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Perturbation determinants for singular perturbations

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Abstract

Let A be a densely defined symmetric operator and let $\{\widetilde{A}', \widetilde{A}\}$ be an ordered pair of proper extensions of A such that their resolvent difference is of trace class. We study the perturbation determinant $\Delta_{\widetilde{A}'/\widetilde{A}}(\cdot)$ of the singular pair $\{\widetilde{A}', \widetilde{A}\}$ by using the boundary triplet approach. We show that under additional mild assumptions on $\{\widetilde{A}', \widetilde{A}\}$ the perturbation determinant $\Delta_{\widetilde{A}'/\widetilde{A}}(\cdot)$ is a ratio of two ordinary determinants involving Weyl function and boundary operators. In particular, if the deficiency indices of A are finite, then

$$\Delta_{\widetilde{A}'/\widetilde{A}}(z) = \frac{\det(B' - M(z))}{\det(B - M(z))}, \quad z \in \rho(\widetilde{A}),$$

where $M(\cdot)$ is the Weyl function and B' and B the boundary operators corresponding to $\widetilde{A}'$ and $\widetilde{A}$ with respect to a chosen boundary triplet Π. The results are applied to ordinary differential operators and to second order elliptic operators.

1 Introduction

The perturbation determinant was introduced by Krein [39] and, independently, by Kuroda in [45]. It is an important tool in studying the spectral shift functions and trace formulas for pairs of self-adjoint operators [9, 10, 39, 42, 43] and non-selfadjoint operators [44] as well. It was also used to analyze certain other properties of non-selfadjoint operators as the completeness of the root vectors, estimates for resolvents of operators with discrete spectrum, etc, cf. [40, 41]. During three last decades the perturbation determinants attract certain attention in connection with spectral shift functions and (higher order) trace formulas for self-adjoint and dissipative operators (see [3, 4, 6, 24, 58, 59, 60, 61, 62, 63, 64, 65]). Applications of perturbation determinants to Schrödinger operators (especially in connection with Jost-Pais formulas) have intensively been studied in [23, 24, 25, 26, 27, 28, 29, 30] and [46, 47, 48].

The following definition goes back to M.G. Krein [42, 44] (see also [31, 69]).

Definition 1.1. An ordered pair $\{H', H\}$ of densely defined closed operators defined on a separable Hilbert space $\mathfrak{H}$ is put to the class $\mathfrak{D}$ if

(i) $\rho(H') \cap \rho(H) \neq \emptyset$,

(ii) $\mathrm{dom}\,(H') = \mathrm{dom}\,(H)$,

(iii) $(H' - H)(H - z)^{-1} \in \mathfrak{S}_1(\mathfrak{H})$ for $z \in \rho(H)$.

If $\{H', H\} \in \mathfrak{D}$, then the scalar-valued function $\Delta_{H'/H}(\cdot)$

$$\Delta_{H'/H}(z) := \det(I + (H' - H)(H - z)^{-1}), \quad z \in \rho(H), \tag{1.1}$$

is called the perturbation determinant of the pair $\{H', H\}$.

Clearly, $\Delta_{H'/H}(\cdot)$ is a holomorphic function. The properties of perturbation determinants are summarized in [12, 31, 69] (see also Appendix). In what follows pairs $\{H', H\}$ satisfying conditions (i) and (ii) are called *regular*. However, non-regular pairs are typical for boundary value problems of differential operators. In the sequel a pair $\{\widetilde{A}', \widetilde{A}\}$ of operators is called *singular* if condition (i) is satisfied and both operators $\widetilde{A}'$ and $\widetilde{A}$ are proper extensions of a densely defined symmetric operator A. In this case condition (ii) is always violated: $\mathrm{dom}\,(\widetilde{A}') = \mathrm{dom}\,(\widetilde{A})$ if and only if $\widetilde{A}' = \widetilde{A}$.

In what follows we consider only *singular* pairs $\{H', H\}$ satisfying

$$R_\xi(H', H) := (H' - \xi)^{-1} - (H - \xi)^{-1} \in \mathfrak{S}_1(\mathfrak{H}), \quad \xi \in \rho(H') \cap \rho(H), \tag{1.2}$$

and denote by $\widetilde{\mathfrak{D}}$ the class of such pairs. For the pair $\{H', H\} \in \widetilde{\mathfrak{D}}$ using the Cayley transform instead of (1.1) a family of perturbation determinants depending on parameter $\xi \in \rho(H') \cap \rho(H)$ is introduced by

$$\begin{aligned}\widetilde{\Delta}_{H'/H}(\xi, z) &:= \det\left((H' - z)(H' - \xi)^{-1}(H - \xi)(H - z)^{-1}\right) \\ &= \det\left(I + (\xi - z)R_\xi(H', H)(H - \xi)(H - z)^{-1}\right), \quad z \in \rho(H),\end{aligned} \tag{1.3}$$

see [43, 12, 69]. Notice that $\widetilde{\Delta}_{H'/H}(\xi, \xi) = 1$. Moreover, if $\{H', H\} \in \mathfrak{D}$, then $\{H', H\} \in \widetilde{\mathfrak{D}}$ and the following representation holds [69, Chapter 8.1.3]

$$\widetilde{\Delta}_{H',H}(\xi, z) = \frac{\Delta_{H',H}(z)}{\Delta_{H',H}(\xi)}, \qquad z, \xi \in \rho(H') \cap \rho(H), \tag{1.4}$$

i.e. for any fixed ξ the determinants $\widetilde{\Delta}_{H',H}(\xi, \cdot)$ and $\Delta_{H',H}(\cdot)$ coincide up to a multiplicative constant $c(\xi) := (\Delta_{H',H}(\xi))^{-1} \in \mathbb{C}$. However, definition (1.3) has a few drawbacks. The main of them is that, in fact, it is not suitable for applications to boundary value problems. A different approach to the (symmetrized) perturbation determinants for singular perturbations has been proposed by Gesztesy and Zinchenko [30]. It is based on the use of positive-type operators and its applicability requires that one of the square root domains of H and H' contains the other instead of condition (ii) of Definition 1.1. Note also that our definition of singular pairs is in accordance with the notion of singularly perturbed operators from [38].

Our aim is *to extend Krein's theory of perturbation determinants to the case of singular pairs and apply it to boundary value problems*. Our approach substantially uses the machinery of boundary triplets and the corresponding Weyl functions (see Section 2 for precise definitions). This new approach to extension theory of symmetric operators has been appeared and elaborated during the last three decades (see [15, 17, 18, 32, 50, 54, 53, 16] and references therein).

In what follows A denotes a closed densely defined symmetric operator in $\mathfrak{H}$ with equal deficiency indices $n_+(A) = n_-(A) \le \infty$.

Recall that a triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is an auxiliary separable Hilbert space and $\Gamma_0, \Gamma_1 : \operatorname{dom}(A^*) \to \mathcal{H}$ are linear mappings, is called a boundary triplet for A^* if the äbstract Green's identity"

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \qquad f, g \in \operatorname{dom}(A^*), \tag{1.5}$$

holds and the mapping $\Gamma := (\Gamma_0, \Gamma_1)^\top : \operatorname{dom}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

A boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* always exists, though it is not unique. Its role in extension theory is similar to that of a coordinate system in analytic geometry. It leads to a natural parameterization of the set Ext_A of proper extensions $\widetilde{A}$ of A ($A \subset \widetilde{A} \subset A^*$) by means of the set $\widetilde{\mathcal{C}}(\mathcal{H})$ of linear relations (multi-valued operators) in $\mathcal{H}$, see [15, 18, 32] for details. In this paper we mostly consider boundary relations Θ being the graph $\operatorname{gr}(B)$ of a closed linear operator B in $\mathcal{H}$ ($B \in \mathcal{C}(\mathcal{H})$). In this case the extension $\widetilde{A}$ is given by

$$\widetilde{A} := A_B := A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0), \tag{1.6}$$

where B is called *the boundary operator* of the extension $\widetilde{A}$ with respect to Π.

The main analytical tool in this approach is the abstract Weyl function $M(\cdot)$ (see Definition 2.4) introduced and studied in [18, 22] which is holomorphic on the resolvent set $\rho(A_0)$ of $A_0 := A^* \upharpoonright \ker(\Gamma_0) = A_0^*$. Its role in the theory of boundary triplets is similar to that of the classical Weyl-Titchmarsh function in the theory of scalar Sturm-Liouville operators (see [13, 18, 54]). For instance,

$$\rho(A_B) \cap \rho(A_0) = \{z \in \rho(A_0) : 0 \in \rho(B - M(z))\}. \tag{1.7}$$

Within the framework of boundary triplets approach our definition of the perturbation determinant of a pair $\{\widetilde{A}', \widetilde{A}\} \subset \operatorname{Ext}_A$ reads as follows.

Definition 1.2. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and $M(\cdot)$ the corresponding Weyl function. We put the ordered pair $\{\widetilde{A}', \widetilde{A}\}$ of proper extensions of A to the class $\mathfrak{D}^\Pi$ if $\widetilde{A}'$ and $\widetilde{A}$ admit representations* (1.6) *with boundary operators B' and B, respectively, satisfying the following conditions:*

(i) $0 \in \rho(B' - M(z)) \cap \rho(B - M(z))$ *for some* $z \in \mathbb{C}$,

(ii) $\operatorname{dom}(B') = \operatorname{dom}(B)$,

(iii) $(B' - B)(B - M(z))^{-1} \in \mathfrak{S}_1(\mathfrak{H})$ *for* $\{z \in \mathbb{C} : 0 \in \rho(B - M(z))\}$.

If $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^\Pi$, then the scalar-valued function

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z) := \det\Big(I_{\mathcal{H}} + \big(B' - B\big)\big(B - M(z)\big)^{-1}\Big), \quad 0 \in \rho(B - M(z)), \tag{1.8}$$

is called the perturbation determinant of the pair $\{\widetilde{A}', \widetilde{A}\}$ with respect to Π.

Note that, according to (1.7) the condition (i) is equivalent to $\rho(A_B)\cap\rho(A'_B)\cap\rho(A_0)\neq\emptyset$ and condition (iii) is valid for $z\in\rho(A_0)\cap\rho(A_B)$.

We show that the implication $\{\widetilde{A}',\widetilde{A}\}\in\widetilde{\mathfrak{D}}\Longrightarrow\{\widetilde{A}',\widetilde{A}\}\in\mathfrak{D}^\Pi$ holds whenever

$$\rho(\widetilde{A}')\cap\rho(\widetilde{A})\cap\mathbb{C}_+\neq\emptyset \quad \text{and} \quad (\rho(\widetilde{A})\cup\sigma_c(\widetilde{A}))\cap\mathbb{C}_-\neq\emptyset, \tag{1.9}$$

where $\sigma_c(\cdot)$ is the continuous spectrum of an operator (see below). Clearly, both conditions (1.9) are satisfied if $\rho(\widetilde{A}')\cap\rho(\widetilde{A})\cap\mathbb{R}\neq\emptyset$. Moreover, under assumptions (1.9) on the pair $\{\widetilde{A}',\widetilde{A}\}$ a boundary triplet Π for A^* can be chosen to be regular, i.e. such that the parameterization $\widetilde{A}':=A_{B'}$ and $\widetilde{A}:=A_B$ (cf. (1.6)) holds with *bounded boundary operators* B' and B. The latter remains true without the second condition in (1.9) whenever $n_\pm(A)=n<\infty$.

Comparing definitions (1.8) and (1.1) we see that the class $\mathfrak{D}$ is transformed into the class $\mathfrak{D}^\Pi$ by means of the "transformation"

$$H\longleftrightarrow B \quad \text{and} \quad z\longleftrightarrow M(z). \tag{1.10}$$

These correspondences appears natural if one allows A to be non-densely defined. In fact, the boundary triplet approach allows an extension to such symmetric operators. For instance, let $A=0$ with $\mathrm{dom}\,(A)=\{0\}$. For this trivial non-densely defined symmetric operator there is an appropriate boundary triplet Π such that the corresponding Weyl function is $M(z)=zI_{\mathfrak{H}}$ and $A_{B'}=B'$, $A_B=B$ (see [20]). Hence, $\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(\cdot)$ given by (1.8) coincides with (1.1).

It follows from the Krein type formula (see (2.7) below) that the inclusion $\{\widetilde{A}',\widetilde{A}\}\in\mathfrak{D}^\Pi$ implies $\{\widetilde{A}',\widetilde{A}\}\in\widetilde{\mathfrak{D}}$, i.e., it implies condition (1.2) with H' and H replaced by $\widetilde{A}'$ and $\widetilde{A}$, respectively. Thus, definition (1.3) can also be applied to the pair $\{\widetilde{A}',\widetilde{A}\}\in\mathfrak{D}^\Pi\subseteq\widetilde{\mathfrak{D}}$. We show (see Theorem 4.1) that in this case the perturbation determinants $\widetilde{\Delta}_{\widetilde{A}',\widetilde{A}}(\xi,\cdot)$ and $\Delta^\Pi_{\widetilde{A}',\widetilde{A}}(\cdot)$ are connected by

$$\widetilde{\Delta}_{\widetilde{A}',\widetilde{A}}(\xi,z)=\frac{\Delta^\Pi_{\widetilde{A}',\widetilde{A}}(z)}{\Delta^\Pi_{\widetilde{A}',\widetilde{A}}(\xi)}, \qquad z,\xi\in\rho(\widetilde{A})\cap\rho(A_0). \tag{1.11}$$

In other words, for any fixed ξ these determinants coincide up to a multiplicative constant $c(\xi)=(\Delta^\Pi_{\widetilde{A}',\widetilde{A}}(\xi))^{-1}$. Clearly, representation (1.11) is similar to representation (1.4) and is in accordance with the correspondence principle (1.10).

To demonstrate the advantage of our approach we note that definition (1.8) allows one to express $\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(\cdot)$ as *a ratio of two ordinary determinants involving only boundary operators and the corresponding Weyl function*. Firstly, we consider the case of the operator A with finite deficiency indices $n_\pm(A)=n<\infty$. In this case, as an immediate consequence of (1.8) one gets that

$$\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(z):=\frac{\det(B'-M(z))}{\det(B-M(z))}, \qquad z\in\rho(\widetilde{A}')\cap\rho(\widetilde{A}). \tag{1.12}$$

For instance, let $A:=A_{\min}$ be the minimal symmetric operator generated in $L^2(\mathbb{R}_+)$ by Sturm-Liouville differential expression $\mathcal{L}=-D^2+q,\ q=\overline{q}\in L^1_{\mathrm{loc}}[0,\infty)$. Assuming the limit point case at infinity, one can choose a boundary triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^* as follows (see Section 7.2)

$$\mathcal{H}=\mathbb{C},\quad \Gamma_0 f=f(0),\quad \Gamma_1 f=f'(0),\quad f\in\mathrm{dom}\,(A^*).$$

Let also $L_j := A_{h_j}, j \in \{1,2\}$, be a proper extension of A given by (cf. (1.6))

$$\mathrm{dom}\,(A_{h_j}) = \{y \in \mathrm{dom}\,(A^*) :\ y'(0) = h_j y(0)\}, \quad j \in \{1,2\}.$$

Then according to (1.12) the perturbation determinant $\Delta^{\Pi}_{L_2/L_1}(\cdot)$ is

$$\Delta^{\Pi}_{L_2/L_1}(z) = \frac{h_2 - m(z)}{h_1 - m(z)},$$

where $m(\cdot)$ is the Weyl function (of the Dirichlet realization) corresponding to the boundary triplet Π. A similar representation for perturbation determinants is valid for two realizations of the Sturm-Liouville operator with a matrix-valued potential as well as of Dirac type operators (see Sections 7.1, 7.2, and 7.3).

Secondary, we consider the case of an operator A with infinite deficiency indices $n_\pm(A) = \infty$. To obtain an analog of formula (1.12) in this case we slightly strength the assumption $R_\xi(\widetilde{A}', \widetilde{A}) \in \mathfrak{S}_1(\mathfrak{H})$ (cf. (1.2)). Namely, let us assume that for some self-adjoint operator $A_0 = A_0^* \in \mathrm{Ext}\,_A$ the conditions

$$\begin{matrix} (\widetilde{A}' - \xi)^{-1} - (A_0 - \xi)^{-1} \in \mathfrak{S}_1(\mathfrak{H}) \\ (\widetilde{A} - \xi)^{-1} - (A_0 - \xi)^{-1} \in \mathfrak{S}_1(\mathfrak{H}) \end{matrix}, \qquad \xi \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0), \tag{1.13}$$

are satisfied. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* such that $A_0 := A^* \upharpoonright \ker(\Gamma_0)$, $\widetilde{A}' = A_{B'}$, and $\widetilde{A} = A_B$ with $B', B \in \mathcal{C}(\mathcal{H})$. From (1.13) it follows that $(B' - \mu)^{-1},\ (B - \mu)^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for some $\mu \in \rho(B') \cap \rho(B) \cap \mathbb{R}$. Moreover, denoting by $M(\cdot)$ the Weyl function of Π we get

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z) = \Delta_{B'/B}(\mu) \frac{\det(I - (\mu - B')^{-1}(\mu - M(z)))}{\det(I - (\mu - B)^{-1}(\mu - M(z)))} \tag{1.14}$$

for $z \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$, where $\Delta_{B'/B}(\cdot)$ is given in accordance with (1.1).

Formula (1.14) can be applied to boundary value problems for partial differential equations. For instance, consider the symmetric Schrödinger operator in domain $\Omega \subset \mathbb{R}^2$ with smooth compact boundary $\partial\Omega$,

$$\mathcal{A} := -\Delta + q(x) = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + q(x), \quad q = \overline{q} \in C^\infty(\overline{\Omega}). \tag{1.15}$$

Furthermore, consider Robin-type realizations of the expression $\mathcal{A}$,

$$\begin{aligned} \widehat{A}_{\sigma_j} &:= A_{\max} \upharpoonright \mathrm{dom}\,(\widehat{A}_{\sigma_j}), \\ \mathrm{dom}\,(\widehat{A}_{\sigma_j}) &:= \{f \in H^2(\Omega) : G_1 f = \sigma G_0 f\}, \quad j \in \{1,2\}, \end{aligned} \tag{1.16}$$

and denote by A_0 the Dirichlet realization of $\mathcal{A}$ given by $\mathrm{dom}\,(A_0) = \{f \in H^2(\Omega) : G_0 f = 0\}$. Here G_0 and G_1 are trace operators, $G_0 u := \gamma_0 u := u|_{\partial\Omega}$ and $G_1 u := \gamma_0\big(\partial u/\partial\nu\big),\ u \in \mathrm{dom}\,(A_{\max})$. It is known that $A_0 = A_0^*$ and the realization $\widehat{A}_{\sigma_j}$ is closed whenever $\sigma_j \in C^2(\partial\Omega)$ and self-adjoint if σ is real.

Denote by $\widehat{\sigma}_j$ the multiplication operator induced by σ_j in $L^2(\partial\Omega)$. Assuming that $0 \in \rho(\widehat{A}_{\sigma_1}) \cap \rho(\widehat{A}_{\sigma_2}) \cap \rho(A_0)$, we indicate a boundary triplet Π for $A_{\max}$ such that $\{\widehat{A}_{\sigma_j}, A_0\} \in \mathfrak{D}^\Pi$ and the corresponding perturbation determinants $\Delta^\Pi_{\widehat{A}_{\sigma_j}/A_0}(\cdot)$ and $\Delta^\Pi_{\widehat{A}_{\sigma_2}/A_{\sigma_1}}(\cdot)$ are given by

$$\Delta^\Pi_{\widehat{A}_{\sigma_j}/A_0}(z) = \det{}_{L^2(\partial\Omega)}\left(I - (\Lambda_{0,0}(z) - \Lambda_{0,0}(0))(\widehat{\sigma}_j - \Lambda_{0,0}(0))^{-1}\right), \tag{1.17}$$

$j \in \{1,2\}$, and

$$\Delta^\Pi_{\widehat{A}_{\sigma_2}/A_{\sigma_1}}(z) = \frac{\det_{L^2(\partial\Omega)}\left(I - (\Lambda_{0,0}(z) - \Lambda_{0,0}(0))(\widehat{\sigma}_2 - \Lambda_{0,0}(0))^{-1}\right)}{\det_{L^2(\partial\Omega)}\left(I - (\Lambda_{0,0}(z) - \Lambda_{0,0}(0))(\widehat{\sigma}_1 - \Lambda_{0,0}(0))^{-1}\right)}, \tag{1.18}$$

$z \in \rho(\widehat{A}_{\sigma_1}) \cap \rho(\widehat{A}_{\sigma_2}) \cap \rho(A_0)$, respectively. Here $\Lambda_{0,0}(\cdot)$ is the Dirichlet to Neumann map restricted to $H^0(\partial\Omega) := L^2(\partial\Omega)$ (see Section 6.3 for details).

The paper is organized as follows. In Section 2 we give a brief introduction into the boundary triplet approach. In Section 3 we introduce a concept of jointly almost solvable extensions $\{\widetilde{A}_j\}_{j=1}^N \subset \mathrm{Ext}_A$ and discuss their properties. It is proved in Theorem 3.5 that under assumption (1.9) on $\widetilde{A}$ there is a boundary triplet Π for A^* which is regular for the pair $\{\widetilde{A}', \widetilde{A}\}$ and, in particular, the implication $\{\widetilde{A}', \widetilde{A}\} \in \widetilde{\mathfrak{D}} \Longrightarrow \{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^\Pi$ holds. In Section 4 we prove the main results on connection between two definitions (1.3) and (1.8) of determinants. In particular, we prove representation (1.11) for $\widetilde{\Delta}_{H'/H}(\xi, \cdot)$ (Theorem 4.2) and formula (1.14). It is also shown here that if $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^\Pi$ and $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi'}$, then the perturbation determinants $\Delta^\Pi_{\widetilde{A}',\widetilde{A}}(\cdot)$ and $\Delta^{\Pi'}_{\widetilde{A}',\widetilde{A}}(\cdot)$ corresponding to the triplets Π and Π' coincide up to a multiplicative constant.

Certain properties of the perturbation determinant $\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(\cdot)$ are discussed in Section 5. In Section 6 we show that under certain additional assumptions the determinant $d(\cdot) := \Delta^\Pi_{\widetilde{A}/\widetilde{A}^*}(\cdot)$ is an annihilation function (in the sense of [67]) for a m-dissipative operator $\widetilde{A}$. We also indicate conditions guaranteeing that $d(\cdot)$ is the minimal annihilation function. Finally, in Section 7 the abstract theory is applied to some ordinary differential operators as well as to elliptic operators on domains with compact boundary. In particular, formulas (1.17) and (1.18) are established there. To make the paper self-contained an appendix is added.

In the forthcoming paper we apply our results to trace formulas. A preliminary version of the paper has been published as a preprint [52].

Notation. Let $\mathfrak{H}$ and $\mathcal{H}$ be separable Hilbert spaces. The set of bounded linear operators from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ is denoted by $[\mathfrak{H}_1, \mathfrak{H}_2]$; $[\mathfrak{H}] := [\mathfrak{H}, \mathfrak{H}]$. By $\mathfrak{S}_p(\mathfrak{H})$, $p \in (0, \infty]$, we denote the Schatten-v.Neumann ideals of compact operators on $\mathfrak{H}$; in particular, $\mathfrak{S}_\infty(\mathfrak{H})$ denotes the ideal of compact operators in $\mathfrak{H}$.

By $\mathrm{dom}\,(T)$, $\mathrm{ran}\,(T)$ and $\sigma(T)$ we denote the domain, range and spectrum of the operator T, respectively. The symbols $\sigma_p(\cdot)$, $\sigma_c(\cdot)$ and $\sigma_r(\cdot)$ stand for the point, continuous and residual spectrum of a linear operator. Recall that $z \in \sigma_c(H)$ if $\ker(H - z) = \{0\}$ and $\mathrm{ran}\,(H - z) \neq \overline{\mathrm{ran}\,(H - z)} = \mathfrak{H}$; $z \in \sigma_r(H)$ if $\ker(H - z) = \{0\}$ and $\overline{\mathrm{ran}\,(H - z)} \neq \mathfrak{H}$.

