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# Brownian occupation measures, compactness and large deviations: Pair interaction


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## 1. INTRODUCTION AND MOTIVATION.

**1.1 Motivation.** Let $\mathbb{P}$ be the Wiener measure on $C([0,\infty),\mathbb{R}^d)$ corresponding to a $d$- dimensional Brownian motion $W = (W_t)_{t\geq 0}$ starting from the origin. We continue with the study of compactness issues pertaining to large deviation theory developed in [MV14]. One of the motivations of our earlier work came from studying the asymptotic behavior of Brownian occupation measures $L_t = 1/t\int_0^t \delta_{W_s}\mathrm{d}s$ under mean-field type interaction expressed by the Gibbs measures of the form

$$\widehat{\mathbb{P}}_t(\mathrm{d}\omega) = \frac{1}{Z_t}\exp\big\{tH(L_t)\big\}\mathbb{P}(\mathrm{d}\omega)$$

where

$$H(\mu) = \int\int_{\mathbb{R}^d\times\mathbb{R}^d} V(x-y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \tag{1.1}$$

for some continuous function $V$ vanishing at infinity and $\mu\in\mathcal{M}_1(\mathbb{R}^d)$, the space of probability measures in $\mathbb{R}^d$. Here $Z_t$ denotes the usual normalization constant or the partition function whose exponential growth rate was determined by Donkser and Varadhan ([DV83]) resulting in the variational formula

$$\lim_{t\to\infty}\frac{1}{t}\log Z_t = \sup_{\substack{\psi\in H^1(\mathbb{R}^d)\\ \|\psi\|_2=1}}\bigg\{\int\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{d}x\mathrm{d}y\; V(x-y)\psi^2(x)\psi^2(y) - \frac{1}{2}\|\nabla\psi\|_2^2\bigg\}. \tag{1.2}$$

If $d=3$ and $V(x) = 1/|x|$, by a result of Lieb ([L76]), it is well-known that the above variational formula admits a rotationally symmetric maximizer $\psi_0$ which is unique modulo spatial translations. Hence, the maximizing set can be written as $\mathfrak{m} = \{\mu_x\colon x\in\mathbb{R}^3\}$ where $\mu_x$ denotes the probability measure with density $\psi_x^2 = \psi_0^2\star\delta_x$.

In [MV14] it was shown that, the distributions $\widehat{\mathbb{P}}_t\circ L_t^{-1}$ concentrate locally around an (infinite) neighborhood of the set of maximizers. More precisely,

$$\limsup_{t\to\infty}\frac{1}{t}\log\widehat{\mathbb{P}}_t\big\{L_t\notin U(\mathfrak{m})\big\} < 0, \tag{1.3}$$

with $U(\mathfrak{m})$ denoting any neighborhood of $\mathfrak{m}$ in the weak topology. Although, (1.3) requires only the weak topology in $\mathcal{M}_1(\mathbb{R}^d)$ in its statement, a rigorous justification is based on a different topology imposed in a "lifted spaceänd in this context, the following simple fact is of crucial significance: For any $x\in\mathbb{R}^d$,

$$H(\mu) = H(\mu\star\delta_x). \tag{1.4}$$

In other words, $H(\mu)$ is a function only of the orbit $\widetilde{\mu} = \{\mu\star\delta_x\colon x\in\mathbb{R}^d\}$ under spatial shifts. This inherent shift invariance of the model naturally leads to a study of the quotient space of orbits $\widetilde{\mathcal{M}}_1(\mathbb{R}^d) = \{\mu\star\delta_x\colon x\in\mathbb{R}^d\}$ and an embedding

$$\widetilde{\mathcal{M}}_1(\mathbb{R}^d)\hookrightarrow\widetilde{\mathcal{X}}.$$

Here $\widetilde{\mathcal{X}}$ is the space of *all* collections of equivalence classes of sub-probability measures and is the "translation-invariant compactificationöf $\widetilde{\mathcal{M}}_1$. This is also the space where a *strong large deviation principle* for the distributions of $\widetilde{L}_t\in\widetilde{\mathcal{M}}_1(\mathbb{R}^d)$ holds and this allows a direct analysis of the path measures $\widehat{\mathbb{P}}_tL_t^{-1}$, which goes far beyond analysis of the partition function $Z_t$ which can be handled by classical *weak* large deviation theory.

In this article we are interested in a variant of the above model with two independent Brownian motions interacting through a mutual self-attraction potential. This can be written as the transformed path measure

$$\begin{aligned}\widehat{\mathbb{P}}_t^{\otimes}\big(\mathrm{d}\omega^{(1)}\otimes\mathrm{d}\omega^{(2)}\big) &= \frac{1}{Z_t^{\otimes}}\exp\bigg\{\frac{1}{t}\int_0^t\int_0^t V\big(\omega_\sigma^{(1)}-\omega_s^{(2)}\big)\bigg\}\,\mathbb{P}^{\otimes}(\mathrm{d}\omega^{(1)}\otimes\mathrm{d}\omega^{(2)}) \\ &= \frac{1}{Z_t^{\otimes}}\exp\Big\{tH^{\otimes}\big(L_t^{(1)}\otimes L_t^{(2)}\big)\Big\}\mathbb{P}^{\otimes}(\mathrm{d}\omega^{(1)}\otimes\mathrm{d}\omega^{(2)})\end{aligned} \tag{1.5}$$

where

$$H^{\otimes}\big(\mu\otimes\nu\big) = \int\int_{\mathbb{R}^d\times\mathbb{R}^d} V(x-y)\mu(\mathrm{d}x)\nu(\mathrm{d}y) \qquad \mu,\nu\in\mathcal{M}_1(\mathbb{R}^d). \tag{1.6}$$

Here $V(\cdot)$ is a continuous function vanishing at infinity as before and $\mathbb{P}^{\otimes}=\mathbb{P}^{(1)}\otimes\mathbb{P}^{(2)}$ denotes the product of two Wiener measures $\mathbb{P}^{(1)}$ and $\mathbb{P}^{(2)}$ corresponding to two independent Brownian motions $W^{(1)}$ and $W^{(2)}$ both starting at the origin. Also $L_t^{(1)}$ and $L_t^{(2)}$ are the corresponding normalized occupations measures and

$$Z_t^{\otimes}=\mathbb{E}^{\otimes}\bigg[\exp\Big\{tH^{\otimes}(L_t^{(1)}\otimes L_t^{(2)})\Big\}\bigg]$$

is the partition function. Let us denote by $L_t^{\otimes}$ the product measure $L_t^{(1)}\otimes L_t^{(2)}$. We are interested in studying the *joint* asymptotic behavior of the product measure $L_t^{\otimes}$ under the transformed path measures $\widehat{\mathbb{P}}_t^{\otimes}$ as $t\to\infty$. It will turn out that, when $V(x)=1/|x|$ in $d=3$ or if $V$ satisfies a mild technical condition (e.g., rotational invariance), then the asymptotic distribution of $L_t^{\otimes}$ under $\widehat{\mathbb{P}}_t^{\otimes}$ concentrate around an “infinite tube"comprising of spatial shifts of the maximizers of variational formula 1.2, being in complete analogy to (1.3). The model (1.5) for $V(x)=1/|x|$ in $d=3$ is related to the mean-field bipolaron problem arising in statistical mechanics. In this context, our first main result, Theorem 3.1, is a rigorous justification of the anticipated localization of path measures, carried out for this model for the first time, to the best of our knowledge.

Another motivation of our work comes from studying the moments of the approximating solutions to the ill-defined stochastic partial differential equation

$$\partial_t Z=\frac{1}{2}\Delta Z+\xi Z \tag{1.7}$$

for spatial white noise potential $\xi$ in $\mathbb{R}^d$. This is called the spatial parabolic Anderson problem and in $d=3$, a rigorous construction of this ill-used equation has been carried out in [HL15] based on the robust theory of regularity structures ([H14]). On the other hand, motivated by different reasons based on our work on large deviations, we prove that, due to spatial shift-invariance of the Gaussian noise, our analysis allows a direct computation of the annealed (i.e., averaged over the noise) Lyapunov exponents of a smoothened and rescaled version of (1.7) as the smoothing parameter goes to zero. These exponents admit explicit variational formulas.

It also seems that our strong large deviation principle for products of occupation measures are related to *mutual intersection local times* of two independent Brownian motions in $\mathbb{R}^3$ (note that due to recurrence, the intersection local times of arbitrarily many Brownian paths in $\mathbb{R}^2$ is infinity, while in $\mathbb{R}^3$, at most two Brownian paths have a non-trivial intersection). Intersection local times, due to singularity of occupation measures in high dimensions, are notoriously difficult to handle. For example, they can be given a rigorous meaning only in a limiting sense, after a suitable mollification procedure. In an asymptotic sense (i.e., in the language of large deviations), they however admit an interpretation as the pointwise product of individual occupation measures. Such a large deviation analysis was carried out in a *bounded domain* in [KM13] based on classical weak Donsker-Varadhan large deviations. It seems that the methods developed in this article can be used to derive similar asymptotic behavior for the intersection local times in the *whole space* $\mathbb{R}^3$ corresponding to two independent Brownian motions.

For these assertions, *strong large deviation principle* for the distributions for $L_t^{\otimes}$ seem to be desirable and it is tempting to appeal to the methods developed in [MV14]. However, due to the “mixed productöf two different measures in (1.6), the crucial translation-invariance of (1.4) is somewhat lost: $H^{\otimes}(\mu \otimes \nu)$ might fail to be equal to

$$H^{\otimes}\big( (\mu \star \delta_x) \otimes (\nu \star \delta_y) \big)$$

if $x \neq y$. Hence, we are not entitled to invoke the shift-invariant theory developed in [MV14] directly and we are led to a slightly different “compactfication"for product measures. Here is the key intuitive idea.

