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MARC KESSEBÖHMER AND BERND O. STRATMANN

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HÖLDER-DIFFERENTIABILITY OF GIBBS DISTRIBUTION FUNCTIONS

MARC KESSEBÖHMER AND BERND O. STRATMANN

ABSTRACT. In this paper we give non-trivial applications of the thermodynamic formalism to the theory of distribution functions of Gibbs measures (devil's staircases) supported on limit sets of finitely generated conformal iterated function systems in $\mathbb{R}$. For a large class of these Gibbs states we determine the Hausdorff dimension of the set of points at which the distribution function of these measures is not α-Hölder-differentiable. The obtained results give significant extensions of recent work by Darst, Dekking, Falconer, Li, Morris, and Xiao. In particular, our results clearly show that the results of these authors have their natural home within thermodynamic formalism.

1. INTRODUCTION

In this paper we study the limit set $\mathscr{L}$ of an iterated function system generated by a finite set of conformal contractions $\{f_a : a \in A\}$ in $\mathbb{R}$ satisfying the strong separation condition. It is well known that each suitably chosen potential function ψ on $\mathscr{L}$ gives rise to a Gibbs measure ν_ψ supported on $\mathscr{L}$. For instance, for the geometric potential $\varphi(x) := \log f_a'\left(f_a^{-1}(x)\right)$ for $x \in f_a(\mathscr{L})$, and with δ referring to the Hausdorff dimension of $\mathscr{L}$, we have that the Gibbs measure $\nu_{\delta\varphi}$ is in the same measure class as the δ-dimensional Hausdorff measure on $\mathscr{L}$. In this paper we concentrate on Gibbs measures ν_ψ associated with Hölder-continuous potential functions ψ for which $P(\psi) = 0$ and $\varphi < \psi < 0$. Here, P refers to the usual pressure function associated with $\mathscr{L}$ (see Section 2 for the definition). For potentials of this type, we consider the set Λ_ψ^α of points at which the α-Hölder derivative of the distribution function F_ψ of ν_ψ does not exist in the generalized sense (note, F_ψ is an 'ordinary devil's staircase'). That is, for $\alpha \in \mathbb{R}_+$ we consider the set

$$\Lambda_\psi^\alpha := \{\xi \in \mathscr{L} : \left(D^\alpha F_\psi\right)(\xi) \text{ does neither exist nor is equal to infinity}\},$$

where D^α refers to the α-Hölder derivative defined for functions F on $\mathscr{L}$ by (given that the limit exists)

$$\left(D^\alpha F\right)(\xi) := \lim_{\eta \to \xi} \frac{|F(\xi) - F(\eta)|}{|\xi - \eta|^\alpha}, \quad \text{for } \xi \in \mathbb{R}.$$

We show that for suitable values of α the Hausdorff dimension $\dim_H(\Lambda_\psi^\alpha)$ of Λ_ψ^α can be determined by employing the thermodynamic formalism. The main results of the paper are summarized in the following theorem.

Main Theorem. *Let $\mathscr{L}$ and ψ be given as above. For all $\alpha \in \mathbb{R}_+$ such that $\psi > \alpha\varphi$, we then have that the Hausdorff dimension of Λ_ψ^α is given by*

$$\dim_H(\Lambda_\psi^\alpha) = s, \tag{1}$$

where s is the unique solution of the equation

$$\beta_\alpha(s) + s \cdot \min\{\varphi(\underline{i})/\psi(\underline{i}) : i \in \{0,1\}\} = 0. \tag{2}$$

Here, β_α is determined implicitly by the pressure equation

$$P((t - \alpha\beta_\alpha(t))\varphi + \beta_\alpha(t)\psi) = 0 \, \textit{for } t \in \mathbb{R}, \tag{3}$$

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and the symbol $i = 0$ (1 resp.) refers to the letter in the alphabet used to code the utter left (right resp.) interval in the geometric representation of the iterated function system, and $\underline{i}$ denotes the infinite word which exclusively contains the letter i.

Remarks.

I. Let us remark that our Main Theorem generalizes recent work in [Dar95, DL03, Fal04, LXD02, Li07, Mo02]. In comparison to the approaches of these authors, with the slight exception of [Fal04] who at least employed multifractal analysis in his study of the Ahlfors regular case, in this paper we develop a completely different and much more general approach which gives these results their natural home within the conceptionally wider frame of the thermodynamical formalism. In fact, we combine certain techniques from this formalism (see Section 2 for the details) with certain other techniques which have their origins in metric Diophantine analysis. By the latter we mean those techniques which were derived through generalizations of results by Jarník [Jar29] and Besicovitch [Bes34] on well-approximable irrational numbers to cuspital excursions on hyperbolic manifolds (see e.g. [Str95, HV98, Str99]), and to Julia sets of parabolic rational maps (see e.g. [SU02]).

II. Let us also remark that the results in this paper can be expressed in terms of so called 'ordinary devil's staircases' as follows. For this recall that the distribution function of a non-atomic positive finite Borel measure μ on a compact interval in $\mathbb{R}$ is a non-increasing continuous function which is constant on the complement of $\text{supp}(\mu)$, the support of μ. Such a distribution function is called an *ordinary devil's staircase* if the 1-dimensional Lebesgue measure $\lambda(\text{supp}(\mu))$ of $\text{supp}(\mu)$ vanishes. Obviously, the distribution functions F_ψ which we consider in this paper *are* ordinary devil's staircases. Interesting sets for devil's staircases are the set Δ_0 of points where the staircase has derivative equal to zero, the set Δ_∞ where the derivative is equal to infinity, and the set $\Delta_\sim$ where the derivative does not exist. Clearly, for the type of staircases in this paper we trivially have that λ-almost every point is in Δ_0, and hence $\dim_H(\Delta_0) = 1$ (in fact, this also holds for the slippery devil's staircases below (see e.g. [Bil79, section 31])). Also, combining our Main Theorem and Corollary 2.4 in this paper, we (almost) immediately have $\dim_H(\Delta_\infty) = \dim_H(\mathscr{L})$. Therefore, for the type of ordinary devil's staircases in this paper we have

$$\dim_H(\Delta_\sim) < \dim_H(\Delta_\infty) < \dim_H(\Delta_0) = 1. \tag{4}$$

Here the question arises of how the Hausdorff dimension $\dim_H(\nu_\psi)$ of the measure ν_ψ fits into this picture. In fact, for the Morris self-similar case (see Remark III. below) we found numerically that there are cases in which $\dim_H(\nu_\psi) > \dim_H(\Delta_\sim)$ (cf. Fig. 1) as well as cases where $\dim_H(\nu_\psi) < \dim_H(\Delta_\sim)$ (cf. Fig. 2) . Therefore, there is no hope to include $\dim_H(\nu_\psi)$ into the hierarchy of dimensions in (4) in general.

Note that these results are in slight contrast to our results for a certain slippery devil's staircase in [KS07]. For a *slippery devil's staircase* we have that although the underlying measure is still singular with respect to λ, the support of the measure is equal to an interval. As was shown in [KS07], the measure of maximal entropy m_U for the Farey map U has a distribution function which is a slippery devil's staircase. In fact, this distribution function is equal to Minkowski's Question Mark Function. More precisely, the main results in [KS07] for this particular slippery devil's staircase are, and the reader is asked to compare these with the outcome for ordinary devil's staircases in (4),

$$\dim_H(m_U) < \dim_H(\Lambda_\sim) = \dim_H(\Lambda_\infty) < \dim_H(\Lambda_0) = 1.$$

III. Let us end this introduction with a brief discussion of three special cases of our Main Theorem, namely the case in which ν_ψ is an Ahlfors regular measure, as well as two cases considered by Morris and Li in which ν_ψ is a self-similar measure.

