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Equidistribution of Elements of Norm 1 in Cyclic Extensions

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Abstract

Upon quotienting by units, the elements of norm 1 in a number field K form a countable subset of a torus of dimension $r_1 + r_2 - 1$ where r_1 and r_2 are the numbers of real and pairs of complex embeddings. When K is Galois with cyclic Galois group we demonstrate that this countable set is equidistributed in this torus with respect to a natural partial ordering.

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1 Introduction

Let K be a number field of degree d over $\mathbb{Q}$. We will eventually restrict ourselves to Galois extensions with cyclic Galois group, but for the moment we maintian generality. Let $\mathcal{N}$ be the subset of elements in K with norm ($N = N_{K/\mathbb{Q}}$) equal to 1. We also define $\mathfrak{o}, U$ and W to be, respectively, the ring of integers, the group of units and the group of roots of unity of K.

The set of archimedean places is denoted S_∞, and has cardinality $r_1 + r_2$ where r_1 is the number of real places and r_2 is the number of complex places. For each place $v \in S_\infty$ we denote by $\|\cdot\|_v$ either the usual absolute value if v is real, or the usual absolute value squared if v is complex. We define the regulator map $\log : K^\times \to \mathbb{R}^{r_1+r_2}$ by

$$\alpha \mapsto (\log \|\alpha\|_v)_{v \in S_\infty} .$$

We also define $\Sigma : \mathbb{R}^{r_1+r_2} \to \mathbb{R}$ by $\mathbf{x} \mapsto x_1 + x_2 + \cdots + x_{r_1+r_2}$. Then, by the proof of Dirchlet's Unit Theorem, $\log U$ is a lattice in $\ker \Sigma$ and the kernel of $\log$ restricted to U is W. This is most usually stated as

$$U \cong W \times \mathbb{Z}^{r_1+r_2-1}.$$

Since $\log U$ is a lattice in $\ker \Sigma$,

$$T := \ker \Sigma / \log U, \tag{1.1}$$

is isomorphic to the torus $\mathbb{T}^{r_1+r_2-1}$. It follows from the definition of the norm that $\log \mathcal{N}$ lies in $\ker \Sigma$ and hence $\widetilde{\mathcal{N}} := \log \mathcal{N} / \log U$ is a countable subset of T.

The point of this paper is to show that when K is a cyclic extension of $\mathbb{Q}$, there is a canonical partial ordering on $\widetilde{\mathcal{N}}$ for which it is equidistributed in T.

1.1 An Ordering on $\widetilde{\mathcal{N}}$

From here forward we will assume that K is Galois over $\mathbb{Q}$ with cyclic Galois group G generated by σ. Note that this implies that K is either totally real, or totally imaginary, but we will still write, for instance, $U \cong W \times \mathbb{Z}^{r_1+r_2-1}$ with the understanding that one of r_1 and r_2 is 0.

We will define an ordering on $\widetilde{\mathcal{N}}$ by devising an ordering on $h : \mathcal{N} \to \mathbb{Z}^{>0}$ with the property that if $\beta \in \mathcal{N}$ and $\upsilon \in U$ then $h(\upsilon\beta) = h(\beta)$.

By Hilbert's Theorem 90, if $\beta \in \mathcal{N}$, there exists $\alpha \in \mathfrak{o}$ such that $\beta = \alpha/\sigma(\alpha)$. Define $\pi : K^\times \to \mathcal{N}$ by $\pi(\alpha) = \alpha/\sigma(\alpha)$. We will say $\alpha \in \mathfrak{o}^\times$ is a *visible point* for β if $\pi(\alpha) = \beta$ and $|N(\alpha)|$ is minimal for all integers with this property. That is, Hilbert's Theorem 90 implies that for every $\beta \in \mathcal{N}$ there is a visible point α, and in this situation, we define

$$h(\beta) = |N(\alpha)|.$$

Claim 1.1. *Assume that $\alpha \in \mathfrak{o}$ is a visible point for β, and $\gamma \in \mathfrak{o}^\times$. Then $\beta = \pi(\gamma)$ if and only if there exists a non-zero rational integer n such that $\gamma = n\alpha$.*

Proof. If $\pi(\alpha) = \pi(\gamma)$ then

$$\frac{\alpha}{\gamma} = \frac{\sigma(\alpha)}{\sigma(\gamma)} = \sigma\left(\frac{\alpha}{\gamma}\right),$$

and hence α/γ is fixed by G. It follows that there exist relatively prime rational integers m and n such that $\alpha/\gamma = m/n$, and by the minimality of $|N(\alpha)|$, $|m| < |n|$. Thus both α and $\gamma = n\alpha/m$ are algebraic integers which map onto β.