**1.2 The main idea.** As usual we denote by $\mathcal{M}_1 = \mathcal{M}_1(\mathbb{R}^d)$ the space of probability measures in $\mathbb{R}^d$ and by $\mathcal{M}_1^{\otimes} = \mathcal{M}_1 \times \mathcal{M}_1$ the space of product measures. For brevity, let us write $\mu\nu$ for the product measure $\mu \otimes \nu \in \mathcal{M}_1^{\otimes}$. Note that, when equipped with the usual weak topology, both $\mathcal{M}_1$ and $\mathcal{M}_1^{\otimes}$ are non-compact. This can be attributed to several reasons. A Gaussian measure with a very large variance spreads thinly into dust and thus, a sequence constructed from the product of two such Gaussian measures disintegrates completely in the product topology. Also, a mixture of the form $\frac{1}{2}(\delta_{a_n} + \delta_{-a_n})$ splits into two (or more) widely separated pieces and escape to infinity as $a_n \to \infty$. Likewise, the product of mixtures $\frac{1}{2}(\delta_{a_n} + \delta_{-a_n})$ and $\frac{1}{2}(\delta_{b_n} + \delta_{-b_n})$ can not converge in the weak topology as $a_n, b_n \to \infty$. Starting with any sequence of probability measures $(\mu_n)_n$ in $\mathcal{M}_1$, the intuitive idea is to “centereäch piece separately as well as to allow some mass to be “thinly spread and disappear". For this, the first step is to identify regions in $\mathbb{R}^d$ where $\mu_n$ has its accumulations of masses. This can be written by the function $q_{\mu_n}(r) = \sup_{x \in \mathbb{R}^d} \mu_n\big(B_r(x)\big)$. By choosing subsequences, we can assume that $q_{\mu_n}(r) \to q_\mu(r)$ as $n \to \infty$ and $q_\mu(r) \to p_1 \in [0,1]$ as $r \to \infty$. Then there is a shift $\mu_n \star \delta_{a_n}$ which converges vaguely along some subsequence to a sub-probability measure $\alpha_1$ of mass $p_1$. We can peel off a measure of mass $\approx p_1$ from $\mu_n$ and repeat the same process for the leftover to get convergence again along a further subsequence. We can go on recursively and end up with a picture where $\mu_n$ concentrates roughly on *widely separated* compact pieces of masses $\{p_j\}_{j \in \mathbb{N}}$ while the rest of the mass $1 - \sum_j p_j$ leaks out.

In an exactly similar manner, starting with another independent sequence of probability measures $(\nu_n)_n$, we can visualize a similar concentration in widely separated compact pieces of masses $\{q_l\}_{l \in \mathbb{N}}$ with the remaining mass $1 - \sum_l q_l$ being dissipated. Note that, regions of concentration of $\mu_n$ and $\nu_n$ are completely independent and hence, are also possibly mutually widely separated. However, for the product $\mu_n \otimes \nu_n$ to converge, we can hope to recover any partial mass $p_j q_l$ only if *some* concentration region of $\mu_n$ happens to have be in $\mathcal{O}(1)$ distance from *some* concentration region of $\nu_n$. Since there is wide separation between individual components of $\mu_n$ and individual components of $\nu_n$, such “matchings"could take place only pairwise and the order of the matchings do not matter. Finally, since we will be interested in a topology inherited from test functions, which, among other properties, admit to vanish at infinity, any other mutually distant (and “unmatched") pair of components will not contribute. We end with a picture where the product sequence $\mu_n \nu_n$ roughly concentrates on matched pairs of islands, so that in each pair, two components are within bounded distance, while the pairs are mutually widely separated, and there is a certain dissipation of mass from unmatched pairs. This intuition again leads to a compactfication of the quotient space of product measures.

In such a compactification, due to mutual attraction, two independent Brownian paths tend to find such matched pairs of islands and stick together by treating each pair as one bigger island. Since such bigger islands are mutually distant, an optimal strategy rules out any interaction between them, leading to asymptotic independence and a full large deviation principle for distribution of orbits $\widetilde{L}_t^{\otimes}$ embedded in the compactfication, the rate function simply being the sum of classical Donsker-Varadhan rate functions on each such island. This is the key heuristic idea behind the aforementioned localization property.

Let us now briefly summarize the organization of the rest of the article. In the next section we collect some topological facts, define a class of relevant test functions, introduce a metric which will lead to the desired compactfication. In section 3, we prove a large deviation principle for the distribution of the orbits $\widetilde{L}_t^{\otimes}$ and give applications to the aforementioned localization of path measures as well as present a direct computation of the annealed Lyapunov exponents of the spatial parabolic Anderson model with white noise potential in $\mathbb{R}^d$. In several parts in our proofs, we will draw heavily on the methods developed in [MV14].

## 2. Shift invariant compactification for product measures

The space $\mathcal{M}_1 = \mathcal{M}_1(\mathbb{R}^d)$ of probability distributions on $\mathbb{R}^d$ carries two natural topologies. A sequence $\mu_n$ in $\mathcal{M}_1$ converges weakly to $\mu$, denoted by $\mu_n \Rightarrow \mu$, if

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x)\mu_n(\mathrm{d}x) = \int_{\mathbb{R}^d} f(x)\mu(\mathrm{d}x), \tag{2.1}$$

for all bounded continuous functions on $\mathbb{R}^d$. On the other hand, in the *vague topology* for the convergence of $\mu_n$ to $\mu$, denoted by $\mu_n \hookrightarrow \mu$, we only require (2.1) for continuous functions with compact support. It continues to hold for continuous functions that tend to $0$ as $|x| \to \infty$. Note that the total mass of probability measures, which is conserved in the weak convergence, is not necessarily conserved under vague convergence– a salient feature which distinguishes these two topologies. If we denote by $\mathcal{M}_{\le 1} = \mathcal{M}_{\le 1}(\mathbb{R}^d)$ the space of all sub-probability measures (non-negative measures with total mass less than or equal to one), then both topologies carry over to $\mathcal{M}_{\le 1}$ with the same requirements. Note that, $\mathcal{M}_{\le 1}$ is compact in the vague topology. The following fact, whose proof is elementary and is omitted, will be useful later.

**Lemma 2.1.** *Let $\mu_n \hookrightarrow \alpha$ in $\mathcal{M}_{\le 1}$. Then $\mu_n$ can be written as $\mu_n = \alpha_n + \beta_n$ where $\alpha_n \Rightarrow \alpha$ and $\beta_n \hookrightarrow 0$.*

**2.1 Test functions on product spaces and diagonal shift-invariance.** Let us denote by

$$\mathcal{M}_1^{\otimes} = \mathcal{M}_1 \otimes \mathcal{M}_1$$

the space of products of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$. We are interested in the action of $\mathbb{R}^d$ as an additive group of translations on $\mathcal{M}_1^{\otimes}$ leading to the orbit space

$$\begin{aligned}\widetilde{\mathcal{M}}_1^{\otimes} &= (\mathcal{M}_1 \otimes \mathcal{M}_1)\big/ \sim \\ &= \big\{(\mu \star \delta_x) \otimes (\nu \star \delta_x) \colon x \in \mathbb{R}^d\big\}\end{aligned}$$

Typical elements of this space will be denoted by $\widetilde{\mu\nu}$ for $\mu, \nu \in \mathcal{M}_1$. Note that $\widetilde{\mathcal{M}}_1^{\otimes}$ fails to be compact when $\mathcal{M}_1$ is equipped with the weak topology. This is the space we would like to compactify and for this purpose we need to identify a class of “continuous functionals". The following class provides a rich class of test functions in this regard.

For any $k \ge 1$, we denote by $\mathcal{F}_k^{\otimes}$ the space of continuous functions $f : (\mathbb{R}^{2d})^k \to \mathbb{R}$ that vanish at infinity in the sense

$$\lim_{r\to\infty} f\big(x_1, y_1, \dots, x_k, y_k\big) = 0. \tag{2.2}$$

where $r$ is the diameter given by $\max_{i,j=1,\dots,k}\big\{|x_i-y_j|,|x_i-x_j|,|y_i-y_j|\big\}$. Furthermore, these functions should be *diagonally translation-invariant* in the sense

$$f\big(x_1,y_1,\dots,x_k,y_k\big)=f\big(x_1+a,y_1+a,\dots,x_k+a,y_k+a\big)$$
$$\forall\ (x_j,y_j)\in\mathbb{R}^{2d},\ a\in\mathbb{R}^d,\ j=1,\dots,k$$

We will often use a typical function $f\in\mathcal{F}_1^{\otimes}$ of the form $f(x_1,y_1)=V(x_1-y_1)$ for some continuous function $V(\cdot)$ vanishing at infinity.

