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THE MINIMAL RESOLUTION CONJECTURE ON A GENERAL QUARTIC SURFACE IN $\mathbb{P}^3$

M. BOIJ, J. MIGLIORE, R.M. MIRÓ-ROIG, AND U. NAGEL

Abstract. Mustaţă has given a conjecture for the graded Betti numbers in the minimal free resolution of the ideal of a general set of points on an irreducible projective algebraic variety. For surfaces in $\mathbb{P}^3$ this conjecture has been proven for points on quadric surfaces and on general cubic surfaces. In the latter case, Gorenstein liaison was the main tool. Here we prove the conjecture for general quartic surfaces. Gorenstein liaison continues to be a central tool, but to prove the existence of our links we make use of certain dimension computations. We also discuss the higher degree case, but now the dimension count does not force the existence of our links.

1. Introduction

The shape of the minimal free resolution (MFR) of a general set, X, of points in the projective plane has been known for many years, probably due to Gaeta. See [G] for a more recent description by the same author. For points in $\mathbb{P}^3$ the shape of the MFR was discovered and shown by Ballico and Geramita [BG]. In her Ph.D. thesis and later in [L], Lorenzini conjectured the shape of the MFR for a general set of points in $\mathbb{P}^n$. Roughly speaking, if we denote by R the polynomial ring $k[x_0, \dots, x_n]$ (k is an algebraically closed field) and we set

$$\beta_{i,j} = \operatorname{Tor}^i(R/I_X, k)_{i+j},$$

then the Betti diagram $\{\beta_{i,j}\}$ consists of two non-trivial rows and we have $\beta_{i,j} \cdot \beta_{i+1,j} = 0$ for all i and j. The latter condition says that there are no redundant terms in the MFR. This conjecture was proven for points in $\mathbb{P}^4$ by Walter [W], and for large numbers of points in any projective space by Hirschowitz and Simpson [HS]. However, Schreyer experimentally found a probable counterexample in the case of 11 general points in $\mathbb{P}^6$ (and a few others), and in a stunning development, a proof was given by Eisenbud and Popescu [EP] that these were, in fact, counterexamples and that there is a much larger class of counterexamples. Further work in this direction was done by Eisenbud, Popescu, Schreyer and Walter [EPSW].

A new Minimal Resolution Conjecture (MRC) was formulated by Mustaţă in [M] concerning a general set, X, of sufficiently many points on an irreducible algebraic subvariety, S, of $\mathbb{P}^n$. Essentially the MRC says that the top part of the Betti diagram for R/I_X consists of the Betti diagram for R/I_S, and that below this part there are only two nonzero rows, again with no redundant terms in the MFR. In that paper he stated the MRC and proved

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Key words and phrases. minimal resolution conjecture; Mustaţă conjecture; Betti numbers; Gorenstein ideals; liaison; linkage; Hilbert scheme.

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several initial results, including a nice periodicity theorem in the case where S is an integral curve. Later, Farkas, Mustaţă and Popa [FMP] proved the MRC for the case where S is a nonhyperelliptic canonically embedded curve. On the other hand, they proved that the MRC fails when S is a curve of large degree.

Turning to the case that S is a surface in $\mathbb{P}^3$, several cases of the MRC have been proven. Giuffrida, Maggioni and Ragusa [GMR] proved it for a general set of points on a smooth quadric surface in $\mathbb{P}^3$. Casanellas [C] proved it for certain cardinalities of points on a smooth cubic surface, and the result was extended to any number of points on a smooth cubic surface more or less at the same time by Migliore and Patnott [MP] and by Miró-Roig and Pons-Llopis [MP2]. The same latter two authors also prove the MRC for certain cardinalities on del Pezzo surfaces [MP1]. Casanellas introduced the use of Gorenstein liaison theory in this question, and the latter two papers continued this approach.

The conjecture includes, as special cases, the so-called Ideal Generation conjecture and the Cohen-Macaulay Type Conjecture. These govern the expected number of minimal generators of I_X and the expected Cohen-Macaulay type of R/I_X; in other words, they govern the beginning and the end of the MFR. For general points in $\mathbb{P}^2$, either the Ideal Generation Conjecture or the Cohen-Macaulay Type Conjecture imply the other. For general points in $\mathbb{P}^3$, knowing both the Ideal Generation Conjecture and the Cohen-Macaulay Type Conjecture gives the MRC as a consequence. For higher projective spaces, this is not true. In [MP1], Miró-Roig and Pons-Llopis also address the latter two conjectures for zero-dimensional schemes on del Pezzo surfaces. In [MP2] the authors also use the known work on ideals generated by general forms of fixed degree in $k[x, y, z]$.

The goal in this paper is to understand the minimal free resolution of a general set of points, X, of fixed cardinality on a general surface, S, of degree d. We will assume that the socle degree of X is larger than d, so that S is the unique surface of degree d containing X. By semicontinuity, it is enough to produce one set of points with the desired resolution, on any surface of degree d. Our strategy is to use Gorenstein liaison to control the resolutions, extending the known methods. We develop a general framework and then specialize to quartics.

A general set of points on an irreducible surface S is *relatively compressed* (see Definition 2.3), and this allows us to partition the cardinalities of the general finite subsets according to the *socle degree*, e, of their artinian reductions. Our approach to the MRC is along the lines of this partition, using induction on e. We use liaison theory to describe certain links that, if they exist, are enough to give the conjectured resolution for any given $e \geq d + 1$. (Lower values of e come for free from work of Ballico and Geramita [BG].) This is done in Section 3.