If we assume $n/m > 0$, then for $j = \lfloor n/m \rfloor$ we have $\gamma - j\alpha$ is an algebraic integer, with

$$\pi(\gamma - j\alpha) = \pi\left(\left(\frac{n}{m} - j\right)\alpha\right) = \frac{(n/m - j)\alpha}{\sigma((n/m - j)\alpha)} = \pi(\alpha),$$

and

$$|N(\gamma - j\alpha)| = \left|N\left(\left(\frac{n}{m} - j\right)\alpha\right)\right| = \left(\frac{n}{m} - j\right)^d |N(\alpha)| < |N(\alpha)|.$$

The only possible situation which does not violate the minimality of $|N(\alpha)|$ is that when $n/m = j$. That is, when n/m is an integer. In this situation $\gamma = n\alpha$.

The case when $n/m < 0$ is similar.

Conversely, if $\gamma = n\alpha$ and alpha is a visible point for β we have

$$\beta = \pi(\alpha) = \frac{\alpha}{\sigma(\alpha)} = \frac{n\alpha}{n\sigma(\alpha)} = \frac{n\alpha}{\sigma(n\alpha)} = \pi(n\alpha) = \pi(\gamma). \qquad \square$$

This proves in particular, that α is a visible point for β if and only if $-\alpha$ is.

Claim 1.2.

- *If $\upsilon \in U$ then there exists $u \in U$ such that $\upsilon = \pi(u)$.*
- *If $\beta \in \mathcal{N}$ and $\upsilon \in U$, then $h(\upsilon\beta) = h(\beta)$.*

Proof. Assume that $\upsilon \in U$. First, we show that $\upsilon = \pi(u)$ where $u = \sigma(\upsilon)$. As σ^d is the identity, it follows that $\upsilon = \sigma^{d-1}(u)$. Therefore

$$\upsilon = \sigma^{d-1}(u) = \frac{u\sigma^{d-1}(u)}{u} = \frac{u\sigma^{d-1}(u)}{\sigma(u)\sigma^{d-1}(u)} = \frac{u}{\sigma(u)} = \pi(u).$$

For the second assertion, assume that α is the visible point corresponding to β. Then $\pi(\alpha) = \alpha/\sigma(\alpha) = \beta$. Moreover, if u be a visible point for $\upsilon \in U$, then

$$\pi(u\alpha) = \frac{u\alpha}{\sigma(u\alpha)} = \frac{u}{\sigma(u)}\frac{\alpha}{\sigma(\alpha)} = \upsilon\beta.$$

Therefore $h(\upsilon\beta) = |N(u\alpha)| = |N(\alpha)| = h(\beta)$. □

Given $\beta \in \mathcal{N}$ we define $\widetilde{\beta} = \log\beta + \log U \in \widetilde{\mathcal{N}}$. Here and throughout, tildes will mark quantities which are invariant under multiplication by a unit. By Claim 1.2 we may define $h : \widetilde{\mathcal{N}} \to \mathbb{Z}^{>0}$ by $h(\widetilde{\beta}) = h(\beta)$, and we may finally explicitly state our main theorem.

Theorem 1.3. *With with the partial ordering specified by h, $\widetilde{\mathcal{N}}$ is equidistributed in T.*

1.2 Equidistribution of $\widetilde{N}$ in T

Given $r > 0$ we define

$$\widetilde{\mathcal{N}}(r) = \{\widetilde{\beta} \in \widetilde{N} : h(\widetilde{\beta}) < r\}.$$

