$ and since

$\ker \gamma$ is pure of weight 1, we conclude that $H^1(\bar{X}) \cap H^1(B) = H^1(\bar{B})$. This ends our proof since $\bar{B} = \bar{C}$ by definition.

□

The following useful result is a simple consequence of the Stein factorisation.

Lemma 2.5. *Let $\bar{X} \to \bar{C}$ be a regular map between smooth projective varieties, where $\bar{X}$ is irreducible and $\bar{C}$ is a curve. Let $D_0 \subset \bar{X}$ be a horizontal curve. Then, for every $p \in \bar{C}$, D_0 meets every connected component of the fibre $\bar{F}_p$.* □

Definition 2.6. We say that f is *simple* if $\deg \bar{f}_{|D_i} = 1$ for all horizontal D_i. This does not depend on the choice of the resolution $\bar{f}$.

We draw some consequences of the triviality of the monodromy groups $\mathrm{Mon}^k f$ and $\mathrm{Mon}^k \bar{f}$, respectively. In the particular case of $f : \mathbb{C}^2 \to \mathbb{C}$, a version of the following Propositions 2.7 and 2.8 has been proved in [Di], [NN1]. The notation Mon without upper index means all $k = 0, 1, 2$.

Proposition 2.7. *Let* $\mathrm{Mon} f = 1$. *Then:*
(a) f *is primitive.*
(b) $\mathrm{Mon}\bar{f} = 1$
(c) f *is simple.*

Proof. (a). If f was not primitive then the *Stein factorisation* allows one to write $f = g \circ h$, where $h : X \to C'$ and $g : C' \to C$ is a covering of degree $d > 1$. Since C' is connected, $\mathrm{Mon}^0 f$ identifies to the monodromy group of the covering g. The later is a subgroup of the permutation group of d points. This permutation subgroup is not trivial if $d > 1$. This actually shows that (a) is equivalent to $\mathrm{Mon}^0 f = 1$.

(b). We have the short exact sequence induced by the inclusion $F \xrightarrow{i} \bar{F}$:

$$0 \to H^1(\bar{F}) \xrightarrow{i^*} H^1(F) \to \operatorname{coker} i^* \to 0 \tag{4}$$

where $\operatorname{coker} i^* \simeq \tilde{H}_0(\bar{F} \setminus F)$. The later is of rank $\delta - 1$, where δ is the number of points at infinity $\bar{F} \setminus F$. Moreover the geometric monodromy of $\bar{f}$ described in §1 acts on F. This sequence shows that the triviality of the monodromy group of f implies the triviality of the monodromy group of $\bar{f}$.

(c). Let us assume that there is at least one horizontal component, otherwise we have nothing to prove. Let $\{p_1, \ldots, p_\delta\} = D \cap \bar{F}$ and let Δ_j denote some small closed disk around the point p_j. The complex normal bundle of some horizontal divisor D_0 within X has a well defined orientation given by the complex orientation of X and that of D_0. The general fibres of $\bar{f}$ are transversal to D_0 and their orientation coincides with the one of the normal bundle. This implies in particular that all the small circles $\partial \Delta_j$ have a canonical orientation.

Let us consider the homology exact sequence of the pair of general fibres $(\bar{F}, F)$:

$$0 \to H_2(\bar{F}) \to H_2(\bar{F}, F) \xrightarrow{\nu} H_1(F) \to H_1(\bar{F}) \to 0$$

where one may identify $H_2(\bar{F}, F)$ with $\oplus_{j=1}^{\delta} H_1(\partial\Delta_j)$ by excision and boundary isomorphism, and where $H_2(\bar{F}) = \mathbb{Z}$. Denote by $e_j := \nu([\Delta_j, \partial\Delta_j])$ the generators of $\mathrm{Im}\,\nu \subset H_1(F)$. We may assume that all the points p_j are in some disc $\Delta \subset \bar{F}$ containing also the small disks Δ_j. Then the unique relation among the e_j's in $H_1(F)$ is precisely $e_1 + \cdots + e_\delta = \nu([\Delta_j, \partial\Delta_j]) = 0$.

If f is not simple then there exists some horizontal component D_0 of the divisor $D = \bar{X} \setminus X$ such that $\deg(\bar{f}_{|D_0}) \geq 2$. The geometric monodromy along any loop in $\bar{C}$ which avoids bifurcation values acts on the generators e_j corresponding to the points $\bar{F} \cap D_0$ by precisely an orientation preserving permutation of these e_j's. Say we have $e_j \mapsto e_k$ for some $k \neq j$. But $e_j = e_k$ cannot be a relation since this would be different from the unique one presented above. It therefore follows that the monodromy is not the identity, which is a contradiction. □

We have the following partial reciprocal of Proposition 2.7.

Proposition 2.8. *Let f be simple. If $\mathrm{Mon}\bar{f} = 1$ then $\mathrm{Mon}f$ is unipotent.*

Proof. If $\mathrm{Mon}^0\bar{f} = 1$ then $\bar{F}$ is connected (since X is smooth and irreducible), hence F is connected, so $\mathrm{Mon}^0 f = 1$. We also have that the general fibre $\bar{F}$ is smooth and irreducible. Then either F is an open set and then $H^2(F) = 0$, or $F = \bar{F}$ and then $\mathrm{Mon}f = \mathrm{Mon}\bar{f} = 1$. In the exact sequence (4) the monodromy acts as the identity on both sides. This proves the unipotency claim for $\mathrm{Mon}^1 f$. □

Question 2.9. Can one find an example of f simple and with $\mathrm{Mon}\bar{f} = 1$ but such that $\mathrm{Mon}f \neq 1$?

3. Topology of H^i-pure surfaces

Definition 3.1. A quasi-projective surface X is called *H^i-pure* for $i \geq 1$ if the mixed Hodge structure on the i-th cohomology $H^i(X)$ is pure.

