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Existence and stability of solutions with periodically moving weak internal layers

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Abstract

We consider the periodic parabolic differential equation $\varepsilon^2\Big(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}\Big) = f(u,x,t,\varepsilon)$ under the assumption that ε is a small positive parameter and that the degenerate equation $f(u,x,t,0) = 0$ has two intersecting solutions. We derive conditions such that there exists an asymptotically stable solution $u_p(x,t,\varepsilon)$ which is T-periodic in t, satisfies no-flux boundary conditions and tends to the stable composed root of the degenerate equation as $\varepsilon \to 0$.

1 Formulation of the problem. Main results

We consider the singularly perturbed parabolic differential equation

$$L_\varepsilon u := \varepsilon^2\Big(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}\Big) = f(u,x,t,\varepsilon) \quad \text{for} \quad (x,t) \in \mathcal{D} \tag{1}$$

with

$$\mathcal{D} := \{(x,t) \in \mathbb{R}^2 : -1 < x < 1, t \in \mathbb{R}\},$$

and

$$\varepsilon \in I_{\varepsilon_1} := \{\varepsilon \in \mathbb{R} : 0 < \varepsilon < \varepsilon_1\}, \quad 0 < \varepsilon_1 \ll 1.$$

We suppose f to be T-periodic in t

$$f(u,x,t+T,\varepsilon) = f(u,x,t,\varepsilon), \quad T > 0 \tag{2}$$

and look for a solution $u(x,t,\varepsilon)$ of equation (1) satisfying the boundary conditions

$$\frac{\partial u}{\partial x}(\pm 1, t, \varepsilon) = 0 \tag{3}$$

and the periodicity condition

$$u(x,t+T,\varepsilon) = u(x,t,\varepsilon). \tag{4}$$

The boundary value problem (1)–(4) has been treated in [3] in the case that the degenerate equation

$$f(u,x,t,0) = 0 \tag{5}$$

which we get from (1) by setting $\varepsilon = 0$, has a root

$$u = \varphi(x,t), \quad (x,t) \in \overline{\mathcal{D}}$$

satisfying the stability condition

$$\frac{\partial f}{\partial u}(\varphi(x,t),x,t,0) > 0 \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}. \tag{6}$$

In that case, for sufficiently small ε, the periodic boundary value problem (1)–(4) has a T-periodic solution $u_p(x,t,\varepsilon)$ with the asymptotic representation

$$u_p(x,t,\varepsilon) = \varphi(x,t) + O(\varepsilon) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}$$

yielding the limit relation

$$\lim_{\varepsilon \to 0} u_p(x,t,\varepsilon) = \varphi(x,t) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}.$$

In this paper we investigate the situation when the degenerate equation (5) has two roots

$$u = \varphi_1(x,t) \quad \text{and} \quad u = \varphi_2(x,t)$$

intersecting along some curve whose projection into the (x,t)-plane is located in $\overline{\mathcal{D}}$. We note that several distinct singularly perturbed problems have been investigated in the last years under the condition that the degenerate equation (5) has intersecting roots. A survey of related results can be found in [2]. They have been derived under the assumption that the intersection of the roots is connected with an exchange of stability in the sense that the stable root, for which the inequality (6) holds, becomes unstable (that is the sign in (6) changes), and vice versa, the unstable root becomes stable. We mention also that this situation occurs in different areas of applications, for example in some problems of chemical kinetics [4].
In what follows we formulate the assumptions under which we investigate the periodic boundary value problem (1)–(4).

(A_1). $f \in C^2(\mathcal{G} \times I_{\varepsilon_1}, \mathbb{R})$, where f is T-periodic in the third variable. Here, $\mathcal{G}$ is defined by

$$\mathcal{G} := \{(u,x,t) \in \mathbb{R}^3 : \ \underline{u}(x,t) \le u \le \overline{u}(x,t), \ (x,t) \in \overline{\mathcal{D}}\},$$

where $\underline{u}$ and $\overline{u}$ are certain given smooth functions mapping $\overline{\mathcal{D}}$ into $\mathbb{R}$, T-periodic in t and satisfy

$$\underline{u}(x,t) < \overline{u}(x,t) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}.$$

For the sequel we represent f in the form

$$f(u,x,t,\varepsilon) = f(u,x,t,0) - \varepsilon f_1(u,x,t) + \varepsilon^2 f_2(u,x,t,\varepsilon). \tag{7}$$