The Ahlfors regular case. Recall that a measure μ is called t-Ahlfors regular if and only if $\mu(B(\xi,r)) \asymp r^t$, for all $0 < r < r_0$ and for all ξ in the support of μ, for some fixed

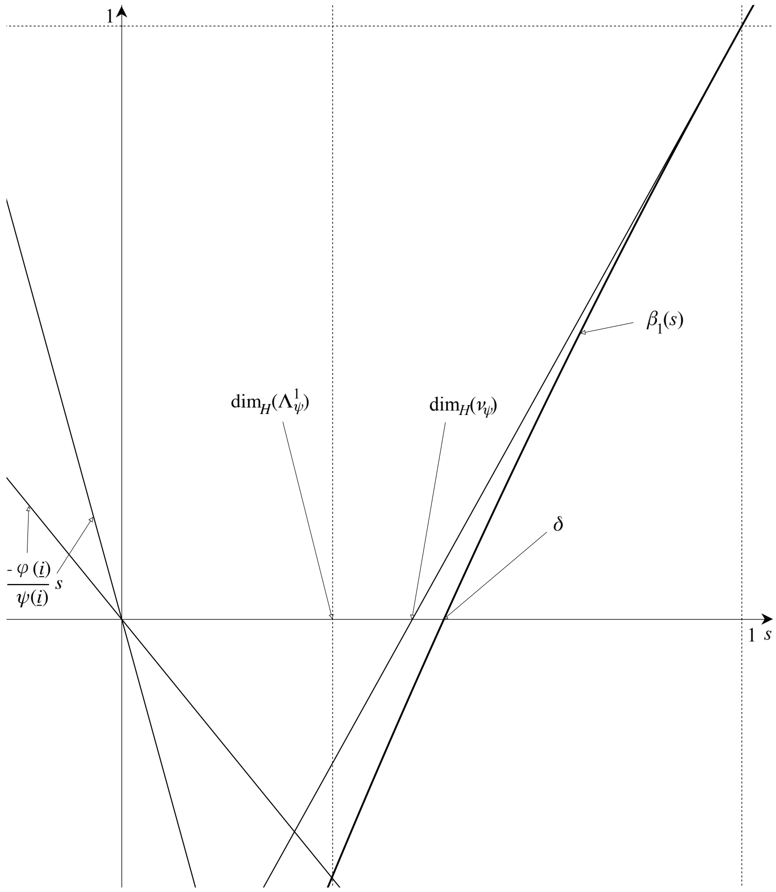

FIGURE 1. The graph of β_α for the Morris case with $a_0 = 0.1$ and $a_1 = 0.5$ and $\alpha = 1$. In this case $\dim_H(\nu_\psi) > \dim_H(\Lambda^1_\psi)$ (cf. Main Theorem and Proposition 2.1).

$r_0, t > 0$. In the situation of the Main Theorem, it is well-known that if the Gibbs measure ν_ψ is t-Ahlfors regular, then t is equal to the Hausdorff dimension δ of $\mathscr{L}$ and $\psi = \delta\varphi$. In this case, the Hausdorff dimension s of Λ^α_ψ can be calculated explicitly for $\alpha > \delta$ as follows. Namely, here (3) implies $P((s + (\delta - \alpha)\beta_\alpha(s))\varphi) = 0$, which immediately gives $s + (\delta - \alpha)\beta_\alpha(s) = \delta$, and hence,

$$\beta_\alpha(s) = \frac{\delta - s}{\delta - \alpha}.$$

Inserting this into (2), we obtain $s + \delta(\delta - s)/(\delta - \alpha) = 0$. Solving the latter for s, one rediscovers Falconer's result [Fal04] on the Hausdorff dimension of $\Lambda^\alpha_{\delta\varphi}$, namely

$$\dim_H(\Lambda^\alpha_{\delta\varphi}) = \frac{\delta^2}{\alpha}, \text{ for all } \alpha > \delta. \tag{5}$$

Let us point out that, as also noted in [Fal04], the equality in (5) remains to be true for $\alpha = \delta$. However, this case requires some additional care, such as for instance the use of ergodicity of the measure $\nu_{\delta\varphi}$ or alternatively some Khintchine-type argument, and hence let us not go into the details here. Also, note that (5) in particular includes the result of Darst [Dar93], who only considered the case $\alpha = 1$ for Cantor sets and showed that in this special linear situation one has

$$\dim_H(\Lambda^1_{\delta\varphi}) = \delta^2.$$

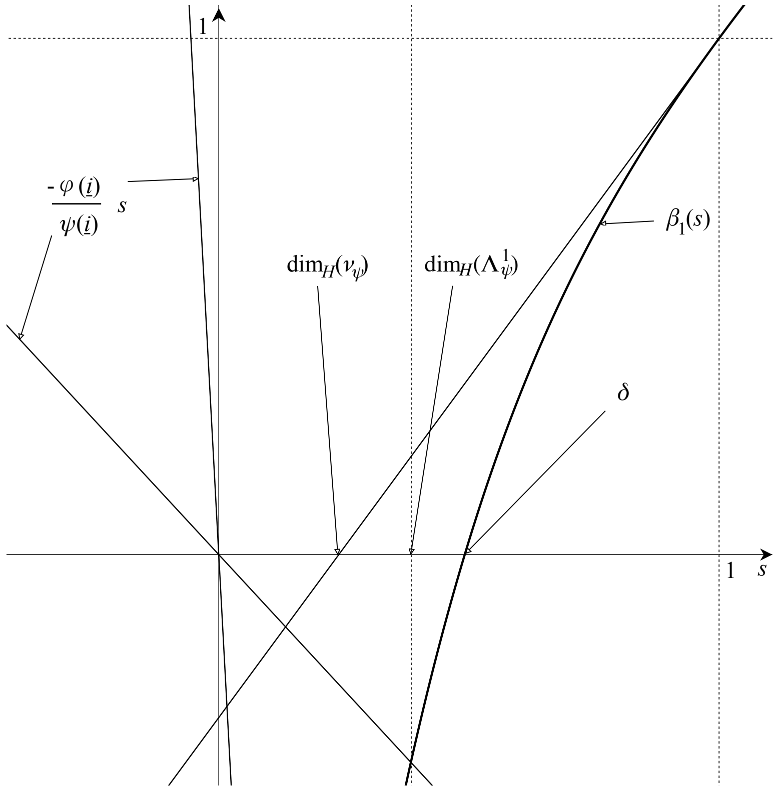

FIGURE 2. The graph of β_α for the Morris case with $a_0 = 0.01$, $a_1 = 0.8$ and $\alpha = 1$. In this case $\dim_H(\nu_\psi) < \dim_H(\Lambda^1_\psi)$ (cf. Main Theorem and Proposition 2.1).