Example 3.2. These conditions are extending the case $X = \mathbb{C}^2$. For instance $\mathbb{Q}$-homology planes are H^i-pure as they have vanishing cohomology. Also, if $\bar{X}$ is a smooth projective surface and D a connected divisor whose components form a tree of rational curves, then $X := \bar{X} \setminus D$ is a H^2- and H^3-pure surface.

Lemma 3.3. *Let X be a smooth quasi projective surface. Let $\bar{X}$ be a connected smooth compactification of X with $D := \bar{X} \setminus X$ a divisor with simple normal crossing. Then:*

(a) *X is H^3-pure if and only if the restriction map $H^0(\bar{X}) \to H^0(D)$ is surjective, i.e., if and only if D is connected.*
(b) *X is H^2-pure if and only if $H^1(\bar{X}) \to H^1(D)$ is surjective.*
(c) *X is H^1-pure if and only if $H^2(\bar{X}) \to H^2(D)$ is surjective, i.e., if and only if the components of D are linearly independent in $H_2(\bar{X})$.*

Proof. Consider the following long exact sequence of the pair $(\bar{X}, D)$

$$\cdots \to H^i(\bar{X}) \to H^i(D) \to H^{i+1}(\bar{X}, D) \to H^{i+1}(\bar{X}) \to \cdots$$

for $i = 0, 1, 2$. We have the following sequence of equivalences: $H^i(\bar{X}) \to H^i(D)$ is surjective $\Leftrightarrow H^{i+1}(\bar{X}, D) \to H^{i+1}(\bar{X})$ is injective $\Leftrightarrow$ the Lefschetz dual map induced by inclusion $H_{4-i-1}(X) \to H_{4-i-1}(\bar{X})$ is injective $\Leftrightarrow H^{4-i-1}(\bar{X}) \to H^{4-i-1}(X)$ is surjective (by the universal coefficient theorem on cohomology with complex coefficients) $\Leftrightarrow X$ is H^{4-i-1}-pure, by Lemma 2.4.

To finish the proof of (c), let us note that the surjectivity of $H^2(\bar{X}) \to H^2(D)$ is equivalent to the injectivity of the dual map $H_2(D) \to H_2(\bar{X})$, which just means the independence of the irreducible components of D in the 2-homology of $\bar{X}$. □

This result and its proof can easily be extended to any dimension n, as follows:

Proposition 3.4. *Assume X is a smooth connected complex variety and $\bar{X}$ is a completion to a smooth projective (complete) variety with the complement $D := \bar{X} \setminus X$. Then X is H^{2n-i-1}-pure if and only if $H^i(\bar{X}) \to H^i(D)$ is surjective.* □

Corollary 3.5. *If X is a H^2-pure surface, then $H^1(D)$ has a pure Hodge structure and hence the dual graph of D is a tree.*

Proof. Since $\bar{X}$ is a smooth complete variety, $H^1(\bar{X})$ has a pure Hodge structure. We have shown above that $H^1(\bar{X}) \to H^1(D)$ is surjective, hence $H^1(D)$ has a pure structure too. By the classical theory of weights it follows that the weight 0 part of H^1 of a normal crossing curve arises from the loops in the dual graph of D. Hence a simple normal crossing curve has H^1 pure if and only if its dual graph is a tree. □

Theorem 3.6. *Let X be a H^2-pure and H^3-pure surface and let $f : X \to C$ be a map onto an affine curve. Let $\bar{f} : \bar{X} \to \bar{C}$ be some compact resolution of f.*

(a) *If f is proper then there is no horizontal component, $\bar{C} \setminus C = \{p\}$ and $D = \bar{F}_p$.*
(b) *If $h^3(X) > 0$, then f is proper.*
(c) *If there is more than one horizontal component, then all horizontal components are rational, and $C \simeq \mathbb{C}$.*
(d) *If $\bar{f}^{-1}(\bar{C} \setminus C)$ is not connected, then there is a unique horizontal component, and $\bar{f}$ has no invariant cycles.*

Proof. (a). We have the equivalence f is proper $\Leftrightarrow$ there is no horizontal component. From the H^3-purity we get that D is connected and therefore $\bar{C} \setminus C = \{p\}$ and $D = \bar{F}_p$. Note that we have not used the H^2-purity here.
(b). The H^2-purity and H^3-purity yield the exactness of the following sequence

$$0 \to H^1(\bar{X}, D) \to H^1(\bar{X}) \to H^1(D) \to 0$$

which implies the equality $h^1(\bar{X}) = h^3(X) + h^1(D)$, where h^i denotes the dimension of the corresponding cohomology H^i.

Let us first assume that f is primitive. Let's then fix some $p \in \bar{C} \setminus C$. From Proposition 2.2 we have an injection $H^1(\bar{X})/H^1(\bar{C}) \to H^1(\bar{F}_p)$. It then follows:

$$h^1(\bar{F}_p) \geq h^1(\bar{X}) - h^1(\bar{C}) = h^3(X) + h^1(D) - h^1(\bar{C}). \tag{5}$$

Let us write additively $D = \bar{F}_p + \sum_{q\in\bar{C}\setminus(C\cup\{p\})} \bar{F}_q + D_0 + \sum_{i\neq 0} D_i + D_v$ as a divisor, without counting the multiplicities, where D_i for $i \geq 0$ denote the horizontal components, D_v is the sum of the vertical components over points of C, $\bar{F}_p$ and $\bar{F}_q$ are the fibres of $\bar{f}$ over $p, q \in \bar{C} \setminus C$. By substituting in (5), we obtain:

$$h^1(\bar{F}_p) \geq h^3(X) + [h^1(D_0) - h^1(C)] + h^1(\bar{F}_p) + \sum_{q\neq p} h^1(\bar{F}_q) + \sum_{i\neq 0} h^1(D_i) + h^1(D_v).$$

