Weierstraß-Institut
für Angewandte Analysis und Stochastik

Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

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submitted: January 13, 2022

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No. 2912
Berlin 2022

2020 *Mathematics Subject Classification.* 65N30.

Key words and phrases. Convection-diffusion equations, discontinuous Galerkin methods, spurious oscillations, post-processing techniques, slope limiter.

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Abstract

This note studies a generalization of a post-processing technique and a novel method inspired by the same technique which significantly reduce spurious oscillations in discontinuous Galerkin solutions of convection-diffusion equations in the convection-dominated regime.

1 Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, with polyhedral Lipschitz boundary $\Gamma = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$ with $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$. A steady-state convection-diffusion equations reads as follows:

$$\begin{aligned} -\varepsilon\Delta u + \boldsymbol{b}\cdot\nabla u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma_{\mathrm{D}}, \\ -\varepsilon\nabla u\cdot\boldsymbol{n} &= 0 \quad \text{on } \Gamma_{\mathrm{N}}. \end{aligned} \tag{1}$$

Convection-diffusion equations model the physical behavior of scalar quantities inside a flowing medium, like temperature or concentration. In (1), $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, is a constant diffusion coefficient, $\boldsymbol{b}(\boldsymbol{x})$ is the convection field, and the sources are denoted by $f(\boldsymbol{x})$. The prescribed Dirichlet boundary conditions on Γ_{D} are denoted by g. At the inflow boundary, Dirichlet boundary conditions have to be prescribed. Finally, $\boldsymbol{n}$ is the outward pointing unit normal vector on Γ.

In practice, the convection-dominated regime $\varepsilon \ll L\|\boldsymbol{b}\|_{L^\infty(\Omega)}$, where L is a characteristic length scale of the problem, is of particular interest. In this regime, (weak) solutions of (1) possess typically layers, which are thin regions with very large gradients, e.g., compare Fig 1. Usually, available grids cannot resolve layers and it is well known that so-called stabilized discretizations have to be utilized [5]. One method of this type is the discontinuous Galerkin (DG) finite element method, whose stability was proved, e.g., in [4]. However, numerical solutions computed with the DG method still exhibit spurious oscillations in a vicinity of layers. In [3], several post-processing methods were investigated that reduce the size of spurious oscillations. This note addresses an open question formulated in [3] concerning the so-called *ConstJumpMod* method, which was the best performing method for triangular grids. In addition, a simplified variant of this method is proposed in this note. Numerical studies at two-dimensional problems show that both methods reduce the spurious oscillations of solutions from the standard DG method significantly.

2 DG method and slope limiting with *ConstJumpMod*

Equation (1) is transferred in a standard way to a weak formulation. Then, a finite element method with discontinuous basis functions is applied for computing a numerical approximation of the solution.

DOI 10.20347/WIAS.PREPRINT.2912 Berlin 2022

Since it is quite lengthy, it requires the introduction of many notations, and it is not important for the topic of this note, we like to refer to the literature for a detailed presentation of the DG method, e.g., see [3]. In our simulations, the DG method was applied with the same parameters as in [3] and numerical solutions are denoted by u_h.

The method *ConstJumpMod* is a modification of a post-processing technique proposed in [1, 2], which is based on estimating the order of local convergence. It is the best performing method on triangular grids and among the best methods on quadrilateral grids in the numerical studies of [3]. Based on computing an estimate of the order α_E for each facet of a mesh cell, cells where one value is sufficiently small, which indicates that the solution is not smooth, are selected. The DG solution in these cells is replaced by its integral mean value in order to reduce spurious oscillations, a process which is called slope limiting. In this note, two methods derived from *ConstJumpMod* are proposed and studied.

As already observed in [3], the definition of α_E in this paper is not scaling invariant. It might lead to meaningless values or might even not be defined at all. To address this difficulty, we study those definition of α_E from the point of view of physical units. Since α_E is a power, it has to be dimensionless, hence the involved numerator and denominator must have the same units. These considerations lead to the following new definition

$$\alpha_E := \ln\left(\frac{1}{C_0 L u_0^2}\int_E [\![u_h]\!]_E^2 \,\mathrm{d}s\right)\Big/\ln\left(\frac{h_E}{L}\right). \tag{2}$$

In (2), $C_0 \in \mathbb{R}$, $C_0 > 0$, is a constant, $L > h_E$ is a characteristic length scale of the problem, $u_0^2 \neq 0$ is a characteristic scale of the solution, h_E is the length of the facet E, and $[\![u_h]\!]_E$ denotes the jump of u_h across E. The parameters C_0, L, and u_0 have to be chosen by the user and have to be adapted to the problem. The former version of *ConstJumpMod* corresponds to the choices $L = 1\ \mathrm{m}$ and $u_0 = 1\ \mathrm{K}$ (if u is a temperature). A cell is marked if $\min_{E\in\mathcal{E}(K)} \alpha_E \leq \alpha_{\mathrm{ref}}$ for some $\alpha_{\mathrm{ref}} \in \mathbb{R}$, $\alpha_{\mathrm{ref}} > 0$, where $\mathcal{E}(K)$ is the set of facets of the mesh cell K. In the numerical examples this method is again called *ConstJumpMod*.

