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Global existence analysis of energy-reaction-diffusion systems

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ABSTRACT. We establish global-in-time existence results for thermodynamically consistent reaction-(cross-)diffusion systems coupled to an equation describing heat transfer. Our main interest is to model species-dependent diffusivities, while at the same time ensuring thermodynamic consistency. A key difficulty of the non-isothermal case lies in the intrinsic presence of cross-diffusion type phenomena like the Soret and the Dufour effect: due to the temperature/energy dependence of the thermodynamic equilibria, a nonvanishing temperature gradient may drive a concentration flux even in a situation with constant concentrations; likewise, a nonvanishing concentration gradient may drive a heat flux even in a case of spatially constant temperature. We use time discretisation and regularisation techniques and derive a priori estimates based on a suitable entropy and the associated entropy production. Renormalised solutions are used in cases where non-integrable diffusion fluxes or reaction terms appear.

1. INTRODUCTION

The purpose of this paper is to establish global-in-time existence results for a class of reaction-diffusion systems arising in the modelling of non-isothermal chemical reactions. We consider thermodynamically consistent models that are based on the total entropy

$$\mathcal{S}(c,u) := \int_\Omega S(c(x),u(x))\,\mathrm{d}x \tag{1.1}$$

as a Lyapunov functional. The point here is that the entropy density S is given in terms of the vector $c(x) \in [0,\infty)^I$ of the concentrations and the internal energy density $u(x)$ rather than the more commonly used temperature $\theta(x)$. Hence, we will be able to rely on concavity of S in (c,u), while concavity in (c,θ) does not hold in general, cf. e.g. [1]. Even more, following [42, 49, 50] the total energy-reaction-diffusion system may be written as a gradient-flow equation for $-\mathcal{S}$. For details on the derivation of our models, see Section 1.1.

Even without accounting for temperature dependence, developing an existence theory for solutions to entropy-driven reaction-diffusion systems has proven challenging: for instance, even for the simple example of a reaction-diffusion system with Fick-law diffusion

$$\dot{c}_i = a_i \Delta c_i + R_i(c) \tag{1.2}$$

(with $a_i > 0$) and entropy-producing chemical reactions $R_i(c)$, the global-in-time existence of generalised solutions has only been shown rather recently [34] and relies on the concept of renormalised solutions; weak (or even strong) solutions are only known to exist under more restrictive assumptions on the reactions [51, 9].

The existence analysis for thermodynamically realistic non-isothermal reaction-diffusion systems involves further challenges: due to the temperature-dependence of the thermodynamic equilibrium, a nonvanishing temperature gradient may drive a concentration flux even in situations with a vanishing concentration gradient, a phenomenon known as the *Soret effect*. Similarly, a nonvanishing concentration gradient may drive a heat flux even if the temperature is spatially constant (the *Dufour effect*). Thermodynamically realistic models for reaction-diffusion systems must therefore allow for *cross-diffusion* effects between internal energy density u and the concentrations c_i. The methods in [34] (respectively [13]) rely heavily on the diagonal structure (respectively the dominantly diagonal structure) of the diffusion; without substantial new ideas, they do not apply to a setting with strong cross-diffusion.

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In the present work, we resolve these mathematical difficulties and provide an existence analysis for generalised solutions to a nontrival class of reaction-(cross-)diffusion systems modelling non-isothermal chemical reactions. Our class of thermodynamically consistent models is derived as a gradient flow in Onsager form (see Section 1.1 below); the resulting equations are typically of the form

$$\dot{c}_i = \operatorname{div}\left(m_i(c,u)\nabla \log \frac{c_i}{w_i(u)} + a(c,u)c_i\frac{w_i'(u)}{w_i(u)}\nabla u\right) + R_i(c,u), \tag{1.3a}$$

$$\dot{u} = \operatorname{div}\big(a(c,u)\nabla u\big), \tag{1.3b}$$

$$\text{where } a(c,u) = -\pi_1(c,u)\Big(\hat{\sigma}''(u) + \sum_{i=1}^{I} c_i\tfrac{w_i''(u)}{w_i(u)}\Big) > 0.$$

Here, $w_i(u)$ denotes the equilibrium concentration of the chemical species $\mathcal{A}_i$ (which may depend on the internal energy density u), the function $m_i(c,u) \geq 0$ describes the diffusive mobility of $\mathcal{A}_i$, the function $\pi_1(c,u)$ describes the heat conductivity, and $\hat{\sigma}(u)$ is related to the thermal part of the entropy density.

It is rather immediate that the system (1.3a)–(1.3b) accounts for the Soret effect: a nonvanishing temperature gradient may drive a concentration flux even for spatially constant concentrations c_i. At first glance, the Dufour effect may appear to not be accounted for by the model, as a vanishing internal energy gradient $\nabla u \equiv 0$ entails the absence of a heat flux. However, even for a constant temperature θ the internal energy density u may be non-constant, as the thermodynamic relation of temperature and internal energy density $\frac{1}{\theta} = \partial_u S(c,u)$ also involves the concentrations.

1.1. **Modelling.** Following [49, 42, 50], we consider energy-reaction-diffusion systems that are motivated by the thermodynamically consistent models that are obtained as gradient-flow equations written in *Onsager form*. Given a state space $\mathbf{Q}$ as a convex subset of a Banach space X with states $Z \in \mathbf{Q}$, an Onsager operator $\mathbb{K} = \mathbb{K}(Z)$ is a possibly unbounded, symmetric and positive semi-definite operator, which may be seen as a generalisation of the inverse of the Riemannian metric tensor in a smooth manifold $\mathbf{Q}$. With a differentiable driving functional $-\mathcal{S} : \mathbf{Q} \to \mathbb{R}$ (typically a convex functional on X), the associated evolution problem reads

$$\dot{Z} = \mathbb{K}(Z)D\mathcal{S}(Z), \tag{1.4}$$

where here $D\mathcal{S}$ denotes the Fréchet derivative of the entropy functional $\mathcal{S}$, and $\dot{Z}$ is the time derivative of the state variable $Z = Z(t)$.

In this framework, different physical phenomena can easily be coupled by taking $\mathbb{K}$ as a sum of operators corresponding to individual processes. In this paper, we let $\mathbb{K} := \mathbb{K}_{\mathrm{diff}} + \mathbb{K}_{\mathrm{react}}$, where $\mathbb{K}_{\mathrm{diff}}$ accounts for diffusion and $\mathbb{K}_{\mathrm{react}}$ for the reactions. In our application, the vector Z of state variables consists of the concentrations $(c_i)_{i=1}^I$ of the I species $\mathcal{A}_i$ diffusing and reacting on the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ and another suitable variable modelling variations in temperature. The latter could be temperature $\theta(t,x)$ itself, the internal energy density $u(t,x)$ or some other suitable scalar quantity [47]. As in [50, 42] we model changes in temperature using the internal energy density $u \geq 0$ (see also [1, 48, 49]). The major advantage of this choice is that the entropy density $S(c,u)$ determining the entropy functional (1.1) is concave in (c,u), wich is a vector of extensive variables, see [46, 1]. Also note that the total energy $\mathcal{E}(c,u) = \int_\Omega u(x)\,\mathrm{d}x$ takes the most simple form, and while chemical reactions typically induce changes in temperature, the internal energy is left invariant and the total energy is a conserved quantity along solutions of (1.3). This also means that the Onsager operator $\mathbb{K}_{\mathrm{react}}(c,u)$ has a nontrivial kernel, namely span$\{(0,1)^T\}$.

The basis of our model are entropy densities $S(c,u)$ of the form (cf. [42, 50])

$$S(c,u) = \sigma(u) - B(c|w(u)) =: \hat{\sigma}(u) + \sum_{i=1}^{I} \Big(c_i \log w_i(u) - \lambda(c_i) \Big), \tag{1.5}$$

where we continue to use the Boltzmann function

$$\lambda(s) := s\log(s) - s + 1, \tag{1.6}$$

the relative Boltzmann entropy $B(c|w) := \sum_{i=1}^{I} w_i \lambda(c_i/w_i)$, and the thermal part $\sigma(u)$ of the entropy density when the concentrations $c = (c_i)_{i=1}^{I}$ are in equilibrium. From the formula

$$D_c S(c,u) = \big(\log w_i(u) - \lambda'(c_i) \big)_i = -\Big(\log \frac{c_i}{w_i(u)} \Big)_i$$

we see that the vector $w(u) = (w_i(u))_i$ defines the thermodynamic equilibria $w = (w_i)_{i=1}^{I}$ of the concentrations as function of the internal energy u. We generally assume that $w_i \in C([0,\infty)) \cap C^2((0,\infty))$ are positive, non-decreasing and concave. Moreover, the C^2 function $\hat{\sigma}(s) := \sigma(s) - \sum_{i=1}^{I} w_i(s) + I$ is supposed to be strictly concave and increasing. These properties ensure that S is strictly concave and that $u \mapsto S(c,u)$ is increasing (see the proof of [50, Prop. 2.1]). The temperature θ can then be recovered via $\theta = \frac{1}{\partial_u S(c,u)}$ and is per se non-negative.

The diffusive part $\mathbb{K}_{\mathrm{diff}}(Z)$ with $Z = (c,u)$ of the Onsager operator is assumed to be of the form

$$\mathbb{K}_{\mathrm{diff}}(Z)W := -\operatorname{div}\big(\mathbb{M}(Z)\nabla W\big), \quad W(x) \in \mathbb{R}^{I+1},$$

for a symmetric and positive semi-definite matrix $\mathbb{M}(Z) \in \mathbb{R}^{(I+1)\times(I+1)}$, the so-called mobility matrix, satisfying suitable additional conditions. Here and below, $\nabla = \nabla_x$ denotes the gradient with respect to $x \in \Omega$. The operator $\mathbb{K}_{\mathrm{diff}}$ will be complemented with the no-flux boundary condition $(\mathbb{M}(Z)\nabla W)\cdot \nu = 0$ on $\partial\Omega$, where ν denotes the outer unit normal to $\partial\Omega$. The precise structure of the reactive part

$$\mathbb{K}_{\mathrm{react}}(Z) = \begin{pmatrix} \mathbb{L}(Z) & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{(I+1)\times(I+1)}$$

will be less relevant in this work, the main hypothesis being that it leaves the energy equation unchanged. We will therefore directly formulate our hypotheses in terms of $R(Z) = (R_i(Z))_{i=1}^{I}$, where $R_i(Z) = (\mathbb{L}(Z)D_cS(Z))_i$ for $i \in \{1,\dots,I\}$. The positive semi-definiteness of $\mathbb{K}_{\mathrm{react}}$ then means that $D_cS(Z)\cdot R(Z) \geq 0$ (see condition (R1) below). We refer to [48, 49] for concrete choices of $\mathbb{L}$ for realising reactions following mass-action kinetics as in (1.11).

The above choices for $\mathbb{K}_{\mathrm{diff}}$ and $\mathbb{K}_{\mathrm{react}}$ of the Onsager operators encode conservation of the total energy $\mathcal{E}(c,u) = \int_\Omega u(x)\,\mathrm{d}x$ since $D\mathcal{E}(c,u) \equiv 1$, hence $\mathbb{K}(Z)D\mathcal{E}(Z) \equiv 0$ and thus, thanks to symmetry, $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(Z) \equiv 0$ along any curve $Z = Z(t)$ satisfying (1.4). For more information on the (modelling) background of such ERDS, we refer to [49, 50, 42].

Thus, letting $A(Z) := -\mathbb{M}(Z)D^2S(Z)$ and $R^\circ(Z) := (R(Z),0)^T$, the *energy-reaction-diffusion system* (ERDS) we consider takes the form

$$\dot{Z} = \operatorname{div}\big(A(Z)\nabla Z\big) + R^\circ(Z), \qquad t>0,\ x\in\Omega, \tag{1.7a}$$

$$0 = \nu\cdot A(Z)\nabla Z, \qquad t>0,\ x\in\partial\Omega. \tag{1.7b}$$

This system will be supplemented with suitable initial conditions

$$Z_{|t=0} = Z^0 \text{ in } \Omega.$$

Without loss of generality, we assume that the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ has unit volume, i.e. $|\Omega| = 1$.

The analytical results in the references [50, 42] primarily concern entropy-entropy production inequalities and address the question of decay rates to equilibrium assuming the existence of suitably regular

global-in-time solutions. To the authors' knowledge, the only global existence result for an ERDS of the form (1.7a)–(1.7b) (with semiconductor-like reactions) appears in [42, Section 6] for the special choice $\mathbb{M}(c,u) = -k(D^2S(c,u))^{-1}$, where $k > 0$ is a positive constant, and under the assumption of bounded initial data (c^0, u^0) with $\inf u^0 > 0$. In this case, species' diffusivities are all equal allowing the authors to infer global existence from maximum type principles.

Mobility matrix. Let us note that owing to the entropic coupling between c_i and u, the Hessian of the entropy density is non-diagonal and takes the form

$$D^2S(c,u) = \begin{pmatrix} \ddots & & & \vdots \\ & -\frac{1}{c_i} & & \frac{w_i'}{w_i} \\ & & \ddots & \vdots \\ \dots & \frac{w_i'}{w_i} & \dots & \partial_u^2 S \end{pmatrix},$$

where we abbreviated $w_i = w_i(u)$ and

$$\partial_u^2 S(c,u) = \hat\sigma''(u) + \sum_{i=1}^I c_i \tfrac{w_i''(u)}{w_i(u)} - \sum_{i=1}^I c_i \left(\tfrac{w_i'(u)}{w_i(u)}\right)^2 < 0,$$

because of the concavity of $\hat\sigma$ and w_i.

Except for the restrictive case $\mathbb{M}(c,u) = -k(D^2S(c,u))^{-1}$ considered in [42], the matrix $A(c,u) = -\mathbb{M}(c,u)D^2S(c,u)$ is not diagonal, usually not symmetric, and positive semi-definiteness cannot be expected. However, thanks to the formal entropy structure and the concavity of S, problem (1.7) is still parabolic in a certain sense (cf. Section 2.1). In our approach, control of the solution is obtained via entropy estimates, coercivity bounds for the entropy production and an L^2-energy estimate for the thermal part, where we also exploit the absence of source terms in the energy equation (see page 11 for more details on our strategy.)

Our existence analysis focuses on mobility matrices of the form

$$\mathbb{M}(c,u) := \operatorname{diag}\big(m_1,\dots,m_I,m_{I+1}\big) + \pi_1(c,u)\mu\otimes\mu \tag{1.8a}$$

$$\text{with } \mu_{I+1}(c,u) = 1 \text{ and } \mu_i(c,u) := c_i\,\frac{w_i'(u)}{w_i(u)} \quad \text{for } \{1,\dots,I\}, \tag{1.8b}$$

where the mobilities $m_i(c,u) \ge 0$, $i \in \{1,\dots,I\}$, and the coupling coefficient $\pi_1(c,u) \ge 0$ are specified below, whereas $m_{I+1} \equiv 0$ throughout (but see Remark 1.4). This choice of $\mathbb{M}$ leads to equations of the form (1.3) (see also (1.10)).

Loosely speaking, this ansatz for $\mathbb{M}$ can be seen as a thermodynamically consistent generalisation of the above-mentioned choice $\mathbb{M}(c,u) = -k(D^2S(c,u))^{-1}$ to allow for species-dependent diffusivities. In fact, when choosing $m_i := c_i$ and $\pi_1(c,u) := 1/\gamma(c,u)$ with

$$\gamma(c,u) := -\partial_u^2 S(c,u) - \sum_{i=1}^I c_i\big(\tfrac{w_i'(u)}{w_i(u)}\big)^2 = -\hat\sigma''(u) - \sum_{i=1}^I c_i \frac{w_i''(u)}{w_i(u)}, \tag{1.9}$$

then $\mathbb{M}(c,u)$ becomes the inverse Hessian of $S(c,u)$ (see [50, p. 777]). We would like to emphasize that a diagonal diffusion matrix $A = \operatorname{diag}(\dots) \in \mathbb{R}^{(I+1)\times(I+1)}$ cannot be produced in a thermodynamically consistent setting unless its entries on the diagonal are all equal, since otherwise $\mathbb{M}$ lacks symmetry.

In this paper, $m_i(c,u)$ is assumed to take the form

(M1) $m_i(c,u) = c_i a_i(c,u)$ for $i \in \{1,\dots,I\}$, $m_{I+1} \equiv 0$, where the functions $a_i \in C([0,\infty)^{I+1})$ satisfy for certain constants $\kappa_{1,i}, \kappa_{0,i} \geq 0$

$$a_i(c,u) \sim \kappa_{0,i} + \kappa_{1,i} c_i \quad \text{for } (c,u) \in [0,\infty)^{I+1}.$$

(See *Notations* on page 12 for the meaning of the symbol $\sim$.) Here, $\kappa_{0,i}$ is typically called the diffusion coefficient of the species $\mathcal{A}_i$, while the $\kappa_{1,i}$ are sometimes referred to as self-diffusion coefficients. The motivation for this choice of $a_i(c,u)$ comes from the observation that, to some extent, models with positive coefficients $\kappa_{1,i}, \kappa_{0,i} > 0$ may be seen as a regularisation of the model where $\kappa_{1,i} = 0$ and $\kappa_{0,i} > 0$ for all i since then entropy estimates may allow to control not only $\nabla\sqrt{c_i}$ but also ∇c_i in L^2, which via Sobolev embeddings leads to better integrability properties of c_i. One should caveat though that, owing to the entropic coupling, the problem is more complex and a positive $\kappa_{1,i}$ does not necessarily ensure an L^2 a priori bound of ∇c_i by entropy production estimates. In fact, for the special choice $\pi_1 \sim 1/\gamma$ such a regularisation is neither needed nor helpful, and we will construct global-in-time weak solutions provided that $\kappa_{1,i} = 0$ for all i.

However, we are interested in more general choices of coefficients π_1 in (1.8a) allowing for strong cross-diffusion due to energy gradients, as alluded to in the beginning. Here, strong cross-diffusion manifests itself in the fact that the flux terms generated by off-diagonal entries in the diffusion matrix $A(Z)$ cannot be controlled in $L^1_{t,x}$ by means of natural entropy estimates associated with the system. Surprisingly (when compared to existing literature), for such problems, we are still able to show an existence result: we will first construct global-in-time weak solutions for a family of 'regularised' models with $\kappa_{1,i} > 0$ for all $i \in \{1,\dots,I\}$. These approximate solutions will then enable an existence analysis in the case of strong cross-diffusion (arising due to vanishing self-diffusion) on the basis of the concept of renormalised solutions, as carried out in Sections 5–6.

Observe that, by the choice of μ_i in (1.8b), cross-diffusion between c_i and c_j, $i \neq j$, does not arise in our models. More precisely, our definition $A(c,u) := -\mathbb{M}(c,u)D^2S(c,u)$ and the fact that $-\frac{1}{c_i}\mu_i + \frac{w_i'}{w_i}\mu_{I+1} \equiv 0$ for $i \in \{1,\dots,I\}$ imply that the submatrix $(A_{ij})_{1\leq j\leq I}^{1\leq i\leq I+1}$ only depends on the diagonal part $\operatorname{diag}(m_i)$ of $\mathbb{M}$. We would like to note that in the self-diffusive case with $\kappa_{1,i} > 0$ for all i, our methods extend to situations modelling cross-diffusion between species, see Remark 1.3.

For the above choice of $\mathbb{M}$, a direct computation of A yields the coefficients

$$\begin{aligned}
A_{ii}(c,u) &= \tfrac{m_i(c,u)}{c_i} =: a_i(c,u) \qquad \text{for } i = 1,\dots,I,\\
A_{i,j}(c,u) &= 0 \qquad \text{for } i \neq j \text{ and } i,j \in \{1,\dots,I\},\\
A_{i,I+1}(c,u) &= -m_i\tfrac{w_i'}{w_i} - \pi_1 c_i \tfrac{w_i'}{w_i}\Big(\hat\sigma''(u) + \sum_{j=1}^{I} c_j \tfrac{w_j''(u)}{w_j(u)}\Big)\\
&= \big(-a_i(c,u) + \pi_1(c,u)\gamma(c,u)\big) c_i \tfrac{w_i'(u)}{w_i(u)} \qquad \text{for } i = 1,\dots,I,\\
A_{I+1,I+1}(c,u) &= -\pi_1\Big(\hat\sigma''(u) + \sum_{i=1}^{I} c_i \tfrac{w_i''(u)}{w_i(u)}\Big) = \pi_1(c,u)\gamma(c,u) =: a(c,u).
\end{aligned} \tag{1.10}$$

In particular, $\sum_j A_{ij}(Z)\nabla Z_j = a_i(c,u)\nabla c_i + A_{i,I+1}(c,u)\nabla u$ if $i \neq I+1$, and $A_{I+1,j}(Z) = a(c,u)\delta_{I+1,j}$.

Reactions. For reaction-diffusion equations with physically realistic reaction rates, the available global existence results in the literature often rely on renormalised solutions. To illustrate the underlying reason, consider for example a single reversible chemical reaction of the form

$$\alpha_1\mathcal{A}_1 + \ldots + \alpha_I\mathcal{A}_I \rightleftharpoons \beta_1\mathcal{A}_1 + \ldots + \beta_I\mathcal{A}_I.$$

The reaction rates according to mass-action kinetics are then given by

$$R_i(c,u) = \kappa(c,u)\left(\prod_{j=1}^{I}\Big(\frac{c_j}{w_j(u)}\Big)^{\alpha_j} - \prod_{k=1}^{I}\Big(\frac{c_k}{w_k(u)}\Big)^{\beta_k}\right)(\beta_i-\alpha_i), \tag{1.11}$$

for some non-negative reaction coefficient $\kappa(c,u)\geq 0$. At the same time, the only known energy estimate for the basic reaction-diffusion system (1.2) is the entropy estimate; it merely provides control of quantities of the form $\sup_t \int_\Omega c_i\log c_i\,\mathrm{d}x$ or $\int_0^T\int_\Omega |\nabla\sqrt{c_i}|^2\,\mathrm{d}x\mathrm{d}t$. Thus, without unphysically strong assumptions on the coefficient $\kappa(c,u)$, the available energy estimates are not sufficient to ensure $L^1([0,T]\times\Omega)$ integrability of the reaction terms (1.11), which would be required for standard weak solutions concepts.

Nevertheless, reactions of the type (1.11) satisfy the entropy inequality

$$\begin{aligned} D_cS(c,u)\cdot R(c,u) &= \kappa(c,u)\,\big(c_w^\alpha - c_w^\beta\big)\,\sum_{i=1}^{I}(\beta_i-\alpha_i)\log\big(\frac{w_i(u)}{c_i}\big)\\ &= \kappa(c,u)\big(c_w^\alpha - c_w^\beta\big)\big(\log c_w^\alpha - \log c_w^\beta\big)\geq 0, \end{aligned}$$

where $c_w^\alpha = \prod_{j=1}^I\big(\frac{c_j}{w_j(u)}\big)^{\alpha_j}$ and similarly for c_w^β. Thus, it is reasonable to impose the *entropy inequality*

(R1) $$D_cS(c,u)\cdot R(c,u)\geq 0 \quad \text{for all } (c,u)\in(0,\infty)^{I+1}.$$

This condition together with $\mathbb{M}=\mathbb{M}^T\geq 0$ ensure that, formally, the entropy functional $\mathcal{S}(c,u)$ is non-decreasing along trajectories of system (1.7). We note that, since $\partial_{c_i}S(c,u) = -\log\big(\frac{c_i}{w_i(u)}\big)$, the condition $[c_i=0 \implies R_i\geq 0]$, which is necessary for the positivity of c_i, is implicitly contained in hypothesis (R1). For more background, applications and specific examples for admissible choices of $R(c,u)$ we refer to [50, Sec. 2.3] and [42, Sec. 3.2] and references therein.