Denote by $\mathcal{F}^{\otimes}=\cup_{k\geq 1}\mathcal{F}_k^{\otimes}$ the countable union and for any $f\in\mathcal{F}^{\otimes}$ and for any $\mu,\nu\in\mathcal{M}_1$, we set

$$\Lambda^{\otimes}(f,\mu\otimes\nu)=\int_{(\mathbb{R}^{2d})^k}f(x_1,y_1,\dots,x_k,y_k)\prod_{j=1}^k\mu(\mathrm{d}x_j)\prod_{j=1}^k\nu(\mathrm{d}y_j).$$

For any fixed $f\in\mathcal{F}^{\otimes}$, this is clearly a function of the orbit $\widetilde{\mu\nu}=\big\{(\mu\star\delta_x)\otimes(\nu\star\delta_x)\colon x\in\mathbb{R}^d\big\}$. Such objects are natural continuous functionals on $\widetilde{\mathcal{M}}_1^{\otimes}$. Note that each $\mathcal{F}_k^{\otimes}$ admits a countable dense subset under the uniform metric and for the countable union $\mathcal{F}^{\otimes}$, we can order them all as a single countable sequence $\{f_r(x_1,y_1,\dots,x_k,y_k)\}_{r\in\mathbb{N}}$. Hence, for any sequences $(\mu_n)_n$ and $(\nu_n)_n$ in $\mathcal{M}_1$, the limit

$$\Lambda^{\otimes}(f)=\lim_{n\to\infty}\Lambda^{\otimes}(f,\mu_n\otimes\nu_n)\tag{2.3}$$

exists along some subsequence by diagonalization. The set of all possible limit points of (2.3) will comprise of the desired compactification. But we need some more useful facts pertaining to the the space $\mathcal{F}^{\otimes}$.

We say that a sequence $(\mu_n)$ in $\mathcal{M}_{\leq 1}$ *totally disintegrates* if for any positive $r<\infty$,

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}^d}\mu_n\big(B(x,r)\big)=0.$$

A typical example of a totally disintegrating sequence $\mu_n$ of measures is a centered Gaussian with covariance matrix $n\,\mathbf{Id}$. The following fact is useful and its proof can be found in Lemma 2.3 in [MV14].

**Lemma 2.2.** *If the sequences $(\mu_n)_n$ totally disintegrates. Then, for any sequence $(\nu_n)_n$ in $\mathcal{M}_{\leq 1}$ and any continuous function $V(x)$ with $\lim_{|x|\to\infty}V(x)=0$,*

$$\lim_{n\to\infty}\int\int_{\mathbb{R}^{2d}}V(x-y)\mu_n(\mathrm{d}x)\nu_n(\mathrm{d}y)=0.$$

*Furthermore, for any $k\geq 1$ and $f\in\mathcal{F}_k$,*

$$\lim_{n\to\infty}\int\cdots\int_{\mathbb{R}^{2dk}}f(x_1,y_1\dots,x_k,y_k)\prod_{i=1}^k\mu_n(\mathrm{d}x_i)\prod_{i=1}^k\nu_n(\mathrm{d}y_i)=0.$$

We say that two sequences $(\alpha_n)_n$ and $(\beta_n)_n$ in $\mathcal{M}_{\leq 1}$ are *widely separated*, if for some function $V$ on $\mathbb{R}^d$ which is continuous and vanishes at infinity,

$$\lim_{n\to\infty}\int V(x-y)\alpha_n(\mathrm{d}x)\beta_n(\mathrm{d}y)=0.\tag{2.4}$$

Note that if a sequence $(\mu_n)_n$ in $\mathcal{M}_{\leq 1}$ totally disintegrates, then it is widely separated from any arbitrary sequence of measures in $\mathcal{M}_{\leq 1}$.

**Lemma 2.3.** *Suppose a sequence $\alpha_n^{(1)}$ is widely separated from $\alpha_n^{(2)}$, while the sequence $\beta_n^{(1)}$ is widely separated from $\beta_n^{(2)}$ in $\mathcal{M}_{\le 1}$. Then the product $\alpha_n^{(1)}\beta_n^{(1)}$ is widely separated from the product $\alpha_n^{(2)}\beta_n^{(2)}$. Furthermore, for every $k \ge 1$ and $f \in \mathcal{F}_k^{\otimes}$,*

(2.5)
$$\lim_{n\to\infty}\Bigg|\int f(x_1,y_1,\dots,x_k,y_k)\prod_{i=1}^k\Big(\big(\alpha_n^{(1)}\beta_n^{(1)}\big)(\mathrm{d}x_i\mathrm{d}y_i)+\big(\alpha_n^{(2)}\beta_n^{(2)}\big)(\mathrm{d}x_i\mathrm{d}y_i)\Big)$$
$$-\int f(x_1,y_1,\dots,x_k,y_k)\prod_{i=1}^k\big(\alpha_n^{(1)}\beta_n^{(1)}\big)(\mathrm{d}x_i\mathrm{d}y_i)\;-\int f(x_1,y_1,\dots,x_k,y_k)\prod_{i=1}^k\big(\alpha_n^{(2)}\beta_n^{(2)}\big)(\mathrm{d}x_i\mathrm{d}y_i)$$

*Proof.* The proof of the first claim is easy. For the second part, for $k=2$, if we can expand the product, we see that all the mixed terms disappear by the first claim and the definition of the class of functions $\mathcal{F}_k^{\otimes}$. The general case follows from an induction argument. □

### 2.2 Compactification of $\widetilde{\mathcal{M}}_1^{\otimes}$.

Let us denote the space

$$\widetilde{\mathcal{X}}^{\otimes}=\Big\{\xi^{\otimes}:\ \xi^{\otimes}=\big\{\widetilde{\alpha_i\beta_i}\big\}_{i\in I},\ \ \alpha_i,\beta_i\in\mathcal{M}_{\le 1}.\sum_i\alpha_i(\mathbb{R}^d)\le 1,\sum_i\beta_i(\mathbb{R}^d)\le 1\Big\}$$

We remark that, in order to keep notation short, we suppressed the fact that the index set $I$ above ranges over empty, finite or countably many collections. We will write any typical element $\xi^{\otimes}\in\widetilde{\mathcal{X}}^{\otimes}$ as $\xi^{\otimes}=\{\widetilde{\alpha_i\beta_i}\}$ with the understanding that either the collection is empty or $i$ ranges over a finite or countable set.

Note that, we have a canonical embedding of $\widetilde{\mathcal{M}}_1^{\otimes}$ in the space $\widetilde{\mathcal{X}}^{\otimes}$. If we write $\langle f,\mu\rangle$ for the integral $\int f\mathrm{d}\mu$ for any function $f$ and any measure $\mu$, then we want a sequence $(\xi_n^{\otimes})_n$ to converge to $\xi^{\otimes}$ in the space $\widetilde{\mathcal{X}}^{\otimes}$ under the desired metric, if the sequence

$$\Lambda^{\otimes}(f,\xi_n^{\otimes})=\sum_{\widetilde{(\alpha_n\beta_n)}\in\xi_n^{\otimes}}\int f(x_1,y_1\dots,x_k,y_k)\prod_{i=1}^n\big(\alpha_n\beta_n\big)(\mathrm{d}x_i\mathrm{d}y_i)=\sum_{\widetilde{(\alpha_n\beta_n)}\in\xi_n^{\otimes}}\big\langle f,\alpha_n\beta_n\big\rangle,$$

converges to the corresponding expression

$$\Lambda^{\otimes}(f,\xi)=\sum_{\widetilde{\alpha\beta}\in\xi^{\otimes}}\int f(x_1,y_1\dots,x_k,y_k)\prod_{i=1}^n\big(\alpha\beta\big)(\mathrm{d}x_i\mathrm{d}y_i)=\sum_{\widetilde{\alpha\beta}\in\xi^{\otimes}}\big\langle f,\alpha\beta\big\rangle,$$

for every $f\in\mathcal{F}^{\otimes}$. Note that the integral

$$\int f(x_1,y_1\dots,x_k,y_k)\prod_{i=1}^n\big(\alpha\beta\big)(\mathrm{d}x_i\mathrm{d}y_i)$$

depends only on the orbit $\widetilde{\alpha\beta}$ of the product measure $\alpha\beta$, by translation invariance of $f\in\mathcal{F}^{\otimes}$.

We fix a countable sequence of functions $\{f_r(x_1y_1,\dots,x_{k_r},y_{k_r})\}_{r\in\mathbb{N}}$ which is dense in $\mathcal{F}^{\otimes}$. For any $\xi_1^{\otimes},\xi_2^{\otimes}\in\widetilde{\mathcal{X}}^{\otimes}$, we define

$$\mathbf{D}^{\otimes}\big(\xi_1^{\otimes},\xi_2^{\otimes}\big)=\sum_{r=1}^{\infty}\frac{1}{2^r}\frac{1}{1+\|f_r\|_\infty}\,\Bigg|\sum_{\widetilde{\alpha\beta}\in\xi_1^{\otimes}}\big\langle f_r,\alpha\otimes\beta\big\rangle-\sum_{\widetilde{\alpha\beta}\in\xi_2^{\otimes}}\big\langle f_r,\alpha\otimes\beta\big\rangle\Bigg|. \tag{2.6}$$

**Theorem 2.4.** $\mathbf{D}^{\otimes}$ *is a metric on* $\widetilde{\mathcal{X}}^{\otimes}$.

*Proof.* The proof of the theorem is lengthy and non-trivial, but follows the same line of arguments modulo slight modifications as Theorem 3.1 in [MV14]. We drop the details to avoid repetition. □

**Theorem 2.5.** *The set of orbits* $\widetilde{\mathcal{M}}_1^{\otimes}$ *is dense in* $\widetilde{\mathcal{X}}^{\otimes}$. *Furthermore, given any sequence* $(\widetilde{\mu_n\nu_n})_n$ *in* $\widetilde{\mathcal{M}}_1^{\otimes}$, *there is a subsequence that converges to a limit in* $\widetilde{\mathcal{X}}^{\otimes}$. *Hence* $\widetilde{\mathcal{X}}^{\otimes}$ *is a compactification of* $\widetilde{\mathcal{M}}_1^{\otimes}$. *It is then also the completion under the metric* $\mathbf{D}^{\otimes}$ *of the totally bounded space* $\widetilde{\mathcal{M}}^{\otimes}$.