Concerning the function $f(u,x,t,0)$ we suppose

(A_2). The function $f(u,x,t,0)$ can be represented in the form

$$f(u,x,t,0) = h(u,x,t)(u - \varphi_1(x,t))(u - \varphi_2(x,t)), \tag{8}$$

where $h \in C^2(\mathcal{G}, \mathbb{R}), \varphi_1, \varphi_2 \in C^2(\overline{\mathcal{D}}, \mathbb{R})$, all functions are T-periodic in t. There is a positive number m such that

$$h(u,x,t) \geq m > 0 \quad \text{for} \quad (u,x,t) \in \mathcal{G}. \tag{9}$$

Condition (A_2) implies that the degenerate equation (5) has exactly two roots in $\mathcal{G}$. From the hypotheses (A_1) and (A_2) it follows that there is a positive number M such that

$$|h_u(u,x,t)| \leq M \quad \text{for} \quad (u,x,t) \in \mathcal{G}. \tag{10}$$

The next condition describes the intersection of the surfaces $u = \varphi_1(x,t)$ and $u = \varphi_2(x,t)$.

(A_3). There exists a smooth T-periodic function $x_0 : \mathbb{R} \to \mathbb{R}$ with

$$-1 < x_0(t) < 1 \quad \text{for} \quad t \in \mathbb{R} \tag{11}$$

such that

$$\begin{gathered} \varphi_1(x_0(t),t) \equiv \varphi_2(x_0(t),t) \quad \text{for} \quad t \in \mathbb{R}. \\ \varphi_1(x,t) > \varphi_2(x,t) \quad \text{for} \quad -1 \leq x < x_0(t), t \in \mathbb{R}, \\ \varphi_1(x,t) < \varphi_2(x,t) \quad \text{for} \quad x_0(t) < x \leq 1, t \in \mathbb{R}. \end{gathered}$$

We denote by Γ_0 the curve defined by

$$\Gamma_0 := \{(x,t) \in \mathcal{D} : x = x_0(t),\ t \in \mathbb{R}\}.$$

By (11) there is a small positive number ω such that Γ_0 is located in the strip

$$\mathcal{S} := \{(x,t) \in \mathcal{D} : -1 + \omega \leq x \leq 1 - \omega,\ t \in \mathbb{R}\}.$$

By means of the roots φ_1 and φ_2 we construct the following composed roots of equation (5).

$$\check{u}(x,t) = \begin{cases} \varphi_1(x,t) & \text{for} \quad -1 \leq x \leq x_0(t), \quad t \in \mathbb{R}, \\ \varphi_2(x,t) & \text{for} \quad x_0(t) \leq x \leq 1, \quad t \in \mathbb{R}, \end{cases}$$

$$\hat{u}(x,t) = \begin{cases} \varphi_2(x,t) & \text{for} \quad -1 \leq x \leq x_0(t), \quad t \in \mathbb{R}, \\ \varphi_1(x,t) & \text{for} \quad x_0(t) \leq x \leq 1, \quad t \in \mathbb{R}. \end{cases}$$

It is obvious that the functions $\check{u}$ and $\hat{u}$ are continuous but in general not smooth on the curve Γ_0.
From the hypotheses (A_2) and (A_3) we get

$$\check{u}(x,t) > \hat{u}(x,t) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}} \setminus \Gamma_0,$$

$$\check{u}(x,t) \equiv \hat{u}(x,t) \quad \text{for} \quad (x,t) \in \Gamma_0,$$

$$\left.\begin{array}{l} \frac{\partial f}{\partial u}(\check{u}(x,t),x,t,0) > 0 \\ \frac{\partial f}{\partial u}(\hat{u}(x,t),x,t,0) < 0 \end{array}\right\} \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}} \setminus \Gamma_0, \tag{12}$$

$$\left.\begin{array}{l} \frac{\partial f}{\partial u}(\check{u}(x,t),x,t,0) = 0 \\ \frac{\partial f}{\partial u}(\hat{u}(x,t),x,t,0) = 0 \end{array}\right\} \quad \text{for} \quad (x,t) \in \Gamma_0. \tag{13}$$