A close look at equation (2) reveals that the marking criterion is based essentially on the magnitude of the square of the integral of the jump. Therefore, with some $p \in [1,\infty]$, another idea consists in evaluating the $L^p(E)$-norm of the jump directly. Hence, for each interior facet

$$\beta_E := \left\|[\![u_h]\!]_E\right\|_{L^p(E)}\Big/|E|^{1/p} \tag{3}$$

can be computed, where $|E|$ is the area of E, and a cell is finally marked if $\max_{E\in\mathcal{E}(K)} \beta_E \geq \beta_{\mathrm{ref}}$ for some user-chosen $\beta_{\mathrm{ref}} \in \mathbb{R}$. In the numerical studies, this approach is called *ConstJumpNorm*.

3 Numerical studies

In the simulations, the results obtained with *ConstJumpMod* and *ConstJumpNorm* are compared with the standard DG method. For *ConstJumpMod*, $\alpha_{\mathrm{ref}} = 4$, $C_0 = 1$, and $u_0 = 1$ are chosen. The parameter β_{ref} in *ConstJumpNorm* is chosen to be the arithmetic mean of all β_E and the $L^\infty(E)$-norm is used, such that the denominator in (3) is one. The $L^1(E)$- and the $L^2(E)$-norm have been investigated as well, but significant differences concerning the results could not be observed. To approximate the $L^\infty(E)$-norm, u_h is evaluated at the vertices of the edges and at the quadrature points of a quadrature rule on the edge of degree $2r$. Furthermore, if $\beta_{\mathrm{ref}} \leq 10^{-13}$ *ConstJumpNorm* does

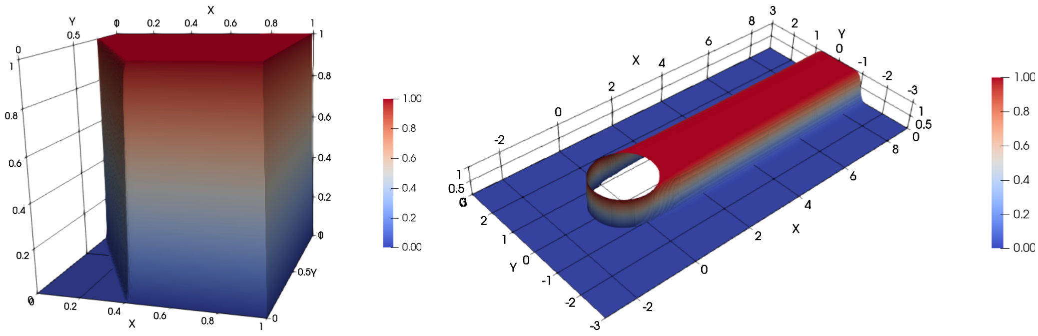

Figure 1: Solution of Example 1 (left) and Example 2 (right).

not mark any cell at all, to prevent limiting of continuous solutions. The simulations are performed with ParMooN [6] for the polynomial degrees $r = 1, 2, 3, 4$.

As in [3], the measures $\mathrm{osc}_{\mathrm{max}}(u_h)$ for the maximal size of spurious oscillations and $\mathrm{osc}_{\mathrm{mean}}(u_h)$ for a mean value of spurious oscillations are investigated. To compute these values, u_h is evaluated at the points of the nodal functionals defining continuous P_r/Q_r finite elements of the same order. The studies consider two standard test problems, whose description can be found in [3] and whose solutions are depicted in Fig. 1.