It is known by work of Beauville [B] that the equation of a general surface of degree $d \leq 15$ can be written as the pfaffian of a skew-symmetric matrix of linear forms, and by work of Faenzi [F] that when $d \leq 14$ is even, S can also be written as the pfaffian of a skew-symmetric matrix of quadratic forms. This is necessary in order for our Gorenstein links to exist, but it is not sufficient.

In order to produce our links we use a dimension argument based on a theorem of Kleppe and the third author [KM] (see Theorem 4.4). Unfortunately it only works for $d = 4$. Nevertheless, it is enough to give us our main theorem, Theorem 4.5, that the MRC is true on a general quartic surface.

For surfaces of higher degree, it is not necessary to use "general" surfaces right from the start. Instead, one could produce sets of points with the expected resolution on a "less general" surface and rely on the semicontinuity for the result. This approach might allow one to use different tricks to produce the Gorenstein links.

2. Background

We let $R = k[x_0, x_1, x_2, x_3]$ where k is an algebraically closed field of characteristic $\neq 2$ (to allow the use of the results of [KM]).

If R/I is a standard graded algebra over R, it admits a minimal free resolution

$$\mathbb{F}: \quad 0 \to \mathbb{F}_4 \to \mathbb{F}_3 \to \mathbb{F}_2 \to \mathbb{F}_1 \to \mathbb{R} \to R/I \to 0.$$

If $\mathbb{F}_i = \bigoplus_{j\in\mathbb{N}} R(-j)^{\beta_{i,j}}$ then $\beta_{i,j} = \dim \operatorname{Tor}_i(\mathbb{F}, k)_j$ and the *Betti table* for R/I is the array

$$\begin{bmatrix} 1 & \beta_{1,1} & \beta_{2,2} & \beta_{3,3} & \beta_{4,4} \\ - & \beta_{1,2} & \beta_{2,3} & \beta_{3,4} & \beta_{4,5} \\ - & \beta_{1,3} & \beta_{2,4} & \beta_{3,5} & \beta_{4,6} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

(In general, when $R = k[x_0, \dots, x_n]$ then there are $n + 2$ columns in the Betti table.) Let $X \subset \mathbb{P}^3$ be a general set of points. The Minimal Resolution Conjecture (MRC) of Lorenzini [L] was given for points in $\mathbb{P}^n$, and says that there are only two non-zero rows of the associated Betti table, and that in these two rows it always holds that $\beta_{i,j} \cdot \beta_{i+1,j} = 0$. As already mentioned, this was shown for $n = 3$ by Ballico and Geramita [BG].

Mustaţă's version of the MRC deals with a general set of points, X, on a fixed variety V in projective space. It says, essentially, that the top part of the Betti table is the Betti table of R/I_V, and that below this there are again at most two rows, which again satisfy $\beta_{i,j} \cdot \beta_{i+1,j} = 0$. In this paper we will take V to be a general surface, S, of degree d in $\mathbb{P}^3$, and we will state the conjecture more precisely in a moment.

As noted in the introduction, the Minimal Resolution Conjecture is known for sets of points on a smooth quadric and on a general cubic surface in $\mathbb{P}^3$. Thus from now on in this paper, we will assume that the degree of our surface is $d \geq 4$. We will use the following notation.

Notation 2.1. Let S be an irreducible surface of degree $d \geq 4$ in $\mathbb{P}^3$. We denote by H_S the Hilbert function of S, and by h_S the first difference $h_S = \Delta H_S := H_S(x) - H_S(x-1)$.

Remark 2.2. We note that

$$\begin{aligned} H_S(x) &= \binom{x+3}{3} - \binom{x-d+3}{3} \\ &= \frac{d}{2}x^2 - \left(\frac{d^2-4d}{2}\right)x + \binom{d-1}{3} + 1 \quad \text{for } x \geq d-1 \end{aligned}$$

and

$$h_S(x) = dx - \binom{d-1}{2} + 1 \qquad \text{for } x \geq d-1.$$

Of course the latter is the Hilbert polynomial of a plane curve of degree d.

Definition 2.3. Let S be an irreducible surface in $\mathbb{P}^3$ and let $X \subset S$ be a finite set of points. We will say that the Hilbert function of X on S is *relatively compressed* if one of the following two situations holds.

(a) If X is arithmetically Gorenstein of socle degree n with h-vector h_X then $h_X(x) = h_S(x)$ for $0 \le x \le \lfloor \frac{n}{2} \rfloor$, and the rest of the h-vector is determined by symmetry.

(b) Otherwise, X is relatively compressed on S if there is are integers $e \ge 0$ and $1 \le t \le h_S(e)$ such that

$$h_X(x) = \begin{cases} h_S(x) & \text{for } 0 \le x \le e-1; \\ t & \text{for } x = e; \\ 0 & \text{for } x > e. \end{cases}$$

We usually simply say that X itself is relatively compressed if its Hilbert function is.

In particular, a general set of points on S is relatively compressed. The following is an equivalent formulation and additional terminology.