Inequality (12) yields a justification to call the root $\check{u}$ stable (and to call the root $\hat{u}$ unstable, see (6)). The fact that inequality

$$\frac{\partial f}{\partial u}(\check{u}(x,t),x,t,0) > 0$$

does not hold on the curve Γ_0 is some obstacle to give a unique answer to the question whether there exists a solution $u_p(x,t,\varepsilon)$ to the problem (1)–(4) converging to the composed stable root $\check{u}(x,t)$ in $\overline{\mathcal{D}}$ as ε tends to zero. We will show that the sign of the function $f_1(\check{u}(x,t),x,t)$ (see (7)) on the curve Γ_0 plays a crucial role in answering the posed question. Therefore, we require

(A_4). $\quad f_1(\check{u}(x,t),x,t) > 0 \quad \text{for} \quad (x,t) \in \Gamma_0.$

The main result of this paper is the following one:

Theorem 1.1 *Suppose the hypotheses* $(A_1)-(A_4)$ *hold. Then, for sufficiently small* ε*, the periodic boundary value problem* (1)–(4) *has a solution* u_p *satisfying*

$$\lim_{\varepsilon\to 0} u_p(x,t,\varepsilon)) = \check{u}(x,t) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}, \tag{14}$$

and this solution is asymptotically stable in the sense of Lyapunov.

The existence result (including the limit relation) follows from Theorem 3.1, the stability result is the content of Theorem 3.2.

2 Lower and upper solutions for the periodic boundary value problem (1)–(4)

2.1 Regularization of the degenerate equation

As we already noticed, the solution $\check{u}(x,t)$ of the degenerated equation (5) is in general not smooth on the curve Γ_0. To overcome the difficulties connected with this fact, in the paper [3] a smoothing procedure was applied for singularly perturbed problems in case that the degenerate equation has intersecting solutions. Recently, a new approach has been established (see [1]) which is based on a special regularization of the degenerate equation and permits to derive a more detailed asymptotics. In the frame of this method, the degenerate equation (5) is replaced by the equation

$$f(u,x,t,0) - \varepsilon f_1(u,x,t) = 0 \tag{15}$$

which takes into account also first order terms in ε and where f_1 is defined in (7). Using the representation (8) and exploiting the relation (9), we rewrite equation (15) in the form

$$(u - \varphi_1(x,t))(u - \varphi_2(x,t)) - \varepsilon a(u,x,t)) = 0, \tag{16}$$

where $a(u,x,t) \equiv h^{-1}(u,x,t) f_1(u,x,t)$. According to assumption (A_4) we have

$$a(\check{u}(x,t),x,t) > 0 \quad \text{for} \quad (x,t) \in \Gamma_0 \tag{17}$$

such that for sufficiently small $\varepsilon > 0$ equation (16) has two roots in u which are smooth in $\overline{\mathcal{D}}$. We denote these roots by $u = \varphi(x,t,\varepsilon)$ and $u = \psi(x,t,\varepsilon)$. From (16) we get

$$\begin{aligned}
\varphi(x,t,\varepsilon) = \frac{1}{2}\Big\{ \quad & \varphi_1(x,t) + \varphi_2(x,t) + [(\varphi_1(x,t) - \varphi_2(x,t))^2 + \\
& 4\varepsilon a(\varphi(x,t,\varepsilon),x,t)]^{1/2}\Big\}, \\
\psi(x,t,\varepsilon) = \frac{1}{2}\Big\{ \quad & \varphi_1(x,t) + \varphi_2(x,t) - [(\varphi_1(x,t) - \varphi_2(x,t))^2 + \\
& 4\varepsilon a(\psi(x,t,\varepsilon),x,t)]^{1/2}\Big\}
\end{aligned} \tag{18}$$

which imply the asymptotic expressions

$$\varphi(x,t,\varepsilon) = \check{u}(x,t) + [\varepsilon a(\check{u},x,t)]^{1/2} + O(\varepsilon) \quad \text{for} \quad (x,t) \in \Gamma_0,$$

$$\psi(x,t,\varepsilon) = \hat{u}(x,t) - [\varepsilon a(\hat{u},x,t)]^{1/2} + O(\varepsilon) \quad \text{for} \quad (x,t) \in \Gamma_0,$$

$$\begin{aligned}
\varphi(x,t,\varepsilon) = \quad & \check{u}(x,t) + O(\sqrt{\varepsilon}) \quad \text{for} \quad (x,t) \in \Gamma_{0,\delta}, \\
\psi(x,t,\varepsilon) = \quad & \hat{u}(x,t) + O(\sqrt{\varepsilon}) \quad \text{for} \quad (x,t) \in \Gamma_{0,\delta},
\end{aligned} \tag{19}$$