*Proof.* **Step 1:** We first prove that, given any $\xi^{\otimes} \in \widetilde{\mathcal{X}}^{\otimes}$, there is a sequence $(\widetilde{\mu_n\nu_n})_n$ in $\widetilde{\mathcal{M}}_1^{\otimes}$ which converges to $\xi^{\otimes}$. Indeed, let $\xi^{\otimes} = (\widetilde{\alpha_j\beta_j})_j$ such that $\alpha_j(\mathbb{R}^d) = p_j$, $\beta_j(\mathbb{R}^d) = q_j$. Then $\sum_j p_j \leq 1$ and $\sum_j q_j \leq 1$. If it is an infinite collection, we choose a finite sub-collection $\{\alpha_1, \dots, \alpha_n\}$ and $\{\beta_1, \dots, \beta_n\}$ so that $\sum_{j>n} p_j$ as well as $\sum_{j>n} q_j$ add up to at most $\varepsilon > 0$. Then, for any $f \in \mathcal{F}_k^{\otimes}$,

(2.7)

$$\sum_{j>n} \int f(x_1, y_1, \dots, x_k, y_k) \prod_{i=1}^k \alpha_j(\mathrm{d}x_i) \prod_{i=1}^k \beta_j(\mathrm{d}y_i) \leq \|f\|_\infty \sum_{j>n} p_j q_j \leq \|f\|_\infty \frac{1}{2} \sum_{j>n} (p_j^2 + q_j^2) \leq \varepsilon \|f\|_\infty.$$

Any centered Gaussian measure $\lambda_M$ with covariance matrix $M\mathbf{Id}$ totally disintegrates as $M \to \infty$. Hence, by Lemma 2.2,

$$\lim_{M\to\infty} \int f(x_1, y_1, \dots, x_k, y_k) \prod_{i=1}^k \lambda_M(\mathrm{d}x_i) \prod_{i=1}^k \lambda_M(\mathrm{d}y_i) = 0. \tag{2.8}$$

We choose a sequence of spatial points $a_1, \dots, a_n \in \mathbb{R}^d$ so that $\inf_{i\neq j} |a_i - a_j| \to \infty$.

Then for the convex combinations

$$\begin{aligned} \mu_n &= \sum_{j=1}^n \alpha_j \star \delta_{a_j} + \Big(1 - \sum_{j=1}^n p_j\Big)\lambda_M \\ \nu_n &= \sum_{j=1}^n \beta_j \star \delta_{a_j} + \Big(1 - \sum_{j=1}^n q_j\Big)\lambda_M, \end{aligned} \tag{2.9}$$

for any $k \geq 1$ and $f \in \mathcal{F}_k^{\otimes}$, we have

$$\int f(x_1, y_1, \dots, x_k, y_k) \prod_{i=1}^k \mu_n(\mathrm{d}x_i) \prod_{i=1}^k \nu_n(\mathrm{d}y_j) \to \sum_{j=1}^n \int f(x_1, y_1, \dots, x_k, y_k) \prod_{i=1}^k \alpha_j(\mathrm{d}x_i) \prod_{i=1}^k \beta_j(\mathrm{d}y_i).$$

as $\inf_{i\neq j} |a_i - a_j| \to \infty$ and $M \to \infty$, by (2.8) and Lemma 2.3. By the definition of the metric $\mathbf{D}^{\otimes}$ in (2.6), clealy $\widetilde{\mu_n\nu_n}$ converges in $\widetilde{\mathcal{X}}^{\otimes}$ to $\xi^{\otimes} = \{\widetilde{\alpha_j\beta_j}\}$.

**Step 2:** We now show that, given any sequence $\widetilde{\mu_n\nu_n}$ in $\widetilde{\mathcal{M}}_1^{\otimes}$, there is a subsequence which converges to some $\xi^{\otimes}$. Let us start with the *concentration functions* of $\mu_n$ and $\nu_n$ given by

$$Q_{\mu_n}(r) = \sup_{x\in\mathbb{R}^d} \mu_n(B_r(x)) \qquad Q_{\nu_n}(r) = \sup_{x\in\mathbb{R}^d} \nu_n(B_r(x)).$$

We can assume that along some subsequences, $Q_{\mu_n}(r) \to Q_\mu(r)$ and $Q_{\nu_n}(r) \to Q_\nu(r)$ as $n \to \infty$. Furthermore, $Q_\mu(r) \to p_1$ and $Q_\nu(r) \to q_1$ as $r \to \infty$.

Let us first consider the case $p_1 = 0$ or $q_1 = 0$. If $p_1 = 0$, then, for every $r > 0$,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \mu_n(B_r(x)) = 0.$$

In other words, the sequence $\mu_n$ totally disintegrates. Then by Lemma 2.2 and the definition of the metric $\mathbf{D}^{\otimes}$ in (2.6), the sequence $\widetilde{\mu_n\nu_n} \to 0$ in $\widetilde{\mathcal{X}}^{\otimes}$.

Let us now assume that $p_1, q_1 \in (0,1]$. Then there is a sequence of shifts $(a_n) \subset \mathbb{R}^d$ such that, for some $r > 0$ and $n$ sufficiently large,

$$\mu_n\big(B_r(a_n)\big) \geq p/2. \tag{2.10}$$

Then, along some subsequence, $\mu_n \star \delta_{a_n} \hookrightarrow \alpha'$ for some $\alpha' \in \mathcal{M}_{\leq 1}$. By lemma 2.1, we can write

$$\mu_n = \alpha_n + \mu_n^{(1)}$$

so that $\alpha_n \star \delta_{a_n} \Rightarrow \alpha'$ and $\mu_n^{(1)} \hookrightarrow 0$. Note that $\alpha_n$ and $\mu_n^{(1)}$ are widely separated.

We repeat the procedure with $\mu_n^{(1)} \star \delta_{a_n}$. Since $\big(\mu_n^{(1)} \star \delta_{a_n}\big)(\cdot) \leq \mu_n(\cdot)$, we conclude, by (2.10), that for every $r > 0$,

$$\lim_{n\to\infty} Q_{(\mu_n^{(1)}\star\delta_{a_n})}(r) \leq \min\big\{1 - p_1/2, p_1\big\}.$$

This iterative process could go on forever, or it might stop at a finite stage (i.e., when the recovered mass $p_{k+1}$, after stage $k$, happens to be $0$). If it stops at a finite stage, then we can write

$$\mu_n = \sum_{j=1}^{k} \alpha_n^{(j)} + \gamma_n$$

such that, for each $j = 1, \ldots, k$, along some subsequence,

$$\alpha_n^{(j)} \star \delta_{a_n^{(j)}} \Rightarrow \alpha_j', \tag{2.11}$$

and for $i \neq j$, $|a_n^{(i)} - a_n^{(j)}| \to \infty$, while the sequence $\gamma_n$ totally disintegrates (i.e., for every $r > 0$, $Q_{\gamma_n}(r) \to 0$). We remark that if $p_1 = 1$, then there is disintegration of mass.

For the sequence $(\nu_n)$, in an exactly similar manner, we can write,

$$\nu_n = \sum_{l=1}^{m} \beta_n^{(l)} + \lambda_n$$

such that, for each $l = 1, \ldots, m$, along some subsequence,

$$\beta_n^{(l)} \star \delta_{b_n^{(l)}} \Rightarrow \beta_l', \tag{2.12}$$

and for $l \neq u$, $|b_n^{(l)} - b_n^{(u)}| \to \infty$, while the sequence $\lambda_n$ also totally disintegrates (again, if $q_1 = 1$, there is no disintegration).

Let us now turn to the product

$$\mu_n \otimes \nu_n = \sum_{j=1}^{k}\sum_{l=1}^{m} \alpha_n^{(j)} \otimes \beta_n^{(l)} + \sum_{l=1}^{m} \beta_n^{(l)} \otimes \gamma_n + \sum_{j=1}^{k} \alpha_n^{(j)} \otimes \lambda_n \tag{2.13}$$

Since both $\gamma_n$ and $\lambda_n$ totally disintegrate, by Lemma 2.2, for any $V \in \mathcal{F}_1^{\otimes}$,

$$\begin{aligned} &\lim_{n\to\infty} \int\int V(x-y)\alpha_n^{(j)}(\mathrm{d}x)\lambda_n(\mathrm{d}y) = 0 \qquad \forall j = 1, \ldots, k \\ &\lim_{n\to\infty} \int\int V(x-y)\beta_n^{(l)}(\mathrm{d}x)\gamma_n(\mathrm{d}y) = 0 \qquad \forall l = 1, \ldots, m. \end{aligned} \tag{2.14}$$