1.2. **Main results.**

General Hypotheses. *In all of our results, the entropy function S_0 is assumed to have the form*

$$S_0(c,u) = \hat\sigma_0(u) + \sum_{i=1}^{I}\Big(c_i\log w_i(u) - \lambda(c_i)\Big),$$

where λ is as in (1.6)*, and $\mathbb{M}(c,u)$ is supposed to be given by* (1.8a)–(1.8b) *with* (M1) *being satisfied. Regarding the coefficient functions $\hat\sigma_0, w_i, \pi_1$, the following basic regularity and qualitative properties will be supposed:*

- (B1) *Suppose that $\hat\sigma_0\in C^2((0,\infty))$ is a strictly concave, non-decreasing function.*
- (B2) *Let $w_i\in C([0,\infty))\cap C^2((0,\infty))$ with $w_i(0)>0$ for $i\in\{1,\dots,I\}$ be concave and non-decreasing functions.*
- (B3) *Assume that $\pi_1\in C([0,\infty)^{I+1},[0,\infty))$ with $\sqrt{\pi_1(u,c)}w_i'(u)\in C([0,\infty)^{I+1})$ for all i.*
- (B4) $\lim_{u\downarrow 0}\hat\sigma_0'(u)=+\infty$, $\lim_{u\uparrow\infty}\hat\sigma_0'(u)=0$
- (B5) $\exists\beta\in(0,1)$ *such that* $w_i(u)\lesssim(1+u)^\beta$ *for all* $i\in\{1,\dots,I\}$
- (B6) $0\leq\pi_1^{\frac12}(c,u)\lesssim 1+u$.

Finally, we generally assume the reaction rates $R\in C([0,\infty)^{I+1},\mathbb{R}^I)$ to be continuous and to satisfy the entropic production estimate (R1)*.*

The above collection of assumptions will be referred to below as General Hypotheses.

The two models (H) and (H'). In our analysis, we consider two 'models', which mainly differ in the choice of the coefficient function π_1 of the rank-one part of $\mathbb{M}$. The first model is determined by the collection of hypotheses **(H)** consisting of the following three conditions (H1)–(H3)

(H1) $\pi_1\gamma \gtrsim 1$ (with γ given by (1.9) for $S = S_0$, i.e. $\gamma = -\hat\sigma_0''(u) - \sum_{i=1}^I c_i \frac{w_i''(u)}{w_i(u)} > 0$)
(H2) $w_i'(u) \lesssim -w_i''(u)\pi_1^{\frac12}(c,u)$
(H3) $\pi_1^{\frac12}(c,u)\frac{w_i'}{w_i} \lesssim 1$.

The second model is determined by hypotheses **(H')** consisting of the following two conditions

(H1') $\pi_1\gamma \sim 1$
(H2') $(w_i')^2 \lesssim -w_i''w_i$.

Let us briefly comment on the hypotheses concerning $\hat\sigma_0, w_i$ and π_1 formulated in the last two paragraphs. The assumptions on the concave functions $\hat\sigma_0$ and w_i are not very restrictive and allow to include essentially all examples typically used in the modelling such as $\hat\sigma_0(s) = b\log(s)$ or $\hat\sigma_0(s) = bs^\alpha$ for $\alpha \in (0,1)$, where $b > 0$ is a positive constant. Typical admissible choices of w_i are $w_i(u) = b_{0,i} + b_{1,i}u^{\beta_i}$ or $w_i(u) = b_{0,i}(1 + b_{1,i}u)^{\beta_i}$ for $\beta_i \in (0,1)$ and $b_{0,i} > 0, b_{1,i} \geq 0$, in which case one can choose $\pi_1^{\frac12} \sim u$ and $\pi_1^{\frac12} \sim 1 + u$ respectively, when assuming **(H)**. When considering **(H')** instead, we should note the compatibility of hypotheses (B6) and (H1') for any power-law ansatz of $\hat\sigma_0$. Indeed, we then have $\gamma(c,u) \geq -\hat\sigma_0''(u) \gtrsim u^{\alpha-2}$ and hence $\pi^{\frac12} \lesssim u + 1$ whenever $\alpha \in [0,1)$ (with $\alpha = 0$ corresponding to $\hat\sigma_0(u) = \log(u)$). Hypotheses (H2) of **(H)** and (H2') of **(H')** can be regarded as concavity conditions on the equilibria w_i and rule out, for instance, that $w_i(u) = b_1u + b_0$ for $b_1 > 0, b_0 \geq 0$ for some $i \in \{1,\dots,I\}$. In view of (B2) and (B6), Hypothesis (H3) can be shown to be always fulfilled if $u \geq 1$, and should thus be understood as a condition for small arguments u close to zero.

Our first main result assumes the following hypotheses.

Hypotheses 1.1. *Let the* General Hypotheses *be satisfied and assume that either hypotheses* **(H)** *or hypotheses* **(H')** *are fulfilled.*

In case **(H)***, we additionally assume that $\kappa_{1,i} > 0$ for all $i \in \{1,\dots,I\}$ and that there exists $0 \leq q_1 < 2 + \frac{2}{d}$ and $0 \leq q_2 < 2 + \frac{4}{d}$ such that $|R(c,u)| \lesssim 1 + |c|^{q_1} + |u|^{q_2}$.*

Under hypotheses **(H')***, we assume that $\kappa_{1,i} = 0$ and $\kappa_{0,i} > 0$ for all $i \in \{1,\dots,I\}$, and impose, for simplicity, a suboptimal growth condition of the form $|R(c,u)| \lesssim 1 + |c| + |u|^2$.*

The proof of the following main result will be completed in Section 4.2.

Theorem 1.2 (Global existence of weak solutions)**.** *Let Hypotheses 1.1 hold true. Let $Z^0 = (c^0, u^0)$ have non-negative components satisfying $c_i^0 \in L\log L(\Omega)$, $i \in \{1,\dots,I\}$, $u^0 \in L^2(\Omega)$ and $\hat\sigma_{0,-}(u^0) \in L^1(\Omega)$. Then there exist non-negative functions*

$$c_i \in L^\infty(0,\infty; L\log L(\Omega)), \quad i \in \{1,\dots,I\},$$
$$u \in L^\infty(0,\infty; L^2(\Omega))$$

such that $Z := (c_1,\dots,c_I,u)$ has the regularity

$$\mathbb{M}(Z)D^2S_0(Z)\nabla Z \in L^s_{\mathrm{loc}}([0,\infty)\times\overline{\Omega})^{d(I+1)}, \text{ where } s = \tfrac{2d+2}{2d+1}, \tag{1.12}$$
$$\partial_t Z_i \in L^r_{\mathrm{loc}}(0,\infty; W^{1,r'}(\Omega)^*), \quad \text{where } r := \min\left\{\tfrac{d+2}{dq_1}, \tfrac{2d+4}{dq_2}, s\right\}, \ \tfrac{1}{r'} + \tfrac{1}{r} = 1,$$

and satisfies for all $T > 0$ *and all* $\phi = (\phi', \phi_{I+1}) \in L^\infty(0,T;W^{1,\infty}(\Omega))^{I+1}$ *the equation*

$$\begin{aligned}\int_0^T \langle \partial_t Z, \phi\rangle \,\mathrm{d}t - \int_0^T \int_\Omega (\mathbb{M}(Z) D^2 S_0(Z) \nabla Z) : \nabla \phi \,\mathrm{d}x\mathrm{d}t \\ = \int_0^T \int_\Omega R(Z)\cdot \phi' \,\mathrm{d}x\mathrm{d}t\end{aligned} \tag{1.13}$$

and the identity $Z(t=0,\cdot) = (c^0, u^0)$ *as an equality in* $(W^{1,\infty}(\Omega)^*)^{I+1}$.

Furthermore, the internal energy is conserved, i.e. for all $t > 0$,

$$\int_\Omega u(t,x)\,\mathrm{d}x = \int_\Omega u^0(x)\,\mathrm{d}x,$$

and the solution satisfies the bound

$$\begin{aligned}\|c\|_{L^\infty(L\log L)} + \|u\|_{L^\infty L^2} + \|\hat\sigma_{0,-}(u)\|_{L^\infty L^1} \\ + \int_0^\infty \int_\Omega \pi_1 \gamma |\nabla u|^2 \,\mathrm{d}x\mathrm{d}t + \int_0^\infty \int_\Omega P(c,u)\,\mathrm{d}x\mathrm{d}t \\ \le C(\|c^0\|_{L\log L}, \|u^0\|_{L^2}, \|\hat\sigma_{0,-}(u^0)\|_{L^1})\end{aligned}$$

for $P(c,u)$ *given by formula* (2.5).

We note that, by density, the equation (1.13) holds true for a somewhat larger set of test functions ϕ, and in case **(H')** the regularity of $\partial_t Z$ can be slightly improved. We should further note that in this work we have not aimed at optimising the regularity of the initial energy density u_0. The choice $u_0 \in L^2$ has been made for simplicity.

At this stage we can comment on the major interplay between the choice of the entropy density S in (1.5) and the mobility matrix $\mathbb{M}$ in (1.8). At a formal level, we obtain along solutions the conservation of energy and the entropy entropy-production balance:

$$\begin{aligned}\mathcal{E}(c(t),u(t)) &= \int_\Omega u(t,x)\,\mathrm{d}x = \int_\Omega u^0(x)\,\mathrm{d}x = \mathcal{E}(c^0,u^0),\\ \mathcal{S}(c(t),u(t)) &= \mathcal{S}(c^0,u^0) + \int_0^t \big(\mathcal{P}_{\mathrm{diff}}(c(r),u(r)) + \mathcal{P}_{\mathrm{react}}(c(r),u(r))\big)\mathrm{d}r\end{aligned}$$

with $\mathcal{P}_{\mathrm{diff}}(Z) = \int_\Omega \nabla Z : D^2S(Z)\mathbb{M}(Z)D^2S(Z)\nabla Z\,\mathrm{d}x$ and $\mathcal{P}_{\mathrm{react}}(c,u) = \int_\Omega D_c S(c,u)\cdot R(c,u)\,\mathrm{d}x$. Using $\mathcal{P}_{\mathrm{diff}}(Z) \ge 0$ and $\mathcal{P}_{\mathrm{react}}(Z) \ge 0$ we conclude $\mathcal{S}(Z(t)) \ge \mathcal{S}(Z^0)$. Combining the trivial bound of $u(t)$ in $L^1(\Omega)$ with the bounds (B5) on w_i, this implies a uniform a priori bound for $\int_\Omega \lambda(c_i(t))\,\mathrm{d}x$, see Lemma 2.2. However, the main difficulty in justifying eq. (1.13) is to show that the flux $A(Z)\nabla Z$ lies in $L^1_{\mathrm{loc}}([0,\infty)\times\overline{\Omega})$, based on the fact that the entropy production $\int_0^\infty \mathcal{P}_{\mathrm{diff}}(Z(r))\mathrm{d}r$ is finite. Because of $A(Z) = -\mathbb{M}(Z)D^2S(Z)$, this means that a bound of the form

$$|A(Z)\nabla Z|^{\widetilde{s}} \le C\Big(1 + |Z| + |u|^\nu + \nabla Z : D^2S(Z)\mathbb{M}(Z)D^2S(Z)\nabla Z\Big)$$

would be desirable for some $\widetilde{s} \ge 1$. The trivial case would be $\nu = 1$; however, using the special structure (1.3b) allows us to derive simple a priori bounds in $L^\nu(\Omega)$, $\nu > 1$, as well. We refer to the estimates (2.10) and (2.11) in Lemma 2.3. At the end, the situation is somewhat more involved and we can exploit Galiardo–Nirenberg estimates as well, which will lead us to the dimension-dependent exponent $s = 1 + 1/(2d+1)$ in (1.12).

Remark 1.3 (Cross-diffusion between species)**.** *In case* **(H)**, *where* $\kappa_{1,i} > 0$ *for all* i, *Theorem 1.2 can be extended to a situation where cross-diffusion between species does occur. Indeed, in this case the coercivity estimate for the entropy production in Lemma 2.1 remains valid when replacing the Onsager matrix* $\mathbb{M}$ *by* $\mathbb{M} + \tilde\mu \otimes \tilde\mu$, *where the continuous function* $\tilde\mu = (\tilde\mu_1, \dots, \tilde\mu_I, 0)$ *is supposed*

to satisfy $|\tilde{\mu}_i(c,u)| \lesssim c_i$ *for all* i*. In this case,* $A_{ij} = \delta_{ij}\frac{m_i}{c_i} + \tilde{\mu}_i\frac{\tilde{\mu}_j}{c_j}, i,j \in \{1,\dots,I\}$*, implying that* $|A_{ij}| \lesssim c_i$*. The additional thermodiffusion-type coefficient in front of* ∇u *in the* i*-th component is given by* $-\tilde{\mu}_i\sum_{j=1}^I \tilde{\mu}_j\frac{w_j'(u)}{w_j(u)}$*. One can now see that, under the assumption* $\kappa_{1,i} \gtrsim 1$*, a flux bound of the form* (2.11) *can still be guaranteed.*

Remark 1.4 (Cross terms in the energy equation)**.** *It is possible in Theorem 1.2 to allow for non-trivial, non-negative continuous coefficients* $m_{I+1}(Z)$ *in the definition of* $\mathbb{M}$ *(cf.* (1.8)*) satisfying suitable growth conditions. Note that while non-trivial coefficients* m_{I+1} *may alter heat conductivity, the Dufour coefficients remain unchanged since* $\partial_u S(Z) = 1/\theta$*. For general* m_{I+1}*, the energy equation in the* Z*-variables takes the form*

$$\dot{u} = \operatorname{div}\left(\big(a(Z) - m_{I+1}\partial_u^2 S(Z)\big)\nabla u + \sum_{j=1}^I d_j(Z)\nabla c_j\right),$$

where $d_j(Z) = -m_{I+1}\frac{w_j'}{w_j}$*. The non-negativity of* m_{I+1} *ensures that the coercivity bounds for the entropy production are preserved. Besides the entropy production estimate, the main a priori estimate for* u *is the energy estimate* (3.10) *in Lemma 3.4. As will become clear in the proof of Theorem 1.2, our analysis can deal with generalisations of estimate* (3.10) *of the form*

$$\epsilon_1\tau\int_\Omega a_\delta(Z)|\nabla u|^2\mathrm{d}x + \int_\Omega u^2\,\mathrm{d}x \le \int_\Omega (u^{k-1})^2\,\mathrm{d}x + C\tau\int_\Omega P(Z)\,\mathrm{d}x, \tag{1.14}$$

provided $\epsilon_1 > 0$*. For instance, under the assumptions of Theorem 1.2, in case* **(H)** *an a priori estimate of the form* (1.14) *can be obtained if* $m_{I+1}(Z) \ge 0$ *satisfies the bound* $\frac{w_j'}{w_j}\,m_{I+1} \lesssim \sqrt{\pi_1}\gamma + \sqrt{\pi_1\gamma}$*. For the case* **(H')** *the corresponding inequality* (1.14) *can be obtained if* $m_{I+1} \lesssim 1$*. (Some additional conditions may have to be imposed to make the full construction work.)*

Remark 1.5. *The hypothesis* $\kappa_{1,i} > 0$ *for all* $i \in \{1,\dots,I\}$ *in case* **(H)** *is essential in our proof of Theorem 1.2. If* $\kappa_{1,i} = 0$*, entropy production bounds and Sobolev type estimates do not generally provide* L^2*-integrability of the concentrations, which we need in case* **(H)** *in order to ensure at least* L^1*-integrability of the flux term associated with thermodiffusion (see Lemmas 2.1 and 2.3).*

Our second result is motivated by the question of global existence in the absence of self-diffusion, i.e. in the case when $\kappa_{1,i} = 0$ for all $i \in \{1,\dots,I\}$. In the setting of **(H)**, choosing $\kappa_{1,i} = 0$ leads to strong cross-diffusion effects and is not covered by Theorem 1.2. Here, entropy (and energy) estimates in general fail to ensure integrability of the thermodiffusive flux terms. We therefore use a weaker concept of solution similar to the notion of renormalised solutions introduced in [34], which allows us at the same time to drop the growth condition on the reaction rates in Theorem 1.2. Weak or no growth restrictions on $|R_i(\cdot)|$ are often desirable when interested in physically realistic reactions. The concept of renormalised solutions utilized in [34] originates from the studies of DiPerna and Lions on the global existence of solutions to Boltzmann and transport equations [27, 28, 29]. During the last decades, various notions of renormalised solutions have been employed in the literature; see e.g. [17, 23, 53]. In [23] the authors present some classes of mass-action kinetics models which admit global weak solutions for reaction rates with at most quadratic growth and which allow for global renormalised solutions with defect measure for at most quartic growth.

As pointed out in Remark 1.5, we are generally lacking an L^2 a priori bound on c_i in the case $\kappa_{1,i} = 0$. For certain classes of models, L^2 bounds for the species can be obtained using duality estimates. See, e.g., the references [51, 25, 8, 22, 45], which include cross-diffusive models. However, without very specific assumptions on the structure of the diffusion operator such duality arguments are not applicable in our setting.

Our definition of renormalised solutions adapts [34, 13].

Definition 1.6 (Renormalised solution)**.** *Let $J = (0,\infty)$ and let $Z_0 = (c_0, u_0) : \Omega \to \mathbb{R}^{I+1}_{\geq 0}$ be measurable. We call a function $Z = (c,u)$ with non-negative components a (global) renormalised solution of* (1.7) *with initial data Z^0 if $Z_i \in L^2_{\mathrm{loc}}(\overline{J}; H^1(\Omega))$ or $\sqrt{Z_i} \in L^2_{\mathrm{loc}}(\overline{J}; H^1(\Omega))$ for each $i \in \{1,\dots,I+1\}$, if furthermore*

$$\chi_{\{|Z|\leq E\}} A(Z)\nabla Z \in L^2_{\mathrm{loc}}(\overline{J}; L^2(\Omega))$$

for all $E \geq 1$, and if for every $\xi \in C^\infty([0,\infty)^{I+1})$ with compactly supported derivative $D\xi$, every $T > 0$, and every $\psi \in C^\infty([0,T]\times\overline{\Omega})$ with $\psi(\cdot,T) = 0$ the following identity is satisfied:

$$\begin{aligned}
&-\int_\Omega \xi(Z^0)\psi(\cdot,0)\,\mathrm{d}x - \int_0^T\int_\Omega \xi(Z)\frac{d}{dt}\psi\,\mathrm{d}x\mathrm{d}t \\
&\quad = -\sum_{i,j,k=1}^{I+1}\int_0^T\int_\Omega \psi\,\partial_i\partial_k\xi(Z)A_{ij}(Z)\nabla Z_j\cdot\nabla Z_k\,\mathrm{d}x\mathrm{d}t \\
&\qquad -\sum_{i,j=1}^{I+1}\int_0^T\int_\Omega \partial_i\xi(Z)A_{ij}(Z)\nabla Z_j\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}t + \sum_{i=1}^{I}\int_0^T\int_\Omega \psi\,\partial_i\xi(Z)R_i(Z)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Our second main result assumes the following conditions.

Hypotheses 1.7. *Let the* General Hypotheses *be satisfied and assume hypotheses* **(H)**. *Further suppose that $\kappa_{1,i} = 0$ and $\kappa_{0,i} > 0$ for all $i \in \{1,\dots,I\}$.*

Note that Hypotheses 1.7 do not impose any growth conditions on $R(c,u)$. Let us further point out that while the restriction to the (arguably more interesting) case $\kappa_{1,i} = 0$ for all i is not necessary and we could have equally treated Onsager matrices of the form considered in Theorem 1.2, our second result appears to require that either $\kappa_{1,i} > 0$ for all i or $\kappa_{1,i} = 0$ for all i if one wants to admit reactions with arbitrarily fast growth.

Theorem 1.8 (Global existence of renormalised solutions)**.** *Let Hypotheses 1.7 hold true. Let $Z^0 = (c^0, u^0)$ have non-negative components satisfying $c_i^0 \in L\log L(\Omega)$, $i \in \{1,\dots,I\}$, $u^0 \in L^2(\Omega)$ and $\hat{\sigma}_{0,-}(u^0) \in L^1(\Omega)$. Then, there exists a global renormalised solution $Z = (c,u)$ to* (1.7) *with initial data Z^0 having the additional regularity*

$$\begin{aligned}
&c_i \in L^\infty(0,\infty; L\log L(\Omega)), \quad i \in \{1,\dots,I\},\\
&u \in L^\infty(0,\infty; L^2(\Omega)).
\end{aligned}$$

For all $T > 0$, u satisfies $\partial_t u \in L^s(0,T; W^{1,s'}(\Omega)^*)$ (with $s = \frac{2d+2}{2d+1}$, $s' = 2d+2$) and

$$\int_0^T \big\langle \partial_t u, \phi\big\rangle\,\mathrm{d}t + \int_0^T\int_\Omega a(c,u)\nabla u\cdot\nabla\phi\,\mathrm{d}x\mathrm{d}t = 0$$

for all $\phi \in L^\infty(0,T; W^{1,\infty}(\Omega))$. Moreover, the internal energy is conserved, i.e. for all $t > 0$,

$$\int_\Omega u(t,x)\,\mathrm{d}x = \int_\Omega u^0(x)\,\mathrm{d}x,$$

and the following bounds are valid:

$$\|c\|_{L^\infty(0,\infty;L\log L)} + \|u\|_{L^\infty(0,\infty;L^2)} + \int_0^\infty\int_\Omega P(c,u)\,\mathrm{d}x\mathrm{d}t + \int_0^\infty\int_\Omega \pi_1\gamma|\nabla u|^2\,\mathrm{d}x\mathrm{d}t \leq C(\mathrm{data}),$$

where $P(c,u)$ is given by (2.5) *and* $\mathrm{data} = (\|c^0\|_{L\log L}, \|u^0\|_{L^2}, \|\hat{\sigma}_{0,-}(u^0)\|_{L^1})$.

For results on the global existence and the asymptotic behaviour of reaction–diffusion systems coupled to Poisson's equation, which is the typical setting for semiconductor models, we refer to [37, 40, 41]. While uniqueness is in general a difficult question for coupled systems, weak-strong uniqueness of renormalised solutions to entropy-dissipating reaction-diffusion systems has been established in [35] in the weakly coupled case, and in [14] for a class of population models featuring weak cross-diffusion. Within the last couple of years, various kinds of entropy estimates have been successfully applied for studying the long-time behavior of reaction–diffusion equations and to typically obtain exponential convergence to equilibrium [3, 19, 20, 21, 26]. Finally, we point out that exponential convergence to equilibrium has also been shown for renormalised solutions in the framework of detailed and complex-balanced chemical reaction networks [24, 32, 33].

Strategy of the proof. The main mathematical difficulty of the problem lies in the fact that the equilibria w_i of the concentrations c_i depend on the internal energy leading to a strong coupling in the entropy and in the associated evolution system. Let us point out that for strongly coupled systems the design of suitable (structure-preserving) approximation schemes may in general be quite tricky, even if suitable formal a priori estimates are available. Our main a priori estimates are obtained from the entropy structure of the coupled system together with the scalar-like structure of the heat equation (we choose L^2 for simplicity). Our approximation scheme for Theorem 1.2 adapts ideas developed in [44, 12] and references therein for reaction-diffusion systems with cross-diffusion. A key aspect of this scheme lies in a transformation to the so-called entropy variables, upon which the semi-definiteness of the mobility matrix $\mathbb{M}$ is exploited to construct approximate solutions to an elliptically regularised problem. In our case, a slight complication arises due to the circumstance that regularisations in the entropy variables interfere with the L^2-structure of the heat equation. We by-pass this issue by suitable additional approximation/regularisation procedures.

The proof of Theorem 1.8 uses the global weak solutions obtained in Theorem 1.2 as approximate solutions and adapts the construction of renormalised solutions to entropy-dissipating reaction-diffusion systems in [34, 13]. In these two references, the main difficulty comes from the lack of control of the reaction rates, whereas in our situation new difficulties arise owing to thermodiffusive cross effects. While the article [13] does consider cross-diffusion, it strongly relies on the presence of self-diffusion, and for suitably controlled reaction rates the existence of global-in-time weak solutions has previously been established in [12]. In contrast, in our situation a mere regularisation of the reaction rates is insufficient for constructing approximate weak solutions, which was one of our motivations for establishing Theorem 1.2, case **(H)**.