Definition 2.4. Let S be an irreducible surface of degree d in $\mathbb{P}^3$. Let $X \subset S$ be a general set of points of fixed cardinality. Define the integers e and t by $H_S(e-1) < |X| = H_S(e-1)+t \le H_S(e)$ for some $e \ge 3$ and $1 \le t \le h_S(e)$. Then e is called the *socle degree* of X and we will call t the *surplus* of X. We will denote by $X_{e,t}$ a general set of points on S with socle degree e and surplus t. Note that the h-vector of $X_{e,t}$ is

$$(1, 3, 6, \dots, h_S(e-1), t)$$

and that

$$|X_{e,t}| = H_S(e-1) + t.$$

The *Minimal Resolution Conjecture (MRC) for surfaces in* $\mathbb{P}^3$ says that the shape of the Betti table for $R/I_{X_{e,t}}$ is

$$\begin{array}{cccc}
1 & - & - & - \\
- & 0 & 0 & 0 \\
 & & \vdots & \\
- & 0 & 0 & 0 \\
- & 1 & - & - \\
- & 0 & 0 & 0 \\
 & & \vdots & \\
- & 0 & 0 & 0 \\
- & \beta_{1,e} & \beta_{2,e+1} & \beta_{3,e+2} \\
- & \beta_{1,e+1} & \beta_{2,e+2} & \beta_{3,e+3} \\
- & 0 & 0 & 0
\end{array}$$

with $\beta_{1,e+1} \cdot \beta_{2,e+1} = 0$ and $\beta_{2,e+2} \cdot \beta_{3,e+2} = 0$. To more easily visualize the needed mapping cones in Section 3, we will use the following notation for this conjectured minimal free resolution.

$$0 \to \begin{array}{c} R(-e-3)^{c_2} \\ \oplus \\ R(-e-2)^{c_1} \end{array} \to \begin{array}{c} R(-e-2)^{b_2} \\ \oplus \\ R(-e-1)^{b_1} \end{array} \to \begin{array}{c} R(-e-1)^{a_2} \\ \oplus \\ R(-e)^{a_1} \\ \oplus \\ R(-d) \end{array} \to I_{X_{e,t}} \to 0$$

where $a_2 \cdot b_1 = 0$, $b_2 \cdot c_1 = 0$, $a_1 = h_S(e) - t$ and $c_2 = t$. More precisely, we will say that the *Ideal Generation Conjecture* holds if $a_2 \cdot b_1 = 0$ and we will say that the *Cohen-Macaulay Type Conjecture* holds if $b_2 \cdot c_1 = 0$.

Mustaţă showed in [M] (Examples 1 and 2) that the MRC holds when $t = h_S(e)$ or $t = h_S(e) - 1$.

Proposition 2.5. *Let S be an irreducible surface of degree d in $\mathbb{P}^3$. For a given socle degree $e \geq d$, there are at most four values of the surplus t needed in order to prove the MRC for all general sets of points on S with socle degree e. More precisely, let*

$$\begin{array}{rcl} m_1(e) & = & \max\{t \mid 3t \leq h_S(e-1)\} \\ m_2(e) & = & \min\{t \mid 3t \geq h_S(e-1)\} \\ m_3(e) & = & \max\{t \mid 3(h_S(e)-t) \geq h_S(e+1)\} \\ m_4(e) & = & \min\{t \mid 3(h_S(e)-t) \leq h_S(e+1)\} \end{array}$$

Notice that $m_1(e) \leq m_2(e) \leq m_3(e) \leq m_4(e)$.

(a) If $X_{e,m_1(e)}$ and $X_{e,m_2(e)}$ both satisfy the Cohen-Macaulay Type Conjecture then so does $X_{e,t}$ for all t.
(b) If $X_{e,m_3(e)}$ and $X_{e,m_4(e)}$ both satisfy the Ideal Generation Conjecture then so does $X_{e,t}$ for all t.
(c) If d is divisible by 3 then $m_1(e) = m_2(e)$ and $m_3(e) = m_4(e)$.

Proof. This has been observed before – see for instance [M] Proposition 1.7(i) or [MP] section 3 (for the case $d = 3$). The idea is to pass to the artinian reduction of $R/I_{X_{e,t}}$. For $t \leq m_1(e)$ the generators of least degree of the canonical module have no linear syzygies, and for $t \geq m_2(e)$ the canonical module is generated in its least degree. Similarly, for $t \geq m_3(e)$ the generators of degree e in the ideal have no linear syzygies, and for $t \leq m_4(e)$ the ideal is generated only in degree e. □

3. Liaison considerations

In this paper we will make extensive use of Gorenstein liaison. The arithmetically Gorenstein sets of points on our surface S of degree d that we will consider are *relatively compressed*. In the case of even socle degree there is one peak in the h-vector, and in the case of odd socle degree there are two.

In this section we will assume that we can always find Gorenstein sets of points containing our general points $X_{e,t}$ and show how liaison is used to build larger sets with the desired resolution. This is the basis for our induction on e to prove the MRC.

3.1. **Links of type 1.** Let S be a surface of degree d (either even or odd) and let $X_{e-2,t} \subset S$ with $e \geq d+1$. Assume that $X_{e-2,t}$ satisfies the MRC. The h-vector of $X_{e-2,t}$ is

$$(1, 3, 6, \ldots, h_S(e-3), t)$$

and the minimal free resolution of $X_{e-2,t}$ is

$$0 \to \begin{array}{c} R(-e-1)^{c_2} \\ \oplus \\ R(-e)^{c_1} \end{array} \to \begin{array}{c} R(-e)^{b_2} \\ \oplus \\ R(-e+1)^{b_1} \end{array} \to \begin{array}{c} R(-e+1)^{a_2} \\ \oplus \\ R(-e+2)^{a_1} \\ \oplus \\ R(-d) \end{array} \to I_{X_{e-2,t}} \to 0$$

where $a_2 \cdot b_1 = 0$, $b_2 \cdot c_1 = 0$, $a_1 = h_S(e-2) - t$ and $c_2 = t$. Notice that necessarily we have $1 \le t \le h_S(e-2) - 1$ by definition of e.