Finally for the products $\alpha_n^{(j)} \otimes \beta_n^{(l)}$, if for some $j \in \{1, \dots, k\}$ and $l \in \{1, \dots, m\}$, the distance of the shifts $|a_n^{(j)} - a_n^{(l)}|$ remains bounded, i.e,

$$|a_n^{(j)} - a_n^{(l)}| \le c^{(jl)} = c < \infty, \tag{2.15}$$

then we can find some common spatial shift $c_n^{(jl)} = c_n$ so that, again along sone subsequence,

$$\begin{aligned}\big(\alpha_n^{(j)} \otimes \beta_n^{(l)}\big) \star \delta_{c_n} &= \big(\alpha_n^{(j)} \star c_n\big) \otimes \big(\beta_n^{(l)} \star c_n\big) \\ &\Rightarrow \alpha_j \otimes \beta_l,\end{aligned}$$

for some $\alpha_j, \beta_l \in \mathcal{M}_{\le 1}$. In other words, for any such pair $j \in \{1, \dots, k\}$ and $l \in \{1, \dots, m\}$, for every $V \in \mathcal{F}_1^{\otimes}$,

$$\begin{aligned}\int\int V(x-y)\alpha_n^{(j)}(\mathrm{d}x)\beta_n^{(l)}(\mathrm{d}y) &= \int\int V(x-y)\big(\alpha_n^{(j)} \star \delta_{c_n}\big)(\mathrm{d}x)\big(\beta_n^{(l)} \star \delta_{c_n}\big)(\mathrm{d}y) \\ &\to \int\int V(x-y)\alpha_j(\mathrm{d}x)\beta_l(\mathrm{d}y).\end{aligned} \tag{2.16}$$

On the other hand, if $|a_n^{(j)} - b_n^{(l)}| \to \infty$,

$$\begin{aligned}\int\int V(x-y)\alpha_n^{(j)}(\mathrm{d}x)\beta_n^{(l)}(\mathrm{d}y) = \int\int V\big(x-y+a_n^{(j)} - b_n^{(l)}\big)\big(\alpha_n^{(j)} \star \delta_{a_n^{(j)}}\big)(\mathrm{d}x)\big(\beta_n^{(l)} \star \delta_{b_n^{(l)}}\big)(\mathrm{d}y) \\ \to 0\end{aligned} \tag{2.17}$$

for any $V \in \mathcal{F}_1^{\otimes}$, by and (2.2), (2.11) and (2.12). Summaraizing (2.14), (2.16) and (2.17), we conclude that the sequence $\widetilde{\mu_n\nu_n}$ converges to some element $\xi^{\otimes} = \big\{\big(\widetilde{\alpha_j\beta_l}\big)_{j,l}\big\} \in \widetilde{\mathcal{X}}^{\otimes}$.[1]

Finally, if the process continues forever, then an induction argument combined with the finite step recursion scheme leads to the same conclusion. ☐

**Corollary 2.6.** *For any $V \in \mathcal{F}_1^{\otimes}$, the functional $H : \widetilde{\mathcal{X}}^{\otimes} \to \mathbb{R}$ defined by*

$$H(\xi^{\otimes}) = \sum_j \int\int V(x-y)\alpha_j(\mathrm{d}x)\beta_j(\mathrm{d}y) \qquad \xi^{\otimes} = (\widetilde{\alpha_j\beta_j})_j$$

*is continuous.*

*Proof.* For any $\mu_n, \nu_n \in \mathcal{M}_1$ if the sequence $\widetilde{\mu_n\nu_n}$ converges to $\xi^{\otimes} = (\widetilde{\alpha_j\beta_j})_j$, then

$$\int\int V(x-y)\mu_n(\mathrm{d}x)\nu_n(\mathrm{d}y) \to \sum_j \int\int V(x-y)\alpha_j(\,dx)\beta_j(\mathrm{d}y).$$

Since $\widetilde{\mathcal{M}}_1^{\otimes}$ is dense in $\widetilde{\mathcal{X}}^{\otimes}$, the corollary is proved. ☐

[01] For example, let $\mu_n$ be a sequence which is a mixture of three Gaussians, one with mean $0$ and variance $1$, one with mean $n$ and variance $1$ and one with mean $0$ and variance $n$, each with equal weights $1/3$. On the other hand, let $\nu_n$ also be a mixture of three Gaussians, one with mean $n^2$ and variance $1$, one with mean $n+1$ and variance $1$ and one with mean $0$ and variance $n$, also with equal weights $1/3$. Then the limiting object for $\widetilde{\mu_n\nu_n}$ is the single orbit $\{\widetilde{\alpha_1\beta_2}\}$, where $\alpha_1$ is the a Gaussian with mean $0$, variance $1$ and mass $1/3$, $\beta_2$ is a Gaussian with mean $1$, variance $1$ and mass $1/3$, while $\widetilde{\alpha_1\beta_2}$ is the equivalence class of the product of these two Gaussians with mass $1/9$.

## 3. Applications

### 3.1 Pair interaction under mean-field path measures.

We now focus on the model introduced in Section 1. $\mathbb{P}^{(1)}$ and $\mathbb{P}^{(2)}$ will denote two Wiener measures corresponding to two independent Brownian motions $W^{(1)}$ and $W^{(2)}$ starting from the origin and $\mathbb{P}^{\otimes} = \mathbb{P}^{(1)} \otimes \mathbb{P}^{(2)}$ will denote the product. We will be interested in the Gibbs measure

$$\begin{aligned}\widehat{\mathbb{P}}_t^{\otimes}\big(\mathrm{d}\omega^{(1)} \otimes \mathrm{d}\omega^{(2)}\big) &= \frac{1}{Z_t^{\otimes}} \exp\left\{\frac{1}{t}\int_0^t \int_0^t V\big(\omega_\sigma^{(1)} - \omega_s^{(2)}\big)\right\} \mathbb{P}^{\otimes}(\mathrm{d}\omega^{(1)} \otimes \mathrm{d}\omega^{(2)}) \\ &= \frac{1}{Z_t^{\otimes}} \exp\left\{tH\big(L_t^{\otimes}\big)\right\} \mathbb{P}^{\otimes}(\mathrm{d}\omega^{(1)} \otimes \mathrm{d}\omega^{(2)}).\end{aligned}$$

Here $L_t^{\otimes} = L_t^{(1)} \otimes L_t^{(2)}$ where $L_t^{(1)}$ and $L_t^{(2)}$ are the normalized occupation measures of $W^{(1)}$ and $W^{(2)}$ respectively and

$$H(\mu \otimes \nu) = \int\int V(x-y)\mu(\mathrm{d}x)\nu(\mathrm{d}y) \qquad \mu, \nu \in \mathcal{M}_1,$$

for $V \in \mathcal{F}_1^{\otimes}$. Note that, via

$$\Omega \to \mathcal{M}_1^{\otimes}(\mathbb{R}^d) \to \widetilde{\mathcal{M}}_1^{\otimes}(\mathbb{R}^d) \subset \widetilde{\mathcal{X}}^{\otimes}$$

we have a distribution $\widehat{\mathbb{Q}}_t^{\otimes}$ of $\widetilde{L}_t^{\otimes}$ in $\widetilde{\mathcal{X}}^{\otimes}$ under $\widehat{\mathbb{P}}_t^{\otimes}$. In other words,

$$\widehat{\mathbb{Q}}_t^{\otimes} = \widehat{\mathbb{P}}_t^{\otimes} \circ \big(\widetilde{L}_t^{\otimes}\big)^{-1}$$

Note that since $V$ vanishes at infinity, the interaction in $\widehat{\mathbb{P}}_t^{\otimes}$ is mutually attractive and we will be interested in the asymptotic behavior of the distributions $\widehat{\mathbb{Q}}_t^{\otimes}$ as $t \to \infty$.

In what follows, an important role will be played by the variational formula

$$\rho = \sup_{\substack{\psi \in H^1(\mathbb{R}^d \\ \|\psi\|_2 = 1}} \left\{\int\int V(x-y)\psi^2(x)\psi_2^2(y)\mathrm{d}x\mathrm{d}y - \|\nabla\psi\|_2^2\right\} \tag{3.1}$$

It is well-known ([L76]) that if $V(x) = 1/|x|$ in $d=3$, then the variational problem (3.1) has a rotationally symmetric maximizer $\psi_0$ which is unique except for spatial translations, despite $1/|x|$ failing to be continuous due to the Coulomb singularity. Let $\mu_0$ be the measure with the maximizing density $\psi_0^2$. We will write

$$\widetilde{\mu}_0 \ = \widetilde{\mu_0\mu_0} \in \widetilde{\mathcal{M}}_1^{\otimes} \subset \widetilde{\mathcal{X}}^{\otimes}.$$

Here is our next main result.

**Theorem 3.1.** *Let $V(x) = 1/|x|$ in $d=3$ or let $V(\cdot)$ be any continuous, positive definite function vanishing at infinity such that the variational problem* (3.1) *has a rotationally symmetric maximizer $\psi_0$ which is unique except for spatial translations. Then the family of probability measures $\widehat{\mathbb{Q}}_t^{\otimes}$ converges weakly to $\delta_{\widetilde{\mu}_0}$.*

We will prove Theorem 3.1 in few steps. Let us introduce the Donsker-Varadhan rate function

$$I(\mu) = \begin{cases}\frac{1}{2}\|\nabla f\|_2^2 & \text{if } f = \sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}x}} \in H^1(\mathbb{R}^d) \\ \infty & \text{else.}\end{cases} \tag{3.2}$$

Here $H^1(\mathbb{R}^d)$ is the usual Sobolev space of square integrable functions with square integrable derivatives. Note that the function $\mu \mapsto I(\mu)$ is translation invariant and depends only on the orbit $\widetilde{\mu}$. Furthermore, this map is convex and homogenous of degree $1$. It is well-known ([DV75]) that the family of distributions of any occupation measure $L_t = 1/t \int_0^t \delta_{W_s} \mathrm{d}s$ under any Wiener measure $\mathbb{P}$ satisfies a “weak"large deviation principle in the space probability measures on $\mathcal{M}_1(\mathbb{R}^d)$ with the rate function $I$. This means, under the weak topology, for every compact subset $K \subset \mathcal{M}_1(\mathbb{R}^d)$ and for every open subset $G \subset \mathcal{M}_1(\mathbb{R}^d)$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(L_t \in K) \leq -\inf_{\mu\in K} I(\mu) \tag{3.3}$$

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}(L_t \in G) \geq -\inf_{\mu\in G} I(\mu), \tag{3.4}$$

If for any family of distributions the upper bound (3.3) holds for any closed set, we say that the family satisfies a strong large deviation principle, or just large deviation principle.

