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Quasistatic damage evolution with spatial BV-regularization

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Abstract

An existence result for energetic solutions of rate-independent damage processes is established. We consider a body consisting of a physically linearly elastic material undergoing infinitesimally small deformations and partial damage. In [TM10] an existence result in the small strain setting was obtained under the assumption that the damage variable z satisfies $z \in W^{1,r}(\Omega)$ with $r \in (1,\infty)$ for $\Omega \subset \mathbb{R}^d$. We now cover the case $r = 1$. The lack of compactness in $W^{1,1}(\Omega)$ requires to do the analysis in $BV(\Omega)$. This setting allows it to consider damage variables with values in $\{0,1\}$. We show that such a brittle damage model is obtained as the Γ-limit of functionals of Modica-Mortola type.

1 Introduction

Damage means the creation and growth of cracks and voids on the micro-level of a solid material. Based on the method of Continuum Damage Mechanics this process is modeled by an internal variable, the damage variable $z : [0,T] \times \Omega \to [0,1]$, which is incorporated to the constitutive law in order to reflect the changes of the elastic behavior due to damage. As in [MR06, TM10] $z(t,x) = 1$ stands for no damage and $z(t,x) = 0$ for maximal damage in the material point x of the body $\Omega \subset \mathbb{R}^d$ at time $t \in [0,T]$.

The damage process is treated within the so-called energetic formulation. This ansatz solely uses an energy functional $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R} \cup \{\infty\}$ and a dissipation potential $\mathcal{R} : \mathcal{Z} \to [0,\infty]$. Here, $\mathcal{Z}$ denotes the set of damage variables and together with the set of displacements $\mathcal{U}$ it defines the state space $\mathcal{Q} := \mathcal{U} \times \mathcal{Z}$, which here is a Banach space. The triple $(\mathcal{Q},\mathcal{E},\mathcal{R})$ is called a (rate-independent) system. The rate-independence of $(\mathcal{Q},\mathcal{E},\mathcal{R})$ is reflected by the positive-1-homogeneity of $\mathcal{R}$, i.e. $\mathcal{R}(0) = 0$ and $\mathcal{R}(\alpha v) = \alpha\mathcal{R}(v)$ for all $\alpha > 0$ and all $v \in \mathcal{Z}$. Moreover, the damage process is assumed to be unidirectional. With a constant $\varrho > 0$ this is modeled by $\mathcal{R}$ being of the following form:

$$\mathcal{R}(v) := \int_\Omega R(v(x))\,\mathrm{d}x\,, \quad \text{where } R(v) := \begin{cases} \varrho|v| & \text{if } v \in (-\infty,0], \\ +\infty & \text{if } v > 0. \end{cases} \tag{1}$$

With $v = \dot z$ as the partial time derivative of z, the dissipation potential accounts for the evolution of the damage process. Moreover, due to its positive-1-homogeneity the convex potential $\mathcal{R}$ generates a dissipation distance between all $z_1, z_2 \in \mathcal{Z}$, which is given by $\mathcal{R}(v)$ from (1) with $v = z_2 - z_1$, i.e. $\mathcal{R}(z_2 - z_1)$ for all $z_1, z_2 \in \mathcal{Z}$; see e.g. [Mie05] for more details. This fact is used within the energetic approach to define a concept of solution that does not involve the partial time derivative of z. These are the so-called energetic solutions:

Definition 1.1 (Energetic solution) *A function $q = (u,z) : [0,T] \to \mathcal{Q}$ is called an energetic solution for the system $(\mathcal{Q},\mathcal{E},\mathcal{R})$, if $t \mapsto \partial_t\mathcal{E}(t,q) \in L^1((0,T))$ and if for all $s,t \in [0,T]$ we have $\mathcal{E}(t,q(t)) < \infty$, global stability* (2(S)) *and global energy balance* (2(E))*:*

$$\textit{for all } \tilde q = (\tilde u, \tilde z) \in \mathcal{Q} \textit{ holds}: \quad \mathcal{E}(t,q(t)) \le \mathcal{E}(t,\tilde q) + \mathcal{R}(\tilde z - z(t))\,, \tag{2(S)}$$

$$\mathcal{E}(t,q(t)) + \mathrm{Diss}_{\mathcal{R}}(z,[s,t]) = \mathcal{E}(s,q(s)) + \textstyle\int_s^t \partial_\xi\mathcal{E}(\xi,q(\xi))\,\mathrm{d}\xi \tag{2(E)}$$

with $\mathrm{Diss}_{\mathcal{R}}(z,[s,t]) := \sup\big\{\sum_{j=1}^N \mathcal{R}(z(\xi_j) - z(\xi_{j-1}))\,|\, s = \xi_0 < \ldots < \xi_N = t,\ N \in \mathbb{N}\big\}$.

In the style of [FN96] the energy functional for our setting is set up as follows:

$$\mathcal{E}(t,u,z) := \int_\Omega f(z)e(u{+}g(t)){:}\mathbb{C}{:}e(u{+}g(t))\,\mathrm{d}x + \mathcal{G}(z) + \int_\Omega \delta_{[0,1]}(z)\,\mathrm{d}x\,. \tag{3}$$

Here, $u:\Omega\to\mathbb{R}^d$ denotes the displacement and $e(u){:=}\frac{1}{2}(\nabla u+\nabla u^\top)$ the linearized strain tensor. The first term in (3) represents the stored elastic energy with the tensor $\mathbb{C}\in\mathbb{R}^{(d\times d)\times(d\times d)}$ being symmetric and positive definite. We assume that

$$\begin{aligned}&f:[0,1]\to[a,b] \text{ for constants } 0{<}a{<}b\\ &\text{is continuous and monotonously increasing on } [0,1].\end{aligned} \tag{4}$$

Moreover, $\delta_{[0,1]}$ is the characteristic function of the interval $[0,1]$, i.e. $\delta_{[0,1]}(z){=}0$, if $z\in[0,1]$ and $\delta_{[0,1]}(z)=\infty$, if $z\not\in[0,1]$. Although $z=0$ is allowed, (3) only models partial damage, since we assume $0<a<b$ in (4). Hence, the coercivity of the energy functional is ensured, so that, in validity of Korn's inequality, the displacements are well-defined allover in Ω. The damage process is driven by the time-dependent external loadings $g:[0,T]\times\Omega\to\mathbb{R}^d$ modeled by a given extension of time-dependent Dirichlet data. Finally, the term $\mathcal{G}(z)$ has regularizing effects. It is used in mathematical literature, see e.g. [FN96], but also in engineering contributions [HS03], where it is considered to account for microscopic interactions. In [MR06] the existence of energetic solutions for the system $(\mathcal{Q},\mathcal{E},\mathcal{R})$ was proven for $\mathcal{G}(z):=\int_\Omega\frac{\kappa}{r}|\nabla z|^r\,\mathrm{d}x$ with $r{>}d$. This restriction was necessary in an essential step of the proof, namely for the construction of a so-called mutual recovery sequence (MRS), where the compact embedding $W^{1,r}(\Omega)\Subset C(\overline{\Omega})$ was exploited. More precisely, the existence of a MRS is used to verify (2(S)) for an energetic solution, see Def. 2.2. The difficulties in the construction lie in the discontinuity of $\mathcal{R}$ and the gradient term $\mathcal{G}$. In [Tho10, TM10] the existence result was extended to $r{\in}(1,\infty)$ by introducing a new technique for the construction of the MRS, which does not use the compact embedding. Instead, the construction is based on the chain rule for $W^{1,r}$-functions composed with Lipschitz functions and on a cancellation argument for the resulting terms. Moreover, a model for partial damage without regularization is treated in [FKS10]. The absence of the gradient causes a lack of compactness, so that one resorts to the framework of Young measures.

In this contribution we focus on the limit case $r{=}1$. In contrast to $r\in(1,\infty)$ the space $W^{1,1}(\Omega)$ lacks sequential compactness. For this reason we extend the functionals to the space $BV(\Omega)$ of functions with bounded variation, which consists of all the functions $z{\in}L^1(\Omega)$, whose distributional derivatives D_iz, $i{=}1,\ldots,d$, can be represented by a finite Radon measure in Ω. Hence, with $\mathrm{D}z$ as the distributional gradient and $|\mathrm{D}z|(\Omega)$ as the variation of z in Ω (see e.g. [AFP05, Def. 3.4]), we set

$$\mathcal{G}(z) := |\mathrm{D}z|(\Omega) \quad \text{for all } z\in BV(\Omega)\,. \tag{5}$$

This covers the intermediate case inbetween damage evolution in Sobolev spaces [MR06, TM10] and the much weaker case of damage evolution in terms of Young measures [FKS10].

In Section 2 the proof of the existence result in the BV-setting will be carried out and the MRS will be constructed in detail by transferring the arguments of the case $r\in(1,\infty)$ to the BV-setting. This involves results from the theory of BV-spaces, which are provided in Section 2.1. The most important tool is the decomposability of BV-functions, see Lemma 2.12 and [AFP05, Th. 3.84], which allows it to compose the elements of the recovery sequence $\hat{z}_k$ piecewise in Ω by the elements of the stable sequence z_k and a testfunction $\hat{z}$ using indicator functions of suitable level sets in order to ensure that $\mathcal{R}(\hat{z}_k - z_k) < \infty$. This construction replaces the chain rule for the composition of $W^{1,r}$-functions with the Lipschitz-function $\min : [0,1] \times [0,1] \to [0,1]$ used in the setting of Sobolev spaces.

In Section 3 we treat a so-called brittle damage model, which accounts for two material states only, the undamaged and a damaged one. This is mathematically modeled by considering the damage variable as an indicator function of a set with finite perimeter. Due to this assumption the BV-regularization is given by the perimeter $P(E, \Omega)$, which is the variation of the indicator function:

$$P(E, \Omega) := |\mathrm{D}I_E|(\Omega) < \infty \,. \tag{6}$$

This regularization is coupled to a stored energy which can be used for the modeling of concrete, see (51). In Section 3.1 it will be shown that the rate-independent brittle damage model can be approximated by functionals of Modica-Mortola type. Having in mind the works [Alb98, MM77], where classical Γ-convergence of the static Modica-Mortola energy to the static perimeter energy term was proven in the context of phase transitions, this convergence seems to be obvious on the first glance. But one must be aware that the present work deals with Γ-convergence of rate-independent systems, where the energy functionals and the dissipation potential interplay because of the conditions (2). In particular, the proof of the upper Γ-limit gets more involved due to the unidirectionality of the dissipation potential, see Section 3.2.2.

2 Existence of energetic solutions for the BV-model

The aim of this section is to prove the existence of energetic solutions for the rate-independent system $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ given by (1), (3) with the regularization (5) in the state space

$$\mathcal{Q} = \mathcal{U} \times \mathcal{Z} \quad \text{with} \quad \mathcal{U} := \left\{ v \in H^1(\Omega, \mathbb{R}^d) \,,\, v = 0 \text{ on } \Gamma_{\mathrm{D}} \right\} \quad \text{and} \quad \mathcal{Z} := \mathrm{BV}(\Omega) \,. \tag{7}$$

The procedure to prove our main result is based on the abstract theory developed in [MM05, Mie05, FM06, MRS08]. In particular, the proof can be carried out by verifying the conditions of [Mie09, Th. 3.4]. Moreover, most of the steps to do are similar to the ones in [TM10, Sect. 3], since the stored energy density $f(z)e : \mathbb{C} : e$ considered here is a special case of [TM10]. The main difference arises from the BV-regularization. Of course, $\mathcal{G}$ defined as the variation of BV-functions is lower semicontinuous and guarantees sequential compactness in $\mathrm{BV}(\Omega)$ with respect to strong $L^1(\Omega)$-convergence, i.e. there holds

$$\sup_{k \in \mathbb{N}} \left(\|z_k\|_{L^1(\Omega)} + \mathcal{G}(z_k) \right) \leq c \;\Rightarrow\; \exists \text{ subseq. } z_k \to z \text{ in } L^1(\Omega) \text{ and } z \in \mathrm{BV}(\Omega) \,, \tag{8a}$$

$$z_k \to z \text{ in } L^1(\Omega) \;\Rightarrow\; \mathcal{G}(z) \leq \liminf_{k \to \infty} \mathcal{G}(z_k) \,, \tag{8b}$$

see [AFP05, Rem. 3.5, Th. 3.23]. The convergence $z_k \to z$ in $L^1(\Omega)$ with $\mathcal{G}(z_k) \leq c$ for all $k \in \mathbb{N}$ is equivalent to weak*-convergence in $\mathrm{BV}(\Omega)$, denoted by $z_k \overset{*}{\rightharpoonup} z$ in $\mathrm{BV}(\Omega)$ [AFP05, Prop. 3.13]. Because of this, the topology of convergence is specified as follows

$$(u_k, z_k) \xrightarrow{\mathcal{T}} (u, z) \;\Leftrightarrow\; \begin{cases} u_k \rightharpoonup u \text{ in } H^1(\Omega, \mathbb{R}^d)\,, \\ z_k \overset{*}{\rightharpoonup} z \text{ in } \mathrm{BV}(\Omega)\,. \end{cases} \tag{9}$$

Properties (8) help to ensure the existence of minimizers at each time step. The main difficulty arises when passing from the time-discretized model to the time-continuous one, in particular, when proving the closedness of stable sets. Similarly to [TM10, Sect. 3.4] we thereto construct a MRS, which requires to transfer the ansatz used for the $W^{1,r}$-regularization for $r \in (1, d]$ to the BV-setting. In the following we present the existence result and we briefly address the nonproblematic steps of the proof. As it is the main issue of the proof, the focus of this section lies in the construction of the MRS. For this, we introduce the relevant tools from the theory of BV-spaces in Section 2.1 and establish the MRS in Section 2.2.

