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Non-Nested Multi-Grid Solvers for Mixed Divergence-free Scott–Vogelius Discretizations

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We apply the general framework developed by John et al. in [15] to analyze the convergence of multi-level methods for mixed finite element discretizations of the generalized Stokes problem using the Scott–Vogelius element. Having in mind that semi-implicit operator splitting schemes for the Navier–Stokes equations lead to this class of problems, we take symmetric stabilization operators into account. The use of the class of Scott–Vogelius elements seems to be promising since discretely divergence-free functions are pointwise divergence-free. However, to satisfy the Ladyzhenskaya–Babuška–Brezzi stability condition, we have to deal in the multi-grid analysis with non-nested families of meshes which are derived from nested macro element triangulations.

1 Introduction

The numerical solution of the instationary, incompressible, and isothermal Navier–Stokes equations

$$\begin{aligned} \mathbf{u}_t - \frac{1}{\mathrm{Re}}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \qquad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times(0,T), \\ \mathbf{u} = 0 \quad \text{on } \partial\Omega\times(0,T], \qquad \mathbf{u}\big|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega, \end{aligned}$$

on a space-time cylinder $\Omega\times(0,T]$ is a challenging task, in particular at higher Reynolds numbers Re. Here, $\Omega\subset\mathbb{R}^d$, $d\in\{2,3\}$, denotes a polyhedral domain. Different discretizations schemes have been proposed in the literature in order to proceed efficiently in time and to reduce this nonlinear problem to a sequence of linearized subproblems. We only mention the semi-implicit operator splitting scheme introduced in [8] and fully implicit time stepping schemes which have to solve in each time step a sequence of linearized Navier–Stokes problems [12]. In the semi-implicit approach, we have to solve efficiently two generalized Stokes problems in each time step. Although the fully implicit approach leads to a sequence of nonsymmetric Oseen type problems for which the multi-grid analysis is not established, we know from numerical experiments that these methods behave well if the analysis holds for the symmetric part of the stabilized operator. Thus, in our analysis we take the use of symmetric stabilizations into account and consider stabilized generalized Stokes problems.

In this paper, we investigate the convergence of multi-grid methods for the recently proposed stabilized Scott–Vogelius element $P_k/P_{k-1}^{\mathrm{disc}}$ with $k\geq d$, see [11]. The lowest order Scott–Vogelius element for $d=2$ consists of continuous, piecewise quadratic velocities and discontinuous, piecewise linear pressures. The analogous lowest order element in the case $d=3$ consists of continuous, piecewise cubic velocities and discontinuous, piecewise quadratic pressures. For a long time, it has been well known that the two-dimensional Scott–Vogelius element is LBB-stable on certain meshes which are derived from macro element triangulations [1, 22]. Recently, an extension to the three-dimensional case has been proven [29]. The most promising property of the Scott–Vogelius element is its pointwise fulfillment of the incompressibility constraint. Indeed, since $\nabla\cdot(P_k)^d\subset P_{k-1}^{\mathrm{disc}}$ holds, the usual weak mass conservation is transformed into a strong mass conservation and the discrete velocities are not

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only discretely divergence-free but also pointwise divergence-free. Moreover, the convergence of the discrete velocities does not depend on the regularity of the pressure. How these properties can be preserved for the discrete Oseen equations and how reaction and convection terms can be stabilized by symmetric stabilization operators has been shown in [11]. In the present paper, we consider abstract stabilization operators having certain properties which guarantee the convergence of our multi-grid method. Note that the edge stabilization method by Burman and Hansbo [10, 9], the two-level local projection method by Braack and Becker [3, 4], and the one-level enriched local projection method proposed in [20, 13] fulfill all necessary properties.

The analysis of the multi-grid algorithm is based on the theory developed in [15] applied to the linear algebraic systems arising from the proposed stabilized Scott–Vogelius discretization for the Navier–Stokes equations. Since proofs of the LBB-stability in 3D are only known for meshes, which are derived from macro element triangulations, we will restrict our considerations to such meshes. Then, the corresponding multi-grid hierarchy is non-nested. But be aware that in the 2D case, for polynomial velocity spaces with $k \geq 4$ and meshes without so-called singular points, the entire multi-grid scheme below works also without macro element meshes, leading then to a nested grid hierarchy [26, 28].

The outline of the paper is as follows. We introduce in Section 2 the discretization and state the approximation properties needed for the multi-grid convergence. Section 3 is devoted to the prolongation and restriction operators of the multi-grid algorithm and to the approximation and smoothing property. Numerical examples are given in Section 4 which illustrate the theoretical results.

Notation. Throughout the paper, C will denote a generic positive constant which is independent of the mesh. Our generalized Stokes problem will be considered in the domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, which is assumed to be a polygonal or polyhedral domain with boundary $\partial\Omega$. For a measurable subset G of Ω, the usual Sobolev spaces $W^{m,p}(G)$ with norm $\|\cdot\|_{m,p,G}$ and semi-norm $|\cdot|_{m,p,G}$ are used. In the case $p = 2$, we have $H^m(G) = W^{m,2}(G)$ and the index p will be omitted. The L^2 inner product on G is denoted by $(\cdot,\cdot)_G$. Note that the index G will be omitted for $G = \Omega$. This notation of norms, semi-norms, and inner products is also used for the vector-valued and tensor-valued case.

2 Continuous and Discrete Problem

2.1 Generalized Stokes Problem and Weak Formulation

We consider the generalized Stokes equations for $(\mathbf{u}, p)$ in a domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$,

$$\begin{aligned} -\Delta\mathbf{u} + \alpha\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\mathbf{u}$ and p denote the velocity and the pressure, respectively, α is a non-negative constant, and $\mathbf{f}$ is a given source term.

Let $V := H_0^1(\Omega)$ and $Q := L_0^2(\Omega) := \{q \in L^2(\Omega) \,:\, (q, 1) = 0\}$. A weak formulation of problem (1) reads

Find $(\mathbf{u}, p) \in V^d \times Q$ such that

$$A[(\mathbf{u}, p), (\mathbf{v}, q)] = (\mathbf{f}, \mathbf{v}) \qquad \forall(\mathbf{v}, q) \in V^d \times Q \tag{2}$$

where

$$\begin{aligned} A[(\mathbf{u}, p), (\mathbf{v}, q)] &:= a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) - b(q, \mathbf{u}), \\ a(\mathbf{u}, \mathbf{v}) &:= (\nabla\mathbf{u}, \nabla\mathbf{v}) + \alpha(\mathbf{u}, \mathbf{v}), \\ b(p, \mathbf{v}) &:= -(p, \nabla\cdot\mathbf{v}). \end{aligned}$$

We can formulate problem (2) also as an elliptic one for the velocity $\mathbf{u}$ in the space

$$\mathbf{H}(\Omega) := \{\mathbf{v} \in V^d \,:\, \nabla\cdot\mathbf{v} = 0\}$$

of divergence-free functions. Indeed, choosing divergence-free test functions $\mathbf{v} \in \mathbf{H}(\Omega)$ leads to the problem

Find $\mathbf{u} \in \mathbf{H}(\Omega)$ such that

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{H}(\Omega). \tag{3}$$

Note that the pressure drops out completely from the equation but it can be reconstructed due to the continuous inf-sup condition, see [14].