The Morris self-similar case. Here, only the case $\alpha = 1$ has previously been considered in the literature. For instance, in his studies of Cantor sets Morris [Mo02] considered self-similar measures with probabilities $p_1 := c_1/(c_1+c_2)$ and $p_2 := c_2/(c_1+c_2)$, where c_1 and c_2 refer to the contraction rates of the two similarities generating the underlying Cantor set. This Morris scenario is contained as a special case in our Main Theorem. Namely, here the f_a are linear contractions and the potential function ψ is equal to $\varphi - P(\varphi)$. We then have that the Hausdorff dimension s of $\Lambda^1_{\delta\varphi}$ and $\beta_1(s)$ can be calculated explicitly. Indeed, here (3) implies $P(s\varphi - \beta_1(s)P(\varphi)) = 0$, which gives

$$\beta_1(s) = \frac{P(s\varphi)}{P(\varphi)}.$$

Inserting this into (2) gives that s is the unique solution of the pressure equation

$$sP(\varphi) = P(s\varphi)\left(\frac{P(\varphi)}{\min\{\varphi(\underline{i}) : i \in \{0,1\}\}} - 1\right).$$

The Li self-similar case. The self-similar case was investigated also by Li [Li07] in greater generality. Li considered an affine iterated function system $\{f_a : x \mapsto c_a x + d_a \,|\, a \in A\}$ fulfilling the strong separation condition, together with a self-similar measure μ given by a probability vector $(p_1, \ldots, p_{\mathrm{card}(A)})$ for which $p_a > c_a$, for all $a \in A$. In terms of our paper here, we then have $\varphi((x_1x_2\ldots)) = \log c_{x_1}$ and $\psi((x_1x_2\ldots)) = \log p_{x_1}$. In this situation our Main Theorem then immediately gives that the value of $\beta_\alpha(t)$ is uniquely determined by the Hutchinson like formula

$$\sum_{a\in A} p_a^{\beta_\alpha(t)}\, c_a^{t-\alpha\cdot\beta_\alpha(t)} = 1,$$

for each $\alpha > 0$ such that $c_a^\alpha < p_a$ for all $a \in A$.

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2. Preliminaries

2.1. **Thermodynamic formalism for iterated function systems.** Throughout, we will consider the following type of conformal iterated function systems $\mathscr{F}$. For some compact connected set $X \subset \mathbb{R}$ and with $A := \{0, 1, \ldots, d\}$, let $\mathscr{F} = \{f_a : a \in A\}$ be generated by differentiable contractions $f_a : X \to \mathrm{Int}X$ such that the following two conditions hold.

Strong separation condition. $f_a(\mathrm{Int}(X)) \subset \mathrm{Int}(X)$ for all $a \in A$, and $f_a(X) \cap f_b(X) = \emptyset$, for each pair of distinct $a, b \in A$.

Hölder condition. There exists $\varepsilon > 0$ and an open interval $Y \supset X$ such that f_a has a $C^{1+\varepsilon}$−continuation $\widetilde{f_a}$ to Y for which $\widetilde{f_a}(Y) \subset Y$ and $\widetilde{f_a} : Y \to \widetilde{f_a}(Y)$ is a diffeomorphism, for each $a \in A$.

Note that the Hölder condition immediately implies the bounded distortion property. That is, we in particular have that $\mathscr{F}$ has the following property.

Bounded distortion property. For each $\omega \in A^n, n \in \mathbb{N}$ and $\xi, \eta \in X$, we have

$$|f'_\omega(\xi)| \asymp |f'_\omega(\eta)|.$$

Here we have used the notation $f_\omega := f_{x_1} \circ f_{x_2} \circ \ldots \circ f_{x_n}$ for $\omega = x_1 x_2 \ldots x_n \in A^n$. Without loss of generality, we will always assume that the intervals $\{f_a(X) : a \in A\}$ are labeled as follows. If $\xi \in f_0(X)$ and $\eta \in f_a(X)$ for some $a \in A \setminus \{0\}$, then $\xi < \eta$. Likewise, if $\xi \in f_1(X)$ and $\eta \in f_a(X)$ for some $a \in A \setminus \{1\}$, then $\xi > \eta$. In other words, $f_0(X)$ ($f_1(X)$ resp.) is assumed to be the utter left (right resp.) interval in the first iteration level $\{f_a(X) : a \in A\}$ of $\mathscr{F}$.
Recall that the limit set $\mathscr{L}$ of $\mathscr{F}$ is the unique non-empty compact subset of $\mathbb{R}$ which satisfies $\mathscr{L} = \bigcup_{a\in A} f_a(\mathscr{L})$. Equivalently, $\mathscr{L}$ is given by

$$\mathscr{L} := \bigcap_{n\in\mathbb{N}} \bigcup_{\omega\in A^n} f_\omega(X).$$

Clearly, the latter description of $\mathscr{L}$ immediately shows that each element of $\mathscr{L}$ can be coded in a unique way by an infinite word with letters chosen from the alphabet A. That is, there is a bijective coding map $\Phi : A^{\mathbb{N}} \to \mathscr{L}$, which is given by

$$\Phi : (x_1 x_2 \ldots) \mapsto \bigcap_{n\in\mathbb{N}} f_{x_1\ldots x_n}(X).$$

For ease of exposition, we will make no explicit distinction between $(x_1 x_2 \ldots) \in A^{\mathbb{N}}$ and $\xi := \Phi((x_1 x_2 \ldots)) \in \mathscr{L}$. Also, throughout we assume that the reader is familiar with the following basic concepts of the thermodynamic formalism (see e.g. [Bow75], [Den05], [Pes97], [Rue78]), where we use the common notation for cylinder sets $[x_1 \ldots x_n] := \{y = (y_1 y_2 \ldots) \in \mathscr{L} : y_i = x_i, \text{ for all } i \in \{1, \ldots, n\}\}$, as well as the notation for Birkhoff sums $S_n g := \sum_{k=0}^{n-1} g \circ \sigma^k$ with σ referring to the usual left-shift map on $A^{\mathbb{N}}$.

- The canonical geometric potential $\varphi : \mathscr{L} \to \mathbb{R}$ associated with $\mathscr{F}$ is given by
$$\varphi(\xi) := \log f'_{x_1}\left(f_{x_1}^{-1}(\xi)\right), \text{ for all } \xi = (x_1x_2\ldots) \in \mathscr{L}.$$
Note that, since $\mathscr{F}$ satisfies the Hölder condition, φ is Hölder-continuous.
- The pressure function $P : C(\mathscr{L},\mathbb{R}) \to \mathbb{R}$ is given for continuous potential functions $g : \mathscr{L} \to \mathbb{R}$ by
$$P(g) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in A^n} \exp(\sup_{\xi\in[\omega]} S_n g(\xi)).$$
- Throughout, let $\psi \in C(\mathscr{L},\mathbb{R})$ refer to a given Hölder-continuous function for which $\psi < 0$, $P(\psi) = 0$, and $\psi > \alpha\varphi$. Then it is well-known that there is a Gibbs measure ν_ψ associated with ψ such that
$$\nu_\psi([\omega]) \asymp e^{S_n\psi(\xi)}, \text{ for all } \xi \in [\omega], \omega \in A^n, n \in \mathbb{N}.$$
- Also, throughout we let $\chi_\alpha := \psi - \alpha\varphi > 0$, for some $\alpha \in \mathbb{R}$, and then consider the potential function
$$s\varphi + t\chi_\alpha, \text{ for } s,t \in \mathbb{R}.$$
By standard thermodynamic formalism, there then exists a strictly increasing, concave, and real-analytic function $\beta_\alpha : \mathbb{R} \to \mathbb{R}$ such that
$$P(s\varphi + \beta_\alpha(s)\chi_\alpha) = 0, \text{ for all } s \in \mathbb{R}.$$
The Gibbs measure associated with the potential function $s\varphi + \beta_\alpha(s)\chi_\alpha$ will be denoted by μ_s. Note that the measure μ_s satisfies
$$\mu_s([\omega]) \asymp e^{sS_n\varphi(\xi)+\beta_\alpha(s)S_n\chi_\alpha(\xi)}, \text{ for all } \xi \in [\omega], \omega \in A^n, n \in \mathbb{N}.$$

The following proposition in particular shows in which way the values of δ and $\dim_H(\nu_\psi)$ can be obtained from the graph of the function β_α. We refer to Fig. 1 and Fig. 2 for illustrations of this proposition for the special case $\alpha = 1$.