Theorem 2.1 (Existence of energetic solutions for the BV-model) *Let $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ be given by (7), (3) and (1) with the regularization (5). Let (4) hold and let the tensor $\mathbb{C}$ in (3) be symmetric and positive definite, i.e. there are constants $0 < c_1^{\mathbb{C}} \leq c_2^{\mathbb{C}}$ such that $c_1^{\mathbb{C}}|e|^2 \leq e : \mathbb{C} : e \leq c_2^{\mathbb{C}}|e|^2$. Moreover, assume that $\Omega \subset \mathbb{R}^d$ is an open, bounded Lipschitz domain, that the Dirichlet boundary $\Gamma_{\mathrm{D}} \neq \emptyset$ and that the extension g of the Dirichlet-datum satisfies $g \in C^1\big([0,T], H^1(\Omega, \mathbb{R}^d)\big)$.*

Then, for any initial value $(u_0, z_0) \in \mathcal{Q}$, which satisfies (2(S)) at $t = 0$, there exists an energetic solution $(u, z) : [0, T] \to \mathcal{Q}$ for the system $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$.

Proof: Let $W(e, z) := f(z)e : \mathbb{C} : e$ such that f and $\mathbb{C}$ satisfy the assumptions of Theorem 2.1. Then, $W : \mathbb{R}^{d\times d} \times [0, 1] \to \mathbb{R}$ enjoys the following properties

(P1) Continuity: $W : \mathbb{R}^{d\times d} \times [0, 1] \to \mathbb{R}$ is continuous.

(P2) Convexity: $\forall z \in [0, 1] : W(\cdot, z) : \mathbb{R}^{d\times d} \to \mathbb{R}$ strictly convex.

(P3) Coercivity: $\exists\, c_1, c_2 > 0\ \forall (e, z) \in \mathbb{R}^{d\times d} \times [0, 1] : c_1|e|^2 \leq W(e, z) \leq c_2|e|^2$.

(P4) Stress control: $\exists\, c_3 > 0\ \forall (e, z) \in \mathbb{R}^{d\times d} \times [0, 1] : |\partial_e W(e, z)| \leq c_3|e|$.

(P5) Lipschitz continuity of the stresses: $\exists\, c_4 > 0\ \forall (e_1, z), (e_2, z) \in \mathbb{R}^{d\times d} \times [0, 1]$: $|\partial_e W(e_1, z) - \partial_e W(e_2, z)| \leq c_4|e_1 - e_2|$.

(P6) Monotonicity: $\forall (e, z_1), (e, z_2) \in \mathbb{R}^{d\times d} \times [0, 1]$ with $z_1 \leq z_2$: $W(e, z_1) \leq W(e, z_2) \leq b/aW(e, z_1)$.

Properties (P1)-(P3) together with (8) imply that $\mathcal{E}(t, \cdot, \cdot)$ is sequentially lower semicontinuous and that its sublevels are compact in the topology $\mathcal{T}$ from (9). Hence, the existence of a minimizer $(u(t_k), z(t_k))$ for $\mathcal{E}(t_k, \cdot, \cdot) + \mathcal{R}(\cdot - z(t_{k-1}))$ is guaranteed for all $0 \leq t_{k-1} < t_k \leq T$. For all $k \in \mathbb{N}$ these minimizers $(u(t_k), z(t_k))$ satisfy (2(S)) at time t_k. Property (P4) together with the assumptions on g enables us to show the existence of

$$\partial_t \mathcal{E}(t, u, z) := \int_\Omega \partial_e W(e(u + g(t), z) : e(\dot{g}(t))\, \mathrm{d}x \quad \text{for all } t \in [0, T]\,. \tag{10}$$

Additionally, it leads to the control of $\partial_t \mathcal{E}(t, u(t_k), z(t_k))$ by $\mathcal{E}(t, u(t_k), z(t_k))$ uniformly in $[0, T]$. Then, a Gronwall argument yields the boundedness of the energy uniformly in time. This implies that $(u(t_k), z(t_k))_{k\in\mathbb{N}}$ is uniformly bounded in $\mathcal{Q}$.

As $t_k \to t$, i.e. when passing to 0 with the step size of the partitions of the time interval $[0,T]$, we therefore have a subsequence $(u(t_k), z(t_k)) \xrightarrow{\mathcal{T}} (u_t, z_t)$.

Properties (P5) and (P6) are used to prove that $\partial_t \mathcal{E}(t, u(t_k), z(t_k)) \to \partial_t \mathcal{E}(t, u_t, z_t)$ for every $(t_k, u(t_k), z(t_k)) \overset{[0,T]\times\mathcal{T}}{\to} (t, u_t, z_t)$ with $(t_k, u(t_k), z(t_k))$ satisfying (2(S)). This allows it to verify the energy balance (2(E)). It remains to show that the limit (t, u_t, z_t) satisfies (2(S)), i.e. the closedness of stable sets must be shown. This will be carried out in detail below. ∎

The proof of the closedness of stable sets is not straight forward due to the unidirectionality of $\mathcal{R}$. Consider $(t_k, u_k, z_k)_{k\in\mathbb{N}}$ satisfying (2(S)) with $(t_k, u_k, z_k) \overset{[0,T]\times\mathcal{T}}{\to} (t, u, z)$ and $\hat{z} \in \mathcal{Z}$. Then we have to prove that (t, u, z) satisfies (2(S)) as well. But since $\mathcal{R}(\hat{z} - z_k) = \infty$ whenever $\hat{z} > z_k$ on a set of positive $\mathcal{L}^d$-measure, we cannot simply pass to the limit in (2(S)). Instead we use the following condition.

Definition 2.2 (MRS-condition) *The system $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ satisfies the mutual recovery sequence condition if for all sequences $(t_k, q_k)_{k\in\mathbb{N}} = (t_k, u_k, z_k)_{k\in\mathbb{N}}$ with (t_k, q_k) satisfying* (2(S)) *for all $k \in \mathbb{N}$ and with $(t_k, q_k) \overset{[0,T]\times\mathcal{T}}{\to} (t, q)$ and for every $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$ there is a sequence $(\hat{q}_k)_{k\in\mathbb{N}} = (\hat{u}_k, \hat{z}_k)_{k\in\mathbb{N}}$ with $\hat{q}_k \xrightarrow{\mathcal{T}} \hat{q}$ in $\mathcal{Q}$, so that*

$$\limsup_{k\to\infty} \big(\mathcal{E}(t_k, \hat{q}_k) + \mathcal{R}(\hat{z}_k - z_k) - \mathcal{E}(t_k, q_k)\big) \leq \mathcal{E}(t, \hat{q}) + \mathcal{R}(\hat{z} - z) - \mathcal{E}(t, q)\,. \tag{11}$$

Note that $\mathcal{E}(t_k, \hat{q}_k) + \mathcal{R}(\hat{z}_k - z_k) - \mathcal{E}(t_k, q_k) \geq 0$ for all $k \in \mathbb{N}$ due to (2(S)) for (t_k, q_k). Hence the MRS-condition implies (2(S)) for (t, q).

The property $\mathcal{R}(\hat{z}_k - z_k) < \infty$ requires that $0 \leq \hat{z}_k \leq z_k$ $\mathcal{L}^d$-a.e. in Ω. In [TM10, Sect. 3.2.5] for the setting of $W^{1,r}$-functions this was achieved by the ansatz $\hat{z}_k := \max\{0, \min\{\hat{z} - \delta_k, z_k\}\}$ using that the superposition of the Lipschitz continuous function min with a $W^{1,r}$-function generates a $W^{1,r}$-function and its gradient can be calculated by a chain rule. Then, the proof of inequality (11) exploited the cancellation of $\mathcal{G}(\hat{z}_k) - \mathcal{G}(z_k)$ on the subsets $[z_k \leq \hat{z} - \delta_k]$, where $\delta_k \to 0$ was determined such that $\mathcal{L}^d([z_k \leq \hat{z} - \delta_k]) \to 0$. In the BV-setting we also want to take advantage of this cancellation argument. A chain rule for BV-functions superposed with Lipschitz continuous functions was established in [ADM90]. Since it may happen that a Lipschitz continuous function l is nowhere differentiable on the range of a BV-function z this general chain rule involves a tangential differential of l to the range of z. However, for our problem we can replace the superposition using indicator functions of suitable level sets, i.e. $\hat{z}_k := (\hat{z} - \delta_k) I_{A_k} + z_k I_{B_k} + 0 \cdot I_{C_k}$, where $A_k := [0 \leq \hat{z} - \delta_k \leq z_k]$, $B_k := [0 \leq z_k < \hat{z} - \delta_k]$ and $C_k := \Omega \backslash (A_k \cup B_k)$. Intuitively (but sloppily), the distributional gradient $\mathrm{D}\hat{z}_k$ is given by $\mathrm{D}\hat{z}$ in A_k, by $\mathrm{D}z_k$ in B_k and additionally by the jumps across the (reduced) boundaries of these sets. In order to ensure that $|\mathrm{D}\hat{z}_k|(\Omega) < \infty$, i.e. that $\hat{z}_k$ composed in this way indeed is a BV-function, requires that A_k and B_k have finite perimeter and that the traces of the functions $\hat{z}$ and z_k on the (reduced) boundaries of A_k and B_k are welldefined and bounded. This relation is stated by the theorem on the decomposability of BV-functions [AFP05, Th. 3.84] (Lemma 2.12, here). For our problem, this can be achieved by choosing δ_k suitably, which is possible due to the coarea formula. Moreover, $\delta_k \to 0$ can be determined such that $\mathcal{L}^d(B_k) \to 0$. But this does not imply that also $P(B_k, \Omega) \to 0$, which would make the jump parts converge suitably. Therefore, we have to evaluate the BV-traces of $\hat{z}$ and $\hat{z}_k$ carefully on the reduced boundaries of A_k and B_k. In order to make the convergence proof of the MRS as readable as possible all the required BV-terminology is provided beforehand in Section 2.1. The MRS is then established in Section 2.2.

2.1 Tools from BV-spaces for the construction of the MRS

This section is a collection of tools from the theory of BV-spaces, which are used for the construction of the MRS in Section 2.2. The notation and the results are taken from [AFP05, Sect. 3] and readers who are familiar with BV-theory may skip this present section.

Proposition 1 ([AFP05, Prop. 3.38] Properties of the perimeter)

1. *The mapping $E \mapsto P(E,\Omega)$ is lower semicontinuous with respect to local convergence in measure in Ω.*
2. *The mapping $E \mapsto P(E,\Omega)$ is local, i.e. $P(E,\Omega) = P(F,\Omega)$ whenever $\mathcal{L}^d(\Omega \cap ((E\backslash F) \cup (F\backslash E))) = 0$.*
3. *It holds $P(E,\Omega) = P(\mathbb{R}^d\backslash E,\Omega)$ and*

$$P(E \cup F,\Omega) + P(E \cap F,\Omega) \leq P(E,\Omega) + P(F,\Omega)\,. \tag{12}$$

Theorem 2.3 ([AFP05, Th. 3.40] Coarea formula in BV) *For any open set $\Omega \subset \mathbb{R}^d$ and $v \in L^1_{\mathrm{loc}}(\Omega)$ one has*

$$|\mathrm{D}v|(\Omega) := \int_{-\infty}^{\infty} P(\{x \in \Omega \,|\, v(x) > t\},\Omega)\,\mathrm{d}t\,. \tag{13}$$

If $v \in BV(\Omega)$ the set $\{v > t\}$ has finite perimeter for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ and

$$|\mathrm{D}v|(B) = \int_{-\infty}^{\infty} |\mathrm{D}I_{\{v>t\}}|(B)\,\mathrm{d}t\,, \quad \mathrm{D}v(B) = \int_{-\infty}^{\infty} \mathrm{D}I_{\{v>t\}}(B)\,\mathrm{d}t \tag{14}$$

for any Borel set $B \subset \Omega$.

Definition 2.4 ([AFP05, Def. 3.54] Reduced boundary) *Let E be an $\mathcal{L}^d$-measurable subset of $\mathbb{R}^d$ and Ω the largest open set such that E is locally of finite perimeter in Ω. The* reduced boundary *$\mathfrak{F}E$ is defined as the collection of all points $x \in \operatorname{supp}|\mathrm{D}I_E| \cap \Omega$ such that the limit*

$$\nu_E(x) := \lim_{\varrho\to 0} \frac{\mathrm{D}I_E(B_\varrho(x))}{|\mathrm{D}I_E|(B_\varrho(x))} \tag{15}$$

exists in $\mathbb{R}^d$ and satisfies $\nu_E(x)$=1. The function $\nu_E : \mathfrak{F}E \to \mathbb{S}^{d-1}$ is called the generalized inner normal *to E.*

Definition 2.5 ([AFP05, Def. 3.60]Points of density t, essential boundary) *For all $t \in [0,1]$ and every $\mathcal{L}^d$-measurable set $E \subset \mathbb{R}^d$ we introduce*

$$E^t := \left\{x \in \mathbb{R}^d \,|\, \lim_{\varrho\to 0} \frac{\mathcal{L}^d(E \cap B_\varrho(x))}{\mathcal{L}^d(B_\varrho(x))} = t\right\} \quad \textit{and} \quad \partial^* E := \mathbb{R}^d\backslash(E^0 \cup E^1)\,. \tag{16}$$

E^t denotes the set of all points where E has density t and ∂^*E is the essential boundary *of E. Moreover, E^1 can be considered as the* measure theoretic interior *and E^0 as the* measure theoretic exterior *of the set E.*

The next properties of the measure theoretic interior directly follow from (16).

Corollary 1 *The measure theoretic interior has the following properties:*
1. *Let $N \subset \Omega$ with $\mathcal{L}^d(N) = 0$. Then $N^1 = \emptyset$ and $(\Omega\backslash N)^1 = \Omega^1$.*
2. *Let $A \subset B \subset \Omega$. Then $A^1 \subset B^1 \subset \Omega^1$.*

The next theorem, which is due to Federer, states that $\mathfrak{F}E$ is the important part of the boundary, since $\Omega\backslash(E^0 \cup \mathfrak{F}E \cup E^1)$ is a $\mathcal{H}^{d-1}$-negligible set.