Theorem 1 (H1)**.** *The generalized Stokes problem* (2) *is well-posed. For any given data* $\mathbf{f} \in \left(L^2(\Omega)\right)^d$ *there is a unique solution* $(\mathbf{u}, p) \in V^d \times Q$ *such that*

$$\|\nabla \mathbf{u}\|_0 + \sqrt{\alpha}\|\mathbf{u}\|_0 + \|p\|_0 \leq C\|\mathbf{f}\|_0.$$

Proof. The theorem is proven by applying the Cauchy-Schwarz inequality, the Poincaré inequality, and the continuous LBB-condition. □

We make in the following always a regularity assumption on problem (1).

Assumption 1. *Whenever the right hand side* $\mathbf{f}$ *belongs to the space* $\left(L^2(\Omega)\right)^d$*, the solution* $(\mathbf{u}, p)$ *satisfies* $\mathbf{u} \in \left(V \cap H^2(\Omega)\right)^d$ *and* $p \in Q \cap H^1(\Omega)$*. Moreover, the estimate*

$$\|\mathbf{u}\|_2 + \|p\|_1 \leq C\|\mathbf{f}\|_0$$

holds true.

This assumption holds if Ω is of class C^2 or Ω is a plane convex polygon.

2.2 Stabilized Scott–Vogelius Discretization

We are given a family $\{\widetilde{\mathcal{T}}_h\}$ of simplicial triangulations of the domain Ω without hanging nodes. The simplices $\widetilde{T} \in \widetilde{\mathcal{T}}_h$ are supposed to be open. Let $h_{\widetilde{T}}$ denote the diameter of the simplex $\widetilde{T} \in \widetilde{\mathcal{T}}_h$ and $h := \max_{\widetilde{T} \in \widetilde{\mathcal{T}}_h} h_{\widetilde{T}}$. Moreover, we assume that the mesh is shape regular, i.e., there exists a constant C independent of h such that

$$\frac{h_{\widetilde{T}}}{\rho_{\widetilde{T}}} \leq C \qquad \forall \widetilde{T} \in \widetilde{\mathcal{T}}_h,\ \forall h > 0$$

where $\rho_{\widetilde{T}}$ is the diameter of the largest ball which can be inscribed into $\widetilde{T}$.

The mesh $\widetilde{\mathcal{T}}_h$ is called *macro triangulation*. The triangulation $\mathcal{T}_h$ which will be the base of our discretization is derived from $\widetilde{\mathcal{T}}_h$ as follows. We connect for each macro simplex $\widetilde{T} \in \widetilde{\mathcal{T}}_h$ its barycenter with its vertices in order to construct a new triangulation. Hence, we get three triangles from each macro triangle in two space dimensions and four tetrahedra from each macro tetrahedron in three space dimensions. This new triangulation $\mathcal{T}_h$ is also shape regular in the above sense.

We consider the Scott–Vogelius element (V_h^k, Q_h^{k-1}) defined by

$$\begin{aligned} V_h^k &:= \left\{v \in H_0^1(\Omega)\ :\ v|_T \in P_k(T)\ \ \forall T \in \mathcal{T}_h\right\}, \\ Q_h^{k-1} &:= \left\{q \in L_0^2(\Omega)\ :\ q|_T \in P_{k-1}(T)\ \ \forall T \in \mathcal{T}_h\right\}. \end{aligned}$$

Hence, each velocity component is approximated by continuous, piecewise polynomials of degree k while the pressure is approximated by discontinuous, piecewise polynomials of degree $k-1$. Since the triangulations $\mathcal{T}_h$ are derived from a macro triangulation and since we assume that $k \geq d$, the pair (V_h^k, Q_h^{k-1}) is LBB-stable, see [22, 1, 29].

Using the Scott–Vogelius element, we propose the following method for discretizing (2)

Find $(\mathbf{u}_h, p_h) \in (V_h^k)^d \times Q_h^{k-1}$ such that

$$A_h[(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)] = (f, \mathbf{v}_h) \qquad \forall (\mathbf{v}_h, q_h) \in \left(V_h^k\right)^d \times Q_h^{k-1} \tag{4}$$

where

$$A_h[(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)] := A[(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)] + \mathbf{S}_h(\mathbf{u}_h, \mathbf{v}_h).$$

Here, $\mathbf{S}_h(\cdot, \cdot)$ is an abstract stabilization operator which might be needed in the case of a dominant reaction term $\alpha\mathbf{u}$ in the generalized Stokes problem or in the case of non-neglectable convection in the non-symmetric part of the generalized Oseen problem. The required properties of $\mathbf{S}_h$ will be discussed later on.

The Scott–Vogelius element has the important property

$$\nabla \cdot [V_h^k]^d \subset Q_h^{k-1} \tag{5}$$

which enforces pointwise mass conservation for the discrete solution $\mathbf{u}_h$ of (4). Indeed, we derive from (4) that

$$(\nabla \cdot \mathbf{u}_h, q_h) = 0 \qquad \forall q_h \in Q_h^{k-1}.$$

Now, it follows from $\mathbf{u}_h \in \left(H_0^1(\Omega)\right)^d$ that the function $\nabla \cdot \mathbf{u}_h$ belongs to $L_0^2(\Omega)$. Due to (5) and since $\mathbf{u}_h \in (V_h^k)^d$, the function $\nabla \cdot \mathbf{u}_h$ belongs also to Q_h^{k-1}. Thus, $\nabla \cdot \mathbf{u}_h$ can be taken as a test function q_h. This results in exact mass conservation in the L^2-sense. Moreover, since the discrete solution $\mathbf{u}_h$ is piecewise polynomial, we conclude that $\nabla \cdot \mathbf{u}_h = 0$ holds pointwise on the closure of each simplex of the triangulation. Hence, the scalar function $\nabla \cdot \mathbf{u}_h$ is pointwise 0 on $\overline{\Omega}$.

The mixed problem (4) can be formulated equivalently as an elliptic one in the space of discretely divergence-free functions. Since discretely divergence-free functions of the considered discretization are divergence-free in the strong sense, the problem (4) is equivalent to

Find $\mathbf{u}_h \in \left(V_h^k\right)^d \cap \mathbf{H}(\Omega)$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \left(V_h^k\right)^d \cap \mathbf{H}(\Omega) \tag{6}$$

where

$$a_h(\mathbf{u}_h, \mathbf{v}_h) := a(\mathbf{u}_h, \mathbf{v}_h) + \mathbf{S}_h(\mathbf{u}_h, \mathbf{v}_h).$$

Note that instead of the usual Galerkin orthogonality, we only have

$$a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) = \mathbf{S}_h(\mathbf{u}, \mathbf{v_h}) \qquad \forall \mathbf{v}_h \in \left(V_h^k\right)^d \cap \mathbf{H}(\Omega) \tag{7}$$

and the consistency error has to be estimated additionally.