Hence by cancelling $h^1(\bar{F}_p)$ we get:

$$(6) \qquad 0 \geq h^3(X) + [h^1(D_0) - h^1(\bar{C})] + \sum_{i\neq 0} h^1(D_i) + h^1(D_v) + \sum_{q\neq p} h^1(\bar{F}_q).$$

By Riemann-Hurwitz, the term $h^1(D_0) - h^1(\bar{C})$ of (6) is non-negative whenever D_0 is not empty, i.e., D has at least one horizontal component, equivalently, $f : X \to C$ is not proper. Since the other terms are obviously non-negative, the above inequality implies that all the terms are 0. Then $h^3(X) = 0$, which is a contradiction. We also must have $h^1(\bar{C}) > 0$ since this term is the only one with negative sign in the inequality (6).

Now if f is not primitive, then we apply Stein factorisation and deduce that this is proper, hence f is proper by the commutativity of the factorisation diagram.

(c). First apply the Stein factorisation and use the inequality (6) for $\bar{f}'$. Let $D_1 \neq D_0$ be some other horizontal component. Then $h^1(D_1) = 0$, hence $h^1(\bar{C}') = 0$, so $h^1(D_0) = 0$. Moreover, since D is a tree, we get that there is no point $q \neq p$, hence $|\bar{C}' \setminus C'| = 1$, and thus $C \simeq \mathbb{C}$. Indeed, by Lemma 2.5, D_0 and D_1 would produce a cycle in the dual graph of D together with some connected component of $\bar{F}_p$ and some connected component of $\bar{F}_q$, and this contradicts Corollary 3.5.

(d). Two horizontal components would produce a loop in the dual graph of D together with two of the connected components of $\bar{f}^{-1}(\bar{C} \setminus C)$, which contradicts the fact that D is a tree, by Lemma 2.5.

After taking the Stein factorisation $\bar{f}'$, the inequality (6) shows that the fibre $\bar{F}'_p$ of $\bar{f}'$ may have nontrivial H^1 for at most one point $p \in \bar{C}' \setminus C'$. But since $|\bar{C}' \setminus C'| \geq 2$ it follows from (6) that $H^1(\bar{F}'_p) = 0$ for all points $p \in \bar{C}' \setminus C'$. By Proposition 2.2, $H^1(\bar{X}) \simeq H^1(\bar{C}')$ and the map $H^1(\bar{X}) \to H^1(\bar{F})$ is zero over $\bar{C}'$, hence over $\bar{C}$ too. Therefore $\bar{f}$ has no global invariant cycles, by Deligne's theorem [De1].

□

Theorem 3.7. *Let X be a H^2-pure and H^3-pure surface and let $f : X \to C$ be a non proper map onto an affine curve. Let $\bar{f} : \bar{X} \to \bar{C}$ be some resolution of f.*

If $g(\bar{C}) > 0$ or if $|\bar{C} \setminus C| \geq 2$ then f has a unique horizontal component.

If moreover f is primitive and either $g(\bar{C}) > 0$ or $|\bar{C} \setminus C| > 2$, then f is also simple.

Proof. If $|\bar{C} \setminus C| \geq 2$ then $\bar{f}^{-1}(\bar{C} \setminus C)$ is not connected and one applies Theorem 3.6(d) to get that there is a unique horizontal component.

Let us assume now $g(\bar{C}) > 0$. We consider the Stein factorisation $\bar{f}' : \bar{X} \to \bar{C}'$. From (6) and by the inequality $g(D_0) \geq g(\bar{C}') \geq g(\bar{C}) > 0$ which is a consequence of the Riemann-Hurwitz formula, it follows from Theorem 3.6(c) that there is only one horizontal component D_0 for $\bar{f}'$, hence a unique horizontal component for $\bar{f}$.

For our second claim, let us now assume that f is primitive.

If the degree d of $D_0 \to \bar{C}$ is > 1 then either $g(\bar{C}) = 1$ or $g(\bar{C}) = 0$. If $g(\bar{C}) = 1$ then there are no ramifications. But on the other hand, since D is a tree (Corollary 3.5), the horizontal component must be totally ramified over any point in $\bar{C} \setminus C$ since otherwise it will produce cycles in the dual graph of D. This gives a contradiction to $d > 1$. Thus the degree d must be 1, hence f is simple.

If $g(\bar{C}) = 0$ then, by Riemann-Hurwitz, either $d = 1$ hence f is simple, or $d > 1$ and there must be at least a ramification. In the later case, since the dual graph of D is a connected tree, all such ramifications over some point of $\bar{C} \setminus C$ must be total and by Riemann-Hurwitz we get $|\bar{C} \setminus C| \leq 2$, which contradicts our assumption $|\bar{C} \setminus C| > 2$. □

Example 3.8. This shows that if f is not primitive then it might be not simple. Let C_1 and C_2 be two smooth projective curves of genus > 1. Let $h : C_1 \to \mathbb{P}^1$ be some regular map such that $\deg h > 1$ and $h^{-1}(\infty) = p$, for some $p \in C_1$. Let $\bar{X} := C_1 \times C_2$ and $D := C_1 \times \{q\} \cup \{p\} \times C_2$ be the union of sections of different projections. Since the dual graph of D is a connected tree, $X := \bar{X} \setminus D$ is H^2-pure and H^3-pure. Let $\pi : C_1 \times C_2 \to C_1$ denote the projection and let $f := h \circ \pi : X \to \mathbb{C}$. Then f is not primitive and not simple. One may produce other examples taking instead of h some map $C_1 \to C_3$ with $g(C_3) > 0$ which is totally ramified at p.