Let us also introduce the functional $\mathfrak{I}^{\otimes} : \widetilde{\mathcal{X}}^{\otimes} \to [0,\infty]$ given by

$$\mathfrak{I}^{\otimes}\big(\xi^{\otimes}\big) = \sum_{\widetilde{\alpha\beta}\in\xi^{\otimes}} \big[I(\alpha) + I(\beta)\big] \tag{3.5}$$

where $I$ is defined in (3.2) and $\alpha, \beta \in \mathcal{M}_{\leq 1}$ so that the product $\alpha\beta$ is any arbitrary element of the orbit $\widetilde{\alpha\beta}$. Also recall that, $I(\cdot)$ is translation invariant. Let us also note that $\mathfrak{I}^{\otimes}(\cdot)$ is a lower semicontinuous functional on $\widetilde{\mathcal{X}}^{\otimes}$.

**Lemma 3.2.** *The distributions of $\widetilde{L}_t^{\otimes}$ under $\mathbb{P}^{\otimes}$ satisfies a large deviation principle in the compact metric space $\widetilde{\mathcal{X}}^{\otimes}$ with rate function $\mathfrak{I}$.*

*Proof.* **The Lower bound.** If $\mathbb{Q}_t^{\otimes}$ denotes the distribution of $\widetilde{L}_t^{\otimes}$ under $\mathbb{P}^{\otimes}$, then it suffices to show that for any $\xi^{\otimes} \in \widetilde{\mathcal{X}}^{\otimes}$ with $\mathfrak{I}^{\otimes}(\xi^{\otimes}) < \infty$ and any neighborhood $U$ of $\xi^{\otimes}$,

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{Q}_t^{\otimes}(U) \geq -\mathfrak{I}^{\otimes}(\xi^{\otimes}) \tag{3.6}$$

First let us note that given any $\xi^{\otimes} \in \widetilde{\mathcal{X}}^{\otimes}$ with $\mathfrak{I}^{\otimes}(\xi^{\otimes}) < \infty$, there is a sequence $(\xi_n^{\otimes})_n$ in $\widetilde{\mathcal{X}}^{\otimes}$ which converges to $\xi^{\otimes}$ and

$$\limsup_{n\to\infty} \mathfrak{I}^{\otimes}(\xi_n^{\otimes}) \leq \mathfrak{I}^{\otimes}(\xi^{\otimes}). \tag{3.7}$$

This is essentially due to the convex decomposition of $\mu_n$ and $\nu_n$ constructed in (2.9). Furthermore, since $I(\cdot)$ is convex and 1-homogeneous,

$$\begin{aligned}\limsup_{n\to\infty} \mathfrak{I}^{\otimes}(\xi_n^{\otimes}) = \limsup_{n\to\infty} \big[I(\mu_n) + I(\nu_n)\big] &\leq \sum_j \big[I(\alpha_j \star \delta_{a_j}) + I(\beta_j \star \delta_{b_j})\big] \\ &= \sum_j \big[I(\alpha_j) + I(\beta_j)\big]\end{aligned}$$

Now for (3.6), we can get the single orbit sequence $\widetilde{\mu_n\nu_n}$ converging to $\xi^{\otimes}$. Then, also invoking independence of $L_t^{(1)}$ and $L_t^{(2)}$,

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{Q}_t^{\otimes}(U) \geq \liminf_{n\to\infty} \bigg[\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}^{(1)}\big(L_t^{(1)} \in U_w(\mu_n)\big) + \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}^{(2)}\big(L_t^{(2)} \in U_w(\nu_n)\big)\bigg]$$

where $U_w(\mu_n), U_w(\nu_n)$ denote some neighborhoods of $\mu_n$ and $\nu_n$ in the usual weak topology in $\mathcal{M}_1$. By the classical lower bound (3.4) and (3.7), we conclude

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{Q}_t^{\otimes}(U) \geq \liminf_{n\to\infty} \bigg\{ - I(\mu_n) - I(\nu_n)\bigg\} \geq -\mathfrak{I}(\xi^{\otimes}).$$

**The upper bound.** Since $\widetilde{\mathcal{X}}^{\otimes}$ is a compact metric space, we only need to prove a local upper bound of the form

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{Q}_t^{\otimes}(U)\le-\mathfrak{I}(\xi^{\otimes})=\sum_j\big[I(\alpha_j)+I(eta_j)\big] \tag{3.8}$$

for any neighborhood $U$ of any $\xi^{\otimes}=\widetilde{(\alpha_j\beta_j)}_j$ in a given closed set $F\subset\widetilde{\mathcal{X}}^{\otimes}$ in the metric $\mathbf{D}^{\otimes}$. By the definition set in (2.6) of $\mathbf{D}^{\otimes}$, the decomposition (2.13), and (2.14)-(2.17) then imply that for (3.8), it is enough to estimate the probability

$$\mathbb{P}^{\otimes}\bigg\{\exists c_1,\dots,c_k\in\mathbb{R}^d,k\in\mathbb{N},r>0\colon\;\;|c_i-c_j|\ge4r\;\forall i\ne j,$$
$$L_t^{(1)}\big|_{B_r}\in U_w\big(\alpha_j\star\delta_{c_j}\big),\;L_t^{(2)}\big|_{B_r}\in U_w\big(\beta_j\star\delta_{c_j}\big)\bigg\},$$

Note that the requirement (2.15) plays an important role in the above statement. Again by independence, the above probability splits into the product

$$\mathbb{P}^{(1)}\bigg\{\exists c_1,\dots,c_k\in\mathbb{R}^d,\;k\in\mathbb{N},r>0\colon\;\;|c_i-c_j|\ge4r,\;\;L_t^{(1)}\big|_{B_r}\in U_w\big(\alpha_j\star\delta_{c_j}\big)\bigg\}$$
$$\times\,\mathbb{P}^{(2)}\bigg\{\exists c_1,\dots,c_k\in\mathbb{R}^d,\;k\in\mathbb{N},r>0\colon\;\;|c_i-c_j|\ge4r,\;\;L_t^{(2)}\big|_{B_r}\in U_w\big(\beta_j\star\delta_{c_j}\big)\bigg\}.$$

Proposition 4.4 in [MV14] identifies $\sum_j I(\alpha_j)$ and $\sum_j I(\beta_j)$ as the exponential decay rates of the probabilities above and finishes the proof of (3.8). $\square$

**Lemma 3.3.** *The family of distributions $\widehat{\mathbb{Q}}_t^{\otimes}$ satisfies a large deviation principle in $\widetilde{\mathcal{X}}^{\otimes}$ with rate function*

$$J(\xi^{\otimes})=\widetilde{\rho}-\sum_j\bigg\{\int V(x-y)\alpha_j(\mathrm{d}x)\beta_j(\mathrm{d}y)-I(\alpha_j)-I(\beta_j)\bigg\}\quad\xi^{\otimes}=\widetilde{(\alpha_j\beta_j)},$$

*and*

$$\widetilde{\rho}=\sup_{\xi^{\otimes}\in\widetilde{\mathcal{X}}^{\otimes}}\sum_j\bigg\{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}V(x-y)\psi_j^2(x)\phi_j^2(y)\mathrm{d}x\mathrm{d}y-\frac{1}{2}\sum_j\big\|\nabla\psi_j\big\|_2^2-\frac{1}{2}\sum_j\big\|\nabla\phi_j\big\|_2^2\bigg\}$$

*and $\alpha_j$ and $\beta_j$ have densities $\psi_j^2$ and $\phi_j^2$ such that $\sum_j\int_{\mathbb{R}^3}\psi_j^2(x)\mathrm{d}x\le1$ and $\sum_j\int_{\mathbb{R}^d}\phi_j^2(y)\mathrm{d}y\le1$.*

*Proof.* Suppose $V$ is continuous and vanishes at infinity (i.e., $V\in\mathcal{F}_1^{\otimes}$). Let us write, for any $A\subset\widetilde{\mathcal{X}}$

$$\begin{aligned}\widehat{\mathbb{Q}}_t^{\otimes}(A)&=\widehat{\mathbb{P}}_t^{\otimes}\big\{\widetilde{L}_t^{\otimes}\in A\big\}\\&=\frac{\mathbb{E}^{\mathbb{Q}_t}\Big\{\exp\big\{\frac{1}{t}\int_0^t\int_0^tV(W_\sigma^{(1)}-W_s^{(2)})\mathrm{d}\sigma\mathrm{d}s\big\}\,1\!\!1_A\Big\}}{\mathbb{E}^{\mathbb{Q}_t^{\otimes}}\Big\{\exp\big\{\frac{1}{t}\int_0^t\int_0^tV(W_\sigma^{(1)}-W_s^{(2)})\mathrm{d}\sigma\mathrm{d}s\big\}\Big\}}\\&=\frac{1}{Z_t^{\otimes}}\mathbb{E}^{\mathbb{Q}_t}\Big\{\exp\big\{\frac{1}{t}\int_0^t\int_0^tV(W_\sigma^{(1)}-W_s^{(2)})\mathrm{d}\sigma\mathrm{d}s\big\}\,1\!\!1_A\Big\}\end{aligned}\tag{3.9}$$

where $\mathbb{Q}_t^{\otimes}$ is the distribution of $\widetilde{L}_t^{\otimes}$ in $\widetilde{\mathcal{X}}^{\otimes}$. To handle the required large deviation upper bound, we can take $A=F\subset\widetilde{\mathcal{X}}^{\otimes}$ to be a closed set. Then the upper bound proved in Lemma 3.2, the continuity assertion proved in Corollary 2.6 and Varadhan's lemma,