Proposition 2.1. *We have that the unique zero of β_α is at δ and that $\beta_\alpha(\alpha) = 1$. Moreover, $\dim_H(\nu_\psi)$ is the point of intersection of the x–axis with the tangent of the graph of β_α at the point $(\alpha,1)$.*

Proof. Since $P(\delta\varphi + 0\chi_\alpha) = P(\delta\varphi) = 0$, we necessarily have $\beta_\alpha(\delta) = 0$. Also, since $P(\alpha\varphi + \chi_\alpha) = P(\alpha\varphi + \psi - \alpha\varphi) = P(\psi) = 0$, we necessarily have that $\beta_\alpha(\alpha) = 1$. Therefore, by employing well-known identities from the thermodynamic formalism (see e.g. [Den05]), it follows
$$\beta'_\alpha(1) = -\frac{\int \varphi\, d\nu_\psi}{\int \chi_\alpha\, d\nu_\psi} = \frac{1}{\alpha - \int \psi\, d\nu_\psi / \int \varphi\, d\nu_\psi} = \frac{1}{\alpha - \dim_H(\nu_\psi)}.$$
This shows that the tangent $L_{(\alpha,1)}$ of the graph of β_α at $(\alpha,1) \in \mathbb{R}^2$ is given by
$$L_{(\alpha,1)}(x) := \frac{1}{\alpha - \dim_H(\nu_\psi)} x - \frac{\alpha}{\alpha - \dim_H(\nu_\psi)} + 1.$$
By solving the equation $L_{(\alpha,1)}(x) = 0$ for x, the assertion of the proposition follows. □

2.2. **The derivative of the Gibbs distribution function.** As noted in the introduction, let
$$F_\psi : \mathbb{R} \to [0,1], x \mapsto \nu_\psi((-\infty,x]),$$
refer to the distribution function of the Gibbs measure ν_ψ associated with the iterated function system $\mathscr{F}$ and the potential function ψ. We always assume that $\mathscr{F}$ and ψ are chosen as specified in the previous section. Also, define
$$\mathscr{E} := \{(x_1x_2\ldots) \in \mathscr{L} : \text{there exist } i \in \{0,1\}, n \in \mathbb{N} \text{ such that } x_k = i \text{ forall } k \geq n\},$$
and let $\mathscr{L}^* := \mathscr{L} \setminus \mathscr{E}$ refer to the limit set of $\mathscr{F}$ without the countable set of 'end points' whose code eventually has only either 0's or 1's.

Lemma 2.2. *For the upper α-Hölder derivative of F_ψ we have*

$$\limsup_{\eta\to\xi}\frac{|F_\psi(\xi)-F_\psi(\eta)|}{|\xi-\eta|^\alpha}=\infty,\ \textit{for all } \xi\in\mathscr{L}^*.$$

Proof. Let $\xi=(x_1x_2\ldots)\in\mathscr{L}^*$ be given. We then have $\xi\in[x_1\ldots x_n]$, for each $n\in\mathbb{N}$. Also, since ξ is not in $\mathscr{E}$, there exists $(n_m)_{m\in\mathbb{N}}$ such that $x_{n_m+1}\neq 1$, for all $m\in\mathbb{N}$. In this situation we then have $\xi\notin[x_1\ldots x_{n_m}1]$. Moreover, note that $\nu_\psi([x_1\ldots x_{n_m}1])\asymp\nu_\psi([x_1\ldots x_{n_m}])$. Using this and the bounded distortion property, it follows, with η_m referring to the right endpoint of $[x_1\ldots x_{n_m}]$, that is $\eta_m:=(x_1\ldots x_{n_m}\underline{1})\in\mathscr{E}$,

$$\begin{aligned}\frac{|F_\psi(\xi)-F_\psi(\eta_m)|}{|\xi-\eta_m|^\alpha} &\geq \frac{\nu_\psi([x_1\ldots x_{n_m}1])}{\mathrm{diam}([x_1\ldots x_{n_m}])^\alpha}\gg\frac{\nu_\psi([x_1\ldots x_{n_m}])}{\mathrm{diam}([x_1\ldots x_{n_m}])^\alpha}\\ &\gg \frac{\exp(S_{n_m}\psi(\xi))}{\exp(S_{n_m}\alpha\varphi(\xi))}=e^{S_{n_m}\chi_\alpha(\xi)}\geq e^{n_m\inf_{\eta\in\mathscr{L}}\chi_\alpha(\eta)}.\end{aligned}$$

Since $\chi_\alpha>0$, the result follows. □

Definition. Let us say that $\xi=(x_1x_2\ldots)\in\mathscr{L}$ has an *i-block of length k at the n-th level*, for $n,k\in\mathbb{N}$ and $i\in\{0,1\}$, if $x_n,x_{n+k+1}\in A\setminus\{i\}$ and $x_{n+m}=i$, for all $m\in\{1,\ldots,k\}$.

Proposition 2.3. *If for $i\in\{0,1\}$ we have that $\xi=(x_1x_2\ldots)\in\mathscr{L}$ has an i-block of length k at the n-th level, then there exists $\eta\in\mathscr{L}$ such that $|\xi-\eta|\asymp\exp(S_n\varphi(\xi))$, and*

$$\frac{|F_\psi(\xi)-F_\psi(\eta)|}{|\xi-\eta|^\alpha}\asymp \mathrm{e}^{S_n\chi_\alpha(\xi)}\cdot\mathrm{e}^{k\psi(\underline{i})}.$$

Proof. Let $\xi=(x_1x_2\ldots)\in\mathscr{L}$ be given as stated in the lemma. Let us only consider the case $i=1$. The case $i=0$ can be dealt with in a similar way, and this is left to the reader. We then have that there exists $j\in A$ such that the interval J bounded by the points $\eta:=(x_1\ldots x_{n-1}j\underline{0})\in\mathscr{E}$ and $\widehat{\eta}:=(x_1\ldots x_n\underline{1})\in\mathscr{E}$ is a 'gap interval' in the construction of $\mathscr{L}$ such that $J\cap\mathscr{L}^*=\emptyset$ (that is, J denotes the gap interval in the construction of $\mathscr{L}$ separating $[x_1\ldots x_n]$ and its right neighbour in the n-th level). Using the bounded distortion property and the strong separation condition, we then have

$$|\xi-\eta|\asymp\mathrm{diam}(J)\asymp\mathrm{diam}([x_1\ldots x_n])\asymp\mathrm{e}^{S_n\varphi(\xi)}.$$

Moreover, since $\nu_\psi(J)=0$, we have

$$\begin{aligned}|F_\psi(\xi)-F_\psi(\eta)| &= \nu_\psi((\xi,\eta])=\nu_\psi((\xi,\widehat{\eta}])\leq\nu_\psi([x_1\ldots x_{n+k}])\\ &\ll \mathrm{e}^{S_{n+k}\psi(\xi)}\ll\mathrm{e}^{S_n\psi(\xi)}\cdot\mathrm{e}^{k\psi(\underline{1})}.\end{aligned}$$

Finally, by noting that $\xi\notin[x_1\ldots x_{n+k}1]$ and $[x_1\ldots x_{n+k}1]\subset(\xi,\widehat{\eta}]$, we similarly have

$$|F_\psi(\xi)-F_\psi(\eta)|=\nu_\psi((\xi,\widehat{\eta}])\geq\nu_\psi([x_1\ldots x_{n+k}1])\gg\mathrm{e}^{S_{n+k+1}\psi(\xi)}\gg\mathrm{e}^{S_n\psi(\xi)}\cdot\mathrm{e}^{k\psi(\underline{1})}.$$

Combining these observations, it follows

$$\frac{|F_\psi(\xi)-F_\psi(\eta)|}{|\xi-\eta|^\alpha}\asymp\frac{\exp(S_n\psi(\xi)+k\psi(\underline{1}))}{\exp(S_n\alpha\varphi(\xi))}=e^{S_n\chi_\alpha(\xi)}\cdot e^{k\psi(\underline{1})}.$$

□

The following corollary gives a generalization of a classical result of Gilman [Gil32], who showed for Cantor sets that if the derivative of the Cantor function exists in the generalized sense at some point in the Cantor set, then it has to be equal to infinity.