The next result extends Dimca's [Di, Theorem 1(i-iii)] for polynomials of two variables.

Theorem 3.9. *Let X be an H^2 and H^3-pure surface and $f : X \to C$ be a regular map onto an affine curve C which is not proper. Let $p \in \bar{C} \setminus C$ and γ_p be a 'small' loop around p. Let T_p be the monodromy of f defined by γ_p and assume that $\bar{F}_p$ is connected.*

Then:

(a) *If $T_p = 1$ then f is simple and* $\mathrm{Mon}\bar{f} = 1$. *If f is simple and* $\mathrm{Mon}\bar{f} = 1$ *then T_p is unipotent.*

(b) *If $\bar{F}_p$ is a tree of rational curves then the eigenvalue 1 of the matrix of T_p occurs in Jordan blocks of size 1 and its multiplicity is the number of horizontal divisors minus one.*

(c) *If the fibre $\bar{F}$ is rational then T_p is finite.*

Proof. (a). The fact that D is a connected tree and that $\bar{F}_p$ is connected implies that every horizontal component of D is totally ramified over the point p. The triviality of T_p implies in addition that the local degree of $\bar{f}_{|D_i}$ over p must be 1, for any horizontal component D_i, i.e., f must be locally simple over p and therefore simple.

If $T_p = 1$ then $\bar{T}_p : H^1(\bar{F}) \to H^1(\bar{F})$ is the identity. Then the *local invariant cycle theorem* [Cl] tells that the image of the map induced by the restriction from a small tubular neighbourhood $H^1(U_p) \cong H^1(\bar{F}_p) \to H^1(\bar{F})$ is surjective, where $\bar{F}$ denotes here the general fibre in U_p. Therefore one has an isomorphism $H^1(\bar{F}_p) \cong H^1(\bar{F})$. Since $\bar{F}_p \subset D$, by the assumption H^2-pure we obtain that both following maps $H^1(\bar{X}) \to H^1(D) \to H^1(\bar{F}_p)$ are surjective, hence their composition too. Then the global invariant cycle theorem [De1] yields that $\mathrm{Mon}\bar{f} = 1$.

The last statement follows from the proof of Proposition 2.8.
(b). and (c). follow by the same arguments of [Di], from the exact sequence (4) and the analysis given in the proof of Proposition 2.7 using the condition that D is a connected tree and that $\bar{F}_p$ is connected which imply that any horizontal divisor is totally ramified over each point $p \in \bar{C} \setminus C$.

□

From the statement and proof of Theorem 3.9 we get now easily the following, which recovers the results by Miyanishi-Sugie [MS] and Dimca [Di] cited in the Introduction.

Corollary 3.10. *Let X be H^2 and H^3-pure. Let $C = \mathbb{C}$ and let $\bar{F}_\infty$ be connected. If $T_\infty = 1$ and $b_1(\bar{X}) = 0$ then f is rational and* $\mathrm{Mon} f = 1$. *Reciprocally, if f is rational and $T_\infty = 1$ then* $\mathrm{Mon} f = 1$. □

4. Bounding the number of reducible fibres

In this section we focus to another aspect, investigated by Kaliman [Kal] in case of polynomials $f : \mathbb{C}^2 \to \mathbb{C}$, namely the the number of reducible fibres. Kaliman proved that this number is at most $\delta - 1$, where $\delta = \#$horizontal components of D, and remarked that this holds (with almost the same proof) for an acyclic surface X instead of $\mathbb{C}^2$. Our aim is to extend the setting to more general surfaces X. Let us remark that the number of reducible fibres is finite only if f is primitive.

Theorem 4.1. *Let $f : X \to C$ be a primitive morphism from a H^2-pure and H^3-pure surface onto an affine curve. Then the number of reducible fibres of f is at most $b_2(X) + \delta - 1$.*

We need two lemmas. Let $\bar{f} : \bar{X} \to \bar{C}$ be a compact resolution of f. We use the following notations: $\bar{F}$ is the generic fibre of $\bar{f}$, $\bar{F}_p := \bar{f}^{-1}(p)$ (and not the closure of the fibre $f^{-1}(p)$).

Lemma 4.2. *Let $f : X \to C$ be a primitive morphism from a H^2-pure surface onto an affine curve C.*

(a) *If f is non-proper then $\sum_{p\in C} b_2(F_p) \le b_2(X)$.*
(b) *If f is proper then $\sum_{p\in C}(b_2(F_p) - 1) \le b_2(X) - 1$.*

Proof. (a). Let A_p denote the union of all the affine irreducible components of F_p. Then the intersection matrix of the components of the divisor $\sum_{p\in C} \mathrm{cl}(F_p \setminus A_p)$ (where cl means taking the closure in $\bar{X}$) is negative definite, by Zariski's Lemma, see [BPV, Lemma 8.2, (9) and (10)]. Therefore these components are linearly independent in $H_2(\bar{X})$. It follows that they are independent in $H^2(X)$ since they have support in X and since the inclusion $X \subset \bar{X}$ induces an inclusion in H_2 due to the assumed H^2-purity. The independence of these components yields now the desired inequality $\sum_{p\in C} b_2(F_p) \leq b_2(X)$.
(b). Let A_p be some irreducible component of F_p. Then the same proof as above yields that the components of the divisor $\sum_{p\in C} \mathrm{cl}(F_p \setminus A_p)$ are linearly independent in $H_2(X)$. Moreover, the general fibre F is independent of these components □

Let us set the notations $k_p := b_1(\bar{F}) - b_1(\bar{F}_p) \geq 0$ (see the proof of Proposition 2.2), $v_p :=$ #vertical irreducible components of $\bar{F}_p$ in D, $a_p :=$ #irreducible components of $\bar{F}_p$ which are not in D.