Theorem 2.6 ([AFP05, Th. 3.61] Federer) *Let E be a set of finite perimeter in Ω. Then*

$$\mathfrak{F}E \cap \Omega \subset E^{1/2} \subset \partial^* E \quad \textit{and} \quad \mathcal{H}^{d-1}(\Omega\backslash(E^0 \cup \mathfrak{F}E \cup E^1)) = 0\,. \tag{17}$$

In particular, E has density either 0 or $1/2$ or 1 at $\mathcal{H}^{d-1}$-a.e. $x \in \Omega$ and $\mathcal{H}^{d-1}$-a.e. $x \in \partial^* E \cap \Omega$ belongs to $\mathfrak{F}E$.

Definition 2.7 ([AFP05, Def. 3.63] Approximate limit) *Let $v \in L^1_{\mathrm{loc}}(\Omega)^d$. We say that v has an approximate limit at $x \in \Omega$ if there exists $\bar{v} \in \mathbb{R}^d$ such that*

$$\lim_{\varrho \to 0} \fint_{B_{\varrho(x)}} |v(y) - \bar{v}| \, \mathrm{d}y = 0\,. \tag{18}$$

The set S_v of points where this property does not hold is called the approximate discontinuity set*. For any $x \in \Omega\backslash S_v$ the vector $\bar{v}$, uniquely determined by (18), is called* approximate limit *of v at x and denoted by $\tilde{v}(x)$.*

Definition 2.8 ([AFP05, Def. 3.67] Approximate jump points) *Let*

$$B^{\pm}_{\varrho}(x, \nu) := \{y \in B_{\varrho}(x) \mid \pm \langle y - x, \nu \rangle > 0\}\,. \tag{19}$$

Let $v \in L^1_{\mathrm{loc}}(\Omega)^d$ and $x \in \Omega$. We say that x is an approximate jump point *of v if there exist a, $b \in \mathbb{R}^d$ and $\nu \in \mathbb{S}^{d-1}$ such that $a \neq b$ and*

$$\lim_{\varrho \to 0} \fint_{B^+_{\varrho}(x,\nu)} |v(y) - a| \, \mathrm{d}y = 0\,, \quad \lim_{\varrho \to 0} \fint_{B^-_{\varrho}(x,\nu)} |v(y) - b| \, \mathrm{d}y = 0\,. \tag{20}$$

The triple (a, b, ν), uniquely determined by (20) up to a permutation of (a, b) and a change of sign of ν, is denoted by $(v^+, v^-, \nu_v(x))$. The set of approximate jump points of v is denoted by J_v.

Definition 2.9 ([AFP05, Def. 2.57] Rectifiable sets) *Let $E \subset \mathbb{R}^d$ be an $\mathcal{H}^k$-measurable set. We say that E is* countably k-rectifiable *if there exists countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^d$ such that*

$$E \subset \cup_{i=0}^{\infty} f_i(\mathbb{R}^k)\,. \tag{21}$$

We say that E is countably $\mathcal{H}^k$-rectifiable *if there exists countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^d$ such that*

$$\mathcal{H}^k\big(E \backslash \cup_{i=0}^{\infty} f_i(\mathbb{R}^k)\big) = 0\,. \tag{22}$$

Clearly, k-rectifiability implies $\mathcal{H}^k$-rectifiability.

Theorem 2.10 ([AFP05, Th. 3.59] De Giorgi) *Let E be an $\mathcal{L}^d$-measurable subset of $\mathbb{R}^d$. Then $\mathfrak{F}E$ is countably $(d-1)$-rectifiable and $|\mathrm{D}I_E| = \mathcal{H}^{d-1}\lfloor\mathfrak{F}E$.*

Due to Th. 2.10 the perimeter of E can be computed by

$$P(E,\Omega) = \mathcal{H}^{d-1}(\Omega \cap \partial^* E) = \mathcal{H}^{d-1}(\Omega \cap E^{1/2})\,. \tag{23}$$

This can be used to rewrite the coarea formula (13) using the essential boundary of level sets

$$|\mathrm{D}u|(B) = \int_{-\infty}^{\infty} \mathcal{H}^{d-1}\big(B \cap \partial^*\{u > t\}\big)\,\mathrm{d}t \quad \text{for all Borel sets } B \subset \Omega\,. \tag{24}$$

Theorem 2.11 ([AFP05, Th. 3.77] Traces on interior rectifiable sets) *Let v be a function in $BV(\Omega)^d$ and let $\Gamma \subset \Omega$ be a countably $\mathcal{H}^{d-1}$-rectifiable set oriented by ν. Then, for $\mathcal{H}^{d-1}$-a.e. $x \in \Gamma$ there exist $v_\Gamma^+(x)$, $v_\Gamma^-(x) \in \mathbb{R}^d$ such that*

$$\lim_{\varrho\to 0} \fint_{B_\varrho^+(x,\nu(x))} |v(y) - v_\Gamma^+(x)|\,\mathrm{d}y = 0\,, \quad \lim_{\varrho\to 0} \fint_{B_\varrho^-(x,\nu(x))} |v(y) - v_\Gamma^-(x)|\,\mathrm{d}y = 0\,. \tag{25}$$

Moreover, $\mathrm{D}v\lfloor\Gamma = (v_\Gamma^+ - v_\Gamma^-) \otimes \nu\mathcal{H}^{d-1}\lfloor\Gamma$.

2.2 Existence of mutual recovery sequences

The construction of a MRS in the BV-setting will be based on the following lemma on the decomposability of BV-functions. Using complete induction, it can deduced from [AFP05, Th. 3.84], which gives the statement of the lemma for $N = 2$.

Lemma 2.12 (Decomposability of BV-functions) *For all $i \in \{1,\ldots,N\}, N \in \mathbb{N}$, let $v_i \in BV(\Omega)$ and $A_i \subset \Omega$ with finite perimeter and the generalized inner normal ν_i to the reduced boundary $\mathfrak{F}A_i$, such that $\cup_{i=1}^N A_i = \Omega$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$. For all $i \in \{1,\ldots,N\}$ and all $j \in \{i+1,\ldots,N\}$ let $\mathfrak{F}A_i \cap \mathfrak{F}A_j$ be oriented by ν_i. Let I_{A_i} denote the indicator function of the set A_i and $v_{i\,\mathfrak{F}A_i}^{\pm}$ the traces on $\mathfrak{F}A_i$. Then*

$$w := \sum_{i=1}^N v_i I_{A_i} \in BV(\Omega) \iff \sum_{\substack{i=1\\ j=i+1}}^N \int_{\mathfrak{F}A_i\cap\mathfrak{F}A_j\cap\Omega} |v_{i\,\mathfrak{F}A_i}^{+} - v_{j\,\mathfrak{F}A_j}^{-}|\,\mathrm{d}\mathcal{H}^{d-1} < \infty\,. \tag{26}$$

If $w \in BV(\Omega)$, the measure $\mathrm{D}w$ is representable by

$$\mathrm{D}w := \sum_{i=1}^N \Big(\mathrm{D}v_i\lfloor A_i^1 + \sum_{j=i+1}^N (v_{i\,\mathfrak{F}A_i}^{+} - v_{j\,\mathfrak{F}A_j}^{-}) \otimes \nu_i\mathcal{H}^{d-1}\lfloor(\mathfrak{F}A_i \cap \mathfrak{F}A_j \cap \Omega)\Big)\,, \tag{27}$$

where A_i^1 is the measure theoretic interior of A_i, as in Def. 2.5.

Moreover, we will exploit that the BV-traces of a function, which is bounded $\mathcal{L}^d$-a.e., are bounded $\mathcal{H}^{d-1}$-a.e. by the same constants. This can be proven by contradiction using formula (25).

Corollary 2 *Let $v \in BV(\Omega)$ with $a \leq v \leq b$ $\mathcal{L}^d$-a.e. in Ω for constants a, $b \in \mathbb{R}$. assume that Γ is a $\mathcal{H}^{d-1}$-rectifiable set oriented by ν. Then $a \leq v_\Gamma^{\pm}(x) \leq b$ for $\mathcal{H}^{d-1}$-a.e. $x \in \Omega$.*

With these tools at hand we are in a position to verify the MRS-condition.

Lemma 2.13 *Let the assumptions of Theorem 2.1 hold. Then* $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ *satisfies the MRS-condition from Definition 2.2.*

Proof: Let $(t_k, u_k, z_k)_{k\in\mathbb{N}} \subset [0,T] \times \mathcal{Q}$ with $(t_k, u_k, z_k) \overset{[0,T]\times\mathcal{T}}{\to} (t, u, z)$. Choose now $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$ such that $\mathcal{E}(t, \hat{q}) < E$ for some $E \in \mathbb{R}$, otherwise (11) trivially holds. Now we distinguish between the following two cases:

Case A: Let $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$ be such that there exists a $\mathcal{L}^d$-measurable set $B \subset \Omega$ with $\mathcal{L}^d(B) > 0$ and $\hat{z} > z$ on B. Then $\mathcal{R}(\hat{z} - z) = \infty$ and (11) holds.

Case B: Let $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$ so that $\hat{z} \le z$ a.e. in Ω. Then $\mathcal{R}(\hat{z} - z)$ is finite, i.e. $\mathcal{R}(\hat{z}-z) = \int_\Omega \varrho(z-\hat{z})\mathrm{d}x < \infty$. To construct a MRS we set $\hat{u}_k := \hat{u}$ for every $k \in \mathbb{N}$ and

$$\hat{z}_k := (\hat{z}-\delta_k)I_{A_k} + z_k I_{B_k} + 0\cdot I_{C_k}, \text{ where} \tag{28}$$

$$A_k := [0 \le \hat{z}-\delta_k \le z_k],\ B_k := [0 \le z_k < \hat{z}-\delta_k],\ C_k = \Omega\backslash(A_k \cup B_k). \tag{29}$$

With this choice we ensure that $0 \le \hat{z}_k \le z_k$ a.e. in Ω. We show now that the sequence $\delta_k \overset{\mathbb{R}}{\to} 0$ can be determined in such a way that $\hat{z}_k \in BV(\Omega)$, so that (27) is applicable, such that $\big(\mathcal{L}^d(B_k)+\mathcal{L}^d(C_k)\big) \to 0$ and $\hat{z}_k \to \hat{z}$ in $L^1(\Omega)$ as $k \to \infty$. Because of $\hat{z} \le z$ in Ω we obtain

$$B_k \subset [z_k < \hat{z}-\delta_k] \subset [z_k < z-\delta_k] \subset [\delta_k < |z-z_k|]. \tag{30}$$

Using Markov's inequality (M) in the last estimate of (31) below we conclude

$$\mathcal{L}^d(B_k) \le \mathcal{L}^d([\delta_k < |z-z_k|]) \overset{\text{(M)}}{\le} \delta_k^{-1}\|z-z_k\|_{L^1(\Omega)} \tag{31}$$

and to ensure that the right-hand side of (31) tends to 0, we may e.g. choose any $\delta_k \in [m_k^{1/2}, m_k^{1/4}]$ with $m_k := \max\{k^{-1}, \|z-z_k\|_{L^1(\Omega)}\}$.

Moreover, to make Cor. 2.12 applicable we have to choose $\delta_k \in [m_k^{1/2}, m_k^{1/4}]$ such that the sets A_k, B_k and C_k have finite perimeter and that the right-hand side of (26) is finite for all $k \in \mathbb{N}$. For this, we rewrite $A_k = [\delta_k \le \hat{z}] \cap [-\delta_k \le z_k - \hat{z}]$ and $B_k = [0 \le z_k] \cap [\delta_k < \hat{z} - z_k]$ as the intersections of levels sets of the functions $\hat{z}$, $(z_k - \hat{z})$, z_k and $(\hat{z} - z_k) \in BV(\Omega)$. By formula (12) and the coarea formulas (13), (24) we conclude that $\delta_k \in [m_k^{1/2}, m_k^{1/4}]$ can be chosen such that A_k, B_k and C_k have finite perimeter. It remains us to verify that the right-hand side of (26) is finite. Coarea formula (24) yields $\mathcal{H}^{d-1}(\mathfrak{F}A_k \cap \mathfrak{F}B_k \cap \Omega) \le 3|\mathrm{D}\hat{z}|(\Omega) + 3|\mathrm{D}z_k|(\Omega)$, $\mathcal{H}^{d-1}(\mathfrak{F}A_k \cap \mathfrak{F}C_k \cap \Omega) \le 3|\mathrm{D}\hat{z}|(\Omega) + |\mathrm{D}z_k|(\Omega)$ and thirdly $\mathcal{H}^{d-1}(\mathfrak{F}B_k \cap \mathfrak{F}C_k \cap \Omega) \le 2|\mathrm{D}\hat{z}|(\Omega) + 2|\mathrm{D}z_k|(\Omega)$, where $|\mathrm{D}z_k|(\Omega) \le C$ for all $k \in \mathbb{N}$ by the properties of stable sequences. Additionally, Cor. 2 implies that $|\hat{z}^\pm - \delta_k - z_k^\pm| \le 1 - \delta_k$, $|\hat{z}^\pm - \delta_k| \le 1 - \delta_k$ as well as $|z_k^\pm| \le 1$ $\mathcal{H}^{d-1}$-a.e. on the respective reduced boundaries. Hence, the right-hand side of (26) is finite and Cor. 2.12 can be applied.