$$\begin{aligned}\varphi(x,t,\varepsilon) &= \quad \check{u}(x,t)+O(\varepsilon) \quad \text{for} \quad (x,t)\in\overline{\mathcal{D}}\setminus\Gamma_{0,\delta},\\ \psi(x,t,\varepsilon) &= \quad \hat{u}(x,t)+O(\varepsilon) \quad \text{for} \quad (x,t)\in\overline{\mathcal{D}}\setminus\Gamma_{0,\delta},\end{aligned} \tag{20}$$

where $\Gamma_{0,\delta}$ is any small δ-neighborhood of Γ_0 which does not depend on ε.

The procedure to replace the degenerate equation (5) by equation (15) represents a regularization. By means of this procedure we approximate the non-smooth functions $\check{u}$ and $\hat{u}$ by functions φ and ψ, which are smooth.

2.2 Auxiliary estimates

In the sequel we need estimates of some derivatives of the function $\varphi(x,t,\varepsilon)$. Straightforward but cumbersome calculations show that the first derivatives φ_x and φ_t are uniformly bounded with respect to ε in $\overline{\mathcal{D}}$, that is, there are positive constants c_1 and c_2 such that

$$|\varphi_x(x,t,\varepsilon)|\le c_1, |\varphi_t(x,t,\varepsilon)|\le c_2 \text{ for } (x,t)\in\overline{\mathcal{D}},\ \varepsilon\in I_{\varepsilon_1}. \tag{21}$$

For the derivative φ_{xx} we get

$$\varphi_{xx}(x,t,\varepsilon)=\frac{2a(\varphi(x,t),x,t)(\varphi_{1x}-\varphi_{2x})^2\varepsilon}{\Big[(\varphi_1-\varphi_2)^2+4a(\varphi(x,t),x,t)\varepsilon\Big]^{3/2}}+O(1).$$

From this representation we obtain the estimates

$$|\varphi_{xx}(x,t,\varepsilon)|\le\frac{c}{\sqrt{\varepsilon}} \quad \text{for} \quad (x,t)\in\Gamma_{0,\delta}, \tag{22}$$

$$|\varphi_{xx}(x,t,\varepsilon)|\le c \quad \text{for} \quad (x,t)\in\overline{\mathcal{D}}\setminus\Gamma_{0,\delta}, \tag{23}$$

where c is some positive constant independent of ε.

We need also an estimate for $f_u(\varphi,x,t,0)$. From (8) we get

$$f_u(\varphi,x,t,0)=h_u(\varphi,x,t)(\varphi-\varphi_1)(\varphi-\varphi_2)+h(\varphi,x,t)(2\varphi-\varphi_1-\varphi_2).$$

Since $\varphi(x,t,\varepsilon)$ is a root of equation (16) we obtain from (16) the relation

$$(\varphi-\varphi_1)(\varphi-\varphi_2)=\varepsilon a(\varphi,x,t)=O(\varepsilon),$$

and from (18) we get

$$2\varphi-\varphi_1-\varphi_2=\Big[(\varphi_1-\varphi_2)^2+4\varepsilon a(\varphi,x,t)\Big]^{1/2}.$$

Taking into account (9) and (10) we have

$$f_u(\varphi, x, t, 0) \geq m\Big[(\varphi_1 - \varphi_2)^2 + 4\varepsilon a(\varphi, x, t)\Big]^{1/2} + O(\varepsilon). \tag{24}$$

In a sufficiently small δ-neighborhood $\Gamma_{0,\delta}$ of the curve Γ_0 it holds by (17)

$$a(\varphi, x, t) \geq a_\delta > 0 \quad \text{for} \quad (x,t) \in \Gamma_{0,\delta},$$

but outside this neighborhood we have

$$|\varphi_1 - \varphi_2| \geq 2c_\delta > 0.$$

Here, a_δ and c_δ are some positive numbers, depending on δ but not on ε. Thus, for sufficiently small ε, we get from (24)

$$f_u(\varphi, x, t, 0) \quad \geq 2m(a_\delta \varepsilon)^{1/2} + O(\varepsilon) \geq m\sqrt{a_\delta}\sqrt{\varepsilon} \quad \text{for} \quad (x,t) \in \Gamma_{0,\delta}, \tag{25}$$

$$f_u(\varphi, x, t, 0) \geq 2mc_\delta + O(\varepsilon) \geq mc_\delta \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}} \setminus \Gamma_{0,\delta}. \tag{26}$$

2.3 Definition of lower and upper solution

The proofs of our results are based on the method of differential inequalities. For this reason we will construct for the problem (1)–(4) lower and upper solutions. We recall their definitions.

