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Urn Models & Operator Ordering Procedures

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Abstract

Ordering of operators is purely combinatorial task involving a number of commutators shuffling components of operator expression to desired form. Here we show how it can be illustrated by simple urn models in which normal ordering procedure is equivalent to enumeration of urn histories.

1 Introduction

Operator algebras constitute mathematical framework within which many modern theories are built. Probably the most spectacular one is Quantum Mechanics with operator formalism at the very heart of the theory [1, 2]. The most unexpected, yet unavoidable, characteristic that makes it so strange and successful at the same time is non-commutativity. It has been realized very early since the advent of the theory non-commutativity has many important consequences [3], such as Bose-Einstein condensation, superconductivity, photon correlations, etc. At the same time, the new quality caused by the fact that the order of operators is relevant stimulated development of novel methods capable of tracing the order of components in operator expressions.

A common realization of operator algebra in quantum theory is the occupation number representation in which the fundamental role is played by the annihilation a and creation $a^\dagger$ operators acting in the infinite dimensional Hilbert space spanned by vectors $|n\rangle$ labeling some characteristic of a system $N|n\rangle = n|n\rangle$. Operators a and $a^\dagger$ are interpreted as the operations shifting these characteristic by one, which is embodied in the algebraic relations $[a, N] = a$ and $[a^\dagger, N] = -a^\dagger$. Conventionally, they are required to satisfy the canonical Heisenberg-Weyl commutation relation

$$[a, a^\dagger] = 1\,, \tag{1}$$

being the hallmark of non-commutativity in Quantum Theory. This causes ambiguities in the representation of an operator as an expression in a and $a^\dagger$, *e.g.* $aa^\dagger = a^\dagger a + 1$, and one needs to standardize the notation by fixing the preferred order of operators. An important practical example of operator ordering is the normally ordered form in which all annihilation operators stand to the right of the creation operators. Shuffling operators into this form comes down to a number of commutations of type Eq. (1) which, in general, is a highly nontrivial problem of genuine combinatorial origin [4, 5].

Here we construct an elementary urn model illustrating the normal ordering procedure for operators satisfying Eq. (1). It's convenience comes from intuitive concept of enumeration of urn histories providing a simple picture of this, after all, abstract mathematical construction. We also develop an elegant resolution to the problem based on generating functions methodology [6]. The primary interest of this paper is attached to intriguing analogy between combinatorial urn models used in description of various discrete phenomena [7, 8] and ordering procedures forced by operator formalism of quantum physics.

2 Urn model

2.1 Urns and processes

We consider here *urns* $\mathcal{U}$ containing balls which are strictly distinguishable between each other, *e.g.* numbered with different integers. For short, $\mathcal{U}_n$ denotes urn with n balls. One can modify contents of the urn by *elementary operations* of removing a ball out or adding one into the urn, see Fig. 1. We assume here that these operations are done one-by-one, *i.e.* only one ball is taken out or put in at a time, and denote them by $\mathcal{D}$ and $\mathcal{X}$ respectively. A *basic process* is just the composition of the elementary ones, *e.g.* $\mathcal{X}^2\mathcal{D}^3\mathcal{X}^4\mathcal{D} \equiv \mathcal{X}\mathcal{X}\mathcal{D}\mathcal{D}\mathcal{D}\mathcal{X}\mathcal{X}\mathcal{X}\mathcal{X}\mathcal{D}$ represents composite process meaning: "take a ball out, put four balls in, take three balls out, and then put two balls in". Each process may be realized in many ways since there are different choices of the balls in the urn. Note also that the order in which elementary operations are made is crucial for the number of possible histories that may occur – there is one more possibility for $\mathcal{D}\mathcal{X}$ than for $\mathcal{X}\mathcal{D}$ – being a sign of non-commutativity in the model.

We shall further need to capture indeterminacy of the process itself. In other words, we assume that each time it is chosen at random from a given repertoire denoted by $\mathcal{H}$. To account for different probabilities with which basic processes may occur we allow for their copies in $\mathcal{H}$. The numbers h_k counting copies of the same basic process $\mathcal{H}_k$ in $\mathcal{H}$ describe their relative probabilities, shortly denoted as $\mathcal{H} = \sum_k h_k\,\mathcal{H}_k$. For example, in process $\mathcal{H} = 2\,\mathcal{X}^3D + 5\,\mathcal{X}\mathcal{D}\mathcal{X}$ probability that occurs $\mathcal{X}^3\mathcal{D}$ or $\mathcal{X}\mathcal{D}\mathcal{X}$ is as $2:5$.