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\mathbb{Q}_t}\Big\{\exp\big\{\frac{1}{t}\int_0^t\int_0^t V(W_\sigma^{(1)}-W_s^{(2)})\mathrm{d}\sigma\mathrm{d}s\big\}\,1\!\!1_F\Big\}$$
$$\leq \sup_{\xi^\otimes\in F}\sum_j\Big\{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}V(x-y)\psi_j^2(x)\phi_j^2(y)\mathrm{d}x\mathrm{d}y-\frac{1}{2}\sum_j\big\|\nabla\psi_j\big\|_2^2-\frac{1}{2}\sum_j\big\|\nabla\phi_j\big\|_2^2\Big\}$$

where $\xi^\otimes=\widetilde{(\alpha_j\beta_j)}$ and $\alpha_j$ and $\beta_j$ have densities $\psi_j^2$ and $\phi_j^2$ such that $\sum_j\int_{\mathbb{R}^3}\psi_j^2(x)\mathrm{d}x\leq 1$ and $\sum_j\int_{\mathbb{R}^d}\phi_j^2(y)\mathrm{d}y\leq 1.$

Similarly for the desired large deviation lower bound, for any open set $A=G\subset\widetilde{\mathcal{X}}^\otimes$, by the lower bound proved in Lemma 3.2, the continuity assertion proved in Corollary 2.6 and Varadhan's lemma,

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}^{\mathbb{Q}_t}\Big\{\exp\big\{\frac{1}{t}\int_0^t\int_0^t V(W_\sigma^{(1)}-W_s^{(2)})\mathrm{d}\sigma\mathrm{d}s\big\}\,1\!\!1_G\Big\}$$
$$\geq \sup_{\xi^\otimes\in F}\sum_j\Big\{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}V(x-y)\psi_j^2(x)\phi_j^2(y)\mathrm{d}x\mathrm{d}y-\frac{1}{2}\sum_j\big\|\nabla\psi_j\big\|_2^2-\frac{1}{2}\sum_j\big\|\nabla\phi_j\big\|_2^2\Big\}$$

For the total mass $Z_t^\otimes$ in (3.9), we can take $G=F=\widetilde{\mathcal{X}}^\otimes$ in the two bounds proved above and conclude (3.10)

$$\lim_{t\to\infty}\frac{1}{t}\log Z_t^\otimes=\sup_{\xi^\otimes\in\widetilde{\mathcal{X}}^\otimes}\sum_j\Big\{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}V(x-y)\psi_j^2(x)\phi_j^2(y)\mathrm{d}x\mathrm{d}y-\frac{1}{2}\sum_j\big\|\nabla\psi_j\big\|_2^2-\frac{1}{2}\sum_j\big\|\nabla\phi_j\big\|_2^2\Big\}$$

If $V(x)=1/|x|$ in $d=3$, then we can follow a truncation argument with replacing the singular function $V(x)=1/|x|$ by $V_\delta(x)=1/(\delta^2+|x|^2)^{1/2}\in\mathcal{F}_1^\otimes$ and invoke the super exponential estimate

$$\limsup_{\delta\to 0}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}^\otimes\bigg[\exp\bigg\{\frac{\lambda}{t}\int_0^t\int_0^t Y_\delta(W_s^{(1)}-W_\sigma^{(2)})\mathrm{d}s\mathrm{d}\sigma\bigg\}\bigg]=0$$

for any $\lambda>0$ to control the difference $Y_\delta=V-V_\delta$. Such an estimate above is a routine check and its proof is omitted. $\square$

**Lemma 3.4.** *If $V(\cdot)$ is positive definite, then*

(3.11)
$$\sup_{\alpha,\beta\in\mathcal{M}_{\leq 1}}\Big\{\int\int V(x-y)\alpha(\,dx)\beta(\mathrm{d}y)-I(\alpha)-I(\beta)\Big\}$$
$$=\sup_{\alpha\in\mathcal{M}_{\leq 1}}\Big\{\int\int V(x-y)\alpha(\,dx)\alpha(\mathrm{d}y)-2I(\alpha)\Big\}$$

*Hence,*

(3.12)
$$\widetilde{\rho}=\sup_{(\widetilde{\alpha}_j)_j\in\widetilde{\mathcal{X}}^\otimes}\sum_j\Big\{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}V(x-y)\psi_j^2(x)\psi_j^2(y)\mathrm{d}x\mathrm{d}y-\big\|\nabla\psi_j\big\|_2^2\Big\}$$

*so that $\alpha_j$ has density $\psi_j^2$ and $\sum_j\int\psi_j^2(x)\mathrm{d}x\leq 1.$*

*Proof.* The lower bound in (3.11) is trivial. For the upper bound, note that, if $V$ is positive definite, then

$$2\int\int V(x-y)\alpha(\mathrm{d}x)\beta(\mathrm{d}y)\leq\int\int V(x-y)\alpha(\mathrm{d}x)\alpha(\mathrm{d}y)+\int\int V(x-y)\beta(\mathrm{d}x)\beta(\mathrm{d}y).$$

$\square$

The following lemma underlines the stability of the variational problem (3.12). This lemma, combined with Lemma 3.3 and Lemma 3.4, will also finish the proof of Theorem 3.1.

**Lemma 3.5.** *Let $V(\cdot)$ be any continuous, positive definite function vanishing at infinity such that the variational problem* (3.1) *has a rotationally symmetric maximizer $\psi_0$ which is unique except for spatial translations. Then the supremum in* (3.12) *is attained only when we have a single orbit $\widetilde{\mu}_0$ with $\mu_0(\mathrm{d}x) = \psi_0^2(x)\mathrm{d}x$ for a unique radially symmetric $\psi_0$ and $\int_{\mathbb{R}^3} \psi_0(x)^2 \mathrm{d}x = 1$*

*Proof.* The proof can be found in Lemma 5.4 in [MV14]. □

### 3.2 Lyapunov exponents of the parabolic Anderson problem in $\mathbb{R}^d$.

Let us now consider the stochastic partial differential equation written formally as

$$\partial_t Z = \frac{1}{2}\Delta Z + Z\xi, \tag{3.13}$$

with a prescribed initial condition. Here $\xi$ denotes white noise in $\mathbb{R}^d$, which is a centered Gaussian process with covariance kernel $\mathbb{E}\big(\xi(x)\xi(y)\big) = \delta_0(x-y)$. Since the random field $\xi = \{\xi(x)\}_{x\in\mathbb{R}^d}$ can not be defined pointwise and the product $Z\xi$ is ill-defined, we need a smoothing procedure leading to a mollified and well defined version of (3.13). We are interested in the asymptotic growth rate of the moments of the smoothened solution under a suitable “rescaling". In other words, we study the *annealed Lyapunov exponents* of the smoothened model as the smoothing parameter is turned off. The The large deviation theory developed in Section 2 and Section 3.1 allows a direct and explicit computation of the objects under interest and implies, in particular, a certain *intermittency* effect exhibited by the smoothened model. Let us now turn to a forma definition of the model.

Let us fix $d \geq 3$ and for each $\varepsilon > 0$, let $\phi_\varepsilon$ be a smooth mollifier in $\mathbb{R}^d$, i.e., $\phi_\varepsilon(x) = \varepsilon^{-d}\phi(x/\varepsilon)$ for some smooth, positive definite, even function $\phi$ with compact support and $\int_{\mathbb{R}^d} \phi = 1$. Then $\int_{\mathbb{R}^d} \phi_\varepsilon = 1$ and $\phi_\varepsilon \Rightarrow \delta_0$ as $\varepsilon \to 0$, where $\Rightarrow$ stands for the weak convergence of probability measures as before. Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space of rapidly decreasing functions. On a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we denote by $\xi = \{\xi(f)\}_{f\in\mathcal{S}}$ a centered Gaussian field with covariance $\mathbb{E}\{\xi(f)\xi(g)\} = \int_{\mathbb{R}^d} f(x)g(x)\mathrm{d}x$. This can also be defined pointwise in $\mathbb{R}^d$ by setting

$$\xi_\varepsilon(x) = \xi\big(\phi_\varepsilon(x-\cdot)\big) = \big(\xi \star \phi_\varepsilon\big)(x).$$

Note that $\xi_\varepsilon = \{\xi_\varepsilon(x)\}_{x\in\mathbb{R}^d}$ is also a centered Gaussian process with covariance

$$\mathbb{E}\big\{\xi_\varepsilon(x)\xi_\varepsilon(y)\big\} = \int_{\mathbb{R}^d} \phi_\varepsilon(x-z)\phi_\varepsilon(y-z)\,\mathrm{d}z = \big(\phi_\varepsilon \star \phi_\varepsilon\big)(x-y), = V_\varepsilon(x-y). \tag{3.14}$$

and we denoted $V_\varepsilon = \phi_\varepsilon \star \phi_\varepsilon$. We want to consider the mollified version of (3.13) and study the asymptotic growth rate of the moments of the solution as $\varepsilon \to 0$. For our purposes, this leads to a rescaled equation

$$\begin{aligned} &\partial_t Z_\varepsilon = \frac{1}{2}\Delta Z_\varepsilon + C(\varepsilon)\, Z_\varepsilon \xi_\varepsilon \\ &Z_\varepsilon(0,x) = 1 \end{aligned} \tag{3.15}$$

where

$$C(\varepsilon) = \varepsilon^{\frac{d-2}{2}} \qquad d \geq 3 \tag{3.16}$$

Then the Feynman-Kac solution is given by

$$Z_\varepsilon(t,x) = E_x\bigg\{\exp\bigg\{C(\varepsilon)\int_0^t \xi_\varepsilon(W_s)\,\mathrm{d}s\bigg\}\bigg\}. \tag{3.17}$$

where $E_x$ refers to the expectation with respect to the Wiener measure $P_x$ for a $d$- dimensional Brownian motion starting at $x \in \mathbb{R}^d$. Since we are interested in the behavior of $Z_\varepsilon(t,x)$ as $\varepsilon \to 0$ for fixed $t$, we will write $Z_\varepsilon(x) = Z_\varepsilon(1,x)$ and study the asymptotic behavior of

$$m_p(\varepsilon,x) = \mathbb{E}\big[Z_\varepsilon(x)^p\big] p \in \mathbb{N}$$

as $\varepsilon \to 0$.