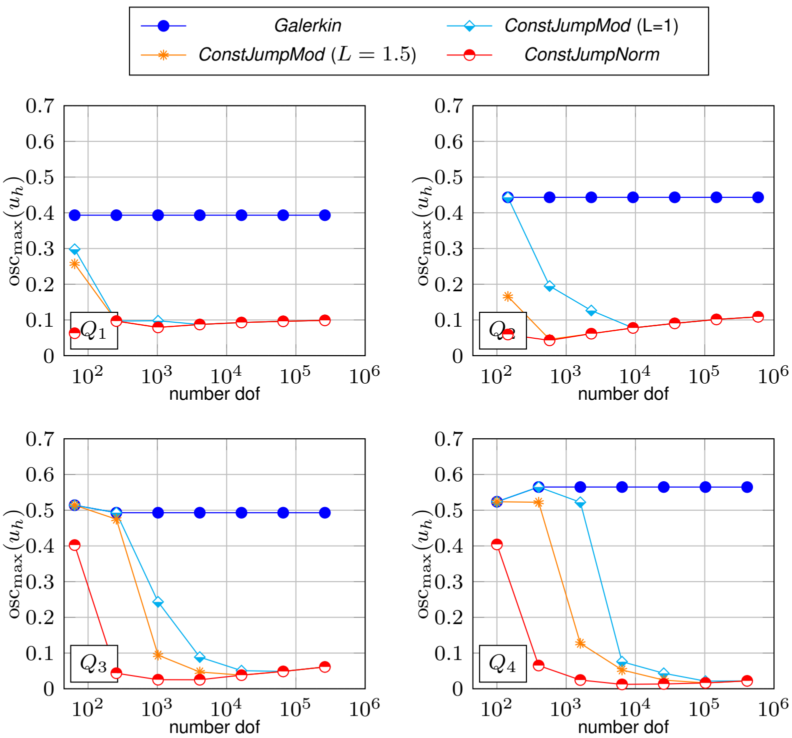

Figure 2: Results of $\mathrm{osc}_{\mathrm{max}}$ for Example 1.

Example 1. A strongly convection-dominated regime is chosen with $\varepsilon = 10^{-8}$. The characteristic length is set to $L = 1.5 > \mathrm{diam}(\Omega)$. For the sake of brevity, results are presented for quadrilateral grids, whose coarsest one consists of a single cell. For triangular grids, a qualitatively similar behavior can be observed.

The results in Figs. 2 and 3 show that all slope limiting techniques significantly reduce the spurious oscillations of *Galerkin*. On coarser grids *ConstJumpNorm* performs better than *ConstJump-*

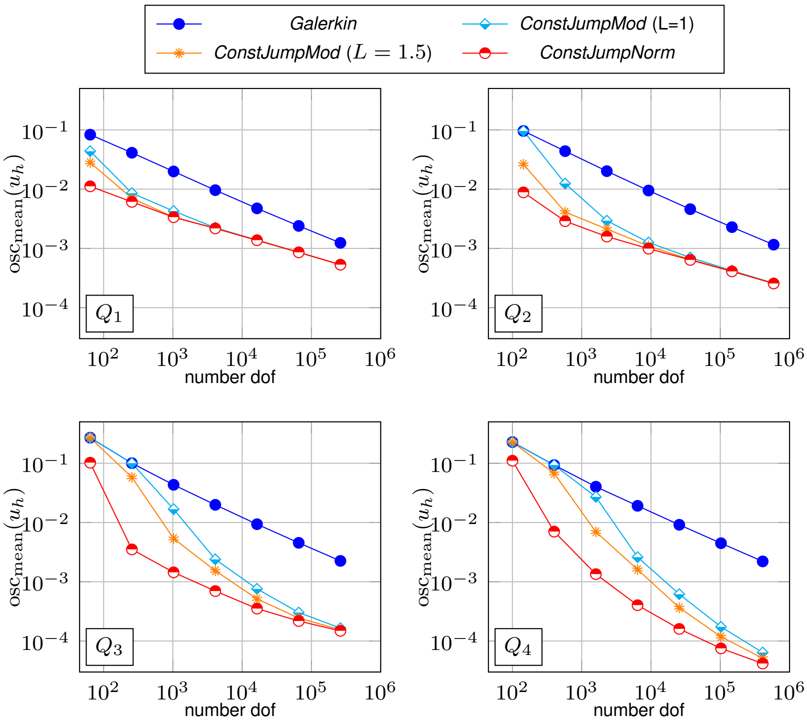

Figure 3: Results of $\mathrm{osc}_{\mathrm{mean}}$ for Example 1.

Mod ($L = 1.5$), especially for $r = 3, 4$. Usually, the results obtained with *ConstJumpMod* ($L = 1.5$) are notably better than those from *ConstJumpMod* (L=1), in particular on coarser grids for higher order elements. For $\mathrm{osc}_{\mathrm{max}}(u_h)$, the curves for all methods coincide on finer meshes and there is a slight increase the finer the mesh becomes.

Example 2. Again, $\varepsilon = 10^{-8}$ is chosen and the characteristic length is set to $L = 13.5 > \mathrm{diam}(\Omega)$. The results are presented for triangular grids, where the initial grid can be seen in [3, Fig. 16]. The results for both quantities of interest are shown in Figs. 4 and 5.

As for Example 1, *ConstJumpMod* ($L = 13.5$) and *ConstJumpNorm* are able to reduce spurious oscillations significantly compared to *Galerkin*. *ConstJumpMod* ($L = 13.5$) performs considerably better than the version *ConstJumpMod* (L=1) used in [3]. Concerning $\mathrm{osc}_{\mathrm{max}}(u_h)$, the values for *ConstJumpMod* ($L = 13.5$) and *ConstJumpNorm* increase marginally if the grids become finer. *ConstJumpMod* ($L = 13.5$) shows a little smaller values for $\mathrm{osc}_{\mathrm{mean}}(u_h)$ on coarser grids than *ConstJumpNorm*, but on finer grids both methods behave similarly.