2.3 Finite Element Analysis

Let $\pi_h : Q \to Q_h^{k-1}$ be the L^2-projection into Q_h^{k-1}such that

$$\|q - \pi_h(q)\|_0 \le Ch\|q\|_1 \qquad\qquad \forall q \in Q \cap H^1(\Omega).$$

We further assume that there exists a projection operator $\pi_h^{div} : V^d \cap \mathbf{H} \to \left(V_h^k\right)^d \cap \mathbf{H}$ which maps divergence-free functions to divergence-free functions and which satisfies

$$\|\mathbf{v} - \pi_h^{div}(\mathbf{v})\|_0 + h\|\nabla(\mathbf{v} - \pi_h^{div}(\mathbf{v}))\|_0 \le Ch^2\|\mathbf{v}\|_2 \tag{8}$$

for all $\mathbf{v} \in \left(V \cap H^2(\Omega)\right)^d \cap \mathbf{H}(\Omega)$. The existence of those interpolation operators is shown in [23].

We now come back to the abstract stabilization operator $\mathbf{S}_h$ and postulate the following properties

- (linearity) for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \left((V \cap H^2(\Omega)) + V_h^k\right)^d$ and $\lambda, \mu \in \mathbb{R}$:

$$\mathbf{S}_h(\lambda\mathbf{u} + \mu\mathbf{v}, \mathbf{w}) = \lambda\mathbf{S}_h(\mathbf{u}, \mathbf{w}) + \mu\mathbf{S}_h(\mathbf{v}, \mathbf{w}); \tag{9}$$

- (symmetry) for all $\mathbf{u}, \mathbf{v} \in \left((V \cap H^2(\Omega)) + V_h^k\right)^d$:

$$\mathbf{S}_h(\mathbf{u}, \mathbf{v}) = \mathbf{S}_h(\mathbf{v}, \mathbf{u});$$

- (non-negativity) for all $\mathbf{u} \in \left((V \cap H^2(\Omega)) + V_h^k\right)^d$:

$$\mathbf{S}_h(\mathbf{u},\mathbf{u}) \geq 0; \tag{10}$$

- (weak consistency) for all $\mathbf{u} \in \left(V \cap H^2(\Omega)\right)^d$:

$$\begin{aligned} |\mathbf{S}_h(\mathbf{u},\mathbf{u})|^{1/2} &\leq Ch^{3/2}\|\mathbf{u}\|_2, \\ |\mathbf{S}_h(\pi_h^{div}\mathbf{u},\pi_h^{div}\mathbf{u})|^{1/2} &\leq Ch^{3/2}\|\mathbf{u}\|_2. \end{aligned} \tag{11}$$

The properties (9) and (10) ensure that for all $\mathbf{u},\mathbf{v} \in \left((V \cap H^2(\Omega)) + V_h^k\right)^d$ the inequality

$$\left|\mathbf{S}_h(\mathbf{u},\mathbf{v})\right| \leq \mathbf{S}_h(\mathbf{u},\mathbf{u})^{1/2}\,\mathbf{S}_h(\mathbf{v},\mathbf{v})^{1/2} \tag{12}$$

holds. The edge stabilization by Burman and Hansbo [10] and the local projection method by Becker and Braack [3] fulfill all properties of this abstract setting. We also refer to [20, 13] for a more general framework of local projection stabilization. Further information on necessary properties for the stabilization operator in the case of dominant convection can be found in [11].

In order to derive the approximation property of our multi-grid scheme, we will give an L^2-estimate for the velocity of the considered stabilized Scott–Vogelius discretization. First, we prove an estimate in the corresponding error norm of the problem, and then we apply an Aubin–Nitsche argument to get the desired L^2-estimate for the velocity error. An estimate of the energy norm for a more general case is given in [11] where also error estimates for the pressure can be found.

The energy norm of the continuous elliptic problem (3) given in the space of divergence-free functions is equivalent to the H^1-norm $\|\cdot\|_1$. The energy norm of the discrete elliptic problem (6) is defined as

$$|\!|\!|\mathbf{v}|\!|\!|_h := \left(|\mathbf{v}|_1^2 + \alpha\|\mathbf{v}\|_0^2 + \mathbf{S}_h(\mathbf{v},\mathbf{v})\right)^{1/2}$$

which is well defined for all $\mathbf{v} \in \left((V\cap H^2(\Omega))+V_h^k\right)^d$. Note that $|\!|\!|\cdot|\!|\!|_h$ is a norm on $\left((V\cap H^2(\Omega))+V_h^k\right)^d$ due to the assumptions on $\mathbf{S}_h$.

To study the unique solvability of (6), we start with the coercivity of the bilinear form a_h.

Lemma 2 (Discrete coercivity)**.** *The stabilized bilinear form a_h satisfies*

$$a_h(\mathbf{v}_h,\mathbf{v}_h) \geq |\!|\!|\mathbf{v}_h|\!|\!|_h^2$$

for all $\mathbf{v}_h \in \left(V_h^k\right)^d \cap \mathbf{H}(\Omega)$.

Proof. The coercivity follows directly from the definitions of the bilinear form and the discrete energy norm. □

We proceed with an approximation property in the discrete energy norm.

Lemma 3 (Approximation)**.** *Suppose* $\mathbf{v} \in \left(V \cap H^2(\Omega)\right)^d$*, then there holds*

$$|\!|\!|\mathbf{v} - \pi_h^{div}(\mathbf{v})|\!|\!|_h \leq Ch\|\mathbf{v}\|_2.$$

Proof. The lemma is a direct consequence of (8) and the property (11) of the stabilizing term $\mathbf{S}_h$. □

We are now able to state an a-priori energy estimate.

Lemma 4 (A-priori energy estimate)**.** *Let* $\mathbf{u}$ *and* $\mathbf{u}_h$ *be the solutions of* (3) *and* (6)*, respectively. Under the additional smoothness assumption* $\mathbf{u} \in \left(H^2(\Omega)\right)^d$*, we have*

$$|\!|\!|\mathbf{u} - \mathbf{u}_h|\!|\!|_h \leq Ch\|\mathbf{u}\|_2.$$

Proof. We get for any $\mathbf{v}_h \in \left(V_h^k\right)^d \cap \mathbf{H}(\Omega)$

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq \|\mathbf{u} - \mathbf{v}_h\|_h + \|\mathbf{u}_h - \mathbf{v}_h\|_h$$

by the triangle inequality. To estimate the second term, we start with

$$\begin{aligned}\|\mathbf{u}_h - \mathbf{v}_h\|_h^2 &\leq a_h(\mathbf{u}_h - \mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h)\\ &= a_h(\mathbf{u} - \mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h) + a_h(\mathbf{u}_h - \mathbf{u}, \mathbf{u}_h - \mathbf{v}_h)\\ &= a_h(\mathbf{u} - \mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h) - \mathbf{S}_h(\mathbf{u}, \mathbf{u}_h - \mathbf{v}_h)\end{aligned}$$

where the representation of the consistency error from (7) was applied. Using (12) and the definition of $\|\cdot\|_h$, we end up with

$$\|\mathbf{u}_h - \mathbf{v}_h\|_h \leq \|\mathbf{u} - \mathbf{v}_h\|_h + \mathbf{S}_h(\mathbf{u}, \mathbf{u})^{1/2}.$$