Corollary 2.4. *If $D^\alpha F_\psi$ exists in the generalized sense at $\xi\in\mathscr{L}^*$, then $(D^\alpha F_\psi)(\xi)=\infty$. On the other hand, there* are *(plenty of) points in $\mathscr{L}^*$ at which $D^\alpha F_\psi$ does not exist in the generalized sense.*

Proof. The first assertion is an immediate consequence of Lemma 2.2. For the second assertion, choose strictly increasing sequences $(n_m)_{m\in\mathbb{N}}$ and $(k_m)_{m\in\mathbb{N}}$ such that $n_{m+1} > n_m + k_m$, and let ξ be an element of $\mathscr{L}$ which has an i-block of length k_m at the n_m-th level. Moreover, assume that (n_m) and (k_m) are chosen such that $\exp(S_{n_m}\chi_\alpha(\xi)+k_m\psi(\underline{i})) \ll 1$. Using Lemma 2.3, it follows that there exists $(\eta_m)_{m\in\mathbb{N}}$ such that $\lim_{m\to\infty}\eta_m = \xi$ and $|F_\psi(\xi)-F_\psi(\eta_m)|/|\xi-\eta_m|^\alpha \ll 1$, for all $m\in\mathbb{N}$. From this we immediately deduce that for the lower α-Hölder derivative of F_ψ we have

$$\liminf_{\eta\to\xi}\frac{|F_\psi(\xi)-F_\psi(\eta)|}{|\xi-\eta|^\alpha} < \infty.$$

By combining the latter with Lemma 2.2, it follows that $(D^\alpha F_\psi)(\xi)$ does not exist in the generalized sense. This finishes the proof. □

For later use we also state the following immediate consequence of Proposition 2.3.

Corollary 2.5. *Let $\xi = (x_1x_2\ldots)\in\mathscr{L}$ be given such that for some $n,k\in\mathbb{N}$ and $i\in\{0,1\}$ we have $x_{n+m} = i$, for all $m\in\{1,\ldots,k\}$. Then there exists $\eta\in\mathscr{L}$ such that $|\xi-\eta| \asymp \exp(S_n\varphi(\xi))$ and*

$$\frac{|F_\psi(\xi)-F_\psi(\eta)|}{|\xi-\eta|^\alpha} \ll \mathrm{e}^{S_n\chi_\alpha(\xi)}\cdot\mathrm{e}^{k\psi(\underline{i})}.$$

Proposition 2.6. *We have that $D^\alpha F_\psi$ does not exist in the generalized sense at $\xi\in\mathscr{L}^*$ if and only if there exists $i\in\{0,1\}$ and strictly increasing sequences $(n_m)_{m\in\mathbb{N}}$ and $(k_m)_{m\in\mathbb{N}}$ of positive integers such that ξ has an i-block of length k_m at the n_m-th level for each $m\in\mathbb{N}$, and*

$$e^{S_{n_m}\chi_\alpha(\xi)+k_m\psi(\underline{i})} \ll 1, \textit{ for each } m\in\mathbb{N}.$$

Proof. The 'if-part' follows immediately from combining Lemma 2.2 and Proposition 2.3. For the 'only-if-part', assume by way of contradiction that $\xi = (x_1x_2\ldots)\in\mathscr{L}^*$ is given such that if ξ has a strictly increasing sequences of i-blocks, say of length k_m at the n_m-th level, for some $i\in\{0,1\}$, then

$$\liminf_{m\to\infty} e^{S_{n_m}\chi_\alpha(\xi)+k_m\psi(\underline{i})} = \infty.$$

Let $(\xi_m)_{m\in\mathbb{N}}$ be any sequence in $X\setminus\{\xi\}$ such that $\lim_{m\to\infty}\xi_m = \xi$. The aim is to show that under these assumptions we necessarily have

$$\lim_{m\to\infty}\frac{|F_\psi(\xi)-F_\psi(\xi_m)|}{|\xi-\xi_m|} = \infty.$$

For this, first note that we can assume without loss of generality that $\xi_m\in\mathscr{L}$. Indeed, if $\xi_m\notin\mathscr{L}$, then move ξ_m away from ξ until one first hits $\mathscr{L}$, say at the point $\xi_m'\in\mathscr{L}$ (note that by choosing ξ_m sufficiently close to ξ, we can assume without loss of generality that such a ξ_m' always exists, since $\xi\in\mathscr{L}^*$). Clearly, we then have $|F_\psi(\xi)-F_\psi(\xi_m)| = |F_\psi(\xi)-F_\psi(\xi_m')|$ as well as $|\xi-\xi_m|\le|\xi-\xi_m'|$, and hence,

$$\frac{|F_\psi(\xi)-F_\psi(\xi_m)|}{|\xi-\xi_m|^\alpha} \ge \frac{|F_\psi(\xi)-F_\psi(\xi_m')|}{|\xi-\xi_m'|^\alpha}.$$

For ease of exposition, let us now only consider the case in which $\xi_m > \xi$ for all $m\in\mathbb{N}$. The case $\xi_m < \xi$ can be dealt with in the same way, and this is left to the reader. Now, let $(n_m)_{m\in\mathbb{N}}$ be the sequence such that $\xi_m\in[x_1\ldots x_{n_m-1}]$ and $\xi_m\notin[x_1\ldots x_{n_m-1}x_{n_m}]$. Then there are two cases to consider. The first case is that $x_{n_m+1}\neq 1$, and the second case is that ξ has a 1-block of length k_m at the n_m-th level. For these two cases one argues as follows.

Case 1: Here we have that $[x_1\ldots x_{n_m}1]$ separates the points ξ and ξ_m, and hence,

$$\frac{|F_\psi(\xi)-F_\psi(\xi_m)|}{|\xi-\xi_m|^\alpha} \ge \frac{\nu_\psi([x_1\ldots x_{n_m}1])}{\mathrm{diam}([x_1\ldots x_{n_m-1}])^\alpha} \gg e^{S_{n_m}\chi_\alpha(\xi)}.$$

Case 2: Here we have that $\left[x_1 \ldots x_{n_m} \underline{1}_{k_m+1}\right]$ separates the points ξ and ξ_m, where $\underline{1}_k$ refers to the word of length k containing exclusively the letter 1. In this situation we obtain

$$\frac{|F_\psi(\xi) - F_\psi(\xi_m)|}{|\xi - \xi_m|^\alpha} \geq \frac{\nu_\psi\left(\left[x_1 \ldots x_{n_m} \underline{1}_{k_m+1}\right]\right)}{\operatorname{diam}\left([x_1 \ldots x_{n_m-1}]\right)^\alpha} \gg e^{S_{n_m}\chi_\alpha(\xi) + k_m \psi(\underline{1})}.$$

In both cases we have that the right hand side is unbounded. This proves that the right α-Hölder derivative of F_ψ at ξ does exist in the generalized sense. By proceeding similarly for the left α-Hölder derivative of F_ψ (where one essentially has to take the 'mirror image' of the above argument and to replace 1 by 0), the statement of the proposition follows. □

3. PROOF OF THE MAIN THEOREM

In this section we give the proof of the Main Theorem. We have split the proof up by first giving the proof for the upper bound of the Hausdorff dimension of Λ_ψ^α, and this is then followed by the proof of the lower bound.