Let G be an arithmetically Gorenstein set of points containing $X_{e-2,t}$ that is relatively compressed of socle degree $2e-2$. G links $X_{e-2,t}$ to a residual set Z with h-vector

$$(1, 3, 6, \ldots, h_S(e-1), h_S(e-2) - t).$$

We will call this a *link of type 1.* The minimal free resolution of I_G is

$$0 \to R(-2e-1) \to \begin{array}{c} R(-2e-1+d) \\ \oplus \\ R(-e-1)^{2d} \end{array} \to \begin{array}{c} R(-e)^{2d} \\ \oplus \\ R(-d) \end{array} \to I_G \to 0. \tag{3.1}$$

Splitting off the two copies of $R(-d)$, the mapping cone gives the following minimal free resolution for I_Z:

$$0 \to \begin{array}{c} R(-e-3)^{a_1} \\ \oplus \\ R(-e-2)^{a_2} \end{array} \to \begin{array}{c} R(-e-2)^{b_1} \\ \oplus \\ R(-e-1)^{b_2+2d} \end{array} \to \begin{array}{c} R(-e-1)^{c_1} \\ \oplus \\ R(-e)^{2d+c_2} \\ \oplus \\ R(-d) \end{array} \to I_Z \to 0$$

We make the following observations.

1. The condition $a_2 \cdot b_1 = 0$ guarantees that there are no redundant copies of $R(-e-2)$.
2. If $c_1 = 0$ then there are no redundant copies of $R(-e-1)$. Equivalently, this holds if $t \ge m_2(e-2)$.
3. The socle degree of Z is e and the surplus is $s = h_S(e-2) - t = a_1$.

We have shown:

Proposition 3.1. *Assume that a relatively compressed arithmetically Gorenstein set of points G with socle degree $2e-2$ can be found containing $X_{e-2,t}$. Assume $e \ge d+1$. If $t \ge m_2(e-2)$ then G links $X_{e-2,t}$ to a set of points Z with socle degree e, relatively compressed h-vector, and having the minimal free resolution predicted by the MRC. The surplus, s, of Z satisfies*

$$1 \le s = h_S(e-2) - t \le \frac{2}{3} \cdot h_S(e) - \frac{5d}{3}.$$

Proof.

$$\begin{array}{rcl} 1 \le s & = & h_S(e-2) - t \\ & \le & h_S(e-2) - \frac{1}{3} h_S(e-3) \\ & = & \dfrac{3h_S(e-2) - h_S(e-3)}{3} \\ & = & \dfrac{2h_S(e-2) - d}{3} \\ & = & \dfrac{2(h_S(e) - 2d) - d}{3} \end{array}$$

from which the result follows. The only comment is that we used the definition of $m_2(e-2)$, Proposition 2.5 and the fact that $h_S(e-2) - h_S(e-3) = d$. □

3.2. **Links of type 2.** As explained in Subsection 3.4, we will now assume that d is even. Let S be a surface of even degree d and let $X_{e-1,t} \subset S$ with $e \geq d+1$. As we will explain shortly, we will also assume that S is defined by the pfaffian of a $(d \times d)$ skew-symmetric matrix of quadrics.

Assume that $X_{e-1,t}$ satisfies the MRC. The h-vector of $X_{e-1,t}$ is

$$(1, 3, 6, \ldots, h_S(e-2), t)$$

and the minimal free resolution of $X_{e-1,t}$ is

$$0 \to \begin{array}{c} R(-e-2)^{c_2} \\ \oplus \\ R(-e-1)^{c_1} \end{array} \to \begin{array}{c} R(-e-1)^{b_2} \\ \oplus \\ R(-e)^{b_1} \end{array} \to \begin{array}{c} R(-e)^{a_2} \\ \oplus \\ R(-e+1)^{a_1} \\ \oplus \\ R(-d) \end{array} \to I_{X_{e-1,t}} \to 0$$

where $a_2 \cdot b_1 = 0$, $b_2 \cdot c_1 = 0$, $a_1 = h_S(e-1) - t$, and $c_2 = t$.

Let G be an arithmetically Gorenstein set of points on S that is relatively compressed of socle degree $2e-1$, having $d+1$ minimal generators (recall that d is even). Note that if such G exists on S then S is defined by the pfaffian of a $(d \times d)$ skew-symmetric matrix of quadrics (since S is one of the minimal generators of I_G). Conversely, given such a pfaffian defining S, it can be extended by adding a sufficiently general row of forms of degree $e+2-d$ and a corresponding column to produce a skew-symmetric $(d+1) \times (d+1)$ matrix whose $(d \times d)$ pfaffians generate I_G.