Lemma 4.3. *Let $f : X \to C$ be a non proper, primitive morphism from a H^2-pure surface onto an affine curve C. Then:*

$$\sum_{p\in\bar{C}}(a_p - 1) + \sum_{p\in\bar{C}} k_p = \delta + 2 + b_2(X) - b_1(X) - b_1(\bar{X}) - \chi(\bar{F})\chi(\bar{C}). \tag{7}$$

Proof. One has the following relation between Euler characteristics, which holds for any regular map from a projective surface to a projective curve:

$$\chi(\bar{X}) - \chi(\bar{F})\chi(\bar{C}) = \sum_{p\in\bar{C}}[\chi(\bar{F}_p) - \chi(\bar{F})]. \tag{8}$$

Using the primitivity assumption, formula (8) yields:

$$\begin{gathered} 2 - 2b_1(\bar{X}) + b_2(\bar{X}) - \chi(\bar{F})\chi(\bar{C}) = \\ \textstyle\sum_{p\in\bar{C}}[1 - b_1(\bar{F}_p) + b_2(\bar{F}_p) - 2 + b_1(\bar{F})] = \sum_{p\in\bar{C}}[k_p + v_p + a_p - 1]. \end{gathered} \tag{9}$$

Let us now consider the exact sequence of the pair $(\bar{X}, D)$:

$$0 \to H^2(\bar{X}, D) \to H^2(\bar{X}) \to H^2(D) \to H^3(\bar{X}, D) \to H^3(\bar{X}) \to 0 \tag{10}$$

where the 0 at the left side is due to the H^2-purity of X and the 0 at the right hand side comes from $H^3(D) = 0$. By duality we have the isomorphisms $H^2(\bar{X}, D) \simeq H_2(X)$, $H^2(\bar{X}) \simeq H_2(\bar{X})$, $H^3(\bar{X}, D) \simeq H_1(X)$, $H^3(\bar{X}) \simeq H_1(\bar{X})$. We also have $b_2(D) = \delta + \sum_{p\in\bar{C}} v_p$, since D is a tree with vetical and horizontal components. We obtain the following relation from (10):

$$b_2(\bar{X}) = b_2(X) + \delta + \sum_{p\in\bar{C}} v_p - b_1(X) + b_1(\bar{X}) \tag{11}$$

Substituting this expression of $b_2(\bar{X})$ in (9) yields (7). □

4.1. **Proof of Theorem 4.1.** The total number of reducible fibres of f is at most $\sum_{p\in C}(a_p-1)$.

If f is proper then F_p is compact and $a_p = b_2(F_p)$. Then the result follows directly from Lemma 4.2(b), since $\delta = 0$.

Let us therefore assume in the following that f is non-proper. The proof follows from a case-by-case study.

Case $g(\bar{C}) > 0$. By Theorem 3.7, f has a unique horizontal component, thus $\delta = 1$, and is simple. This implies that there is a single puncture at infinity for all fibres. Since the components of the fibres are separated by the resolution, it follows that any fibre of f has a single affine irreducible component. We therefore have $a_p - 1 = b_2(F_p)$. Taking the sum over p we get $\sum_{p\in C}(a_p-1) = \sum b_2(F_p) \le b_2(X)$ from Lemma 4.2(a). Our Theorem follows.

Case $g(\bar{C}) = 0$ *and* $|\bar{C}\setminus C| > 2$. By Theorem 3.7, we get that f is simple. Then as in the previous paragraph, we coclude the requered inequality.

Case $g(\bar{C}) = 0$ *and* $|\bar{C}\setminus C| = 2$. By definition, $a_p = 0$ if $p \in \bar{C}\setminus C$. Under the hypotheses of H^2- and H^3-pure surface X and $|\bar{C}\setminus C| = 2$, by the proof of Theorem 3.6(d), we have $\delta = 1$, $b_1(\bar{F}_p) = 0$ for $p \in \bar{C}\setminus C$, $C \cong \mathbb{C}^*$ and $b_1(\bar{X}) = 0$. Moreover, $b_3(X) = 0$ by Theorem 3.6(b). Moreover if $\bar{X}_C := \bar{f}^{-1}(C)$ denote the inverse image C, then we have $1 = H^1(C) \subset H^1(\bar{X}_C) \subset H^1(\bar{X})$, hence $b_1(X) \ge 1$.

Substituting these in (7) we get :

$$\sum_{p\in C}(a_p-1) + \sum_{p\in C} k_p + b_1(X) - 1 = b_2(X).$$

Now we have $k_p \ge 0$ for any $p \in C$. Hence the inequality: $\sum_{p\in C}(a_p-1) \le b_2(X)$ which proves our theorem in this case.

Case $C \simeq \mathbb{C}$. Let us denote by ∞ the point $\bar{C}\setminus C$. Substituting in (7) the terms $\chi(\bar{C}) = 2$, $a_\infty = 0$, $k_\infty = b_1(\bar{F}) - b_1(\bar{F}_\infty)$, we get:

$$\sum_{p\in C}(a_p-1) + \sum_{p\in C} k_p - b_1(\bar{F}) + b_1(X) = \delta - 1 + b_2(X) + [b_1(\bar{F}_\infty) - b_1(\bar{X})]. \tag{12}$$

Let us first show that the last couple of terms on the right hand side is cancelling. The following exact sequence is exact by using the H^3-purity (left) and the H^2-purity (right):

$$0 \to H^1(\bar{X}, D) \to H^1(\bar{X}) \to H^1(D) \to 0,$$

where $H^1(\bar{X}, D) \simeq H^3(X) = 0$ by Theorem 3.6(b). But $H^1(D) \simeq H^1(\bar{F}_\infty)$ since D is a tree and since $h^1(D_i) = 0$ for any the horizontal or vertical remaining divisor D_i, by the relation (6). Hence $b_1(\bar{X}) = b_1(\bar{F}_\infty)$.