By Weyl's criteria, $\widetilde{\mathcal{N}}$ is equidistributed with respect to h if, for any character $\chi : T \to \mathbb{T}$,

$$\lim_{r\to\infty} \frac{1}{\#\widetilde{\mathcal{N}}(r)} \sum_{\widetilde{\beta}\in\widetilde{\mathcal{N}}(r)} \chi(\widetilde{\beta}) = \int_T \chi\, d\mu,$$

where μ is Haar (probability) measure on T. This is equivalent to

$$\lim_{r\to\infty} \frac{1}{\#\widetilde{\mathcal{N}}(r)} \sum_{\widetilde{\beta}\in\widetilde{\mathcal{N}}(r)} \chi(\widetilde{\beta}) = \begin{cases} 1 & \text{if } \chi \text{ is trivial;} \\ 0 & \text{otherwise.} \end{cases}$$

The function

$$r \mapsto \sum_{\widetilde{\beta}\in\widetilde{\mathcal{N}}(r)} \chi(\widetilde{\beta})$$

is the summatory function of the L-series

$$L(\chi; s) = \sum_{\widetilde{\beta}\in\widetilde{\mathcal{N}}} \frac{\chi(\widetilde{\beta})}{h(\widetilde{\beta})^s}.$$

This observation is useful since asymptotics of the summatory function (as a function of r) follow from analytic properties of $L(\chi; s)$ using standard Tauberian theorems. Specifically, if we can show that, for the trivial character χ_0, $L(s, \chi_0)$ has a pole at $s = \sigma_0$ and there exists $\epsilon > 0$ such that for all other characters, $L(s, \chi)$ is analytic for $\mathrm{Re}(s) > \sigma_0 - \epsilon$, then there exists a non-zero constant C such that as $r \to \infty$,

$$\#\widetilde{\mathcal{N}}(r) \sim Cr^{\sigma_0} \qquad \text{and} \qquad \sum_{\widetilde{\beta}\in\widetilde{\mathcal{N}}(r)} \chi(\widetilde{\beta}) = o(r^{\sigma_0}),$$

and Weyl's criteria will be satisfied. See [3, Ch.VII §3].

1.3 Rewriting the L-series

We begin with a lemma summarizing useful facts about h and π.

Lemma 1.4. *Suppose α is a visible point for β, u is a unit and n a natural number, then*

- $n^d h(\widetilde{\beta}) = |N(n\alpha)|$,
- $h(\widetilde{u\beta}) = h(\widetilde{\beta})$
- $\widetilde{\pi(n\alpha)} = \widetilde{\pi(\alpha)}$
- $\widetilde{\pi(u\alpha)} = \widetilde{\pi(\alpha)}$.

Proof. The first is simply the observation that

$$n^d h(\widetilde{\beta}) = n^d |N(\alpha)| = |N(n\alpha)|.$$

The second is a restatement of Claim 1.2 using the fact that $h(\widetilde{\beta}) = h(\beta)$. For the third item,

$$\pi(n\alpha) = \frac{n\alpha}{\sigma(n\alpha)} = \frac{n}{\sigma(n)}\frac{\alpha}{\sigma(\alpha)} = \frac{\alpha}{\sigma(\alpha)} = \pi(\alpha),$$

since σ acts trivially on $\mathbb{Q}$. The final statement follows since

$$\widetilde{\pi(u\alpha)} = \log \pi(u) + \log \pi(\alpha) + \log U = \log \pi(\alpha) + \log U = \widetilde{\pi(\alpha)}. \qquad \square$$

Given $\alpha \in \mathcal{V}$, we write $\widetilde{\alpha} = \alpha U$ to be a coset of U, and we define

$$\widetilde{\mathcal{V}} = \bigcup_{\alpha \in \mathcal{V}} \widetilde{\alpha}.$$

Lemma 1.4 and Claim 1.1 imply that π extends to an injective map from $\widetilde{\mathcal{V}}$ onto $\widetilde{\mathcal{N}}$. It follows that

$$L(\chi; s) = \sum_{\widetilde{\beta} \in \widetilde{\mathcal{N}}} \frac{\chi(\widetilde{\beta})}{h(\widetilde{\beta})^s} = \sum_{\widetilde{\alpha} \in \widetilde{\mathcal{V}}} \frac{\chi(\widetilde{\pi(\alpha)})}{|N(\alpha)|^s}.$$