Ignoring for a moment the question of containing $X_{e-1,t}$, the minimal free resolution of such I_G is

$$0 \to R(-2e-2) \to \begin{array}{c} R(-2e-2+d) \\ \oplus \\ R(-e-2)^d \end{array} \to \begin{array}{c} R(-e)^d \\ \oplus \\ R(-d) \end{array} \to I_G \to 0. \tag{3.2}$$

If such G exists containing $X_{e-1,t}$, we will call this a *link of type 2*. Then G links $X_{e-1,t}$ to a residual set Z with h-vector

$$(1, 3, 6, \ldots, h_S(e-1), h_S(e-1) - t).$$

Splitting off the two copies of $R(-d)$, the mapping cone gives the following minimal free resolution for I_Z:

$$0 \to \begin{array}{c} R(-e-3)^{a_1} \\ \oplus \\ R(-e-2)^{a_2} \end{array} \to \begin{array}{c} R(-e-2)^{d+b_1} \\ \oplus \\ R(-e-1)^{b_2} \end{array} \to \begin{array}{c} R(-e-1)^{c_1} \\ \oplus \\ R(-e)^{d+c_2} \\ \oplus \\ R(-d) \end{array} \to I_Z \to 0$$

We make the following observations.

1. The condition $b_2 \cdot c_1 = 0$ guarantees that there are no redundant copies of $R(-e-1)$.
2. If $a_2 = 0$ then there are no redundant copies of $R(-e-2)$. Equivalently, this holds if $t \leq m_3(e-1)$.
3. If $b_1 = 0$, then we again get no redundant copies of $R(-e-2)$ provided we can split off all the degree e minimal generators of I_G. However, in general this is not possible.
4. The socle degree of Z is e and the surplus is $s = h_S(e-1) - t$.

We have shown:

Proposition 3.2. *Assume that d is even and that $e \geq d+1$. Assume that a relatively compressed arithmetically Gorenstein set of points G with socle degree $2e-1$ and resolution (3.2) can be found containing $X_{e-1,t}$. If $t \leq m_3(e-1)$ then G links $X_{e-1,t}$ to a set of points Z with socle degree e, having a relatively compressed h-vector, and having the minimal free resolution predicted by the MRC. The surplus, s, of Z satisfies*

$$h_S(e-1) - 1 \geq s = h_S(e-1) - t \geq \frac{h_S(e)}{3}.$$

Proof.

$$\begin{array}{rcl} s & = & h_S(e-1) - t \\ & \geq & h_S(e-1) - [h_S(e-1) - \frac{1}{3}h_S(e)] \end{array}$$

from which the result follows. The only comment is that we used the definition of $m_3(e-1)$ and Proposition 2.5. □

3.3. **Consequences.**

Lemma 3.3. *Assume that $d \geq 4$ and $(d,e) \neq (4,5)$.*

$$\frac{2}{3} \cdot h_S(e) - \frac{5d}{3} \geq \frac{h_S(e)}{3}.$$

Proof. Notice that $e \geq \frac{d+7}{2}$, from which it follows that $10 \leq 2e - d + 3$. Then

$$\frac{5}{3} \leq \frac{2e-d+3}{6}$$

so

$$\frac{5}{3}d \leq \frac{2de - d^2 + 3d}{6} = \frac{2de - [d^2 - 3d + 2] + 2}{6} = \frac{1}{3}\left[de - \binom{d-1}{2} + 1\right] = \frac{1}{3}h_S(e).$$

This means

$$\frac{2}{3}h_S(e) - \frac{5}{3}d \geq \frac{1}{3}h_S(e)$$

as desired. □

Lemma 3.4.

$$h_S(e-1) - 1 \geq m_4(e).$$

Proof. We have to show that

$$3\,[h_S(e) - (h_S(e-1) - 1)] \leq h_S(e+1).$$

Using Remark 2.2 and the fact that $h_S(e) - h_S(e-1) = d$, we have to show that

$$3d + 3 \leq d(e+1) - \binom{d-1}{2} + 1.$$

This reduces to showing that

$$\frac{d-3}{2} + \frac{3}{d} \leq e - 2,$$

which is clearly true. □

Combining Proposition 3.1, Proposition 3.2, Lemma 3.3 and Lemma 3.4, we obtain the following.

Theorem 3.5. *Assume that there is some irreducible surface S of degree d such that*

(a) d is even;
(b) for $e \leq d$, $X_{e,t}$ satisfies the MRC for any t;
(c) for $e \geq d+1$ and t satisfying the conditions of Propositions 3.1 and 3.2, Gorenstein links of types 1 and 2 can be made for $X_{e,t}$.

Then the MRC holds on S, hence on a general surface of degree d.

Proof. The proof is by induction on e. For $e \leq d$, assumption (b) begins the induction.

We first consider the case $d = 4, e = 5$. In this case $m_2(e-2) = 2$ and $m_3(e-1) = 8$. Proposition 3.1 shows that $X_{e,s}$ has the desired resolution for $1 \leq s \leq 10-t \leq 8$. Proposition 3.2 shows that $X_{e,s}$ has the desired resolution for $13 \geq s = 14-t \geq 6$. Hence the result holds for $d = 4, e = 5$. From now on in this proof we assume that $(d, e) \neq (4, 5)$.

Thanks to Proposition 3.1, Proposition 3.2, Lemma 3.3 and semicontinuity, the MRC is true for $X_{e,s}$ for

$$1 \leq s \leq h_S(e-1) - t.$$

Thanks to Lemma 3.4 and Proposition 2.5, this is enough to deduce the truth of the MRC for socle degree e. □

3.4. **Surfaces of odd degree.** In using links of type 2 we were forced to assume that d is even. In this subsection we discuss the problems with surfaces of odd degree.