2 Preliminaries

2.1 Linear relations

A linear relation Θ in $\mathcal{H}$ is a closed linear subspace of $\mathcal{H}\oplus\mathcal{H}$. The set of all linear relations in $\mathcal{H}$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{H})$. Denote also by $\mathcal{C}(\mathcal{H})$ the set of all closed linear (not necessarily densely defined) operators in $\mathcal{H}$. Identifying each operator $T\in\mathcal{C}(\mathcal{H})$ with its graph $\mathrm{gr}\,(T)$ we regard $\mathcal{C}(\mathcal{H})$ as a subset of $\widetilde{\mathcal{C}}(\mathcal{H})$.

The role of the set $\widetilde{\mathcal{C}}(\mathcal{H})$ in extension theory becomes clear from Proposition 2.3. However, its role in the operator theory is substantially motivated by the following circumstances: in contrast to $\mathcal{C}(\mathcal{H})$, the set $\widetilde{\mathcal{C}}(\mathcal{H})$ is closed with respect to taking inverse and adjoint relations Θ^{-1} and Θ^*. The latter are given by: $\Theta^{-1}=\{\{g,f\}:\{f,g\}\in\Theta\}$ and

$$\Theta^* = \left\{\begin{pmatrix}k\\k'\end{pmatrix} : (h',k)=(h,k') \text{ for all } \begin{pmatrix}h\\h'\end{pmatrix}\in\Theta\right\}.$$

A linear relation Θ is called symmetric if $\Theta\subset\Theta^*$ and self-adjoint if $\Theta=\Theta^*$.

2.2 Boundary triplets and proper extensions

Let A be a densely defined closed symmetric operator in $\mathfrak{H}$ with equal deficiency indices $n_\pm(A)=\dim(\mathfrak{N}_{\pm i})$, $\mathfrak{N}_z:=\ker(A^*-z)$, $z\in\mathbb{C}_\pm$.

Definition 2.1.

(i) *A closed extension $\widetilde{A}$ of A is called a proper extension if $A\subseteq\widetilde{A}\subseteq A^*$;*

(ii) *Two proper extensions $\widetilde{A}'$, $\widetilde{A}$ are called disjoint if $\mathrm{dom}\,(\widetilde{A}')\cap\mathrm{dom}\,(\widetilde{A})=\mathrm{dom}\,(A)$ and transversal if in addition $\mathrm{dom}\,(\widetilde{A}')+\mathrm{dom}\,(\widetilde{A})=\mathrm{dom}\,(A^*)$.*

Denote by $\widetilde{A}\in\mathrm{Ext}\,_A$, the set of proper extensions of A completed by non-proper extensions A and A^*. Any self-adjoint or maximal dissipative (accumulative) extension is proper.

Definition 2.2 ([33]). *A triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$, where $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0,\Gamma_1:\ \mathrm{dom}\,(A^*)\to\mathcal{H}$ are linear mappings, is called a boundary triplet for A^* if the äbstract Green's identity"*

$$(A^*f,g)-(f,A^*g)=(\Gamma_1 f,\Gamma_0 g)_{\mathcal{H}}-(\Gamma_0 f,\Gamma_1 g)_{\mathcal{H}},\qquad f,g\in\mathrm{dom}\,(A^*), \tag{2.1}$$

holds and the mapping $\Gamma:=(\Gamma_0,\Gamma_1)^\top:\mathrm{dom}\,(A^*)\to\mathcal{H}\oplus\mathcal{H}$ is surjective.

A boundary triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^* always exists whenever $n_+(A)=n_-(A)$. Note also that $n_\pm(A)=\dim(\mathcal{H})$ and $\ker(\Gamma_0)\cap\ker(\Gamma_1)=\mathrm{dom}\,(A)$.

With any boundary triplet Π one associates two canonical self-adjoint extensions $A_j:=A^*\upharpoonright\ker(\Gamma_j)$, $j\in\{0,1\}$. Conversely, for any extension $A_0=A_0^*\in\mathrm{Ext}\,_A$ there exists a (non-unique) boundary triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^* such that $A_0:=A^*\upharpoonright\ker(\Gamma_0)$.

Using the concept of boundary triplets one can parameterize all proper extensions of A in the following way.

Proposition 2.3 ([18, 50]). *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then the mapping*

$$\mathrm{Ext}_A \ni \widetilde{A} \to \Gamma \mathrm{dom}\,(\widetilde{A}) = \{\{\Gamma_0 f, \Gamma_1 f\} :\ f \in \mathrm{dom}\,(\widetilde{A})\} =: \Theta \in \widetilde{\mathcal{C}}(\mathcal{H}) \tag{2.2}$$

establishes a bijective correspondence between the sets Ext_A and $\widetilde{\mathcal{C}}(\mathcal{H})$. We write $\widetilde{A} = A_\Theta$ if $\widetilde{A}$ corresponds to Θ by (2.2)*. Moreover, the following holds:*

(i) $A^*_\Theta = A_{\Theta^*}$*, in particular, $A^*_\Theta = A_\Theta$ if and only if $\Theta^* = \Theta$.*

(ii) A_Θ *is symmetric (self-adjoint) if and only if Θ is symmetric (self-adjoint).*

(iii) A_Θ *is m-dissipative (m-accumulative) if and only if so is Θ.*

(iv) *The extensions A_Θ and A_0 are disjoint (transversal) if and only if $\Theta \in \mathcal{C}(\mathcal{H})$ $(\Theta \in [\mathcal{H}])$. In this case* (2.2) *takes the form*

$$A_\Theta := A_{\mathrm{gr}\,(\Theta)} = A^* \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0). \tag{2.3}$$

In particular, $A_j := A^* \upharpoonright \ker(\Gamma_j) = A_{\Theta_j},\ j \in \{0,1\}$, where $\Theta_0 := \{0\} \times \mathcal{H}$ and $\Theta_1 := \mathcal{H} \times \{0\} = \mathrm{gr}\,(\mathbb{O})$ where $\mathbb{O}$ denotes the zero operator in $\mathcal{H}$. Note also that $\widetilde{\mathcal{C}}(\mathcal{H})$ contains the trivial linear relations $\{0\} \times \{0\}$ and $\mathcal{H} \times \mathcal{H}$ parameterizing the extensions A and A^*, respectively, for any boundary triplet Π.

2.3 Weyl functions and spectra of proper extensions

It is well known that Weyl functions are an important tool in the direct and inverse spectral theory of singular Sturm-Liouville operators. In [17, 18, 22] the concept of Weyl function was generalized to the case of an arbitrary symmetric operator A with $n_+(A) = n_-(A) \le \infty$. Following [18] we briefly recall basic facts on Weyl functions and γ-fields associated with a boundary triplet Π.

Definition 2.4 ([17, 18, 22]). Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and $A_0 = A^* \upharpoonright \ker(\Gamma_0)$. The operator valued functions $\gamma(\cdot) :\ \rho(A_0) \to [\mathcal{H}, \mathfrak{H}]$ and $M(\cdot) :\ \rho(A_0) \to [\mathcal{H}]$ defined by

$$\gamma(z) := \bigl(\Gamma_0 \upharpoonright \mathfrak{N}_z\bigr)^{-1} \qquad \text{and} \qquad M(z) := \Gamma_1 \gamma(z), \quad z \in \rho(A_0), \tag{2.4}$$

are called the γ-field and the Weyl function, respectively, corresponding to the boundary triplet Π.

Clearly, the Weyl function can equivalently be defined by

$$M(z)\Gamma_0 f_z = \Gamma_1 f_z, \qquad f_z \in \mathfrak{N}_z, \quad z \in \rho(A_0).$$

The γ-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ in (2.4) are well defined. Moreover, both $\gamma(\cdot)$ and $M(\cdot)$ are holomorphic on $\rho(A_0)$ and the following relations

$$\gamma(z) = \big(I + (z-\zeta)(A_0 - z)^{-1}\big)\gamma(\zeta), \qquad z,\zeta \in \rho(A_0), \tag{2.5}$$

and

$$M(z) - M(\zeta)^* = (z-\overline{\zeta})\gamma(\zeta)^*\gamma(z), \qquad z,\zeta \in \rho(A_0), \tag{2.6}$$

hold. Identity (2.6) yields that $M(\cdot)$ is $[\mathcal{H}]$-valued Nevanlinna function ($M(\cdot) \in R[\mathcal{H}]$), i.e. $M(\cdot)$ is $[\mathcal{H}]$-valued holomorphic function on $\mathbb{C}\backslash\mathbb{R}$ satisfying

$$M(z) = M(\overline{z})^* \quad \text{and} \quad \frac{\mathrm{Im}\,(M(z))}{\mathrm{Im}\,(z)} \geq 0, \qquad z \in \mathbb{C}_+ \cup \mathbb{C}_-.$$

It follows also from (2.6) that $0 \in \rho(\mathrm{Im}\,(M(z)))$ for all $z \in \mathbb{C}\backslash\mathbb{R}$.

One easily verifies that if $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^*, then $\Pi^\top = \{\mathcal{H}, -\Gamma_1, \Gamma_0\}$ is also a boundary triplet for A^* which is called the transposed one. A straightforward computation shows that

$$M^\top(z) = -M(z)^{-1} \quad \text{and} \quad \gamma^\top(z) = -\gamma(z)M(z)^{-1}, \qquad z \in \mathbb{C}_\pm.$$

Proposition 2.5 ([17, 18]). *Let A be a simple closed densely defined symmetric operator in $\mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function. Assume that $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ and $z \in \rho(A_0)$. Then the following holds:*

(i) *$z \in \rho(A_\Theta)$ if and only if $0 \in \rho(\Theta - M(z))$;*

(ii) *$z \in \sigma_\tau(A_\Theta)$ if and only if $0 \in \sigma_\tau(\Theta - M(z))$, $\tau = p, c, r$. Moreover, $\dim(\ker(A_\Theta - z)) = \dim(\ker(\Theta - M(z)))$.*

2.4 Krein-type formula for resolvents and comparability

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ and $\gamma(\cdot)$ the corresponding Weyl function and γ-field, respectively. For any proper (not necessarily self-adjoint) extension $\widetilde{A}_\Theta \in \mathrm{Ext}\,_A$ with non-empty resolvent set $\rho(\widetilde{A}_\Theta)$ the following Krein-type formula holds (cf. [17, 18, 21, 22])

$$(A_\Theta - z)^{-1} - (A_0 - z)^{-1} = \gamma(z)(\Theta - M(z))^{-1}\gamma^*(\overline{z}), \quad z \in \rho(A_0) \cap \rho(A_\Theta). \tag{2.7}$$

Formula (2.7) extends the known Krein formula for canonical resolvents to the case of any $A_\Theta \in \mathrm{Ext}\,_A$ with $\rho(A_\Theta) \neq \emptyset$. Moreover, due to relations (2.2), (2.3) and (2.4) formula (2.7) is connected with the boundary triplet Π. Emphasize, that this connection makes it possible to apply the Krein-type formula (2.7) to boundary value problems. The following result is deduced from formula (2.7).

Proposition 2.6 ([18, Theorem 2]). *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $\Theta', \Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ and $\rho(A_{\Theta'}) \cap \rho(A_\Theta) \neq \emptyset$. If $\rho(\Theta') \cap \rho(\Theta) \neq \emptyset$, then for any Neumann-Schatten ideal $\mathfrak{S}_p$, $p \in (0, \infty]$, the following holds:*

(i) *The inclusion*

$$(A_{\Theta'} - z)^{-1} - (A_\Theta - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}), \quad z \in \rho(A_{\Theta'}) \cap \rho(A_\Theta), \tag{2.8}$$

is equivalent to the inclusion

$$\big(\Theta' - \zeta\big)^{-1} - \big(\Theta - \zeta\big)^{-1} \in \mathfrak{S}_p(\mathcal{H}), \quad \zeta \in \rho(\Theta') \cap \rho(\Theta). \tag{2.9}$$

In particular, $(A_\Theta - z)^{-1} - (A_0 - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H})$ *for* $z \in \rho(A_\Theta) \cap \rho(A_0)$ *if and only if* $\big(\Theta - \zeta\big)^{-1} \in \mathfrak{S}_p(\mathcal{H})$ *for* $\zeta \in \rho(\Theta)$.

(ii) *If* $B', B \in \mathcal{C}(\mathcal{H})$ *and* $\mathrm{dom}\,(B') = \mathrm{dom}\,(B)$, *then the implication*

$$\overline{B' - B} \in \mathfrak{S}_p(\mathcal{H}) \Longrightarrow (A_{B'}-z)^{-1}-(A_B-z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}), \quad z \in \rho(A_{\Theta'})\cap\rho(A_\Theta), \tag{2.10}$$

holds. Moreover, if $B', B \in [\mathcal{H}]$, *then* (2.8) *is equivalent to* $B' - B \in \mathfrak{S}_p(\mathcal{H})$.

(iii) *The extensions* $A_{\Theta'}$ *and* A_Θ *are transversal if and only if* $0 \in \rho\Big(\big(\Theta' - \zeta\big)^{-1} - \big(\Theta - \zeta\big)^{-1}\Big)$ *for some* $\zeta \in \rho(\Theta') \cap \rho(\Theta)$.

2.5 Determinants

Following [31] let us briefly recall some basic facts on infinite determinants.

Definition 2.7. *Let* T *be a trace class operator, i.e.* $T \in \mathfrak{S}_1(\mathcal{H})$, *and let* $\{\lambda_j(T)\}_{j=1}^\infty$ *be its eigenvalues counted with respect to their algebraic multiplicities. The determinant* $\det(I+T)$ *is defined by* $\det(I+T) := \Pi_{j=1}^\infty\big(1+\lambda_j(T)\big)$.

The determinants have the following interesting properties.

Proposition 2.8 ([31, Section 4.1]). *Let* $T_1 \in [\mathcal{H}_1, \mathcal{H}_2]$ *and* $T_2 \in [\mathcal{H}_2, \mathcal{H}_1]$.

(i) *If* $T_1T_2 \in \mathfrak{S}_1(\mathcal{H}_2)$ *and* $T_2T_1 \in \mathfrak{S}_1(\mathcal{H}_1)$, *then*

$$\det\nolimits_{\mathcal{H}_2}(I + T_1T_2) = \det\nolimits_{\mathcal{H}_1}(I + T_2T_1). \tag{2.11}$$

(ii) *If* $\mathcal{H} := \mathcal{H}_1 = \mathcal{H}_2$ *and* $T_1, T_2 \in \mathfrak{S}_1(\mathcal{H})$, *then*

$$\det[(I+T_1)(I+T_2)] = \det(I+T_1)\cdot\det(I+T_2). \tag{2.12}$$

(iii) *If* $T \in \mathfrak{S}_1(\mathcal{H})$, *then* $\det(I+T^*) = \overline{\det(I+T)}$.

In the sequel we will also need a slightly improved version of the property(i).

Lemma 2.9. *Let* K *be a bounded operator. Further, let* C *be linear operator such that* $\mathrm{dom}\,(C) \supseteq \mathrm{ran}\,(K)$. *If* $\overline{KC} \in \mathfrak{S}_1(\mathfrak{H})$ *and* $CK \in \mathfrak{S}_1(\mathfrak{H})$, *then*

$$\det(I + \overline{KC}) = \det(I + CK). \tag{2.13}$$

The proof is left to the reader.

3 Almost solvable extensions and class $\mathfrak{D}^{\Pi}$

3.1 Almost solvable extensions and class $\mathfrak{D}^{\Pi}$

Assume that $\{\widetilde{A}', \widetilde{A}\} \subset \mathrm{Ext}_A$ and the resolvent difference is trace class,

$$(\widetilde{A}' - \xi)^{-1} - (\widetilde{A} - \xi)^{-1} \in \mathfrak{S}_1(\mathfrak{H}), \qquad \xi \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}), \tag{3.1}$$

i.e. $\{\widetilde{A}', \widetilde{A}\} \in \widetilde{\mathfrak{D}}$. Here we show that under additional (not too restrictive) assumptions on the operator $\widetilde{A}$ there exists a boundary triplet Π such that the implication $\{\widetilde{A}', \widetilde{A}\} \in \widetilde{\mathfrak{D}} \Longrightarrow \{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$ holds. Moreover, we show that the boundary operators B' and B in Definition 1.2 can be chosen to be bounded. This naturally leads to a concept of "jointly almost solvable extensions".

Definition 3.1.

(i) *An extension $\widetilde{A} \in \mathrm{Ext}_A$ is called almost solvable if there exists a self-adjoint extension $\widehat{A}$ of A such that $\widehat{A}$ and $\widetilde{A}$ are transversal, see Definition 2.1(ii).*

(ii) *The family $\{\widetilde{A}_j\}_{j=1}^N \subset \mathrm{Ext}_A$, $2 \le N \le \infty$, is called jointly almost solvable if there exists a self-adjoint extension $\widehat{A} \in \mathrm{Ext}_A$ such that $\widehat{A}$ is transversal to each $\widetilde{A}_j,\ j \in \{1, \ldots, N\}$.*

The class of almost solvable extensions of A was introduced and investigated in [22] (see also [19, 20, 21]).

Definition 3.2. *Let $\{\widetilde{A}_j\}_{j=1}^N \subset \mathrm{Ext}_A$. A boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^** will be called regular *for $\{\widetilde{A}_j\}_{j=1}^N$ if there exist operators $B_j \in [\mathcal{H}]$, $j \in \{1, \ldots, N\}$, such that $\widetilde{A}_j = A_{B_j} := A^* \upharpoonright \ker(\Gamma_1 - B_j\Gamma_0)$, $j \in \{1, \ldots, N\}$.*

Proposition 3.3. *Let $\{\widetilde{A}_j\}_{j=1}^N \subset \mathrm{Ext}_A$. The family $\{\widetilde{A}_j\}_{j=1}^N$ is jointly almost solvable if and only if there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* which is regular for the family $\{\widetilde{A}_j\}_{j=1}^N$.*

Proof. The proof follows immediately by combining Proposition 2.3(iv) with [21, Proposition 7.1]. □

Proposition 3.4 ([19],[20])**.** *Let $\widetilde{A} \in \mathrm{Ext}_A$ such that $z_1 \in \rho(\widetilde{A})$. If there is $z_2 \in \rho(\widetilde{A}) \cup \sigma_c(\widetilde{A})$ such that $\mathrm{Im}\,(z_1)\mathrm{Im}\,(z_2) < 0$, then $\widetilde{A}$ is almost solvable. In particular, $\widetilde{A}$ is almost solvable if $\rho(\widetilde{A}) \cap \mathbb{R} \neq \emptyset$.*

Proof. Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function. Let for definiteness $z_1 \in \mathbb{C}_+$, hence $z_2 \in (\rho(\widetilde{A}) \cup \rho_c(\widetilde{A})) \cap \mathbb{C}_-$. Without loss of generality we can assume that $M(z_1) = iI$. Further, let $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ be a boundary relation of $\widetilde{A}$ with respect to Π, i.e. $\widetilde{A} = A_\Theta$. By Proposition 2.5, $0 \in \rho(\Theta - M(z_1))$ and either $0 \in \rho(\Theta - M(z_2))$ or $0 \in \sigma_c(\Theta - M(z_2))$. We set

$$X := I + (M(z_1) - M(\overline{z}_1))(\Theta - M(z_1))^{-1} = I + 2i(\Theta - i)^{-1} \tag{3.2}$$

and

$$Y := I + (M(z_1) - M(z_2))(\Theta - M(z_1))^{-1} = I + (iI - M(z_2))(\Theta - i)^{-1}.$$

The assumption $0 \in \rho(\Theta - M(z_2)) \cup 0 \in \sigma_c(\Theta - M(z_2))$ yields $\ker(W) = \{0\}$ and $\overline{\operatorname{ran}(W)} = \mathcal{H}$. A straightforward computation shows that

$$X = T_1 + T_2 \tag{3.3}$$

where

$$T_1 := (M(z_2) + i)(M(z_2) - i)^{-1} \quad \text{and} \quad T_2 := 2i(iI - M(z_2))^{-1}Y. \tag{3.4}$$

Since $z_2 \in \mathbb{C}_-$, the imaginary part of $M(z_2)$ is strictly negative, $\operatorname{Im} M(z_2) < -\varepsilon I$. Hence T_1 is a strict contraction, $\|T_1\| < 1$. Further, since $\ker(Y) = \{0\}$ and $\overline{\operatorname{ran}(Y)} = \mathcal{H}$ we get $\ker(T_2) = \{0\}$ and $\overline{\operatorname{ran}(T_2)} = \mathcal{H}$. Hence the polar decomposition $T_2 = U|T_2|$ holds with the unitary U. We set $U(\varphi) := e^{i\varphi}U$, $\varphi \in (0, 2\pi)$. Clearly,

$$X - U(\varphi) = T_1 + U(|T_2| - e^{\varphi}) = (I + T_1(|T_2| - e^{i\varphi})^{-1}U^*)U(|T_2| - e^{i\varphi}).$$

If $\varphi \in (\frac{\pi}{2}, \frac{3\pi}{2})$, then $\|(|T_2| - e^{i\varphi})^{-1}\| \le 1$. Hence $\|T_1(|T_2| - e^{i\varphi})^{-1}U^*\| \le \|T_1\| < 1$ and $0 \in \rho(X - U(\varphi))$. Further, since $\mathcal{H}$ is separable, the point spectrum $\sigma_p(U)$ is at most countable, hence

$$\{e^{-i\varphi} : \varphi \in (\tfrac{\pi}{2}, \tfrac{3\pi}{2})\} \bigcap (\sigma_c(U) \cup \rho(U)) \neq \emptyset.$$

Choose any $\varphi \in (\frac{\pi}{2}, \frac{3\pi}{2})$ such that $e^{-i\varphi} \in \sigma_c(U) \cup \rho(U)$. Then $1 \in \sigma_c(U(\varphi)) \cup \rho(U(\varphi))$ and the operator $\widehat{B} := -iI + 2i(U(\varphi) - I)^{-1}$ is self-adjoint and

$$U(\varphi) = (\widehat{B} + i)(\widehat{B} - i)^{-1} = I + 2i(\widehat{B} - i)^{-1}. \tag{3.5}$$

Since $0 \in \rho(X - U(\varphi))$, one has $0 \in \rho\left(((\Theta - i))^{-1} - (\widehat{B} - i)^{-1}\right)$. By Proposition 2.6(iii), the extensions $A_{\widehat{B}} = A^*_{\widehat{B}}$ and $\widetilde{A}$ are transversal, i.e. $\widetilde{A}$ is almost solvable. □

Emphasize however, that the sufficient conditions of Theorem 3.5 are not necessary. It might even happen that $n_+(A) = n_-(A) < \infty$ and $\widetilde{A}$ is almost solvable although $(\rho(\widetilde{A}) \cup \sigma_c(\widetilde{A})) \cap \mathbb{C}_\pm = \emptyset$. Such extensions can easily be found for $A = A_+ \oplus A_-$ where $A_\pm$ are simple symmetric operators with deficiency indices $n_+(A_+) = n_-(A_-) = 1$ and $n_-(A_+) = n_+(A_-) = 0$.

Theorem 3.5. *Let A be a densely defined closed symmetric operator and let $\{\widetilde{A}_j\}_{j=1}^N \subset \operatorname{Ext}_A$ and $\widetilde{A} =: \widetilde{A}_{N+1} \in \operatorname{Ext}_A$. Let also $\bigcap_{j=1}^{N+1} \rho(\widetilde{A}_j) \neq \emptyset$ and*

$$(\widetilde{A} - z_1)^{-1} - (\widetilde{A}_j - z_1)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}), \quad z_1 \in \bigcap_{j=1}^{N+1} \rho(\widetilde{A}_j), \quad j \in \{1, \ldots, N\}. \tag{3.6}$$

If there is $z_2 \in \rho(\widetilde{A}) \cup \sigma_c(\widetilde{A}_1)$ such that $\operatorname{Im}(z_1)\operatorname{Im}(z_2) < 0$, then the family $\{\widetilde{A}, \widetilde{A}_1, \ldots, \widetilde{A}_N\}$ is jointly almost solvable.