Note that, since $N(\alpha) = \pm N(u\alpha)$ for $u \in U$, $|N(\alpha)|$ is well-defined for any $\alpha \in \widetilde{\alpha}$ and hence the latter sum is well-defined. Multiplying and dividing by $\zeta(ds)$, we have

$$L(\chi; s) = \frac{1}{\zeta(ds)} \sum_{n=1}^{\infty} \sum_{\widetilde{\alpha} \in \widetilde{\mathcal{V}}} \frac{\chi(\widetilde{\pi(\alpha)})}{n^{ds}|N(\alpha)|^s} = \frac{1}{\zeta(ds)} \sum_{n=1}^{\infty} \sum_{\widetilde{\alpha} \in \widetilde{\mathcal{V}}} \frac{\chi(\widetilde{\pi(n\alpha)})}{|N(n\alpha)|^s}$$

Since each non-zero algebraic integer γ can be written uniquely as $n\alpha$ for visible point α and positive integer n, by Lemma 1.4 we have

$$L(\chi; s) = \frac{1}{\zeta(ds)} \sum_{\gamma \in \widetilde{\mathcal{O}}} \frac{\chi(\widetilde{\pi(\gamma)})}{|N(\gamma)|^s}$$

where $\widetilde{\mathcal{O}} = \mathfrak{o}^{\times}/U$. The set $\widetilde{\mathcal{O}}$ corresponds to the non-zero principal (integral) ideals P. Therefore, we can write this as a sum over P by setting $\widehat{\chi}(\gamma\mathfrak{o}) := \chi(\widetilde{\pi(\gamma)})$, so that

$$L(\chi; s) = \frac{1}{\zeta(ds)} \sum_{\mathfrak{a} \in P} \frac{\widehat{\chi}(\mathfrak{a})}{\mathbb{N}\mathfrak{a}^s}.$$

2 Analytic Properties of $L(\chi; s)$

It is convenient to define

$$\Xi_1(\widehat{\chi}; s) = \sum_{\mathfrak{a} \in P} \frac{\widehat{\chi}(\mathfrak{a})}{\mathbb{N}\mathfrak{a}^s},$$

so that $L(\chi; s) = \Xi_1(\widehat{\chi}; s)/\zeta(ds)$. The 1 subscript reflects the fact that we wish to see Ξ_1 as a *partial* zeta function—that is a subsum over the ideal class of principal ideals—of some *complete* zeta function given as a sum over *all* non-zero integral ideals. We will call this complete zeta function, once it is suitably defined, $\Xi(\widehat{\chi}; s)$ and we will show that it is a Hecke zeta function. Equidistribution will follow from known analytic properties of partial Hecke zeta functions, together with standard Tauberian theorems.

To see $\Xi_1(\widehat{\chi}; s)$ as a partial zeta function of a Hecke zeta function we must extend $\widehat{\chi}$ to a Hecke character, that is to a character on the idéle class group. Recall that for each place v of K, K_v is the completion of K with respect to v and, if v is non-archimedean U_v is the subset of K_v with absolute value less than 1 (the choice of representative absolute value for the place is irrelevant). The group of idéles is the restricted direct product over all places v of $K_v^\times$ with respect to U_v (since we have not defined U_v for archimedean places, $K_v^\times$ always appears for these terms in the product). For a finite set of non-archimedean places S (which we index by their corresponding prime ideals), we denote by

$$J_S = \prod_{\mathfrak{p} \in S} K_{\mathfrak{p}}^\times \times \prod_{\mathfrak{p} \notin S} U_{\mathfrak{p}},$$

and we denote the product of $K_v^\times$ over archimedean places by

$$E_\infty = \prod_{v|\infty} K_v^\times \cong (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}.$$

Then the group of idéles of K is

$$J = \bigcup_S E_\infty \times J_S,$$

where the union is over all finite sets of non-archimedean places and multiplication is component-wise. The set $K^\times$ embeds in J via the diagonal map, and a Hecke character (for our purposes) is a continuous homomorphism $J \to \mathbb{T}$ which is trivial on $K^\times$ as identified with its image under this embedding.[1] We call the restriction of a Hecke character to E_∞, the *infinite component* of the character. We will make use of the fact that any character on E_∞ appears as the infinite component of a Hecke character if and only if it is trivial on units [2, p.78].