Example 3.6. Let $d = 5$, $e = 6$ and consider the set $X_{6,t}$. Its h-vector is $(1, 3, 6, 10, 15, 20, t)$. A relatively compressed arithmetically Gorenstein set of points G with socle degree 13 and containing $X_{6,t}$ links $X_{6,t}$ to a set Z with h-vector $(1, 3, 6, 10, 15, 20, 25, 25-t)$ so one could hope that the latter set is "general enough." But consider the minimal free resolutions

$$0 \to \begin{matrix} R(-9)^{c_2} \\ \oplus \\ R(-8)^{c_1} \end{matrix} \to \begin{matrix} R(-8)^{b_2} \\ \oplus \\ R(-7)^{b_1} \end{matrix} \to \begin{matrix} R(-7)^{a_2} \\ \oplus \\ R(-6)^{a_1} \\ \oplus \\ R(-5) \end{matrix} \to I_{X_{6,t}} \to 0$$

where $c_2 = t$, $b_2 \cdot c_1 = 0$, $a_2 \cdot b_1 = 0$, $a_1 = 25 - t$, and

$$0 \to R(-16) \to \begin{matrix} R(-11) \\ \oplus \\ R(-9)^5 \\ \oplus \\ R(-8) \end{matrix} \to \begin{matrix} R(-8) \\ \oplus \\ R(-7)^5 \\ \oplus \\ R(-5) \end{matrix} \to I_G \to 0$$

(even supposing that G could be found with only one generator of degree 8, which is not obvious). The free resolution for Z then has the form

$$0 \to \begin{matrix} R(-10)^{a_1} \\ \oplus \\ R(-9)^{a_2} \end{matrix} \to \begin{matrix} R(-9)^{5+b_1} \\ \oplus \\ R(-8)^{1+b_2} \end{matrix} \to \begin{matrix} R(-8)^{1+c_1} \\ \oplus \\ R(-7)^{5+c_2} \\ \oplus \\ R(-5) \end{matrix} \to I_Z \to 0$$

Assume that all possible splitting has been performed. The copy of $R(-8)$ coming from the minimal generators of I_G does not split with anything, so the summand $R(-8)^{1+b_2}$ does not reduce to zero in the minimal free resolution of I_Z. On the other hand, it is not at all clear that it is possible to split off the $R(-8)$ corresponding to a first syzygy of I_G, and even if it were possible, if $t < m_1(6)$ then $c_1 > 0$ and the summands $R(-8)^{c_1}$ do not split. Thus the latter minimal free resolution is probably not of the desired type in general, and certainly not if $t < m_1(6)$.

4. Finding Gorenstein links

In the previous section we showed what we can say about the MRC assuming that Gorenstein links of a certain kind could be found. In this section we address this problem.

There are two approaches that suggest themselves. One is to find a special surface S of degree d where we have enough control to construct many Gorenstein sets of points, and try to inductively produce sets X for any socle degree and surplus, having the desired resolution. Then semicontinuity gives the result for a general set of points of the same cardinality on a general surface of degree d. The other approach is to consider general S and use dimension counts to force the existence of the two types of Gorenstein links for $X_{e,t}$. In this section we take the latter approach.

In the situation where $e \geq d+1$, S is the unique surface of degree d containing $X_{e,t}$, and more importantly it is the unique surface of degree d containing the Gorenstein sets G that we want to use for our links. Hence it corresponds to a minimal generator of I_G. By the main result of [BE], this means that S is Pfaffian. In the case of type 1 links (see below), it is the Pfaffian of a skew symmetric matrix of linear forms. In the case of type 2 links, the forms are quadratic.

We recall the following results.

Proposition 4.1 ([B] Proposition 7.2). *Let S be a smooth surface of degree d in $\mathbb{P}^3$. Then the following are equivalent.*

- *(i) S can be defined by an equation $pf(M) = 0$, where M is a skew-symmetric linear $(2d) \times (2d)$ matrix;*
- *(ii) S contains a finite arithmetically Gorenstein reduced subscheme Z of socle degree $2d-4$, not contained in any surface of degree $d-2$ (in particular it is compressed).*

Theorem 4.2 ([B] Proposition 7.6). *A general surface of degree d in $\mathbb{P}^3$ can be defined by a linear pfaffian if and only if $d \leq 15$.*

Theorem 4.3 ([F]). *A general surface of even degree d in $\mathbb{P}^3$ is the pfaffian of a skew-symmetric matrix with quadratic entries if and only if $d \leq 14$.*

The following result gives the dimension of the Hilbert scheme containing a fixed arithmetically Gorenstein scheme X in $\mathbb{P}^3$ (which was shown by Diesel [D] to be irreducible).