Proof. We keep notations of Proposition 3.4. Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and let $\widetilde{A}_j := A_{\Theta_j}$, $\Theta_j \in \widetilde{\mathcal{C}}(H)$. Similarly to (3.2) we set

$$X_j := I + (M(z_1) - M(\overline{z}_1))(\Theta_j - M(z_1))^{-1} = I + 2i(\Theta_j - i)^{-1}, \tag{3.7}$$

$j \in \{1, 2 \ldots, N\}$. By Proposition 2.6(i), inclusion (3.6) is equivalent to

$$(\Theta - i)^{-1} - (\Theta_j - i)^{-1} \in \mathfrak{S}_\infty(\mathcal{H}), \qquad j \in \{1 \ldots, N\}.$$

Combining this relation with (3.2) and (3.7) yields $K_j := X_j - X \in \mathfrak{S}_\infty(\mathcal{H})$, $j \in \{1 \ldots, N\}$. Moreover, using the polar decomposition $T_2 = U|T_2|$, formulas (3.3), (3.4), (3.7), and setting $T := U^* T_1 + |T_2|$, one gets

$$\begin{aligned} X_j - U(\varphi) &= K_j + X - U(\varphi) \\ &= K_j + T_1 + U(|T_2| - e^{i\varphi}) = U(U^* K_j + T - e^{i\varphi} I), \quad j \in \{1, \ldots, N\}. \end{aligned} \tag{3.8}$$

Since $\|T_1\| < 1$, there is $\varepsilon \in (0, 1)$ such that $\Re(T) \geq -1 + \varepsilon$ and hence

$$\mathbb{C}_{l,\varepsilon} := \{z \in \mathbb{C} : \mathrm{Re}\,(z) < -1 + \varepsilon\} \subseteq \rho(T).$$

Since the half-plane $\mathbb{C}_{l,\varepsilon}$ belongs to the infinite component of $\rho(T)$ and $U^* K_j \in \mathfrak{S}_\infty(\mathcal{H})$, the spectrum of each operator $T + U^* K_j$ within $\mathbb{C}_{l,\varepsilon}$ consists only of isolated eigenvalues of finite multiplicities (cf. [31, Lemma I.5.2]). Hence,

$$\{e^{i\varphi} : \varphi \in (\tfrac{\pi}{2}, \tfrac{3\pi}{2})\} \bigcap \mathbb{C}_{l,\varepsilon} \setminus \left(\bigcup_1^N \sigma_p(U^* K_j + T) \right) \neq \emptyset.$$

Thus there is $\varphi_0 \in (\frac{\pi}{2}, \frac{3\pi}{2})$ such that $e^{i\varphi_0} \in \bigcap_{j=1}^N \rho(U^* K_j + T)$ and $e^{-i\varphi_0} \in \sigma_c(U) \cup \rho(U)$. Therefore, by (3.8), $0 \in \bigcap_1^N \rho(X_j - U(\varphi_0))$. Moreover, $U(\varphi_0)$ admits representation (3.5) with $\widehat{B} := -iI + 2i(U(\varphi_0) - I)^{-1} = \widehat{B}^*$. Combining these relations with (3.7) yields $0 \in \bigcap_{j=1}^N \rho((\Theta_j - i)^{-1} - (\widehat{B} - i)^{-1})$. One completes the proof by applying Proposition 2.6(iii). □

Remark 3.6. (i) Proposition 3.4 and Theorem 3.5 remain valid if we replace the condition $z_2 \in \rho(\widetilde{A}) \cup \sigma_c(\widetilde{A})$ by

$$\dim \ker(\widetilde{A} - z_2) = \dim \ker(\widetilde{A}^* - \bar{z}_2).$$

(ii) Different proofs of Proposition 3.4 can be found in [22, 19, 21]. The given proof does not use Krein space theory and fits better to our exposition because of direct generalization for a system of operators given in Theorem 3.5.

Corollary 3.7. *Let A be a densely defined closed symmetric operator with finite deficiency indices. Further, let $\{\widetilde{A}_j\}_{j=1}^N \subset \mathrm{Ext}_A$ and $\bigcap_{j=1}^N \rho(\widetilde{A}_j) \neq \emptyset$. Then $\{\widetilde{A}_j\}_{j=1}^N$ is jointly almost solvable.*

Proof. Let $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$. Then there is non-real $z_1 \in \bigcap_{j=1}^N \rho(\widetilde{A}_j)$ such that

$$(\widetilde{A}_j - z_1)^{-1} - (\widetilde{A} - z_1)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}), \quad j \in \{1, \dots, N\}.$$

Moreover, there is $z_2 \in \rho(\widetilde{A})$ such that $\mathrm{Im}\,(z_1)\mathrm{Im}\,(z_2) < 0$. By Theorem 3.5, the family $\{\widetilde{A}, \widetilde{A}_1, \dots, \widetilde{A}_N\}$ is jointly almost solvable. Hence $\{\widetilde{A}_j\}_{j=1}^N$ is jointly almost solvable. □

The following corollary is an immediate consequence of Theorem 3.5.

Corollary 3.8. *Let $\widetilde{A}', \widetilde{A} \in \mathrm{Ext}_A$ and let condition* (3.1) *be satisfied. If there exists $\xi' \in \rho(\widetilde{A}) \cup \sigma_c(\widetilde{A})$ such that $\mathrm{Im}\,(\xi)\mathrm{Im}\,(\xi') < 0$, then $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^\Pi$.*

3.2 Almost solvable extensions and characteristic function

It is known several approaches to the definition of the characteristic function (CF) of an unbounded operator with non-empty resolvent set. The most relevant to our considerations definitions have been proposed in [66] and [19, 20]. In general, the CF might have some exotic properties. However, it was shown in [19, 20, 22] that the CF of an almost solvable extension of A takes values in $[\mathcal{H}]$ and has some nice properties similar to those of bounded operators (cf. [14]). We will not present a strict definition of CF since in what follows we need only a representation of CF by means of the Weyl function and boundary operator.

Proposition 3.9 ([19, Theorem 2])**.** *Let A be a densely defined closed symmetric operator and let $\widetilde{A}$ be an almost solvable extension of A. Let also $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* which is regular for $\widetilde{A}$, i.e. $\widetilde{A} = A_B = A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$ and $B \in [\mathcal{H}]$. Then the characteristic function of the operator A_B admits the representation*

$$W^\Pi_{\widetilde{A}}(z) = I + 2i|B_I|^{1/2}\big(B^* - M(z)\big)^{-1}|B_I|^{1/2}J, \quad z \in \rho(\widetilde{A}^*) \cap \rho(A_0), \tag{3.9}$$

where $B_I = J|B_I|$, $J = \mathrm{sign}(B_I)$, is the polar decomposition of $B_I := \mathrm{Im}\,(B)$.

It follows from (3.9) that $W^\Pi_{\widetilde{A}}(\cdot)$ takes values in $[\mathcal{H}]$ and is J-contractive (J-expansive) in $\mathbb{C}_+$ (resp., in $\mathbb{C}_-$). If $\widetilde{A} = A_B$ is m-dissipative, then, by Proposition 2.3(iii), B is m-dissipative, $J = I$ and $W^\Pi_{\widetilde{A}}(\cdot)$ is contractive in $\mathbb{C}_+$.

4 Main formulas for perturbation determinants

Here we establish a connection between our definition (1.8) of the perturbation determinant of a pair $\{\widetilde{A}', \widetilde{A}\} \subset \mathrm{Ext}_A$ and the classical one given by formula (1.3). In particular we prove representation (1.11) for $\widetilde{\Delta}_{\widetilde{A}',\widetilde{A}}(\xi, z)$ as well as formulas (1.12) and (1.14) mentioned in the introduction.

Theorem 4.1. *Let A be a densely defined closed symmetric operator and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function. Further, let $\widetilde{A}', \widetilde{A} \in \mathrm{Ext}_A$ and let $\Theta', \Theta \in \widetilde{C}(\mathcal{H})$ be the corresponding boundary relations, i.e. $\widetilde{A}' = A_{\Theta'}$ and $\widetilde{A} = A_\Theta$. Assume also that $\{\widetilde{A}', \widetilde{A}\} \in \widetilde{\mathfrak{D}}$ and $\xi \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$. Then:*

(i) *The linear relations* $\Theta' - M(\xi)$ *and* $\Theta - M(\xi)$ *are boundedly invertible.*

(ii) *The following condition is satisfied*

$$R_\xi^M(\Theta', \Theta) := (\Theta' - M(\xi))^{-1} - (\Theta - M(\xi))^{-1} \in \mathfrak{S}_1(\mathcal{H}); \tag{4.1}$$

(iii) *For any* $\xi \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$ *and* $z \in \rho(\widetilde{A}) \cap \rho(A_0)$ *the perturbation determinant* $\widetilde{\Delta}_{\widetilde{A}',\widetilde{A}}(\xi, \cdot)$ *admits the following representation*

$$\begin{aligned} \widetilde{\Delta}_{\widetilde{A}'/\widetilde{A}}(\xi, z) = \det \big(I_{\mathcal{H}} + R_\xi^M(\Theta', \Theta) \times \\ [I_{\mathcal{H}} - (M(\xi) - M(z))(\Theta - M(z))^{-1}](M(\xi) - M(z)) \big) . \end{aligned} \tag{4.2}$$

Proof. (i) By Proposition 2.5, $\xi \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$ if and only if $0 \in \rho(\Theta' - M(\xi)) \cap \rho(\Theta - M(\xi))$. Hence the inverse relations $(\Theta' - M(\xi))^{-1}$ and $(\Theta - M(\xi))^{-1}$ exist and are bounded operators.

(ii) According to (2.7)

$$(\widetilde{A}' - \xi)^{-1} - (\widetilde{A} - \xi)^{-1} = \gamma(\xi) R_\xi^M(\Theta', \Theta) \gamma(\overline{\xi})^* \tag{4.3}$$

where $R_\xi^M(\Theta', \Theta)$ is given by (4.1). Since $\gamma(\xi)$ and $\gamma(\overline{\xi})^*$ isomorphically map $\mathcal{H}$ onto $\mathfrak{N}_{\overline{\xi}}$ and $\mathfrak{N}_\xi$ onto $\mathcal{H}$, respectively, the inclusion (3.1) yields (4.1).

(iii) According to (1.3) for any $z \in \rho(\widetilde{A}) \setminus \{\xi\}$

$$\widetilde{\Delta}_{\widetilde{A}'/\widetilde{A}}(\xi, z) = \det \Big(I + (\xi - z) \Big((\widetilde{A}' - \xi)^{-1} - (\widetilde{A} - \xi)^{-1} \Big) (\widetilde{A} - \xi)(\widetilde{A} - z)^{-1} \Big) .$$

Combining this formula with (4.3) and using the property (2.11) we get

$$\widetilde{\Delta}_{\widetilde{A}'/\widetilde{A}}(\xi, z) = \det \Big(I + (\xi - z)(\widetilde{A} - \xi)(\widetilde{A} - z)^{-1} \gamma(\xi) R_\xi^M(\Theta', \Theta) \gamma(\overline{\xi})^* \Big) . \tag{4.4}$$

Next we transform the expression $(\widetilde{A} - \xi)(\widetilde{A} - z)^{-1}\gamma(\xi)$. Noting that $\widetilde{A} = A_\Theta$ and applying the Krein type formula (2.7) we obtain

$$\begin{aligned} (\widetilde{A} - \xi)(\widetilde{A} - z)^{-1}\gamma(\xi) &= \Big(I + (z - \xi)(\widetilde{A} - z)^{-1} \Big) \gamma(\xi) \\ &= \big(I + (z - \xi)(A_0 - z)^{-1} + (z - \xi)\gamma(z)(\Theta - M(z))^{-1}\gamma(\overline{z})^* \big) \gamma(\xi) \\ &= \big(I + (z - \xi)(A_0 - z)^{-1} \big)\gamma(\xi) + (z - \xi)\gamma(z)(\Theta - M(z))^{-1}\gamma(\overline{z})^* \gamma(\xi). \end{aligned}$$

On the other hand, by (2.5), $(I + (z - \xi)(A_0 - z)^{-1})\gamma(\xi) = \gamma(z)$ Hence

$$\begin{aligned} (\widetilde{A} - \xi)(\widetilde{A} - z)^{-1}\gamma(\xi) &= (I + (z - \xi)(\widetilde{A} - z)^{-1})\gamma(\xi) \\ &= \gamma(z) \big(I + (z - \xi)(\Theta - M(z))^{-1}\gamma(\overline{z})^*\gamma(\xi) \big) . \end{aligned}$$

Further, rewriting identity (2.6) in the form

$$(z - \xi)\gamma(\overline{z})^*\gamma(\xi)) = M(z) - M(\xi) = (z - \xi)\gamma(\overline{\xi})^*\gamma(z) \tag{4.5}$$

and combining it with the previous one we derive

$$(\widetilde{A}-\xi)(\widetilde{A}-z)^{-1}\gamma(\xi)=\gamma(z)\left(I+(\Theta-M(z))^{-1}(M(z)-M(\xi))\right).$$

Substituting this identity in (4.4) and applying property (2.11) we get

$$\begin{aligned}\widetilde{\Delta}_{\widetilde{A}'/\widetilde{A}}(\xi,z)&=\det\big(I+\\ &\quad[I+(\Theta-M(z))^{-1}(M(z)-M(\xi)]R^M_\xi(\Theta',\Theta)(\xi-z)\gamma(\overline{\xi})^*\gamma(z)\big).\end{aligned}$$

Finally, combining this identity with the second equality in (4.5) and taking property (2.11) into account we arrive at (4.2). □

Next we simplify representation (4.2) assuming that $\{\widetilde{A}',\widetilde{A}\}\in\mathfrak{D}^\Pi$. In particular, we prove representation (1.11) as well as establish a connection between two determinants for the pair $\{\widetilde{A}',\widetilde{A}\}$ corresponding to different boundary triplets.

Theorem 4.2. *Let A be a densely defined closed symmetric operator and let $\widetilde{A}',\widetilde{A}\in\mathrm{Ext}_A$. Assume also that $\{\widetilde{A}',\widetilde{A}\}\in\mathfrak{D}^\Pi$ for a boundary triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^*. Then the following holds:*

(i) $\{\widetilde{A}',\widetilde{A}\}\in\widetilde{\mathfrak{D}}$ *and the representation*

$$\widetilde{\Delta}_{\widetilde{A}'/\widetilde{A}}(\xi,z)=\frac{\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(z)}{\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(\xi)} \tag{4.6}$$

holds for $\xi\in\rho(\widetilde{A}')\cap\rho(\widetilde{A})\cap\rho(A_0)$ and $z\in\rho(\widetilde{A})\cap\rho(A_0)$.

(ii) *The perturbation determinant $\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(z)$ admits a holomorphic continuation from $\rho(\widetilde{A})\cap\rho(A_0)$ to the domain $\rho(\widetilde{A})$.*

(iii) *If $\Pi'=\{\mathcal{H}',\Gamma_0',\Gamma_1'\}$ is another boundary triplet for A^* such that $\{\widetilde{A}',\widetilde{A}\}\in\mathfrak{D}^{\Pi'}$, then for any $\xi\in\rho(\widetilde{A}')\cap\rho(\widetilde{A})\cap\rho(A_0)\cap\rho(A_0')$ the following identity holds*

$$\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(z)=c\,\Delta^{\Pi'}_{\widetilde{A}'/\widetilde{A}}(z)\quad\textit{with}\quad c=\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(\xi)\left(\Delta^{\Pi'}_{\widetilde{A}'/\widetilde{A}}(\xi)\right)^{-1},\quad z\in\rho(\widetilde{A}).$$

In particular, this identity is valid for some non-real $\xi\in\rho(\widetilde{A}')\cap\rho(\widetilde{A})$.

Proof. (i) If $\{\widetilde{A}',\widetilde{A}\}\in\mathfrak{D}^\Pi$ then, by Proposition 2.5(i), $\rho(\widetilde{A}')\cap\rho(\widetilde{A})\neq\emptyset$, hence $\rho(\widetilde{A}')\cap\rho(\widetilde{A})\cap\rho(A_0)\neq\emptyset$. Moreover, since $\{\widetilde{A}',\widetilde{A}\}\in\mathfrak{D}^\Pi$, $\widetilde{A}'$ and $\widetilde{A}$ admit representations $\widetilde{A}'=A_{B'}$ and $\widetilde{A}=A_B$ with $B',B\in\mathcal{C}(\mathcal{H})$ and satisfying $\mathrm{dom}\,(B')=\mathrm{dom}\,(B)$. Combining the later condition with condition (iii) of Definition 1.2 yields

$$\begin{aligned}R^M_\xi(B',B)&=(B'-M(\xi)^{-1}-(B-M(\xi))^{-1}\\ &=(B'-M(\xi))^{-1}(B-B')(B-M(\xi)^{-1}\in\mathfrak{S}_1(\mathcal{H})\end{aligned} \tag{4.7}$$

for $\xi \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$. In turn combining this inclusion with the Krein type formula (2.7) implies condition (3.1), meaning that $\{\widetilde{A}', \widetilde{A}\} \in \widetilde{\mathfrak{D}}$.

Further, by Theorem 4.1(iii), the perturbation determinant $\widetilde{\Delta}_{\widetilde{A}'/\widetilde{A}}(\xi, z)$ admits the representation

$$\widetilde{\Delta}_{\widetilde{A}'/\widetilde{A}}(\xi, z) = \det\left(I_{\mathcal{H}} + R_\xi^M(B', B)(B - M(\xi))(B - M(z))^{-1}(M(\xi) - M(z))\right)$$

for $\xi \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$ and $z \in \rho(\widetilde{A}) \cap \rho(A_0)$. Combining this relation with representation (4.7) for $R_\xi^M(B', B)$ and using the cyclicity property (2.11) we obtain

$$\begin{aligned} &\widetilde{\Delta}_{\widetilde{A}'/\widetilde{A}}(\xi, z) \\ &\quad = \det(I_{\mathcal{H}} + (B - B')(B - M(z))^{-1}(M(\xi) - M(z))(B' - M(\xi))^{-1}). \end{aligned}$$

On the other hand, one easily verifies by a straightforward computation that

$$\begin{aligned} &I_{\mathcal{H}} + (B - B')(B - M(z))^{-1}(M(\xi) - M(z))(B' - M(\xi))^{-1} \\ &\quad = (I_{\mathcal{H}} + (B' - B)(B - M(z))^{-1})(I_{\mathcal{H}} + (B - B')(B' - M(\xi))^{-1}) \end{aligned}$$

for $\xi \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$ and $z \in \rho(\widetilde{A}) \cap \rho(A_0)$. By the multiplicative property (2.12) it follows that

$$\begin{aligned} &\widetilde{\Delta}_{\widetilde{A}'/\widetilde{A}}(\xi, z) \\ &\quad = \det\left(I_{\mathcal{H}} + (B - B')(B' - M(\xi))^{-1}\right) \det\left(I_{\mathcal{H}} + (B' - B)(B - M(z))^{-1}\right) \\ &\quad = \Delta^{\Pi}_{\widetilde{A}/\widetilde{A}'}(\xi)\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z), \qquad z \in \rho(\widetilde{A}) \cap \rho(A_0). \end{aligned}$$

Now (4.6) is implied by combining the later relation with the identity

$$\Delta^{\Pi}_{\widetilde{A}/\widetilde{A}'}(\xi) = (\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(\xi))^{-1}, \qquad \xi \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0).$$

(ii) Obviously, the generalized perturbation determinant $\widetilde{\Delta}_{\widetilde{A}'/\widetilde{A}}(\xi, z)$ is holomorphic in $z \in \rho(\widetilde{A})$. If $\xi \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$ is fixed, then it follows from (4.6) that $\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(\cdot)$ admits a holomorphic continuation to the domain $\rho(\widetilde{A})$.

(iii) Let $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$ and $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi'}$. Then writing down representation (4.6) for both boundary triplets Π and Π', we arrive at the identity

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z) = \frac{\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(\xi)}{\Delta^{\Pi'}_{\widetilde{A}'/\widetilde{A}}(\xi)} \Delta^{\Pi'}_{\widetilde{A}'/\widetilde{A}}(z), \quad z \in \rho(\widetilde{A}) \cap \rho(A_0) \cap \rho(A_0').$$

Taking into account (ii) one completes the proof. □

Proposition 4.3. *Let A be a densely defined closed symmetric operator in $\mathfrak{H}$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, and $M(z)$ the corresponding Weyl function. Let $\widetilde{A} = A_\Theta \in \mathrm{Ext}_A$ where $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$. Then the following holds:*

(i) *If* $\{\widetilde{A}, A_0\} \in \widetilde{\mathfrak{D}}$*, then for* $\xi \in \rho(\widetilde{A}) \cap \rho(A_0)$ *and* $z \in \rho(A_0)$

$$\widetilde{\Delta}_{\widetilde{A}/A_0}(\xi, z) = \det\left(I_{\mathcal{H}} + (\Theta - M(\xi))^{-1}(M(\xi) - M(z))\right). \tag{4.8}$$

Moreover, the spectrum of Θ *is discrete,* $(\Theta - \mu)^{-1} \in \mathfrak{S}_1(\mathcal{H})$*,* $\mu \in \rho(\Theta)$*, and*

$$\widetilde{\Delta}_{\widetilde{A}/A_0}(\xi, z) = \frac{\det(I_{\mathcal{H}} - (\mu - \Theta)^{-1}(\mu - M(z)))}{\det(I_{\mathcal{H}} - (\mu - \Theta)^{-1}(\mu - M(\xi)))} \tag{4.9}$$

for $\xi \in \rho(\widetilde{A}) \cap \rho(A_0)$ *and* $z \in \rho(A_0)$*.*

(ii) *If* $\{\widetilde{A}', A_0\} \in \widetilde{\mathfrak{D}}$ *and* $\{\widetilde{A}, A_0\} \in \widetilde{\mathfrak{D}}$*, then* $\{\widetilde{A}', \widetilde{A}\} \in \widetilde{\mathfrak{D}}$ *and*

$$\widetilde{\Delta}_{\widetilde{A}'/\widetilde{A}}(\xi, z) = c_\xi \frac{\det\left(I_{\mathcal{H}} - (\mu - \Theta')^{-1}(\mu - M(z))\right)}{\det\left(I_{\mathcal{H}} - (\mu - \Theta)^{-1}(\mu - M(z))\right)} \tag{4.10}$$

for $\mu \in \rho(\Theta') \cap \rho(\Theta)$*,* $\xi \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$ *and* $z \in \rho(\widetilde{A}) \cap \rho(A_0)$ *where*

$$c_\xi := \frac{\det\left(I_{\mathcal{H}} - (\mu - \Theta)^{-1}(\mu - M(\xi))\right)}{\det\left(I_{\mathcal{H}} - (\mu - \Theta')^{-1}(\mu - M(\xi))\right)}.$$

Proof. (i) The extension A_0 is given by A_{Θ_0} with $\Theta_0 := \{0\} \times \mathcal{H}$. Clearly, $\Theta_0 - M(\xi) = \{0\} \times \mathcal{H}$ and $(\Theta_0 - M(\xi))^{-1} = 0$. Therefore $R^M_\xi(\Theta, \Theta_0) := (\Theta - M(\xi))^{-1} - (\Theta_0 - M(\xi))^{-1} = (\Theta - M(\xi))^{-1}$. Substituting this expression in formula (4.2) we obtain (4.8). By Proposition 2.6, Θ has discrete spectrum and $(\mu - \Theta)^{-1} \in \mathfrak{S}_1(\mathcal{H}), \mu \in \rho(\Theta)$.

Let us prove (4.9). It is easily seen that for $\xi \in \rho(\widetilde{A}) \cap \rho(A_0)$ and $\mu \in \rho(\Theta)$

$$\frac{1}{\det\left(I_{\mathcal{H}} - (\mu - \Theta)^{-1}(\mu - M(\xi))\right)} = \det\left(I_{\mathcal{H}} - (\Theta - M(\xi))^{-1}(\mu - M(\xi))\right).$$

Hence for any $\xi \in \rho(\widetilde{A}) \cap \rho(A_0)$ and $z \in \rho(A_0)$ one has

$$\begin{aligned}&\frac{\det(I_{\mathcal{H}} - (\mu - \Theta)^{-1}(\mu - M(z)))}{\det(I_{\mathcal{H}} - (\mu - \Theta)^{-1}(\mu - M(\xi)))} \\ &\quad = \det\left(I_{\mathcal{H}} - (\Theta - M(\xi))^{-1}(\mu - M(\xi))\right)\det\left(I_{\mathcal{H}} - (\mu - \Theta)^{-1}(\mu - M(z))\right).\end{aligned}$$

A straightforward computation shows that

$$\begin{aligned}&I_{\mathcal{H}} + (\Theta - M(\xi))^{-1}(M(\xi) - M(z)) \\ &\quad = \left(I_{\mathcal{H}} - (\Theta - M(\xi))^{-1}(\mu - M(\xi))\right)\left(I_{\mathcal{H}} - (\mu - \Theta)^{-1}(\mu - M(z))\right)\end{aligned}$$

for $\xi \in \rho(\widetilde{A}) \cap \rho(A_0)$ and $z \in \rho(A_0)$. Combining two last identities with relation (4.8) we arrive at (4.9).