Throughout, let us fix for each $n \in \mathbb{N}$ a partition $\mathscr{C}_n$ of $\mathscr{L}$ by cylinder sets such that the following holds.

$$\text{For each } [\omega] \in \mathscr{C}_n \text{ and } \xi \in [\omega], \text{ we have } |S_{|\omega|}\chi_\alpha(\xi) - n| \ll 1,$$

where $|\omega|$ refers to the word length of ω. In the following we also consider the 'stopping time' $T_t : \mathscr{L} \to \mathbb{R}$, which is defined by

$$T_t(\xi) := \sup\{k \in \mathbb{N} : S_k\chi_\alpha(\xi) < t\}, \text{ for all } t > 0, \xi \in \mathscr{L}.$$

Moreover, for $i \in \{0,1\}$ and $\varepsilon > 0$ we define

$$\mathscr{C}_n^{(i)}(\varepsilon) := \left\{[\omega \underline{i}_{n_\varepsilon}] : [\omega] \in \mathscr{C}_n\right\},$$

where $\underline{i}_{n_\varepsilon}$ refers to the word of length $n_\varepsilon := \lfloor -n(1-\varepsilon)/\psi(\underline{i}) \rfloor$ containing exclusively the letter i.

Finally, for each $i \in \{0,1\}$ let s_i be the unique solution of the equation

$$s_i \varphi(\underline{i})/\psi(\underline{i}) + \beta_\alpha(s_i) = 0. \tag{6}$$

Throughout, let us always assume without loss of generality that $\max\{s_0, s_1\} = s_0$. For the proof of our Main Theorem it is then sufficient to show that

$$\dim_H(\Lambda_\psi^\alpha) = s_0. \tag{7}$$

Indeed, for this note that (6) has the following interpretation. Namely, s_i is equal to the x-coordinate of the point of intersection of the graph of β_α with the straight line L_i through the origin of slope $-\varphi(\underline{i})/\psi(\underline{i})$. Since β_α is increasing and $\beta_\alpha(t) < 0$ for $t < \delta$, the assumption $s_0 = \max\{s_0, s_1\}$ gives that the slope of L_1 has to be less than or equal to the slope of L_0. Hence, since $\varphi(\underline{i})/\psi(\underline{i}) > 0$, it follows that the minimum of $\varphi(\underline{i})/\psi(\underline{i})$ is attained for $i = 0$. Combining this observation with (6) and assuming that (7) holds, the implicit characterization of $\dim_H(\Lambda_\psi^\alpha)$ in (2) follows.

Therefore, we are now left with proving the statement in (7), which will be done in the following two remaining sections.

3.1. The upper bound.

In this section we give the proof for the upper bound of the Hausdorff dimension of Λ_ψ^α as stated in the Main Theorem. In a nutshell, the idea is to show that there is a suitable covering of Λ_ψ^α which allows to apply the Borel-Cantelli Lemma in order to derive the desired upper bound.

Recall that $s_0 = \max\{s_0, s_1\}$, and let $\varepsilon > 0$ be given. For $\kappa > 0$ such that $(1-\varepsilon)(s_0+\kappa) = s_0 + \tau$, for some $\tau > 0$ (note that by choosing ε sufficiently small, κ can be made arbitrary small), we then have

$$
\begin{aligned}
\sum_{n\in\mathbb{N}}\sum_{C\in\mathscr{C}_n^{(0)}(\varepsilon)}(\operatorname{diam}(C))^{s_0+\kappa} \asymp \sum_{n\in\mathbb{N}}\sum_{C\in\mathscr{C}_n^{(0)}(\varepsilon)} & \mathrm{e}^{\sup_{\xi\in C}(s_0+\kappa)S_{T_n(\xi)+n_\varepsilon}\varphi(\xi)} \\
\asymp \quad & \sum_{n\in\mathbb{N}}\sum_{C\in\mathscr{C}_n^{(0)}(\varepsilon)}\mathrm{e}^{\sup_{\xi\in C}(s_0+\kappa)S_{T_n(\xi)+n_\varepsilon}\varphi(\xi)} \\
\ll \quad & \sum_{n\in\mathbb{N}}\mathrm{e}^{-n(1-\varepsilon)(s_0+\kappa)\varphi(\underline{0})/\psi(\underline{0})}\sum_{C\in\mathscr{C}_n^{(0)}}\mathrm{e}^{s_0\sup_{\xi\in C}S_{T_n(\xi)}\varphi(\xi)} \\
\asymp \quad & \sum_{n\in\mathbb{N}}\mathrm{e}^{-n(1-\varepsilon)(s_0+\kappa)\varphi(\underline{0})/\psi(\underline{0})-n\beta_\alpha(s_0)}\sum_{C\in\mathscr{C}_n}\mathrm{e}^{\sup_{\xi\in C}S_{T_n(\xi)}(s_0\varphi(\xi)+\beta_\alpha(s_0)\chi_\alpha(\xi))} \\
\asymp \quad & \sum_{n\in\mathbb{N}}\left(\mathrm{e}^{-\tau\varphi(\underline{0})/\psi(\underline{0})}\right)^n<\infty.
\end{aligned}
$$

Here, we have used the Gibbs property $\sum_{C\in\mathscr{C}_n}\exp(\sup_{\xi\in C}S_{T_n(\xi)}(s_0\varphi(\xi)+\beta_\alpha(s_0)\chi_\alpha(\xi)))\asymp 1$ of the Gibbs measure μ_{s_0}. Similar, one finds (using the fact that $(1-\varepsilon)(s_0+\kappa)\geq s_1+\tau$),

$$
\begin{aligned}
\sum_{n\in\mathbb{N}}\sum_{C\in\mathscr{C}_n^{(1)}(\varepsilon)}(\operatorname{diam}(C))^{s_0+\kappa} \ll & \sum_{n\in\mathbb{N}}\mathrm{e}^{-n(1-\varepsilon)(s_0+\kappa)\varphi(\underline{1})/\psi(\underline{1})}\sum_{C\in\mathscr{C}_n}\mathrm{e}^{s_0\sup_{\xi\in C}S_{T_n(\xi)}\varphi(\xi)} \\
\ll \quad & \sum_{n\in\mathbb{N}}\mathrm{e}^{-n(1-\varepsilon)(s_0+\kappa)\varphi(\underline{1})/\psi(\underline{1})-n\beta_\alpha(s_1)}\sum_{C\in\mathscr{C}_n}\mathrm{e}^{\sup_{\xi\in C}S_{T_n(\xi)}(s_1\varphi(\xi)+\beta_\alpha(s_1)\chi_\alpha(\xi))} \\
\ll \quad & \sum_{n\in\mathbb{N}}\mathrm{e}^{-n(s_1+\tau)\varphi(\underline{1})/\psi(\underline{1})-n\beta_\alpha(s_1)}<\infty.
\end{aligned}
$$