After setting $\mathbf{v}_h := \pi_h^{div}(\mathbf{u})$, we apply Lemma 3 and (11) to yield the assertion of the Lemma. □

Additionally, we prove an a-priori L^2-estimate using a duality argument. To this end, we consider the following continuous adjoint problem for a given $\mathbf{g} \in \left(L^2(\Omega)\right)^d$

Find $\mathbf{w_g} \in V^d \cap \mathbf{H}(\Omega)$ such that

$$a(\mathbf{v}, \mathbf{w_g}) = (\mathbf{g}, \mathbf{v}) \qquad \forall \mathbf{v} \in V^d \cap \mathbf{H}(\Omega). \tag{13}$$

This problem is well-posed. Due to Assumption 1 on the regularity of the Stokes problem, we conclude $\mathbf{w_g} \in H^2(\Omega)$ and $\|\mathbf{w_g}\|_2 \leq C\|\mathbf{g}\|_0$.

Theorem 5 (A-priori L^2-estimate, H2)**.** *Let* $\mathbf{u}$ *and* $\mathbf{u}_h$ *be the solutions of* (3) *and* (6)*, respectively. If in addition* $\mathbf{u} \in \left(H^2(\Omega)\right)^d$*, we have the a-priori* L^2*-estimate*

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq Ch^2\|\mathbf{u}\|_2.$$

Proof. We start the proof by using $\mathbf{v} := \mathbf{u} - \mathbf{u}_h$ as a test function in (13) to obtain

$$(\mathbf{g}, \mathbf{u} - \mathbf{u}_h) = a(\mathbf{u} - \mathbf{u}_h, \mathbf{w_g}) = a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{w_g}) - \mathbf{S}_h(\mathbf{u} - \mathbf{u}_h, \mathbf{w_g})$$

where the definition of a and a_h were used. Let for a moment $\mathbf{v}_h \in \left(V_h^k\right)^d \cap \mathbf{H}(\Omega)$ be arbitrary. Using (7) in the form $\mathbf{S}_h(\mathbf{u}, \mathbf{v}_h) - a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) = 0$, we get

$$\begin{aligned}(\mathbf{g}, \mathbf{u} - \mathbf{u}_h) &= a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{w_g}) - a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + \mathbf{S}_h(\mathbf{u}, \mathbf{v}_h) - \mathbf{S}_h(\mathbf{u} - \mathbf{u}_h, \mathbf{w_g})\\ &= a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{w_g} - \mathbf{v}_h) + \mathbf{S}_h(\mathbf{u}, \mathbf{v}_h) - \mathbf{S}_h(\mathbf{u} - \mathbf{u}_h, \mathbf{w_g}).\end{aligned}$$

By setting $\mathbf{g} := \mathbf{u} - \mathbf{u}_h$, we obtain

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h\|_0^2 &= (\mathbf{g}, \mathbf{u} - \mathbf{u}_h)\\ &\leq \|\mathbf{u} - \mathbf{u}_h\|_h \|\mathbf{w_g} - \mathbf{v}_h\|_h + |\mathbf{S}_h(\mathbf{u}, \mathbf{u})|^{1/2} |\mathbf{S}_h(\mathbf{v}_h, \mathbf{v}_h)|^{1/2}\\ &\qquad + |\mathbf{S}_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h)|^{1/2} |\mathbf{S}_h(\mathbf{w_g}, \mathbf{w_g})|^{1/2}\\ &\leq \|\mathbf{u} - \mathbf{u}_h\|_h \left(\|\mathbf{w_g} - \mathbf{v}_h\|_h + |\mathbf{S}_h(\mathbf{w_g}, \mathbf{w_g})|^{1/2}\right)\\ &\qquad + |\mathbf{S}_h(\mathbf{u}, \mathbf{u})|^{1/2} |\mathbf{S}_h(\mathbf{v}_h, \mathbf{v}_h)|^{1/2}.\end{aligned}$$

Choosing $\mathbf{v}_h := \pi_h^{div}(\mathbf{w_g})$, we get by applying Lemmata 3 and 4 the stated estimate where (11) and $\|\mathbf{w_g}\|_2 \leq C\|\mathbf{g}\|_0$, which follows from Assumption 1, were used. □

Remark 6. *We can see from the proof above that the expected asymptotic convergence order of the scheme does not deteriorate by adding the symmetric stabilization operator.*

3 Multi-Level Approach

In [15], five sufficient conditions (**H1**)–(**H5**) have been identified which allow to conclude that some multi-level solvers for quite general discretizations of mixed problems converge at optimal convergence rates. The considerations in [15] include non-nested discretizations and even discretizations with different finite element ansatz functions on different levels. The Theorems 1 and 5 which were proved in the previous section are the the conditions (**H1**) and (**H2**) for the special case of the proposed stabilized Scott–Vogelius scheme. We will outline in the following section a multi-level approach for the proposed stabilized Scott–Vogelius element and we will apply the general framework given in [15] to this situation. For the sake of completeness, we will repeat some of the arguments used in [15]. We also mention that hints to a multi-level convergence analysis for the unstabilized Scott–Vogelius element are already given in [29].

3.1 Multi-Level Discretization

Let $\widetilde{\mathcal{T}}_0$ denote the coarse macro triangulation. The finer macro triangulations $\widetilde{\mathcal{T}}_l$, $l \geq 1$, are obtained by successive regular refinement. Since the Scott–Vogelius element is not necessarily LBB-stable on such meshes, we construct triangulations $\mathcal{T}_l$ from the macro triangulations $\widetilde{\mathcal{T}}_l$ as described in Sect. 2.2. Note that the mesh size of $\mathcal{T}_l$ is just the half of the mesh size of $\mathcal{T}_{l-1}$. Let V_l and Q_l denote the spaces V_h^k and Q_h^{k-1} with respect to the triangulation $\mathcal{T}_l$. Note that both the sequence $\{V_l^d\}_{l\geq 0}$ of velocity spaces and the sequence $\{Q_l\}_{l\geq 0}$ of pressure spaces are non-nested. This is caused by the non-nested triangulations which are, however, derived from nested triangulations. Figure 3.1 shows two

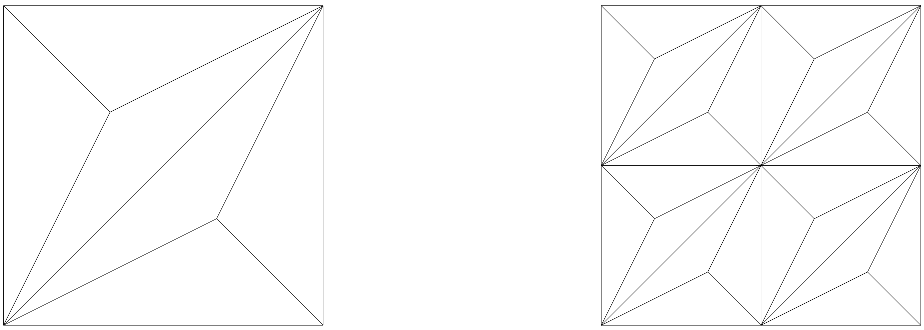

Figure 1: Two subsequent triangulations of the unit square with two and eight macro elements which result in six and 24 elements, respectively.

subsequent triangulation of the unit square into two and eight macro triangles which are refined into 6 and 24 triangles, respectively. The non-nested character of the grid hierarchy is clearly demonstrated.