(ii) From $\{\widetilde{A}', A_0\} \in \widetilde{\mathfrak{D}}$ and $\{\widetilde{A}, A_0\} \in \widetilde{\mathfrak{D}}$ we get that the spectra of $\widetilde{A}'$ and $\widetilde{A}$ are discrete in $\mathbb{C}_\pm$. Hence $\{\widetilde{A}', \widetilde{A}\} \in \widetilde{\mathfrak{D}}$ and

$$\widetilde{\Delta}_{\widetilde{A}'/\widetilde{A}}(\xi, z) = \widetilde{\Delta}_{\widetilde{A}'/A_0}(\xi, z)\widetilde{\Delta}_{A_0/\widetilde{A}}(\xi, z)$$

for $\xi \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$ and $z \in \rho(A_0) \cap \rho(\widetilde{A})$. Combining this identity with (4.9) and applying the known identity $\widetilde{\Delta}_{A_0/\widetilde{A}}(\xi, z) = (\widetilde{\Delta}_{\widetilde{A}/A_0}(\xi, z))^{-1},\ \xi, z \in \rho(\widetilde{A}) \cap \rho(A_0)$, (cf. (a.1)), we arrive at (4.10). □

Corollary 4.4. *Assume the conditions of Proposition 4.3. Suppose in addition that $\widetilde{A}$ is disjoint with A_0, i.e. $\widetilde{A} = A_B$ with $B \in \mathcal{C}(\mathcal{H})$. Then for $\xi \in \rho(\widetilde{A}) \cap \rho(A_0)$ the perturbation determinant $\widetilde{\Delta}_{\widetilde{A}/A_0}(\xi, \cdot)$ is given by*

$$\widetilde{\Delta}_{\widetilde{A}/A_0}(\xi, z) = \det\left((B - M(z))(B - M(\xi))^{-1}\right). \tag{4.11}$$

If, in addition, $\dim \mathcal{H} < \infty$, then

$$\widetilde{\Delta}_{\widetilde{A}/A_0}(\xi, z) = \frac{\det(B - M(z))}{\det(B - M(\xi))}. \tag{4.12}$$

Remark 4.5. Combining the chain rule with formula (4.12) we arrive at formula (1.12) for $\widetilde{\Delta}_{\widetilde{A}'/\widetilde{A}}(\xi, \cdot)$. Note that formula (4.12) up to a constant was established in [7] for the first time.

Corollary 4.6. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function, and $\widetilde{A}' = A_{B'}$, $\widetilde{A} = A_B$ where $B', B \in \mathcal{C}(\mathcal{H})$. If, in addition, $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$, then for any $\mu \in \rho(B') \cap \rho(B)$*

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z) = \Delta_{B'/B}(\mu) \frac{\det\left(I_{\mathcal{H}} - (\mu - B')^{-1}(\mu - M(z))\right)}{\det\left(I_{\mathcal{H}} - (\mu - B)^{-1}(\mu - M(z))\right)}, \tag{4.13}$$

$z \in \rho(\widetilde{A}) \cap \rho(A_0)$. Here $\Delta_{B/B'}(\mu)$ is the classical perturbation determinant defined by (1.1).

Proof. Let us show that the inclusion $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$ yields $(B' - B)(B - \mu)^{-1} \in \mathfrak{S}_1(\mathcal{H})$, $\mu \in \rho(B)$. Since $(B' - B)(B - M(\xi))^{-1} \in \mathfrak{S}_1(\mathcal{H})$, by assumption, we get

$$(B' - B)(B - \mu)^{-1} = (B' - B)(B - M(\xi))^{-1}(B - M(\xi))(B - \mu)^{-1} \in \mathfrak{S}_1(\mathcal{H}).$$

By Theorem 4.2(i) (see (4.6)), for any $\xi \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z) = \Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(\xi) \widetilde{\Delta}_{\widetilde{A}'/\widetilde{A}}(\xi, z), \qquad z \in \rho(\widetilde{A}) \cap \rho(A_0).$$

Combining this identity with (4.10) yields

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z) = c'_{\xi} \frac{\det\left(I_{\mathcal{H}} - (\mu - B')^{-1}(\mu - M(z))\right)}{\det\left(I_{\mathcal{H}} - (\mu - B)^{-1}(\mu - M(z))\right)}, \ z \in \rho(\widetilde{A}) \cap \rho(A_0), \tag{4.14}$$

for $\mu \in \rho(B') \cap \rho(B)$, $\xi \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$ where

$$c'_{\xi} := \Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(\xi)\ c_{\xi} = \Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(\xi) \frac{\det\left(I_{\mathcal{H}} - (\mu - B)^{-1}(\mu - M(\xi))\right)}{\det\left(I_{\mathcal{H}} - (\mu - B')^{-1}(\mu - M(\xi))\right)}.$$

To compute this constant recall that, by definition,

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(\xi) = \det(I_{\mathcal{H}} + (B' - B)(B - M(\xi))^{-1}).$$

A straightforward computation shows that

$$[I_{\mathcal{H}}+(B'-B)(B-M(\xi))^{-1}]\cdot[I_{\mathcal{H}}-(\mu-M(\xi))(\mu-B)^{-1}]=(M(\xi)-B')(\mu-B)^{-1}$$

and

$$[I_{\mathcal{H}}-(\mu-M(\xi))(\mu-B')^{-1}]^{-1}=(\mu-B')(M(\xi)-B')^{-1}.$$

Taking the product of these identities and applying the properties (2.11), (2.12), of the determinants we finally get

$$c'_\xi=\det(I_{\mathcal{H}}+(B'-B)(B-\mu)^{-1})=\Delta_{B'/B}(\mu).$$

Combining this relation with (4.14) we arrive at (4.13). □

5 Properties of $\Delta^{\Pi}_{\widetilde{A}',\widetilde{A}}(\cdot)$

Here we extend basic properties of the classical perturbation determinants (cf. Appendix) to the determinants $\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(\cdot)$ defined by Definition 1.2.

Proposition 5.1. *Let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function and $A_1:=A^*\upharpoonright\ker(\Gamma_1)$. Assume that $\widetilde{A}=A_B$, $\widetilde{A}'=A_{B'}$, $\widetilde{A}''=A_{B''}$ with $B,B',B''\in\mathcal{C}(\mathcal{H})$.*

(i) *Let $\rho(\widetilde{A}'')\cap\rho(\widetilde{A}')\cap\rho(\widetilde{A})\neq\emptyset$. If $\{\widetilde{A}'',\widetilde{A}'\}\in\mathfrak{D}^{\Pi}$ and $\{\widetilde{A}',\widetilde{A}\}\in\mathfrak{D}^{\Pi}$, then $\{\widetilde{A}'',\widetilde{A}\}\in\mathfrak{D}^{\Pi}$ and the multiplicative property holds*

$$\Delta^{\Pi}_{\widetilde{A}''/\widetilde{A}'}(z)\ \Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z)=\Delta^{\Pi}_{\widetilde{A}''/\widetilde{A}}(z),\qquad z\in\rho(\widetilde{A}')\cap\rho(\widetilde{A}).\tag{5.1}$$

(ii) *If $\{\widetilde{A}',\widetilde{A}\}\in\mathfrak{D}^{\Pi}$, then $\{\widetilde{A},\widetilde{A}'\}\in\mathfrak{D}^{\Pi}$ and*

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z)\Delta^{\Pi}_{\widetilde{A}/\widetilde{A}'}(z)=1,\quad z\in\rho(\widetilde{A}')\cap\rho(\widetilde{A}).\tag{5.2}$$

(iii) *Let $\{\widetilde{A}',\widetilde{A}\}\in\mathfrak{D}^{\Pi}$ and let z_0 be either a regular point or a normal eigenvalue of the operators $\widetilde{A}'$ and $\widetilde{A}$ of algebraic multiplicities $m_{z_0}(\widetilde{A}')$ and $m_{z_0}(\widetilde{A})$. Then $\operatorname{ord}\left(\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z_0)\right)=m_{z_0}(\widetilde{A}')-m_{z_0}(\widetilde{A})$. In particular, $\operatorname{ord}\left(\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z_0)\right)=m_{z_0}(\widetilde{A}')$ for any $z_0\in\rho(A_0)\cap\rho(\widetilde{A})$.*

(iv) *If $\{\widetilde{A}',\widetilde{A}\}\in\mathfrak{D}^{\Pi}$, then for $z\in\rho(\widetilde{A}')\cap\rho(\widetilde{A})$ one has*

$$\frac{1}{\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z)}\frac{d}{dz}\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z)=\operatorname{tr}((\widetilde{A}-z)^{-1}-(\widetilde{A}'-z)^{-1}).\tag{5.3}$$

(v) *If $\{\widetilde{A}',\widetilde{A}\}\in\mathfrak{D}^{\Pi}$ and $\{\widetilde{A}'^*,\widetilde{A}^*\}\in\mathfrak{D}^{\Pi}$, then*

$$\Delta^{\Pi}_{\widetilde{A}'^*/\widetilde{A}^*}(z)=\overline{\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(\overline{z})},\quad z\in\rho(\widetilde{A}^*).\tag{5.4}$$

(vi) *If* $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$*, then for* $z \in \rho(\widetilde{A})$ *and* $\zeta \in \rho(\widetilde{A}') \cap \rho(\widetilde{A})$

$$\frac{\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z)}{\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(\zeta)} = \det(I_{\mathcal{H}} + (M(z) - M(\zeta))(B - M(z))^{-1}(B' - B)(B' - M(\zeta))^{-1}).$$

Proof. (i) If the condition $\rho(\widetilde{A}'') \cap \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \neq \emptyset$ is satisfied, then $\rho(\widetilde{A}'') \cap \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0) \neq \emptyset$. Hence, by Proposition 2.5(i) one has $\{z \in \rho(A_0) : 0 \in \rho(B'' - M(z)) \cap \rho(B' - M(z)) \cap \rho(B - M(z))\} \neq \emptyset$. To check Definition 1.2(iii) we note that due to the property $\mathrm{dom}\,(B'') = \mathrm{dom}\,(B') = \mathrm{dom}\,(B)$

$$(B'' - B)(B - M(z))^{-1} = (B'' - B')(B - M(z))^{-1} + (B' - B)(B - M(z))^{-1}$$

for $z \in \rho(\widetilde{A}') \cap \rho(A_0)$. By the assumption, $(B' - B)(B - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$. It remains to check that $(B'' - B')(B - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for $z \in \rho(\widetilde{A}) \cap \rho(A_0)$. Clearly,

$$\begin{aligned}(B'' - B')(B - M(z))^{-1} &= (B'' - B')(B' - M(z))^{-1} \\ &+ (B'' - B')(B' - M(z))^{-1}(B' - B)((B - M(z))^{-1}\end{aligned}$$

for $z \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$. Since $(B'' - B')(B' - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for $z \in \rho(\widetilde{A}') \cap \rho(A_0)$ and $(B' - B)(B - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for $z \in \rho(\widetilde{A}) \cap \rho(A_0)$, by the assumption, we find $(B'' - B)(B - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for $z \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$. Now the multiplicative property (5.1) for $z \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$ immediately follows from definition (1.8). Finally, applying Theorem 4.2(ii) one extends identity (5.1) to the domain $\rho(\widetilde{A}') \cap \rho(\widetilde{A})$.

(ii) Clearly, conditions (i) and (ii) of Definition 1.2 are satisfied for the pairs $\{\widetilde{A}, \widetilde{A}'\}$ and $\{\widetilde{A}', \widetilde{A}\}$, simultaneously. Let us check condition (iii) of Definition 1.2. Since $\mathrm{dom}\,(B) = \mathrm{dom}\,(B')$, the operator $(B - M(z))(B' - M(z))^{-1}$ is bounded. Combining this fact with the assumption $(B - B')(B - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$ and using the representation

$$(B - B')(B' - M(z))^{-1} = -(B' - B)(B - M(z))^{-1}(B - M(z))(B' - M(z))^{-1}$$

for $z \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$ we get $(B - B')(B' - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for $z \in \rho(\widetilde{A}') \cap \rho(A_0)$. To prove (5.2) it suffices to set $\widetilde{A}'' = \widetilde{A}$ in (5.1).

(iii) Combining formula (4.6) with Proposition A.1(iii) we get the statement.

(iv) Combining formula (4.6) with Proposition A.1(iv) it yields (5.3).

(v) If $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^{\Pi}$, then $C(B - M(z))^{-1} \in \mathfrak{S}_1(\mathcal{H})$ for $z \in \rho(\widetilde{A}) \cap \rho(A_0)$ where $C := B' - B, \mathrm{dom}\,(C) := \mathrm{dom}\,(B') = \mathrm{dom}\,(B)$. Since B' and B are densely defined and closed, the operators B'^{*}, B^{*} and C^{*} are well defined and $\overline{(B^{*} - M(z)^{*})^{-1}C^{*}} \in \mathfrak{S}_1(\mathcal{H})$. Setting $C_{*} := B'^{*} - B^{*}$, $\mathrm{dom}\,(C_{*}) = \mathrm{dom}\,(B'^{*}) = \mathrm{dom}\,(B^{*})$ we get $\mathrm{dom}\,(C^{*}) \supseteq \mathrm{dom}\,(C_{*})$. Moreover, we have

$$C^{*}(B^{*} - M(z)^{*})^{-1} = C_{*}(B^{*} - M(z)^{*})^{-1} \in \mathfrak{S}_1(\mathcal{H}), \quad z \in \rho(\widetilde{A}) \cap \rho(A_0). \tag{5.5}$$

Combining this relation with Lemma A.2 one gets

$$\det(I_{\mathcal{H}} + \overline{(B^{*} - M(z)^{*})^{-1}C^{*}}) = \det(I_{\mathcal{H}} + C_{*}(B^{*} - M(z)^{*})^{-1}) \tag{5.6}$$

for $z \in \rho(\widetilde{A}) \cap \rho(A_0)$. Since $((B' - B)(B - M(z))^{-1})^* = \overline{(B^* - M(z)^*)^{-1}C^*}$, applying Proposition 2.8(iii) and using (5.5) and the identity $M(z)^* = M(\overline{z})$, $z \in \rho(A_0)$, we obtain

$$\overline{\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z)} = \det(I_{\mathcal{H}} + \overline{(B^* - M(z)^*)^{-1}C^*}) = \Delta^{\Pi}_{\widetilde{A}'^*/\widetilde{A}^*}(\overline{z}), \tag{5.7}$$

for $z \in \rho(\widetilde{A}) \cap \rho(A_0)$. Replacing here $\overline{z}$ by z we arrive at (5.4).

(vi) This statement follows from (ii) and (iii). □

6 Determinants and annihilation functions

Following [67] we briefly recall some basic concepts and facts on C_0-contractions. Using dilation theory Foias and Nagy [67] have extended the Riesz-Dunford functional calculus for a contraction T to the class $H^\infty_T(\mathbb{D})$ (see [67, Section 3.2] for precise definitions). If a contraction T is completely non-unitary, then $H^\infty_T(\mathbb{D}) = H^\infty(\mathbb{D})$ is just the Hardy class in the unit disc $\mathbb{D}$. The extended functional calculus makes it possible to introduce concepts of C_0-contractions and minimal annihilation function.

Definition 6.1 ([67])**.**

(i) *A contraction T in $\mathfrak{H}$ is put in the class $C_{0\cdot}$ $(C_{\cdot 0})$ if $s - \lim_{n\to\infty} T^n = 0$ s-$(\lim_{n\to\infty} T^{*n} = 0)$. It is set $C_{00} := C_{\cdot 0} \cap C_{0\cdot}$.*

(ii) *It is said that a completely non-unitary operator T belongs to the class C_0 if there exists a function $u(\cdot) \in H^\infty(\mathbb{D}) \setminus \{0\}$ such that $u(T) = 0$. The function $u(\cdot)$ is called an annihilation function for T.*

(iii) *An annihilation function $u_0(\cdot)$ is called minimal if it is a divisor in $H^\infty(\mathbb{D})$ of any other annihilation function $u(\cdot)$ for T.*

It is well known that $C_0 \subset C_{00}$. Moreover, it is known [67, Proposition 3.4.4] that for any $T \in C_0$ the minimal function exists and is denoted by $m_T(\cdot)$. It is unique up to a multiplicative constant and is always an inner function.

Any m-dissipative operator D in $\mathfrak{H}$ is an infinitesimal operator of a contractive semigroup $U_D(t) = \exp(itD)$ and vise versa (see [67, Chapter 3])) where $\exp(itD)$, $t \in \mathbb{R}_+$, can be defined by means of the Nagy-Foias calculus. The main properties of D are closely related with the corresponding properties of its Cayley transform T_D, the contraction given by

$$T := T_D := (D - i)(D + i)^{-1} = I - 2i(D + i)^{-1}.$$

For instance, $s - \lim_{t\to\infty} \exp(itD) = 0$, i.e. the semigroup $U_D(\cdot)$ is stable, if and only if $T_D \in C_{0\cdot}$, [67, Proposition 3.9.2]. Moreover, T_D is completely non-unitary if and only if D is completely non-selfadjoint.

For any function $v(\cdot) \in H^\infty(\mathbb{C}_+)$ and any completely non-selfadjoint m-dissipative operator D we set $v(D) := \widetilde{v}(T_D)$ where

$$H^\infty(\mathbb{D}) \ni \widetilde{v}(\zeta) := v\left(i\frac{1+\zeta}{1-\zeta}\right), \quad \zeta \in \mathbb{D}.$$

We say the m-dissipative operator D belongs to the class $C_{0\cdot}$ ($C_{\cdot 0},\ C_0$) if T_D belongs to $C_{0\cdot}$ (resp. $C_{\cdot 0},\ C_0$). In other words, $D \in C_0$ if it is completely non-selfadjoint and $v(D) = \widetilde{v}(T_D) = 0$ with certain $v(\cdot) \in H^\infty(\mathbb{C}_+)$. Clearly, there always exists a minimal function $m_D(\cdot) \in H^\infty(\mathbb{C}_+)$ which is an inner function.

Here we demonstrate a role of perturbation determinants $\Delta^{\Pi}_{\widetilde{A}/\widetilde{A}^*}(\cdot)$ in the Nagy-Foias theory of annihilation functions of m-dissipative extensions $\widetilde{A}$ of a symmetric operator A with finite deficiency indices. Recall that a point $z \in \mathbb{C}$ is called a point of regular type for A (in short $z \in \widehat{\rho}(A)$), cf. [1, Section VIII.100], if there is a constant $c > 0$ such that $\|(A - z)f\|^2 \geq c\|f\|^2$, $f \in \mathrm{dom}\,(A)$. Recall that an unbounded operator $T \in \mathcal{C}(\mathfrak{H})$ with the compact resolvent is called complete in $\mathfrak{H}$ if the system of its root vectors is complete in $\mathfrak{H}$.

Proposition 6.2. *Let A be a* simple *closed symmetric operator in $\mathfrak{H}$ with finite deficiency indices $n := n_\pm(A) < \infty$ and $\widetilde{A} \in \mathrm{Ext}\,_A$ a maximal dissipative extension of A. Let also $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* and $\widetilde{A}$ disjoint with A_0, i.e. $\widetilde{A} = A_B$ with $B \in [\mathcal{H}]$. If $\widehat{\rho}(A) = \mathbb{C}$, then the following holds:*

(i) *The resolvent of $\widetilde{A}$ is compact, i.e. the spectrum of $\widetilde{A}$ is discrete.*

(ii) *If $\ker(B_I) = \{0\}$, $B_I = \mathrm{Im}\,(B)$, then $\widetilde{A}$ is completely non-selfadjoint operator with discrete spectrum. In particular, $\mathbb{R} \subset \rho(\widetilde{A})$.*

(iii) *If $\widetilde{A}$ is completely non-selfadjoint, then $\widetilde{A}$ belongs to the class C_0 and the perturbation determinant $d(\cdot) := \Delta^{\Pi}_{\widetilde{A}/\widetilde{A}^*}(\cdot)$ is an inner function holomorphic in $\mathbb{C}_+ \cup \mathbb{R}$. Moreover, $d(\cdot)$ is an annihilation function for $\widetilde{A}$.*

(iv) *If $\widetilde{A}$ is completely non-selfadjoint and complete, then the annihilation function $d(\cdot)$ is minimal for $\widetilde{A}$ if and only if the geometric multiplicity of any eigenvalue z of $\widetilde{A}$ is one, i.e. $\dim(\ker(\widetilde{A} - z)) = 1$ or, equivalently, $\dim(\ker(B - M(z))) = 1$. In particular, $d(\cdot)$ is minimal if $n_\pm(A) = 1$.*

Proof. (i) According to [1, Section 105] $z \in \mathbb{C}$ belongs to $\sigma_c(A)$, the continuous spectrum of A, if $z \in \mathbb{C} \setminus \widehat{\rho}(A)$ and $\mathrm{ran}\,(A - z)$ is not closed. Since $n < \infty$ and $\widehat{\rho}(A) = \mathbb{C}$, then, by [1, Theorem 100.1], $\sigma_c(\widehat{A}) = \emptyset$ for any $\widehat{A} = \widehat{A}^* \in \mathrm{Ext}\,_A$. Hence, the spectrum of any $\widehat{A} = \widehat{A}^* \in \mathrm{Ext}\,_A$ is discrete, i.e. its resolvent is compact. By the Krein type formula (2.7), the resolvent of any other extension $\widetilde{A}$ with $\rho(\widetilde{A}) \neq \emptyset$ is compact too, i.e. $\widetilde{A}$ has discrete spectrum.

(ii) Let us show that $\widetilde{A} = A_B$ is completely non-selfadjoint. By Proposition 2.3(iii), B is dissipative, i.e. $\mathrm{Im}\,(B) \geq 0$. Since the spectrum of $\widetilde{A}$ is discrete, it suffices to show that $\widetilde{A}$ has no real eigenvalues. It follows from the Green formula (2.1) that for any $x \in \mathbb{R}$ the following identity holds

$$\begin{aligned}
&\mathrm{Im}\,\big((A_B - x)f, f\big) = -i[(B\Gamma_0 f, \Gamma_0 f)_{\mathcal{H}} - (\Gamma_0 f, B\Gamma_0 f)_{\mathcal{H}}] \\
&= 2(B_I\Gamma_0 f, \Gamma_0 f)_{\mathcal{H}} = 2\left\|\sqrt{B_I}\Gamma_0 f\right\|^2_{\mathcal{H}}, \qquad f \in \mathrm{dom}\,(A^*).
\end{aligned}$$

Let $f \in \ker(A_B - x)$. Since, by the assumption, $\ker(B_I) = \{0\}$, the above identity yields $\Gamma_0 f = 0$ and $\Gamma_1 f = B\Gamma_0 f = 0$. Thus, $f \in \mathrm{dom}\,(A)$ and $f \in \ker(A - x)$. This contradicts the simplicity of A. Thus, $\mathbb{R} \subset \rho(\widetilde{A}))$.