2.2 Urn histories

Clearly, applying process $\mathcal{H}$ to urn $\mathcal{U}$ (possibly many times) one ends with outcome which has some *history* behind, *i.e.* the record of events which happened in a series of steps. We shall be interested in enumerating all possible histories with given input and output, *i.e.*

$$G^{(n)}_{l\to k} \;:=\; \left\{ \begin{array}{c} \text{number of histories} \\ \text{in } n \text{ steps from urn } \mathcal{U}_l \text{ to } \mathcal{U}_k \end{array} \right\}. \qquad (2)$$

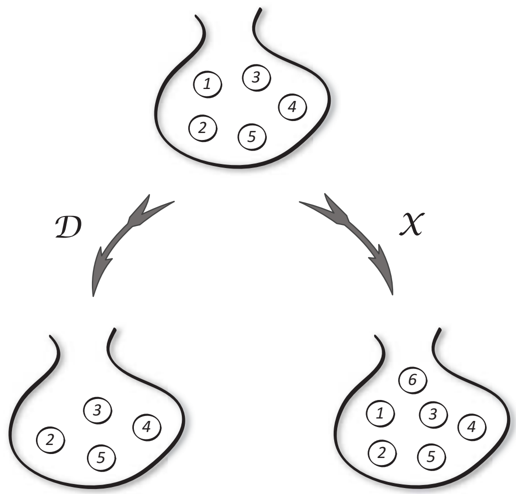

Figure 1: Urn containing 5 balls and elementary operations of removal $\mathcal{D}$ and addition $\mathcal{X}$ of a ball. There are 5 possibilities to take a ball out, and only 1 to put a ball in.

Enumeration of histories is a nontrivial task, especially if one needs to do it for general n. An elegant and efficient way of sorting and tackling information about sequences is attained through their generating functions. Hence, for each n we define the multivatiate generating functions

$$G^{(n)}(x,y) = \sum_{k,l} G^{(n)}_{l\to k}\; x^k \frac{y^l}{l!}, \tag{3}$$

and the exponential generating function

$$G(x,y,z) = \sum_{n} G^{(n)}(x,y)\; \frac{z^n}{n!}. \tag{4}$$

In practice one often faces the problem of explicit calculation or at least studying properties of these objects. A typical issue addressed in this context concerns finding distribution of probabilities $\mathbb{P}^{(n)}_{l\to k}$ of ending with urn $\mathcal{U}_k$ if started from $\mathcal{U}_l$ as the results of n iterations of a given process $\mathcal{H}$, defined by

$$\mathbb{P}^{(n)}_{l\to k} \;=\; \frac{G^{(n)}_{l\to k}}{\sum_k G^{(n)}_{l\to k}}. \tag{5}$$

Probability generating function is then simply expressed trough $G(x,y,z)$ as

$$\sum_{k,l} \mathbb{P}^{(n)}_{l\to k}\, x^k \frac{y^l}{l!} = \frac{[z^n]G(x,y,z)}{[z^n]G(1,y,z)} \tag{6}$$

$$= \frac{\partial_z^n G(x,y,z)|_{z=0}}{\partial_z^n G(1,y,z)|_{z=0}}, \tag{7}$$

and all statistical properties, such as moments, asymptotic, etc., can be conveniently studied with the methods of generating functions [6].

In Section 3 we shall provide a simple scheme of calculating generating functions of Eqs. (3) and (4) deriving from the operator ordering methodology.