A new aspect of our approximation scheme is the idea of approximating the problem with vanishing self-diffusion and uncontrolled flux (treated in Theorem 1.8) by (thermodynamically consistent) models featuring self-diffusion (as considered in case **(H)** of Theorem 1.2). This construction relies on a second key ingredient: a stability result for the thermal part, namely the strong convergence of the gradient in L^2. Such stability results for elliptic and parabolic equations are classical, and have been used, for instance, in the existence analysis of elliptic equations with measure data [5, 6, 4, 17].

Outline and notations. The remaining part of this paper is devoted to the proofs of Theorems 1.2 and 1.8. In Section 2 we introduce the transformation to the entropy variables and a regularised entropy (Sec. 2.1), establish formal coercivity bounds for the entropy production as well as an estimate on the flux term. In Section 3 we use elliptic methods and a fixed point theorem to construct a solution to a time-discrete nonlinear regularised system, remove the elliptic regularisation (Prop. 3.3) and perform an L^2 estimate at the level of u (Lemma 3.4). In Section 4 we construct a global weak solution to the time-continuous problem.

Some preliminary tools for proving Theorem 1.8 are gathered in Section 5. In particular, we provide a weak chain rule for the time derivative of truncated solutions. The construction of a global renormalised solution is carried out in Section 6 by first proving some compactness properties to obtain a limiting candidate. This limit is then shown to be a global renormalised solution by deriving an approximate equation for appropriately truncated solutions and subsequently passing to the limit of infinite truncation height. Some auxiliary results from the literature are recalled in the appendix (see page 39).

Notations. We use the following notations and conventions.

- For $s \in \mathbb{R}$, we use the convention $s_+ = \max\{s, 0\}$ and $s_- = \min\{s, 0\}$ so that $s = s_+ + s_-$.
- In the notation of L^p and Sobolev spaces we usually do not explicitly state the underlying domain Ω. For a Banach space X, we sometimes write $L^p(X)$ to denote the Bochner space $L^p(0, T; X)$.
- We abbreviate $\Omega_T := (0, T) \times \Omega$.
- Unless otherwise stated, ∇ denotes the gradient with respect to the space variable $x \in \Omega$, while for a general function $F(a_i, \dots, a_N)$ (in several variables) we denote by DF its total derivative.
- The notation $A \lesssim B$ for non-negative quantities A, B means that there exists a constant $C \in (0, \infty)$ (only depending on fixed parameters) such that $A \le CB$; $A \gtrsim B$ is defined as $B \lesssim A$, and by $A \sim B$ we mean that both $A \lesssim B$ and $A \gtrsim B$ hold true.
- Unless specified otherwise, C denotes a positive constant that may change from line to line.
- We usually neither indicate nor mention the dependence of constants on fixed parameters such as the number I of species or the space dimension.
- $(u, v)_{L^2(\Omega)} = \int_\Omega uv \,\mathrm{d}x$ denotes the standard $L^2(\Omega)$ inner product of $u, v \in L^2(\Omega)$.
- Given a Banach space V, V^* denotes its topological dual and $\langle u, v\rangle_{V^*,V}$ the dual pairing between $u \in V^*$ and $v \in V$.
- For real matrices $A, B \in \mathbb{R}^{k\times l}$ we let $A : B = \sum_{i=1}^k \sum_{j=1}^l A_{ij}B_{ij}$.
- For a vector $\phi = (\phi_1, \dots, \phi_I, \phi_{I+1}) \in \mathbb{R}^{I+1}$ we let $\phi' = (\phi_1, \dots, \phi_I)$.
- Abbreviate $\overline{\kappa} := \max_{i,j} \kappa_{j,i} \in (0, \infty)$. The dependence of our estimates on $\overline{\kappa}$ is typically not indicated since $\overline{\kappa}$ remains uniformly bounded throughout our analysis. To indicate this, we occasionally also write $\kappa_{j,i} \lesssim 1$.

2. Entropy tools

2.1. **Entropy variables.** Here, we consider entropies of the form (1.5), where the equilibria $w_i, i \in \{1, \dots, I\}$, are as specified in (B2), and $\hat\sigma \in C^2((0, \infty))$ is supposed to satisfy $\hat\sigma''(s) < 0$ for all $s > 0$. This property ensures that the matrix $D^2S(Z)$ is negative definite for all $Z \in (\mathbb{R}_{>0})^{I+1}$, see the proof of [50, Prop. 2.1]. We can then define a change of variables to the so-called entropy variables

$$W := \begin{pmatrix} y \\ v \end{pmatrix} := -\begin{pmatrix} D_c S \\ \partial_u S \end{pmatrix}_{|(c,u)} = -DS(Z). \tag{2.1}$$

The regularity and strict concavity property of the entropy density S ensure that the transformation

$$-DS: \quad \mathcal{U} := (\mathbb{R}_{>0})^{I+1} \to -DS(\mathcal{U}), \qquad Z \mapsto -DS(Z) = W$$

is well-defined and invertible. The choice of this transformation is motivated by the fact that it allows us to rewrite the system $\dot Z = -\operatorname{div}(\mathbb{M}(Z)D^2S(Z)\nabla Z)$ as

$$\dot Z = \operatorname{div}\big(\mathbb{M}(Z)\nabla W\big),$$

where here $(c, u) = (-DS)^{-1}(y, v)$ is to be understood as a function of (y, v). Such a transformation has proved useful in the construction of solutions to cross-diffusion systems with a formal entropy structure (see e.g. [38, 10, 11, 30, 25, 44, 12, 13]). The main motivation for this transformation lies in the fact that, thanks to the positivity of $(-D^2S)$ and the semi-positivity of $\mathbb{M}$, the system in the entropy variables is parabolic (in the sense of [2]): letting $H = H(W)$ denote the Legendre transform

of the convex function $-S$, we have $Z = DH(W)$, and hence the above system can be written as $\frac{\mathrm{d}}{\mathrm{d}t}(DH(W)) = \mathrm{div}(\mathbb{M}(Z)\nabla W)$ with convex H.

Definition (2.1) can be written more explicitly as

$$y_i = \log\big(\tfrac{c_i}{w_i(u)}\big), \qquad v = -\hat\sigma'(u) - \sum_{i=1}^{I} c_i \tfrac{w_i'(u)}{w_i(u)}.$$

If the range of $-DS$ equals $\mathbb{R}^{I+1}$, we can compute for any given $W \in \mathbb{R}^{I+1}$

$$Z = Z(W) = (-DS)^{-1}(W).$$

In fact, the densities c_i, u can be recovered from y_i, v fairly explicitly: substituting $\frac{c_i}{w_i(u)}$ for $\exp(y_i) = \frac{c_i}{w_i(u)}$, we deduce $v = -\hat\sigma'(u) - \sum_{i=1}^{I} \exp(y_i) w_i'(u)$. Thanks to strict concavity, for given $y \in \mathbb{R}^I$ the function

$$(0,\infty) \ni s \mapsto \Psi(s,y) := -\hat\sigma'(s) - \sum_{i=1}^{I} \exp(y_i) w_i'(s)$$

is strictly increasing. Denoting by $\Phi(\cdot, y)$ its inverse, we have $u = \Phi(v,y)$ and

$$c_i = w_i(u)\exp(y_i) = w_i(\Phi(v,y))\exp(y_i).$$

In our application (where $S = S_0$ as in General Hypotheses), the range of $-DS_0$ equals $\mathbb{R}^I \times \mathbb{R}_{<0}$ (cf. (B4)) and thus fails to coincide with $\mathbb{R}^{I+1}$. To circumvent this issue we introduce an approximation S_δ of S_0 such that $\mathrm{im}(DS_\delta) = \mathbb{R}^{I+1}$. This can be achieved by setting

$$S_\delta(c,u) = S_0(c,u) - \delta\lambda(u) \qquad \delta \in (0,1], \tag{2.2}$$

where $\lambda \geq 0$ is as in (1.6). Since $\mathrm{im}(\lambda') = \mathbb{R}$, $-DS_\delta : \mathbb{R}^{I+1}_{>0} \mapsto \mathbb{R}^{I+1}$ is indeed onto. Similar approximations have been used in the analysis of population models [25, 44, 12] to deal with sublinear transition rates. For the choice (2.2) of the entropy, the associated function $\Phi = \Phi_\delta$ is locally bounded, i.e. $(0 \leq)\Phi_\delta(v,y) \leq C(|(v,y)|)$ for suitable $C = C_\delta$ and hence

$$0 \leq Z_i(W) \leq C(|W|) \tag{2.3}$$

for all $i \in \{1,\dots,I+1\}$. Here, we can see another advantage of this approach: non-negativity of the original variables is guaranteed by construction. Finally, note that since $D_c S_0 = D_c S_\delta$, the Lyapunov type hypothesis (R1) is preserved

$$D_c S_\delta(c,u)\cdot R(c,u) \geq 0 \quad \text{ for all } (c,u) \in (0,\infty)^{I+1}.$$

2.2. **A priori estimates.** Here, we gather a class of a priori estimates fundamental to our problem. Since we want to apply these estimates also to certain regularised models to gain control on our approximate solutions, the results will be stated in a somewhat more general form.

2.2.1. *Entropy production estimates.*

Lemma 2.1. *Let S be of the form* (1.5) *for some strictly concave function $\hat\sigma \in C^2((0,\infty))$ and equilibrium functions w_i as in hp.* (B2)*. Let $\mathbb{M}$ be given by* (1.8) *and fulfill* (M1) *and* (B3)*. Further suppose hypotheses* **(H)** *or, alternatively, suppose hypothesis* **(H')** *with $\kappa_{1,i} = 0$ for all i. Abbreviate $M(Z) := D^2S(Z)\mathbb{M}(Z)D^2S(Z)$. There exists $\epsilon > 0$ depending only on fixed parameters such that for all $\zeta := (\xi,\eta) \in \mathbb{R}^{I+1}$ and all $(c,u) \in \mathbb{R}^{I+1}_{\geq 0}$*

$$\zeta^T M(c,u)\zeta \geq \epsilon\Big[\sum_{i=1}^{I}\Big(\kappa_{1,i}|\xi_i|^2 + \kappa_{0,i}|\tfrac{1}{\sqrt{c_i}}\xi_i|^2\Big) + \pi_1\gamma^2|\eta|^2\Big],$$

where, as before, $\gamma(c,u) := -\big(\hat\sigma''(u) + \sum_{l=1}^I \frac{w_l''}{w_l} c_l\big)$.

Proof. We compute

$$\zeta^T M(c,u)\zeta = \sum_{i=1}^I m_i |\tfrac{1}{c_i}\xi_i - \tfrac{w_i'}{w_i}\eta|^2 + \pi_1 \Big(\hat\sigma''(u) + \sum_{i=1}^I \tfrac{w_i''}{w_i} c_i\Big)^2 |\eta|^2,$$

where $w_i = w_i(u), \pi_1 = \pi_1(c,u)$.

Using the bound $|n+\tilde n|^2 \geq \frac12 |n|^2 - |\tilde n|^2$, we estimate

$$m_i |\tfrac{1}{c_i}\xi_i - \tfrac{w_i'}{w_i}\eta|^2 \geq \tfrac12 m_i |\tfrac{1}{c_i}\xi_i|^2 - m_i |\tfrac{w_i'}{w_i}\eta|^2.$$

By (M1), we have on the one hand

$$m_i |\tfrac{1}{c_i}\xi_i|^2 \gtrsim \kappa_{1,i}|\xi_1|^2 + \kappa_{0,i}|\tfrac{1}{\sqrt{c_i}}\xi_i|^2.$$

On the other hand, supposing **(H')** with $\kappa_{1,i} = 0$ for all i allows us to estimate

$$\sum_{i=1}^I m_i |\tfrac{w_i'}{w_i}\eta|^2 \lesssim -\sum_{i=1}^I \kappa_{0,i} \tfrac{w_i''}{w_i} c_i |\eta|^2 \lesssim \pi_1 \gamma^2 |\eta|^2 \leq \zeta^T M(c,u)\zeta,$$

while under Hypotheses **(H)** we find using (H2) resp. (H3) and (H1)

$$\begin{aligned} |c_i \tfrac{w_i'}{w_i}\eta|^2 &\lesssim |\pi_1^{\frac12} \tfrac{w_i''}{w_i} c_i \eta|^2, \\ |\sqrt{c_i}\tfrac{w_i'}{w_i}\eta|^2 &\lesssim |\pi_1^{\frac12} \tfrac{w_i''}{w_i} c_i \eta|^2 + |\pi_1^{\frac12}\gamma\eta|^2, \end{aligned}$$

where the right-hand sides of the last two estimates are bounded above by $|\pi_1^{\frac12}\gamma\eta|^2$.

By our convention that $\max_{j,i} \kappa_{j,i} = \overline{\kappa} \lesssim 1$, this completes the proof of the assertion. □

Let us now suppose that $Z = (c_1, \dots, c_I, u)$ is a measurable function and that the gradients ∇Z and ∇W are defined in a suitable sense, where $W = -DS(c,u)$. Then, defining (formally)

$$\begin{aligned} Q(c,u) &:= \nabla W : (\mathbb{M}(Z)\nabla W), \text{ where } W = -DS(c,u), \\ \mathcal{Q}(c,u) &:= \int_\Omega Q(c,u)\,\mathrm{d}x, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} P(c,u) &:= \sum_{i=1}^I \Big(\kappa_{1,i}|\nabla c_i|^2 + \kappa_{0,i}|\nabla\sqrt{c_i}|^2\Big) + \pi_1\gamma^2|\nabla u|^2, \\ \mathcal{P}(c,u) &:= \int_\Omega P(c,u)\,\mathrm{d}x, \end{aligned} \tag{2.5}$$

Lemma 2.1 implies the existence of $\epsilon_* > 0$ such that

$$\epsilon_* \mathcal{P}(c,u) \leq \mathcal{Q}(c,u). \tag{2.6}$$

(Observe that $\mathcal{Q}$ agrees with $\mathcal{P}_{\mathrm{diff}}$ on page 8.)

2.2.2. *Entropy and flux bounds.* To derive an upper bound for c_i using the total entropy $\mathcal{S}(c,u) = \int_\Omega S(c,u)\,\mathrm{d}x$, it is convenient to use the entropy relative-entropy estimate derived in [36, App. A.2]. For $p > 1$ we define the p-entropy function $U_p(w) = \frac{1}{p(p-1)}\big(w^p - pw + p - 1\big) \geq 0$, which is characterised by $U_p''(w) = w^{p-2}$ and $U_p(1) = U_p'(1) = 1$. Recalling the Boltzmann function $\lambda(z) = z\log z - z + 1$, we have, for all $p > 1$ and $c, w \geq 0$,

$$w\,\lambda\big(\tfrac{c}{w}\big) \;\geq\; \tfrac{p-1}{p}\lambda(c) - (p-1)U_p(w) \;\geq\; \tfrac{p-1}{p}\lambda(c) - \tfrac{1}{p}\,w^p - 1. \tag{2.7}$$

Lemma 2.2 (Upper and lower entropy bounds)**.** *Let $S = S(c,u)$ have the form* (1.5) *and satisfy Hypotheses 1.1 with $S = S_0$, $\hat\sigma = \hat\sigma_0$, and $0 < w_0 \leq w_i(u) \leq C_w^\beta(1+u)^\beta$, where $\beta \in (0,1)$ is from* (B5)*. Then, for all $(c,u) \in [0,\infty)^{I+1}$,*

$$S(c,u) \leq \hat\sigma(u) + 2I\beta C_w u + I(2\beta C_w + 1) - (1-\beta)\sum_{i=1}^I \lambda(c_i), \tag{2.8}$$

$$S(c,u) \geq \hat\sigma(u) + I - \frac{I}{\min\{w_0,1\}} - 2\sum_{i=1}^I \lambda(c_i). \tag{2.9}$$

Proof. The upper estimate follows by applying (2.7) with $p = 1/\beta > 1$ and adding over $i = 1,\dots,I$, namely

$$\begin{aligned} S(c,u) &\leq \sigma(u) - \sum\nolimits_1^I \big(\tfrac{p-1}{p}\lambda(c_i) - (p-1)U_p(w_i(u))\big) \\ &\leq \hat\sigma(u) - I + \sum\nolimits_1^I w_i(u) - \sum_1^I \big((1-\beta)\lambda(c_i) - \beta C_w(1+u) - 1\big), \end{aligned}$$

which gives (2.8) if we invoke $w_i(u) \leq \beta C_w(1+u) + 1 - \beta$.

The lower estimate (2.9) is obtained by using $w_i(u) \geq w_0 > 0$, which leads to

$$S(c,u) \geq \hat\sigma(u) + \sum\nolimits_1^I \big(c_i \log w_0 - \lambda(c_i)\big).$$

For $w_0 \geq 1$ we may simply drop the term $c_i \log w_0 \geq 0$ and obtain (2.9). For $w_0 \in (0,1)$ we use the the Young–Fenchel inequality

$$-c_i \log w_0 = c_i \log\big(\tfrac{1}{w_0}\big) \leq \lambda(c_i) + \lambda^*\big(\log\big(\tfrac{1}{w_0}\big)\big) = \lambda(c_i) + \tfrac{1}{w_0} - 1,$$

where λ^* denotes the Legendre transform of λ. Estimate (2.9) now easily follows. □

Estimates qualitatively equivalent to (2.8), (2.9) can also easily be obtained elementarily.

In the following lemma, we establish general bounds on the flux term. Before stating the assertion, let us recall our convention that $0 \leq \kappa_{j,i} \lesssim 1$ for all j, i.

Lemma 2.3 (Control of the flux)**.** *Let $A(Z) := -\mathbb{M}(Z)D^2S(Z)$, where S is of the general form* (1.5) *for some strictly concave function $\hat\sigma \in C^2((0,\infty))$ and equilibrium functions w_i as in hp.* (B2)*. Let $\mathbb{M}$ have the form* (1.8) *with* (M1)*,* (B3) *and* (B6)*, and assume that $Z = (c,u)$ has non-negative components and is such that $P(c,u)$ given by* (2.5) *is a well-defined function.*

If **(H')** *holds true and $\kappa_{1,i} = 0$ for all i, then*

$$|A(Z)\nabla Z| \lesssim \max_{i=1,\dots,I}\Big(\sqrt{c_i} + \sqrt{\pi_1(Z)}\Big)P^{\frac12}(Z). \tag{2.10}$$

If instead hypotheses **(H)** *are fulfilled, then*

$$|A(Z)\nabla Z| \lesssim \max_{i=1,\dots,I}\Big(c_i + \kappa_{0,i} + \sqrt{\pi_1(Z)}\Big)P^{\frac12}(Z). \tag{2.11}$$

Thus, in both cases, for any $\xi \in C([0,\infty)^{I+1})$ *with* $\operatorname{supp}\xi \subseteq \{|Z| \leq E\}$

$$|\xi(Z)A(Z)\nabla Z| \lesssim (E+1)P^{\frac{1}{2}}(Z). \tag{2.12}$$

Proof. We first note that

$$\begin{aligned} |A_{ii}(c,u)\nabla c_i| &\lesssim \kappa_{1,i}c_i|\nabla c_i| + \kappa_{0,i}\sqrt{c_i}|\nabla\sqrt{c_i}| \\ &\lesssim \big(\kappa_{1,i}^{\frac{1}{2}}c_i + \kappa_{0,i}^{\frac{1}{2}}\sqrt{c_i}\big)P^{\frac{1}{2}}(Z). \end{aligned} \tag{2.13}$$

Let us next turn to the coefficients $A_{i,I+1}(Z)$ (given by (1.10)). Supposing **(H')**, we recall that $\kappa_{1,i} = 0$ in this case, and estimate using (H2') and (H1')

$$\begin{aligned} |A_{i,I+1}(Z)| &\lesssim \kappa_{0,i}\sqrt{c_i}\big(\tfrac{w_i''}{w_i}c_i\big)^{\frac{1}{2}} + \sqrt{c_i}\big(\tfrac{w_i''}{w_i}c_i\big)^{\frac{1}{2}} \\ &\lesssim (\kappa_{0,i}+1)\sqrt{c_i}\pi_1^{\frac{1}{2}}\gamma \end{aligned}$$

and hence

$$|A_{i,I+1}(Z)\nabla u| \lesssim (\kappa_{0,i}+1)\sqrt{c_i}P^{\frac{1}{2}}(Z).$$

By our convention that $\kappa_{j,i} \lesssim 1$ for all j, i, we see that the RHS is bounded above by the RHS of (2.10).

Supposing instead **(H)**, we estimate using (H2), (H3)

$$\begin{aligned} |A_{i,I+1}(Z)| &\lesssim -a_i\pi_1^{\frac{1}{2}}\tfrac{w_i''}{w_i}c_i + \pi_1^{\frac{1}{2}}\gamma c_i \\ &\lesssim (a_i(c,u)+c_i)\pi_1^{\frac{1}{2}}\gamma \end{aligned}$$

and deduce

$$|A_{i,I+1}(Z)\nabla u| \lesssim ((\kappa_{1,i}+1)c_i + \kappa_{0,i})P^{\frac{1}{2}}(Z).$$

Finally, we recall (cf. eq. (1.10)) that $0 \leq A_{I+1,I+1}(Z) = \pi_1\gamma$ to infer

$$|A_{I+1,I+1}(Z)\nabla u| \lesssim \pi_1^{\frac{1}{2}}(Z)P^{\frac{1}{2}}(Z). \tag{2.14}$$

Put together, we obtain estimate (2.10) resp. (2.11). □

3. Approximation scheme

In this and the subsequent section (Sections 3–4), we assume that all hypotheses of Theorem 1.2 hold true, and the main purpose of these sections is to prove Theorem 1.2. In the current section, we construct solutions to an approximate time-discrete, regularised problem. In our approximate equations we replace the time derivative by a backward difference quotient $\dot{Z} \approx \tau^{-1}(Z - Z^{k-1})$ with $0 < \tau \ll 1$ denoting the size of the time step and Z^{k-1} corresponding to the solution at the previous time step, and introduce an elliptic regularisation at the level of the entropy variables. Schemes of that type have been proposed in [44] (see also [7, 43, 18] for a selection of earlier works).

Throughout this section, $\delta > 0$ is kept fixed and we assume that $S = S_\delta$ is given by (2.2). Then, the associated $\gamma = \gamma_\delta$ (see def. (1.9)) depends on δ, and we have $\gamma_\delta \geq \gamma$. To obtain good a priori estimates, we will also consider a modified coefficient $\pi_{1,\delta}$ converging, as $\delta \to 0$, to the function π_1 considered in Theorem 1.2. The specific choice of $\pi_{1,\delta}$ depends on whether **(H')** or **(H)** are considered:

- Under assumption **(H')**, we choose $\pi_{1,\delta} = \pi_1\gamma\frac{1}{\gamma_\delta}$, which ensures that (H1') is preserved for the δ-model. Condition (H2') remains obviously valid.

- Let us now instead suppose **(H)**. In this case, we simply let $\pi_{1,\delta}(c,u) = \pi_1(c,u) + \delta u^2$. This ensures that $\pi_{1,\delta}^{\frac{1}{2}}(c,u)\lambda''(u) \gtrsim_\delta 1$, where λ denotes the Boltzmann function (so that $\lambda''(u) = \frac{1}{u}$). Here, we need to point out that hp. (H3) remains true since $u\frac{w_i'(u)}{w_i(u)} \lesssim 1$ as a consequence of hp. (B2). (Indeed, $w_i(u) \geq w_i(u) - w_i(0) = \int_0^u w_i'(v)\,\mathrm{d}v \geq u w_i'(u)$ since w_i is concave.) The bounds (H1), (H2) for the corresponding δ-quantities are obvious.