Here, we have used the Gibbs property $\sum_{C\in\mathscr{C}_n}\exp(\sup_{\xi\in C}S_{T_n(\xi)}(s_1\varphi(\xi)+\beta_\alpha(s_1)\chi_\alpha(\xi)))\asymp 1$ of the Gibbs measure μ_{s_1}. Therefore, we now have

$$
\dim_H(\{\xi\in X:\xi\in\bigcup_{i=0,1}\mathscr{C}_n^{(i)}(\varepsilon)\text{ for infinitely many }n\in\mathbb{N}\})\leq s_0+\kappa.
$$

Hence, it remains to show that for all $\varepsilon>0$,

$$
\Lambda_\psi^\alpha\subset\bigcup_{i=0,1}\bigcup_{n\in\mathbb{N}}\mathscr{C}_n^{(i)}(\varepsilon).
$$

For this, let $\xi\in\Lambda_\psi^\alpha$ be given. By Proposition 2.6, there exists $i\in\{0,1\}$ and strictly increasing sequences $(n_m)_{m\in\mathbb{N}}$ and $(k_m)_{m\in\mathbb{N}}$ of positive integers, such that ξ has an i-block of length k_m at the n_m-th level, for each $m\in\mathbb{N}$, and

$$
e^{S_{n_m}\chi_\alpha(\xi)+k_m\psi(\underline{i})}\ll 1,\text{ for each }m\in\mathbb{N}.
$$

By setting $\ell(n_m):=\lfloor S_{n_m}\chi_\alpha(\xi)\rfloor$, it follows $\exp(k_m)\gg\exp(-\ell(n_m)/\psi(\underline{i}))$. Hence, for each $\varepsilon>0$ and for each m sufficiently large, we have $k_m\geq-\ell(n_m)(1-\varepsilon)/\psi(\underline{i}))$. It follows that $\xi\in\mathscr{C}_{n_m}^{(i)}(\varepsilon)$, which finishes the proof of the upper bound.

3.2. **The lower bound.** In this section we give the proof for the lower bound of the Hausdorff dimension of Λ_ψ^α as stated in the Main Theorem. For this, we use the usual strategy and define a probability measure m supported on a certain Cantor-like set $\mathscr{M}$ contained in Λ_ψ. We then show that the Mass Distribution Principle is applicable to $(\mathscr{M},m)$, and this will then eventually lead to the desired estimate from below.
For the construction of $\mathscr{M}$, let $(n_k)_{k\in\mathbb{N}}$ denote a rapidly increasing sequence (to be specified later) of positive integers. With this sequence at hand, consider the sequences $(m_k)_{k\in\mathbb{N}}$ and $(N_k)_{k\in\mathbb{N}}$ which are given inductively as follows.

$$N_1 := n_1 \text{ and } N_k := \left\lfloor \sum_{j=1}^{k} n_j + \chi_\alpha(\underline{0}) \sum_{j=1}^{k-1} m_j \right\rfloor \text{ for all } k \geq 2,$$

$$\text{where } m_j := \lfloor -N_j/\psi(\underline{0}) \rfloor \text{ for all } j \in \mathbb{N}.$$

Now, consider the Cantor-like set $\mathscr{M}$ given by

$$\mathscr{M} := \{(x_1x_2\ldots) \in \mathscr{L} : (x_1x_2\ldots) = \omega_1\underline{0}_{m_1}\omega_2\underline{0}_{m_2}\ldots, \text{ with } \omega_j \in \mathscr{C}_{n_j} \text{ for all } j \in \mathbb{N}\},$$

where $\underline{0}_{m_j}$ refers to the word consisting of m_j times the letter 0. In order to see that $\mathscr{M} \subset \Lambda_\psi^\alpha$, let $\xi = (\omega_1\underline{0}_{m_1}\omega_2\underline{0}_{m_2}\ldots) \in \mathscr{M}$ be given. With ℓ_k referring to the word length of $\omega_1\underline{0}_{m_1}\ldots\omega_k$, we then have by construction $\exp(S_{\ell_k}\chi_\alpha(\xi)) \asymp \exp(N_k)$. Therefore, ξ has a 0-block of length m_k at the ℓ_k-th level for each $k \in \mathbb{N}$, and

$$\mathrm{e}^{S_{\ell_k}\chi_\alpha(\xi)+m_k\psi(\underline{0})} \asymp \mathrm{e}^{N_k+\lfloor -N_k/\psi(\underline{0})\rfloor\psi(\underline{0})} \ll 1.$$

An application of Lemma 2.6 then gives $\xi \in \Lambda_\psi^\alpha$, showing that $\mathscr{M} \subset \Lambda_\psi^\alpha$.
The next step is to define a measure m on X. This can be done as follows.

(C1): For cylinder sets of the form $[\omega_1\underline{0}_{m_1}\ldots\omega_{u-1}\underline{0}_{m_{u-1}}\omega_u\underline{0}_k]$ with $k < m_u$ and $\omega_j \in \mathscr{C}_{n_j}$ for each $j \in \{1,\ldots,u\}$, put

$$m([\omega_1\underline{0}_{m_1}\ldots\omega_{u-1}\underline{0}_{m_{u-1}}\omega_u\underline{0}_k]) := \prod_{j=1}^{u}\mu_{s_0}([\omega_j]).$$

(C2): For cylinder sets of the form $[\omega_1\underline{0}_{m_1}\ldots\omega_u\underline{0}_{m_u}x_1\ldots x_l]$ with $\omega_j \in \mathscr{C}_{n_j}$ for all $j \in \{1,\ldots,u\}$, and with $[x_1\ldots x_l]$ containing some cylinder set from $\mathscr{C}_{m_{u+1}}$, put

$$m([\omega_1\underline{0}_{m_1}\ldots\omega_u\underline{0}_{m_u}x_1\ldots x_l]) := \mu_{s_0}([x_1\ldots x_l)\cdot\prod_{j=1}^{u}\mu_{s_0}([\omega_j]).$$

By Kolmogorov's Consistency Theorem this defines a measure which strictly speaking is first only defined on $A^{\mathbb{N}}$. However, we then identify this measure with the push-down to $\mathscr{L}$ via the coding map Φ. One immediately verifies that for the so obtained measure m on $\mathscr{L}$ we have $m(\mathscr{M}) = 1$. In order to complete the proof for the lower bound, it is now sufficient to show that m satisfies the 'Frostman condition' for cylinder sets of the type in (C1) as well as for cylinder sets of the type in (C2). For this, first note that for $\omega \in \mathscr{C}_{n_1}, k \leq m_1$, and with ξ referring to some arbitrary element in $[\omega\underline{0}_{m_1}]$, we have the following estimate.

$$\begin{aligned} m([\omega\underline{0}_k]) &= \mu_{s_0}([\omega]) \asymp \mathrm{e}^{s_0 S_{T_{n_1}(\xi)}\varphi(\xi)+n_1\beta_\alpha(s_0)} \asymp \left(\mathrm{e}^{S_{T_{n_1}(\xi)}\varphi(\xi)-n_1\varphi(\underline{0})/\psi(\underline{0})}\right)^{s_0} \\ &\asymp \left(\mathrm{e}^{S_{T_{n_1}(\xi)}\varphi(\xi)-n_1\varphi(\underline{0})/\psi(\underline{0})}\right)^{s_0} \asymp \left(\mathrm{e}^{S_{T_{n_1}(\xi)+\lfloor -n_1/\psi(\underline{0})\rfloor}\varphi(\xi)}\right)^{s_0} \\ &\asymp (\mathrm{diam}([\omega\underline{0}_{m_1}]))^{s_0} \leq (\mathrm{diam}([\omega\underline{0}_k]))^{s_0}, \end{aligned}$$

where we have used the fact $\beta_\alpha(s_0) = -s_0\varphi(\underline{0})/\psi(\underline{0})$. Using the latter estimate, we can now check the Frostman condition for each of the two types of cylinder sets (C1) and (C2) separately as follows.