Theorem 4.4 ([KM] Theorem 2.6 and Remark 2.7). *Let $X \subset \mathbb{P}^3$ be an arithmetically Gorenstein zero-dimensional scheme whose homogeneous ideal I_X has minimal free resolution*

$$0 \to F_2 = R(-f) \to F_1 = \bigoplus_{i=1}^{r} R(-n_{2,i}) \to F_0 = \bigoplus_{i=1}^{r} R(-n_{1,i}) \to I_X \to 0.$$

Assume without loss of generality that

$$n_{1,1} \leq \cdots \leq n_{1,r} \quad \text{and} \quad n_{2,1} \geq \cdots \geq n_{2,r}.$$

Let H_X be the Hilbert function of X. Then

$$\begin{aligned}\dim Hilb_{[X]}\mathbb{P}^3 &= \sum_{i=1}^{r} H_X(n_{1,i}) + \sum_{1\le i<j\le r}\binom{-n_{1,i}+n_{2,j}+3}{3} \\ &\quad - \sum_{1\le i<j\le r}\binom{n_{1,i}-n_{2,j}+3}{3} - \sum_{i=1}^{r}\binom{n_{1,i}+3}{3}.\end{aligned}$$

Let S be a general surface of degree d and let $X_{e-2,t} \subset S$ be a general set of points of socle degree $e-2$ and surplus t. Note that $|X_{e-2,t}| = H_S(e-3)+t$. A link of type 1 for $X_{e-2,t}$ is provided by a relatively compressed Gorenstein set of points G of socle degree $2e-2$ and minimal free resolution (3.1). We have

$$\begin{aligned}&r = 2d+1\\ &n_{1,1} = d,\\ &n_{1,i} = e \quad \text{for } 2 \le i \le 2d+1\\ &n_{2,1} = 2e+1-d\\ &n_{2,j} = e+1 \quad \text{for } 2\le j\le 2d+1\\ &f = 2e+1.\end{aligned}$$

We obtain for links of type 1 that

$$\begin{aligned}\dim Hilb_{[G]}\mathbb{P}^3 &= \sum_{i=1}^{2d+1}\left[H_G(n_{1,i}) - \binom{n_{1,i}+3}{3}\right] + \sum_{1\le i<j\le 2d+1}\binom{n_{2,j}-n_{1,i}+3}{3} \\ &= -1-2d\left[2d+\binom{e-d+3}{3}\right] + 2d\binom{e-d+4}{3} + 4\binom{2d}{2} \qquad (4.1) \\ &= de^2 + (5-2d)de + d^3 - d^2 + 2d - 1.\end{aligned}$$

Similarly, when d is even, a link of type 2 for $X_{e-1,t}$ is provided by a relatively compressed Gorenstein set of points G of socle degree $2e+1$ and minimal free resolution (3.2). We have

$$\begin{aligned}&r = d+1\\ &n_{1,1} = d,\\ &n_{1,i} = e \quad \text{for } 2 \le i \le d+1\\ &n_{2,1} = 2e+2-d\\ &n_{2,j} = e+2 \quad \text{for } 2\le j\le d+1\\ &f = 2e+2.\end{aligned}$$

Then one similarly computes for Gorenstein sets of type 2 that

$$\dim Hilb_{[G]}\mathbb{P}^3 = de^2 + de(6-2d) + d^3 - 2d^2 + 4d - 1. \qquad (4.2)$$

We now apply these observations to the case of quartics ($d = 4$).

Theorem 4.5. *Let S be a general surface of degree 4 in $\mathbb{P}^3$. Then the MRC holds for points on S.*

Proof. By Theorems 4.2 and 4.3 respectively, we can assume that S is both the pfaffian of an 8×8 skew-symmetric matrix of linear forms and a 4×4 skew-symmetric matrix of quadratic forms.

We first consider Gorenstein links of type 1 as described in Subsection 3.1, so in particular we will be considering general sets of points of socle degree $e-2$. Recall $e \geq 5$. In this case formula (4.1) gives, after a routine calculation,

$$\dim Hilb_{[G]}\mathbb{P}^3 = 4e^2 - 12e + 55.$$

Since each such G lies on a unique surface of degree 4, we have a natural projection

$$Hilb_{[G]}\mathbb{P}^3 \to \mathbb{P}[R]_4.$$

It follows from Proposition 4.1 and Theorem 4.2 that this map is dominant. The generic fibre of this map has dimension $4e^2 - 12e + 21$.

On a surface of degree 4 it is not hard to see that

$$m_1(e) = \left\lfloor \frac{4e-6}{3} \right\rfloor, \quad m_2(e) = \left\lceil \frac{4e-6}{3} \right\rceil, \quad m_3(e) = \left\lfloor \frac{8e-8}{3} \right\rfloor, \quad m_4(e) = \left\lfloor \frac{8e-8}{3} \right\rfloor.$$

We also note that

$$|X_{e-2,t}| = H_S(e-3) + t = 2(e-3)^2 + 2 + t.$$

Using Proposition 3.1, we observe that a link of type 1 links $X_{e-2,t}$ to a set with the same resolution as $X_{e,m_1(e)}$ or $X_{e,m_2(e)}$ when

$$t = \left\lfloor \frac{8e-24}{3} \right\rfloor \quad \text{or} \quad \left\lceil \frac{8e-24}{3} \right\rceil.$$

We focus on $X_{e-2,t}$ for these two values of t. On S, the condition of containing a given general point imposes at most 2 conditions. One observes that

$$4e^2 - 12e + 21 - 2\left[2(e-3)^2 + 2 + \frac{8e-24}{3}\right] = \frac{20e-9}{3} > 0.$$

Thus we can find a Gorenstein set of points of type 1 containing $X_{e-2,t}$, so Proposition 3.1 applies.

Now we consider Gorenstein sets of type 2 on our surface of degree $d = 4$. Formula (4.2) gives

$$\dim Hilb_{[G]}\mathbb{P}^3 = 4e^2 - 8e + 47.$$

Since each such G lies on a unique surface of degree 4, we still have a natural projection

$$Hilb_{[G]}\mathbb{P}^3 \to \mathbb{P}[R]_4,$$

which is again dominant thanks to Theorem 4.3. The generic fibre of this map has dimension at least $4e^2 - 8e + 13$.