Of course, the δ-dependence of $\pi_1 = \pi_{1,\delta}$ implies that $\mathbb{M} = \mathbb{M}_\delta$ is also δ-dependent. Crucial for our subsequent analysis is the observation that thanks to the properties listed above, hypotheses **(H)** resp. **(H')** and thus the a priori estimates in Lemma 2.1 and Lemma 2.3 are equally true for the δ-regularised model. The δ-dependencies will therefore not be explicitly indicated in the current section. We also assume, in this section, that the reaction rates $R = (R_i(Z))_{i=1}^I$ are globally bounded. Possible dependencies on $\|R_i\|_{L^\infty}$ will always be indicated. These hypotheses regarding S and R will be removed in Section 4 by approximation.

Given an approximate solution $Z^{k-1} = (c^{k-1}, u^{k-1})$ (at the previous time step) we are concerned with constructing a solution at the subsequent time step. This amounts to solving a problem of elliptic type obtained by the backward time discretisation. We therefore suppose in this section that

$$Z^{k-1} = (c^{k-1}, u^{k-1}) \in (L\log L(\Omega))^I \times L^2(\Omega)$$

is a given vector-valued function with non-negative components. We will further require the hypothesis

$$\hat{\sigma}_-(u^{k-1}) \in L^1(\Omega),$$

which is non-redundant if $\hat{\sigma}(0+) = -\infty$. It ensures that the entropy $\int_\Omega S(Z^{k-1})\,\mathrm{d}x$ is finite (see Lemma 2.2).

To construct a solution to the nonlinear elliptic problem in the entropy variables, it is convenient to introduce a higher order regularisation of order $m > \frac{d}{2}$, so that $H^m(\Omega) \overset{\mathrm{c}}{\hookrightarrow} C_b(\Omega)$. The associated regularisation parameter is denoted by $\varepsilon > 0$. To simplify notation we will often drop superscripts like $(\cdot)^{I+1}$ etc. when denoting spaces of vector-valued functions.

Lemma 3.1. *Given $\tilde{W} \in L^\infty$ and letting $\tilde{Z} := (-DS)^{-1}(\tilde{W})$, there exists a unique $W = (y,v) \in (H^m)^{I+1}$ satisfying for all $\phi = (\phi', \phi_{I+1}) \in (H^m)^{I+1}$*

$$\begin{aligned}\tau\int_\Omega (\mathbb{M}(\tilde{Z})\nabla W) : \nabla\phi\,\mathrm{d}x + \tau\varepsilon\left(\sum_{|\alpha|=m}\int_\Omega \partial^\alpha W\cdot\partial^\alpha\phi\,\mathrm{d}x + \int_\Omega W\cdot\phi\,\mathrm{d}x\right)\\ = -\int_\Omega(\tilde{Z}-Z^{k-1})\cdot\phi\,\mathrm{d}x + \tau\int_\Omega R(\tilde{Z})\cdot\phi'\,\mathrm{d}x.\end{aligned}$$

Proof. This follows from the Lax–Milgram theorem using the local boundedness of the functions $\mathbb{M}(\cdot)$ and $R(\cdot)$, and the fact that by (2.3), $\|\tilde{Z}\|_{L^\infty} \leq C_\delta(\|\tilde{W}\|_{L^\infty})$. (See e.g. [44, 13] for the proofs of rather similar assertions). □

We now construct a solution to the nonlinear problem.

Lemma 3.2. *There exists $W \in H^m$ such that for all $\phi \in H^m$:*

$$\begin{aligned}\tau\int_\Omega (\mathbb{M}(Z)\nabla W) : \nabla\phi\,\mathrm{d}x + \tau\varepsilon\big(\sum_{|\alpha|=m}\int_\Omega \partial^\alpha W\cdot\partial^\alpha\phi\,\mathrm{d}x + \int_\Omega W\cdot\phi\,\mathrm{d}x\big)\\ = -\int_\Omega(Z-Z^{k-1})\cdot\phi\,\mathrm{d}x + \tau\int_\Omega R(Z)\cdot\phi'\,\mathrm{d}x.\end{aligned} \tag{3.1}$$

Here $Z = (-DS)^{-1}(W)$.

Proof. We want to apply the Leray–Schauder theorem (see [39, Theorem 11.3]). For this purpose, we define a fixed point map

$$\Gamma : L^\infty \to L^\infty, \qquad \tilde{W} \mapsto W,$$

where $W \in H^m \overset{c}{\hookrightarrow} L^\infty$ denotes the solution obtained in Lemma 3.1.

We proceed in three steps.

Step 1: Γ is a continuous operator. Suppose that $\tilde{W}_j \to \tilde{W}$ in L^∞ and let $W_j = \Gamma(\tilde{W}_j)$ and $\tilde{Z}_j := (-DS)^{-1}(\tilde{W}_j)$. Then $L := \sup_j \|\tilde{W}_j\|_{L^\infty} < \infty$ and hence

$$\|\tilde{Z}_j\|_{L^\infty} \le C(L).$$

Choosing $\phi = W_j$ in the equation for W_j we therefore get for $\theta > 0$

$$\tau\varepsilon \|W_j\|_{H^m}^2 \le C(L, \theta, \|Z^{k-1}\|_{L^1}) + \theta \|W_j\|_{L^\infty}^2. \tag{3.2}$$

Using the embedding $H^m \hookrightarrow L^\infty$, an absorption argument gives

$$\|W_j\|_{H^m}^2 \le C(L, \tau, \varepsilon, \|Z^{k-1}\|_{L^1}).$$

Hence, there exists $W_\infty \in H^m$ such that, along a subsequence, $W_j \rightharpoonup W_\infty$ in H^m, $W_j \to W_\infty$ in $C_b(\Omega)$. Observe that, after possibly passing to a subsequence, we can assume that $\tilde{Z}_j \to \tilde{Z}$ a.e. in Ω and, by dominated convergence, also $\tilde{Z}_j \to \tilde{Z}$ in L^p for any $p \in [1, \infty)$. It is now easy to see that W_∞ satisfies for all $\phi \in H^m$

$$\begin{aligned}
&\tau \int_\Omega (\mathbb{M}(\tilde{Z}) \nabla W_\infty) : \nabla \phi \,\mathrm{d}x + \tau\varepsilon \left(\sum_{|\alpha|=m} \int_\Omega \partial^\alpha W_\infty \cdot \partial^\alpha \phi' \,\mathrm{d}x + \int_\Omega W_\infty \cdot \phi \,\mathrm{d}x \right) \\
&= -\int_\Omega (\tilde{Z} - Z^{k-1}) \cdot \phi \,\mathrm{d}x + \tau \int_\Omega R(\tilde{Z}) \cdot \phi' \,\mathrm{d}x.
\end{aligned}$$

By the uniqueness of solutions to the linear equation, we infer that $W_\infty = \Gamma(\tilde{W})$.

The above reasoning shows that any subsequence of $(\tilde{W}_j)$ has a subsequence along which Γ converges to $\Gamma(\tilde{W})$ in L^∞. This implies that $\Gamma(\tilde{W}_j) \to \Gamma(\tilde{W})$ in L^∞. Hence Γ is continuous.

Step 2: Γ is a compact operator. Arguments similar to the proof of Step 1 show that the image of any bounded set in L^∞ under the map Γ is bounded in H^m (cf. (3.2)). Taking also into account the compactness of the embedding $H^m \hookrightarrow L^\infty$, we infer that the operator Γ is compact.

Step 3: A priori bound. Suppose that $W = \gamma_1 \Gamma(W)$ for some $\gamma_1 \in (0, 1]$. By hypothesis, W satisfies for all $\phi \in H^m$ the equation

$$\begin{aligned}
&\tau \int_\Omega (\mathbb{M}(Z) \nabla W) : \nabla \phi \,\mathrm{d}x + \tau\varepsilon \Big(\sum_{|\alpha|=m} \int_\Omega \partial^\alpha W \cdot \partial^\alpha \phi \,\mathrm{d}x + \int_\Omega W \cdot \phi \,\mathrm{d}x \Big) \\
&= -\gamma_1 \int_\Omega (Z - Z^{k-1}) \cdot \phi \,\mathrm{d}x + \gamma_1 \tau \int_\Omega R(Z) \cdot \phi' \,\mathrm{d}x.
\end{aligned}$$

Choosing $\phi = W = -DS(Z)$ we obtain

$$\begin{aligned}
&\tau \int_\Omega (\mathbb{M}(Z) \nabla W) : \nabla W \,\mathrm{d}x + \tau\varepsilon \Big(\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha W|^2 \,\mathrm{d}x + \int_\Omega |W|^2 \,\mathrm{d}x \Big) \\
&\qquad = \gamma_1 \int_\Omega (Z - Z^{k-1}) \cdot DS(Z) \,\mathrm{d}x - \gamma_1 \tau \int_\Omega R(Z) \cdot D_c S(Z) \,\mathrm{d}x \\
&\qquad \le \gamma_1 (\mathcal{S}(Z) - \mathcal{S}(Z^{k-1})).
\end{aligned} \tag{3.3}$$

Choosing $\phi = (0, \dots, 0, 1)^T$ further yields

$$\gamma_1 \int_\Omega u \,\mathrm{d}x = \gamma_1 \int_\Omega u^{k-1} \,\mathrm{d}x - \varepsilon\tau \int_\Omega v \,\mathrm{d}x \le \int_\Omega u^{k-1} \,\mathrm{d}x + \varepsilon\tau C + \tfrac{1}{2}\varepsilon\tau \|v\|_{L^2}^2.$$

Adding the last inequality to (3.3) yields upon absorption

$$\gamma_1 \int u \,\mathrm{d}x + \tau \mathcal{Q}(c,u) + \tfrac{1}{2}\tau\varepsilon\Big(\sum_{|\alpha|=m} \|\partial^\alpha W\|^2 + \|W\|_{L^2}^2\Big) \\ \le \gamma_1\big(\mathcal{S}(Z) - \mathcal{S}(Z^{k-1})\big) + \int_\Omega u^{k-1} \,\mathrm{d}x + \varepsilon\tau C,$$

where $\mathcal{Q}(c,u)$ is defined in (2.4). Using Lemma 2.2 to estimate the RHS and applying standard Hölder's and Young's inequality (to deal with integrals of sublinear functions of u), we can infer after absorption (possibly increasing $\beta < 1$ in Lemma 2.2)

$$\begin{aligned} \gamma_1 c_\beta \sum_{i=1}^I \int_\Omega c_i \log_+(c_i) \,\mathrm{d}x + \tfrac{\gamma_1}{2} \int_\Omega u \,\mathrm{d}x + \gamma_1 \|\hat\sigma_-(u)\|_{L^1} + \tau \mathcal{Q}(c,u) \\ + \tfrac{1}{2}\tau\varepsilon \left(\sum_{|\alpha|=m} \|\partial^\alpha W\|_{L^2}^2 + \|W\|_{L^2}^2 \right) \qquad (3.4) \\ \le \|\hat\sigma_-(u^{k-1})\|_{L^1} + C \sum_{i=1}^I \int_\Omega c_i^{k-1} \log_+(c_i^{k-1}) \,\mathrm{d}x + \|u^{k-1}\|_{L^2}^2 + C. \end{aligned}$$

Here, we used the rough estimate $\delta \|\lambda(u^{k-1})\|_{L^1} \lesssim \|u^{k-1}\|_{L^2}^2 + 1$.

In particular, we have obtained the bound

$$\|W\|_{L^\infty} \le C\|W\|_{H^m} \le C(\|c^{k-1}\|_{L\log L}, \|u^{k-1}\|_{L^2}, \|\hat\sigma_-(u^{k-1})\|_{L^1}, \tau, \varepsilon).$$

Theorem 11.3 in [39] now yields the existence of a fixed point $W = \Gamma(W)$. □

Letting $\gamma_1 = 1$ in Step 3 of the proof Lemma 3.2 (see (3.3), (3.4)), we infer using the entropy production estimate (2.6)

$$\epsilon_* \tau \mathcal{P}(c,u) + \tau\varepsilon\Big(\sum_{|\alpha|=m} \|\partial^\alpha W\|_{L^2}^2 + \|W\|_{L^2}^2 \Big) \le \mathcal{S}(Z) - \mathcal{S}(Z^{k-1}), \qquad (3.5)$$

and

$$\|(c,u)\|_{L^1} + \|\hat\sigma_-(u)\|_{L^1} + \epsilon_*\tau\mathcal{P}(c,u) + \tau\varepsilon\Big(\sum_{|\alpha|=m} \|\partial^\alpha W\|_{L^2}^2 + \|W\|_{L^2}^2\Big) \\ \le C(\|c^{k-1}\|_{L\log L}, \|u^{k-1}\|_{L^2}, \|\hat\sigma_-(u^{k-1})\|_{L^1}). \qquad (3.6)$$

To proceed, we need to distinguish the cases **(H)** and **(H')**.

Let us first assume that hypotheses **(H)** hold true. Then, by the Sobolev embedding, the fact that $\pi_1^{\frac{1}{2}} = \pi_{1,\delta}^{\frac{1}{2}} \gtrsim_\delta u$, $\gamma_\delta \gtrsim_\delta \frac{1}{u}$ and the definition (2.5) of $\mathcal{P}(c,u) = \mathcal{P}_\delta(c,u)$, we have

$$\begin{aligned} \|u\|_{L^2} &\lesssim_\delta \|\nabla u\|_{L^2} + \|u\|_{L^1} \qquad (3.7) \\ &\lesssim C(\|c^{k-1}\|_{L\log L}, \|u^{k-1}\|_{L^2}, \|\hat\sigma_-(u^{k-1})\|_{L^1}, \tau, \epsilon_*, \delta). \end{aligned}$$

Under hypotheses **(H')** it suffices to note that

$$|\nabla\sqrt{u}| \lesssim_\delta \sqrt{\gamma_\delta}|\nabla u| \lesssim P_\delta^{\frac{1}{2}}(c,u). \qquad (3.7')$$

We are now in a position to pass to the limit $\varepsilon \to 0$ in problem (3.1).

Proposition 3.3. *Let $A(Z) = -\mathbb{M}(Z)D^2S(Z)$. There exists $Z = (c_1, \dots, c_I, u)$ with $c_i, u \geq 0$, $\kappa_{0,i}\sqrt{c_i} \in H^1(\Omega)$, $\kappa_{1,i}c_i \in H^1(\Omega)$ and $u \in H^1(\Omega)$ in case* **(H)**, *$\sqrt{u} \in H^1(\Omega)$ in case* **(H')** *such that for all $\phi = (\phi', \phi_{I+1}) \in W^{1,\infty}(\Omega)^{I+1}$*

$$\tau \int_\Omega A(Z)\nabla Z : \nabla\phi \,\mathrm{d}x = -\int_\Omega (Z - Z^{k-1}) \cdot \phi \,\mathrm{d}x + \tau \int_\Omega R(Z) \cdot \phi' \,\mathrm{d}x. \tag{3.8}$$

Furthermore,

$$\tau\epsilon_* \mathcal{P}(Z) \leq \mathcal{S}(Z) - \mathcal{S}(Z^{k-1}). \tag{3.9}$$

Proof. By (3.6), (3.7) resp. (3.7'), and Sobolev embeddings there exists a (non-negative) vector-valued function $Z = (c, u)$ and a sequence $\varepsilon_j \searrow 0$ such that the associated solutions $W_j = -DS(Z_j) = -DS(c_j, u_j)$ of Lemma 3.2 (with $\varepsilon = \varepsilon_j$) satisfy for suitable $\widetilde{\epsilon} = \widetilde{\epsilon}(d) > 0$:

(i) $c_{j,i} \rightharpoonup c_i$ in $H^1(\Omega)$, $c_{j,i} \to c_i$ in $L^{2+\widetilde{\epsilon}}(\Omega)$ and a.e. in Ω provided $\kappa_{1,i} > 0$
(ii) $\sqrt{c_{j,i}} \rightharpoonup \sqrt{c_i}$ in $H^1(\Omega)$, $\sqrt{c_{j,i}} \to \sqrt{c_i}$ in $L^{2+\widetilde{\epsilon}}(\Omega)$ and a.e. in Ω if $\kappa_{0,i} > 0$
(iii) $\nabla u_j \rightharpoonup \nabla u$ in $L^2(\Omega)$, $u_j \to u$ in $L^{2+\widetilde{\epsilon}}(\Omega)$ and a.e. in Ω under hp. **(H)**

(iv) $\nabla\sqrt{u_j} \rightharpoonup \nabla\sqrt{u}$ in $L^2(\Omega)$, $\sqrt{u_j} \to \sqrt{u}$ in $L^{2+\widetilde{\epsilon}}(\Omega)$ and a.e. in Ω under hp. **(H')**.

Thanks to Lemma 2.3, hp. (B6) and the above convergence results (using in case **(H)** also the elementary Lemma 6.6), we further deduce

$$A_{\varepsilon_j}(Z_j)\nabla Z_j \rightharpoonup A(Z)\nabla Z \text{ in } L^{1+\epsilon}(\Omega)$$

for some $\epsilon = \epsilon(d) > 0$. Observe that in case **(H)**, owing to estimate (2.11), we here require the hypothesis that $\kappa_{1,i} > 0$ for all i to ensure $L^{2+\widetilde{\epsilon}}(\Omega)$-integrability of c_i (and not only of $\sqrt{c_i}$). By boundedness and continuity, the passage to the limit in the reaction rates is immediate. The fact that $\varepsilon_j\big(\sum_{|\alpha|=m} \|\partial^\alpha W\|_{L^2} + \|W\|_{L^2}\big) \lesssim \varepsilon_j^{\frac{1}{2}}$ and the above convergence results allow us to pass to the limit in eq. (3.1) to obtain (3.8) for all $\phi \in H^m(\Omega)^{I+1}$. A density argument then allows to extend this identity in particular to all $\phi \in W^{1,\infty}(\Omega)^{I+1}$.

Inequality (3.5) and weak lower semi-continuity in L^2 yield (3.9). Here, we also used the fact that $\limsup_{j\to\infty} \mathcal{S}(c_j, u_j) \leq \mathcal{S}(c, u)$, which follows from (i)–(iv)) and Fatou's lemma applied to $\hat{\sigma}_-(u_j)$ giving $\limsup_{j\to\infty} \int_\Omega \hat{\sigma}_-(u_j) \,\mathrm{d}x \leq \int_\Omega \hat{\sigma}_-(u) \,\mathrm{d}x$. □

Having removed the regularisation in the entropy variables, we derive an L^2-energy estimate to upgrade the regularity of u.

Lemma 3.4 (Energy estimate)**.** *The solution $Z = (c, u)$ obtained in Prop. 3.3 satisfies the bound*

$$2\tau \int_\Omega a(c, u)|\nabla u|^2 \mathrm{d}x + \int_\Omega u^2 \,\mathrm{d}x \leq \int_\Omega (u^{k-1})^2 \,\mathrm{d}x. \tag{3.10}$$

Proof. It follows from (3.8) that for all $\psi \in W^{1,\infty}(\Omega)$

$$\tau \int_\Omega a(c, u)\nabla u \cdot \nabla\psi \,\mathrm{d}x = -\int_\Omega (u - u^{k-1})\psi \,\mathrm{d}x.$$

For $L \in \mathbb{N}$ consider $u_L := \min\{u, L\}$. Then, in view of estimate (2.14), (3.9) and hp. (B6), we have $\int_\Omega a(c, u)|\nabla u_L|^2 \mathrm{d}x < \infty$, both in case **(H)** and in case **(H')**. Using an approximation argument, one can now show that the above identity also holds for $\psi = u_L$. (See for instance the proof of the

L^2-energy identity (6.6) for a detailed argument in a related, but somewhat more complex situation.) Thus,

$$\tau \int_\Omega a(c,u)|\nabla u_L|^2 \mathrm{d}x + \int_\Omega u u_L \,\mathrm{d}x = \int_\Omega u^{k-1} u_L \,\mathrm{d}x.$$

Sending $L \to \infty$, we infer

$$\tau \big\| a^{\frac{1}{2}}(c,u)\nabla u \big\|_{L^2}^2 + \|u\|_{L^2}^2 \le \int_\Omega u^{k-1} u \,\mathrm{d}x \le \tfrac{1}{2}\|u^{k-1}\|_{L^2}^2 + \tfrac{1}{2}\|u\|_{L^2}^2,$$

which yields the asserted bound (3.10). □

4. Global weak solutions

We recall that $\mathcal{S} = \mathcal{S}_\delta$ (see eq. (2.2)), $\mathcal{P} = \mathcal{P}_\delta$, $\mathbb{M} = \mathbb{M}_\delta$ etc. (see the second paragraph of the introductory part in Section 3), and apply Proposition 3.3 with $R = R_\varrho$ defined by

$$R_\varrho(Z) = \frac{R(Z)}{\varrho |R(Z)| + 1}. \tag{4.1}$$

In order to emphasise the δ-dependence of quantities like $\mathcal{S}(Z), \mathcal{P}(Z)$, we will add the subscript $(\cdot)_\delta$ and write $\mathcal{S}_\delta(Z), \mathcal{P}_\delta(Z)$ etc.

Given a vector $Z^{k-1} = (c^{k-1}, u^{k-1}) \in (L\log L)^I \times L^2$, $k \in \mathbb{N}$, with non-negative components and such that $\|\hat\sigma_{0,-}(u^{k-1})\|_{L^1} < \infty$, we let $(c^k, u^k) = Z^k$ denote the solution Z of eq. (3.8) (with $S = S_\delta$, $R = R_\varrho$) obtained in Prop. 3.3. We also use the notation $W^k = (y^k, v^k) = -DS_\delta(c^k, u^k)$. By construction and Lemma 3.4, we have $(c^k, u^k) \in (L\log L)^I \times L^2$ with non-negative components. As a consequence of estimate (3.9), we further have $\|\hat\sigma_{0,-}(u^k)\|_{L^1} < \infty$. Hence, given an initial datum $Z^0 \in (L\log L)^I \times L^2$ with $\|\hat\sigma_{0,-}(u^0)\|_{L^1} < \infty$ and a time step size $\tau > 0$, this allows us to iteratively construct a sequence $(Z^k)_{k\in\mathbb{N}}$, where $Z_i^k \ge 0$ for all i and $Z^k \in (L\log L)^I \times L^2$ with $\|\hat\sigma_{0,-}(u^k)\|_{L^1} < \infty$ for all $k \in \mathbb{N}$. Our goal is now to send $\tau, \varrho, \delta \to 0$ in order to construct a solution to the original time-continuous problem.