4 Summary and outlook

A generalization of the slope limiting technique *ConstJumpMod* from [3] and a novel method inspired by this technique have been proposed and studied. Both led to similar results and they significantly reduced spurious oscillations compared to the standard DG method. In particular, such oscillations were reduced often considerably better than with the original method from [3]. Future studies on the impact of the methods' parameters on the numerical solutions are planned.

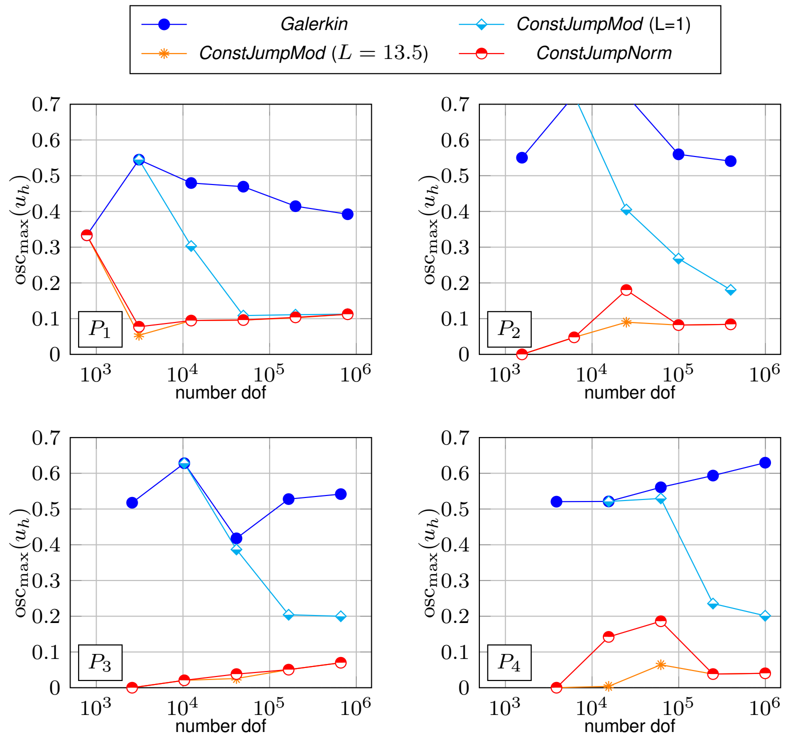

Figure 4: Results of $\mathrm{osc}_{\max}$ for Example 2.

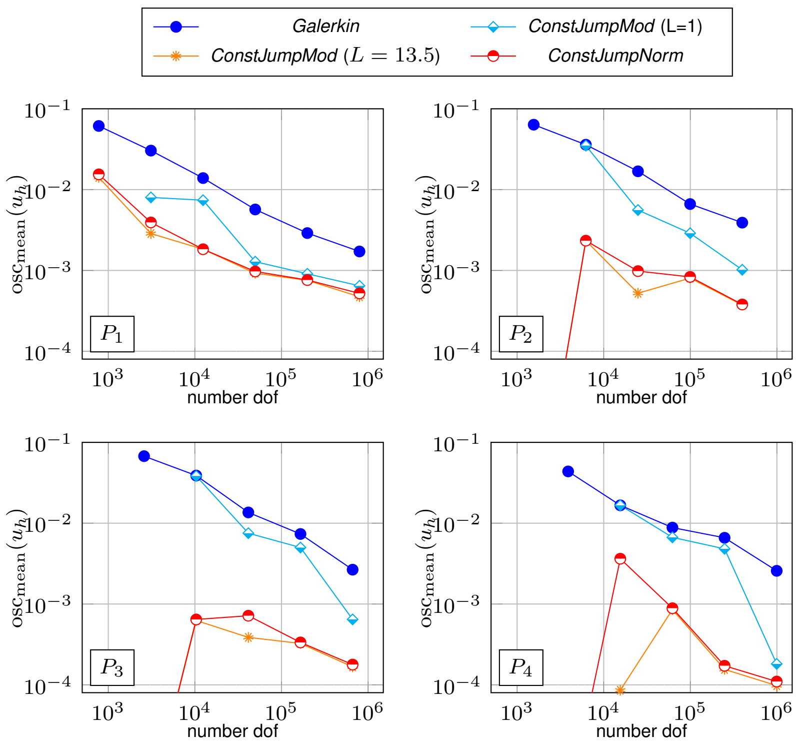

Figure 5: Results of $\mathrm{osc}_{\mathrm{mean}}$ for Example 2.

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