Now we verify that $\hat{z}_k \to \hat{z}$ in $L^1(\Omega)$. For this we use that

$$C_k = [\hat{z} - \delta_k < 0] \cup [z_k < 0], \tag{32}$$

where the second set is $\mathcal{L}^d$-negligible. Moreover, we have $[\hat{z} - \delta_k < 0] \to [\hat{z} < 0]$ pointwise $\mathcal{L}^d$-a.e., which again is an $\mathcal{L}^d$-negligible set. This shows that $\mathcal{L}^d(C_k) \to 0$ and together with (31) we have obtained that $\big(\mathcal{L}^d(B_k)+\mathcal{L}^d(C_k)\big) \to 0$ as $k \to \infty$. From this, we infer

$$\begin{aligned} \|\hat{z}_k - \hat{z}\|_{L^1(\Omega)} &= \delta_k \mathcal{L}^d(A_k) + \|z_k - \hat{z}\|_{L^1(B_k)} + \|\hat{z}\|_{L^1(C_k)} \\ &\le \delta_k \mathcal{L}^d(\Omega) + \mathcal{L}^d(B_k) + \mathcal{L}^d(C_k)\ \to 0 \quad \text{as } \delta_k \to 0. \end{aligned} \tag{33}$$

Now we are in a position to verify the lim sup estimate (11). For this we use that

$$\begin{aligned}&\limsup_{k\to\infty}\left(\mathcal{E}(t_k,\hat q_k)+\mathcal{R}(\hat z_k-z_k)-\mathcal{E}(t_k,q_k)\right)\\&\leq\limsup_{k\to\infty}\mathcal{I}(t,\hat q_k)-\liminf_{k\to\infty}\mathcal{I}(t,q_k)\\&\quad+\limsup_{k\to\infty}\big(|\mathrm{D}\hat z_k|(\Omega)-|\mathrm{D}z_k|(\Omega)\big)+\limsup_{k\to\infty}\mathcal{R}(\hat z_k-z_k)\,,\end{aligned}\tag{34}$$

where, we introduced $\mathcal{I}(t,q):=\int_\Omega f(z)e(u+g(t)):\mathbb{C}:e(u+g(t))\,\mathrm{d}x$ for $q=(u,z)$. In the following, we estimate the different terms in (34) separately.

Due to the strong L^1-convergence obtained in (33) and the fact that $\hat z_k\leq z_k$ for all $k\in\mathbb{N}$ by construction we conclude that

$$\mathcal{R}(\hat z_k-z_k)\to\mathcal{R}(\hat z-z)\quad\text{as } k\to\infty\,.\tag{35}$$

Moreover, since $\hat u_k=\hat u$ and $\hat z_k\leq\hat z$ for all $k\in\mathbb{N}$ by construction we infer from the monotonicity of $f:[0,1]\to[a,b]$ together with the continuity of the given data $g\in C^1([0,T],H^1(\Omega,\mathbb{R}^d))$ that

$$\limsup_{k\to\infty}\mathcal{I}(t_k,\hat q_k)\leq\limsup_{k\to\infty}\mathcal{I}(t_k,\hat q)=\mathcal{I}(t,\hat q)\,.\tag{36}$$

Furthermore, the weak sequential lower semicontinuity of $\mathcal{I}$ implies that

$$-\liminf_{k\to\infty}\mathcal{I}(t,q_k)\leq-\mathcal{I}(t,q)\,.\tag{37}$$

Thus, it remains to show that

$$\limsup_{k\to\infty}\big(|\mathrm{D}\hat z_k|(\Omega)-|\mathrm{D}z_k|(\Omega)\big)\leq|\mathrm{D}\hat z|(\Omega)-|\mathrm{D}z|(\Omega)\,.\tag{38}$$

For this, we use $\hat z_k=(\hat z-\delta_k)I_{A_k}+z_kI_{B_k}+0\cdot I_{C_k}$ as well as $z_k=z_k(I_{A_k}+I_{B_k}+I_{C_k})$ and we express their derivatives with the aid of formula (27). Hence, we obtain

$$\begin{aligned}|\mathrm{D}\hat z_k|(\Omega)=&|\mathrm{D}\hat z|(A_k^1)+|\mathrm{D}z_k|(B_k^1)+\int_{\mathfrak{F}A_k\cap\mathfrak{F}B_k\cap\Omega}|\hat z^+-\delta_k-z_k^-|\,\mathrm{d}\mathcal{H}^{d-1}\\&+\int_{\mathfrak{F}A_k\cap\mathfrak{F}C_k\cap\Omega}|\hat z^+-\delta_k|\,\mathrm{d}\mathcal{H}^{d-1}+\int_{\mathfrak{F}B_k\cap\mathfrak{F}C_k\cap\Omega}|z_k^+|\,\mathrm{d}\mathcal{H}^{d-1}\,,\end{aligned}\tag{39}$$

where we applied Cor. 2 to determine the traces $\hat z_k^\pm$ on the different parts of the reduced boundaries. Similarly we find

$$\begin{aligned}-|\mathrm{D}z_k|(\Omega)=&-|\mathrm{D}z_k|(A_k^1)-|\mathrm{D}z_k|(B_k^1)-|\mathrm{D}z_k|(C_k^1)-\int_{\mathfrak{F}A_k\cap\mathfrak{F}B_k\cap\Omega}|z_k^+-z_k^-|\,\mathrm{d}\mathcal{H}^{d-1}\\&-\int_{\mathfrak{F}A_k\cap\mathfrak{F}C_k\cap\Omega}|z_k^+-z_k^-|\,\mathrm{d}\mathcal{H}^{d-1}-\int_{\mathfrak{F}B_k\cap\mathfrak{F}C_k\cap\Omega}|z_k^+-z_k^-|\,\mathrm{d}\mathcal{H}^{d-1}\,.\end{aligned}\tag{40}$$

We note that $|\mathrm{D}\hat z_k|(B_k^1)-|\mathrm{D}z_k|(B_k^1)$ cancels out in (38). Moreover, $-|\mathrm{D}z_k|(C_k^1)\leq0$ in (40). Thus, to establish (38) we have to show $-\liminf_{k\to\infty}|\mathrm{D}z_k|(A_k^1)\leq-|\mathrm{D}z|(\Omega)$

and that the boundary terms in (39)+(40) can be estimated as follows for all $k \in \mathbb{N}$:

$$\begin{aligned}
&\int_{\mathfrak{F}A_k\cap\mathfrak{F}B_k\cap\Omega} |\hat{z}^+ - \delta_k - z_k^-| \,\mathrm{d}\mathcal{H}^{d-1} + \int_{\mathfrak{F}A_k\cap\mathfrak{F}C_k\cap\Omega} |\hat{z}^+ - \delta_k| \,\mathrm{d}\mathcal{H}^{d-1} + \int_{\mathfrak{F}B_k\cap\mathfrak{F}C_k\cap\Omega} |z_k^+| \,\mathrm{d}\mathcal{H}^{d-1} \\
&- \int_{\mathfrak{F}A_k\cap\mathfrak{F}B_k\cap\Omega} |z_k^+ - z_k^-| \,\mathrm{d}\mathcal{H}^{d-1} - \int_{\mathfrak{F}A_k\cap\mathfrak{F}C_k\cap\Omega} |z_k^+ - z_k^-| \,\mathrm{d}\mathcal{H}^{d-1} - \int_{\mathfrak{F}B_k\cap\mathfrak{F}C_k\cap\Omega} |z_k^+ - z_k^-| \,\mathrm{d}\mathcal{H}^{d-1} \\
&\le \Big(\int_{\mathfrak{F}A_k\cap\mathfrak{F}B_k\cap\Omega} + \int_{\mathfrak{F}A_k\cap\mathfrak{F}C_k\cap\Omega} + \int_{\mathfrak{F}B_k\cap\mathfrak{F}C_k\cap\Omega} \Big) |\hat{z}^+ - \hat{z}^-| \,\mathrm{d}\mathcal{H}^{d-1} .
\end{aligned} \tag{41}$$

To verify estimate (41) we use the information on the traces stated in Cor. 2 and distinguish between all possible relations. On $\mathfrak{F}A_k\cap\mathfrak{F}B_k\cap\Omega$ it holds $0 \le \hat{z}^+ - \delta_k \le z_k^+$ and $0 \le z_k^- < \hat{z}^- - \delta_k$ $\mathcal{H}^{d-1}$-a.e.. Hence, for $\mathcal{H}^{d-1}$-a.e. $x \in \mathfrak{F}A_k \cap \mathfrak{F}B_k \cap \Omega$ with

$$\begin{array}{llll}
z_k^+ \le z_k^- & \text{it is} & \hat{z}^+ - \delta_k \le z_k^+ \le z_k^- < \hat{z}^- - \delta_k, & \text{i.e. } |\hat{z}^+ - \delta_k - z_k^-| \le |\hat{z}^+ - \hat{z}^-| , \\
z_k^+ > z_k^- & \text{it is either} & \hat{z}^+ - \delta_k \le z_k^- < z_k^+ \le \hat{z}^- - \delta_k, & \text{i.e. } |\hat{z}^+ - \delta_k - z_k^-| \le |\hat{z}^+ - \hat{z}^-| , \\
& \text{or} & \hat{z}^+ - \delta_k \le z_k^- < \hat{z}^- - \delta_k \le z_k^+, & \text{i.e. } |\hat{z}^+ - \delta_k - z_k^-| \le |\hat{z}^+ - \hat{z}^-| , \\
& \text{or} & z_k^- < \hat{z}^- - \delta_k \le \hat{z}^+ - \delta_k \le z_k^+, & \text{i.e. } |\hat{z}^+ - \delta_k - z_k^-| \le |z_k^+ - z_k^-| , \\
& \text{or} & z_k^- < \hat{z}^+ - \delta_k \le z_k^+ \le \hat{z}^- - \delta_k, & \text{i.e. } |\hat{z}^+ - \delta_k - z_k^-| \le |z_k^+ - z_k^-| , \\
& \text{or} & z_k^- < \hat{z}^+ - \delta_k < \hat{z}^- - \delta_k \le z_k^+, & \text{i.e. } |\hat{z}^+ - \delta_k - z_k^-| \le |z_k^+ - z_k^-| .
\end{array}$$

Using these estimates and denoting the set of points, where one of the last three relations holds by E, we find that

$$\begin{aligned}
&\int_{\mathfrak{F}A_k\cap\mathfrak{F}B_k\cap\Omega} |\hat{z}^+ - \delta_k - z_k^-| \,\mathrm{d}\mathcal{H}^{d-1} - \int_{\mathfrak{F}A_k\cap\mathfrak{F}B_k\cap\Omega} |z_k^+ - z_k^-| \,\mathrm{d}\mathcal{H}^{d-1} \le \int_{\mathfrak{F}A_k\cap\mathfrak{F}B_k\cap\Omega\setminus E} |\hat{z}^+ - \hat{z}^-| \,\mathrm{d}\mathcal{H}^{d-1} - 0 \\
&\le \int_{\mathfrak{F}A_k\cap\mathfrak{F}B_k\cap\Omega} |\hat{z}^+ - \hat{z}^-| \,\mathrm{d}\mathcal{H}^{d-1} .
\end{aligned} \tag{42}$$

On $\mathfrak{F}A_k\cap\mathfrak{F}C_k\cap\Omega$ it holds $0 \le \hat{z}^+ - \delta_k \le z_k^+$ and $\hat{z}^- - \delta_k < 0 \le z_k^-$ $\mathcal{H}^{d-1}$-a.e.. Thus, for $\mathcal{H}^{d-1}$-a.e. $x \in \mathfrak{F}A_k \cap \mathfrak{F}C_k \cap \Omega$ with

$$\begin{array}{llll}
z_k^+ \le z_k^- & \text{it is} & \hat{z}^- - \delta_k < 0 \le \hat{z}^+ - \delta_k \le z_k^+ \le z_k^-, & \text{i.e. } |\hat{z}^+ - \delta_k| \le |\hat{z}^+ - \hat{z}^-| , \\
z_k^+ > z_k^- & \text{it is either} & \hat{z}^- - \delta_k < 0 \le z_k^- \le \hat{z}^+ - \delta_k \le z_k^+, & \text{i.e. } |\hat{z}^+ - \delta_k| \le |\hat{z}^+ - \hat{z}^-| , \\
& \text{or} & \hat{z}^- - \delta_k < 0 \le \hat{z}^+ - \delta_k \le z_k^- \le z_k^+, & \text{i.e. } |\hat{z}^+ - \delta_k| \le |\hat{z}^+ - \hat{z}^-| .
\end{array}$$

Thus, we have

$$\int_{\mathfrak{F}A_k\cap\mathfrak{F}C_k\cap\Omega} |\hat{z}^+ - \delta_k| \,\mathrm{d}\mathcal{H}^{d-1} - \int_{\mathfrak{F}A_k\cap\mathfrak{F}C_k\cap\Omega} |z_k^+ - z_k^-| \,\mathrm{d}\mathcal{H}^{d-1} \le \int_{\mathfrak{F}A_k\cap\mathfrak{F}C_k\cap\Omega} |\hat{z}^+ - \hat{z}^-| \,\mathrm{d}\mathcal{H}^{d-1} - 0 . \tag{43}$$

On $\mathfrak{F}B_k\cap\mathfrak{F}C_k\cap\Omega$ it holds $0 \le z_k^+ < \hat{z}^+ - \delta_k$ and $\hat{z}^- - \delta_k < 0 \le z_k^-$ $\mathcal{H}^{d-1}$-a.e.. Hence, for $\mathcal{H}^{d-1}$-a.e. $x \in \mathfrak{F}B_k \cap \mathfrak{F}C_k \cap \Omega$ with

$$\begin{array}{llll}
z_k^+ \le z_k^- & \text{it is either} & \hat{z}^- - \delta_k < 0 \le z_k^+ \le \hat{z}^+ - \delta_k \le z_k^-, & \text{i.e. } |\hat{z}^+ - \delta_k| \le |\hat{z}^+ - \hat{z}^-| , \\
& \text{or} & \hat{z}^- - \delta_k < 0 \le z_k^+ \le z_k^- \le \hat{z}^+ - \delta_k, & \text{i.e. } |\hat{z}^+ - \delta_k| \le |\hat{z}^+ - \hat{z}^-| , \\
z_k^+ > z_k^- & \text{it is} & \hat{z}^- - \delta_k < 0 \le z_k^- \le z_k^+ \le \hat{z}^+ - \delta_k, & \text{i.e. } |\hat{z}^+ - \delta_k| \le |\hat{z}^+ - \hat{z}^-| ,
\end{array}$$

which yields

$$\int_{\mathfrak{F}B_k\cap\mathfrak{F}C_k\cap\Omega}|z_k^+|\,\mathrm{d}\mathcal{H}^{d-1}-\int_{\mathfrak{F}B_k\cap\mathfrak{F}C_k\cap\Omega}|z_k^+-z_k^-|\,\mathrm{d}\mathcal{H}^{d-1}\le\int_{\mathfrak{F}B_k\cap\mathfrak{F}C_k\cap\Omega}|\hat z^+-\hat z^-|\,\mathrm{d}\mathcal{H}^{d-1}-0\,. \tag{44}$$