**Theorem 3.6.** *For any $p \in \mathbb{N}$ and $x \in \mathbb{R}^d$,*

(3.18)

$$\begin{aligned}\lim_{\varepsilon\to 0} \varepsilon^2 \log m_p(\varepsilon,x) &= m_p \\ &= 2^{p-1} \sup_{\substack{\psi\in H^1(\mathbb{R}^d)\\ \|\psi\|_2=1}} \bigg\{2^{p-2}\int\int_{\mathbb{R}^d\times\mathbb{R}^d} V(x-y)\psi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y - \frac{1}{2}\|\nabla\psi\|_2^2\bigg\},\end{aligned}$$

*where $V = V_\phi = \phi \star \phi$.*

***Remark* 1.** *It follows directly from Theorem 3.6 that*

$$m_1 < \frac{m_2}{2} < \frac{m_3}{3} < \dots.$$

*This strict ordering is related to what is known as intermittency. As explained in* [GM90]*, intermittent random fields are distinguished by formation of a peculiar spatial structure of strong pronounced "islands"(such as sharp peaks) which determine the main contribution to the physical process in such media.*

***Remark* 2.** *Recently Hairer and Labbe (*[HL15]*) have carried out a robust construction of the ill-posed equation* (3.13) *in $\mathbb{R}^3$ based on the seminal work on the theory of regularly structures (*[H14]*). This result shows that if we set the rescaling $C(\varepsilon) = 1$ in the mollified equation* (3.15)*, then the approving solution diverges like $\mathrm{e}^{1/\varepsilon}$ (in leading order). Hence for a well-posed limit one must consider the renormalized equation $\partial_t X_\varepsilon = \frac{1}{2}\Delta X_\varepsilon + \xi_\varepsilon(X_\varepsilon - \kappa_\varepsilon)$ where $\kappa_\varepsilon \sim 1/\varepsilon + |\log\varepsilon|$. Note that this nature of renormalization is different from our scaling, subtracting the "infinite"constant $\kappa_\varepsilon$ is crucial for the convergence to a well-defined solution in a robust sense (*[HL15]*). On the other hand, although our purpose is significantly different and modest, a simple scaling argument in our proof shows that in $d = 3$, if we set $C(\varepsilon) = 1$ in* (3.15)*, then the first moment of the solution is the integral*

$$E_x\bigg[\exp\bigg\{\varepsilon\int_0^{1/\varepsilon^2}\int_0^{1/\varepsilon^2} V(W_\sigma - W_s)\mathrm{d}\sigma\mathrm{d}s\bigg\}\bigg].$$

*Note that the above double integral diverges like $\varepsilon^{-2}$ along the diagonal and hence divergence of the first moment of the solution coincides with the aforementioned $\mathrm{e}^{1/\varepsilon}$ divergence.*

Let us now turn to the short proof of Theorem 3.6.

**Proof of Theorem 3.6.** We fix any starting point $x \in \mathbb{R}^d$ and handle the case $p = 1$ first. Then,

$$\begin{aligned} m_1(\varepsilon,x) = \mathbb{E}\big(Z_\varepsilon(x)\big) &= \mathbb{E}\bigg\{E^{(x)}\bigg(\mathrm{e}^{\varepsilon^{\frac{d}{2}-1}\int_0^1 \xi_\varepsilon(W_s)\mathrm{d}s}\bigg)\bigg\} \\ &= E^{(x)}\bigg[\exp\bigg\{\frac{1}{2}\varepsilon^{d-2}\int_0^1\int_0^1 \mathrm{d}\sigma\mathrm{d}s\, V_\varepsilon(W_\sigma - W_s)\bigg\}\bigg],\end{aligned}$$

since $\{\xi_\varepsilon(x)\}_{x\mathbb{R}^d}$ is centered Gaussian with covariance given by (3.14). But,

$$V_\varepsilon(x-y) = \big(\phi_\varepsilon \star \phi_\varepsilon\big)(x-y) = \varepsilon^{-d} V\bigg(\frac{x-y}{\varepsilon}\bigg),$$

where $V = \phi \star \phi$. By Brownian scaling,

$$\begin{aligned} m_1(\varepsilon,x) &= E_x\bigg[\exp\bigg\{\frac{1}{2}\varepsilon^{d-2}\,\varepsilon^{-d}\int_0^1\int_0^1 \mathrm{d}\sigma\mathrm{d}s\,V(\varepsilon^{-1}(W_\sigma-W_s))\bigg\}\bigg] \\ &= E_x\bigg[\exp\bigg\{\frac{1}{2}\varepsilon^{d-2}\,\varepsilon^{-d}\int_0^1\int_0^1 \mathrm{d}\sigma\mathrm{d}s\,V\big(W_{\sigma/\varepsilon^2}-W_{s/\varepsilon^2}\big)\bigg\}\bigg] \\ &= E_x\bigg[\exp\bigg\{\frac{1}{2}\varepsilon^{2}\int_0^{1/\varepsilon^2}\int_0^{1/\varepsilon^2} \mathrm{d}\sigma\mathrm{d}s\,V\big(W_\sigma-W_s\big)\bigg\}\bigg] \\ &= E_x\bigg\{\exp\bigg\{\frac{1}{2\tau}\int_0^\tau\int_0^\tau \mathrm{d}\sigma\mathrm{d}s\,V(W_\sigma-W_s)\bigg\}\bigg\} \end{aligned}$$

for $\tau = \varepsilon^{-2}$. Since the first moment involves only one path, by (5.10) in Theorem 5.3 in our earlier work ([MV14]) and Lemma 3.5,

$$\begin{aligned} \lim_{\varepsilon\to0}\varepsilon^2\log m_1(\varepsilon,x) &= \lim_{\tau\to\infty}\frac{1}{\tau}\log E_x\bigg\{\exp\bigg\{\frac{1}{2\tau}\int_0^\tau\int_0^\tau \mathrm{d}\sigma\mathrm{d}s\,V(W_\sigma-W_s)\bigg\}\bigg\} \\ &= \sup_{\|\psi\|_2=1}\bigg\{\frac{1}{2}\int\int_{\mathbb{R}^d\times\mathbb{R}^d}V(x-y)\psi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y-\frac{1}{2}\|\nabla\psi\|_2^2\bigg\} = m_1, \end{aligned}$$

proving Theorem 3.6 for $p=1$.

Let us now turn to the case $p \geq 2$ and focus on the case $p=2$ for simplicity. Let $W^{(1)}, W^{(2)}$ be two independent Brownian motions both starting at $x \in \mathbb{R}^d$ with $E_x^{\otimes}$ denoting their joint distribution. Then, using similar scaling relations as before,

$$\begin{aligned} m_p(\varepsilon,x) &= \mathbb{E}\bigg[E_x^{\otimes}\bigg\{\mathrm{e}^{\sum_{i=1}^2\varepsilon^{d/2-1}\int_0^1\xi_\varepsilon\left(W_s^{(i)}\right)\mathrm{d}s}\bigg\}\bigg] \\ &= E_x^{\otimes}\bigg[\exp\bigg\{\frac{1}{2}\sum_{i,j=1}^2\varepsilon^{d-2}\int_0^1\int_0^1 V_\varepsilon\big(W_\sigma^{(i)}-W_s^{(j)}\big)\mathrm{d}\sigma\mathrm{d}s\bigg\}\bigg] \\ &= E_x^{\otimes}\bigg[\exp\bigg\{\frac{1}{2}\sum_{i,j=1}^2\varepsilon^{2}\int_0^{1/\varepsilon^2}\int_0^{1/\varepsilon^2} V\big(W_\sigma^{(i)}-W_s^{(j)}\big)\mathrm{d}\sigma\mathrm{d}s\bigg\}\bigg]. \end{aligned}$$

We combine (3.10) in Lemma 3.3, Lemma 3.4 (recall $V$ is positive definite) and Lemma 3.5,

$$\begin{aligned} \limsup_{\varepsilon\to0}\varepsilon^2\log\mathbb{E}\big(m_2(\varepsilon,x)\big) &= 2\sup_{\substack{\psi\in H^1(\mathbb{R}^d)\\ \|\psi\|_2=1}}\bigg\{\int\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{d}x\mathrm{d}y\,V(x-y)\psi^2(x)\psi^2(y)-\frac{1}{2}\|\nabla\psi\|_2^2\bigg\} \\ &= m_2. \end{aligned}$$

□

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