3.2 Matrix Representation

Let $\{\varphi_{l,i} : i \in I_l\}$ and $\{\psi_{l,j} : j \in J_l\}$ be bases of the spaces V_l^d and Q_l, respectively, where I_l, J_l denote the corresponding index sets. The solution $(\mathbf{u}_h, p_h)$ of (4) with V_h^k and Q_h^{k-1} based on the triangulation $\mathcal{T}_h = \mathcal{T}_l$ will be denoted by $(\mathbf{u}_l, p_l)$. The unique representations

$$\mathbf{u}_l = \sum_{i\in I_l} u_{l,i}\, \varphi_{l,i}, \qquad p_l = \sum_{j\in J_l} p_{l,j}\, \psi_{l,j}$$

define the finite element isomorphisms $\Phi_l : \mathcal{U}_l \to V_l^d$, $\Psi_l : \mathcal{P}_l \to Q_l$ between the vector spaces $\mathcal{U}_l = \mathbb{R}^{\dim V_l^d}$, $\mathcal{P}_l = \mathbb{R}^{\dim Q_l}$ of coefficient vectors $\underline{u}_l = (u_{l,i})_{i\in I_l}$, $\underline{p}_l = (p_{l,j})_{j\in J_l}$ and the finite element spaces V_l^d and Q_l, respectively. Let a_l be the bilinear form a_h based on $\mathcal{T}_h = \mathcal{T}_l$. We introduce the finite element matrices A_l and B_l having the entries $a_{l,ij} = a_l(\varphi_{l,j}, \varphi_{l,i})$ and $b_{l,ij} = b(\psi_{l,i}, \varphi_{l,j})$. Now the discrete problem (4) is equivalent to

$$\begin{pmatrix} A_l & B_l^T \\ B_l & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_l \\ \underline{p}_l \end{pmatrix} = \begin{pmatrix} \underline{f}_l \\ 0 \end{pmatrix} \tag{14}$$

with $f_{l,i} = (\mathbf{f}, \varphi_{l,i})$. Note that A_l is a symmetric matrix. We will use in the vector spaces $\mathcal{U}_l$ and $\mathcal{P}_l$ the usual Euclidean norms scaled by suitable factors such that the following norm equivalences

$$\begin{aligned} C^{-1}\|\underline{v}_l\|_{\mathcal{U}_l} \le \|\mathbf{v}_{h_l}\|_0 \le C\|\underline{v}_l\|_{\mathcal{U}_l} \qquad & \forall \mathbf{v}_{h_l} \in V_l^d, \\ C^{-1}\|\underline{q}_l\|_{\mathcal{P}_l} \le \|q_{h_l}\|_0 \le C\|\underline{q}_l\|_{\mathcal{P}_l} \qquad & \forall q_{h_l} \in Q_l, \end{aligned}$$

are satisfied with a mesh- and level-independent constant C.

3.3 Smoothing Property

For smoothing the error of an approximate solution of (14), we take the basic iteration

$$\begin{pmatrix} \alpha_l D_l & B_l^T \\ B_l & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_l^{j+1} - \underline{u}_l^j \\ \underline{p}_l^{j+1} - \underline{p}_l^j \end{pmatrix} = \begin{pmatrix} \underline{f}_l \\ 0 \end{pmatrix} - \begin{pmatrix} A_l & B_l^T \\ B_l & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_l^j \\ \underline{p}_l^j \end{pmatrix}, \qquad j \ge 0. \tag{15}$$

This can be considered as a special case of the symmetric incomplete Uzawa algorithm proposed by Bank, Welfert, and Yserentant in [2]. The smoothing properties of (15) have been studied in [6] for the special case $D_l = I_l$, in [24] for the general case provided that an additional projection step is performed, and in [30] for a more general setting.

The matrix D_l is a pre-conditioner of A_l such that the linear system (15) is more easily solvable than (14). Note that we have

$$B_l(\underline{u}^{j+1} - \underline{u}_l^j) = -B_l\underline{u}_l^j, \qquad j \ge 0,$$

implying that after one smoothing step the iterate $\underline{u}_l^{j+1}$ is divergence-free, i.e. $B_l\underline{u}_l^{j+1} = 0$.

Theorem 7 (H5)**.** *The matrix $B_lB_l^T$ in (14) is non-singular.*

Proof. The invertibilty of $B_lB_l^T$ is a consequence of the fulfillment of the discrete LBB-condition. □

Remark 8. *It is easy to verify that*

$$\begin{pmatrix} \underline{u}_l - \underline{u}_l^{j+1} \\ \underline{p}_l - \underline{p}_l^{j+1} \end{pmatrix} = \begin{pmatrix} \alpha_l D_l & B_l^T \\ B_l & 0 \end{pmatrix}^{-1} \begin{pmatrix} (\alpha_l D_l - A_l)(\underline{u}_l - \underline{u}_l^j) \\ 0 \end{pmatrix}$$

where $(\underline{u}_l, \underline{p}_l)$ is the solution of (14). This shows that the iteration is a so-called u-dominant method since the new iterate $(\underline{u}_l^{j+1}, \underline{p}_l^{j+1})$ depends on $\underline{u}_l^j$ but not on $\underline{p}_l^j$.