Next, we show that $\sum_{p\in C} k_p - b_1(\bar{F}) + b_1(X)$ is non-negative. Indeed, since $C \simeq \mathbb{C}$, we may contract $\mathbb{C}$ to a union of small disks around the set of critical values $A = \{p_1, \ldots\}$ and union with simple non-intersecting paths connecting them with some exterior point in $\mathbb{C}$. Then the pullback of this total union is homotopy equivalent to $\bar{X}_{\mathbb{C}}$, and by using a Mayer-Vietoris argument, we have the following exact sequence:

$$\oplus_i H_1(\bar{F}) \stackrel{\nu}{\to} H_1(\bar{F}) \oplus \oplus_i H_1(\bar{F}_{p_i}) \to H_1(\bar{X}_{\mathbb{C}}) \to 0$$

which shows that $\sum_{p\in C} k_p - b_1(\bar{F}) + b_1(\bar{X}_{\mathbb{C}}) \geq 0$. We may replace here $b_1(\bar{X}_{\mathbb{C}})$ by $b_1(X)$ since $b_1(X) \geq b_1(\bar{X}_{\mathbb{C}})$. Therefore from (12) we finally get the inequality:

$$\sum_{p\in C}(a_p - 1) \leq \delta - 1 + b_2(X)$$

which proves our claim. This ends the proof of our theorem. □

Note 4.4. In the case $g(\bar{C}) = 0$ one may also build a uniform proof starting from the approach used in the case $C \simeq \mathbb{C}$.

Note 4.5. We may actually prove a sharper statement. Let X be a smooth surface and let $\bar{X}$ be some compactification of X with $D := \bar{X}\setminus X$ a normal crossings divisor. Let $H^1(\Omega^1_{\bar{X}})$ be the cohomology group in $H^2(\bar{X}, \mathbb{C})$ generated by algebraic cycles. Let $\Omega(X)$ denote the vector subspace of $H^2(X)$ generated by complete curves in X, that is $\Omega(X) := \mathrm{Im}\,(H^1(\Omega^1_{\bar{X}}) \to H^1(\Omega^1_{\bar{X}}\langle D\rangle)) \subset H^2(X)$, and let $\omega(X) := \dim \Omega(X)$.

Counting the complete curves in X like done in Lemma 4.2 (a) and (b), we obtain $\sum_p b_2(F_p) \leq \omega(X)$ in the non-proper case and $1 + \sum_p (b_2(F_p) - 1) \leq \omega(X)$ in the proper case. We conclude that one may replace $b_2(X)$ by $\omega(X)$ in Theorem 4.1 and since by definition $\omega(X) \leq b_2(X)$, we get a sharper inequality.

Example 4.6. Kaliman [Kal] also proved that for a polynomial function $\mathbb{C}^2 \to \mathbb{C}$ with rational fibres, at most one among the horizontal components may have degree greater than 1. The following example shows that this statement does not extend in our more general setting, namely that more than one component may have higher degree.

Let $f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the first projection and consider C_1, C_2, two type $(d, 1)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$ such that the restrictions $f_{|C_1}$ and $f_{|C_2}$ of the projection are totally ramified over some point $p \in \mathbb{P}^1$ at two distinct points, $a \in f^{-1}(p)$ and $b \in f^{-1}(p)$, respectively. We have $C_1 \cdot C_2 = 2d$ and let us assume that these intersections are simple, thus at $2d$ distinct points $\{x_1, \ldots, x_{2d}\} \in \mathbb{P}^1 \times \mathbb{P}^1$. Let $\bar{X}$ be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at those points, let $\bar{f} : \bar{X} \to \mathbb{P}^1$, let C_1' and C_2' denote the proper transforms of the curves and let $D := C_1' \cup C_2' \cup \bar{f}^{-1}(p)$. Then D is a connected tree of $\mathbb{P}^1$'s, hence $X := \bar{X} \setminus D$ is a H^2-pure and H^3-pure surface. The horizontal components of D are C_1' and C_2', and the degree of the restriction of $\bar{f}$ is equal to d on both of them.

5. Trivial mondromy with irreducible fibres

In this section we generalize the theorem of Dimca [Di] that if a polynomial in 2 variables has irreducible fibers and trivial monodromy, then it is an algebraically trivial fibration. We will begin with some examples to explain what we may expect in this situation for maps on H^2 and H^3-pure surfaces.

REMARK 5.1. *Even if f has trivial monodromy and irreducible fibres, one may not even get analytic local triviality.* Indeed, consider a curve C of genus $g(C) > 1$ and a point $p \in C$. Let $\bar{X} := C \times C$ and let $D := \Delta_C \cup \{p\} \times C$ be the union of the diagonal Δ_C and a fibre. Then $X := \bar{X} \setminus D$ is an H^2 and H^3-pure smooth surface. The restriction of the projection $f : X \to C$ has trivial monodromy and irreducible fibres. It is not analytically locally trivial since $C \setminus \{x\}$ and $C \setminus \{y\}$ cannot be isomorphic for all points x and y due to the fact that the set of automorphisms of a curve of genus > 1 is finite. However one has only topological local triviality.

REMARK 5.2. *Even if f has trivial monodromy and irreducible fibres, it can still have nonreduced fibres and hence may not be topologically locally trivial.* To show this, consider the quadric hypersurface Z in $\mathbb{C}^3$ defined by th equation $x^2 - yz = 0$. The y-coordinate projection has every fibre isomorphic to $\mathbb{C}$ but the fibre over 0 is multiple, defined by $x^2 = 0$ in the (x, z) plane. This projection $Z \to \mathbb{C}$ has a section $\sigma : \mathbb{C} \to Z$ given by $\sigma(y) = (0, y, 0)$. Then $X := Z \setminus \sigma(\mathbb{C})$ is a smooth H^2 and H^3-pure surface and the restriction of f to X has trivial monodromy since the fibres are affine lines. Each fibre is irreducible, isomorphic $\mathbb{C}^*$, but there is a nonreduced fibre and hence cannot be locally trivial.