Definition 2.1 *Let $\underline{U}(x,t,\varepsilon)$ and $\overline{U}(x,t,\varepsilon)$ be functions continuously mapping $\overline{\mathcal{D}} \times I_{\varepsilon_1}$ into R, twice continuously differentiable in x and continuously differentiable in t, which are T-periodic in t. The functions $\underline{U}$ and $\overline{U}$ are called ordered lower and upper solutions of the periodic boundary value problem (1)–(4), respectively, if they satisfy the inequalities*

$$\underline{U}(t,x,\varepsilon) \leq \overline{U}(x,t,\varepsilon) \quad \textit{for} \quad (x,t) \in \overline{\mathcal{D}}, \tag{27}$$

$$L_\varepsilon \underline{U} - f(\underline{U}, x, t, \varepsilon) \geq 0 \geq L_\varepsilon \overline{U} - f(\overline{U}, x, t, \varepsilon) \quad \textit{for} \quad (x,t) \in \overline{\mathcal{D}}, \tag{28}$$

$$\begin{aligned} &\frac{\partial \underline{U}}{\partial x}(-1,t,\varepsilon) \geq 0 \geq \frac{\partial \underline{U}}{\partial x}(1,t,\varepsilon) \quad \textit{for} \quad t \in \mathbb{R}, \\ &\frac{\partial \overline{U}}{\partial x}(-1,t,\varepsilon) \leq 0 \leq \frac{\partial \overline{U}}{\partial x}(1,t,\varepsilon) \quad \textit{for} \quad t \in \mathbb{R}. \end{aligned} \tag{29}$$

It is well-known [5] that the existence of lower and upper solutions to problem (1)–(4) implies the existence of a solution u_p to (1)–(4) satisfying

$$\underline{U}(x,t,\varepsilon) \leq u_p(t,x,\varepsilon) \leq \overline{U}(x,t,\varepsilon) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}}. \tag{30}$$

2.4 Construction of ordered lower and upper solutions

We construct ordered lower and upper solutions to (1)–(4) in the form

$$\begin{aligned} \underline{U}(x,t,\varepsilon) &= \varphi(x,t,\varepsilon) - \varepsilon(\kappa + z(x,\varepsilon)), \\ \overline{U}(x,t,\varepsilon) &= \varphi(x,t,\varepsilon) + \varepsilon(\kappa + z(x,\varepsilon)), \end{aligned} \tag{31}$$

where φ is the root of equation (16) described in (18), and z is a uniformly bounded function defined by

$$z(x,\varepsilon) = \quad \exp\Big\{-\frac{k}{\varepsilon}(x+1)\Big\} + \exp\Big\{\frac{k}{\varepsilon}(x-1)\Big\} \quad \text{for} \quad (x,\varepsilon) \in [-1,1] \times I_{\varepsilon_1}. \tag{32}$$

Here, κ and k are sufficiently large positive constants which will be chosen later. First we will show that for sufficiently large k the functions $\underline{U}$ and $\overline{U}$ satisfy condition (29) in Definition 2.1.
Indeed, we have

$$\frac{\partial \underline{U}}{\partial x}(-1,t,\varepsilon) = \varphi_x(-1,t,\varepsilon) + k\Big[1 - \exp\Big(-\frac{2k}{\varepsilon}\Big)\Big].$$

According to (21) we have $|\varphi_x(-1,t,\varepsilon)| \le c_1$ for $(t,\varepsilon) \in R \times I_{\varepsilon_1}$. Thus, for sufficiently large k it holds

$$\frac{\partial \underline{U}}{\partial x}(-1,t,\varepsilon) \ge 0 \quad \text{for} \quad (t,\varepsilon) \in \mathbb{R} \times I_{\varepsilon_1}.$$