Lemma 9. *Now we assume that D_l is symmetric and that α_l can be chosen such that*

$$\frac{1}{\delta}\lambda_{\max}(A_l) < \alpha_l\lambda_{\min}(D_l) \le \alpha_l\|D_l\| \le \gamma\lambda_{\max}(A_l)$$

for some level- and mesh-independent constants $\delta \in [1, 2)$ and $\gamma > 0$. Moreover, let the basis of V_l^d be chosen such that $\lambda_{\max}(A_l) = \mathcal{O}(h_l^{-2})$. Then, the basic iteration (15) satisfies the smoothing property

$$\|A_l(\underline{\mathbf{u}}_l - \underline{\mathbf{u}}_l^m) + B_l^T(\underline{p}_l - \underline{p}_l^m)\|_{\mathcal{U}_l} \le \frac{C}{m}h_l^{-2}\|\underline{\mathbf{u}}_l - \underline{\mathbf{u}}_l^0\|_{\mathcal{U}_l}.$$

Proof. See [6, 24, 30]. □

3.4 Prolongation and Restriction

Essential ingredients of a multi-level algorithm for mixed problems are appropriate prolongations

$$P_{l-1,l}^u : \mathcal{U}_{l-1} \to \mathcal{U}_l, \qquad P_{l-1,l}^p : \mathcal{P}_{l-1} \to \mathcal{P}_l$$

and restrictions

$$R_{l,l-1}^u := (P_{l-1,l}^u)^* : \mathcal{U}_l \to \mathcal{U}_{l-1}, \qquad R_{l,l-1}^p := (P_{l-1,l}^p)^* : \mathcal{P}_l \to \mathcal{P}_{l-1}.$$

Since we deal with a non-nested finite element discretization, we define prolongations by

$$P^u_{l-1,l} := \Phi_l^{-1} \circ i_{\mathbf{u}} \circ \Phi_{l-1}, \qquad P^p_{l-1,l} := \Psi_l^{-1} \circ i_p \circ \Psi_{l-1}$$

where $i_{\mathbf{u}} : (V_{l-1} + V_l)^d \to V_l^d$ and $i_p : Q_{l-1} + Q_l \to Q_l$ are suitable prolongation operators.

The convergence analysis in [15] is based on the u-dominance of the smoother, i.e., the new iterate $(\underline{u}_l^{j+1}, \underline{p}_l^{j+1})$ depends on $\underline{u}_l^j$ but not on $\underline{p}_l^j$. Thus, we only have to investigate in depth the velocity prolongation which is in our situation much simpler than in the general framework. First, we restrict our considerations to a scalar prolongation for one velocity component. Second, we can make important simplifications since the velocity spaces V_l are continuous. Third, the local finite element space $V_l|_{\widetilde{T}}$ on any macro simplex $\widetilde{T} \in \widetilde{\mathcal{T}}_l$ is derived from a single corresponding finite element space $\widehat{V}$ on a reference macro simplex $\widehat{T}$ by

$$V_l|_{\widetilde{T}} = \left\{ \hat{v} \circ F_{\widetilde{T}}^{-1} : \hat{v} \in \widehat{V} \right\}$$

where $F_{\widetilde{T}} : \widehat{T} \to \widetilde{T}$ is an affine transformation with $F_{\widetilde{T}}(\widehat{T}) = \widetilde{T}$ and $\widehat{T}$ denotes the unit reference simplex.

We will construct a scalar velocity prolongation $i_u : \Sigma_l \to V_l$ with $\Sigma_l := V_{l-1} + V_l$ which fulfills the necessary properties given in [15]. Note that for macro elements $\widetilde{T} \in \widetilde{\mathcal{T}}_{l-1}$ the local finite element space $\Sigma_l|_{\widetilde{T}} = V_{l-1}|_{\widetilde{T}} + V_l|_{\widetilde{T}}$ can be derived from $\widehat{\Sigma} = \widehat{V}_c + \widehat{V}_f$ where the finite element spaces $\widehat{V}_c$ and $\widehat{V}_f$ on the reference macro element $\widehat{T}$ correspond to the refinement levels $l = 0$ and $l = 1$ of the reference simplex $\widehat{T}$. Let $n := \dim \widehat{V}_f$. We introduce n linear nodal functions $\widehat{N}_i$, $i = 1, \dots, n$, such that

$$\widehat{N}_i : \widehat{\Sigma} \to \widehat{V}_f, \quad \hat{v} \mapsto \hat{v}(\mathbf{x}_i), \qquad i = 1, \dots, n,$$

for the set $\{\mathbf{x}_i,\, i = 1, \dots, n\}$ of Lagrange points with respect to $\widehat{V}_f$. Let $\{\hat{\varphi}_1, \dots, \hat{\varphi}_n\}$ be the basis of $\widehat{V}_f$ which is dual with respect to $\{\widehat{N}_1, \dots, \widehat{N}_n\}$, i.e., $\widehat{N}_i(\hat{\varphi}_j) = \delta_{ij}$, $i, j = 1, \dots, n$, where δ_{ij} denotes the Kronecker delta. We define

$$\hat{\imath}_u : \widehat{\Sigma} \to \widehat{V}_f, \qquad \hat{v} \mapsto \sum_{i=1}^{n} \widehat{N}_i(\hat{v})\, \hat{\varphi}_i.$$

Obviously, $\hat{\imath}_u$ is a continuous linear operator on $\widehat{\Sigma}$ since the nodal functionals $\widehat{N_i}$ are linear and since $\dim \widehat{\Sigma} < \infty$. Hence, the estimate

$$\|\hat{\imath}_u \hat{v}\|_{0,\widehat{T}} \le C \|\hat{v}\|_{0,\widehat{T}} \qquad \forall \hat{v} \in \widehat{\Sigma} \tag{16}$$

holds true. We define for each macro element $\widetilde{T} \in \widetilde{\mathcal{T}}_{l-1}$ a local prolongation operator by

$$i_u^{\widetilde{T}} : \Sigma_l|_{\widetilde{T}} \to V_l|_{\widetilde{T}}, \quad v \mapsto \sum_{i=1}^{n} \widehat{N}_i(v \circ F_{\widetilde{T}})\, (\hat{\varphi}_i \circ F_{\widetilde{T}}^{-1}).$$

Due to the chosen set of Lagrange points, the restriction of $i_u^{\widetilde{T}} v$ to $\partial \widetilde{T}$ depends only on the restriction of v to $\partial \widetilde{T}$. Hence, the local operators can be put together to a global operator $i_u : \Sigma_l \to V_l$ such that

$$i_u v|_{\widetilde{T}} = i_u^{\widetilde{T}}\big(v|_{\widetilde{T}}\big), \qquad \widetilde{T} \in \widetilde{\mathcal{T}}_{l-1}.$$

Theorem 10 (H3)**.** *For the interpolation operator $i_u : \Sigma_l \to V_l$, there holds for all $v \in V_l$*

$$i_u v = v.$$

Proof. The statement of this theorem is a direct consequence of the definition of the operator i_u. □

Theorem 11 (H4)**.** *For all $v_l \in \Sigma_l$, it holds*

$$\|i_u v\|_0 \le C \|v\|_0.$$

Proof. Setting in the following $v := \hat{v} \circ F_{\widetilde{T}}^{-1}$ and $\hat{\mathbf{x}} := F_{\widetilde{T}}^{-1}(\mathbf{x})$, we compute

$$\begin{aligned}\|i_u v\|_0^2 &= \sum_{\widetilde{T} \in \widetilde{\mathcal{T}}_{l-1}} \int_{\widetilde{T}} \left\{(i_u^{\widetilde{T}} v)(\mathbf{x})\right\}^2 d\mathbf{x} = \sum_{\widetilde{T} \in \widetilde{\mathcal{T}}_{l-1}} d!\, |\widetilde{T}| \int_{\widehat{T}} \left\{(i_u^{\widetilde{T}} v)(F_{\widetilde{T}}(\hat{\mathbf{x}}))\right\}^2 d\hat{\mathbf{x}} \\ &= \sum_{\widetilde{T} \in \widetilde{\mathcal{T}}_{l-1}} d!\, |\widetilde{T}| \int_{\widehat{T}} \left\{(\hat{i}_u \hat{v})(\hat{\mathbf{x}})\right\}^2 d\hat{\mathbf{x}} = \sum_{\widetilde{T} \in \widetilde{\mathcal{T}}_{l-1}} d!\, |\widetilde{T}|\, \|\hat{i}_u \hat{v}\|_{0,\widehat{T}}^2.\end{aligned}$$