These remarks support the sharpness of the following theorem, which is the main result of this section:

Theorem 5.3. *Let $f : X \to C$ be a morphism from a H^2 and H^3 pure surface onto an affine curve, with trivial monodromy group. If all fibres of f are irreducible then all fibres of f are diffeomorphic. If moreover all fibres are assumed to be reduced, then f is a locally trivial fibration.*

Proof. Consider $\bar{f} : \bar{X} \to \bar{C}$, a compact resolution of f with a smooth projective completion $\bar{X}$ of X and $D := \bar{X} \setminus X$, a divisor with simple normal crossings. For $p \in \bar{C} \setminus C$, we have $\bar{F}_p \subset D$. Since f has trivial mondromy, f is simple and primitive by Proposition 2.7. Also Proposition 2.2 implies that $H^1(\bar{F}_t) \cong H^1(\bar{F}_p)$ for all $t \in \bar{C}$, where $\bar{F}_t$ is the fibre of $\bar{f}$. Let F_t be the fibre of f and $\mathrm{cl}(F_t)$ denote the closure of F_t in $\bar{X}$. Since F_t is irreducible, every other components of $\bar{F}_t$ are contained in D. Moreover D being a tree by the H^2-purity, the number of points in $\mathrm{cl}(F_t) \setminus F_t$ is the same. Indeed, any reduction in this number would produce the intersection of two horizontal components which creates a loop as they alredy intersect the fibre F_p. On the other hand, an increase in this number would imply that D is not connected. By (5) each vertical component of D that is not contained in $\bar{F}_p$ is a rational curve. Thus we get the isomorphisms $H^1(\bar{F}_t) \cong H^1(\mathrm{cl}(F_t)) \cong H^1(\bar{F}_p)$ for all $t \in C$. This shows that all F_t have the same topology and in particular that all have nonsingular reduced structure. Since open Riemann surfaces, they are also diffeomorphic.

To prove our second statement, let D_t be the union of vertical components of D lying over $t \in C$. Every connected component of D_t intersects $\mathrm{cl}(F_t)$ in a unique point and distinct connected components of D_t intersect $\mathrm{cl}(F_t)$ in distinct points. Each connected component of D_t is negative definite and hence blows down to a normal analytic singularity by Grauert [Gr, Satz 3.1]. After blowing down all connected components of D_t, for points $t \in C$, we obtain a normal compact analytic surface X' together with the induced map $f' : X' \to \bar{C}$. Now for each $t \in C$, the arithmetic genus $p_a(\mathrm{cl}(F_t))$ of the closure $\mathrm{cl}(F_t)$ is equal to the arithmetic genus $p_a(\bar{F})$ of the general fibre of $\bar{f}$. This follows as the monodromy is trivial and all other components of $\bar{F}_t$ are contained in D which is a tree of smooth rational curve. Let $\bar{F}'_t$ denote the fibre of f' over $t \in C$ and $\bar{F}'$ denote the general fibre, respectively. Then the map $\mathrm{cl}(F_t) \to \bar{F}'_t$ is bijective and hence we have:

$$p_a(\bar{F}') = p_a(\bar{F}) = p_a(\mathrm{cl}(F_t)) \leq p_a((\bar{F}'_t)_{\mathrm{red}})$$

We also have

$$p_a(\bar{F}'_t) = p_a(\bar{F}') = p_a(\bar{F}).$$

If $\bar{F}'_t$ is reduced, then all these will be equal and it follows that $\bar{F}'_t$ is smooth, for all $t \in C$. Since C is also smooth, it implies that X' is smooth. All fibres have the same number of points at infinity and these are smooth points, we conclude that $X \to C$ is a locally trivial fibration by the relative Ehresmann Fibration Theorem.

To show now that $\bar{F}'_t$ is reduced, we note that all fibres of f' are generically reduced as fibres of f are reduced and f and f' coincide on X. So we only need to check that the fibre $\bar{F}'_t$ has no embedded component. On the other hand X' is a normal surface and hence is Cohen Macaulay. On a Cohen Macaulay germ, any Cartier divisor which is generaically reduced is a reduced divisor. Hence $\bar{F}'_t$ is reduced proving the Theorem. □

Corollary 5.4. *Let $f : X \to C$ be a morphism from a H^2 and H^3 pure surface onto an affine curve, with trivial monodromy group. If all fibres of f are irreducible and there is a nonreduced fibre, then the general fibre is isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$.*

Proof. Consider $\bar{f} : \bar{X} \to \bar{C}$, a smooth projective resolution of f like above and let $f' : X' \to \bar{C}$ be the blow down of all vertical components of D over points of C as in the proof of Theorem 5.3. By Theorem 5.3 all fibres of f are smooth (when considered with reduced structure) and diffemorphic. Let $t_1, t_2, \ldots, t_r \in C$ be the points where f has multiple fibres with multiplicities $m_1, m_2, \ldots, m_r > 1$ respectively. Let $g : C' \to C$ be a ramified Galois cover of C of ramification index m_i at $t_i \in C$ for $i = 1, 2, \ldots, r$, which means that at any ramification point $t'_i \in C'$ over t_i the index is the same, equal to m_i. One can always find such a cover, eventually with some more branch points in C. Let $h : Y \to C'$ and $h' : Y' \to C'$ be the normalization of the fibre product of f, g and f', g respectively. This construction implies that all fibres of $h : Y \to C'$ are smooth reduced and the restriction map $h^{-1}(t'_i) \to f^{-1}(t_i)_{\mathrm{red}}$ is étale, which fact will be used later in this proof.