Thus, estimate (41) holds. In total we have up to now obtained that the left-hand side of (38) can be estimated by

$$\begin{aligned}
&\limsup_{k\to\infty}\big(|\mathrm{D}\hat z_k|(\Omega)-|\mathrm{D}z_k|(\Omega)\big)\\
&\le\limsup_{k\to\infty}\Big(|\mathrm{D}\hat z|(A_k^1)+\int_{((\mathfrak{F}A_k\cap\mathfrak{F}B_k)\cup(\mathfrak{F}A_k\cap\mathfrak{F}C_k)\cup(\mathfrak{F}B_k\cap\mathfrak{F}C_k))\cap\Omega}|\hat z^+-\hat z^-|\,\mathrm{d}\mathcal{H}^{d-1}\\
&\qquad\qquad+|\mathrm{D}\hat z|(B_k^1)+|\mathrm{D}\hat z|(C_k^1)-|\mathrm{D}z_k|(A_k^1)\Big)\\
&\le|\mathrm{D}\hat z|(\Omega)-\liminf_{k\to\infty}|\mathrm{D}z_k|(A_k^1)
\end{aligned}\tag{45}$$

To show that $-\liminf_{k\to\infty}|\mathrm{D}z_k|(A_k^1)\le-|\mathrm{D}z|(\Omega)$ in (45) we first choose a subsequence $(z_k)_{k\in\mathbb{N}}$ such that the lim inf is attained. Then, we introduce the sets

$$U_n:=\bigcup_{k=n}^{\infty}(B_k\cup C_k)\,. \tag{46}$$

Since both $\mathcal{L}^d(B_k)\to0$ and $\mathcal{L}^d(C_k)\to0$ as $k\to\infty$ we may choose a further subsequence in such a way that $\sum_{k=1}^\infty\mathcal{L}^d(B_k)+\mathcal{L}^d(C_k)<\infty$. For this subsequence

$$\mathcal{L}^d(U_n)<\infty\quad\text{and}\quad\mathcal{L}^d(U_n)\to0\quad\text{as }n\to\infty\,. \tag{47}$$

We set $\lim_{n\to\infty}U_n=N$ and put $\Omega_n:=\Omega\backslash U_n$, which satisfies $\Omega_n\subset A_n$ for all $k\ge n$. Then, also $\Omega_n^1\subset A_k^1$ as well as $\Omega_n^1\subseteq\Omega_{n+1}^1\subset\Omega^1$ for all $n\in\mathbb{N}$ by Cor. 1, 2.). Since $\mathcal{L}^d(N)=0$ we conclude that $(\Omega\backslash N)^1=\Omega^1$ by Cor. 1, 1.). This proves that $\Omega_n^1\to\Omega^1$. Note that $\Omega\subset\mathbb{R}^d$ is an open set, hence $\Omega^1=\Omega$.

Keep $n\in\mathbb{N}$ fixed. Then the sets $\Omega_n^1\subset A_k^1$ can be used to fix a set independent of $k\ge n$, so that the lower semicontinuity of the variation can be exploited on Ω_n^1 for the sequence $z_k\overset{*}{\rightharpoonup}z$ in $BV(\Omega)$ and we have ensured that $\Omega_n^1\to\Omega$. For all $k\ge n$ we have

$$-\liminf_{k\to\infty}|\mathrm{D}z_k|(A_k^1)\le-\liminf_{k\to\infty}|\mathrm{D}z_k|(\Omega_n^1)\le-|\mathrm{D}z|(\Omega_n^1)\ \to-|\mathrm{D}z|(\Omega)\text{ as }n\to\infty\,.$$

This finishes the proof of estimate (45), so that it is shown that the MRS $(\hat u_k,\hat z_k)_{k\in\mathbb{N}}$ given by $\hat u_k=\hat u$ and $\hat z_k$ from (28) satisfies the lim sup-estimate (11). ∎

3 A brittle damage model and its Modica-Mortola approximation

As an example for the model with BV-regularization we now discuss the special case, when the damage variable attains the values 1 or 0, only. This means that the damage variable $z:\Omega\to\{0,1\}$ only distinguishes between the two situations: locally unbroken for $z(x)=1$ and locally broken for $z(x)=0$. For this reason it is called brittle damage, see [FG06, GL09], or brutal damage in [FM93]. In this setting, the set $\mathcal{Z}$ of admissible damage variables can be considered as the subset of $\mathrm{BV}(\Omega)$ consisting of the indicator functions of sets of finite perimeter, i.e.

$$\mathcal{Z}_{\mathrm{B}} := \{I_Z : \Omega \to \{0,1\} \text{ indicator function of } Z \subset \Omega, P(Z,\Omega) < \infty\} . \tag{48}$$

Compactness properties of $\mathcal{Z}_{\mathrm{B}}$ are discussed in Remark 1 below. Since an indicator function I_Z of such a set Z is simply a jump function, its variation in Ω reduces to the jump part, which is exactly the perimeter of Z in Ω, i.e. $|\mathrm{D}I_Z|(\Omega) = P(Z,\Omega)$. Hence, with a constant $\sigma > 0$, the regularizing BV-gradient term is given by

$$\mathcal{G}(z) := \sigma\mathcal{H}^{d-1}(J_z) = \sigma P(Z,\Omega) . \tag{49}$$

We want to use the above regularization in a model that describes the damage of concrete. From now on we denote with $Z := [z=1]$ the set where the structure is unbroken. Then, $\Omega\backslash Z := [z=0]$ describes the regions where the structure is completely disintegrated. We assume that these regions are filled with pulverized material which is densely packed. For this reason, the region $\Omega\backslash Z$ is able to resist compression as good as the undamaged region Z. Since we only allow for infinitesimally small strains we may expect that the body Ω keeps its outward appearance. We further assume that the concrete structure contains a reinforcement, which ensures that the body Ω can react on tension even in pulverized regions $\Omega\backslash Z$, but no longer as good as the sound material in Z. All these properties are featured by the stored energy density of the form

$$W_{\mathrm{B}}(e,z) := \mu(z+\rho)|e|^2 + \tfrac{\lambda}{2}\big(|(\mathrm{tr}\, e)^-|^2 + (z+\alpha)|(\mathrm{tr}\, e)^+|^2\big) , \tag{50}$$

where $\alpha \in (0,1)$ is constant and μ, $\lambda > 0$ are the Lamé constants. Moreover, also $\rho \in (0,1]$ is constant and clearly, the assumption $\rho > 0$ preserves the coercivity of W_{B} with respect to e. Since the volumetric part of the strain tensor is under control by the term $\frac{\lambda}{2}\big(|(\mathrm{tr}\, e)^-|^2 + (z+\alpha)|(\mathrm{tr}\, e)^+|^2\big)$ it particularly ensures that also the deviatoric part is controlled. This means that finite shear stresses can occur in the pulver $\Omega\backslash Z$.

In the setting of reinforced concrete we define the state space $\mathcal{Q}$ as in (7). With $\mathcal{R} : \mathcal{Z} \to [0,\infty]$ from (1) and $\mathcal{Q}_{\mathrm{B}} := \mathcal{U} \times \mathcal{Z}_{\mathrm{B}}$ from above the system $(\mathcal{Q}, \mathcal{E}_{\mathrm{B}}, \mathcal{R})$ is completed by the energy functional $\mathcal{E}_{\mathrm{B}} : [0,T] \times \mathcal{Q} \to \mathbb{R} \cup \{\infty\}$,

$$\mathcal{E}_{\mathrm{B}}(t,u,z) := \begin{cases} \int_\Omega W_{\mathrm{B}}(e(u+g(t)),z)\,\mathrm{d}x + \sigma\mathcal{H}^{d-1}(J_z) & \text{if } (u,z) \in \mathcal{Q}_{\mathrm{B}} , \\ \infty & \text{otherwise.} \end{cases} \tag{51}$$

Again, the rate-independent damage process is driven by slow time-dependent external loadings induced by time-dependent Dirichlet conditions, which are modeled by the given displacement $g : [0,T] \to H^1(\Omega,\mathbb{R}^d)$, and $\sigma > 0$.

The works [FM93, FG06, GL09] consider brittle damage without any regularization for the damage variable. In these works the density is of the form $\widetilde{W}_{\mathrm{B}}(e,z) = ze:\mathbb{A}:e + (1-z)e:\mathbb{B}:e$, where $\mathbb{A}, \mathbb{B} \in \mathbb{R}^{(d\times d)\times(d\times d)}$ are symmetric and positive definite with constants $c_1, c_2 > 0$ such that $c_1|e|^2 \le e:\mathbb{B}:e \le e:\mathbb{A}:e \le c_2|e|^2$ for all $e \in \mathbb{R}^{d\times d}$. Thus, W_{B} from (50) can be regarded as a special case of $\widetilde{W}_{\mathrm{B}}$. In [FM93, FG06, GL09], minimizing energy plus dissipation in the first time-step means minimizing $\int_\Omega \widetilde{W}_{\mathrm{B}}(e(u),z)+\varrho(1-z)\,\mathrm{d}x$ in $H^1(\Omega,\mathbb{R}^d)\times L^\infty(\Omega)$. Because of the absence of a damage gradient one can immediately eliminate z by performing the minimization of the functional $\int_\Omega \widehat{W}_{\mathrm{B}}(e(u))\,\mathrm{d}x$ in $H^1(\Omega,\mathbb{R}^d)$, where

$\widehat{W}_{\mathrm{B}}(e) = \min\{e : \mathbb{A} : e,\, e : \mathbb{B} : e + \varrho\}$. This density is nonconvex and in order to guarantee the existence of minimizers a relaxation using homogenization tools is required. However, for the brittle damage problem (51) regularized with (49) one cannot remove z from the minimization as easily. For all $z \in \mathcal{Z}_{\mathrm{B}}$ one rather considers the reduced energy functional $\mathcal{E}_R(t,z) = \min_{u\in\mathcal{U}} \mathcal{E}_{\mathrm{B}}(t,u,z)$ and then minimizes $\mathcal{E}_R(t,z) + \mathcal{R}(1-z)$ in $\mathcal{Z}_{\mathrm{B}}$ (at the first time-step). Since for every $z \in [0,1]$ fixed the density $W_{\mathrm{B}}(\cdot,z)$ is convex with respect to the strains the corresponding $\mathcal{E}_R(t,z)$ exists for all $z \in \mathcal{Z}_{\mathrm{B}}$. In order to make sure that also a minimizer of $\mathcal{E}_R(t,\cdot)$ exists, we now discuss the lower semicontinuity and compactness properties of the regularization (49) in $\mathcal{Z}_{\mathrm{B}}$.

Remark 1 (Compactness of $\mathcal{Z}_{\mathrm{B}}$, cf. [AFP05, Chap. 4]) *The distributional gradient* $\mathrm{D}z$ *of any function* $z \in \mathrm{BV}(\Omega)$ *can be uniquely divided into three parts:*

$$\mathrm{D}z = \mathrm{D}^a z + \mathrm{D}^j z + \mathrm{D}^c z\,. \tag{52}$$

Here, $\mathrm{D}^a z$ *denotes the part which is absolutely continuous with respect to the measure* $\mathcal{L}^d$ *and* $(\mathrm{D}^j z + \mathrm{D}^c z)$ *is singular with respect to* $\mathcal{L}^d$*. Moreover,* $\mathrm{D}^j z$ *stands for the jump part and* $\mathrm{D}^c z$ *for the Cantor part. We say that* z *is a special function with bounded variation, i.e.* $z \in \mathrm{SBV}(\Omega)$*, if* $\mathrm{D}^c z = 0$*. The set* $\mathrm{SBV}(\Omega)$ *is an algebraically closed subspace of* $\mathrm{BV}(\Omega)$ *[AFP05, p. 213, Cor. 4.3]. In particular, for any* $z \in \mathrm{SBV}(\Omega)$ *the derivative in* (52) *takes a special structure since it can be recovered from the approximate differential* ∇z*, the approximate one-sided limits* (z^+, z^-) *and the normal* ν_z *to the jump set* J_z*, i.e.*

$$\forall z \in \mathrm{SBV}(\Omega): \quad \mathrm{D}z = \nabla z \mathcal{L}^d + (z^+ - z^-) \otimes \nu_z \mathcal{H}^{d-1} \lfloor J_z\,. \tag{53}$$

According to [AFP05, p. 216, Th. 4.7] the topological closedness of $\mathrm{SBV}(\Omega)$ *is ensured if the following holds:* Let $\phi : [0,\infty) \to [0,\infty]$, $\theta : (0,\infty) \to (0,\infty]$ be lower semicontinuous increasing functions and assume that

$$\lim_{t\to\infty} \frac{\phi(t)}{t} = \infty \quad \text{and} \quad \lim_{t\to 0} \frac{\theta(t)}{t} = \infty\,. \tag{54}$$

Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let $(z_k)_{k\in\mathbb{N}} \subset \mathrm{SBV}(\Omega)$ such that

$$\sup_{k\in\mathbb{N}} \left\{ \int_\Omega \phi(|\nabla z_k|)\,\mathrm{d}x + \int_{J_{z_k}} \theta(|z_k^+ - z_k^-|)\,\mathrm{d}\mathcal{H}^{d-1} \right\} < \infty\,. \tag{55}$$

If $z_k \overset{*}{\rightharpoonup} z$ in $\mathrm{BV}(\Omega)$, then $z \in \mathrm{SBV}(\Omega)$, in particular, $\nabla z_k \rightharpoonup \nabla z$ in $L^1(\Omega)^d$ and $\mathrm{D}^j z_k \overset{*}{\rightharpoonup} \mathrm{D}^j z$ in Ω. Moreover, we have lower semicontinuity of the functionals, i.e.

$$\int_\Omega \phi(|\nabla z|)\,\mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega \phi(|\nabla z_k|)\,\mathrm{d}x \quad \text{if } \phi \text{ is convex}, \tag{56}$$

$$\int_{J_z} \theta(|z^+ - z^-|)\,\mathrm{d}\mathcal{H}^{d-1} \le \liminf_{k\to\infty} \int_{J_{z_k}} \theta(|z_k^+ - z_k^-|)\,\mathrm{d}\mathcal{H}^{d-1} \quad \text{if } \theta \text{ is concave}. \tag{57}$$

The space $\mathrm{SBV}(\Omega)$ is compact with respect to the weak* topology, if (55) holds together with the additional equiboundedness of $\|z_k\|_\infty$, i.e., if $(z_k)_{k\in\mathbb{N}} \subset \mathrm{SBV}(\Omega)$ satisfies (55) and $\|z_k\|_\infty < c$, then there is a subsequence $z_k \overset{*}{\rightharpoonup} z$ in $\mathrm{BV}(\Omega)$ and $z \in \mathrm{SBV}(\Omega)$ [AFP05, p. 216, Th. 4.8].