(iii) Since $\widetilde{A}$ is m-dissipative, the characteristic function $W_{\widetilde{A}}(\cdot) := W^{\Pi}_{\widetilde{A}}(\cdot)$ is given by (3.9) with $J = I$. Further, since $\widetilde{A}$ is m-dissipative, the characteristic function $W_{\widetilde{A}}(\cdot)$ is contractive and holomorphic in $\mathbb{C}_+$. Moreover, since $\mathbb{R} \subset \rho(\widetilde{A})$, $W_{\widetilde{A}}(\cdot)$ is holomorphic and unitary on $\mathbb{R}$, $W^*_{\widetilde{A}}(x)W_{\widetilde{A}}(x) = I_{\mathcal{H}}$, i.e. $W_{\widetilde{A}}(\cdot)$ is an inner matrix-valued function. By [67, Proposition VI.3.5], $T_{\widetilde{A}} \in C_{\cdot 0}$. Similarly, since the operator $-\widetilde{A}^*$ is m-dissipative too, its characteristic function $W_{-\widetilde{A}^*}(\cdot)$ is also inner matrix-valued function in $\mathbb{C}_+$. Hence $T_{-\widetilde{A}^*} = T^*_{\widetilde{A}}$ belongs to the class $C_{\cdot 0}$ which is equivalent to the inclusion $T_{\widetilde{A}} \in C_{0\cdot}$. Therefore $T_{\widetilde{A}} \in C_{\cdot 0} \cap C_{0\cdot} = C_{00}$. Hence the determinant $d_1(\cdot) := \det(W_{\widetilde{A}}(\cdot))$ is an inner function in $\mathbb{C}_+$ holomorphic in $\mathbb{C}_+ \cup \mathbb{R}_+$.

Since $n_\pm(A) = n < \infty$, the contraction $T_{\widetilde{A}} = (\widetilde{A} - i)(\widetilde{A} + i)^{-1}$ has equal finite defect numbers, i.e. $\dim\left(\mathrm{ran}\,(I - T^*_{\widetilde{A}}T_{\widetilde{A}})\right) = \dim\left(\mathrm{ran}\,(I - T_{\widetilde{A}}T^*_{\widetilde{A}})\right) < \infty$. By [67, Theorem VI.5.2], $T_{\widetilde{A}} \in C_0$ and the determinant $d_1(\cdot) = \det(W_{\widetilde{A}}(\cdot))$ defined on $\mathbb{C}_+$, is an annihilation function for $\widetilde{A}$: $d_1(\widetilde{A}) = \widetilde{d}_1(T_{\widetilde{A}}) = 0$.

On the other hand, since $n_\pm(A) < \infty$, $\{\widetilde{A}, \widetilde{A}^*\} \in \mathfrak{D}^{\Pi}$ and the perturbation determinant $d(\cdot) := \Delta^{\Pi}_{\widetilde{A}/\widetilde{A}^*}(\cdot)$ is well defined on $\rho(\widetilde{A}^*)$. Therefore combining (1.8) with (3.9) one gets

$$d(z) = \det\big((B - M(z))\big(B^* - M(z)\big)^{-1}\big) = \det\big(I + 2iB_I^{1/2}\big(B^* - M(z)\big)^{-1}B_I^{1/2}\big),$$

i.e. $d(z) = \det(W_{\widetilde{A}}(z)) = d_1(z)$, $z \in \rho(\widetilde{A}^*) \cap \rho(A_0)$. Thus, $d(\cdot)$ is an inner function in $\mathbb{C}_+$ and annihilates $\widetilde{A}$, $d(\widetilde{A}) = 0$.

(iv) Let $\{e_j\}_{j=1}^n$ be a fixed orthonormal basis in $\mathcal{H}$. Denote by $\Theta_{\widetilde{A}}(\cdot)$ the matrix representation of the characteristic function $W_{\widetilde{A}}(\cdot)$ with respect to the basis $\{e_j\}_{j=1}^n$. By $\mathrm{adj}\,(\Theta_{\widetilde{A}}(\cdot))$ we denote the adjugate matrix of $\Theta_{\widetilde{A}}(\cdot)$. Note that together with $\Theta_{\widetilde{A}}(\cdot)$ the matrix function $\mathrm{adj}\,(\Theta_{\widetilde{A}}(\cdot))$ is holomorphic and contractive in $\mathbb{C}_+$, too (cf. the proof of [67, Proposition V.6.1]). By [67, Theorem VI.5.2], the determinant $d(\cdot) := \det(\Theta_{\widetilde{A}}(\cdot)) = \det(W_{\widetilde{A}}(\cdot))$ of $\Theta_{\widetilde{A}}(\cdot)$ coincides with the minimal annihilation function $m_{\widetilde{A}}(\cdot)$ of $\widetilde{A}$ if and only if the entries of $\mathrm{adj}\,(\Theta_{\widetilde{A}}(\cdot))$ have no common non-trivial inner divisor in the algebra $H^\infty(\mathbb{C}_+)$.

On the other hand, by (iii), $T_{\widetilde{A}} \in C_0$. Therefore the operator $T_{\widetilde{A}}$, hence $\widetilde{A}$, is complete if and only if the determinant $d(\cdot)$ is a Blaschke product (see [57, Section 4.5]). Therefore it follows from the identity $\mathrm{adj}\,(\Theta_{\widetilde{A}}(\cdot)) \cdot \Theta_{\widetilde{A}}(\cdot) = d(\cdot)I_n$ that each common divisor $\varphi(\cdot)$ of the entries of $\mathrm{adj}\,(\Theta_{\widetilde{A}}(\cdot))$ has to be a divisor of $d(\cdot)$. Therefore $\varphi(\cdot)$ always contains a Blaschke factor, i.e., it admits the representation $\varphi(\cdot) = \varphi_1(\cdot)b^{m_0}_{z_0}(\cdot)$ where $b^{m_0}_{z_0}(\cdot)$ is a Blaschke factor $b^{m_0}_{z_0}(z) := (e^{i\alpha_0}(z - z_0)/(z - \overline{z_0}))^{m_0}$, $m_0 \geq 1$, $z_0 \in \mathbb{C}_+$, cf. [37]. Clearly, the latter happens if and only if $\mathrm{adj}\,(\Theta_{\widetilde{A}}(z_0)) = \mathbb{O}_n := 0 \cdot I_n$. However, $\mathrm{adj}\,(\Theta_{\widetilde{A}}(z_0)) = \mathbb{O}_n$ if and only if $\mathrm{rank}\,(\Theta_{\widetilde{A}}(z_0)) \leq n-2$, i.e. $\dim\,(\ker(\Theta_{\widetilde{A}}(z_0))) = \dim\,(\ker(W_{\widetilde{A}}(z_0))) \geq 2$.

Further, by Proposition 2.5(ii), $\dim\,\ker(\widetilde{A} - z) = \dim\,\ker\big(B - M(z)\big)$ for any $z \in \rho(A_0)$. Let us show that

$$\dim\,(\ker(W_{\widetilde{A}}(z_0))) = \dim\,(\ker(B - M(z_0))), \qquad z_0 \in \mathbb{C}_+. \tag{6.1}$$

Indeed, setting $T_1 := 2i(B^* - M(z_0))^{-1}B_I^{1/2}$ one immediately gets $\ker(T_1) = \{0\}$ by $\ker(B_I) = \{0\}$. Further, we note that $W_{A_B}(z_0)h_0 = 0$ if and only if $h_0 \in \ker(I + B_I^{1/2}T_1)$. If $h_0 \in \ker(I+B_I^{1/2}T_1)$, then $T_1 h_0 \in \ker(I+T_1B_I^{1/2})$ which yields $\dim(\ker(I+B_I^{1/2}T_1)) \leq \dim(\ker(I+T_1B_I^{1/2}))$. Conversely, if $h_1 \in \ker(I+T_1B_I^{1/2})$, then $B_I^{1/2}h_1 \in \ker(I+B_I^{1/2}T_1)$ which proves the relation $\dim(\ker(I + T_1B_I^{1/2})) \leq \dim(\ker(I + B_I^{1/2}T_1))$. Hence

$$\dim(\ker(W_{\widetilde{A}}(z_0)) = \dim(\ker(I + B_I^{1/2}T_1)) = \dim(\ker(I + T_1B_I^{1/2})).$$

Combining this relation with the identity $I+T_1B_I^{1/2} = (B^*-M(z_0))^{-1}(B-M(z_0))$ we arrive at (6.1). Thus, $d(\cdot) = \Delta^{\Pi}_{\widetilde{A}/\widetilde{A}^*}(\cdot)$ is a minimal annihilation function if and only if $\dim(\ker(B - M(z))) = 1$ for $z \in \sigma(\widetilde{A})$ which yields $\dim(\ker(\widetilde{A} - z)) = 1$. $\square$

7 Examples

7.1 Sturm-Liouville operators on a finite interval

Consider in $L^2([0,b],\mathbb{C}^n)$ the matrix Sturm-Liouville differential expression

$$(\mathcal{A}f)(x) := -\frac{d^2}{dx^2}f(x) + Q(x)f(x), \quad f = \mathrm{col}\{f_1,\ldots,f_n\}, \tag{7.1}$$

with $n\times n$-matrix potential $Q(\cdot) = Q(\cdot)^* \in L^2([0,b],\mathbb{C}^{n\times n})$. It is well known that the maximal operator $A_{\max}$ associated in $L^2([0,b],\mathbb{C}^n)$ with the differential expression (7.1) is given by

$$(A^*f)(x) := (\mathcal{A}[f])(x), \quad f \in \mathrm{dom}(A^*) = W^{2,2}((0,b),\mathbb{C}^n). \tag{7.2}$$

The minimal operator $A = A_{\min}$ is a closed symmetric operator given by

$$(Af)(x) := (\mathcal{A}[f])(x), \quad f \in \mathrm{dom}(A) = W_0^{2,2}((0,b),\mathbb{C}^n) \tag{7.3}$$

Notice that $A_{\max} = A^*$. Due to the regularity property (7.2) for $\mathrm{dom}(A^*)$ the mappings

$$\Gamma_0 f := \begin{pmatrix} f(b) \\ f(0) \end{pmatrix}, \quad \Gamma_1 f := \begin{pmatrix} -f'(b) \\ f'(0) \end{pmatrix}, \quad f \in W^{2,2}((0,b),\mathbb{C}^n), \tag{7.4}$$

are well defined. Moreover, one easily checks that $\Pi = \{\mathbb{C}^{2n},\Gamma_0,\Gamma_1\}$ forms a boundary triplet for A^*. Notice that $A_0 := A^* \upharpoonright \ker(\Gamma_0)$ and $A_1 := A^* \upharpoonright \ker(\Gamma_1)$ correspond to the Dirichlet and Neumann boundary conditions, respectively.

Let us introduce the $n\times n$ matrix solutions $C(z,x)$ and $S(z,x)$

$$\begin{aligned}
\mathcal{A}[C(z,x)] &= zC(z,x), \quad C(z,0) = I_n, \quad C'(z,0) = \mathbb{O}_n \\
\mathcal{A}[S(z,x)] &= zS(z,x), \quad S(z,0) = \mathbb{O}_n, \quad S'(z,0) = I_n.
\end{aligned}$$

Any $f_z \in \ker(A^* - z)$ admits the representation $f_z(x) = C(z,x)\xi + S(z,x)\eta$ for some $\xi, \eta \in \mathbb{C}^n$. Hence

$$\Gamma_0 f_z = \begin{pmatrix} C(z,b) & S(z,b) \\ I_n & 0_n \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \text{and} \quad \Gamma_1 f_z = \begin{pmatrix} -C'(z,b) & -S'(z,b) \\ 0_n & I_n \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Combining these relations with Definition 2.4 we find that the Weyl function $M(\cdot)$ corresponding to the triplet Π is

$$\begin{aligned} M(z) &= \begin{pmatrix} -C'(z,b) & -S'(z,b) \\ 0_n & I_n \end{pmatrix} \begin{pmatrix} 0_n & I_n \\ S(z,b)^{-1} & -S(z,b)^{-1}C(z,b) \end{pmatrix} \\ &= \begin{pmatrix} -S'(z,b)S(z,b)^{-1} & -C'(z,b) + S'(z,b)S(z,b)^{-1}C(z,b) \\ S(z,b)^{-1} & -S(z,b)^{-1}C(z,b) \end{pmatrix} \\ &= \begin{pmatrix} -S'(z,b)S(z,b)^{-1} & S^*(\bar{z},b)^{-1} \\ S(z,b)^{-1} & -S(z,b)^{-1}C(z,b) \end{pmatrix}, \qquad z \in \mathbb{C}_\pm. \end{aligned} \tag{7.5}$$

If $Q \equiv 0$, then the Weyl function $M(\cdot) = M_0(\cdot)$ takes the form

$$M_0(z) = -\frac{1}{\sin(\sqrt{z}b)} \begin{pmatrix} \sqrt{z}\cos(\sqrt{z}b)I_n & -I_n \\ -I_n & \cos(\sqrt{z}b)I_n \end{pmatrix}. \tag{7.6}$$

Let $B' = \begin{pmatrix} B'_{11} & B'_{12} \\ B'_{21} & B'_{22} \end{pmatrix}$, $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, where B'_{ij}, $B_{ij} \in \mathbb{C}^{n\times n}$, $i,j \in \{1,2\}$. Define proper extensions $\widetilde{A}' := A_{B'}$ and $\widetilde{A} := A_B$ with the boundary operators B' and B, respectively (cf. (2.3)). Using (7.4) one defines $\widetilde{A}'$ and $\widetilde{A}$ explicitly in terms of boundary conditions. For instance,

$$\operatorname{dom}(\widetilde{A}) \;=\; \left\{ f \in W^{2,2}(0,b) : \begin{matrix} -f'(b) = B_{11}f(b) + B_{12}f(0) \\ f'(0) = B_{21}f(b) + B_{22}f(0) \end{matrix} \right\}. \tag{7.7}$$

Notice that $\{\widetilde{A}', \widetilde{A}\} \in \mathfrak{D}^\Pi$. Since $n_\pm(A) = 2n < \infty$, we get (cf. (1.12))

$$\Delta^\Pi_{\widetilde{A}'/\widetilde{A}}(z) = \frac{\det(B' - M(z))}{\det(B - M(z))}, \qquad z \in \rho(\widetilde{A}) \cap \rho(A_0). \tag{7.8}$$

Combining this formula with (4.6) we arrive at the following representation

$$\widetilde{\Delta}_{\widetilde{A}'/\widetilde{A}}(\xi, z) = \frac{\det(B' - M(z))}{\det(B - M(z))} \frac{\det(B - M(\xi))}{\det(B' - M(\xi))}$$

for $z \in \rho(\widetilde{A}) \cap \rho(A_0)$ and $\xi \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$.

If $\widetilde{A}' = A_{B'}$ and $\widetilde{A} = A_0$, then we obtain from (4.12) that

$$\widetilde{\Delta}_{\widetilde{A}'/A_0}(\xi, z) = \frac{\det(B' - M(z))}{\det(B' - M(\xi))}, \qquad z \in \rho(A_0), \quad \xi \in \rho(\widetilde{A}') \cap \rho(A_0).$$

Proposition 7.1. *Let A be the minimal Sturm-Liouville operator on $[0,b]$ defined by (7.3) and let $B \in \mathbb{C}^{2n\times 2n}$, $B_I := \operatorname{Im}(B) \geq 0$, and $\ker B_I = \{0\}$. Let also $A_B = A^* \upharpoonright \operatorname{dom}(A_B)$, $\operatorname{dom}(A_B) := \ker(\Gamma_1 - B\Gamma_0)$. Then the following holds:*

(i) A *is simple and* $\widehat{\rho}(A) = \mathbb{C}$.

(ii) A_B *is a* m*-dissipative and completely non-selfadjoint operator with discrete spectrum such that* $\mathbb{R} \subseteq \rho(A_B)$. *Moreover, the operator* A_B *is complete.*

(iii) A_B *is of the* C_0*–class. Moreover, the perturbation determinant* $d(\cdot) = \Delta^{\Pi}_{A_B/A_B^*}(\cdot)$ *is an annihilation function for* A_B*, that is,* $d(A_B) = 0$.

(iv) *The annihilation function* $d(\cdot)$ *is minimal if and only if* $\dim\,(\ker\,(B - M(z))) = 1$ *for any* $z \in \sigma(A_B) \cap \mathbb{C}_+ = \sigma_p(A_B)$.

Proof. (i) It follows immediately from the Cauchy uniqueness theorem.

(ii) The first claim follows from Proposition 6.2(i) and (ii). Further, it is well known that the resolvent of A_0 is of trace class. It follows from Proposition 2.6(i) that the resolvent of A_B is also of trace class. Since, in addition, A_B is m-dissipative, it follows from [31, Theorem V.6.1]) that A_B is complete.

(iii) and (iv) These statements follow from Proposition 6.2(iii) and (iv). □

7.2 Matrix Sturm-Liouville operators on $\mathbb{R}_+$

Let us consider in $L^2(\mathbb{R}_+, \mathbb{C}^n)$ the matrix Sturm-Liouville differential expression (7.1) with $n \times n$ matrix potential $Q(\cdot) = Q(\cdot)^* \in L^1(\mathbb{R}_+, \mathbb{C}^{n\times n}) \cap L^\infty(\mathbb{R}_+, \mathbb{C}^{n\times n})$. Denote by $A = A_{\min}$ and $A_{\max}$ the corresponding minimal and maximal operators, respectively, associated in $L^2(\mathbb{R}_+, \mathbb{C}^n)$ with expression (7.1). Clearly, A is symmetric. Notice that the deficiency indices are $n_\pm(A) = n$. It is known (see, for instance, [56, Section 5.17.4]) that $A^* = A_{\max}$. The latter means that $\mathrm{dom}\,(A^*) = W^{2,2}(\mathbb{R}_+, \mathbb{C}^n)$. A^* is given by the differential expression (7.1) on the domain $\mathrm{dom}\,(A^*)$. Therefore the trace mappings $\Gamma_0, \Gamma_1 : \mathrm{dom}\,(A^*) \to \mathbb{C}^n$,

$$\Gamma_0 f = f(0), \quad \Gamma_1 f = f'(0), \quad f = \mathrm{col}\,\{f_1, \ldots, f_n\},$$

are well defined and the Green identity (2.1) holds. Moreover, one easily proves that $\Pi = \{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^*. Hence the minimal operator $A = A_{\min}$ is a restriction of A^* to the domain $\mathrm{dom}\,(A) = \ker \Gamma_0 \cap \ker \Gamma_1$,

$$\mathrm{dom}\,(A) = W^{2,2}_0(\mathbb{R}_+, \mathbb{C}^n) := \{f \in W^{2,2}(\mathbb{R}_+, \mathbb{C}^n) : f'(0) = f(0) = 0\}.$$

Due to our assumption on Q the Weyl matrix solution $\Psi(z, x)$ coincides with Jost matrix function $F(z, \cdot)$, being the solution of the integral equation

$$F(z, x) = e^{i\sqrt{z}x} I_n - \int_x^\infty \frac{1}{\sqrt{z}} \sin\left(\sqrt{z}(x - t)\right) Q(t) F(z, t) dt, \quad z \in \mathbb{C} \setminus \{0\}, \tag{7.9}$$

where $\mathrm{Im}\,(\sqrt{z}) \geq 0$. The Weyl $n \times n$ matrix function is given by

$$M(z) = F'(z, 0) F(z, 0)^{-1}, \qquad z \in \mathbb{C}_\pm.$$

Let $\widetilde{A}'$ and $\widetilde{A}$ be proper extensions of A disjoint with $A_0 := A^* \upharpoonright \ker(\Gamma_0)$. By Proposition 2.3, $\widetilde{A}' = A_{B'}$ and $\widetilde{A} = A_B$ with some operators $B', B \in [\mathbb{C}^n]$, i.e. $\mathrm{dom}\,(\widetilde{A}') = \{f \in \mathrm{dom}\,(A^*) : f'(0) = B'f(0)\}$ and $\mathrm{dom}\,(\widetilde{A}) = \{f \in \mathrm{dom}\,(A^*) : f'(0) = Bf(0)\}$. Thus, the boundary triplet Π is regular for the pair $\{\widetilde{A}', \widetilde{A}\}$. Moreover, since $n_\pm(A) = n < \infty$, condition (3.1) is satisfied and

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z) = \frac{\det\big(B' - M(z)\big)}{\det\big(B - M(z)\big)} = \frac{\det\big(B'F(z,0) - F'(z,0)\big)}{\det\big(BF(z,0) - F'(z,0)\big)}, \tag{7.10}$$

for $z \in \rho(\widetilde{A}) \cap \rho(A_0)$. Using (4.6) we find the representation

$$\widetilde{\Delta}_{\widetilde{A}'/\widetilde{A}}(\xi, z) = \frac{\det\big(B'F(z,0) - F'(z,0)\big)}{\det\big(BF(z,0) - F'(z,0)\big)} \frac{\det\big(BF(\xi,0) - F'(\xi,0)\big)}{\det\big(B'F(\xi,0) - F'(\xi,0)\big)}$$

for $z \in \rho(\widetilde{A}) \cap \rho(A_0)$ and $\xi \in \rho(\widetilde{A}') \cap \rho(\widetilde{A}) \cap \rho(A_0)$. In particular, if $Q \equiv 0$, then $M(z) = M_0(z) := i\sqrt{z}I_n$ and

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z) = \frac{\det\big(B' - M_0(z)\big)}{\det\big(B - M_0(z)\big)} = \frac{\det\big(B' - i\sqrt{z}\big)}{\det\big(B - i\sqrt{z}\big)}, \qquad z \in \rho(\widetilde{A}) \cap \rho(A_0).$$

If $\widetilde{A} = A_0$, then $\{\widetilde{A}', A_0\} \notin \mathfrak{D}^{\Pi}$. However, according to (4.12) for any $\xi, z \in \rho(\widetilde{A}) \cap \rho(A_0)$ the generalized perturbation determinant $\widetilde{\Delta}_{\widetilde{A}'/A_0}(\xi, \cdot)$ is

$$\widetilde{\Delta}_{\widetilde{A}'/A_0}(\xi, z) = \frac{\det\big(B'F(z,0) - F'(z,0)\big)}{\det\big(F(z,0)\big))} \frac{\det\big(F(\xi,0)\big)}{\det\big(B'F(\xi,0) - F'(\xi,0)\big)}.$$

7.3 Dirac type operators on a finite interval

Consider in $L^2([0,1], \mathbb{C}^n)$ the first order differential expression

$$(\mathcal{A}[f])(x) := -iB^{-1}\frac{d}{dx}f(x) + Q(x)f(x), \qquad f = \mathrm{col}\,\{f_1, \ldots, f_n\}, \tag{7.11}$$

with a potential matrix $Q(\cdot) = Q(\cdot)^* \in L^2([0,1], \mathbb{C}^{n\times n})$. Here B is $n \times n$ self-adjoint non-singular diagonal matrix with not necessarily different eigenvalues,

$$B = \mathrm{diag}\,(b_1, \ldots, b_n) = B^* \in \mathbb{C}^{n\times n}. \tag{7.12}$$

Denote by $P_-(P_+)$ the ortho-projection onto the eigenvectors of B corresponding to its negative (resp. positive) eigenvalues. We put $\kappa := \dim P_- \in \{0, 1..., n\}$. If $n = 2m$ and $B = (-I_m, I_m)$, then $\mathcal{A}$ is equivalent to the Dirac expression.

The minimal operator $A = A_{\min}$ associated in $L^2\big([0,1], \mathbb{C}^n\big)$ with the differential expression $\mathcal{A}$ is given by

$$\begin{aligned} (Af)(x) &:= (\mathcal{A}[f])(x), \\ \mathrm{dom}\,(A) &= W^{1,2}_0\big([0,1], \mathbb{C}^n\big) := \{f \in W^{1,2}\big([0,1], \mathbb{C}^n\big) : f(0) = f(1) = 0\}. \end{aligned}$$

Clearly, A is a closed symmetric operator with $n_{\pm}(A) = n$. Moreover, the maximal operator $A_{\max} = A^*$ is given by

$$(A^* f)(x) := (\mathcal{A}[f])(x), \qquad \mathrm{dom}\,(A^*) = W^{1,2}([0,1], \mathbb{C}^n). \tag{7.13}$$

Due to the regularity property (7.13) of the domain $\mathrm{dom}\,(A^*)$ the mappings

$$\sqrt{2}\Gamma_0 f = -iB^{-1}\big(f(0) - f(1)\big), \quad \sqrt{2}\Gamma_1 f = f(0) + f(1), \ f \in \mathrm{dom}\,(A^*), \tag{7.14}$$

are well defined. Moreover, one easily checks that the triplet $\Pi = \{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ forms a boundary triplet for A^*.