4.1. Uniform estimates.

Summing estimate (3.9) over the time steps from $k = 1$ to $k = n \in \mathbb{N}$ yields

$$\epsilon_* \tau \sum_{k=1}^n \mathcal{P}_\delta(Z^k) \le \mathcal{S}_\delta(Z^n) - \mathcal{S}_\delta(Z^0).$$

By estimate (3.10) we further have

$$2\tau \sum_{k=1}^n \int_\Omega a_\delta(c^k, u^k)|\nabla u^k|^2 \mathrm{d}x + \int_\Omega (u^n)^2 \,\mathrm{d}x \le \int_\Omega (u^0)^2 \,\mathrm{d}x.$$

Adding the previous two inequalities and recalling estimates (2.8), (2.9) (with $S = S_0$, treating $\delta\lambda(u^0) \lesssim 1 + (u^0)^2$ separately), we deduce upon absorption

$$\begin{aligned} \sup_{n\in\mathbb{N}} \bigg(\sum_{i=1}^I \int_\Omega c_i^n \log c_i^n \,\mathrm{d}x + \|u^n\|_{L^2}^2 + \|\hat\sigma_{0,-}(u^n)\|_{L^1} \bigg) + \epsilon_* \tau \sum_{k=1}^\infty \mathcal{P}_\delta(Z^k) \\ + \tau \sum_{k=1}^\infty \int_\Omega a_\delta(c^k, u^k)|\nabla u^k|^2 \mathrm{d}x \le C\bigg(\sum_{i=1}^I \|c^0\|_{L\log L}, \|u^0\|_{L^2}, \|\hat\sigma_{0,-}(u^0)\|_{L^1} \bigg). \end{aligned} \tag{4.2}$$

To simplify notation, we henceforth abbreviate

$$\mathcal{N}(Z^0) := \sum_{i=1}^{I} \|c_i^0\|_{L\log L} + \|u^0\|_{L^2} + \|\hat{\sigma}_{0,-}(u^0)\|_{L^1}.$$

Lemma 4.1. *Given $T \in (0,\infty)$ and $N \in \mathbb{N}$ suppose that $\tau = T/N$. Let further $\delta \in (0,1]$ and $s = \frac{2d+2}{2d+1}$. Then there exists a finite constant $C = C(\mathcal{N}(Z^0))$ (depending also on ϵ_* and $\kappa_{j,i}$) such that*

$$\tau \sum_{k=1}^{N} \|A_\delta(Z^k)\nabla Z^k\|_{L^s(\Omega)}^s \le C(\mathcal{N}(Z^0))(1+T). \tag{4.3}$$

Proof. Below we abbreviate $\frac{1}{s'} + \frac{1}{s} = 1$, $q = \frac{2s}{2-s}$, $\theta = \frac{1-\frac{1}{q}}{\frac{1}{2}+\frac{1}{d}}$. By Lemma 2.3 (with $A = A_\delta$), hp. (B6), estimate (4.2) and the Gagliardo–Nirenberg inequality (Lemma 6.4 with q, θ as defined here and $p = 1$),

$$\|A_\delta(Z^k)\nabla Z^k\|_{L^s(\Omega)} \le (\mathcal{P}_\delta)^{\frac{1}{2}}(c^k,u^k)\big((\mathcal{P}_\delta)^{\frac{\theta}{2}}(c^k,u^k) + \|\nabla u^k\|_{L^2}^\theta + 1\big)C\big(\mathcal{N}(Z^0)\big).$$

When considering **(H')**, GNS will be applied to $\sqrt{c_i}$ instead of c_i, and thanks to the time-uniform $L^1(\Omega)$ control of c_i we could have chosen $p = 2$ in Lemma 6.4, thus somewhat improving the last estimate. Since this would only lead to a minor improvement of the integrability of the flux term in case **(H')**, we content ourselves with this somewhat suboptimal bound.
Since $\frac{s(1+\theta)}{2} = 1$, the previous estimate yields

$$\|A_\delta(Z^k)\nabla Z^k\|_{L^s(\Omega)}^s \le C(\mathcal{N}(Z^0))\bigg(\mathcal{P}_\delta(c^k,u^k) + \|\nabla u^k\|_{L^2}^2 + 1\bigg).$$

Taking the discrete time integral up to time T and using once more (4.2), we infer (4.3). □

4.2. **Limit $(\tau,\varrho,\delta) \to 0$.**

Proof of Theorem 1.2. To pass to the limit $\tau \to 0$ we can follow the approach in [16, 44, 12, 13]. Given the sequence $(Z^k)_{k\in\mathbb{N}}$ constructed above, we define the piecewise constant interpolant function

$$Z^{(\tau)}(t,\cdot) = Z^k \qquad \text{if } t \in ((k-1)\tau, k\tau].$$

We further let $(c^{(\tau)},u^{(\tau)}) := Z^{(\tau)}$, $W^{(\tau)} = -DS(Z^{(\tau)})$ and define the discrete time derivative

$$\partial_t^{(\tau)} Z^{(\tau)}(t,\cdot) = \tfrac{1}{\tau}(Z^k - Z^{k-1}) \qquad \text{if } t \in ((k-1)\tau, k\tau].$$

Let now $T \in (0,\infty)$ be fixed but arbitrary. Then, by (4.2),

$$\begin{aligned} &\|c^{(\tau)}\|_{L^\infty(0,T;L\log L)} + \|u^{(\tau)}\|_{L^\infty(0,T;L^2)} + \|\hat{\sigma}_{0,-}(u^{(\tau)})\|_{L^\infty(0,T;L^1)} \\ &\quad + \int_{(0,T)} \mathcal{P}_\delta(Z^{(\tau)}(t,\cdot))\,\mathrm{d}t + \|\sqrt{a_\delta(Z^{(\tau)})}\nabla u^{(\tau)}\|_{L^2(\Omega_T)}^2 \le C(\mathcal{N}(Z^0)). \end{aligned} \tag{4.4}$$

Observe that since $a_\delta \gtrsim 1$ (cf. (H1) resp. (H1')), estimate (4.4) implies the bound

$$\|\nabla u^{(\tau)}\|_{L^2(\Omega_T)}^2 \le C(\mathcal{N}(Z^0)).$$

Next, for s as in Lemma 4.1, we have by (4.3)

$$\|A_\delta(Z^{(\tau)})\nabla Z^{(\tau)}\|_{L^s(\Omega_T)} \le C(\mathcal{N}(Z^0),T). \tag{4.5}$$

In the remaining reasoning, we need to distinguish the cases **(H)** and **(H')**. We provide the details only in the case **(H)**, and then briefly describe how to modify the arguments to deal with case **(H')**. Thus, let us first suppose **(H)**.

In order to control the reaction term, we need to upgrade the space-time integrability of (c,u). By the Gagliardo–Nirenberg inequality (Lemma 6.4 with $p=1$, $q_1=2+\frac{2}{d}$, θ as in (6.16) $\rightsquigarrow q_1\theta=2$), we have for $c=c^{(\tau)}$

$$\begin{aligned}\|c\|_{L^{q_1}(\Omega_T)}^{q_1} &\le C_1\int_0^T \|\nabla c\|_{L^2(\Omega)}^{q_1\theta}\|c\|_{L^1(\Omega)}^{q_1(1-\theta)}\,\mathrm{d}t + C_2T\|c\|_{L^\infty(L^1)}^{q_1}\\ &\le C(\mathcal{N}(Z^0),T).\end{aligned} \tag{4.6}$$

Similarly, applying Lemma 6.4 with $p=2$, $q_2=2+\frac{4}{d}$, θ as in (6.16) (so that again $q_2\theta=2$), we find for $u=u^{(\tau)}$

$$\begin{aligned}\|u\|_{L^{q_2}(\Omega_T)}^{q_2} &\le C_1\int_0^T \|\nabla u\|_{L^2(\Omega)}^{q_2\theta}\|u\|_{L^2(\Omega)}^{q_2(1-\theta)}\,\mathrm{d}t + C_2T\|u\|_{L^\infty(L^1)}^{q_2}\\ &\le C(\mathcal{N}(Z^0),T).\end{aligned} \tag{4.7}$$

We next assert that for $r:=\min\left\{\frac{2+\frac{2}{d}}{q_1},\frac{2+\frac{4}{d}}{q_2},s\right\}>1$ and r' given by $\frac{1}{r'}+\frac{1}{r}=1$,

$$\|\partial_t^{(\tau)}Z^{(\tau)}\|_{L^r(0,T;(W^{1,r'}(\Omega))^*)} \le C(\mathcal{N}(Z^0),T). \tag{4.8}$$

To show (4.8), we take $\phi\in W^{1,r'}(\Omega)$ with $\|\phi\|_{W^{1,r'}(\Omega)}\le 1$. The choice of r implies that $r'>d$ and hence $W^{1,r'}(\Omega)\hookrightarrow L^\infty(\Omega)$. Thus, for any $k\in\mathbb{N}$,

$$\Big|\int_\Omega \tau^{-1}(Z^k-Z^{k-1})\cdot\phi\,\mathrm{d}x\Big| \lesssim \|A_\delta(Z^k)\nabla Z^k\|_{L^s(\Omega)} + \|R_\varrho(Z^k)\|_{L^1(\Omega)}$$

and hence $\|\partial_t^{(\tau)}Z^{(\tau)}\|_{(W^{1,r'}(\Omega))^*}\lesssim \|A_\delta(Z^{(\tau)})\nabla Z^{(\tau)}\|_{L^s(\Omega)}+\|R_\varrho(Z^{(\tau)})\|_{L^1(\Omega)}$. Estimate (4.8) then follows upon taking the L^r_t norm and recalling (4.5), (4.6) and (4.7).

We can therefore apply the Aubin–Lions lemma in the version of [31, Theorem 1] for any $T<\infty$. Choosing a sequence $T\to\infty$ and using a diagonal argument, then allows one to infer the existence of a sequence $(\tau,\varrho,\delta)\to 0$ and

$$\begin{aligned}&c\in L^\infty(0,\infty;L\log L) \text{ with } \nabla c\in L^2((0,\infty)\times\Omega),\\ &u\in L^\infty(0,\infty;L^2) \text{ with } \nabla u\in L^2((0,\infty)\times\Omega)\end{aligned}$$

such that for any $T>0$

$$(c^{(\tau)},u^{(\tau)})\to(c,u) \quad \text{in } L^2(\Omega_T), \tag{4.9}$$

$$(\nabla c^{(\tau)},\nabla u^{(\tau)})\rightharpoonup(\nabla c,\nabla u) \quad \text{in } L^2(\Omega_T),$$

$$(c^{(\tau)},u^{(\tau)})\to(c,u) \quad \text{a.e. in } \Omega_T, \tag{4.10}$$

$$\lim_{\tau\to0}\int_0^T\int_\Omega \partial_t^{(\tau)}Z^{(\tau)}\cdot\phi\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_\Omega Z\cdot\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega Z^0\cdot\phi(0,\cdot)\,\mathrm{d}x \tag{4.11}$$

for any $\phi\in C_c^\infty([0,T)\times\bar\Omega)$, and

$$\partial_t^{(\tau)}Z^{(\tau)}\overset{*}{\rightharpoonup}\partial_t Z \quad \text{in } L^r(0,T;(W^{1,r'})^*), \tag{4.12}$$

where the last two assertions are obtained as in [16, p. 2792f.].

Arguing as in the proof of Prop. 3.3 (using in addition (4.5)), one further has

$$A_\delta(Z^{(\tau)})\nabla Z^{(\tau)}\rightharpoonup A(Z)\nabla Z \quad \text{in } L^s(\Omega_T). \tag{4.13}$$

Also, by (4.4) and (4.9), $\nabla\sqrt{c_i^{(\tau)}}\rightharpoonup\nabla\sqrt{c_i}$ in $L^2(\Omega_T)$ for all $i\in\{1,\dots,I\}$ with $\kappa_{0,i}>0$, and (using also (4.13) and Lemma 6.5) $\sqrt{a_\delta(Z^{(\tau)})}\nabla u^{(\tau)}\rightharpoonup\sqrt{a(Z)}\nabla u$ in $L^2(\Omega_T)$.

By (4.4), the above convergence results, and weak(-star) lower semi-continuity, we obtain in the limit $(\tau, \varrho, \delta) \to 0$

$$\|c\|_{L^\infty(0,T;L\log L)} + \|u\|_{L^\infty(0,T;L^2)} + \|\hat\sigma_{0,-}(u)\|_{L^\infty(0,T;L^1)} + \|a^{\frac12}(c,u)\nabla u\|^2_{L^2(\Omega_T)} + \int_0^T \mathcal{P}(Z)\,\mathrm{d}t \le C(\mathcal{N}(Z^0)).$$

Here, the bound for $\|c\|_{L^\infty(0,T;L\log L)}$ and $\|\hat\sigma_{0,-}(u)\|_{L^\infty(0,T;L^1)}$ was obtained using (4.10) and Fatou's lemma.

To infer equation (1.13), we sum eq. (3.8) (with $Z = Z^k$, $S = S_\delta$ and $R = R_\varrho$) from $k = 1$ to $k = N$ (where $\tau N = T$). The resulting equation can be written in the form

$$\int_0^T\int_\Omega A_\delta(Z^{(\tau)})\nabla Z^{(\tau)} : \nabla\phi\,\mathrm{d}x\mathrm{d}t = -\int_0^T\int_\Omega \partial_t^{(\tau)} Z^{(\tau)}\cdot\phi\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega R_\varrho(Z^{(\tau)})\cdot\phi'\,\mathrm{d}x\mathrm{d}t. \tag{4.14}$$

Since this equation holds true for any $T' \in (0,T]$, the density of functions in $L^{r'}(0,T;W^{1,r'}(\Omega))$ which are piecewise constant in time (see [52, Prop. 1.36]) allows us to extend eq. (4.14) to time-dependent test functions $\phi = (\phi', \phi_{I+1}) \in L^{r'}(0,T;W^{1,r'}(\Omega))^{I+1}$. We then obtain in particular

$$\int_0^T \langle \partial_t Z, \phi\rangle\,\mathrm{d}t + \int_0^T\int_\Omega A(Z)\nabla Z : \nabla\phi\,\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega R(Z)\cdot\phi'\,\mathrm{d}x\mathrm{d}t \tag{4.15}$$

for all $\phi \in L^\infty(0,T;W^{1,\infty}(\Omega))^{I+1}$. Since $W^{1,r}(0,T;(W^{1,r'})^*) \hookrightarrow C([0,T];(W^{1,r'})^*)$, we have $Z \in C([0,T];(W^{1,r'})^*)$ and (4.11), (4.12) then imply that $Z(t=0,\cdot) = Z^0$ in $(W^{1,r'}(\Omega))^*$ and thus in particular in the $(W^{1,\infty}(\Omega))^*$-sense.

It remains to prove the conservation of the internal energy. Set $V := W^{1,r'}(\Omega)$ and let $T < \infty$ be arbitrary. The fact that $\partial_t u \in L^r(0,T;V^*)$ (where $\partial_t u$ is understood as the distributional derivative of the Bochner function u) implies that for any $\psi \in V$ the function $t \mapsto (u(t,\cdot),\psi)_{L^2(\Omega)} = \langle u(t,\cdot),\psi\rangle_{V^*,V}$ is absolutely continuous and for all $0 \le t_1 \le t_2 \le T$

$$(u(t_2,\cdot),\psi)_{L^2(\Omega)} - (u(t_1,\cdot),\psi)_{L^2(\Omega)} = \int_{t_1}^{t_2} \langle \partial_t u(t,\cdot),\psi\rangle_{V^*,V}\,\mathrm{d}t. \tag{4.16}$$

On the other hand, choosing $\phi = (0,\dots,0,\phi_{I+1})$ with $\phi_{I+1}(t,\cdot) \equiv \chi_{[t_1,t_2]}(t)$ in equation (4.15) shows that the RHS of (4.16) vanishes for $\psi \equiv 1$. Hence, for any $t_2 > 0$ $(u(t_2,\cdot),1)_{L^2(\Omega)} = (u(0,\cdot),1)_{L^2(\Omega)}$, which concludes the proof of Theorem 1.2 under hypotheses **(H)**.

Let us now sketch the necessary modification under hypotheses **(H')**. Since $\kappa_{1,i} = 0$, only $\nabla\sqrt{c_i^{(\tau)}}$ (but not $\nabla c_i^{(\tau)}$) is controlled in $L^2(\Omega_T)$. In order to obtain compactness, we therefore have to apply a nonlinear version of the Aubin–Lions lemma. Such a result covering our situation has been provided in [15]. Under the (suboptimal) growth hypotheses on the reactions for model **(H')** and the flux bound (4.5), we easily obtain a (τ,ϱ,δ)-uniform bound of the form

$$\|\sqrt{c_i^{(\tau)}}\|_{L^2(0,T;H^1(\Omega))} + \|\partial_t c_i^{(\tau)}\|_{L^1(0,T;H^n(\Omega)^*)} \le C(\mathrm{data})$$

for sufficiently large $n \in \mathbb{N}$ (using the embedding $W^{1,s'}(\Omega)^* \hookrightarrow H^n(\Omega)^*$, which holds true e.g. for $n > \frac{d}{2}+1$). Applying [15, Theorem 3], we deduce that, along a subsequence, $c_i^{(\tau)} \to c_i$ in $L^1(0,T;L^{1+\epsilon}(\Omega))$ for any $\epsilon \in [0, \frac{1}{(1-\frac{2}{d})_+})$ and a.e. in Ω_T. The remaining arguments are as before.

□

5. Preliminaries for Theorem 1.8

From now on, we suppose Hypotheses 1.7 (in place of Hypotheses 1.1). We then let $\varepsilon, \varrho > 0$ denote small positive parameters, consider the Onsager matrix

$$\mathbb{M}^\varepsilon(c,u) := \operatorname{diag}\left(m_1^\varepsilon, \dots, m_I^\varepsilon, 0\right) + \pi_1(Z)\mu \otimes \mu,$$

where $m_i^\varepsilon = c_i a_i^\varepsilon(c,u)$, $a_i^\varepsilon(c,u) \sim \varepsilon c_i + \kappa_{0,i}$, satisfy (M1) with $\kappa_{1,i} = \varepsilon > 0$ for all $i \in \{1, \dots, I\}$, and choose the regularised reactions $R_\varrho(Z) = \frac{1}{1+\varrho|R(Z)|} R(Z)$ (cf. (4.1)). Since the choice $\mathbb{M} = \mathbb{M}^\varepsilon$, $R = R_\varrho$ clearly satisfies Hypotheses 1.1 (with $q_1 = q_2 = 0$), Theorem 1.2 provides us with a family of functions $Z = (c,u)$ (depending on $\varepsilon, \varrho > 0$) with the regularity

$$\begin{aligned} &A^\varepsilon(Z)\nabla Z \in L^s((0,\infty)\times\Omega) \text{ for } s := \tfrac{2d+2}{2d+1}, \text{ where } A^\varepsilon := -\mathbb{M}^\varepsilon D^2 S_0, \\ &\partial_t Z \in L^s_{\mathrm{loc}}(0,\infty; W^{1,s'}(\Omega)^*) \text{ for } s' := 2d+2 \end{aligned}$$

emanating from the initial datum Z^0 (in the $W^{1,\infty}(\Omega)^*$ sense) that satisfy for all $T > 0$ the weak formulation

$$\begin{aligned} \int_0^T \langle \partial_t Z, \phi\rangle \,\mathrm{d}t - \int_0^T \int_\Omega (\mathbb{M}^\varepsilon(Z) D^2 S_0(Z) \nabla Z) : \nabla\phi \,\mathrm{d}x\mathrm{d}t \\ = \int_0^T \int_\Omega R_\varrho(Z)\cdot\phi' \,\mathrm{d}x\mathrm{d}t \end{aligned} \tag{5.1}$$

for any $\phi = (\phi', \phi_{I+1}) \in L^\infty(0,T; W^{1,\infty}(\Omega))^{I+1}$, enjoy the ϱ-independent bound

$$\begin{aligned} &\|c\|_{L^\infty(0,\infty;L\log L)} + \|u\|_{L^\infty(0,\infty;L^2)} \\ &\qquad + \int_0^T\int_\Omega P_\varepsilon(Z)\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega a(Z)|\nabla u|^2\,\mathrm{d}x\mathrm{d}t \le C(\mathrm{data}), \end{aligned} \tag{5.2}$$

where $\mathrm{data} := (\|c^0\|_{L\log L}, \|u^0\|_{L^2}, \|\hat\sigma(u^0)\|_{L^1})$, and where $P_\varepsilon(Z)$ denotes the quantity $P(Z)$ defined (2.5) with $\kappa_{1,i} = \varepsilon$. These solutions conserve the internal energy, i.e. $\int u(t,x)\,\mathrm{d}x = \int u^0(x)\,\mathrm{d}x$ for any $t > 0$, and for later usage, we further note the (ε, ϱ)-uniform bound

$$\|\partial_t u\|_{L^s(0,T;W^{1,s'}(\Omega)^*)} \le C(\mathrm{data}), \tag{5.3}$$

which is a consequence of (5.2) and the bound (2.14) on the heat flux. The functions $Z = Z^{\varepsilon,\varrho}$ now serve as approximate solutions to the model considered in Theorem 1.8.

In this section, we gather some technical tools and derive a renormalised formulation for the weak solutions $Z = (c,u)$. Throughout this section, $\varepsilon, \varrho > 0$ will be kept fixed, and the dependence of Z on ε and ϱ will not be explicitly indicated.

As in [13] and [34], we employ the family of special cut-off functions

$$\varphi_i^E(Z) := Z_i \phi\left(\frac{\sum_{k=1}^{I+1} Z_k - E}{E}\right) + 3E\left(1 - \phi\left(\frac{\sum_{k=1}^{I+1} Z_k - E}{E}\right)\right)$$

for $i \in \{1, \dots, I\}$ and $Z \in \mathbb{R}^{I+1}$ where $\phi \in C^\infty(\mathbb{R}, [0,1])$ is a fixed non-increasing function with the property $\phi = 1$ on $(-\infty, 0]$ and $\phi = 0$ on $[1, \infty)$.

The following properties of φ_i^E are easily verified.

Lemma 5.1. *The truncations $\varphi_i^E(Z)$ fulfill the following conditions.*

(C1) $\varphi_i^E \in C^\infty((\mathbb{R}_0^+)^{I+1})$.

(C2) *There exists a constant $K_1 > 0$ such that for all $E > 0$ and $Z \in (\mathbb{R}_0^+)^{I+1}$*

$$|Z||D^2\varphi^E(Z)| \le K_1.$$

(C3) *For all* $E > 0$*, the set* $\operatorname{supp} D\varphi_i^E$ *is compact. More specifically,* $(D\varphi_i^E)(Z) = 0$ *for all* $Z \in \mathbb{R}^{I+1}$ *satisfying* $\sum_{j=1}^{I+1} Z_j \geq 2E$.

(C4) *For all* $j \in \{1, \dots, I+1\}$ *and* $Z \in (\mathbb{R}_0^+)^{I+1}$*, we have* $\lim_{E\to\infty} \partial_j \varphi_i^E(Z) = \delta_{ij}$.

(C5) *There exists a constant* $K_2 > 0$ *such that* $|D\varphi_i^E(Z)| \leq K_2$ *holds true for all* $E > 0$ *and* $Z \in (\mathbb{R}_0^+)^{I+1}$.

(C6) *For all* $Z \in (\mathbb{R}_0^+)^{I+1}$ *with* $\sum_{j=1}^{I+1} Z_j < E$*, we have* $\varphi_i^E(Z) = Z_i$.

(C7) *For all* $j, k \in \{1, \dots, I+1\}$ *and* $K > 0$*, we have*

$$\lim_{E\to\infty} \sup_{|Z|\leq K} |\partial_j \partial_k \varphi_i^E(Z)| = 0.$$

(C8) *For all* $Z \in (\mathbb{R}_0^+)^{I+1}$*, and* $E \in \mathbb{N}$*, we have*

$$\varphi_i^E(Z) \leq Z_i + 3 \sum_{j=1}^{I+1} Z_j.$$

(C9) *For all* $E \geq E_0 \geq 0$ *and* $Z \in (\mathbb{R}_0^+)^{I+1}$*, we have*

$$\sum_{i=1}^{I} Z_i \geq E_0 \;\Rightarrow\; \sum_{i=1}^{I} \varphi_i^E(Z) \geq E_0.$$

5.1. **A weak chain rule for truncated solutions.** The remaining part of this section is devoted to deriving an evolution equation for $\varphi_i^E(Z)$, as asserted in the following lemma.