ad (C1): For cylinder sets as in (C1), we have for each $\varepsilon > 0$ and with $c > 0$ referring to some constant (which takes care of the comparability constant in the above estimate for $m([\omega\underline{0}_k])$ and of the fact that $\mathrm{diam}([ab]) \ll \mathrm{diam}([a])\cdot\mathrm{diam}([b])$, for all $a,b \in A$),

$$
\begin{aligned}
m([\omega_1 \underline{0}_{m_1} \ldots \omega_{u-1} \underline{0}_{m_{u-1}} \omega_u \underline{0}_k]) &= \prod_{j=1}^{u} \mu_{s_0}([\omega_j]) \\
&\leq c^u \left(\operatorname{diam}([\omega_1 \underline{0}_{\lfloor -n_1/\psi(\underline{0}) \rfloor}]) \ldots \operatorname{diam}([\omega_u \underline{0}_{\lfloor -n_u/\psi(\underline{0}) \rfloor}]) \right)^{s_0} \\
&\ll \frac{c^u \cdot \left(\operatorname{diam}([\underline{0}_{m_u}])\right)^{\varepsilon}}{\left(\operatorname{diam}([\underline{0}_{m_1 + \lfloor -m_1/\psi(\underline{0}) \rfloor}]) \ldots \operatorname{diam}([\underline{0}_{m_{u-1} + \lfloor -m_{u-1}/\psi(\underline{0}) \rfloor}])\right)^{s_0}} \\
&\qquad \cdot \left(\operatorname{diam}([\omega_1 \underline{0}_{m_1} \ldots \omega_u \underline{0}_{m_u}])\right)^{s_0 - \varepsilon},
\end{aligned}
$$

where we have used the fact

$$
\begin{aligned}
m_j - \lfloor -n_j/\psi(\underline{0}) \rfloor &= \left\lfloor -\left(\sum_{r=1}^{j} n_r + \chi_\alpha(\underline{0}) \sum_{r=1}^{j-1} m_r \right) / \psi(\underline{0}) \right\rfloor - \lfloor -n_j/\psi(\underline{0}) \rfloor \\
&= m_{j-1} + \lfloor -m_{j-1} \chi_\alpha(\underline{0}) / \psi(\underline{0}) \rfloor \pm 1.
\end{aligned}
$$

Now the announced growth condition for the sequence $(n_k)_{n \in \mathbb{N}}$ comes into play. Namely, by choosing this sequence to increase sufficiently fast, one immediately verifies that the first factor in the above estimate is uniformly bounded from above. Indeed, we have

$$
\begin{aligned}
&\frac{c^u \cdot \left(\operatorname{diam}([\underline{0}_{m_u}])\right)^{\varepsilon}}{\left(\operatorname{diam}([\underline{0}_{m_1 + \lfloor -m_1/\psi(\underline{0}) \rfloor}]) \ldots \operatorname{diam}([\underline{0}_{m_{u-1} + \lfloor -m_{u-1}/\psi(\underline{0}) \rfloor}])\right)^{s_0}} \\
&\quad \ll c^u \mathrm{e}^{\varepsilon \varphi(\underline{0}) m_u} \mathrm{e}^{-s_0 \varphi(\underline{0}) \sum_{r=1}^{u-1} m_r (1 - 1/\psi(\underline{0}))} \\
&\quad = \exp\left(\varphi(\underline{0}) m_u \left(\varepsilon - \left((1 - 1/\psi(\underline{0})) s_0 / m_u \sum_{r=1}^{u-1} m_r - u \log c / (\varphi(\underline{0}) m_u) \right) \right) \right).
\end{aligned}
$$

Hence, by choosing $(n_k)_{n \in \mathbb{N}}$ appropriately, we can make sure that

$$
(1 - 1/\psi(\underline{0})) \frac{s_0}{m_u} \sum_{r=1}^{u-1} m_r - \frac{u \log c}{\varphi(\underline{0}) m_u} < \varepsilon,
$$

and thus,

$$
\frac{c^u \cdot \left(\operatorname{diam}([\underline{0}_{m_u}])\right)^{\varepsilon}}{\left(\operatorname{diam}([\underline{0}_{m_1 + \lfloor -m_1/\psi(\underline{0}) \rfloor}]) \ldots \operatorname{diam}([\underline{0}_{m_{u-1} + \lfloor -m_{u-1}/\psi(\underline{0}) \rfloor}])\right)^{s_0}} \ll 1.
$$

Therefore, we have now shown that for each $\varepsilon > 0$ and for all $u \in \mathbb{N}$ sufficiently large,

$$
\begin{aligned}
m([\omega_1 \underline{0}_{m_1} \ldots \omega_{u-1} \underline{0}_{m_{u-1}} \omega_u \underline{0}_k]) &\ll \left(\operatorname{diam}([\omega_1 \underline{0}_{m_1} \ldots \omega_u \underline{0}_{m_u}])\right)^{s_0 - \varepsilon} \\
&\leq \left(\operatorname{diam}([\omega_1 \underline{0}_{m_1} \ldots \omega_u \underline{0}_k])\right)^{s_0 - \varepsilon}.
\end{aligned}
$$

This finishes the proof of the 'Frostman condition' for cylinder sets of type (C1).

ad (C2): For cylinder sets as in (C2), first note that since $\beta_\alpha(s_0) < 0$ and $\chi_\alpha > 0$, we have for all $\xi \in [x_1 \ldots x_l]$,

$$
\mu_{s_0}([x_1 \ldots x_l]) \asymp \mathrm{e}^{s_0 S_l \varphi(\xi) + \beta_\alpha(s_0) S_l \chi_\alpha(\xi)} \leq \mathrm{e}^{s_0 S_l \varphi(\xi)} \ll \left(\operatorname{diam}([x_1 \ldots x_l])\right)^{s_0}.
$$

Combining this with the estimate in '$ad\ (C1)$' we obtain for each $\varepsilon > 0$ and for all $u \in \mathbb{N}$ sufficiently large,

$$
\begin{aligned}
m([\omega_1 \underline{0}_{m_1} \ldots \omega_u \underline{0}_{m_u} x_1 \ldots x_l]) &= \mu_{s_0}([x_1 \ldots x_l]) \cdot \prod_{j=1}^{u} \mu_{s_0}([\omega_j]) \\
&\ll \left(\operatorname{diam}([x_1 \ldots x_l])\right)^{s_0} \cdot \left(\operatorname{diam}([\omega_1 \underline{0}_{m_1} \ldots \omega_u \underline{0}_{m_u}])\right)^{s_0 - \varepsilon} \\
&\leq \left(\operatorname{diam}([\omega_1 \underline{0}_{m_1} \ldots \omega_u \underline{0}_{m_u} x_1 \ldots x_l])\right)^{s_0 - \varepsilon}.
\end{aligned}
$$

By combining the results of *'ad (C1)'* and *'ad (C2)'* and using the fact that $\mathscr{M}$ is a subset of Λ_{ψ}^{α}, we have now shown that for each $\varepsilon > 0$,

$$\dim_H(\Lambda_{\psi}^{\alpha}) \geq \dim_H(\mathscr{M}) \geq s_0 - \varepsilon.$$

This finishes the proof of the lower bound, and hence also the proof of the Main Theorem.

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