Let $\Phi(\cdot, z)$ be the (fundamental) $n \times n$ matrix solution of the Cauchy problem

$$-iB^{-1}\frac{d}{dx}\Phi(x,z) + Q(x)\Phi(x,z) = z\Phi(x,z), \quad \Phi(0,z) = I_n, \quad z \in \mathbb{C}. \tag{7.15}$$

Clearly, $\Gamma_0\Phi = -iB^{-1}\left(I_n - \Phi(1,z)\right)$ and $\Gamma_1\Phi = I_n + \Phi(1,z)$. By Definition 2.4, the corresponding Weyl function is

$$M(z) = i\left(I + \Phi(1,z)\right)\left(I - \Phi(1,z)\right)^{-1} B. \tag{7.16}$$

Denote by $\mathcal{A}_{C,D}$ a restriction of the operator $A_{\max}$ to the domain

$$\mathrm{dom}\,(\mathcal{A}_{C,D}) = \{f \in W^{1,2}([0,1], \mathbb{C}^n) : \ Cf(0) + Df(1) = 0\}, \tag{7.17}$$

and assume the condition $\mathrm{rank}\,(C\ D) = n$, i.e. $\ker(CC^* + DD^*) = \{0\}$.

The operator $\widetilde{A} := \mathcal{A}_{C,D}$ is naturally associated with boundary value problem

$$-iB^{-1}\frac{d}{dx}y(x,z) + Q(x)y(x,z) = zy(x,z), \quad Cy(0) + Dy(1) = 0. \tag{7.18}$$

Consider also an operator $\widetilde{A}' := \mathcal{A}_{C',D'}$ given by (7.17)–(7.18) with matrices $C', D' \in \mathbb{C}^{n\times n}$ in place of C and D. Suppose also that $\mathrm{rank}\,(C'\ D') = n$.

Proposition 7.2. *Assume that* $0 \in \rho(C+D) \cap \rho(C'+D')$. *Then the perturbation determinant* $\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(\cdot)$ *is given by*

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z) = \frac{\det\big(C + D\big)}{\det\big(C' + D'\big)}\frac{\det\big(C' + D'\Phi(1,z)\big)}{\det\big(C + D\Phi(1,z)\big)}\,. \tag{7.19}$$

Proof. It is easily seen that with respect to the boundary triplet Π the operators $\widetilde{A}$ and $\widetilde{A}'$ admit representations $\widetilde{A} = A_T = A^* \upharpoonright \ker(\Gamma_1 - T\Gamma_0)$ and $\widetilde{A}' = A_{T'} = A^* \upharpoonright \ker(\Gamma_1 - T'\Gamma_0)$ with the boundary operators T and T' given by

$$T := i(C+D)^{-1}(D-C)B \qquad \text{and} \qquad T' := i(C'+D')^{-1}(D'-C')B. \tag{7.20}$$

Combining (7.16) with (7.20) one easily checks that

$$\begin{aligned}&\det\bigl(T-M(z)\bigr)=\\&\qquad(-2i)^n\det(C+D)^{-1}\det\bigl((I-\Phi(1,z))^{-1}B\bigr)\det\bigl(C+D\Phi(1,z)\bigr).\end{aligned}\tag{7.21}$$

Combining this relation with a similar relation for the determinant $\det\bigl(T'-M(\cdot)\bigr)$ and using

$$\Delta^{\Pi}_{\widetilde{A}'/\widetilde{A}}(z)=\det\bigl(T'-M(z)\bigr)\bigl(\det\bigl(T-M(z)\bigr)\bigr)^{-1},$$

cf. (1.12), one proves (7.19). □

Remark 7.3. Recall that $\det\bigl(C+D\Phi(1,z)\bigr)$ is called the characteristic determinant of the operator $\mathcal{A}_{C,D}$ (cf. [5]) and [55]. Thus, formula (7.19) shows that up to a multiplicative constant the perturbation determinant $\widetilde{\Delta}^{\Pi}_{\widetilde{A}'/\widetilde{A}}(\cdot)$ is a ratio of the two characteristic determinants.

Proposition 7.4. *Let A be the minimal operator associated in $L^2\bigl([0,1],\mathbb{C}^n\bigr)$ with the differential expression $\mathcal{A}$ (7.11) and let $\widetilde{A}:=A_{C,D}\in \mathrm{Ext}_A$ be its proper extension given by (7.17). Then:*

(i) *A is simple and $\widehat{\rho}(A)=\mathbb{C}$.*

(ii) *The operator $\widetilde{A}=A_{C,D}$ is m-dissipative if and only if*

$$K:=DBD^*-CBC^*\geq 0.\tag{7.22}$$

If, in addition, $\ker K=\{0\}$, then $\widetilde{A}=A_{C,D}$ is completely non-selfadjoint operator with discrete spectrum and $\mathbb{R}\subset\rho(\widetilde{A})$.

(iii) *$\widetilde{A}\in C_0$ if (7.22) is satisfied and $\ker K=\{0\}$. Moreover, the perturbation determinant $d(\cdot)=\Delta^{\Pi}_{\widetilde{A}/\widetilde{A}^*}(\cdot)$*

$$d(z)=\Delta^{\Pi}_{\widetilde{A}/\widetilde{A}^*}(z)=\frac{\det(C^*+D^*)}{\det(C+D)}\frac{\det\bigl(C+D\Phi(1,z)\bigr)}{\det\bigl(D^*+B^{-1}\Phi(1,z)BC^*\bigr)}\,.\tag{7.23}$$

is an annihilation function for $\widetilde{A}$, $d(\widetilde{A})=0$.

(iv) *The operator $A_{C,D}$ is complete provided that it is m-dissipative and*

$$\det(CP_++DP_-)\neq 0.\tag{7.24}$$

In this case the perturbation determinant $d(\cdot)$ is the minimal annihilation function if and only if $\dim\,\ker\bigl(C+D\Phi(1,z)\bigr)=1$ for $z\in\sigma(A_{C,D})\cap\mathbb{C}_+=\sigma_p(A_{C,D})$.

Proof. (i) The statement immediately follows from the Cauchy uniqueness theorem.

(ii) We confine ourselves to the case $0\in\rho(C+D)$. Then the corresponding boundary operator T is given by (7.20). By the straightforward computation

$$-i(T-T^*)=2(C+D)^{-1}K(C^*+D^*)^{-1}.\tag{7.25}$$

Hence the dissipativity of T is equivalent to (7.22). In turn, by Proposition 2.3(iii), the m-dissipativity of $A_T = A_{C,D}$ is equivalent to (7.22).

Further, by (7.25), the condition $\ker K = \{0\}$ yields $\ker(T_I) = \{0\}$. Since $T_I \geq 0$ and $\ker(T_I) = \{0\}$, then, by Proposition 6.2(ii), A_T is completely non-selfadjoint operator with discrete spectrum and $\mathbb{R} \subset \rho(A_T)$.

(iii) Let us find the matrices C_*, D_* parameterizing the operator $\mathcal{A}^*_{C,D}$ by means of (7.17). Without loss of generality we can assume that $C + D = I_n$. Otherwise we replace C and D in (7.17) by $(C+D)^{-1}C$ and $(C+D)^{-1}D$, respectively. By (7.20) the boundary operator is $T = i(D-C)B$. Setting

$$C_* = BD^*B^{-1}, \quad D_* = BC^*B^{-1}, \quad C_* + D_* = I_m, \tag{7.26}$$

one easily gets $T_* = i(D_* - C_*)B = -iB(D^* - C^*) = T^*$. Hence $\mathcal{A}_{C_*,D_*} = (\mathcal{A}_{C,D})^*$. It follows from (7.16) and (7.20) that

$$\begin{aligned}\det(T^* - M(z)) &= (-2i)^n \big(\det(I - \Phi(1,z))\big)^{-1} \det(C_* + D_*\Phi(1,z)) \\ &= (-2i)^n \big(\det(I - \Phi(1,z))\big)^{-1} \det\big(B(I - C^*)B^{-1} + BC^*B^{-1}\Phi(1,z)\big) \\ &= (-2i)^n \big(\det(I - \Phi(1,z))\big)^{-1} \det\big(D^* + C^*B^{-1}\Phi(1,z)B\big)\end{aligned}$$

Combining (7.26) with (7.21) we arrive at (7.23).

(iv) According to [55, Corollary 4.3] conditions (7.22), (7.24) imply the completeness of the operator $A_{C,D}$. It remains to apply Proposition 6.2(iv) and note that due to (7.21) $\dim\left(\ker(T - M(z))\right) = \dim\,\ker\big(C + D\Phi(1,z)\big)$. □

Remark 7.5. If $Q = 0$, then the corresponding Weyl function (7.16) is

$$M(z) = \operatorname{diag}\left(-b_1\cot(2^{-1}b_1 z), \ldots, -b_n\cot(2^{-1}b_n z)\right).$$

The annihilation function (7.23) is $\det\big(C + D\Phi(1,z)\big)\big(\det\big(D^* + C^*\Phi(1,z)\big)\big)^{-1}$.

7.4 Second order elliptic operators in domains with compact boundary

7.4.1 Basic facts on elliptic operators

Consider the second-order formally symmetric elliptic operator with smooth real coefficients in a domain $\Omega \subset \mathbb{R}^n$ with smooth compact boundary $\partial\Omega$,

$$\mathcal{A} := -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk}(x) \frac{\partial}{\partial x_k} + q(x), \quad a_{jk} = \overline{a}_{jk}, \;\; q = \overline{q} \in C^\infty(\overline{\Omega}). \tag{7.27}$$

Recall that ellipticity of $\mathcal{A}$ means that its (principle) symbol $a_0(x,\xi)$ satisfies

$$a_0(x,\xi) := \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \neq 0, \qquad (x,\xi) \in \overline{\Omega} \times (\mathbb{R}^n \setminus \{0\}). \tag{7.28}$$

Here basically following [34] and [51] we present some known facts on operators (7.27)-(7.28), needed for computing the perturbation determinants.

Let $A = A_{\min}$ be the minimal elliptic operator associated in $L^2(\Omega)$ with expression (7.27). Due to Green's identity the operator A is symmetric in $L^2(\Omega)$. Any proper extension $\widetilde{A} \in \mathrm{Ext}_A$ of A is called a realization of $\mathcal{A}$. Clearly, any realization $\widetilde{A}$ of $\mathcal{A}$ is closable. We equip $\mathrm{dom}\,(A_{\max})$ with the corresponding graph norm. It is known (cf. [8, 49]) that for bounded domain Ω $\mathrm{dom}\,(A_{\min})$, equipped with the graph norm, coincides with the Sobolev space $H^2_0(\Omega)$ algebraically and topologically, $\mathrm{dom}\,(A_{\min}) = H^2_0(\Omega)$. However, in contrast to the case of ordinary differential operators, $\mathrm{dom}\,(A_{\max}) \neq H^2(\Omega)$ while

$$H^2(\Omega) \subsetneq \mathrm{dom}\,(A_{\max}) \subsetneq H^2_{\mathrm{loc}}(\Omega).$$

Since the symbol $a_0(x,\xi)$ is real the operator $\mathcal{A}$ is properly elliptic (see [49]).

Hypothesis 7.6. $\mathcal{A}$ is uniformly elliptic and its leading coefficients $a_{jk}(\cdot)$ are bounded and uniformly continuous in Ω with all their derivatives. Further, let $q \in L^\infty(\Omega) \cap C^\infty(\Omega)$.

In particular, assuming Hypothesis 7.6 we have $\mathrm{dom}\,(A_{\min}) = H^2_0(\Omega)$ even for unbounded domains. Notice that for bounded Ω any elliptic differential expression $\mathcal{A}$ with $C(\overline{\Omega})$-coefficients is automatically uniformly elliptic in $\overline{\Omega}$.

Denote by $\frac{\partial}{\partial\nu}$ the conormal derivative:

$$\frac{\partial}{\partial\nu} = \sum_{j,k=1}^{n} a_{jk}(x)\cos(n,x_j)\frac{\partial}{\partial x_k} \tag{7.29}$$

and set

$$G_0 u := \gamma_0 u := u|_{\partial\Omega}, \quad G_1 u := \gamma_0\left(\frac{\partial u}{\partial\nu}\right) = \left(\frac{\partial u}{\partial\nu}\right)\Big|_{\partial\Omega}, \qquad u \in \mathrm{dom}\,(A_{\max}).$$

We define the Dirichlet and Neumann realizations $\widehat{A}_{G_0}$ and $\widehat{A}_{G_1}$ by setting

$$\begin{aligned} \widehat{A}_{G_j} &:= A_{\max} \upharpoonright \mathrm{dom}\,(\widehat{A}_{G_j}), \\ \mathrm{dom}\,(\widehat{A}_{G_j}) &:= \{u \in H^2(\Omega) \,|\, G_j u = 0\}, \quad j \in \{0,1\}. \end{aligned} \tag{7.30}$$

It is well known that under Hypothesis 7.6 the realization $\widehat{A}_{G_j}$ is self-adjoint in $H^0(\Omega) := L^2(\Omega)$, i.e. $\widehat{A}_{G_j} = \widehat{A}^*_{G_j}$, $j \in \{0,1\}$.

To apply Proposition 4.3 and Corollary 4.6 we need a boundary triplet for A^*. Note, that the classical Green's formula reeds now as follows

$$\begin{aligned} (\mathcal{A}u, v) - (u, \mathcal{A}v) &= \int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\cdot\overline{v} - u\cdot\overline{\frac{\partial v}{\partial\nu}}\right)ds \\ &= \int_{\partial\Omega}\left(G_1 u\cdot\overline{G_0 v} - G_0 u\cdot\overline{G_1 v}\right)ds, \qquad u, v \in H^2(\Omega). \end{aligned} \tag{7.31}$$

Proposition 7.7 ([34]). *Let the Hypothesis 7.6 be satisfied and let* $0 \in \rho\big(\widehat{A}_{G_0}\big)$*. Then for any* $s \in \mathbb{R}$ *the operator* G_0 *isomorphically maps the set*

$$Z^s_{\mathcal{A}}(\Omega) := \{u \in H^s(\Omega) : A_{\max} u = 0\} \qquad \text{onto} \qquad H^{s-1/2}(\partial\Omega).$$

Definition 7.8 ([34, 68]). *Assume Hypothesis 7.6.*

(i) Let $z \in \rho(\widehat{A}_{G_0})$ *and* $\varphi \in H^{s-1/2}(\partial\Omega)$*,* $s \in \mathbb{R}$*. Then one defines* $P(z)\varphi$ *to be the unique* $u \in Z^s_{\mathcal{A}-zI_{L^2(\Omega)}}(\Omega)$ *satisfying* $G_0 u = \varphi$*.*

(ii) The Poincare-Steklov operator $\Lambda(z)$ *is defined by*

$$\Lambda(z) : H^{s-1/2}(\partial\Omega) \to H^{s-3/2}(\partial\Omega), \quad \Lambda(z)\varphi = G_1 P(z)\varphi. \tag{7.32}$$

To be precise we denote the operator $\Lambda(\cdot)\,:\ H^s(\partial\Omega) \to H^{s-1}(\partial\Omega)$ *by* $\Lambda_s(\cdot)$*. Notice that* $\Lambda_s(z) \in [H^s(\partial\Omega), H^{s-1}(\partial\Omega)]$*.*

Further, let $\Delta_{\partial\Omega}$ be the Laplace-Beltrami operator in $L^2(\partial\Omega)$. Since $-\Delta_{\partial\Omega} \geq 0$, the operator $(-\Delta_{\partial\Omega} + I)^{-s/2}$ isomorphically maps $L^2(\partial\Omega)$ onto $H^s(\partial\Omega)$, $s \in \mathbb{R}$.

Notice that the classical Green formula (7.31) cannot be extended to $u, v \in \operatorname{dom}(A^*)$ since the traces $G_0 u$ and $G_1 u$ belong to the spaces $H^{-1/2}(\partial\Omega)$ and $H^{-3/2}(\partial\Omega)$, respectively (see[49, 35]). A construction of a boundary triplet for A^* as well as the respective regularization of the Green formula (7.31) goes back to the classical papers by Vishik [68] and Grubb [34]. An adaptation of this construction to the case of boundary triplets in the sense of Definition 2.2 was done in [51]. First we recall a result from [51] that modifies and completes [34, Theorem 3.1.2]

Proposition 7.9 ([51, Proposition 5.1]). *Let the Hypothesis 7.6 be satisfied and let* $0 \in \rho\big(\widehat{A}_{G_0}\big)$*. Then the following statements are valid:*

(i) *The totality* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$*, where* $\mathcal{H} := L^2(\partial\Omega)$ *and*

$$\begin{aligned} \Gamma_0 u &:= (-\Delta_{\partial\Omega} + I)^{-1/4} G_0 u, \\ \Gamma_1 u &:= (-\Delta_{\partial\Omega} + I)^{1/4} (G_1 - \Lambda_{-\frac{1}{2}}(0) G_0) u, \end{aligned} \qquad u \in \operatorname{dom}(A_{\max}), \tag{7.33}$$

forms a boundary triplet for A^**. In particular, the Green formula*

$$(A^* u, v)_{L^2(\Omega)} - (u, A^* v)_{L^2(\Omega)} = (\Gamma_1 u, \Gamma_0 v)_{L^2(\partial\Omega)} - (\Gamma_0 u, \Gamma_1 v)_{L^2(\partial\Omega)}, \tag{7.34}$$

$u, v \in \operatorname{dom}(A^*)$*, holds and* $A_0 := A^* \restriction \ker(\Gamma_0) = \widehat{A}_{G_0}$*.*

(ii) *The operator valued function* $\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0)$ *has the regularity property*

$$\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0) \colon\ H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega), \qquad z \in \rho\big(\widehat{A}_{G_0}\big). \tag{7.35}$$

Moreover, $\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0) \in [H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)]$ *for any* $z \in \rho\big(\widehat{A}_{G_0}\big)$*.*

(iii) *The corresponding Weyl function is given by*

$$M(z) = (-\Delta_{\partial\Omega} + I)^{1/4}\big(\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0)\big)(-\Delta_{\partial\Omega} + I)^{1/4}, \quad z \in \mathbb{C}_\pm. \tag{7.36}$$

In contrast to the mapping $\Gamma = \{\Gamma_0, \Gamma_1\} : \mathrm{dom}\,(A_{\max}) \to L^2(\partial\Omega) \times L^2(\partial\Omega)$, the mapping

$$G = \{G_0, G_1\} : \ \mathrm{dom}\,(A_{\max}) \to H^{-1/2}(\partial\Omega) \times H^{-3/2}(\partial\Omega) \tag{7.37}$$

is not surjective. The following statement describes the range $\mathrm{ran}\,(G)$.

Corollary 7.10. *Let the assumptions of Proposition* (7.9) *be satisfied. Then for any pair* $\{h_0, h_1\} \in H^{-1/2}(\partial\Omega) \times H^{-3/2}(\partial\Omega)$ *the system* $G_j f = h_j, j \in \{0,1\}$*, has a solution* $f \in \mathrm{dom}\,(A_{\max})$ *if and only if*

$$h_1 - \Lambda_{-\frac{1}{2}}(0)h_0 \in H^{1/2}(\partial\Omega). \tag{7.38}$$

Proof. If (7.38) is satisfied, then it follows from Proposition (7.9)(i) and the surjectivity of $\Gamma = \{\Gamma_0, \Gamma_1\} : \mathrm{dom}\,(A_{\max}) \to L^2(\partial\Omega) \times L^2(\partial\Omega)$ that the system

$$\begin{cases} \Gamma_0 f = & (-\Delta_{\partial\Omega} + I)^{-1/4} h_0 \\ \Gamma_1 f = & (-\Delta_{\partial\Omega} + I)^{1/4}\big(h_1 - \Lambda_{-\frac{1}{2}}(0)h_0\big) \end{cases}$$

has a (non-unique) solution $f \in \mathrm{dom}\,(A_{\max})$. According to definition (7.33), f also satisfies the system $G_j f = h_j$, $j \in \{0,1\}$.

Conversely, let $f \in \mathrm{dom}\,(A_{\max})$ satisfy $G_j f = h_j$, $j \in \{0,1\}$. Then,

$$h_1 - \Lambda_{-\frac{1}{2}}(0)h_0 = G_1 f - \Lambda_{-\frac{1}{2}}(0)G_0 f = (-\Delta_{\partial\Omega} + I)^{-1/4}\Gamma_1 f \in H^{1/2}(\partial\Omega)$$

which proves (7.38). □

For any operator $K : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ we set

$$\begin{aligned} \widehat{A}_K &:= A_{\max} \upharpoonright \mathrm{dom}\,(\widehat{A}_K), \\ \mathrm{dom}\,(\widehat{A}_K) &:= \{f \in \mathrm{dom}\,(A_{\max}) : G_1 f = K G_0 f\}. \end{aligned} \tag{7.39}$$

Obviously it holds $\mathrm{dom}\,(A) \subseteq \mathrm{dom}\,(\widehat{A}_K) \subseteq \mathrm{dom}\,(A^*) = \mathrm{dom}\,(A_{\max})$ but in general the operator $\widehat{A}_K$ is not closed. That is the reason why $\widehat{A}_K$ is in general not a proper extension of A, cf. Definition 2.1. In the following we denote the set of all not necessarily closed extensions of A with domain between $\mathrm{dom}\,(A)$ and $\mathrm{dom}\,(A^*)$ by $\widetilde{\mathrm{Ext}}_{\mathrm{A}}$. Obviously one has $\mathrm{Ext}\,_A \subset \widetilde{\mathrm{Ext}}_A$.

Lemma 7.11. *Assume the Hypothesis 7.6. Let* $\widetilde{A} \in \widetilde{\mathrm{Ext}}_A$ *and let* G *be the mapping given by* (7.37)*. Then the following three statements are equivalent:*

(i) $\widetilde{A}$ *admits the representation* (7.39)*, i.e.* $\widetilde{A} = \widehat{A}_K$.

(ii) $G\mathrm{dom}\,(\widetilde{A})$ *is a graph of an operator* $K : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$*, i.e.*

$$G\mathrm{dom}\,(\widetilde{A}) = \mathrm{gr}\,(K). \tag{7.40}$$

(iii) $\widetilde{A}$ *and* $A_0 = \widehat{A}_{G_0}$ *are disjoint.*

Combining Proposition 2.3 with Proposition 7.9 this yields the parameterization of realizations $\widetilde{A} = \widehat{A}_K$ by means of boundary operators (see (2.3)). However in contrast to the parameterization (2.3), parameterization (7.39) is not bijective: two different K_1 and K_2 may correspond to the same $\widetilde{A}$, i.e. it might happen that $\widetilde{A} = \widehat{A}_{K_1} = \widehat{A}_{K_2}$. Moreover, an operator $K : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ in (7.39) is not arbitrary, i.e. not each such K can satisfy (7.40). Next we describe those operators K admitting representations (7.40).

Lemma 7.12. *Let Hypothesis 7.6 be satisfied and* $0 \in \rho\big(\widehat{A}_{G_0}\big)$ *let* $K : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$*. There is a not necessarily closed extension* $\widetilde{A} \in \widetilde{\mathrm{Ext}}_A$ *such that* (7.40) *is valid if and only if the regularity condition*

$$\mathrm{ran}\,(K - \Lambda_{-\frac{1}{2}}(0)) \subseteq H^{1/2}(\partial\Omega) \tag{7.41}$$

is satisfied. Moreover, $\widetilde{A}$ *is disjoint with* A_0 *and* $\widetilde{A} = \widehat{A}_K$*.*

Proof. Let $K : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ be a linear operator such that (7.41) is satisfied and let $\{h_0, h_1\} \in \mathrm{gr}\,(K)$. Since $h_1 = Kh_0$ one has $h_1 - \Lambda_{-\frac{1}{2}}(0)h_0 = Kh_0 - \Lambda_{-\frac{1}{2}}(0)h_0 \in H^{1/2}(\partial\Omega)$. By Corollary 7.10, there is $f \in \mathrm{dom}\,(A_{\max})$ such that $h_j = G_j f$, $j \in \{0, 1\}$. Setting $\widetilde{A} := A_{\max} \restriction \mathrm{dom}\,(\widetilde{A})$,

$$\mathrm{dom}\,(\widetilde{A}) := \{f \in \mathrm{dom}\,(A_{\max}) : \; Gf = \{h_0, h_1\} \in \mathrm{gr}\,(K)\},$$

we define a generalized proper extension of A such that $\mathrm{gr}\,(K) = G\mathrm{dom}\,(\widetilde{A})$. By Lemma 7.11 $\widetilde{A}$ is disjoint with A_0 and according to (7.39) $\widetilde{A} = \widehat{A}_K$.