Note that

$$|X_{e-1,t}| = H_S(e-2) + t = 2(e-2)^2 + 2 + t.$$

Using Proposition 3.2, we observe that a link of type 2 links $X_{e-1,t}$ to a set with the same resolution as $X_{e,m_3(e)}$ or $X_{e,m_4(e)}$ when

$$t = \left\lfloor \frac{4e-10}{3} \right\rfloor \quad \text{or} \quad \left\lceil \frac{4e-10}{3} \right\rceil.$$

We focus on $X_{e-1,t}$ for these two values of t. On S, the condition of containing a given general point imposes at most 2 conditions. One observes that

$$4e^2 - 8e + 13 - 2\left[2(e-2)^2 + 2 + \frac{4e-10}{3}\right] = \frac{16e-1}{3} > 0.$$

Thus we can find a Gorenstein set of points of type 2 containing $X_{e-2,t}$, so Proposition 3.2 applies. Since the main result of [BG] applies to a general set of ≤ 34 points in $\mathbb{P}^3$, the result now follows from the main idea of Theorem 3.5. We note, though, that rather than showing that links exist for all t, it was enough to do it for four specific values of t. ☐

Remark 4.6. For surfaces of higher degree, the dimension counts do not work as nicely. For instance, when $d = 6$ the links of type 1 would allow for at least a partial result (MRC for socle degree $e \leq 13$). However, links of type 2 are more delicate, and we were not able to obtain any reasonable result with this approach.

References

[BG] E. Ballico and A. Geramita, *The minimal free resolution of the ideal of s general points in $\mathbb{P}^3$*, Proceedings of the 1984 Vancouver conference on algebraic geometry, CMS Conf. Proc. no. 6 (1986), pgs 1–10. Amer. Math. Soc. Providence, RI.

[B] A. Beauville, *Determinantal hypersurfaces*, Michigan Math. J. **48** (2000), 39–64.

[BE] D. Buchsbaum and D. Eisenbud, *Algebra Structures for Finite Free Resolutions, and some Structure Theorems for Ideals of Codimension 3*, Amer. J. of Math. **99** (1977), 447–485.

[C] M. Casanellas, *The minimal resolution conjecture for points on the cubic surface*, Canad. J. Math. **61** (2009), 29–49.

[D] S. Diesel, *Irreducibility and Dimension Theorems for Families of Height 3 Gorenstein Algebras*, to appear in Pacific J. of Math.

[EP] D. Eisenbud and S. Popescum *Gale duality and free resolutions of ideals of points*, Invent. Math. **136** (1999), 419–449.

[EPSW] D. Eisenbud, S. Popescu, F.-O. Schreyer, C. Walter, *Exterior algebra methods for the minimal resolution conjecture*, Duke Math. J. 112 (2) (2002) 379–395.

[F] D. Faenzi, *A remark on Pfaffian surfaces and aCM bundles*, Vector bundles and low codimensional subvarieties: state of the art and recent developments, 209–217, Quad. Mat., 21, Dept. Math., Seconda Univ. Napoli, Caserta, 2007.

[FMP] G. Farkas, M. Mustaţă, M. Popa, *Divisors on $\mathcal{M}_{g,g+1}$ and the minimal resolution conjecture for points on canonical curves*, Ann. Sci. École Norm. Sup. 36 (4) (2003) 553–581.

[G] F. Gaeta, *A fully explicit resolution of the ideal defining N generic points in the plane*, preprint, 1995.

[GMR] S. Giuffrida, R. Maggioni, A. Ragusa, *Resolutions of generic points lying on a smooth quadric*, Manuscripta Math. 91 (4) (1996) 421–444.

[HS] A. Hirschowitz, C. Simpson, *La résolution minimale de lidéal dun arrangement général dun grand nombre de points dans $\mathbb{P}^n$*, Invent. Math. 126 (3) (1996) 467–503.

[I] A. Iarrobino, *Compressed algebras: Artin algebras having given socle degrees and maximal length*, Trans. Amer. Math. Soc. **285** (1984), 337–378.

[KM] J. Kleppe and R.M. Miró-Roig, *The dimension of the Hilbert scheme of Gorenstein codimension 3 subschemes*, J. Pure Appl. Algebra **127** (1998), 73–82.

[L] A. Lorenzini, *The minimal resolution conjecture*, J. Algebra **156** (1993), 5–35.

[MP] J. Migliore and M. Patnott, *Minimal free resolutions of general points lying on cubic surfaces*, J. Pure Appl. Algebra **215** (2011), 1737–1746.

[MP1] R. Miró-Roig, J. Pons-Llopis, *The minimal resolution conjecture for points on del Pezzo surfaces*, Algebra & Number Theory 6 (2012), 27–46.

[MP2] R. Miró-Roig, J. Pons-Llopis, *Minimal free resolution for points on surfaces*, J. Algebra 357 (2012), 304–318.

[M] M. Mustaţă, *Graded Betti numbers of general finite subsets of points on projective varieties*, Le Matematiche **53** (1998), 53–81.

[W] C. Walter, *The minimal free resolution of the homogeneous ideal of s general points in $\mathbb{P}^4$*, Math. Zeit. 219 (2) (1995) 231–234.

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