Lemma 5.2. *For all* $T > 0$*,* $E \in \mathbb{N}$*,* $i \in \{1, \dots, I\}$*, and* $\psi \in C^\infty([0,T] \times \overline{\Omega})$ *with* $\psi(T, \cdot) = 0$*, the weak solution* $Z = (c, u)$ *of eq.* (5.1) *satisfies the following equation:*

$$\begin{aligned} -\int_\Omega \varphi_i^E(Z^0)\psi(0,\cdot)\,\mathrm{d}x - \int_0^T \int_\Omega \varphi_i^E(Z) \frac{d}{dt}\psi \,\mathrm{d}x\mathrm{d}t \qquad (5.4)\\ = -\sum_{j,k,l=1}^{I+1} \int_0^T \int_\Omega \psi\, \partial_j \partial_k \varphi_i^E(Z) A_{jl}^\varepsilon(Z) \nabla Z_l \cdot \nabla Z_k \,\mathrm{d}x\mathrm{d}t \\ - \sum_{j=1}^{I+1} \int_0^T \int_\Omega \partial_j \varphi_i^E(Z) A_{jl}^\varepsilon(Z) \nabla Z_l \cdot \nabla \psi \,\mathrm{d}x\mathrm{d}t \\ + \sum_{j=1}^{I} \int_0^T \int_\Omega \psi\, \partial_j \varphi_i^E(Z) R_{\varrho,j}(Z) \,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

The proof of this lemma is based on the techniques used in the proof of [13, Lemmas 11 and 12]. It makes use of the mollifier ρ_η defined as follows.

Definition 5.3. *Denote by* $\widetilde{\rho}_\eta$ *the standard mollifier, i.e.*

$$\widetilde{\rho}_\eta(x) := C\eta^{-d} \exp\left(\frac{1}{\eta^{-2}|x|^2 - 1}\right) \qquad (5.5)$$

for $|x| < \eta$*,* $\widetilde{\rho}_\eta(x) := 0$ *for* $|x| \geq \eta$*, which satisfies* $\widetilde{\rho}_\eta \in C_{cpt}^\infty(\mathbb{R}^d)$ *and* $\int_{\mathbb{R}^d} \widetilde{\rho}_\eta(x)\,\mathrm{d}x = 1$*, and set* $\rho_\eta := \widetilde{\rho}_\eta * \widetilde{\rho}_\eta$*. Further define* $\Omega^\vartheta := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \vartheta\}$ *for* $\vartheta > 0$.

We will make use of the following basic properties satisfied by ρ_η and $\widetilde{\rho}_\eta$.

Lemma 5.4 ([13], Lemma 10)**.** *Let ρ_η for $\eta > 0$ be the special mollifier from Definition 5.3, let u, v be locally integrable functions on Ω where $\operatorname{supp} u \subset \Omega^{4\eta}$, and let $x \in \Omega^{3\eta}$ and $\widetilde{x} \in B(x,\eta)$. Then,*

$$\int_\Omega u\, \rho_\eta * v \,\mathrm{d}x = \int_{\Omega^{3\eta}} \big(\widetilde{\rho}_\eta * u\big)\big(\widetilde{\rho}_\eta * v\big)\,\mathrm{d}x, \tag{5.6}$$
$$\int_{B(x,3\eta)} \rho_\eta(\widetilde{x}-y)u(y)\,\mathrm{d}y = \big(\rho_\eta * u\big)(\widetilde{x}), \qquad \int_{B(x,3\eta)} \rho_\eta(\widetilde{x}-y)\,\mathrm{d}y = 1.$$

We can now turn to the proof of Lemma 5.2.

Proof of Lemma 5.2. As in [13] we first observe that for any $i \in \{1,\dots,I+1\}$ the expression $\int_0^T \langle \partial_t Z_i, \psi\rangle \,\mathrm{d}t$ in (5.1) (with $\phi_i = \psi$ vanishing at $t = T$) can be rewritten in the distributional form $-\int_0^T \int_\Omega Z_i \frac{\mathrm{d}}{\mathrm{d}t}\psi \,\mathrm{d}x\mathrm{d}t - \int_\Omega Z_{0,i}\psi(0,\cdot)\,\mathrm{d}x$. We next specify for $i \in \{1,\dots,I\}$ in Lemma 6.3 (Lemma 4 of [34]) the variables $v_i := c_i \in L^2(0,T;H^1(\Omega))$ (which follows together with Lemma 6.4), $v_{I+1} := u \in L^2(0,T;H^1(\Omega))$, $(v_0)_i := c_{0,i} \in L^1(\Omega)$, $(v_0)_{I+1} := u_0 \in L^1(\Omega)$, $\xi := \varphi_i^E$, $\nu_i := 0$, $w_i := R_{\varrho,i}(Z) \in L^1(0,T;L^1(\Omega))$, $w_{I+1} := 0$,

$$z_i := -\sum_{j=1}^{I+1} A_{ij}^\varepsilon(Z)\nabla Z_j \in L^s(0,T;L^s(\Omega,\mathbb{R}^d)),$$

and $z_{I+1} := -A_{I+1,I+1}^\varepsilon \nabla Z_{I+1} \in L^s(0,T;L^s(\Omega,\mathbb{R}^d))$.
In the case of the better regularity $z \in L^2(0,T;L^2(\Omega,\mathbb{R}^d))^{I+1}$, we directly arrive at (5.4) by applying L. 6.3.

In the general case, we proceed as in [13] and follow the steps in the proof of Lemma 4 on page 579 in [34]. In our situation this amounts to selecting a smooth ψ compactly supported in $[0,T)\times\Omega^{4\eta}$, taking $\rho_\eta * \psi$ as a test function in (6.15), and integrating by parts the last term on the right-hand side. This enables us to apply Lemma 5 in [34] with the choices

$$\begin{aligned} v_i &:= \rho_\eta * c, \qquad v_{I+1} := \rho_\eta * u, \\ w_i &:= \rho_\eta * R_{\varrho,i}(Z) + \operatorname{div}\Big(\rho_\eta * \sum_{j=1}^{I+1} A_{ij}^\varepsilon(Z)\nabla Z_j\Big), \\ w_{I+1} &:= \operatorname{div}\big(\rho_\eta * (A_{I+1,I+1}^\varepsilon(Z)\nabla Z_{I+1})\big). \end{aligned}$$

As a consequence, we may replace the second line on page 580 in [34] by

$$-\sum_{j,k,l=1}^{I+1} \int_0^T \int_\Omega \psi\, \partial_j\partial_k\varphi_i^E (\rho_\eta * Z)\, \rho_\eta * \big(A_{jl}^\varepsilon(Z)\nabla Z_l\big)\cdot\nabla(\rho_\eta * Z_k)\,\mathrm{d}x\mathrm{d}t. \tag{5.7}$$

This expression converges for $\eta \to 0$ to the first line on the right-hand side of (5.4), which is proven in Lemma 5.5 below and which concludes the proof for the case of test functions ψ being compactly supported in $[0,T)\times\Omega$. The corresponding situation (involving the fifth line on page 582 in [34]) of test functions ψ being compactly supported in $[0,T)\times U$ where U is a coordinate patch of Ω at the boundary $\partial\Omega$ is treated similarly. □

Lemma 5.5. *The expression* (5.7) *converges for $\eta \to 0$ to the first line on the right-hand side of* (5.4).

Proof. We start by rewriting (5.7) using (5.6) as

$$\sum_{j=1}^{I+1}\int_0^T\int_\Omega \psi\,\nabla\big(\partial_j\varphi_i^E(\rho_\eta * Z)\big)\cdot\Big(\rho_\eta * \sum_{l=1}^{I+1} A_{jl}^\varepsilon(Z)\nabla Z_l\Big)\,\mathrm{d}x\mathrm{d}t$$
$$= \sum_{j=1}^{I+1}\int_0^T\int_{\Omega^{3\eta}} J_\eta^j(t,x)\cdot K_\eta^j(t,x)\,\mathrm{d}x\mathrm{d}t$$

where

$$J_\eta^j(t,x) := \widetilde{\rho}_\eta * \big(\psi\,\nabla\big(\partial_j\varphi_i^E(\rho_\eta * Z)\big)\big), \qquad K_\eta^j(t,x) := \widetilde{\rho}_\eta * \sum_{l=1}^{I+1} A_{jl}^\varepsilon(Z)\nabla Z_l.$$

The limit $\eta\to 0$ is now performed by following the steps in [13].

Step 1: Bound on K_η^j. We first calculate for $(t,x)\in[0,T)\times\Omega^{3\eta}$ by using Lemma 2.3, the hypothesis (B6), and the Cauchy–Schwarz inequality

$$\begin{aligned}
|K_\eta^j(t,x)| &= \Big|\int_{B(x,\eta)}\widetilde{\rho}_\eta(x-y)\Big(\sum_{l=1}^{I+1}A_{jl}^\varepsilon(Z)\nabla Z_l\Big)(y)\,\mathrm{d}y\Big|\\
&\le \sum_{i=1}^{I}\int_{B(x,\eta)}\widetilde{\rho}_\eta(x-\cdot)\big(1+u+c_i\big)P^{\frac12}(Z)\,\mathrm{d}y\\
&\le C\eta^{-d}\bigg[\bigg(\int_{B(x,\eta)}(1+u)^2\,\mathrm{d}y\bigg)^{\frac12}+\sum_{i=1}^{I}\bigg(\int_{B(x,\eta)}c_i^2\,\mathrm{d}y\bigg)^{\frac12}\bigg]\\
&\quad\times\bigg(\int_{B(x,\eta)}P(Z)\,\mathrm{d}y\bigg)^{\frac12}.
\end{aligned}$$

By reusing the arguments on page 5921 in [13], we obtain

$$\begin{aligned}
\eta^{-\frac{d}{2}}\bigg(\int_{B(x,\eta)}(1+u)^2\,\mathrm{d}y\bigg)^{\frac12} &\le C\eta^{1-\frac{d}{2}}\bigg(\int_{B(x,3\eta)}|\nabla u|^2\,\mathrm{d}y\bigg)^{\frac12}\\
&\quad + C\inf_{\widetilde{x}\in B(x,\eta)}|(\rho_\eta*(1+u))(\widetilde{x})|,
\end{aligned}$$

and an analogous relation involving c_k. In particular,

$$\begin{aligned}
|K_\eta^j(t,x)| &\le C\eta^{1-d}\int_{B(x,3\eta)}\big(|\nabla u|^2+P(Z)\big)\,\mathrm{d}y\\
&\quad + C\eta^{-\frac{d}{2}}\Big(\int_{B(x,3\eta)}\big(|\nabla u|^2+P(Z)\big)\,\mathrm{d}y\Big)^{\frac12}\inf_{\widetilde{x}\in B(x,\eta)}|(\rho_\eta*(1+Z))(\widetilde{x})|
\end{aligned}$$

Step 2: Bound on J_η^j. The estimate on J_η^j can be inferred in the same way as in [13]; in particular, the bound on page 5923 therein entails:

$$|J_\eta^j(t,x)|\le C\min\bigg\{\eta^{-1},\eta^{-\frac{d}{2}}\Big(\int_{B(x,3\eta)}\Big(|\nabla u|^2+P(Z)\Big)\,\mathrm{d}y\Big)^{\frac12}\bigg\}.$$

Step 3: Bound on $J_\eta^j\cdot K_\eta^j$. For $(t,x)\in[0,T)\times\Omega^{3\eta}$, we first consider the case that $\sum_{k=1}^{I+1}(\rho_\eta * Z_k)(\widetilde{x})\ge 2E$ holds true for all $\widetilde{x}\in B(x,\eta)$. Then, $\partial_j\varphi_i^E(\rho_\eta*Z)=0$ on $B(x,\eta)$ and, hence,

$J_\eta^j(t,x) = 0 = J_\eta^j(t,x)K_\eta^j(t,x)$. In the opposite case, there exists some $x^* \in B(x,\eta)$ with $\sum_{k=1}^{I+1}(\rho_\eta * Z_k)(x^*) < 2E$. As a result, we deduce

$$\inf_{\widetilde{x}\in B(x,\eta)} |(\rho_\eta * Z)(\widetilde{x})| \le |(\rho_\eta * Z)(x^*)| < 2E.$$

The estimates on $J_\eta^j(t,x)$ and $K_\eta^j(t,x)$ now lead to

$$|J_\eta^j(t,x)\cdot K_\eta^j(t,x)| \le CE\eta^{-d}\int_{B(x,3\eta)} \Big(|\nabla u|^2 + P(Z)\Big)\,\mathrm{d}y. \tag{5.8}$$

Step 4: Bound on $J_0^j \cdot K_0^j$. The pointwise limits of J_η^j and K_η^j read as follows:

$$J_0^j(t,x) := \psi\,\nabla\big(\partial_j\varphi_i^E(Z)\big),$$
$$K_0^j(t,x) := \sum_{l=1}^{I+1} A_{jl}^\varepsilon(Z)\nabla Z_l.$$

We infer from Lemma 2.3 that for all $(t,x)\in[0,T)\times\Omega^{3\eta}$,

$$\begin{aligned}|J_0^{I+1}(t,x)\cdot K_0^{I+1}(t,x)| &\le \left|\psi\sum_{k=1}^{I+1}\partial_j\partial_k\varphi_i^E(Z)\nabla Z_k\right|\sum_{i=1}^{I}(1+u+c_i)P^{\frac12}(Z)\\ &\le C(E)\Big(|\nabla u|^2+P(Z)\Big),\end{aligned}$$

where the E-dependence of the constant $C(E)$ arises from estimating $1+u+c_i\le C(E)$ (note that $\partial_j\partial_k\varphi_i^E(Z)$ is nonzero only for $|Z|\lesssim E$).

Step 5: Limit $\eta\to 0$. We are now able to prove that

$$\int_0^T\int_{\Omega^{3\eta}} \big(J_\eta^j\cdot K_\eta^j - J_0^j\cdot K_0^j\big)(t,x)\,\mathrm{d}x\mathrm{d}t \to 0$$

for all $j\in\{1,\dots,I+1\}$ as $\eta\to 0$. Following [13], we decompose the domain of integration into the sets $\{|J_\eta\cdot K_\eta - J_0\cdot K_0|>\vartheta\}$ and $\{|J_\eta\cdot K_\eta - J_0\cdot K_0|\le\vartheta\}$ for arbitrary $\vartheta>0$. We then infer from (5.8) the bound

$$\begin{aligned}\int_0^T\int_{\Omega^{3\eta}} |J_\eta^j(t,x)\cdot K_\eta^j(t,x)|\chi_{\{|J_\eta\cdot K_\eta - J_0\cdot K_0|>\vartheta\}}(t,x)\,\mathrm{d}x\mathrm{d}t&\\ \le CE\int_0^T\int_\Omega g_\eta\big(|\nabla u|^2+P(Z)\big)\,\mathrm{d}y\mathrm{d}t&\end{aligned}$$

after defining $g_\eta(t,y) := \eta^{-d}\int_\Omega \chi_{\Omega^{3\eta}}(x)\chi_{B(x,3\eta)}(y)\chi_{\{|J_\eta\cdot K_\eta - J_0\cdot K_0|>\vartheta\}}(t,x)\,\mathrm{d}x$. Next, we calculate

$$\int_0^T\int_\Omega |g_\eta(t,y)|\,\mathrm{d}y\mathrm{d}t \le C\mathcal{L}^{d+1}\big(\{|J_\eta\cdot K_\eta - J_0\cdot K_0|>\vartheta\}\big).$$

As the right-hand side tends to zero for $\eta\to 0$ (since $J_\eta^j\cdot K_\eta^j \to J_0^j\cdot K_0^j$ a.e. in $(0,T)\times\Omega$), we know that $g_\eta\to 0$ in $L^1((0,T)\times\Omega)$ and, hence, (up to a subsequence) $g_\eta\to 0$ a.e. in $(0,T)\times\Omega$. We further have

$$g_\eta\big(|\nabla u|^2+P(Z)\big) \le C\big(|\nabla u|^2+P(Z)\big) \in L^1((0,T)\times\Omega)$$

due to the uniform bounds (5.2). Lebesgue's dominated convergence theorem now guarantees that for $\eta \to 0$,

$$\int_0^T \int_{\Omega^{3\eta}} |J_\eta^j(t,x) \cdot K_\eta^j(t,x)| \chi_{\{|J_\eta \cdot K_\eta - J_0 \cdot K_0| > \vartheta\}}(t,x) \,\mathrm{d}x\mathrm{d}t \\ \leq CE \int_0^T \int_\Omega g_\eta \big(|\nabla u|^2 + P(Z)\big) \,\mathrm{d}y\mathrm{d}t \to 0.$$

We now apply Lebesgue's dominated convergence theorem to the parallel situation involving $J_0^j \cdot K_0^j$ for any $j \in \{1, \dots, I+1\}$. To this end, we use the convergence $|(J_0^j \cdot K_0^j)| \chi_{\{|J_\eta \cdot K_\eta - J_0 \cdot K_0| > \vartheta\}} \to 0$ a.e. in $(0,T) \times \Omega$ for $\eta \to 0$ and the uniform bound $|(J_0^j \cdot K_0^j)| \chi_{\{|J_\eta \cdot K_\eta - J_0 \cdot K_0| > \vartheta\}} \leq C(E)\big(|\nabla u|^2 + P(Z)\big) \in L^1((0,T) \times \Omega)$. This yields

$$\int_0^T \int_{\Omega^{3\eta}} |J_0^j(t,x) \cdot K_0^j(t,x)| \chi_{\{|J_\eta \cdot K_\eta - J_0 \cdot K_0| > \vartheta\}}(t,x) \,\mathrm{d}x\mathrm{d}t \to 0$$

for $\eta \to 0$. Finally, we conclude that

$$\begin{aligned} &\left| \int_0^T \int_{\Omega^{3\eta}} \big(J_\eta^j \cdot K_\eta^j - J_0^j \cdot K_0^j\big)(t,x) \,\mathrm{d}x\mathrm{d}t \right| \\ &\quad \leq \int_0^T \int_{\Omega^{3\eta}} |(J_\eta^j \cdot K_\eta^j - J_0^j \cdot K_0^j)(t,x)| \chi_{\{|J_\eta \cdot K_\eta - J_0 \cdot K_0| > \vartheta\}}(t,x) \,\mathrm{d}x\mathrm{d}t + C\vartheta \\ &\quad \to C\vartheta \end{aligned}$$

for $\eta \to 0$. This completes the proof since $\vartheta > 0$ can be chosen arbitrarily small. □

6. Construction of renormalised solutions

To simplify notation, we henceforth let $\varrho = \varepsilon$, where we recall that by $\varrho > 0$ we have denoted the regularisation parameter associated with the reaction rates. The global weak solutions of $\dot{Z} = \operatorname{div}(A^\varepsilon(Z)\nabla Z) + R_\varepsilon^\circ$, where $R_\varepsilon^\circ := (R_\varepsilon, 0)^T$, introduced in Section 5 will henceforth be denoted by Z^ε, and in this section we aim to study the limit $\varepsilon \to 0$.

6.1. Convergence of a subsequence.

In the following lemma, we establish a fundamental compactness result, allowing to deduce weak and pointwise convergence of $(c^\varepsilon, u^\varepsilon)$ along a subsequence. As in [34, Lemma 2], pointwise convergence of c^ε is obtained by applying a version of the Aubin–Lions lemma to $(\varphi_i^E(Z^\varepsilon))_\varepsilon$, $i = 1, \dots, I$, for every $E \in \mathbb{N}$. In contrast to [34], this is, however, not sufficient to deduce the renormalised formulation for the limiting candidate obtained upon $\varepsilon \to 0$. This issue is due to the strong cross-diffusion effects driven by gradients of the internal energy density, which result in a lack of a priori estimates of the first term on the RHS of eq. (5.4) that are uniform in $\varepsilon > 0$ and $E \in \mathbb{N}$. The key to resolve this problem lies in a stability result, namely the strong convergence of ∇u^ε in $L^2(\Omega_T)$, which will be deduced from L^2-energy identities for the internal energy density (cf. (6.5) and (6.6) below).

Lemma 6.1. *Let Hypotheses 1.7 hold true and denote by $Z^\varepsilon = (c^\varepsilon, u^\varepsilon)$ the global solution of eq.* (5.1) *introduced above. Then, along a sequence $\varepsilon \searrow 0$, Z^ε converges a.e. on $[0,\infty) \times \Omega$ to some limit $Z = (c,u)$ with $c_i \log c_i \in L^\infty(0,\infty; L^1(\Omega))$ for all $i \in \{1, \dots, I\}$ and $u \in L^\infty(0,\infty; L^2(\Omega))$. Furthermore, the weak convergence $\sqrt{c_i^\varepsilon} \rightharpoonup \sqrt{c_i}$ and the strong convergence $u^\varepsilon \to u$ in $L^2(0,T; H^1(\Omega))$ hold true for all $T > 0$.*

Proof. In a first step, we recall that due to (5.2)–(5.3), u^ε is uniformly bounded in $L^2(0,T;H^1(\Omega))$, while $\partial_t u^\varepsilon$ is uniformly bounded in $L^s(0,T;W^{1,s'}(\Omega)^*)$. Employing the Aubin–Lions Lemma then gives rise to a subsequence $(u^\varepsilon)_\varepsilon$ converging strongly in $L^2(0,T;L^2(\Omega))$ and pointwise a.e. in Ω_T to some $u \in L^2(0,T;H^1(\Omega))$. The additional regularity $u \in L^\infty(0,\infty;L^2(\Omega))$ follows with Fatou's Lemma.

We now keep E fixed and aim to apply the Aubin–Lions Lemma as stated in [52, Corollary 7.9] for proving the existence of a subsequence $(\varphi_i^E(Z^\varepsilon))_\varepsilon$ converging strongly in $L^2(0,T;L^2(\Omega))$. To this end, we will verify that $\varphi_i^E(Z^\varepsilon)$ is ε-uniformly bounded in the Bochner space $L^2(0,T;H^1(\Omega))$ and that the distributional time derivative $\frac{\mathrm{d}}{\mathrm{d}t}\varphi_i^E(Z^\varepsilon)$ is ε-uniformly bounded in the space of Radon measures $\mathcal{M}([0,T],H^p(\Omega)^*)$ if $p > d/2+1$.

The uniform boundedness of $\varphi_i^E(Z^\varepsilon)$ in $L^2(0,T;H^1(\Omega))$ follows from the boundedness of φ_i^E itself, the ε-uniform bound on $\nabla\sqrt{c_i^\varepsilon}$ in $L^2(\Omega_T)$ and the identity

$$\nabla\big(\varphi_i^E(Z^\varepsilon)\big) = \sum_{j=1}^{I+1} \partial_j\varphi_i^E(Z^\varepsilon)\nabla Z_j^\varepsilon.$$

In fact, keeping in mind that $\operatorname{supp} D\varphi_i^E$ is a compact subset of $\mathbb{R}^{I+1}$, we see that $\varphi_i^E(Z^\varepsilon)$ is uniformly bounded w.r.t. ε in $L^2(0,T;H^1(\Omega))$ for all fixed $T>0$, $E>0$, and $i \in \{1,\dots,I\}$.