Loops in components of fibres of $h' : Y' \to C'$. Note that $h'^{-1}(t') \cong f'^{-1}(g(t'))$ for all $t' \in C'$ such that $g(t') \neq t_i$. The only structural change happens at fibres $h'^{-1}(t'_i)$ for $t'_i \in g^{-1}(t_i)$. For any $t'_i \in g^{-1}(t_i)$, $h'^{-1}(t'_i) \to f'^{-1}(t_i)$ is a ramified cover of degree m_i, which is étale over $f^{-1}(t_i)$. Let $D_1, \cdots, D_k$ be the horizontal components of $X' \to \bar{C}$. By the definition of Y and of Y' the inverse image D'_i of D_i in Y' is a curve isomorphic to C' which forms a section of h'. Hence every component of the fibre ${h'}^{-1}(t'_i)$ has k points at infinity (i.e. points that are not in the fibre of h) and they are the same for any i. Then the projection ${h'}^{-1}(t'_i) \to f'^{-1}(t_i)$ is totally ramified at these points. Consequently, if $k > 1$ and if ${h'}^{-1}(t'_i)$ is reducible then there are loops in the dual graph of this fibre.

Genus of the components of fibres of $h' : Y' \to C'$. Without loss of generality let us choose $i = 1$ and fix a point t'_1 lying over $t_1 \in C$. Let ${h'}^{-1}(t'_1) := C'_1 \cup \cdots \cup C'_r$ be the decomposition of the fibre ${h'}^{-1}(t'_1)$ into its irreducible components. Then $p_a(C_j) \geq p_a(f'^{-1}(t_1))$ for all j, and moreover f' and h' have isomorphic general fibres. Let us denote by $t' \in C'$ be a general point (near t'_1) and let $t := g(t') \in C$. Then we get:

$$p_a({h'}^{-1}(t')) = p_a({h'}^{-1}(t'_1)) \geq \sum_{j=1}^{r} p_a(C_j) \geq r p_a({f'}^{-1}(t_1)) = r p_a({f'}^{-1}(t)) = r p_a({h'}^{-1}(t))$$

The first inequality follows from the fact that the fibre ${h'}^{-1}(t'_1)$ is generaically reduced and hence is reduced as Y' is Cohen Macaulay, and the fact that the arithmetic genus of a reduced curve is at least the sum of the genera of each component. Now we analyse the following cases.

Case 1: $p_a({h'}^{-1}(t')) > 1$. From the above sequence it follows that if $p_a({h'}^{-1}(t')) > 0$ then $r = 1$ and therefore all the inequalities are equalities. But if $p_a({h'}^{-1}(t)) > 1$, the Riemann-Hurwitz theorem implies $p_a(C_1) > p_a(f'^{-1}(t_1))$, contradiction.

Case 2: $p_a({h'}^{-1}(t')) = 1$. Then again $r = 1$ and all inequalities are equalities, thus $p_a({f'}^{-1}(t)) = p_a(C_1) = 1$. By the first part of Theorem 5.3, the special fibre of f' has the same genus, i.e., $p_a(f'^{-1}(t_1)) = 1$. Since $C_1 \to f'^{-1}(t_i)$ is a degree $m_i > 1$ covering of curves of genus 1 with total ramifications at the points at infinity, the Riemann-Hurwitz argument yields again a contradiction.

We therefore conclude that $p_a({h'}^{-1}(t))$ must be equal to 0, hence all the curves in the above displayed sequence are rational.

Case 3: $p_a({h'}^{-1}(t')) = 0$. This implies that the fibre ${h'}^{-1}(t'_1)$ cannot have "loops" in its dual graph. If $k > 1$, the above argument of "loops in components" shows that this fibre is irreducible. Thus the fibres $h^{-1}(t'_1)$ and $f^{-1}(t')$ are a $\mathbb{P}^1$ minus k points and the map between them is étale, which implies that $k = 2$, i.e. exactly two sections at infinity. It follows that the general fibre of f must be $\mathbb{C}^*$.

If $k = 1$ then the general fibre of f is $\mathbb{C}$, and each fibre $h^{-1}(t'_i)$ consists of m_i disjoint copies of $f^{-1}(t_i) \cong \mathbb{C}$, as there are no non-trivial étale covers of $\mathbb{C}$. □

Remark 5.5. From the last part of the above proof, under the same hypotheses, we moreover get the following byproduct:

If the general fibre of f is $\mathbb{C}^*$, then all the fibres of h' are irreducible and, as in the proof of Theorem 5.3, h' is a locally trivial fibration. Since every fibre of h' is a smooth $\mathbb{P}^1$, it follows that Y' is smooth and $Y' \to X'$ is an étale cover of $X' \setminus \cup_{i,j}\{f'^{-1}(t_i) \cap D_j\}$. Hence X' is the quotient of a smooth surface by a finite group (in fact the Galois group of $C' \to C$).

If the general fibre of f is $\mathbb{C}$, then $Y' \to X'$ is an étale cover outside $\cup_i\{f'^{-1}(t_i) \cap D_1\}$, but h' cannot be a fibration and actually Y' cannot be a non-singular variety.

Note 5.6. It is interesting to compare our Corollary 5.4 to Zaidenberg's result [Za1, Lemma 3.2(b)] since the conclusions are the same, namely that f is either a $\mathbb{C}$- or a $\mathbb{C}^*$-family of affine curves. The contexts are however entirely different. Zaidenberg has a local assumption that f is affine or Stein but no other assumption on the local geometry of the divisor at infinity, while we have H^2- and H^3-purity, therefore strong global restrictions on components of the divisor at infinity. On the other hand, he assumes the constancy of the Euler characteristic while we assume here the triviality of the monodromy group.

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