The set $\mathcal{Z}_{\mathrm{B}}$ from (48)*, which consists of the indicator functions I_Z of all the sets Z with finite perimeter in Ω is a subset of* $\mathrm{SBV}(\Omega)$ *having the property* $\mathrm{D}I_Z = (I_Z^+ - I_Z^-)\otimes\nu_{I_Z}\mathcal{H}^{d-1}(J_{I_Z})$*, where $I_Z^+, I_Z^- \in \{0,1\}$. Hence, the function θ from above can be any power law $\theta(t)=t^p$ with $p\in(0,1)$ to ensure* (54) *and the concavity from* (57)*. Thus, for any $I_Z\in\mathcal{Z}_{\mathrm{B}}$ we obtain*

$$\int_{J_{I_Z}} \theta(|I_Z^+ - I_Z^-|)\,\mathrm{d}\mathcal{H}^{d-1} = \mathcal{H}^{d-1}(J_{I_Z}) = P(Z,\Omega)\,.$$

Consider $(I_{Z_k})_{k\in\mathbb{N}}\subset\mathcal{Z}_{\mathrm{B}}$ with $\|I_{Z_k}\|_\infty + P(Z_k,\Omega)\le c$. Then the compactness theorem for piecewise constant functions [AFP05, p. 234, Th. 4.25] guarantees the existence of a subsequence that converges in measure to a piecewise constant function z. Moreover, the lower semicontinuity of the Hausdorff-measure ensures that $\mathcal{H}^{d-1}(J_z)\le\liminf_{k\to\infty}\mathcal{H}^{d-1}(J_{I_{Z_k}})\le c$. Since a sequence that converges in measure contains a subsequence that converges $\mathcal{L}^d$-a.e. we conclude that also $z\in\mathcal{Z}_{\mathrm{B}}$.

In order to address the main issue in the proof of energetic solutions, it should be mentioned that the recovery sequence for $\hat z\in\mathcal{Z}_{\mathrm{B}}$ can be adopted from Section 2.2. Now, one may consider $\hat z_k := \hat z I_{A_k} + z_k I_{B_k} + 0\cdot I_{C_k}$ with $A_k = [0\le \hat z - \delta_k \le z_k]$, $B_k = [0\le z_k < \hat z - \delta_k]$, $C_k = \Omega\backslash(A_k\cup B_k)$ and $\delta_k\to 0$ determined as in Section 2.2. This is due to the fact that $\hat z$ and $\hat z_k$ take the values 0 and 1 only, so that for $\delta_k<1$ the property $\hat z(x)-\delta_k\le z_k(x)$ implies $\hat z(x)\le z_k(x)$ for $\mathcal{L}^d$-a.e. $x\in A_k$ and this can be transferred to the relations for the traces by Cor. 2.

The distributional gradient $\mathcal{H}^{d-1}(J_z)$ may be disadvantageous for numerical computations. Therefore, we would like to approximate it by integral terms via Γ-convergence. Following the ideas of [MM77, Mod87] which originate in modeling of phase transitions, this can be achieved by a term of Modica-Mortola type

$$\mathcal{M}_k(z) := \begin{cases} \int_\Omega \left(k^2 z^2 (1-z)^2 + \frac{1}{k^2}|\nabla z|^2\right)\mathrm{d}x & \text{if } z\in H^1(\Omega,[0,1])\,,\\ \infty & \text{otherwise,}\end{cases} \tag{58}$$

where $H^1(\Omega,[0,1])$ denotes the set of $H^1(\Omega)$-functions with values in the interval $[0,1]$. A detailed proof for the Γ-convergence of $\mathcal{M}_k(z_k)$ to the limit $\sigma\mathcal{H}^{d-1}(J_z)$ with $\sigma := 2\int_0^1 z(1-z)\,\mathrm{d}z$, can be found e.g. in [Alb98]. Intuitively, it seems to be clear that this ansatz also works for the brittle damage model. The only difficulty is given by the unidirectionality of $\mathcal{R}$. Hence, to prove the MRS-condition, the recovery sequence $(\hat z_k)_{k\in\mathbb{N}}$ given in [Alb98] has to be adjusted suitably.

A Modica-Mortola term in the context of damage can also be found in [Gia05]. There, as a part of the Ambrosio-Tortorelli model for volume damage it was used to approximate the Francfort-Marigo model for Griffith cracks [BFM08]. Within this limit passage the (volume) damage variable turns into the $d-1$-dimensional crack set, i.e. into the jump set of the limit displacement. However, here we want to use a functional of Modica-Mortola type to approximate a model for brittle volume damage by a more regular model for volume damage.

3.1 Approximation of $(\mathcal{Q},\mathcal{E}_{\mathrm{B}},\mathcal{R})$ by a Modica-Mortola term

In this section we show that the system $(\mathcal{Q},\mathcal{E}_{\mathrm{B}},\mathcal{R})$ given by (7), (51) and (1) with $\rho>0$ in (50) can be approximated by systems $(\mathcal{Q},\mathcal{E}_k,\mathcal{R})_{k\in\mathbb{N}}$ in the sense of Γ-convergence of rate-independent systems developed in [MRS08]. In this context, for all $k \in \mathbb{N}$ the approximating energy functionals $\mathcal{E}_k : [0,T] \to \mathcal{Q}$ are given by

$$\mathcal{E}_k(t,u,z) := \mathcal{I}_{\mathrm{B}}(t,g_k,u,z) + \mathcal{M}_k(z) \qquad \text{with } \mathcal{M}_k(z) \text{ from (58) and} \tag{59}$$

$$\mathcal{I}_{\mathrm{B}}(t,g_k,u,z) := \int_\Omega W_{\mathrm{B}}(e(u{+}g_k(t)),z)\,\mathrm{d}x \quad \text{with } W_{\mathrm{B}} \text{ from (50).} \tag{60}$$

For the given data we assume $(g_k)_{k\in\mathbb{N}} \subset C^1([0,T],H^1(\Omega,\mathbb{R}^d))$ and

$$\exists\, c_g > 0\ \forall k \in \mathbb{N}: \quad \|g_k\|^2_{C^1([0,T],H^1(\Omega,\mathbb{R}^d))} \le c_g\,. \tag{61}$$

For every $k \in \mathbb{N}$ fixed the rate-independent systems $(\mathcal{Q},\mathcal{E}_k,\mathcal{R})$ fit into the framework discussed in [TM10, Sect. 5.2]. Hence, we may state the existence of energetic solutions for $(\mathcal{Q},\mathcal{E}_k,\mathcal{R})$ as a direct consequence of [TM10, Th. 3.1].

Lemma 3.1 (Existence of energetic solutions for $(\mathcal{Q},\mathcal{E}_k,\mathcal{R})$) *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded Lipschitz domain with a Dirichlet boundary $\Gamma_D \neq \emptyset$. For all $k \in \mathbb{N}$ let the system $(\mathcal{Q},\mathcal{E}_k,\mathcal{R})$ be given by* (7)*,* (59) *and* (1) *with $\rho > 0$ in* (50)*. Let* (61) *hold true. Assume that the initial data $(u_k(0),z_k(0))$ satisfy* (2(S)) *for $\mathcal{E}_k$ and $\mathcal{R}$ at time $t = 0$. Then, for all $k \in \mathbb{N}$ there exists an energetic solution $(u_k,z_k) : [0,T] \to \mathcal{Q}$ for the system $(\mathcal{Q},\mathcal{E}_k,\mathcal{R})$ and the initial datum $(u_k(0),z_k(0))$.*

Our aim is to show that energetic solutions of the systems $(\mathcal{Q},\mathcal{E}_k,\mathcal{R})$ converge to an energetic solution of the brittle damage system $(\mathcal{Q},\mathcal{E}_{\mathrm{B}},\mathcal{R})$, where the convergence of sequences $(u_k,z_k) \xrightarrow{\mathcal{T}} (u,z)$ is to be understood in the sense of (9).

Theorem 3.2 (Modica-Mortola approximation of $(\mathcal{Q},\mathcal{E}_{\mathrm{B}},\mathcal{R})$) *Let the assumptions of Lemma 3.1 hold. For all $k \in \mathbb{N}$ let $(u_k,z_k) : [0,T] \to \mathcal{Q}$ be an energetic solution to the system $(\mathcal{Q},\mathcal{E}_k,\mathcal{R})$ given by* (7)*,* (59) *and* (1)*. If the initial data satisfy $(u_k(0),z_k(0)) \xrightarrow{\mathcal{T}} (u(0),z(0))$ and $\mathcal{E}_k(0,u_k(0),z_k(0)) \to \mathcal{E}_{\mathrm{B}}(0,u(0),z(0))$ then there is a subsequence $(u_k(t),z_k(t)) \xrightarrow{\mathcal{T}} (u(t),z(t))$ for all $t \in [0,T]$ and $(u,z) : [0,T] \to \mathcal{Q}$ is an energetic solution of $(\mathcal{Q},\mathcal{E}_{\mathrm{B}},\mathcal{R})$.*

3.2 Proof of Convergence Theorem 3.2

In the following we show the existence of a subsequence of energetic solutions of $(\mathcal{Q},\mathcal{E}_k,\mathcal{R})_{k\in\mathbb{N}}$ which converges in the topology $\mathcal{T}$ for all $t \in [0,T]$ to an energetic solution of the brittle damage system $(\mathcal{Q},\mathcal{E}_{\mathrm{B}},\mathcal{R})$. This is done following the ideas of [MRS08, Th. 3.1]. To obtain this converging subsequence, it is necessary that the energies are uniformly bounded and that sublevels of the energies are compact in $\mathcal{T}$, which is verified in Section 3.2.1 and particularly in Corollary 3 below. The convergence of the sequence pointwise for all $t \in [0,T]$ can be obtained following the ideas of [MM05, Th. 3.2]. The proof of the energy balance for the limit system further requires that the Γ-lim inf-inequality holds, which is established in Proposition 3 below. Additionally, the partial time derivatives must converge pointwise for all $t \in [0,T]$, i.e. $\partial_t\mathcal{E}_k(t_k,u_k(t),z_k(t)) \to \partial_t\mathcal{E}_{\mathrm{B}}(t,u_k(t),z_k(t))$, where $\partial_t\mathcal{E}_k$ and $\partial_t\mathcal{E}_{\mathrm{B}}$ have the form (10). As in the proof of Theorem 2.1, the above convergence can be deduced from the properties (P4) and (P5) of W_{B}, see e.g. [MRT10] for details. With this convergence a lower energy estimate can be established, see [MRS08, Th. 3.1]. The respective upper energy estimate can be obtained following the ideas of [MRS08, Prop. 2.4], so that the energy balance (2(E)) for (u,z) and $(\mathcal{Q},\mathcal{E}_{\mathrm{B}},\mathcal{R})$ is gained. The stability of (u,z) and $(\mathcal{Q},\mathcal{E}_{\mathrm{B}},\mathcal{R})$ is deduced with the aid of a MRS in Lemma 3.5 in Section 3.2.2.