The other inequalities in (29) can be verified analogously for sufficiently large k.
Now we check the conditions in (28) for sufficiently large κ.
By (1), (31), (32) and (7) we have

$$\begin{aligned} &L_\varepsilon \underline{U} - f(\underline{U}(x,t,\varepsilon),x,t,\varepsilon) = \varepsilon^2(\varphi_{xx} - \varphi_t) - \varepsilon^3 z_{xx} \\ &-\Big[f(\varphi,x,t,\varepsilon) - f_u(\varphi,x,t,\varepsilon)(\kappa+z)\varepsilon + O((\kappa+z)^2\varepsilon^2)\Big] \\ &= \varepsilon^2(\varphi_{xx} - \varphi_t) - \varepsilon k^2 z - \Big[f(\varphi,x,t,0) - \varepsilon f_1(\varphi,x,t) \\ &+\varepsilon^2 f_2(\varphi,x,t,\varepsilon) - f_u(\varphi,x,t,0)(\kappa+z)\varepsilon + O((\kappa+z)^2\varepsilon^2)\Big]. \end{aligned} \tag{33}$$

If we take into account the estimate $|\varphi_t| \le c_2$ from (21) and that φ solves (15) we get from (33)

$$\begin{aligned} &L_\varepsilon \underline{U} - f(\underline{U},x,t,\varepsilon) = \varepsilon^2 \varphi_{xx} - \varepsilon k^2 z \\ &+f_u(\varphi,x,t,0)(\kappa+z)\varepsilon + O(\kappa^2\varepsilon^2) + O(\varepsilon^2). \end{aligned} \tag{34}$$

In a sufficiently small δ-neighborhood of the curve Γ_0, the function $z(x,\varepsilon)$ is of order $o(\varepsilon^N)$ for any positive integer N, and for the expressions φ_{xx} and $f_u(\varphi,x,t,0)$ the relation (22) and (25) are valid. Taking into account these relations we obtain from (34)

$$
\begin{aligned}
&L_\varepsilon \underline{U} - f(\underline{U}, x, t, \varepsilon) \geq -c\varepsilon^{3/2} + m\sqrt{a_\delta}\kappa\varepsilon^{3/2} + O(\kappa^2\varepsilon^2) + O(\varepsilon^2) \\
&= (m\sqrt{a_\delta}\kappa - c)\varepsilon^{3/2} + O(\kappa^2\varepsilon^2) + O(\varepsilon^2) \quad \text{for} \quad (x,t) \in \Gamma_{0,\delta}.
\end{aligned} \tag{35}
$$

The first term in the second line of (35) is positive for sufficiently large κ and dominates for sufficiently small ε. Thus, we have for sufficiently large κ and sufficiently small ε

$$
L_\varepsilon \underline{U} - f(\underline{U}, x, t, \varepsilon) > 0 \quad \text{for} \quad (x,t) \in \Gamma_{0,\delta}. \tag{36}
$$

Outside this neighborhood we get from (34) by using the estimates (23) and (26)

$$
\begin{aligned}
&L_\varepsilon \underline{U} - f(\underline{U}, x, t, \varepsilon) \\
&\geq -c\varepsilon^2 - \varepsilon k^2 z + mc_\delta(\kappa + z)\varepsilon + O(\kappa^2\varepsilon^2) + O(\varepsilon^2) \\
&\geq \Big[mc_\delta(\kappa + z) - k^2 z \Big]\varepsilon + O(\kappa^2\varepsilon^2) + O(\varepsilon^2) \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}} \setminus \Gamma_{0,\delta}.
\end{aligned} \tag{37}
$$

The first term in the third line in (37) is positive for sufficiently large κ and is dominant for sufficiently small ε. Thus, we have for sufficiently large κ and sufficiently small ε

$$
L_\varepsilon \underline{U} - f(\underline{U}, x, t, \varepsilon) > 0 \quad \text{for} \quad (x,t) \in \overline{\mathcal{D}} \setminus \Gamma_{0,\delta}. \tag{38}
$$

The inequalities (36) and (38) imply that the conditions for $\underline{U}$ in (28) are fulfilled. By the same manner we can verify the conditions for $\overline{U}$ in (28) for sufficiently large κ and sufficiently small ε. The validity of the inequality in (27) is obvious. Therefore, the functions $\underline{U}$ and $\overline{U}$ defined in (31) are ordered lower and upper solution of the periodic boundary value problem (1)–(4).

3 Existence of a periodic solution and its asymptotic stability

Theorem 3.1 *Suppose the hypotheses* $(A_1) - (A_4)$ *are fulfilled. Then the periodic boundary value problem* (1)–(4) *has a solution* u_p *with the asymptotic representation*

$$
u_p(x,t,\varepsilon) = \varphi(x,t,\varepsilon) + O(\varepsilon) \quad \textit{for} \quad (x,t) \in \overline{\mathcal{D}}, \tag{39}
$$

where φ *is defined in* (18).