We are left to establish an ε-uniform bound on the time derivative of $\varphi_i^E(Z^\varepsilon)$ in $\mathcal{M}([0,T],H^p(\Omega)^*)$, the topological dual space of $C^0([0,T],H^p(\Omega))$. For this purpose, we first extend (5.4) in Lemma 5.2 to allow for test functions $\psi \in C^\infty([0,T]\times\overline{\Omega})$, which do not necessarily vanish at time $t=T$. This is achieved by a standard approximation procedure carried out on pages 5926–5927 in [13] and leads to

$$\begin{aligned}
&\int_\Omega \varphi_i^E(Z^\varepsilon(T,\cdot))\psi(T,\cdot)\,\mathrm{d}x - \int_\Omega \varphi_i^E(Z^0)\psi(0,\cdot)\,\mathrm{d}x - \int_0^T\int_\Omega \varphi_i^E(Z^\varepsilon)\tfrac{\mathrm{d}}{\mathrm{d}t}\psi\,\mathrm{d}x\mathrm{d}t \qquad (6.1)\\
&\quad= -\sum_{j,k,l=1}^{I+1}\int_0^T\int_\Omega \psi\,\partial_j\partial_k\varphi_i^E(Z^\varepsilon)A_{jl}^\varepsilon(Z^\varepsilon)\nabla Z_l^\varepsilon\cdot\nabla Z_k^\varepsilon\,\mathrm{d}x\mathrm{d}t\\
&\qquad-\sum_{j,l=1}^{I+1}\int_0^T\int_\Omega \partial_j\varphi_i^E(Z^\varepsilon)A_{jl}^\varepsilon(Z^\varepsilon)\nabla Z_l^\varepsilon\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}t + \sum_{j=1}^{I}\int_0^T\int_\Omega \psi\,\partial_j\varphi_i^E(Z^\varepsilon)R_{\varepsilon,j}(Z^\varepsilon)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Observe that the LHS of eq. (6.1) is just the action of the distribution $\frac{\mathrm{d}}{\mathrm{d}t}\varphi_i^E(Z^\varepsilon)$ on $\psi \in C^\infty([0,T]\times\overline{\Omega})$. On the other hand, thanks to Lemma 2.3 (and in particular (2.12)), the compactness of $\operatorname{supp} D\varphi_i^E$ and the continuous embedding $H^p(\Omega)\hookrightarrow W^{1,\infty}(\Omega)$ for $p>\frac{d}{2}+1$, the RHS of (6.1) can be bounded above in modulus by

$$C(E)\left(1+\int_0^T\int_\Omega P_\varepsilon(Z^\varepsilon)\,\mathrm{d}x\mathrm{d}t\right)\|\psi\|_{C([0,T],H^p(\Omega))},$$

where the first term is controlled by the initial data (cf. (5.2)). Since $C^\infty([0,T]\times\overline{\Omega})$ is dense in $C^0([0,T],H^p(\Omega))$, this implies the asserted ε-uniform bound for $\big\|\frac{\mathrm{d}}{\mathrm{d}t}\varphi_i^E(Z^\varepsilon)\big\|_{\mathcal{M}([0,T],(H^p)^*)}$.

Having proven the existence of a subsequence $\varphi_i^E(Z^\varepsilon)$ converging in $L^2(0,T;L^2(\Omega))$ (for both $E\in\mathbb{N}$ and $T>0$ fixed), there exists a further subsequence converging pointwise a.e. to a measurable function v_i^E for all $i\in\{1,\dots,I\}$ and all $E\in\mathbb{N}$. This can be shown by applying a diagonal sequence argument.

We can now follow the reasoning in [34]. Thanks to the uniform bounds on $\sum_{j=1}^I c_j^\varepsilon\log c_j^\varepsilon$ in $L^\infty(0,\infty;L^1(\Omega))$ as well as (C5) and (C6), the functions $\varphi_i^E(Z^\varepsilon)\log\varphi_i^E(Z^\varepsilon)$ are uniformly bounded in $L^\infty(0,\infty;L^1(\Omega))$

w.r.t. $\varepsilon > 0$ and $E \in \mathbb{N}$. The pointwise a.e. convergence of $\varphi_i^E(Z^\varepsilon)$ to v_i^E and Fatou's Lemma now entail an E-uniform bound on $v_i^E \log v_i^E$ in $L^\infty(0,\infty;L^1(\Omega))$. We further proceed as in [34] to establish the pointwise a.e. convergence of $v^E = (v_i^E)_i$ to some measurable $c = (c_i)_i$ for $E \to \infty$ satisfying $c_i \log c_i \in L^\infty(0,\infty;L^1(\Omega))$. We start with the elementary observation

$$\sum_{j=1}^I \varphi_j^E(Z^\varepsilon) + u^\varepsilon < E \;\Rightarrow\; \sum_{j=1}^I Z_j^\varepsilon + u^\varepsilon < E \;\Rightarrow\; \varphi_j^E(Z^\varepsilon) = Z_j^\varepsilon = \varphi_j^{\widetilde{E}}(Z^\varepsilon)$$

for all $j \in \{1,\dots,I\}$ and $\widetilde{E} > E$, which follows from (C9) and (C6). Moreover, if $\sum_{j=1}^I v_j^E + u = \lim_{\varepsilon\to 0} \sum_{j=1}^I \varphi_j^E(Z^\varepsilon) + u^\varepsilon < E$, then $\sum_{j=1}^I \varphi_j^E(Z^\varepsilon) + u^\varepsilon < E$ for ε sufficiently small, and hence, $v_j^E = v_j^{\widetilde{E}}$ for all $\widetilde{E} > E$. By the uniform bound on $\sum_{j=1}^I v_j^E + u$ in $L^1([0,T]\times\Omega)$ for arbitrary but fixed $T > 0$ we know that $\sum_{j=1}^I v_j^E + u \geq E$ can hold true only on a set of points with vanishing measure in the limit $E \to \infty$. As a consequence, $v^E = (v_i^E)_i$ converges a.e. in Ω_T to a measurable function $c = (c_i)_i$ satisfying, thanks to Fatou's lemma, $c_i \log c_i \in L^\infty(0,\infty;L^1(\Omega))$.

Next, we prove that a subsequence c_i^ε pointwise a.e. converges to c_i for $\varepsilon \to 0$. The uniform bound on $\sum_{j=1}^{I+1} Z_j^\varepsilon$ in $L^1([0,T]\times\Omega)$ guarantees that the measure of the subset of Ω_T where $\sum_{j=1}^{I+1} Z_j^\varepsilon \geq E$ holds true tends to zero for $E \to \infty$, uniformly in ε. Therefore, $\varphi_i^E(Z^\varepsilon) \neq c_i^\varepsilon$ can be true only on a set of points with vanishing measure in the limit $E \to \infty$, uniformly in ε. For any $\vartheta > 0$ we now find

$$\begin{aligned}
&\mathcal{L}^{d+1}\bigg(\Big\{(t,x)\in\Omega_T \;:\; |c_i^\varepsilon(t,x) - c_i(t,x)| > \vartheta\Big\}\bigg)\\
&\quad\leq \mathcal{L}^{d+1}\bigg(\Big\{(t,x)\in\Omega_T \;:\; c_i^\varepsilon(t,x) \neq \varphi_i^E(Z^\varepsilon)(t,x)\Big\}\bigg)\\
&\qquad + \mathcal{L}^{d+1}\bigg(\Big\{(t,x)\in\Omega_T \;:\; |\varphi_i^E(Z^\varepsilon)(t,x) - v_i^E(t,x)| > \frac{\vartheta}{2}\Big\}\bigg)\\
&\qquad + \mathcal{L}^{d+1}\bigg(\Big\{(t,x)\in\Omega_T \;:\; |v_i^E(t,x) - c_i(t,x)| > \frac{\vartheta}{2}\Big\}\bigg).
\end{aligned}$$

The first term on the right-hand side tends to zero for $E \to \infty$ as discussed just before. The second term tends to zero for fixed E and $\varepsilon \to 0$ due to the definition of v_i^E, whereas the last term converges to zero for $E \to \infty$ by the definition of c_i. This ensures the convergence of c^ε to c in measure and, hence, convergence a.e. for another subsequence. Combined with the uniform boundedness of $c_i^\varepsilon \log c_i^\varepsilon$ in $L^\infty(0,T;L^1(\Omega))$ for every $T > 0$, this implies that c_i^ε converges strongly to c_i in $L^p(0,T;L^1(\Omega))$ for any $p \in [1,\infty)$ and $i \in \{1,\dots,I\}$. As a result of the strong convergence of c_i^ε to c_i in $L^1(0,T;L^1(\Omega))$, we also obtain the distributional convergence of $\sqrt{c_i^\varepsilon}$ to $\sqrt{c_i}$. In combination with the uniform $L^2(0,T;H^1(\Omega))$ bound on $\sqrt{c_i^\varepsilon}$, we deduce that a subsequence $\sqrt{c_i^\varepsilon}$ weakly converges to $\sqrt{c_i}$ in $L^2(0,T;H^1(\Omega))$. Employing the convergence $u^\varepsilon \to u$ in the distributional sense and the uniform bound (5.3) on $\partial_t u^\varepsilon$ in $L^s(0,T;W^{1,s'}(\Omega)^*)$, where $s = \frac{2d+2}{2d+1}$, we infer that $\partial_t u^\varepsilon \overset{*}{\rightharpoonup} \partial_t u$ in $L^s(0,T;W^{1,s'}(\Omega)^*)$ with the limit $\partial_t u$ again satisfying (5.3) by weak-$*$ lower semi-continuity. The convergence results derived above are sufficient to pass to the limit $\varepsilon \to 0$ in the equation for u^ε to infer

$$\int_0^T \int_\Omega a(c,u)\nabla u \cdot \nabla\phi \,\mathrm{d}x\mathrm{d}t + \int_0^T \big\langle \partial_t u, \phi\big\rangle \,\mathrm{d}t = 0, \tag{6.2}$$

where the distributional convergence of $a(c^\varepsilon,u^\varepsilon)\nabla u^\varepsilon$ to $a(c,u)\nabla u$ can be obtained as before using Lemma 6.6. Using lower semi-continuity arguments and the above convergence results, we infer

from (5.2) in particular the bounds

$$\|\nabla\sqrt{c}\|_{L^2(\Omega_T)} + \Big\|\sqrt{a}\nabla u\Big\|_{L^2(\Omega_T)} + \Big\|\pi_1^{\frac{1}{2}}(\hat\sigma'' + \sum_i c_i\tfrac{w_i''}{w_i})\nabla u\Big\|_{L^2(\Omega_T)} \le C(\mathrm{data}). \tag{6.3}$$

It remains to prove the strong convergence

$$\nabla u^\varepsilon \to \nabla u \quad \text{in } L^2(\Omega_T). \tag{6.4}$$

The key ingredient are the following L^2-energy identities, valid for a.e. $T>0$,

$$\frac12\int_\Omega u^\varepsilon(T,\cdot)^2\,\mathrm{d}x\mathrm{d}t - \frac12\int_\Omega u_0^2\,\mathrm{d}x + \int_0^T\int_\Omega a(c^\varepsilon,u^\varepsilon)|\nabla u^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t = 0, \tag{6.5}$$

$$\frac12\int_\Omega u(T,\cdot)^2\,\mathrm{d}x - \frac12\int_\Omega u_0^2\,\mathrm{d}x + \int_0^T\int_\Omega a(c,u)|\nabla u|^2\,\mathrm{d}x\mathrm{d}t = 0. \tag{6.6}$$

Let us first assume the validity of these equations and show how they entail (6.4).

Since for a.e. $T>0$ the first term on the LHS of (6.5) converges to the corresponding term in (6.6), the above identities imply that

$$\int_0^T\int_\Omega a(c^\varepsilon,u^\varepsilon)|\nabla u^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t \to \int_0^T\int_\Omega a(c,u)|\nabla u|^2\,\mathrm{d}x\mathrm{d}t.$$

Thanks to the weak convergence $\sqrt{a(c^\varepsilon,u^\varepsilon)}\nabla u^\varepsilon \rightharpoonup \sqrt{a(c,u)}\nabla u$ in $L^2(\Omega_T)$, we thus deduce

$$\begin{aligned}\int_0^T\int_\Omega\Big|\sqrt{a(c^\varepsilon,u^\varepsilon)}\nabla u^\varepsilon - \sqrt{a(c,u)}\nabla u\Big|^2\,\mathrm{d}x\mathrm{d}t &= \int_0^T\int_\Omega a(c^\varepsilon,u^\varepsilon)|\nabla u^\varepsilon|^2\,\mathrm{d}x\mathrm{d}t\\ &\quad - \int_0^T\int_\Omega\Big(2\sqrt{a(c^\varepsilon,u^\varepsilon)}\sqrt{a(c,u)}\nabla u^\varepsilon\cdot\nabla u - \Big|\sqrt{a(c,u)}\nabla u\Big|^2\Big)\,\mathrm{d}x\mathrm{d}t\\ &\longrightarrow 0 \quad \text{as } \varepsilon\to 0.\end{aligned}$$

Since $\sqrt{a}\gtrsim 1$ and $(c^\varepsilon,u^\varepsilon)\to(c,u)$ a.e., this easily implies (6.4).

Proof of identities (6.5) and (6.6): we confine ourselves to the proof of (6.6), since (6.5) can be shown along the same lines. Recall that $s=\frac{2d+2}{2d+1}$ along with the conjugate exponent $s'=2d+2$. The function u is then known to satisfy

$$\int_0^T\langle\tfrac{\mathrm{d}u}{\mathrm{d}t},\phi\rangle_{X^*\times X}\,\mathrm{d}t + \int_0^T\int_\Omega a(c,u)\nabla u\cdot\nabla\phi\,\mathrm{d}x\mathrm{d}t = 0 \tag{6.7}$$

for all $\phi\in L^{s'}(0,T;X)$ where $X:=W^{1,s'}(\Omega)$.

In order to prove identity (6.6), we use again the radially symmetric spatial mollifier $\widetilde\rho_\eta$ from (5.5) and let $\rho_\eta=\widetilde\rho_\eta*\widetilde\rho_\eta$. We further take $\zeta\in C_c^\infty([0,\infty))$ such that $\zeta(z)=\frac12 z^2$ if $z\in[0,1]$ and $\zeta(z)=0$ if $z\ge 2$, and then define $\zeta_E(z)=E^2\zeta(\frac{z}{E})$ for $E\ge 1$. Observe that $\zeta_E''(z)=\zeta''(\frac{z}{E})$ and, hence, $\sup_{E,z}|\zeta_E''(z)| = \|\zeta''\|_{C([0,\infty))}<\infty$. As in the proof of [34, Lemma 4], we make use of a partition of unity χ_k on $\overline\Omega$, i.e. we consider functions $\chi_k\in C^\infty(\overline\Omega)$ satisfying $\sum_{k=1}^K\chi_k=1$ on $\overline\Omega$ such that each $\chi=\chi_k$ is either compactly supported in Ω, or supported in some coordinate chart $U\subset\overline\Omega$ that is relatively open in $\overline\Omega$ and has the property that $U\cap\partial\Omega$ can be written as the graph of a suitable Lipschitz function τ.

In the first situation, we have $\operatorname{supp}\chi\subset\Omega^{4\overline\eta}$ for $\overline\eta>0$ sufficiently small. We then choose the test function

$$\phi_\eta^E := \rho_\eta*\big(\zeta_E'(\rho_\eta*u)\chi\big)\in L^\infty(0,T;W^{1,\infty}(\Omega))\subset L^{s'}(0,T;X)$$

in equation (6.7) with $\eta \in (0, \overline{\eta})$. Since χ and ϕ_η^E are compactly supported in Ω, we have

$$\begin{aligned}\int_0^T \left\langle \tfrac{\mathrm{d}}{\mathrm{d}t} u, \phi_\eta^E \right\rangle_{X^* \times X} \mathrm{d}t &= \int_0^T \int_\Omega \tfrac{\mathrm{d}}{\mathrm{d}t} (\rho_\eta * u) \zeta_E'(\rho_\eta * u) \chi \, \mathrm{d}x\mathrm{d}t \\ &= \int_\Omega \zeta_E(\rho_\eta * u(T, \cdot)) \chi \, \mathrm{d}x - \int_\Omega \zeta_E(\rho_\eta * u_0) \chi \, \mathrm{d}x.\end{aligned} \tag{6.8}$$

Notice that the equation for u (i.e. (6.7)) ensures sufficient (time) regularity of $\rho_\eta * u$ to justify the above lines. Hence, by dominated convergence,

$$\int_0^T \left\langle \tfrac{\mathrm{d}}{\mathrm{d}t} u, \phi_\eta^E \right\rangle_{X^* \times X} \mathrm{d}t \to \int_\Omega \zeta_E(u(T, \cdot)) \chi \, \mathrm{d}x - \int_\Omega \zeta_E(u_0) \chi \, \mathrm{d}x \tag{6.9}$$

as $\eta \to 0$.

On the other hand, we assert that

$$\int_0^T \int_\Omega a(c, u) \nabla u \cdot \nabla \phi_\eta^E \, \mathrm{d}x\mathrm{d}t \to \int_0^T \int_\Omega a(c, u) \nabla u \cdot \nabla \big(\zeta_E'(u) \chi \big) \, \mathrm{d}x\mathrm{d}t \tag{6.10}$$

as $\eta \to 0$. Reformulating the LHS as

$$\int_0^T \int_\Omega a(c, u) \nabla u \cdot \nabla \phi_\eta^E \, \mathrm{d}x\mathrm{d}t = \int_0^T \int_{\Omega^{3\eta}} J_\eta(t, x) \cdot K_\eta(t, x) \, \mathrm{d}x\mathrm{d}t$$

with $J_\eta(t, x) := \widetilde{\rho}_\eta * \nabla \big(\zeta_E'(\rho_\eta * u) \chi \big)$ and $K_\eta(t, x) := \widetilde{\rho}_\eta * (a(c, u) \nabla u)$ for all $(t, x) \in (0, T) \times \Omega^{3\eta}$, we can verify (6.10) by using the same arguments as in the proof of Lemma 5.5: in the current situation, Step 1 reads

$$\begin{aligned}|K_\eta(t, x)| \le{}& C \eta^{1-d} \int_{B(x, 3\eta)} \Big(|\nabla u|^2 + P(Z) \Big) \, \mathrm{d}y \\ &+ C \eta^{-\frac{d}{2}} \Big(\int_{B(x, 3\eta)} \Big(|\nabla u|^2 + P(Z) \Big) \, \mathrm{d}y \Big)^{\frac{1}{2}} \inf_{\widetilde{x} \in B(x, \eta)} |(\rho_\eta * (1 + u))(\widetilde{x})|.\end{aligned}$$

Step 2 is slightly more complex. Here, we estimate

$$\begin{aligned}|J_\eta(t, x)| &\le \big| \widetilde{\rho}_\eta * \big(\chi \nabla \zeta_E'(\rho_\eta * u) \big) \big| + \big| \widetilde{\rho}_\eta * \big(\zeta_E'(\rho_\eta * u) \nabla \chi \big) \big| \\ &\le C \min \Big\{ \eta^{-1}, 1 + \eta^{-\frac{d}{2}} \Big(\int_{B(x, 3\eta)} |\nabla u|^2 \, \mathrm{d}y \Big)^{\frac{1}{2}} \Big\},\end{aligned}$$

where the first part of the estimate follows as in [13], while the second part is trivially bounded in terms of a constant that can be controlled by the last line. The upper bound from Step 3 is then given by

$$|J_\eta(t, x) \cdot K_\eta(t, x)| \le C E \eta^{-d} \int_{B(x, 3\eta)} \big(1 + |\nabla u|^2 + P(Z) \big) \, \mathrm{d}y$$

for all $(t, x) \in (0, T) \times \Omega^{3\eta}$. Defining $J_0(t, x) := \nabla \big(\zeta_E'(u) \chi \big)$ and $K_0(t, x) := a(c, u) \nabla u$, Step 4 reduces to $|J_0(t, x) K_0(t, x)| \le C(1 + |\nabla u|) a(c, u) |\nabla u| \in L^1((0, T) \times \Omega)$ for all $(t, x) \in (0, T) \times \Omega^{3\eta}$ and Step 5 can be reused in a one-to-one fashion to establish (6.10).

We are left to deal with the case that $\operatorname{supp} \chi \subset U$ for some coordinate chart $U \subset \overline{\Omega}$ as above that satisfies $U \cap \partial\Omega \neq \emptyset$. Employing the notation of [34, Proof of Lemma 4] and using the fact that Ω is a Lipschitz domain, there exist a set $V \subset \mathbb{R}_0^+ \times \mathbb{R}^{d-1}$ (relatively open in $\mathbb{R}_0^+ \times \mathbb{R}^{d-1}$) and a bi-Lipschitz homeomorphism $\theta : V \to U$ with the property that after appropriate rotations and translations of Ω one has

$$\theta(y) = (y_1 + \tau(y_2, \dots, y_d), y_2, \dots, y_d) \tag{6.11}$$

for some suitable Lipschitz function $\tau : \mathbb{R}^{d-1} \to \mathbb{R}$ and all $y \in V$. Note that $\det D\tau = 1$ a.e. in Ω. We then let $\Theta : [0,T] \times V \to [0,T] \times U$, $\Theta(t,y) := (t, \theta(y))$, and introduce the transformed variables

$$\hat{c} := c \circ \Theta, \quad \hat{u} := u \circ \Theta, \quad \hat{\chi} := \chi \circ \theta.$$

We now extend these transformed variables by mirroring to $[0,T] \times V_a$ and V_a, respectively, where $V_a := V \cup V_m$ and $V_m := \{y \in \mathbb{R}^d \mid (-y_1, y') \in V\}$ using the short hand notation $y' := (y_2, \dots, y_d)$. More precisely,

$$\hat{c}(t, y_1, y') := \hat{c}(t, -y_1, y'), \quad \hat{u}(t, y_1, y') := \hat{u}(t, -y_1, y'), \quad \hat{\chi}(y_1, y') := \hat{\chi}(-y_1, y')$$

for all $(y_1, y') \in V_m$. Furthermore, we extend θ to V_a by imposing the same formula as in (6.11) but now valid for all $(y_1, y') \in V_a$. Observe that for $\hat{\phi} := \phi \circ \Theta$

$$\int_0^T \int_U a(c,u) \nabla u \cdot \nabla \phi \, \mathrm{d}x \mathrm{d}t = \int_0^T \int_V a(\hat{c}, \hat{u}) \big((D\theta)^{-T} \nabla \hat{u} \big) \cdot \big((D\theta)^{-T} \nabla \hat{\phi} \big) \, \mathrm{d}y \mathrm{d}t,$$

and a similar expression holds for the integral involving the time derivative. We will now show that an identity analogous to (6.7) holds true for the extended quantities on V_a in order to essentially reduce the problem to the first situation where $\operatorname{supp} \chi \subset\subset \Omega$. While on the set V one has

$$D\theta = \begin{pmatrix} 1 & \partial_2 \tau & \dots & \partial_d \tau \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \qquad (D\theta)^{-T} = \begin{pmatrix} 1 & & & \\ -\partial_2 \tau & 1 & & \\ \vdots & & \ddots & \\ -\partial_d \tau & & & 1 \end{pmatrix},$$

the corresponding matrices on V_m read

$$D\theta = \begin{pmatrix} -1 & \partial_2 \tau & \dots & \partial_d \tau \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \qquad (D\theta)^{-T} = \begin{pmatrix} -1 & & & \\ \partial_2 \tau & 1 & & \\ \vdots & & \ddots & \\ \partial_d \tau & & & 1 \end{pmatrix}.$$