3.2.1 Compactness of energy sublevels and the lower Γ-limit

From the stability inequality (2(S)) one obtains that the energies $\mathcal{E}_k(t,u_k(t),z_k(t))$ of the energetic solutions $(u_k,z_k):[0,T]\to\mathcal{Q}$ are uniformly bounded for all $t\in[0,T]$. This can be seen from testing (2(S)) with the functions $(\hat u_k,\hat z_k)$ with $\hat u_k=0$ and $z_k=0$:

$$\mathcal{E}_k(t,u_k(t),z_k(t))\le\mathcal{E}_k(t,0,0)+\mathcal{R}(0-z_k)\le C\,. \tag{62}$$

Lemma 3.3 (A priori estimates) *Let* (61) *be satisfied and* $\rho>0$ *in* (50). *For all* $k\in\mathbb{N}$ *let the function* $(u_k,z_k):[0,T]\to\mathcal{Q}$ *be an energetic solution of the system* $(\mathcal{Q},\mathcal{E}_k,\mathcal{R})$. *Then, there is a constant* $C:=\mathcal{L}^d(\Omega)(\varrho+c_g(\mu(1+\rho)+\lambda(2+\alpha)/2))$ *such that for all* $t\in[0,T]$ *the following estimates hold:*

$$\mathcal{E}_k(t,u_k(t),z_k(t))\le C\,, \tag{63a}$$

$$\|e(u_k(t))\|^2_{L^2(\Omega,\mathbb{R}^d)}\le 2C/(\mu\rho)+2c_g\,, \tag{63b}$$

$$\int_\Omega z_k^2(1-z_k)^2\,\mathrm{d}x\le C/k^2\,, \tag{63c}$$

$$\int_\Omega|\nabla H(z_k(t))|\,\mathrm{d}x\le C\,,\quad \textit{where } H(z):=2\int_0^z\xi(1-\xi)\,\mathrm{d}\xi\,. \tag{63d}$$

Proof: An energetic solution satisfies stability inequality (2(S)) for all $t\in[0,T]$. Hence, estimate (63a) and the constant C can be obtained uniformly in time by testing (2(S)) with the functions $\hat u_k=0$ and $\hat z_k=0$. With this choice we find that

$$\begin{aligned}W_{\mathrm{B}}(0,0)&\le\mu(1+\rho)|e(g_k(t))|^2+\tfrac{\lambda}{2}|(\operatorname{tr}e(g_k(t)))^-|^2+\tfrac{\lambda}{2}(1+\alpha)|(\operatorname{tr}e(g_k(t)))^+|^2\\&\le(\mu(1+\rho)+\lambda(2+\alpha)/2)|e(g_k(t))|^2.\end{aligned}$$

Moreover it is $\mathcal{R}(z_k(t)-0)\le\varrho\mathcal{L}^d(\Omega)$. Integrating $W_{\mathrm{B}}(0,0)$ over Ω then yields $C:=\mathcal{L}^d(\Omega)(\varrho+c_g(\mu(1+\rho)+\lambda(2+\alpha)/2))$ and establishes estimate (63a). Then, estimate (63c) is an immediate consequence of (63a), since all the terms in $\mathcal{E}_k(t,u_k(t),z_k(t))$ are positive. In order to obtain estimate (63b) we use the following calculation with $a=e(u_k(t))$, $b=e(g_k(t))$ and Young's inequality in the last estimate:

$$|a+b|^2\ge(|a|-|b|)^2=|a|^2-2(\tfrac12|a|)(2|b|)+|b|^2\ge\tfrac12|a|^2-|b|^2\,. \tag{64}$$

Together with (63a) this implies $\|e(u_k(t))\|^2_{L^2}\le\frac{2C}{\mu\rho}+2\|e(g_k(t))\|^2_{L^2}$, i.e. (63b).

It remains us to verify (63d). For $H(z):=2\int_0^z\xi(1-\xi)\,\mathrm{d}\xi$ it is $H'(z)=2z(1-z)$ and $\nabla H(z)=H'(z)\nabla z$. Applying Young's inequality to $\mathcal{M}_k(z_k(t))$ we find that $\nabla H(z_k(t))$ is uniformly bounded in $L^1(\Omega)$, i.e.

$$\begin{aligned}\mathcal{M}_k(z_k(t))&=\int_\Omega\big(k^2(z_k^2(1-z_k)^2+k^{-2}|\nabla z_k|^2\big)\,\mathrm{d}x\\&\ge2\int_\Omega z_k(1-z_k)|\nabla z_k|\,\mathrm{d}x\ge\int_\Omega|\nabla H(z_k(t))|\,\mathrm{d}x\,.\end{aligned}$$

■

The above a-priori estimates are used to deduce the precompactness of unions of energy sublevels.

Proposition 2 (Precompactness of unions of energy sublevels) *Let the assumptions of Lemma 3.3 hold. Let the energy functionals $\mathcal{E}_k$ be given by* (59). *Assume that $t_k \to t$ and $\mathcal{E}_k(t_k, u_k, z_k) \leq E$ for all $k \in \mathbb{N}$. Then there is a subsequence $(u_k, z_k) \xrightarrow{\mathcal{T}} (u, z)$ and $(u, z) \in \mathcal{Q}_{\mathrm{B}}$.*

Proof: Because of $\mathcal{E}_k(t_k, u_k, z_k) \leq E$ the sequence $(u_k, z_k)_{k\in\mathbb{N}}$ satisfies bounds similar to (63). In particular, we have $\|u_k\|_{H^1_0(\Omega,\mathbb{R}^d)} \leq c_{\mathcal{K}}\|e(u_k)\|_{L^2(\Omega,\mathbb{R}^{d\times d})} \leq \widetilde{E}$ by estimate (63b) and Korn's inequality. Since $H^1_0(\Omega, \mathbb{R}^d)$ is a reflexive Banach space, Banach-Alaoglu's theorem states the existence of a subsequence $u_k \rightharpoonup u$ in $H^1_0(\Omega, \mathbb{R}^d)$.

Now, we prove the existence of a subsequence $z_k \overset{*}{\rightharpoonup} z$ in $\mathrm{BV}(\Omega)$. Estimate (63d) implies that the sequence $(H(z_k))_{k\in\mathbb{N}}$ is uniformly bounded in $\mathrm{BV}(\Omega)$. Hence, there is a subsequence $(H(z_k))_{k\in\mathbb{N}}$ converging strongly in $L^1(\Omega)$, i.e. $(H(z_k))_{k\in\mathbb{N}}$ is precompact in $L^1(\Omega)$. Since $|H(\tilde{z}) - H(\hat{z})| = |\int_{\hat{z}}^{\tilde{z}} H'(\xi)\,\mathrm{d}\xi| \leq |\tilde{z} - \hat{z}|$ we obtain that the operator $H : L^1(\Omega) \to L^1(\Omega)$ is continuous. Hence, also $(z_k)_{k\in\mathbb{N}}$ as the preimage of $(H(z_k))_{k\in\mathbb{N}}$ is precompact in $L^1(\Omega)$. Thus, there is a subsequence $z_k \to z$ in $L^1(\Omega)$ and from the lower semicontinuity of the variation with respect to strong L^1-convergence for $(z_k)_{k\in\mathbb{N}} \subset \mathrm{BV}(\Omega)$ we conclude that $z \in \mathrm{BV}(\Omega)$. Moreover, from estimate (63c) we deduce that $z(x) \in \{0, 1\}$ for a.e. $x \in \Omega$, i.e. $z \in \mathcal{Z}_{\mathrm{B}}$. ∎

Proposition 3 (Lower Γ-limit) *Let the assumptions of Lemma 3.3 hold. Let the energy functionals $\mathcal{E}_k$ be given by* (59). *Let $\sigma := (H(1)-H(0))$ with H from* (63d). *Assume that $t_k \to t$ and $(u_k, z_k) \xrightarrow{\mathcal{T}} (u, z)$. Moreover, let $\mathcal{E}_k(t_k, u_k, z_k) \leq E$ for all $k \in \mathbb{N}$. Then*

$$\mathcal{E}_{\mathrm{B}}(t, u, z) \leq \liminf_{k\to\infty} \mathcal{E}_k(t_k, u_k, z_k)\,. \tag{65}$$

Proof: We first show that $\liminf_{k\to\infty} \mathcal{M}_k(z_k) \geq \sigma\mathcal{H}^{d-1}(J_z)$. Since the operator $H : L^1(\Omega) \to L^1(\Omega)$ from (63d) is continuous, as it was shown in the proof of Proposition 2, we have $H(z_k) \to H(z)$ in $L^1(\Omega)$. Moreover, due to the equiboundedness of the energies, estimate (63d) applies, which states that $(H(z_k))_{k\in\mathbb{N}}$ is uniformly bounded in $\mathrm{BV}(\Omega)$. Hence, the lower semicontinuity of the variation yields

$$\liminf_{k\to\infty} \mathcal{M}_k(z_k) \geq \liminf_{k\to\infty} \int_\Omega |\nabla H(z_k)|\mathrm{d}x \geq |\mathrm{D}H(z)|(\Omega)\,. \tag{66}$$

Because of the equiboundedness of the energies Proposition 2 yields $(u, z) \in \mathcal{Q}_{\mathrm{B}}$. In particular, $z(x) \in \{0, 1\}$ and hence $H(z(x)) \in \{H(0), H(1)\}$ for a.e. $x \in \Omega$. Moreover, $|\mathrm{D}z|(\Omega) = |\mathrm{D}^j z|(\Omega) = \mathcal{H}^{d-1}(J_z)$. The chainrule for BV functions composed with Lipschitz-continuous functions [AFP05, p. 188] then yields

$$|\mathrm{D}^j H(z)|(\Omega) = \Big|\frac{H(z^+) - H(z^-)}{z^+ - z^-}\mathrm{D}^j z\Big|(\Omega) = (H(1) - H(0))|\mathrm{D}^j z|(\Omega)\,, \tag{67}$$

where $(H(1) - H(0)) = \sigma$. ∎

As a direct consequence of the precompactness of unions sublevels proved in Proposition 2 and the lower Γ-limit we may conclude their compactness.

Corollary 3 (Compactness of unions of energy sublevels) *Let the assumptions of Proposition 3 hold. Let $t_k \to t$ and $\mathcal{E}_k(t_k, u_k, z_k) \leq E$ for all $k \in \mathbb{N}$. Then there is a subsequence $(u_k, z_k) \xrightarrow{\mathcal{T}} (u, z)$ and $\mathcal{E}_{\mathrm{B}}(t, u, z) \leq E$.*

3.2.2 Closedness of stable sets via MRS

In this section we show that the limit states of sequences which satisfy (2(S)) for the approximating systems $(\mathcal{Q}, \mathcal{E}_k, \mathcal{R})$ are stable for the limit system $(\mathcal{Q}, \mathcal{E}_{\mathrm{B}}, \mathcal{R})$. As usual, this is done by proving the existence of a MRS.

Definition 3.4 (MRS-condition) *Let $t_k \to t$ and $q_k \xrightarrow{\mathcal{T}} q$ for $q_k := (u_k, z_k)$ and $q := (u, z)$. For all $k \in \mathbb{N}$ assume that q_k satisfies* (2(S)) *for $(\mathcal{Q}, \mathcal{E}_k, \mathcal{R})$. For all $\hat{q} := (\hat{u}, \hat{z}) \in \mathcal{Q}$ there is a sequence $(\hat{q}_k)_{k\in\mathbb{N}} \subset \mathcal{Q}$ with $\hat{q}_k := (\hat{u}_k, \hat{z}_k)$ and $\hat{q}_k \xrightarrow{\mathcal{T}} \hat{q}$ so that*

$$\limsup_{k\to\infty} \big(\mathcal{E}_k(t_k, \hat{q}_k) - \mathcal{E}_k(t_k, q_k) + \mathcal{R}(\hat{z}_k - z_k)\big) \le \mathcal{E}_{\mathrm{B}}(t, \hat{q}) - \mathcal{E}_{\mathrm{B}}(t, q) + \mathcal{R}(\hat{z} - z)\,. \tag{68}$$

Clearly, our problem allows it to set $\hat{u}_k := \hat{u}$. Thus, the main difficulty is hidden in the construction of $(\hat{z}_k)_{k\in\mathbb{N}}$. For this, we will of course resort to the ideas applied in [MM77, Mod87, Alb98]. In particular, in [Alb98], the recovery sequence, which enables to show that $\mathcal{M}_k$ Γ-converges to $\sigma\mathcal{H}^{d-1}$, is constructed for a dense set $\mathcal{D}$ of $\mathcal{Z}_{\mathrm{B}}$, only, namely for the indicator functions of polyhedral sets with finite perimeter in Ω, i.e.

$$\mathcal{D} := \big\{ I_Z : \Omega \to \{0,1\} \text{ indicator function of polyhedron } Z \subset \Omega,\ P(Z, \Omega) < \infty \big\}\,.$$

The density of $\mathcal{D}$ in $\mathcal{Z}_{\mathrm{B}}$ is a direct consequence of the fact that any set $\widehat{Z}$ of finite perimeter can be approximated by open, smooth sets $(S_k)_{k\in\mathbb{N}}$ such that $S_k \to \widehat{Z}$ in $\mathcal{L}^d$-measure and $P(S_k, \Omega) \to P(\widehat{Z}, \Omega)$ [AFP05, p. 147, Th. 3.42].

For our problem, the recovery sequence will be nontrivial if $\hat{z} \le z$ a.e. in Ω, i.e. if $\widehat{Z} \subset Z$ with $\widehat{Z} := [\hat{z} = 1]$ and $Z := [z = 1]$. Only in this case we have $\mathcal{R}(\hat{z} - z) < \infty$. As Z refers to a given state, which is supposed to be stable, we cannot simply replace it by a sequence of polyhedra $(D_j)_{j\in\mathbb{N}}$ in (68). Using the triangle inequality $\mathcal{R}(\hat{z} - z) \le \mathcal{R}(\hat{z} - I_{D_j}) + \mathcal{R}(I_{D_j} - z)$ such that the right-hand side is finite, requires that $\widehat{Z} \subset Z \subset D_j$ for all $j \in \mathbb{N}$. Moreover, if $\widehat{Z}$ shall be approximated by polyhedra $\widehat{D}_j$ such that $\mathcal{R}(I_{\widehat{D}_j} - z) < \infty$ necessitates even $\widehat{D}_j \subset \widehat{Z} \subset Z \subset D_j$ for all $j \in \mathbb{N}$. But the following example similar to [AFP05, p. 154, Ex. 3.53] or [Giu84, p. 24, Rem. 1.27] shows that sets of finite perimeter in general cannot be approximated from inside or outside by smooth open sets.