Proof. The existence of lower and upper solutions constructed in section 2.4 implies the existence of a solution u_p of the periodic boundary value problem (1)–(4), where u_p satisfies the inequalities (30). These inequalities and the expressions in (31) for $\underline{U}$ and $\overline{U}$ yield immediately the representation (39).

Corollary 3.1 *The solution u_p satisfies the limit relation* (14).

Proof. From (19) and (20) we get

$$\lim_{\varepsilon\to 0}\varphi(x,t,\varepsilon)=\check{u}(x,t)\quad\text{for}\quad (x,t)\in\overline{\mathcal{D}}.$$

This relation and (39) imply the validity of (14).

Theorem 3.2 *Under the assumptions of Theorem* 3.1 *and for sufficiently small ε the solution u_p is asymptotically stable in the sense of Lyapunov.*

Proof. We estimate the derivative $f_u(u,x,t,\varepsilon)$ on the solution u_p. Using the representation (39) we get by (25) and (26)

$$f_u(u_p,x,t,\varepsilon)=f_u(\varphi,x,t,0)+O(\varepsilon)\geq m\sqrt{a_\delta\varepsilon}+O(\varepsilon)\text{ for }(x,t)\in\Gamma_{0,\delta},$$

$$f_u(u_p,x,t,\varepsilon)\geq mc_\delta\quad\text{for}\quad (x,t)\in\overline{\mathcal{D}}.$$

This inequality implies the asymptotic stability of the solution u_p for $t\to+\infty$ (see [5], Lemma 14.2, Remark 23.3).

Remark 3.1 *Since the solution u_p is asymptotically stable, there arises the question for the global region of attraction, that is, for the set of initial functions $u^0(x,\varepsilon)$ such that the solution $u(t,x,\varepsilon)$ of equation (1) satisfying the boundary condition* (3) *and the initial condition*

$$u(t_0,x,\varepsilon)=u^0(x,\varepsilon)\quad\textit{for}\quad x\in[-1,1]$$

exists for $t>t_0$ and satisfies for sufficiently small ε the relation

$$\lim_{t\to\infty}[u(x,t,\varepsilon)-u_p(x,t,\varepsilon)]=0\quad\textit{for}\quad x\in[-1,1].$$

The answer to this question will be given in a forthcoming paper.

4 Example

Consider the equation

$$\varepsilon^2\Big(\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}\Big)=(u+x-\sin t)(u-3x+\sin t)-\varepsilon \tag{1}$$

which is a special case of equation (1). Concerning the representations (7) and (8) we have

$$f_1(u,x,t)\equiv 1,f_2(u,x,t)\equiv 0,$$

$$h(u,x,t)\equiv 1,\varphi_1(x,t)\equiv -x+\sin t,\varphi_2(x,t)\equiv 3x-\sin t.$$

The roots $u = \varphi_1(x,t)$ and $u = \varphi_2(x,t)$ intersect in a curve whose projection into the (x,t)- plane is described by

$$x = x_0(t) \equiv \frac{1}{2}\sin t.$$

The corresponding composed stable root is

$$\check{u}(x,t) = \begin{cases} -x + \sin t & \text{for} \quad -1 \leq x \leq x_0(t), \quad t \in \mathbb{R}, \\ -3x - \sin t & \text{for} \quad x_0(t) \leq x \leq 1, \quad t \in \mathbb{R}. \end{cases}$$

The regularized degenerate equation to (1) has the form

$$(u + x - \sin t)(u - 3x + \sin t) - \varepsilon = 0,$$

and the corresponding smooth root $\varphi(x,t)$ reads

$$\varphi(x,t) \equiv x + [(2x - \sin t)^2 + \varepsilon]^{1/2}.$$

Thus, we can conclude that the assumptions $(A_1) - (A_4)$ are satisfied, and we get from Theorem 3.1 the existence of a solution $u_p(x,t,\varepsilon)$ to equation (1) obeying the conditions (3), (4) and having the asymptotic representation

$$u_p(x,t,\varepsilon) = x + [(2x - \sin t)^2 + \varepsilon]^{1/2} + O(\varepsilon), \quad (x,t) \in \overline{\mathcal{D}}.$$

According to Theorem 3.2 this solution is asymptotically stable.

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