Conversely, assume that $G\mathrm{dom}\,(\widetilde{A}) = \mathrm{gr}\,(K)$ for some $\widetilde{A} \in \mathrm{Ext}_A$. Then by Lemma 7.11(ii) for any $\{h_0, h_1\} \in \mathrm{gr}\,(K)$ there is $f \in \mathrm{dom}\,(\widetilde{A})$, such that $Gf = \{h_0, h_1\}$. By Corollary 7.10 one has $h_1 - \Lambda_{-\frac{1}{2}}(0)h_0 \in H^{1/2}(\partial\Omega)$. Hence $Kh_0 - \Lambda_{-\frac{1}{2}}(0)h_0 \in H^{1/2}(\partial\Omega)$ for any $h_0 \in \mathrm{dom}\,(K)$ and condition (7.41) is satisfied. □

Note that any $K : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ defines realization $\widehat{A}_K$ by (7.39). However if K does not satisfy regularity condition (7.41), one has only inclusion $G\mathrm{dom}\,(\widehat{A}_K) \subseteq \mathrm{gr}\,(K)$ instead of equality (7.40). In this case according to Lemma 7.12 alongside the operator K we consider its restriction $K' : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ given by

$$\begin{aligned} K' &:= K \restriction \mathrm{dom}\,(K'), \\ \mathrm{dom}\,(K') &:= \{h \in \mathrm{dom}\,(K) : Kh - \Lambda_{-\frac{1}{2}}(0)h \in H^{1/2}(\partial\Omega)\} \subseteq \mathrm{dom}\,(K). \end{aligned} \tag{7.42}$$

Clearly, $\mathrm{gr}\,(K') = G\mathrm{dom}\,(\widehat{A}_K)$, i.e. $\widehat{A}_K = \widehat{A}_{K'}$. For instance, consider the zero operator $\mathbb{O} : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$. Then $\widehat{A}_{\mathbb{O}} = \widehat{A}_{G_1}$. However, $\mathbb{O}' := \mathbb{O} \restriction \mathrm{dom}\,(\mathbb{O}')$,

$$\mathrm{dom}\,(\mathbb{O}') := \{f \in H^{-1/2}(\partial\Omega) : -\Lambda_{-\frac{1}{2}}(0)f \in H^{1/2}(\partial\Omega)\} = H^{3/2}(\partial\Omega). \tag{7.43}$$

Hence $\widehat{A}_{G_1} = \widehat{A}_{\mathbb{O}'}$ and $\mathrm{dom}\,(\widehat{A}_{G_1}) = \{f \in H^2(\Omega) : G_1 f = 0\}$.

Next we describe certain properties of realizations $\widehat{A}_K$ by means of boundary operators with respect to the boundary triplet Π given in Proposition 7.9.

Proposition 7.13 ([51, Proposition 3.8]). *Assume the conditions of Proposition 7.9. Let* $K : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ *and let* $\Pi = \{L^2(\partial\Omega), \Gamma_0, \Gamma_1\}$ *be the boundary triplet for* A^* *given by* (7.33). *Then the following holds:*

(i) $\widehat{A}_K = A_{B_K}$, *where* $A_{B_K} := A^* \upharpoonright \ker(\Gamma_1 - B_K\Gamma_1)$ *and*

$$B_K := (-\Delta_{\partial\Omega} + I)^{1/4}(K' - \Lambda_{-\frac{1}{2}}(0))(-\Delta_{\partial\Omega} + I)^{1/4} : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega). \quad (7.44)$$

(ii) *The operator* $\widehat{A}_K$ *is closed (and necessarily disjoint with* $A_0 = \widehat{A}_{G_0}$*) if and only if the operator* $K' - \Lambda_{-\frac{1}{2}}(0) : H^{-1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega)$ *is closed.*

(iii) *Let* $\widehat{A}_K$ *be not closed (but necessary closable). Its closure is disjoint with* A_0 *if and only if the operator* $K' - \Lambda_{-\frac{1}{2}}(0) : H^{-1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega)$ *is closable.*

(iv) *If* $z \in \rho(A_0)$, *then* $z \in \rho(\widehat{A}_K)$ *if and only if the operator* $K' - \Lambda_{-\frac{1}{2}}(z)$ *maps* $\mathrm{dom}\,(K') \subset H^{-1/2}(\partial\Omega)$ *onto* $H^{1/2}(\partial\Omega)$ *and its kernel is trivial.*

Proof. (i) Since $\widehat{A}_K = \widehat{A}_{K'}$, we get from (7.42) that

$$G_1 f - \Lambda_{-\frac{1}{2}}(0) G_0 f = (K' - \Lambda_{-\frac{1}{2}}(0)) G_0 f, \quad f \in \mathrm{dom}\,(\widehat{A}_K).$$

Combining this relation with definition (7.33) it yields

$$\begin{aligned}\Gamma_1 f &= (-\Delta_{\partial\Omega} + I)^{1/4}(K' - \Lambda_{-\frac{1}{2}}(0))(-\Delta_{\partial\Omega} + I)^{1/4}(-\Delta_{\partial\Omega} + I)^{-1/4} G_0 f \\ &= (-\Delta_{\partial\Omega} + I)^{1/4}(K' - \Lambda_{-\frac{1}{2}}(0))(-\Delta_{\partial\Omega} + I)^{1/4}\Gamma_0 f, \quad f \in \mathrm{dom}\,(\widehat{A}_K).\end{aligned}$$

Hence, if $f \in \mathrm{dom}\,(\widehat{A}_K)$, then $f \in \ker(\Gamma_1 - B_K\Gamma_0)$. Therefore $\widehat{A}_K \subseteq A_{B_K}$.

Conversely, if $f \in \mathrm{dom}\,(A_{B_K}) = \ker(\Gamma_1 - B_K\Gamma_0)$, then combining (7.33) and (7.42) yields $G_1 f = K' G_0 f$. Hence $\mathrm{dom}\,(A_{B_K}) \subseteq \mathrm{dom}\,(\widehat{A}_K)$ and $\widehat{A}_K = A_{B_K}$.

(ii) $\widehat{A}_K$ is closed and disjoint with A_0 if and only if B_K is closed. In turn, B_K is closed if and only if $K' - \Lambda_{-\frac{1}{2}}(0) : H^{-1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega)$ is closed.

(iii) The closure of $\widehat{A}_K$ is disjoint with A_0 if and only if B_K is closable. In turn, B_K is closable simultaneously with $K' - \Lambda_{-\frac{1}{2}}(0)$.

(iv) By Proposition 2.5, $z \in \rho(\widehat{A}_K)$ if and only if $0 \in \rho(B_K - M(z))$ where $M(\cdot)$ is the Weyl function given by (7.36). Combining (7.44) and (7.36) yields

$$B_K - M(z) = (-\Delta_{\partial\Omega} + I)^{1/4}(K' - \Lambda_{-\frac{1}{4}}(z))(-\Delta_{\partial\Omega} + I)^{1/4}. \quad (7.45)$$

However, $0 \in \rho(B_K - M(z))$ if and only if the operator $K' - \Lambda_{-\frac{1}{2}}(z) : H^{-1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega)$ has a bounded inverse. □

7.4.2 Perturbation determinants

To state the next result we recall the following definition.

Definition 7.14. *Let* $\mathcal{S}_p(\mathfrak{H}) = \{T \in \mathfrak{S}_\infty(\mathfrak{H}) : s_j(T) = O(j^{-1/p}), \text{ as } j \to \infty\}$, $p > 0$, *where* $s_j(T)$, $j \in \mathbb{N}$, *denote the singular values of* T *(i.e., the eigenvalues of* $(T^*T)^{1/2}$ *decreasingly ordered counting multiplicity).*

It is known that $\mathcal{S}_p(\mathfrak{H})$ is a two-sided (non-closed) ideal in $[\mathfrak{H}]$. Clearly, $\mathcal{S}_{p_1} \subset \mathcal{S}_{p_2}$ if $p_1 > p_2$. An important property of the classes $\mathcal{S}_p(\mathfrak{H})$ needed in the sequel is

$$\mathcal{S}_{p_1} \cdot \mathcal{S}_{p_2} \subset \mathcal{S}_p, \quad \text{where} \quad p^{-1} = p_1^{-1} + p_2^{-1}. \tag{7.46}$$

Theorem 7.15 ([51, Theorem 4.13])**.** *Assume the Hypothesis 7.6. Let* $A_0 := \widehat{A}_{G_0}$ *and* $0 \in \rho\big(A_0\big)$. *Further, let* $K : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ *be an operator satisfying* $\operatorname{dom}(K) \subseteq L^2(\partial\Omega)$ *and* $\operatorname{ran}(K) \subseteq L^2(\partial\Omega)$. *Then*

$$(\widehat{A}_K - z)^{-1} - (A_0 - z)^{-1} \in \mathcal{S}_{\frac{2n-2}{3}}(L^2(\Omega)), \quad z \in \rho(\widehat{A}_K) \cap \rho(A_0). \tag{7.47}$$

For $n = 2$ *the resolvent difference in* (7.47) *is of trace class operator.*

Lemma 7.16. *Let* $\mathfrak{X}$, $\mathfrak{X}_0$, *and* $\mathfrak{Y}$ *be Banach spaces and let* $X : \mathfrak{X} \longrightarrow \mathfrak{Y}$ *be a closed operator with bounded inverse. Assume that* $\mathfrak{X}_0$ *is a dense subset of* $\mathfrak{X}$ *and the embedding* $J : \mathfrak{X}_0 \longrightarrow \mathfrak{X}$ *is continuous. If* $\operatorname{dom}(X) \subseteq J\mathfrak{X}_0$, *then the operator* $X_0 := XJ : \mathfrak{X}_0 \longrightarrow \mathfrak{Y}$, $\operatorname{dom}(X_0) := \{f \in \mathfrak{X}_0 : Jf \in \operatorname{dom}(X)\}$ *is well defined, closed, and has a bounded inverse. Moreover,* $X^{-1} = JX_0^{-1}$.

Proof. The first statement is well known since X is closed and J is bounded. The second statement immediately follows from $\operatorname{ran}(X) = \operatorname{ran}(X_0) = \mathfrak{Y}$. □

In what follows we apply Lemma 7.16 with $\mathfrak{X} := H^{-1/2}(\partial\Omega)$, $\mathfrak{X}_0 := H^0(\partial\Omega)$, $\mathfrak{Y} := H^{1/2}(\partial\Omega)$ and $\mathfrak{Y}' := H^{-3/2}(\partial\Omega)$. Denote by J the embedding operator,

$$J : \; H^0(\partial\Omega) = L^2(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega), \qquad Jf = f. \tag{7.48}$$

Let $K : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$. Since $\operatorname{dom}(K) \subseteq JL^2(\partial\Omega)$, we can set

$$K_0 := KJ : H^0(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega),$$
$$\operatorname{dom}(K_0) := \{f \in H^0(\partial\Omega) : Jf \in \operatorname{dom}(K)\}.$$

Clearly, $\Lambda_0(z) = \Lambda_{-\frac{1}{2}}(z)J, \; \operatorname{dom}(\Lambda_0(z)) := JH^0(\partial\Omega)$, and

$$\begin{aligned} K_0' &:= K_0 \upharpoonright \operatorname{dom}(K_0'), \\ \operatorname{dom}(K_0') &:= \{f \in \operatorname{dom}(K_0) : \; (K_0 - \Lambda_0(0))f \in H^{1/2}(\partial\Omega)\}. \end{aligned} \tag{7.49}$$

Clearly, $K_0' = K'J : H^0(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$.

Now we are in position to state the first main result of this section.

Proposition 7.17. *Let the assumptions of Theorem 7.15 be satisfied and let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be the boundary triplet given by Proposition 7.9 (cf.* (7.33)*). If* $0 \in \rho(A_0) \cap \rho(\widehat{A}_K)$*, then the following holds:*

(i) *For any* $z \in \rho(\widehat{A}_K) \cap \rho(A_0)$ *the operator* $K' - \Lambda_{-\frac{1}{2}}(z) : H^{-1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega)$ *is boundedly invertible and*

$$(K' - \Lambda_{-\frac{1}{2}}(z))^{-1} \in \mathcal{S}_{\frac{2n-2}{3}}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)). \tag{7.50}$$

In particular, if $n = 2$ *then*

$$(K' - \Lambda_{-\frac{1}{2}}(z))^{-1} \in \mathfrak{S}_1(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)). \tag{7.51}$$

(ii) *Let* $n = 2$*. Then the transposed boundary triplet* $\Pi^\top = \{\mathcal{H}, -\Gamma_1, \Gamma_0\}$ *is regular for the pair* $\{\widehat{A}_K, A_0\}$*, and the perturbation determinant* $\Delta^{\Pi^\top}_{\widehat{A}_K/\widehat{A}_0}(\cdot)$ *is*

$$\begin{aligned}\Delta^{\Pi^\top}_{\widehat{A}_K/\widehat{A}_0}(z) &= \det{}_{H^{-\frac{1}{2}}}\Big(I - \big(K' - \Lambda_{-\frac{1}{2}}(0)\big)^{-1}\big(\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0)\big)\Big)\\ &= \det{}_{H^{\frac{1}{2}}}\Big(I - \big(\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0)\big)\big(K' - \Lambda_{-\frac{1}{2}}(0)\big)^{-1}\Big).\end{aligned} \tag{7.52}$$

(iii) *Let* $n = 2$*. Then* $(\Lambda_0(z) - \Lambda_0(0))(K_0' - \Lambda_0(0))^{-1} \in \mathfrak{S}_1(H^{1/2}(\partial\Omega))$ *and the perturbation determinant* $\Delta^{\Pi^\top}_{\widehat{A}_K/A_0}(\cdot)$ *admits the representation*

$$\Delta^{\Pi^\top}_{\widehat{A}_K/A_0}(z) = \det{}_{H^{\frac{1}{2}}}\left(I - (\Lambda_0(z) - \Lambda_0(0))(K_0' - \Lambda_0(0))^{-1}\right). \tag{7.53}$$

Proof. (i) Combining Proposition 2.5(i) with Proposition 7.13(i) this yields $0 \in \rho(B_K - M(z))$ for any $z \in \rho(\widehat{A}_K) \cap \rho(A_0)$ where B_K is given by (7.44). In turn, combining Proposition 2.6 with Theorem 7.15 implies $(B_K - M(z))^{-1} \in \mathcal{S}_{\frac{2n-2}{3}}(H^0(\partial\Omega))$. It follows with account of (7.45) that

$$\begin{aligned}&(I - \Delta_{\partial\Omega})^{-1/4}(K' - \Lambda_{-\frac{1}{2}}(z))^{-1}(I - \Delta_{\partial\Omega})^{-1/4}\\ &\qquad = (B_K - M(z))^{-1} \in \mathcal{S}_{\frac{2n-2}{3}}(H^0(\partial\Omega)).\end{aligned}$$

Since $(I - \Delta_{\partial\Omega})^{-1/4}$ isomorphically maps $H^s(\partial\Omega)$ onto $H^{s+1/2}(\partial\Omega)$ for $s \in \mathbb{R}$ we arrive at (7.50). Further, for $n = 2$ inclusion (7.50) implies

$$(K' - \Lambda_{-\frac{1}{2}}(z))^{-1} \in \mathcal{S}_{\frac{2}{3}}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)) \subset \mathfrak{S}_1(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)).$$

This proves the last statement.

(ii) Since $n = 2$ one has $(\widehat{A}_K - z)^{-1} - (A_0 - z)^{-1} \in \mathfrak{S}_1(L^2(\Omega))$ by Theorem 7.15. Further, by Proposition 7.13(i), the condition $0 \in \rho(A_0) \cap \rho(\widehat{A}_K)$ is equivalent to $0 \in \rho(B_K - M(0)) = \rho(B_K)$. By Definition 3.2 one easily checks that the boundary triplet $\Pi^\top = \{\mathcal{H}, -\Gamma_1, \Gamma_0\}$ is regular for the pair $\{\widehat{A}_K, A_0\}$, hence $\{\widehat{A}_K, A_0\} \in \mathfrak{D}^{\Pi^\top}$, and the perturbation determinant $\Delta^{\Pi^\top}_{\widehat{A}_K/A_0}(\cdot)$ is given by (1.8). Inserting in this expression formulas (7.36) and (7.44) we get

$$\Delta^{\Pi^\top}_{\widehat{A}_K/A_0}(z) = \det_{L^2(\partial\Omega)}(I - B_K^{-1}M(z)) =$$
$$\det_{H^0}\Big(I - (I-\Delta_{\partial\Omega})^{-1/4}(K' - \Lambda_{-\frac{1}{2}}(0))^{-1}(\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0))(I-\Delta_{\partial\Omega})^{1/4}\Big)$$

for $z \in \rho(\widehat{A}_K) \cap \rho(A_0)$. Further, according to Proposition 7.9(ii), $\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0) \in [H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)]$. Combining this inclusion with (7.51) we get $T_2(z) \in \mathfrak{S}_1(H^0(\partial\Omega), H^{-1/2}(\partial\Omega))$ where

$$T_2(z) := (K' - \Lambda_{-\frac{1}{2}}(0))^{-1}(\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0))(I-\Delta_{\partial\Omega})^{1/4}, \quad z \in \rho(\widehat{A}_K) \cap \rho(A_0).$$

Noting that $T_1 = (I-\Delta_{\partial\Omega})^{-1/4}$ isomorphically maps $H^{-1/2}(\partial\Omega)$ onto $H^0(\partial\Omega)$ we see that $T_2(z)T_1$ is well defined and $T_2(z)T_1 \in \mathfrak{S}_1(H^{-1/2}(\partial\Omega))$. Moreover, due to the inclusion (7.51), $T_1T_2(z) \in \mathfrak{S}_1(H^0(\partial\Omega))$. Taking both last inclusions into account and applying property (2.11) we arrive at the equality

$$\Delta^{\Pi^\top}_{\widehat{A}_K/A_0}(z) = \det_{L^2(\partial\Omega)}\big(I - T_1T_2(z)\big) = \det_{H^{-1/2}(\partial\Omega)}\big(I - T_2(z)T_1\big)$$

coinciding with the first identity in (7.52). The second identity in (7.52) is implied by combining the first one with the property (2.11). Note that the applicability of (2.11) is possible due to inclusion (7.51) and Proposition 7.9(ii).

(iii) By Lemma 7.16, the operator $(K_0' - \Lambda_0(0))^{-1} : H^{1/2}(\partial\Omega) \longrightarrow L^2(\partial\Omega)$ is bounded and

$$(K' - \Lambda_{-\frac{1}{2}}(0))^{-1} = J(K_0' - \Lambda_0(0))^{-1}. \qquad z \in \rho(\widehat{A}_K) \cap \rho(A_0). \tag{7.54}$$

Combining this formula with (7.51) we get $J(K_0' - \Lambda_0(0))^{-1} \in \mathfrak{S}_1(H^{1/2}(\partial\Omega))$. Therefore inserting (7.54) into the second formula in (7.52) we get

$$\Delta^{\Pi^\top}_{\widehat{A}_K/A_0}(z) = \det_{H^{1/2}}\Big(I - \Big(\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0)\Big)\, J\,(K_0' - \Lambda_0(0))^{-1}\Big).$$

To arrive at (7.53) it remains to note that $\Lambda_{-\frac{1}{2}}(z)J = \Lambda_0(z)$. □

Combining the chain rule (5.1) with Proposition 7.17 one arrives at the following statement.

Corollary 7.18. *Assume the conditions of Proposition 7.17. Further, let* $K_j : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ *satisfy* $\operatorname{dom}(K_j) \subseteq L^2(\partial\Omega)$ *and* $\operatorname{ran}(K_j) \subseteq L^2(\partial\Omega)$, $j \in \{1,2\}$. *Assume also that* $0 \in \rho(A_0) \cap \rho(\widehat{A}_{K_1}) \cap \rho(\widehat{A}_{K_2})$. *Then the boundary triplet* $\Pi^\top = \{\mathcal{H}, -\Gamma_1, \Gamma_0\}$ *for* $A_{\max}$ *is regular for the family* $\{\widehat{A}_{K_1}, \widehat{A}_{K_2}, A_0\}$, *and the perturbation determinant* $\Delta^{\Pi^\top}_{\widehat{A}_2/\widehat{A}_1}(\cdot)$ *where* $\widehat{A}_j := \widehat{A}_{K_j}$ *is*

$$\Delta^{\Pi^\top}_{\widehat{A}_2/\widehat{A}_1}(z) = \frac{\det_{H^{-\frac{1}{2}}}\Big(I - \big(K_2' - \Lambda_{-\frac{1}{2}}(0)\big)^{-1}\big(\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0)\big)\Big)}{\det_{H^{-\frac{1}{2}}}\Big(I - \big(K_1' - \Lambda_{-\frac{1}{2}}(0)\big)^{-1}\big(\Lambda_{-\frac{1}{2}}(z) - \Lambda_{-\frac{1}{2}}(0)\big)\Big)}, \tag{7.55}$$

$z \in \rho(\widehat{A}_{K_1}) \cap \rho(\widehat{A}_{K_2}) \cap \rho(A_0)$.

Our next goal is to show that under additional restrictions on K the perturbation determinant $\Delta^{\Pi^\top}_{\widehat{A}_K/A_0}(\cdot)$ can be computed in $L^2(\partial\Omega)$. To this end we introduce the operator-valued function $\Lambda_{0,0}(\cdot) : H^0(\partial\Omega) \longrightarrow H^0(\partial\Omega)$ by setting

$$\begin{aligned} \Lambda_{0,0}(z) &:= \Lambda_0(z) \upharpoonright \mathrm{dom}\,(\Lambda_{0,0}(z)), \\ \mathrm{dom}\,(\Lambda_{0,0}(z)) &:= \{f \in \mathrm{dom}\,(\Lambda_0(z)) : \Lambda_0(z)f \in H^0(\partial\Omega)\}. \end{aligned} \tag{7.56}$$

Lemma 7.19. *Let* $0 \in \rho(\widehat{A}_{G_0})$. *Then*

$$\mathrm{dom}\,(\Lambda_{0,0}(z)) = H^1(\partial\Omega), \qquad z \in \rho(\widehat{A}_{G_0}), \tag{7.57}$$

and, for any $z \in \rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1})$ *the operator* $(\Lambda_{0,0}(z))^{-1}$ *exits and satisfies* $(\Lambda_{0,0}(z))^{-1} \in \mathcal{S}_1(H^0(\partial\Omega))$. *Moreover, if* $0 \in \rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1})$, *then the operator* $\Lambda_{0,0}(0)$ *is self-adjoint, has discrete spectrum, and* $(\Lambda_{0,0}(0))^{-1} \in \mathcal{S}_1(H^0(\partial\Omega))$.

Proof. It follows from Definition 7.8 that $\mathrm{dom}\,(\Lambda_{0,0}(\cdot)) \supseteq H^1(\partial\Omega)$. Let us prove the equality (7.57). Since both realizations $\widehat{A}_{G_0}$ and $\widehat{A}_{G_1}$ are self-adjoint, $\rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1}) \supset \mathbb{C}_\pm$. Let $z \in \rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1})$. Then $\mathrm{dom}\,(\Lambda_1(z)) = H^1(\partial\Omega)$ and Λ_1 isomorphically maps $H^1(\partial\Omega)$ onto $H^0(\partial\Omega)$ (see [34, Theorem 5.2]). Since $\Lambda_{0,0}(z)h = \Lambda_1(z)h$ for $h \in H^1(\partial\Omega)$ we conclude that $\mathrm{dom}\,(\Lambda_{0,0}(z)) = H^1(\partial\Omega)$ and $\mathrm{ran}\,(\Lambda_{0,0}(z)) = H^0(\partial\Omega)$.