As a consequence, we find

$$\int_0^T \int_U a(c,u) \nabla u \cdot \nabla \phi \, \mathrm{d}x \mathrm{d}t = \frac{1}{2} \int_0^T \int_{V_a} a(\hat{c}, \hat{u}) \big((D\theta)^{-T} \nabla \hat{u} \big) \cdot \big((D\theta)^{-T} \nabla \hat{\phi} \big) \, \mathrm{d}y \mathrm{d}t,$$

by using the symmetry w.r.t. y_1 of the terms inside the gradients and the structure of the first column of $(D\theta)^{-T}$ on V and V_m, respectively. Using similar (but simpler) symmetry properties for the integral involving the time derivative, this allows us to infer

$$\int_0^T \langle \tfrac{\mathrm{d}}{\mathrm{d}t} \hat{u}, \hat{\phi} \rangle \, \mathrm{d}t = \int_0^T \int_{V_a} a(\hat{c}, \hat{u}) \big((D\theta)^{-T} \nabla \hat{u} \big) \cdot \big((D\theta)^{-T} \nabla \hat{\phi} \big) \, \mathrm{d}y \mathrm{d}t,$$

where $\langle \frac{\mathrm{d}}{\mathrm{d}t} \hat{u}, \hat{\phi} \rangle := \langle \frac{\mathrm{d}}{\mathrm{d}t} u, \hat{\phi}(t, (\theta_{|V})^{-1}) \rangle + \langle \frac{\mathrm{d}}{\mathrm{d}t} u, \hat{\phi}(t, O \circ (\theta_{|V})^{-1}) \rangle$, where $O(y) = (-y_1, y')$. We now choose the (symmetric) test function

$$\hat{\phi} := \hat{\phi}_\eta^E := \rho_\eta * (\zeta_E'(\rho_\eta * \hat{u}) \hat{\chi}),$$

where ρ_η, $0 < \eta \ll 1$, and ζ^E, $E \gg 1$, are as before. The symmetry of $\hat{\phi}_\eta^E$ follows from the radial symmetry of $\widetilde{\rho}_\eta$. Moreover, we have $\operatorname{supp} \hat{\chi} \subset\subset V_a$. We can therefore argue essentially as in the first situation (derivation of (6.9) and (6.10)), the main difference being the appearance of the pull-backs of the gradients in the integral on the RHS, which are, however, harmless thanks to the boundedness of $(D\theta)^{-T}$. Concerning the term involving $\frac{\mathrm{d}}{\mathrm{d}t} \hat{u}$, we omit here a standard approximation procedure to

justify the chain rule argument for the time derivative analogous to (6.8) (one can use e.g. the methods in [52, Chapter 7]). In the limit $\eta \to 0$, we then obtain

$$\int_{V_a} \zeta_E(\hat{u}(T,\cdot))\hat{\chi}\,\mathrm{d}y - \int_{V_a} \zeta_E(\hat{u}_0)\hat{\chi}\,\mathrm{d}y \\ = \int_0^T \int_{V_a} a(\hat{c},\hat{u})\big((D\theta)^{-T}\nabla\hat{u}\big)\cdot(D\theta)^{-T}\nabla\big(\zeta_E'(\hat{u})\hat{\chi}\big)\,\mathrm{d}y\mathrm{d}t$$

or equivalently

$$\int_\Omega \zeta_E(u(T,\cdot))\chi\,\mathrm{d}x - \int_\Omega \zeta_E(u_0)\chi\,\mathrm{d}x = \int_0^T \int_U a(c,u)\nabla u\cdot\nabla\big(\zeta_E'(u)\chi\big)\,\mathrm{d}x\mathrm{d}t.$$

In combination, we infer that

$$\int_\Omega \zeta_E(u(T,\cdot))\chi_k\,\mathrm{d}x - \int_\Omega \zeta_E(u_0)\chi_k\,\mathrm{d}x + \int_0^T \int_\Omega a(c,u)\nabla u\cdot\nabla\big(\zeta_E'(u)\chi_k\big)\,\mathrm{d}x\mathrm{d}t = 0$$

for a.e. $T>0$ and all $k=1,\dots,K$, and thus, upon summation over k,

$$\int_\Omega \zeta_E(u(T,\cdot))\,\mathrm{d}x - \int_\Omega \zeta_E(u_0)\,\mathrm{d}x + \int_0^T \int_\Omega a(c,u)|\nabla u|^2\zeta_E''(u)\,\mathrm{d}x\mathrm{d}t = 0.$$

Lebesgue's dominated convergence theorem finally allows us to pass to the limit $E\to\infty$ for a.e. $T>0$ to establish (6.6). □

6.2. **Preliminary PDE for $\varphi_i^E(Z)$.** As it is generally not possible to directly obtain the desired equation for $\xi(Z)$ by passing to the limit $\varepsilon\to 0$ in the equation for $\xi(Z^\varepsilon)$, where $\xi\in C^\infty([0,\infty)^{I+1})$ has compactly supported derivatives, we first pass to the limit $\varepsilon\to 0$ in (6.1). Thanks in particular to the strong convergence of ∇u^ε, this leads to an equation for the truncations $\varphi_i^E(Z)$ involving signed Radon measures μ_i^E that vanish in the limit $E\to\infty$. The Radon measures μ_i^E arise due to the terms in eq. (6.1) involving squared gradients of the form $\nabla c_l\cdot\nabla c_k$ and the lack of strong convergence in L^2 of $\nabla\sqrt{c_i}$ for all i.

Lemma 6.2. *Let Z be the limiting function obtained in Lemma 6.1. Then, for all $E,T>0$ and $\psi\in C^\infty([0,T]\times\overline{\Omega})$ with $\psi(T,\cdot)=0$ the function $\varphi_i^E(Z)$ satisfies the identity*

$$-\int_\Omega \varphi_i^E(Z^0)\psi(0,\cdot)\,\mathrm{d}x - \int_0^T\int_\Omega \varphi_i^E(Z)\tfrac{\mathrm{d}}{\mathrm{d}t}\psi\,\mathrm{d}x\mathrm{d}t \tag{6.12}$$
$$= -\int_0^T\int_\Omega \psi\,\mathrm{d}\mu_i^E(t,x) \\ - \sum_{j,k=1}^{I+1}\int_0^T\int_\Omega \psi\,\partial_j\partial_k\varphi_i^E(Z)A_{j,I+1}(Z)\nabla u\cdot\nabla Z_k\,\mathrm{d}x\mathrm{d}t \\ - \sum_{j,l=1}^{I+1}\int_0^T\int_\Omega \partial_j\varphi_i^E(Z)A_{jl}(Z)\nabla Z_l\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}t \\ + \sum_{j=1}^{I}\int_0^T\int_\Omega \psi\,\partial_j\varphi_i^E(Z)R_j(Z)\,\mathrm{d}x\mathrm{d}t,$$

where μ_i^E is a sequence of signed Radon measures with the property that

$$\lim_{E\to\infty}|\mu_i^E|([0,T)\times\overline{\Omega}) = 0 \tag{6.13}$$

for all $T > 0$ and every $i \in \{1, \dots, I\}$. Moreover, under the same hypotheses we have

$$-\int_\Omega Z^0_{I+1}\psi(0,\cdot)\,\mathrm{d}x - \int_0^T\int_\Omega Z_{I+1}\tfrac{\mathrm{d}}{\mathrm{d}t}\psi\,\mathrm{d}x\mathrm{d}t = -\int_0^T\int_\Omega A_{I+1,I+1}(Z)\nabla Z_{I+1}\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}t. \quad (6.14)$$

Proof. We stress that (6.14) is a reformulation of (6.2), which serves as the analogue of (6.12) for the internal energy but without the need for the functions φ_i^E.

For proving (6.12), we aim to pass to the limit $\varepsilon \to 0$ in (6.1) (or equivalently in (5.4) with $\varrho = \varepsilon$) using the convergence properties of Z^ε to Z established in Lemma 6.1. The terms on the LHS converge to the LHS of (6.12) thanks to dominated convergence. This argument also applies to the last term on the RHS involving the reactions. Next, we make the crucial observation that, thanks to the strong convergence of $\nabla u^\varepsilon \to \nabla u$ in $L^2(\Omega_T)$, the first terms on the RHS of (6.1) with $l = I+1$ (involving the thermodiffusive cross terms) converge to the second term on the RHS of (6.12). Here, one also uses the weak convergence $\nabla\sqrt{c_j^\varepsilon} \rightharpoonup \nabla\sqrt{c_j}$ in $L^2(0,T;L^2(\Omega))$, Lemma 2.3, the uniform bounds (5.2), Lemma 6.5 and the fact that $D\varphi_i^E$ is compactly supported. A similar (but simpler) reasoning allows to pass to the limit in the penultimate line. Regarding the remaining terms in the first line of the RHS of (6.1), we need to take more care since they involve products $\nabla c_j^\varepsilon \cdot \nabla c_k^\varepsilon$ of merely weakly converging functions. To treat these terms we define the signed Radon measures

$$\mu_i^{\varepsilon,E} := \sum_{k=1}^{I+1}\sum_{j=1}^{I}\partial_j\partial_k\varphi_i^E(Z^\varepsilon)A_{jj}^\varepsilon(Z^\varepsilon)\nabla c_j^\varepsilon\cdot\nabla Z_k\,\mathrm{d}x\mathrm{d}t,$$

which are easily seen to be uniformly bounded in $\varepsilon \in (0,1]$ as long as $E > 0$ is kept fixed. A key property is the fact that the measures $\mu_i^{\varepsilon,E}$ can also be controlled uniformly in E, which will allow us to infer (6.13). To see this, we note that thanks to the properties (C2) and (C3) of φ_i^E, the pointwise estimate (2.13) of $A_{jj}^\varepsilon(Z^\varepsilon)\nabla c_j^\varepsilon$, the fact that $P_\varepsilon^{\frac{1}{2}}(Z^\varepsilon)$ controls $\varepsilon^{\frac{1}{2}}|\nabla c_k| + |\nabla\sqrt{c_k}|$ and the ε-uniform control of ∇u^ε in $L^2(\Omega_T)$ (due to (5.2) and (H1)), we have

$$|\mu_i^{\varepsilon,E}|([0,T)\times\overline{\Omega}) \lesssim \int_0^T\int_\Omega \big(P_\varepsilon(Z^\varepsilon) + |\nabla u^\varepsilon|^2\big)\,\mathrm{d}x\mathrm{d}t \le C(\mathrm{data}).$$

By the weak-$*$ compactness of Radon measures, there exists a subsequence $\mu_i^{\varepsilon,E}$ that converges weak-$*$ in measure to some Radon measure μ_i^E as $\varepsilon \to 0$.

For proving that μ_i^E vanishes in the limit $E \to \infty$, we introduce the non-negative measures

$$\nu^{\varepsilon,L} := \chi_{\{|(c^\varepsilon,u^\varepsilon)|\in[L-1,L)\}}\big(P_\varepsilon(Z^\varepsilon) + |\nabla u^\varepsilon|^2\big)\mathrm{d}x\mathrm{d}t, \quad L \in \mathbb{N},$$

which satisfy the bound $\sum_{L=1}^\infty \nu^{\varepsilon,L}([0,T)\times\overline{\Omega}) \lesssim C(\mathrm{data})$.

As before, we employ (C2) to estimate

$$\begin{aligned}|\mu_i^{\varepsilon,E}|([0,T)\times\overline{\Omega}) &\lesssim \sum_{L=1}^\infty\int_0^T\int_\Omega \chi_{\{|(c^\varepsilon,u^\varepsilon)|\in[L-1,L)\}}|Z^\varepsilon||D^2\varphi_i^E(Z^\varepsilon)|\big(P_\varepsilon(Z^\varepsilon)+|\nabla u^\varepsilon|^2\big)\,\mathrm{d}x\mathrm{d}t\\ &\lesssim \sum_{L=1}^\infty \nu^{\varepsilon,L}([0,T)\times\overline{\Omega})\sup_{|\tilde Z|\in[L-1,L)}|\tilde Z||D^2\varphi_i^E(\tilde Z)|.\end{aligned}$$

Observe that since $D^2\varphi_i^E$ is compactly supported, for fixed $E \in \mathbb{N}$ only finitely many terms on the RHS are non-zero. This allows us to estimate

$$\begin{aligned}|\mu_i^E|([0,T)\times\overline{\Omega}) &\le \liminf_{\varepsilon\to 0} |\mu_i^{\varepsilon,E}|([0,T)\times\overline{\Omega}) \\ &\lesssim \sum_{L=1}^{\infty} \liminf_{\varepsilon\to 0} \nu^{\varepsilon,L}([0,T)\times\overline{\Omega}) \sup_{|\tilde{Z}|\in[L-1,L)} |\tilde{Z}||D^2\varphi_i^E(\tilde{Z})|,\end{aligned}$$

where the first inequality is due to the weak-$*$ lower semi-continuity of the total variation of Radon measures on an open set.

Next, applying Fatou's Lemma (for the counting measure on $\mathbb{N}$), we find

$$\sum_{L=1}^{\infty} \liminf_{\varepsilon\to 0} \nu^{\varepsilon,L}([0,T)\times\overline{\Omega}) \le \liminf_{\varepsilon\to 0} \sum_{L=1}^{\infty} \nu^{\varepsilon,L}([0,T)\times\overline{\Omega}) \le C(\text{data}).$$

By using (C2), (C7), the previous estimate, and the dominated convergence theorem (for the counting measure on $\mathbb{N}$), we are led to

$$\begin{aligned}\limsup_{E\to\infty} |\mu_i^E|([0,T)\times\overline{\Omega}) &\lesssim \sum_{L=1}^{\infty} \liminf_{\varepsilon\to 0} \nu^{\varepsilon,L}([0,T)\times\overline{\Omega}) \lim_{E\to\infty} \sup_{|\tilde{Z}|\in[L-1,L)} |\tilde{Z}||D^2\varphi_i^E(\tilde{Z})| \\ &\lesssim \sum_{L=1}^{\infty} \liminf_{\varepsilon\to 0} \nu^{\varepsilon,L}([0,T)\times\overline{\Omega}) \cdot 0 \;=\; 0.\end{aligned}$$

This concludes the proof. □

6.3. **Proof of the existence of global renormalised solutions.** We are now in a position to deduce the equation for $\xi(Z)$ as stated in Definition 1.6. In fact, we derive an approximate expression for the weak time derivative of $\xi(\varphi^E(Z), Z_{I+1})$ from which the equation for $\xi(Z)$ then emerges when passing to the limit $E \to \infty$.

Proof of Theorem 1.8. In order to prove that Z is a global renormalised solution of (1.7), let $T \in (0,\infty)$ be arbitrary. Keeping in mind that Z satisfies (6.12) and (6.14), the hypotheses of Lemma 6.3 are fulfilled if we define

$$\begin{aligned}v_i &:= \varphi_i^E(Z), \quad (v_0)_i := \varphi_i^E(Z^0), \quad v_{I+1} := Z_{I+1}, \quad (v_0)_{I+1} := Z_{I+1}^0, \\ \nu_i &:= -\mu_i^E, \quad \nu_{I+1} := 0, \\ w_i &:= -\sum_{j,k=1}^{I+1} \partial_j\partial_k\varphi_i^E(Z) A_{j,I+1}(Z)\nabla u\cdot\nabla Z_k + \sum_{j=1}^{I} \partial_j\varphi_i^E(Z) R_j(Z), \\ w_{I+1} &:= 0, \\ z_i &:= -\sum_{j,l=1}^{I+1} \partial_j\varphi_i^E(Z) A_{jl}(Z)\nabla Z_l, \quad z_{I+1} := -A_{I+1,I+1}(Z)\nabla Z_{I+1}.\end{aligned}$$

As a consequence, the function $\xi(\varphi^E(Z), Z_{I+1})$ satisfies for all $\psi \in C^\infty([0,T]\times\overline{\Omega})$ with $\psi(T,\cdot) = 0$ the estimate

$$\begin{aligned}
\Big| &- \int_0^T\int_\Omega \xi(\varphi^E(Z), Z_{I+1})\tfrac{\mathrm{d}}{\mathrm{d}t}\psi \,\mathrm{d}x\mathrm{d}t - \int_\Omega \xi(\varphi^E(Z^0), Z^0_{I+1})\psi(0,\cdot)\,\mathrm{d}x \\
&\quad + \sum_{i,k,j,l=1}^{I+1}\int_0^T\int_\Omega \psi\partial_i\partial_k\xi(\varphi^E(Z), Z_{I+1})\partial_j\varphi_i^E(Z)A_{jl}(Z)\nabla Z_l\cdot\nabla\varphi_k^E(Z)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \sum_{k,l=1}^{I+1}\int_0^T\int_\Omega \psi\partial_{I+1}\partial_k\xi(\varphi^E(Z), Z_{I+1})A_{I+1,I+1}(Z)\nabla Z_{I+1}\cdot\nabla\varphi_k^E(Z)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \sum_{i,j,l=1}^{I+1}\int_0^T\int_\Omega \partial_i\xi(\varphi^E(Z), Z_{I+1})\partial_j\varphi_i^E(Z)A_{jl}(Z)\nabla Z_l\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}t \\
&\quad + \sum_{l=1}^{I+1}\int_0^T\int_\Omega \partial_{I+1}\xi(\varphi^E(Z), Z_{I+1})A_{I+1,I+1}(Z)\nabla Z_{I+1}\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}t \\
&\quad + \sum_{i,j,k=1}^{I+1}\int_0^T\int_\Omega \psi\partial_i\xi(\varphi^E(Z), Z_{I+1})\partial_j\partial_k\varphi_i^E(Z)A_{j,I+1}(Z)\nabla u\cdot\nabla Z_k\,\mathrm{d}x\mathrm{d}t \\
&\quad - \sum_{i=1}^{I+1}\sum_{j=1}^{I}\int_0^T\int_\Omega \psi\partial_i\xi(\varphi^E(Z), Z_{I+1})\partial_j\varphi_i^E(Z)R_j(Z)\,\mathrm{d}x\mathrm{d}t\Big| \\
&\qquad\qquad \le C\|\psi\|_{L^\infty}\sup_{\tilde v\in\mathbb{R}^I}|D\xi(\tilde v)|\sum_{i=1}^{I+1}\|\mu_i^E\|_{\mathcal{M}([0,T)\times\overline{\Omega})}.
\end{aligned}$$

In view of (6.13), the RHS converges to zero as $E\to\infty$. To establish the convergence of the LHS, we employ the pointwise a.e. convergence of $\varphi_i^E(Z)$ to Z_i, the boundedness of $D\varphi_i^E$ (cf. (C5)), the compact support of $D\xi$, the regularity $\nabla\sqrt{c_j}, \nabla u \in L^2((0,T)\times\Omega)$, Lemma 2.3 and the uniform bounds (5.2). As in [34] and [13], we utilize the following auxiliary result: there exists a constant $E_0 > 0$ such that for all $E > E_0$ the relation $\sum_{i=1}^{I+1} Z_i \ge E_0$ implies $D\xi(\varphi^E(Z), Z_{I+1}) = D\xi(Z) = 0$ and $D^2\xi(\varphi^E(Z), Z_{I+1}) = D^2\xi(Z) = 0$. This result ensures that all terms involving derivatives of ξ are zero if $\max_i Z_i$ is larger than E_0.

This auxiliary result is easily proven by choosing $E_0 > 0$ such that $\operatorname{supp} D\xi \subset B_{E_0/\sqrt{I+1}}(0)$. For $E > E_0$ and $\sum_{i=1}^{I+1} Z_i \ge E_0$, we then get $\sum_{i=1}^{I}\varphi_i^E(Z) + Z_{I+1} \ge E_0$ from (C9). As a consequence, $Z, (\varphi^E(Z), Z_{I+1}) \notin B_{E_0/\sqrt{I+1}}(0)$ and the claim follows.

As $T > 0$ was chosen arbitrarily, we have thus shown that $Z = (c,u)$ is a global renormalised solution. The weak formulation of the equation for u already appeared in (6.2), while the conservation of the energy results from the energy conservation of u^ε and the convergence $u^\varepsilon \to u$ in $L^2([0,T]\times\Omega)$. Finally, the bounds on the solution follow from Lemma 6.1 and (6.3). □

APPENDIX

The following lemma recalls the approximate chain rule established in [34].

Lemma 6.3 ([34, L. 4]). *Let $\Omega\subset\mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Let $T > 0$, $v \in L^2(0,T;H^1(\Omega)^{I+1})$, and $v_0 \in L^1(\Omega)^{I+1}$. Moreover, let $\nu_i \in \mathcal{M}([0,T)\times\overline{\Omega})$ be a Radon measure, $w_i \in L^1(0,T;L^1(\Omega))$, and $z_i \in L^2(0,T;L^2(\Omega)^d)$ for $i\in\{1,\dots,I+1\}$.*

Suppose that v solves for any $\psi \in C^\infty([0,T]\times\overline{\Omega})$ with $\psi(T,\cdot)=0$ the identity

$$-\int_0^T\int_\Omega v_i(t,\cdot)\frac{d}{dt}\psi(t,\cdot)\,\mathrm{d}x\mathrm{d}t-\int_\Omega (v_0)_i\psi(0,\cdot)\,\mathrm{d}x = \int_{\overline{\Omega}\times[0,T)}\psi\,\mathrm{d}\nu_i+\int_0^T\int_\Omega \psi\, w_i\,\mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega z_i\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}t. \quad (6.15)$$

Let $\xi:\mathbb{R}^{I+1}\to\mathbb{R}$ be a smooth function with compactly supported first derivatives. Then, there exists a constant $C(\Omega)>0$ such that for all $\psi\in C^\infty([0,T]\times\overline{\Omega})$ with $\psi(T,\cdot)=0$,

$$\Big|-\int_0^T\int_\Omega \xi(v)\frac{d}{dt}\psi\,\mathrm{d}x\mathrm{d}t-\int_\Omega\xi(v_0(\cdot))\psi(0,\cdot)\,\mathrm{d}x-\sum_{i=1}^{I+1}\sum_{j=1}^{I+1}\int_0^T\int_\Omega\psi\partial_i\partial_j\xi(v)z_i\cdot\nabla v_j\,\mathrm{d}x\mathrm{d}t$$
$$-\sum_{i=1}^{I+1}\int_0^T\int_\Omega\partial_i\xi(v)z_i\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}t-\sum_{i=1}^{I+1}\int_0^T\int_\Omega\psi\partial_i\xi(v)w_i\,\mathrm{d}x\mathrm{d}t\Big| \le C(\Omega)\|\psi\|_{L^\infty}\sup_{\tilde v\in\mathbb{R}^{I+1}}|D\xi(\tilde v)|\sum_{i=1}^{I+1}\|\nu_i\|_{\mathcal{M}([0,T)\times\overline{\Omega})}.$$

For the reader's convenience, we further recall some classical results frequently used in this paper.

Lemma 6.4 (Gagliardo–Nirenberg inequality)**.** *Let $1\le p<q$, define*

$$\theta=\frac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{p}+\frac{1}{d}-\frac{1}{2}}, \quad (6.16)$$

and suppose that $\theta\in(0,1)$. There exists $C_1\in(0,\infty)$ such that for all $z\in H^1(\Omega)$

$$\|z\|_{L^q(\Omega)}\le C_1\|\nabla z\|^\theta_{L^2(\Omega)}\|z\|^{1-\theta}_{L^p(\Omega)}+C_2\|z\|_{L^1(\Omega)}.$$

The following lemma is a corollary of the Dunford–Pettis theorem if $p=1$. In fact, we only need the simpler version for $p\in(1,\infty)$, which follows from standard results on weak convergence.

Lemma 6.5. *Let $p\in[1,\infty)$ and suppose that $f_j,f:\Omega\to\mathbb{R}$ are measurable functions and $g_j,g\in L^p(\Omega)$. If $f_j\to f$ a.e. in Ω, $\sup_j\|f_j\|_{L^\infty(\Omega)}<\infty$ and $g_j\rightharpoonup g$ in $L^p(\Omega)$, then*

$$f_jg_j\rightharpoonup fg \quad \text{in } L^p(\Omega).$$

The next observation allows us to deal with coefficients in our system that are singular near $u=0$, which may arise in the case of model **(H)**.

Lemma 6.6. *Suppose that $a_j,V_j,j\in\mathbb{N}$, are measurable functions on a bounded domain Ω satisfying*

- *$V_j\rightharpoonup V$ in $L^1(\Omega)$*
- *$a_j\to a$ a.e. in Ω (and $|a_j|\gtrsim 1$ for all j)*
- *$\sup_j\|a_jV_j\|_{L^{1+\epsilon}}<\infty$ for some $\epsilon>0$.*

Then $a_jV_j\rightharpoonup aV$ in $L^{1+\epsilon}(\Omega)$.

We remark that the hypothesis $|a_j|\gtrsim 1$ can be removed by writing $a_jV_j=a_j(\chi_{\tilde\Omega_j}V_j)+(1-\chi_{\tilde\Omega_j})a_jV_j$, where $\tilde\Omega_j=\{a_j\ge 1\}$ a.e.

Proof. By weak compactness, there exists $X\in L^{1+\epsilon}(\Omega)$ such that, along a subsequence, $a_jV_j\rightharpoonup X$ in $L^{1+\epsilon}(\Omega)$. Assuming $|a_j|\gtrsim 1$ a.e., we can invoke Lemma 6.5 to deduce that $V_j\rightharpoonup\frac{1}{a}X$ in $L^{1+\epsilon}(\Omega)$ and hence $X=aV$. The conclusion now follows from the observation that this argument applies to any subsequence. □

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