We can apply the estimate (16) and conclude

$$\|i_u v\|_0^2 \leq C^2 \sum_{\widetilde{T} \in \widetilde{\mathcal{T}}_{l-1}} d!\, |\widetilde{T}|\, \|\hat{v}\|_{0,\widehat{T}}^2 = C^2 \sum_{\widetilde{T} \in \widetilde{\mathcal{T}}_{l-1}} \|v\|_{0,\widetilde{T}}^2 = C^2 \|v\|_0^2.$$

Thus, the L^2-stability of the prolongation is shown with a constant C independent of the level. □

Remark 12. *It would have been possible to retract completely on the much more general proof in [15] about the L^2-stability of a general class of possible velocity prolongation operators. There, two additional assumptions **(H6)** and **(H7)** are introduced which imply **(H3)** and **(H4)**. These assumptions are clearly true in our case but the proof presented above is much more simpler.*

3.5 Approximation Property

Let an approximation $(\tilde{\mathbf{u}}_l, \tilde{p}_l) \in V_l^d \times Q_l$ of the discrete solution $(\mathbf{u}_l, p_l)$ be given. We can think of $(\tilde{\mathbf{u}}_l, \tilde{p}_l)$ as the result after some smoothing steps and consequently assume that

$$\nabla \cdot \tilde{\mathbf{u}}_l = 0.$$

Then, the coarse-level correction is defined as the solution of the following problem

Find $(\mathbf{u}_{l-1}^*, p_{l-1}^*) \in V_{l-1}^d \times Q_{l-1}$ such that for all $(\mathbf{v}_{l-1}, q_{l-1}) \in V_{l-1}^d \times Q_{l-1}$

$$A_{l-1}[(\mathbf{u}_{l-1}^*, p_{l-1}^*), (\mathbf{v}_{l-1}, q_{l-1})] = (\mathbf{f}, i_{\mathbf{u}} \mathbf{v}_{l-1}) - A_l[(\tilde{\mathbf{u}}_l, \tilde{p}_l), (i_{\mathbf{u}} \mathbf{v}_{l-1}, 0)]. \tag{17}$$

The coarse-level correction yields via the transfer operator $i_{\mathbf{u}}$ from Section 3.4 the new velocity approximation

$$\mathbf{u}_l^{new} := \tilde{\mathbf{u}}_l + i_{\mathbf{u}} \mathbf{u}_{l-1}^*. \tag{18}$$

The basic idea for proving the approximation property is to construct a auxiliary continuous problem such that $(\mathbf{u}_{l-1}^*, p_{l-1}^*)$ and $(\mathbf{u}_l - \tilde{\mathbf{u}}_l, p_l - \tilde{p}_l)$ are finite element solutions of the corresponding discrete solutions in the spaces $V_{l-1}^d \times Q_{l-1}$ and $V_l^d \times Q_l$, respectively. This idea has been used for scalar elliptic equations in [7] and has been applied to more general situations in [5] and [18].

The auxiliary problem will be

Find $(\mathbf{z}, w) \in V^d \times Q$ such that for all $(\mathbf{v}, q) \in V^d \times Q$

$$A[(\mathbf{z}, w), (\mathbf{v}, q)] = (\mathbf{F}_l, \mathbf{v}).$$

where $\mathbf{F}_l \in \Sigma_l^d$ is given via the Riesz representation of the residue by

$$(\mathbf{F}_l, \mathbf{s}) := (\mathbf{f}, i_{\mathbf{u}} \mathbf{s}) - A_l[(\tilde{\mathbf{u}}_l, \tilde{p}_l), (i_{\mathbf{u}} \mathbf{s}, 0)] \qquad \forall \mathbf{s} \in \Sigma_l^d.$$

Due to (**H3**), we have for $\mathbf{s} \in V_l^d$ that

$$(\mathbf{F}_l, \mathbf{s}) = (\mathbf{f}, \mathbf{s}) - A_l[(\tilde{\mathbf{u}}_l, \tilde{p}_l), (\mathbf{s}, 0)] = A_l[(\mathbf{u}_l - \tilde{\mathbf{u}}_l, p_l - \tilde{p}_l), (\mathbf{s}, 0)]$$

which means that $(\mathbf{u}_l - \tilde{\mathbf{u}}_l, p_l - \tilde{p}_l)$ is a finite element approximation of $(\mathbf{z}, w)$ in the space $V_l^d \times Q_l$. On the other hand, $(\mathbf{F}_l, \mathbf{s})$ becomes just the right hand side of (17) if $\mathbf{s} \in V_{l-1}^d$, i.e., $(\mathbf{u}_{l-1}^*, p_{l-1}^*)$ is the finite element approximation of $(\mathbf{z}, w)$ in the space $V_{l-1}^d \times Q_{l-1}$. Furthermore, we have for $\mathbf{s} \in \Sigma_l^d$

$$(\mathbf{F}_l, \mathbf{s}) = a_h(\mathbf{u}_l - \tilde{\mathbf{u}}_l, i_{\mathbf{u}} \mathbf{s}) + b(i_{\mathbf{u}} \mathbf{s}, p_l - \tilde{p}_l) = \left(A_l(\underline{u}_l - \underline{\tilde{u}}_l) + B_l^T(\underline{p}_l - \underline{\tilde{p}}_l), \underline{i_{\mathbf{u}} \mathbf{s}}\right)_{\mathcal{U}_l}.$$

Applying the Cauchy–Schwarz inequality and setting $\mathbf{s} = \mathbf{F}_l$, we get

$$\|\mathbf{F}_l\|_0 \le C\|A_l(\underline{u}_l - \underline{\tilde{u}}_l) + B_l^T(\underline{p}_l - \underline{\tilde{p}}_l)\|_{\mathcal{U}_l} \tag{19}$$

where $\|\underline{i_{\mathbf{u}}\mathbf{s}}\|_{\mathcal{U}_l} \le C\|i_{\mathbf{u}}\mathbf{s}\|_0 \le C\|\mathbf{s}\|_0$ was used which follows from the equivalence of norms in V_l^d and $\mathcal{U}_l$ together with (**H4**).