Next let $x_0 = \bar{x}_0 \in \rho(\widehat{A}_{G_0}) \setminus \rho(\widehat{A}_{G_1})$. We can assume without loss of generality that $x_0 = 0$. Otherwise we replace the expression $\mathcal{A}$ by $\mathcal{A} - x_0 I$. Then, by Proposition 7.9(ii), the difference $T(z) := \Lambda_{-\frac{1}{2}}(\cdot) - \Lambda_{-\frac{1}{2}}(0)\colon\ H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is bounded. Hence the operator $\Lambda_{0,0}(\cdot) - \Lambda_{0,0}(0)$ being a closable restriction of $T(\cdot)$ on $H^0(\partial\Omega)$ is bounded on $H^0(\partial\Omega)$. Hence $\mathrm{dom}\,(\Lambda_{0,0}(0)) = \mathrm{dom}\,(\Lambda_{0,0}(z)) = H^1(\partial\Omega)$ for $z \in \rho(\widehat{A}_{G_0})$. Further, since $\mathrm{ran}\,(\Lambda_{0,0}(z))^{-1}) = H^1(\partial\Omega)$ for $z \in \rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1})$, we have $(\Lambda_{0,0}(z))^{-1} \in \mathcal{S}_1(H^0(\partial\Omega))$.

Clearly, for any $x_0 = \bar{x}_0 \in \rho(\widehat{A}_{G_0})$ the operator $\Lambda_{0,0}(x_0)$ is symmetric. If, in addition, $x_0 \in \rho(\widehat{A}_{G_1})$, then the operator $\Lambda_{0,0}(x_0)$ is self-adjoint since $\mathrm{ran}\,(\Lambda_{0,0}(x_0)) = H^0(\partial\Omega)$. If $0 \in \rho(\widehat{A}_{G_0}) \setminus \rho(\widehat{A}_{G_1})$, then the self-adjointness of $\Lambda_{0,0}(0)$ is implied by the self-adjointness of $\Lambda_{0,0}(x_0)$ with $x_0 = \bar{x}_0 \in \rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1})$ and the boundedness of $\Lambda_{0,0}(x_0) - \Lambda_{0,0}(0)$ on $H^0(\partial\Omega)$.

Further, since the boundary $\partial\Omega$ is compact, the spectrum of $\Lambda_{0,0}(0)$ is discrete. Moreover, since by (7.57), $\mathrm{ran}\,(\Lambda_{0,0}(z))^{-1}) = H^1(\partial\Omega))$ for $\rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1})$, we have $(\Lambda_{0,0}(z))^{-1} \in \mathcal{S}_1(H^0(\partial\Omega))$. □

Now we prove the main result of the section. Namely, we show that under additional assumptions on K determinant (7.53) can be computed in $L^2(\partial\Omega)$.

Theorem 7.20. *Assume the Hypothesis 7.6. Let* $K : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ *be an operator satisfying* $\mathrm{dom}\,(K) \subseteq L^2(\partial\Omega)$ *and* $\mathrm{ran}\,(K) \subseteq L^2(\partial\Omega)$. *Let also that* $0 \in \rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1}) \cap \rho(\widehat{A}_K)$ *and let*

$$\widehat{K}_0 := KJ : H^0(\partial\Omega) \longrightarrow H^0(\partial\Omega), \quad \mathrm{dom}\,(K) = J\mathrm{dom}\,(\widehat{K}_0), \tag{7.58}$$

where J is the embedding operator given by (7.48). *If $\widehat{K}_0$ is relatively compact with respect to $\Lambda_{0,0}(0)$, then*

$$\big(\Lambda_{0,0}(z) - \Lambda_{0,0}(0)\big)\big(\widehat{K}_0 - \Lambda_{0,0}(0)\big)^{-1} \in \mathcal{S}_{\frac{1}{2}}(H^0(\partial\Omega)) \subset \mathfrak{S}_1(H^0(\partial\Omega)), \tag{7.59}$$

$z \in \rho(\widehat{A}_K) \cap \rho(\widehat{A}_{G_0})$, and the perturbation determinant $\Delta^{\Pi^\top}_{\widehat{A}_K/A_0}(\cdot)$ given by (7.52) *admits the representation*

$$\Delta^{\Pi^\top}_{\widehat{A}_K/A_0}(z) = \det{}_{L^2(\partial\Omega)}\left(I - \big(\Lambda_{0,0}(z) - \Lambda_{0,0}(0)\big)\big(\widehat{K}_0 - \Lambda_{0,0}(0)\big)^{-1}\right), \tag{7.60}$$

for $z \in \rho(\widehat{A}_K) \cap \rho(\widehat{A}_{G_0})$. In particular, representation (7.60) *holds whenever $\widehat{K}_0$ is bounded, i.e. $\widehat{K}_0 \in [H^0(\partial\Omega)]$.*

Proof. We prove the theorem in two steps.

1. Let us prove the inclusion (7.59). According to (7.35), $\Lambda_0(z) - \Lambda_0(0) : H^0(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega)$. Hence

$$\overline{\Lambda_0(z) - \Lambda_0(0)} \in \mathcal{S}_2(H^0(\partial\Omega)), \qquad z \in \rho(A_0). \tag{7.61}$$

Further, by (7.48), J^* continuously embeds $H^{1/2}(\partial\Omega)$ into $H^0(\partial\Omega)$. Therefore

$$J^*(\Lambda_0(z) - \Lambda_0(0))h = (\Lambda_{0,0}(z) - \Lambda_{0,0}(0))h, \quad h \in H^1(\partial\Omega). \tag{7.62}$$

Combining relations (7.62) and (7.61), using the inclusion $J^* \in \mathcal{S}_2(H^{1/2}, H^0)$ and taking into account property (7.46) of the ideals $\mathcal{S}_p$ we obtain

$$\overline{\Lambda_{0,0}(z) - \Lambda_{0,0}(0)} \in \mathcal{S}_1(H^0(\partial\Omega)), \qquad z \in \rho(A_0). \tag{7.63}$$

By the assumption, $\mathrm{dom}\,(K_0) \subseteq H^0(\partial\Omega)$. Hence and from definitions (7.42), (7.49), one gets $\mathrm{dom}\,(K_0') \subset H^0(\partial\Omega)$. Moreover, since $\mathrm{ran}\,(K) \subseteq H^0(\partial\Omega)$, by the assumption, and, by (7.42), $K_0 h - \Lambda_0(0)h \in H^{1/2}(\partial\Omega)$ for $h \in \mathrm{dom}\,(K_0')$, one has $\Lambda_0(0)h \in H^0(\partial\Omega)$ for $h \in \mathrm{dom}\,(K_0')$. Therefore, by (7.56), $\Lambda_0(0)h = \Lambda_{0,0}(0)h$ for $h \in \mathrm{dom}\,(K_0')$ and $\mathrm{dom}\,(K_0') \subset \mathrm{dom}\,(\Lambda_{0,0}(0))$. Combining this inclusion with Lemma 7.19, yields $\mathrm{dom}\,(K_0') \subset H^1(\partial\Omega)$, hence (7.49) takes the form

$$\mathrm{dom}\,(K_0') := \{h \in \mathrm{dom}\,(K_0) \cap H^1(\partial\Omega) : K_0 h - \Lambda_{0,0}(0)h \in H^{1/2}(\partial\Omega)\}.$$

Besides, since J^* (continuously) embeds $H^{1/2}(\partial\Omega)$ into $H^0(\partial\Omega)$, we get

$$J^*(K_0' - \Lambda_0(0))h = (\widehat{K}_0 - \Lambda_{0,0}(0))h, \qquad h \in \mathrm{dom}\,(K_0'). \tag{7.64}$$

Clearly, $\widehat{A}_K = \widehat{A}_{K_0}$. Therefore, by Proposition 7.13(iv) the condition $0 \in \rho(\widehat{A}_{K_0})$ with account of $\mathrm{dom}\,(K_0') \subset H^0(\partial\Omega)$, yields $\mathrm{ran}\,(K_0' - \Lambda_0(0)) = \mathrm{ran}\,(K' - \Lambda_{-\frac{1}{2}}(0)) = H^{1/2}(\partial\Omega)$. Combining this relation with (7.64) shows that $\mathrm{ran}\,(\widehat{K}_0 - \Lambda_{0,0}(0))(\supset H^{1/2}(\partial\Omega))$ is dense in $H^0(\partial\Omega)$. Let us show that $0 \in \rho(\widehat{K}_0 - \Lambda_{0,0}(0))$.

First, by Lemma 7.19, the operator $\Lambda_{0,0}(0)$ is self-adjoint and its spectrum is discrete. In particular, $\Lambda_{0,0}(0)$ is a Fredholm operator with zero index. Next, since $\widehat{K}_0$ is $\Lambda_{0,0}(0)$-compact, $\mathrm{dom}\,(\widehat{K}_0 - \Lambda_{0,0}(0)) = \mathrm{dom}\,(\Lambda_{0,0}(0)) = H^1(\partial\Omega)$ and $\widehat{K}_0 - \Lambda_{0,0}(0)$ is also Fredholm operator with zero index [36, Theorem 4.5.26] (in fact, it has discrete spectrum too). Therefore the range $\mathrm{ran}\,(\widehat{K}_0 - \Lambda_{0,0}(0))$ is closed and being dense in $H^0(\partial\Omega)$, coincides with $H^0(\partial\Omega)$. Since $\mathrm{ind}(\widehat{K}_0 - \Lambda_{0,0}(0)) = 0$, the latter is equivalent to $0 \in \rho(\widehat{K}_0 - \Lambda_{0,0}(0))$.

Further, due to the assumption $0 \in \rho(\widehat{A}_{G_1})$, $\Lambda_{0,0}(0)$ is invertible and $(\Lambda_{0,0}(0))^{-1} \in \mathcal{S}_1(H^0(\partial\Omega))$. Therefore the operator $\left(I - \widehat{K}_0(\Lambda_{0,0}(0))^{-1}\right)^{-1}$ exists, is bounded on $H^0(\partial\Omega)$, and

$$\left(\widehat{K}_0 - \Lambda_{0,0}(0)\right)^{-1} = -\,(\Lambda_{0,0}(0))^{-1}\left(I - \widehat{K}_0(\Lambda_{0,0}(0))^{-1}\right)^{-1} \in \mathcal{S}_1(H^0(\partial\Omega)).$$

Combining this relation with (7.63) and applying (7.46) we arrive at (7.59).

2. In this step we prove (7.60). Since $\mathrm{ran}\,(K) \subseteq L^2(\partial\Omega)$, the operator $K_0 = KJ : H^0(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ satisfies $\mathrm{ran}\,(K_0) \subseteq L^2(\partial\Omega)$. However, we distinguish between K_0 and $\widehat{K}_0$ defined by (7.58). Note that, by Proposition 7.13(iv), the condition $0 \in \rho(\widehat{A}_{K_0}) = \rho(\widehat{A}_K)$ also implies $\ker(K_0' - \Lambda_0(0)) = \ker(K' - \Lambda_{-\frac{1}{2}}(0)) = \{0\}$, i.e. the inverse $(K_0' - \Lambda_0(0))^{-1} : H^{1/2}(\partial\Omega) \longrightarrow H^0(\partial\Omega)$ exists. Combining Lemma 7.16 with (7.64) yields

$$(\widehat{K}_0 - \Lambda_{0,0}(0))^{-1}J^* = (K_0' - \Lambda_0(0))^{-1}. \tag{7.65}$$

Inserting (7.65) into (7.53) we obtain

$$\Delta^{\Pi^\top}_{\widehat{A}_K/A_0}(z) = \det{}_{H^{\frac{1}{2}}(\partial\Omega)}\left(I - (\Lambda_0(z) - \Lambda_0(0))(\widehat{K}_0 - \Lambda_{0,0}(0))^{-1}J^*\right).$$

Using (7.59) and applying the cyclicity property (see (2.11)) we get

$$\Delta^{\Pi^\top}_{\widehat{A}_K/A_0}(z) = \det{}_{H^0(\partial\Omega)}\left(I - J^*(\Lambda_0(z) - \Lambda_0(0))(\widehat{K}_0 - \Lambda_{0,0}(0))^{-1}\right),$$

$z \in \rho(\widehat{A}_K) \cap \rho(A_0)$. Combining this identity with (7.62) we arrive at (7.60). □

Remark 7.21. Note that though the assumption $0 \in \rho(\widehat{A}_K)$ implies $\ker(K_0' - \Lambda_0(0)) = \{0\}$, we cannot conclude from (7.64) that $\ker(\widehat{K}_0 - \Lambda_{0,0}(0)) = \{0\}$ Indeed, the inclusion $J^*(K_0' - \Lambda_0(0)) \subset \widehat{K}_0 - \Lambda_{0,0}(0)$ is always strict since the range $\mathrm{ran}\,(J^*) = H^{1/2}(\partial\Omega)$ is only dense in $H^0(\partial\Omega)$ in opposite to $\mathrm{ran}\,(\widehat{K}_0 - \Lambda_{0,0}(0)) = H^0(\partial\Omega)$.

Corollary 7.22. *Assume the Hypothesis 7.6. Let $K_j : H^{-1/2}(\partial\Omega) \longrightarrow H^{-3/2}(\partial\Omega)$ be an operator satisfying $\mathrm{dom}\,(K_j) \subseteq L^2(\partial\Omega)$ and $\mathrm{ran}\,(K_j) \subseteq L^2(\partial\Omega)$, $j \in \{1,2\}$. Further, let $0 \in \rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1}) \cap \rho(\widehat{A}_{K_j})$, and*

$$\widehat{K}_{j,0} := K_jJ : H^0(\partial\Omega) \longrightarrow H^0(\partial\Omega), \quad \mathrm{dom}\,(K_j) = J\mathrm{dom}\,(\widehat{K}_{j,0}), \quad j \in \{1,2\}.$$

If the operator $\widehat{K}_{j,0}$ is relatively compact with respect to $\Lambda_{0,0}(0)$, then the perturbation determinant $\Delta^{\Pi^\top}_{\widehat{A}_2/\widehat{A}_1}(\cdot)$ given by (7.55) admits the representation

$$\Delta^{\Pi^\top}_{\widehat{A}_{K_2}/\widehat{A}_{K_1}}(z) = \frac{\det_{L^2}\left(I - \left(\Lambda_{0,0}(z) - \Lambda_{0,0}(0)\right)(\widehat{K}_{2,0} - \Lambda_{0,0}(0))^{-1}\right)}{\det_{L^2}\left(I - \left(\Lambda_{0,0}(z) - \Lambda_{0,0}(0)\right)(\widehat{K}_{1,0} - \Lambda_{0,0}(0))^{-1}\right)},$$

for $z \in \rho(\widehat{A}_{K_2}) \cap \rho(\widehat{A}_{K_1}) \cap \rho(\widehat{A}_{G_0})$.

Proof. The proof follows from Corollary 7.18 and Proposition 7.20. □

Consider Robin-type realizations

$$\begin{aligned}\widehat{A}_\sigma &:= A_{\max} \upharpoonright \mathrm{dom}\,(\widehat{A}_\sigma),\\ \mathrm{dom}\,(\widehat{A}_\sigma) &:= \{f \in H^2(\Omega) : G_1 f = \sigma G_0 f\}.\end{aligned} \tag{7.66}$$

It follows from the classical a priory estimate (see [2, Theorem 15.2]) that the realization $\widehat{A}_\sigma$ is closed whenever $\sigma \in C^2(\partial\Omega)$. Moreover, in this case $\rho(\widehat{A}_\sigma) \neq \emptyset$ and $\widehat{A}_\sigma$ is self-adjoint whenever σ is real.

Let $\widehat{\sigma}$ denote the multiplication operator induced by σ in $L^2(\partial\Omega)$.

Corollary 7.23. *Assume the conditions of Theorem 7.20. Let* $\sigma \in C^2(\partial\Omega)$ $0 \in \rho(\widehat{A}_\sigma) \cap \rho(\widehat{A}_{G_0}) \cap \rho(\widehat{A}_{G_1})$, *and let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be the boundary triplet, given in Theorem 7.9. Then the transposed boundary triplet* $\Pi^\top = \{\mathcal{H}, -\Gamma_1, \Gamma_0\}$ *is regular for the pair* $\{\widehat{A}_\sigma, A_0\}$ *and for* $z \in \rho(\widehat{A}_\sigma) \cap \rho(A_0)$ *the perturbation determinant* $\Delta^{\Pi^\top}_{\widehat{A}_\sigma/A_0}(\cdot)$ *is*

$$\Delta^{\Pi^\top}_{\widehat{A}_\sigma/A_0}(z) = \det{}_{L^2(\partial\Omega)}\left(I - (\Lambda_{0,0}(z) - \Lambda_{0,0}(0))(\widehat{\sigma} - \Lambda_{0,0}(0))^{-1}\right).$$

Proof. Setting $K = \widehat{\sigma}$ and noting that $\sigma \in C^2(\Omega)$ we easily get from (7.66)

$$\mathrm{dom}\left(K - \Lambda_{-1/2}(0)\right) = \mathrm{dom}\,(K) = \mathrm{dom}\,(\widehat{\sigma}) \subset \mathrm{ran}\,(G_0) = H^{3/2}(\partial\Omega).$$

Since $\Lambda_{3/2}(0)$ is a restriction of $\Lambda_{-1/2}(0)$, then according to (7.32) $\mathrm{ran}\left(\Lambda_{-1/2}(0) \upharpoonright H^{3/2}(\partial\Omega)\right) \subset H^{1/2}(\partial\Omega)$. Further, the assumption $\sigma \in C^2(\partial\Omega)$ yields $\mathrm{ran}\left(K \upharpoonright H^{3/2}(\partial\Omega)\right) \subset H^{3/2}(\partial\Omega)$. Combining these inclusions we arrive at the regularity property $\mathrm{ran}\left(K - \Lambda_{-1/2}(0)\right) \subset H^{1/2}(\partial\Omega)$ (see (7.41)).

Hence $K' = K$ (see definition (7.42)) and $\mathrm{dom}\,(\widehat{A}_K) = \mathrm{dom}\,(\widehat{A}_\sigma)$. Since $\mathrm{dom}\,(K),\ \mathrm{ran}\,(K) \subset H^{3/2}(\partial\Omega)$, then, by (7.58), $\widehat{K}_0 = K = \widehat{\sigma}$. Finally, since $\widehat{K}_0 = \widehat{\sigma} \in [H^0(\partial\Omega)]$, one completes the proof by applying Proposition 7.20. □

Appendix

Following [69, Section 8.1] and [11, 31]. we summarize some basic properties of the perturbation determinants $\Delta_{H'/H}(\cdot)$ and $\widetilde{\Delta}_{H',H}(\xi, z)$.

A point $z_0 \in \sigma_p(T)$ is called a normal eigenvalue of T if it is isolated and its algebraic multiplicity $m_{z_0}(T)$ is finite. An isolated eigenvalue $z_0 \in \sigma_p(T)$ is called normal if the Riesz projection $P_{z_0} := -\frac{1}{2\pi i}\int_{|z-\lambda_0|=\delta} R_T(z)dz$ is finite dimensional. In this case $m_{z_0} = \dim\,(P_{z_0})$. We also set $m_{z_0}(T) := 0$ if $z_0 \in \rho(T)$.

Further, if the function $f(\cdot)$ is analytic in a punctured neighborhood of $z_0 \in \mathbb{C}$ and z_0 is not an essential singularity of it, then the order $\mathrm{ord}\,(f(z_0))$ of $f(\cdot)$ at $z_0 \in \mathbb{C}$ is the integer $k \in \mathbb{Z}$ in the representation $f(z) = (z - z_0)^k g(z)$ where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$, [31, Chapter IV.3].

Proposition A.1.

(i) *If* $\{H'', H'\} \in \widetilde{\mathfrak{D}}$ *and* $\{H', H\} \in \widetilde{\mathfrak{D}}$*, then*

$$\widetilde{\Delta}_{H''/H'}(\xi, z)\widetilde{\Delta}_{H'/H}(\xi, z) = \widetilde{\Delta}_{H''/H}(\xi, z), \qquad z \in \rho(H') \cap \rho(H).$$

In particular, if $\{H'', H'\} \in \mathfrak{D}$ *and* $\{H', H\} \in \mathfrak{D}$

$$\Delta_{H''/H'}(z)\Delta_{H'/H}(z) = \Delta_{H''/H}(z), \qquad z \in \rho(H') \cap \rho(H).$$

(ii) *If* $\{H', H\} \in \widetilde{\mathfrak{D}}$*, then* $\{H, H'\} \in \widetilde{\mathfrak{D}}$ *and*

$$\widetilde{\Delta}_{H'/H}(\xi, z)\widetilde{\Delta}_{H/H'}(\xi, z) = 1, \quad z \in \rho(H') \cap \rho(H). \tag{a.1}$$

In particular, if $\{H', H\} \in \mathfrak{D}$*, then* $\{H, H'\} \in \mathfrak{D}$ *and*

$$\Delta_{H'/H}(z)\Delta_{H/H'}(z) = 1, \quad z \in \rho(H') \cap \rho(H).$$

(iii) *Let* $\{H', H\} \in \widetilde{\mathfrak{D}}$*. If* z *is either a regular or a normal eigenvalue for* H' *and* H *with algebraic multiplicities* $m_z(H')$ *and* $m_z(H)$*, respectively, then* $\mathrm{ord}\,(\widetilde{\Delta}_{H'/H}(\xi, z)) = m_z(H') - m_z(H)$*. If in addition* $\{H', H\} \in \mathfrak{D}$*, then* $\mathrm{ord}\,(\Delta_{H'/H}(z)) = m_z(H') - m_z(H)$*.*

(iv) *If* $\{H', H\} \in \widetilde{\mathfrak{D}}$*, then*

$$\frac{1}{\widetilde{\Delta}_{H'/H}(\xi, z)}\frac{d}{dz}\widetilde{\Delta}_{H'/H}(\xi, z) = \mathrm{tr}\,((H - z)^{-1} - (H' - z)^{-1}), \tag{a.2}$$

$\xi, z \in \rho(H') \cap \rho(H)$*. In particular, if* $\{H', H\} \in \mathfrak{D}$*, then*

$$\frac{1}{\Delta_{H'/H}(z)}\frac{d}{dz}\Delta_{H'/H}(z) = \mathrm{tr}\,((H - z)^{-1} - (H' - z)^{-1}),$$

$\xi, z \in \rho(H') \cap \rho(H)$*.*

(v) *If* $\{H', H\} \in \widetilde{\mathfrak{D}}$*, then* $\{H'^*, H^*\} \in \widetilde{\mathfrak{D}}$ *and* $\widetilde{\Delta}_{H'^*/H^*}(\overline{\xi}, z) = \overline{\widetilde{\Delta}_{H'/H}(\xi, \overline{z})}$ *for* $z \in \rho(H^*)$*. In particular, if* $\{H', H\} \in \mathfrak{D}$ *and* $\{H'^*, H^*\} \in \mathfrak{D}$*, then* $\Delta_{H'^*/H^*}(z) = \overline{\Delta_{H'/H}(\overline{z})}$ *for* $z \in \rho(H^*)$*.*

(vi) *If* $\{H', H\} \in \widetilde{\mathfrak{D}}$*, then the following identity holds*

$$\frac{\widetilde{\Delta}_{H'/H}(\xi, z)}{\widetilde{\Delta}_{H'/H}(\xi, \zeta)} = \widetilde{\Delta}_{H'/H}(\zeta, z), \quad z \in \rho(H), \quad \zeta \in \rho(H') \cap \rho(H).$$

In particular, if $\{H', H\} \in \mathfrak{D}$*, then*

$$\frac{\Delta_{H'/H}(z)}{\Delta_{H'/H}(\zeta)} = \det\left(I + (z-\zeta)(H'-\zeta)^{-1}V(H-z)^{-1}\right),$$

for $z \in \rho(H)$ *and* $\zeta \in \rho(H') \cap \rho(H)$.

The proof of second part of point (v) is not obvious and is based on the following lemma.

Lemma A.2. *Let* T *be a densely defined closed operator such that* $0 \in \rho(T)$*. Further, let* C *be a linear operator such that* $\operatorname{dom}(C) \supseteq \operatorname{ran}(T)$*. If* $\overline{T^{-1}C} \in \mathfrak{S}_1(\mathfrak{H})$ *and* $CT^{-1} \in \mathfrak{S}_1(\mathfrak{H})$*, then*

$$\det(I + \overline{T^{-1}C}) = \det(I + CT^{-1}). \tag{a.3}$$

Notice that Lemma A.2 generalizes the following known property of determinants: $\overline{\det(I+T)} = \det(I+T^*)$.

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