Recall that for the principal integral ideal $\mathfrak{a} = \alpha\mathfrak{o}$, we defined

$$\widehat{\chi}(\mathfrak{a}) = \chi(\widetilde{\pi(\alpha)}) = \chi(\log \pi(\alpha) + \log U).$$

This allows us to view χ as a homomorphism from $\mathfrak{o} \setminus \{0\}$ into $\mathbb{T}$ which is trivial on units. Since $\mathfrak{o}$ embeds into $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ as a lattice, we can view $\widehat{\chi}$ as a homomorphism from this lattice (minus the origin) in E_∞ to $\mathbb{T}$. It remains to extend this homomorphism to a continuous homomorphism on all of E_∞.

To do this, we remark that the Galois group of K acts on the infinite places of K. Let w be a (fixed) infinite place, and set $w' = \sigma(w)$. Then σ extends to a continuous map $K_w \to K_{w'}$ [4, §1]. In particular, since K is Galois over $\mathbb{Q}$, we have $K_w = K_{w'}$ (as metric spaces) and hence σ is a

[1] We omit any discussion of the conductor of a Hecke character, since in our situation it will always be $\mathfrak{o}$.

continuous automorphism of K_w. If $a_n \in K_w^\times$ and $\sigma(a_n) \to 0$ then since σ is continuous on K_w and σ^d is the identity, $\sigma(a_n) \to 0$ implies that

$$a_n = \sigma^{d-1} \circ \sigma(a_n) \to \sigma^{d-1}(0) = 0.$$

It follows that, the map $\pi : a \mapsto a/\sigma(a)$ extends to a continuous function on $K_w^\times$. Likewise, the map $\log \circ \pi : E_\infty \to \mathbb{R}^{r_1+r_2-1}$ given by

$$\log \circ \pi(\mathbf{a}) := (\log \|\pi(a_v)\|_v)_{v \in S}$$

is a continuous extension of the map $\alpha \mapsto \log \pi(\alpha)$ to E_∞ by viewing $K^\times$ as a subset of E_∞ using the diagonal (canonical) embedding of $K^\times$ into E_∞. To see that $\log \circ \pi(E_\infty)$ is in $\ker \Sigma$ it suffices to note that $K^\times$ is dense in E_∞ and $\log \circ \pi(K^\times) \subseteq \ker \Sigma$. It follows that

$$\chi(\mathbf{a}) := \chi\left(\log \circ \pi(\mathbf{a}) + \log U\right)$$

is a continuous character on E_∞ which agrees with $\chi\left(\log \circ \pi(\alpha) + \log U\right)$ on canonical embedding of $\mathfrak{o} \setminus \{0\} \hookrightarrow E_\infty$. It follows that there is a unique extension to a Hecke character on the idéles of K. We call this character $\widehat{\chi}$, since it agrees with $\widehat{\chi}$ on $\mathfrak{o} \setminus \{0\}$, and hence on principal ideals. Notice that $\widehat{\chi}$ is trivial exactly when χ is the trivial character on $\mathbb{T}^{r_1+r_2-1}$.

The associated Hecke zeta function for this character is given by

$$\Xi(\widehat{\chi}; s) = \sum_{\mathfrak{a} \in I} \frac{\widehat{\chi}(\mathfrak{a})}{\mathbb{N}\mathfrak{a}^s},$$

where I is the set of non-zero integral ideals of $\mathfrak{o}$. Note that $\Xi_1(\widehat{\chi}; s)$ as defined before, is exactly the partial zeta function of $\Xi(\widehat{\chi}; s)$ over principal ideals.

We approach the end of our journey. In [1] it is proved that $\Xi_1(\widehat{\chi}; s)$ has an analytic continuation (as a function of s) to the entire plane, except when $\widehat{\chi}$ is trivial, in which case it has a meromorphic continuation to a function with a single simple pole at $s = 1$. It follows that, if χ is nontrivial, then so is $\widehat{\chi}$, and $L(\chi; s) = \Xi_1(\widehat{\chi}; s)/2\zeta(ds)$ has an analytic continuation to the half-plane $\mathrm{Re}(s) > 1/d$. If χ is trivial, then $L(\chi; s)$ has a meromorphic continuation to the same half-plane, but with a simple pole at $s = 1$. Weyl's criteria now follows from the standard Tauberian theorems presented in Section 1.2.

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