Example 1 (Topological boundary $\neq$ reduced boundary) *Let $Q := (0,1)^2$. The set of points in Q with rational coordinates $Q \cap \mathbb{Q}^2$ is countable and can be arranged in a sequence $(q_j)_{j\in\mathbb{N}}$. For every $j \in \mathbb{N}$ we define the open ball $B(q_j, r_j)$ with radius $r_j := 1/2^{j+2}$ and center in q_j. Then, $\mathcal{L}^2(B(q_j, r_j)) = r_j^2\pi = \pi/2^{2(j+2)}$ and $P(B(q_j, r_j), Q) = \mathcal{L}(\partial B(q_j, r_j)) = 2\pi r_j = \pi/2^{j+1}$. Let $A := \cup_{j\in\mathbb{N}} B(q_j, r_j)$. Then A is an open set and we obtain that $\mathcal{L}^2(A) \le \sum_{j\in\mathbb{N}} \mathcal{L}^2(B(q_j, r_j)) = \pi/12$ and $P(A, Q) \le \sum_{j\in\mathbb{N}} P(B(q_j, r_j), Q) = \pi$. Moreover, since $\mathbb{Q}^2 \subset A$ we note that A is dense in Q. Let now $E := Q \backslash A$. Since $\mathcal{L}^2(Q) = 1$ we find that $\mathcal{L}^2(E) \ge 1 - \pi/12 > 0$ and hence E is nonempty with $E \subset Q \backslash \mathbb{Q}^2$. Moreover, since A is dense in Q we conclude that every point in E is an accumulation point of A and hence $E = \partial A$. This shows that the topological boundary ∂A has a positive L^2-measure. However, since $\mathcal{H}^{d-1}(\mathfrak{F}A) = P(A, Q) < \infty$ we know that the reduced boundary $\mathfrak{F}A$ has finite $\mathcal{H}^{d-1}$-measure, see Theorem 2.10. Hence $\mathcal{H}^d(\mathfrak{F}A) = \mathcal{L}^d(\mathfrak{F}A) = 0$. Therefore we conclude that the topological boundary ∂A consists of the reduced boundary $\mathfrak{F}A$ and* *the measure-theoretic exterior A^0, see Definition 2.5, with $\mathcal{L}^2(A^0) = \mathcal{L}^2(\partial A)$. Since A is open, E is closed. Moreover, due to $E = \partial A$ it is even nowhere dense. Because of $A^0 = E^1$, this shows that the measure-theoretic interior of a set is in general not an open set in topological sense.*

Neither E nor E^1 has a nonempty interior and therefore it cannot be approximated by open sets contained in E such that the perimeters converge. Moreover, due to $\operatorname{cl} A = Q$, we conclude that A cannot be approximated by open sets from the outside with perimeters converging to $P(A, \Omega)$.

Since the polyhedra might not enjoy the properties required in our setting we cannot directly adopt the recovery sequence from [MM77, Mod87, Alb98]. Instead, we consider the sequence of polyhedra $(\widehat{D}_j)_{j\in\mathbb{N}}$ that approximates $\widehat{Z}$. For each element of the sequence we apply the construction of [MM77, Mod87, Alb98], which involves the solution of the optimal profile problem. We choose a diagonal sequence $(\tilde{z}_k)_{k\in\mathbb{N}}$ with the property $\mathcal{M}_k(\tilde{z}_k) \leq \sigma\mathcal{H}^{d-1}(\mathfrak{F}\widehat{Z}) + o(1)$. Finally, we obtain the recovery sequence $(\hat{z}_k)_{k\in\mathbb{N}}$, which is suitable for our purpose, with an ansatz similar to Section 2.2, namely

$$\forall\, k \in \mathbb{N}: \quad \hat{z}_k := \max\left\{0, \min\{\tilde{z}_k - \delta_k, z_k\}\right\}, \tag{69}$$

where $\delta_k \to 0$ has to be adjusted. With this idea we can verify the MRS-condition.

Lemma 3.5 *Let the assumptions of Proposition 3 hold. Then the MRS-condition from Definition 3.4 is satisfied.*

Proof: Let $(t_k, u_k, z_k)_{k\in\mathbb{N}} \subset [0,T] \times \mathcal{Q}$ with $(t_k, u_k, z_k) \overset{[0,T]\times\mathcal{T}}{\to} (t, u, z)$. Choose $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$ such that $\mathcal{E}_{\text{B}}(t, \hat{q}) \leq E$ for some $E \in \mathbb{R}$, otherwise (68) trivially holds. We distinguish between the following two cases:

Case A: Let $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$ be such that there exists a $\mathcal{L}^d$-measurable set $B \subset \Omega$ with $\mathcal{L}^d(B) > 0$ and $\hat{z} > z$ on B. Then $\mathcal{R}(\hat{z} - z) = \infty$ and (68) holds.

Case B: Let $\hat{q} = (\hat{u}, \hat{z}) \in \mathcal{Q}$ so that $\hat{z} \leq z$ a.e. in Ω. Then, $\mathcal{R}(\hat{z} - z) = \int_\Omega \varrho(z - \hat{z})\mathrm{d}x < \infty$. Let $\widehat{Z} := [\hat{z} = 1]$. For $\widehat{Z}$ we find a sequence of polyhedra $(\widehat{D}_j)_{j\in\mathbb{N}}$ such that $\widehat{D}_j \to \widehat{Z}$ in $\mathcal{L}^d$-measure and $P(\widehat{D}_j, \Omega) \to P(\widehat{Z}, \Omega)$. For all $l \in \mathbb{N}$ we choose a polyhedron $\widehat{D}_j$ with the property $\|I_{\widehat{D}_j} - \hat{z}\|_{L^1(\Omega)} + |P(\widehat{D}_j, \Omega) - P(\widehat{Z}, \Omega)| < 1/l$ and label it $\widehat{D}_l$. For each $\widehat{D}_l$ we now apply the classical construction of [Alb98, p. 16] to obtain the sequence $(\tilde{z}_l^k)_{k\in\mathbb{N}}$. This construction uses the solution of the optimal profile problem in order to approximate $I_{\widehat{D}_l}$ near the boundary of $\widehat{D}_l$ by a smooth function. We refer to [Alb98, p. 16] for the detailed construction. This sequence satisfies $\tilde{z}_l^k \to I_{\widehat{Z}_l}$ and $\mathcal{M}_k(\tilde{z}_l^k) \leq \sigma P(\widehat{D}_l, \Omega) + o(1)$ as $k \to \infty$. Hence, we have $\mathcal{M}_k(\tilde{z}_l^k) \leq \sigma P(\widehat{Z}, \Omega) + 1/l + o(1)$. Moreover, by the lower Γ-limit there is a subsequence with $\sigma P(\widehat{Z}, \Omega) - 1/l \leq \sigma P(\widehat{D}_l, \Omega) \leq \mathcal{M}_k(\tilde{z}_l^k)$. Thus, for all $k \in \mathbb{N}$ we can pick $\tilde{z}_k^k$ with $l = k$ and set $\tilde{z}_k := \tilde{z}_k^k$. We find that

$$\tilde{z}_k \to \hat{z} \text{ in } L^1(\Omega) \text{ and } \lim_{k\in\mathbb{N}} \mathcal{M}_k(\tilde{z}_k) = \sigma P(\widehat{Z}, \Omega). \tag{70}$$

Now we can apply the construction (69) to get $\hat{z}_k$. In a first step we will determine $\delta_k \to 0$ such that $\hat{z}_k \to \hat{z}$ in $L^1(\Omega)$. As a direct consequence we then have

$$\mathcal{I}_{\text{B}}(t_k, g_k, \hat{u}, \hat{z}_k) \to \mathcal{I}_{\text{B}}(t, g, \hat{u}, \hat{z}) \quad \text{and also } \mathcal{R}(\hat{z}_k - z_k) \to \mathcal{R}(\hat{z} - z), \tag{71}$$

since $\hat{z}_k \leq z_k$ by construction and $z_k \to z$ in $L^1(\Omega)$. Thus, as a second step, it remains to prove

$$\limsup_{k\to\infty} \big(\mathcal{M}_k(\hat{z}_k) - \mathcal{M}_k(z_k)\big) \leq \sigma P(\widehat{Z}, \Omega) - \sigma P(Z, \Omega)\,. \tag{72}$$

Step 1 ($\hat{z}_k \to \hat{z}$ in $L^1(\Omega)$): As in Section 2.2 we decompose the domain into three subsets, i.e. $\Omega = A_k \cup B_k \cup C_k$ with

$$A_k := [0 \leq \tilde{z}_k - \delta_k < z_k]\,, \quad B_k := [0 \leq z_k \leq \tilde{z}_k - \delta_k]\,, \quad C_k := \Omega \backslash (A_k \cup B_k)\,. \tag{73}$$

We first determine $\delta_k \to 0$ such that $\mathcal{L}^d(B_k) \to 0$. For this, we use that

$$B_k = [z_k \leq \tilde{z}_k - \delta_k] \subset [\delta_k \leq \tilde{z}_k - \hat{z} + z - z_k] \subset [\delta_k \leq |\tilde{z}_k - \hat{z} + z - z_k|]\,, \tag{74}$$

due to $\hat{z} \leq z$. With Markov's inequality we obtain

$$\mathcal{L}^d(B_k) \leq \mathcal{L}^d([\delta_k \leq |\tilde{z}_k - \hat{z} + z - z_k|]) \leq \delta_k^{-1} \|\tilde{z}_k - \hat{z} + z - z_k\|_{L^1(\Omega)} \overset{!}{\to} 0\,. \tag{75}$$

Because both $\tilde{z}_k \to \hat{z}$ and $z_k \to z$ in $L^1(\Omega)$ we see that $\delta_k \to 0$ can be chosen such that the right-hand-side of (75) tends to 0. This is e.g. the case for $\delta_k := \|\tilde{z}_k - \hat{z} + z - z_k\|_{L^1(\Omega)}^{1/2}$.

We now show that also $\mathcal{L}^d(C_k) \to 0$ and prove that $\hat{z}_k \to \hat{z}$ in $L^1(\Omega)$. For this, we use a sequence $\nu_k \to 0$, similar to $\delta_k \to 0$, and we obtain

$$\begin{aligned} C_k = [\tilde{z}_k - \hat{z} + \hat{z} < \delta_k] &= [\hat{z} < \delta_k + \hat{z} - \tilde{z}_k] \cap \big([|\hat{z} - \tilde{z}_k| < \nu_k] \cup [|\hat{z} - \tilde{z}_k| \geq \nu_k]\big) \\ &\subset [\hat{z} < \delta_k + \nu_k] \cup [|\hat{z} - \tilde{z}_k| \geq \nu_k]\,. \end{aligned}$$

Clearly, $[\hat{z} < \delta_k + \nu_k] \to \emptyset$ as $\delta_k + \nu_k \to 0$. Moreover, by the same procedure as in (75), we can determine ν_k such that $\mathcal{L}^d([|\hat{z} - \tilde{z}_k| \geq \nu_k]) \to 0$, since $\tilde{z}_k \to \hat{z}$ in $L^1(\Omega)$. Hence, $\mathcal{L}^d(C_k) \to 0$ as $k \to \infty$. With the above results and $\tilde{z}_k, \hat{z} \in [0,1]$ we find

$$\|\hat{z}_k - \hat{z}\|_{L^1(\Omega)} \leq \|\tilde{z}_k - \hat{z}\|_{L^1(\Omega)} + \delta_k \mathcal{L}^d(\Omega) + \mathcal{L}^d(B_k) + \mathcal{L}^d(C_k) \to 0\,. \tag{76}$$

Step 2 (Proof of (72)): To shorten notation we write $\mathcal{M}_k(z_k, E)$ to indicate that the Modica-Mortola-term (58) is defined by integration over the set E. Using the decomposition $\Omega = A_k \cup B_k \cup C_k$ and the definition of $\hat{z}_k$ we calculate that

$$\mathcal{M}_k(\hat{z}_k) = \mathcal{M}_k(\hat{z}_k, \Omega) = \mathcal{M}_k(\tilde{z}_k - \delta_k, A_k) + \mathcal{M}_k(z_k, B_k) + \mathcal{M}_k(0, C_k)$$

with $\mathcal{M}_k(0, C_k) = 0$. Thus we have

$$\begin{aligned} \limsup_{k\to\infty} \big(\mathcal{M}_k(\hat{z}_k, \Omega) - \mathcal{M}_k(z_k, \Omega)\big) &= \limsup_{k\to\infty} \big(\mathcal{M}_k(\tilde{z}_k - \delta_k, A_k) - \mathcal{M}_k(z_k, A_k \cup C_k)\big) \\ &\leq \lim_{k\to\infty} \mathcal{M}_k(\tilde{z}_k - \delta_k, \Omega) - \liminf_{k\to\infty} \mathcal{M}_k(z_k, A_k) \\ &\leq \lim_{k\to\infty} \big(\mathcal{M}_k(\tilde{z}_k, \Omega) + (\delta_k^2 + 2\delta_k)\mathcal{L}^d(\Omega)\big) - \liminf_{k\to\infty} \mathcal{M}_k(z_k, A_k) \\ &\leq \sigma \mathcal{H}^{d-1}(J_{\hat{z}}) - \sigma \mathcal{H}^{d-1}(z)\,. \end{aligned}$$

Here, the last estimate holds because of $\mathcal{M}_k(\tilde{z}_k, \Omega) \leq \sigma \mathcal{H}^{d-1}(J_{\hat{z}}) + 1/k + o(1)$ by construction. Moreover, $\liminf_{k\to\infty} \mathcal{M}_k(z_k, A_k) \geq \sigma \mathcal{H}^{d-1}(J_z)$ is obtained by repeating the arguments of Section 2.2 starting from (45). That is, to choose a subsequence which realizes the lim sup and which satisfies $\sum_{k\in\mathbb{N}}\mathcal{L}^d(B_k\cup C_k)<\infty$. Then one can introduce the sets $U_n:=\cup_{k=n}^{\infty}(B_k\cup C_k)$, which satisfy $\mathcal{L}^d(U_n)\to 0$ as $n\to\infty$. For all $k\geq n$ it is $(B_k\cup C_k)\subset U_n$ and hence $\Omega\backslash U_n\subset A_k$. These sets $\Omega\backslash U_n$ are used in order to exploit the lower Γ-limit (65) for $(z_k)_{k\in\mathbb{N}}$ on fixed domains, i.e. for all $k\geq n$ with $n\in\mathbb{N}$ fixed it is $\liminf_{k\to\infty}\mathcal{M}_k(z_k,A_k)\geq\liminf_{k\to\infty}\mathcal{M}_k(z_k,\Omega\backslash U_n)\geq\sigma P(Z,\Omega\backslash U_n)$. Then, for $n\to\infty$ it holds $P(Z,\Omega\backslash U_n)\to P(Z,\Omega)$. This finishes the proof of (72) and hence the MRS-condition is verified. ∎

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