Lemma 13. *The approximation property holds with*

$$\|\mathbf{u}_l - \mathbf{u}_l^{new}\|_0 \le Ch_l^2\|A_l(\underline{u}_l - \underline{\tilde{u}}_l) + B_l^T(\underline{p}_l - \underline{\tilde{p}}_l)\|_{\mathcal{U}_l}.$$

Proof. We get from (**H3**) and (18) that $\mathbf{u}_l - \mathbf{u}_l^{new} = i_{\mathbf{u}}(\mathbf{u}_l - \tilde{\mathbf{u}}_l - \mathbf{u}_{l-1}^*)$. Applying (**H4**), the triangle inequality, and Theorem 5, we get

$$\|\mathbf{u}_l - \mathbf{u}_l^{new}\|_0 \le C\|\mathbf{u}_l - \tilde{\mathbf{u}}_l - \mathbf{u}_{l-1}^*\|_0 \le C\big(\|\mathbf{u}_l - \tilde{\mathbf{u}}_l - \mathbf{z}\|_0 + \|\mathbf{z} - \mathbf{u}_{l-1}^*\|_0\big) \le C(h_l^2 + h_{l-1}^2)\,\|\mathbf{z}\|_2.$$

Using Assumption 1, (19), and $h_{l-1} = 2\,h_l$ yields the statement of this lemma. □

3.6 Multi-Level Convergence

We shortly describe the two-level algorithm using m smoothing steps on the level l, $l \ge 1$, and the coarse-level correction (18). Let $(\mathbf{u}_l^0, p_l^0)$ be an initial guess for the solution $(\mathbf{u}_l, p_l)$ of (4). We apply m smoothing steps of the basic iteration (15) and obtain (u_l^m, p_l^m). Now the coarse-level correction (18) is performed using

$$(\tilde{\mathbf{u}}_l, \tilde{p}_l) = (\mathbf{u}_l^m, p_l^m)$$

as an approximate solution of the discrete problem (4). Finally, the new velocity approximation is obtained by

$$\mathbf{u}_l^{new} := \mathbf{u}_l^m + i_{\mathbf{u}}\mathbf{u}_{l-1}^*.$$

Combining the smoothing and the approximation property, we get the multi-level convergence.

Theorem 14. *Under the assumptions of Lemma 9 and Lemma 13, the two-level method converges for sufficiently many smoothing steps with respect to the L^2- and $\mathcal{U}_l$-norm. In particular, there are level- and mesh-independent constants C and $\widetilde{C}$ such that*

$$\|\underline{u}_l - \underline{u}_l^{new}\|_{\mathcal{U}_l} \le \frac{C}{m}\|\underline{u}_l - \underline{u}^0\|_{\mathcal{U}_l}$$

and

$$\|\mathbf{u}_l - \mathbf{u}_l^{new}\|_0 \le \frac{\widetilde{C}}{m}\|\mathbf{u}_l - \mathbf{u}_l^0\|_0.$$

Once proven the convergence of the two-level method, the convergence of the W-cycle multi-level method follows in a standard way.

4 Numerical Results

This section presents some numerical results for the Scott–Vogelius element applied to the unstabilized Stokes problem, i.e. the case with $\alpha = 0$ and $\mathbf{S}_h \equiv 0$. Numerical results for different mixed finite element pairs applied to the stabilized generalized Stokes problems can be found in [21] where convergence of the stabilized scheme was demonstrated only numerically.

All numerical computations used the software package MooNMD [17] and were performed on a Linux PC (Pentium IV, 2.8 GHz).

Let $\Omega = (0,1)^2$. The right-hand side $\mathbf{f}$ and the inhomogeneous boundary condition in the Stokes problem are chosen such that

$$\mathbf{u}(x,y) = \begin{pmatrix} \sin(x)\sin(y) \\ \cos(x)\cos(y) \end{pmatrix},$$
$$p(x,y) = 2\cos(x)\sin(y) - 2\sin(1)\big(1-\cos(1)\big)$$

is the solution. This example was taken from [6].

We have used the lowest order two-dimensional Scott–Vogelius element pair P_2/P_1^{disc} on the family of non-nested meshes described in Sect. 2.2.

Table 1 shows the number of triangles and the number of degrees of freedom for the discretizations on different refinement levels. The coarsest mesh (level 0) consists of six triangles obtained from a macro decomposition of the unit square into two macro triangles by the diagonal of slope +1, see Fig. 3.1, left. We see that the number of triangles and the number of degrees of freedom increase by a factor of four from one level to the next finer one. Furthermore, the number of degrees of freedom for both velocity components together is approximately 4/3 times the number of pressure degrees of freedom.

Table 1: Number of triangles and number of degrees of freedom (dofs) on different refinement levels.

level	5	6	7	8
number of triangles	6,144	24,576	98,304	393,216
velocity dofs	24,834	98,818	394,242	1,574,914
pressure dofs	18,432	73,728	294,912	1,179,648
total dofs	43,266	172,546	689,154	2,754,562

In our calculation with the Braess–Sarazin smoother [6, 24], we have chosen D_l to be the incomplete LU-decomposition of the matrix A_l and $\alpha_l = 1$.

Furthermore, we carried out calculations with Vanka-type smoothers, see [16, 27]. These smoothers can be seen as block Gauss-Seidel smoothers. We have chosen as blocks in our calculations all degrees of freedom which are connected to a single macro element, i.e., each block contains 20 velocity and 9 pressure degrees of freedom.

Table 2 shows the averaged multigrid rates for a W-cycle with m pre-smoothing and m post-smoothing steps. It can be seen that the averaged multigrid rates are independent of the number of levels within the multigrid hierarchy. Moreover, the rates for the Vanka-type smoother are much better than those for the Braess–Sarazin smoother. However, it should be noted that almost nothing is known about the smoothing properties of multiplicative Vanka-type smoothers which we have used in our calculations. In [25], additive Vanka-type smoothers have been considered and transformed into inexact Uzawa methods under certain conditions. Unfortunately, these conditions cannot be satisfied in general [19]. Nevertheless, in case of the nonconforming Crouzeix–Raviart discretization of lowest order, a convergence rate of $\mathcal{O}(\sqrt{m})$ for the additive Vanka-type smoother has been proven [25].

Table 2: Averaged multigrid rates for a $W(m,m)$-cycle.

	Braess-Sarazin		Vanka-type	
m	7 levels	8 levels	8 levels	9 levels
1	0.8013	0.8056	0.3563	0.3550
2	0.6674	0.6756	0.1600	0.1606
3	0.5553	0.5657	0.0924	0.0911
4	0.4626	0.4749	0.0545	0.0580
5	0.3878	0.3978	0.0455	0.0459
6	0.3235	0.3347	0.0338	0.0345
7	0.2757	0.2861	0.0260	0.0264
8	0.2270	0.2396	0.0250	0.0257
9	0.1922	0.1932	0.0215	0.0221
10	0.1592	0.1702	0.0180	0.0218

Concerning the convergence of Vanka-type solvers for the Stokes and Navier–Stokes problem (in case of small Reynolds numbers), we refer to [19].

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