2.3 Operator representation

This simple urn model can be conveniently described in terms of polynomials and differential operators. Let us represent an urn $\mathcal{U}_n$ containing n balls by monomial x^n, and elementary operations $\mathcal{X}$ and $\mathcal{D}$ by multiplication X and derivative D operators respectively, *i.e.*

$$\begin{array}{ccc} \mathcal{U}_n & \longleftrightarrow & x^n, \\ \mathcal{D} & \longleftrightarrow & D, \\ \mathcal{X} & \longleftrightarrow & X. \end{array}$$

Observe that acting with thus obtained representation of a basic process on x^n we get monomial corresponding to the resulting urn multiplied by the number enumerating all possible histories in which the process could have occurred, *e.g.* $X^2D^3X^3D\ x^n = n(n+2)(n+1)n\ x^{n+1}$. Following up this remark one finds out that correspondence

$$\mathcal{H} \quad \longleftrightarrow \quad H(X,D)$$

holds true for any process $\mathcal{H}$. Accordingly, applying operator $H(X,D)$ to x^n we get polynomial indicating possible results with coefficients enumerating all histories in which the process could have occurred, *e.g.* for $\mathcal{H} = 2\,\mathcal{X}^3\mathcal{D} + 5\,\mathcal{X}\mathcal{D}\mathcal{D}\mathcal{X}$, one gets $H(X,D)\,x^n \equiv (X^3D + 5\,XDDX)\,x^n = 2n\ x^{n+2} + 5(n+1)n\ x^n$.

These observations are a simple consequence of intentional choice of the representation of urns by monomials and basic processes by multiplication and derivation operators respectively, dictated by the relations

$$\begin{aligned} D\ x^n &= n\ x^{n-1}, \\ X\ x^n &= x^{n+1}, \end{aligned} \tag{8}$$

and reflecting the facts that

- there are n ways of removing a ball from urn containing n distinguishable balls,
- a ball can be added to any urn only in *one* way.

This gives surprisingly simple combinatorial insight into the commutator

$$[D, X] = 1. \tag{9}$$

Thus established correspondence translating model of urn processes into the language of polynomials and differential operators will establish equivalence with operator ordering procedures. We will find it useful in enumerating urn histories in Section 3.3.

3 Normal ordering procedure

3.1 Normal order

Motivation for operator ordering procedures has strong grounds in quantum physics where reshuffling of operator expressions to desired form introduces both calculational and interpretative advantages. A typical example is the occupation number representation where the annihilation a and creation $a^\dagger$ operators, satisfying the relation of Eq. (1), are best handled if all annihilators stand to the right of creators. For a gentle introduction to the normal ordering methods as well as physical motivation see [5].

For the purpose at hand we observe that algebra of Eq. (1) may be realized in the space of polynomials (formal power series) as follows

$$\begin{aligned} a^\dagger &\longleftrightarrow X \\ a &\longleftrightarrow D \end{aligned} \tag{10}$$

where X and $D \equiv \partial_x$ are the multiplication and derivative operators defined in Eq. (8). We note that choice of this representation is not accidental as we are interested in combinatorics underpinning the Heisenberg-Weyl commutation relation of Eqs. (1) or (9).

Accordingly, we standardize the notation of any operator H given as expression in operators X and D to its *normally ordered* form in which all multiplication operators X stand to the left of derivative operators D. Clearly, each H can be unambiguously put in this form, *i.e.*

$$H(X, D) = \sum_{k,l \geq 0} h_{kl}\ X^k D^l. \tag{11}$$

By the *normal ordering* of an operator we mean procedure of moving all the derivatives D to the right using commutation relation of Eq. (9). Below we shall be interested in normal ordering of powers and exponential of a given operator.

3.2 Formal resolution

Suppose we are given operator $H(X, D)$ and look for the normally ordered form of its n-th power, *i.e.*

$$(H(X, D))^n = \sum_{k,l \geq 0} h_{kl}^{(n)}\ X^k D^l. \tag{12}$$

(If $H(X, D)$ is as in Eq. (11) one has $h_{kl}^{(0)} = \delta_{k0}\delta_{0l}$ and $h_{kl}^{(1)} = h_{kl}$.)
The normally ordered form of operator in Eq.(12) can be conveniently encoded into polynomial

$$B_n(x, y) = \sum_{k,l \geq 0} h_{kl}^{(n)}\ x^k y^l, \tag{13}$$

Observe that it satisfies the identity

$$B_n(x, y) = e^{-xy} H(X, D)^n e^{xy}. \tag{14}$$

Now we are ready to derive the recurrence for the polynomials $B_n(x, y)$. By Eq. (14) we have

$$B_{n+1}(x, y) \quad = \quad e^{-xy} H(X, D) e^{xy} B_n(x, y),$$

and using property $D^l e^{xy} = e^{xy}(D+y)^l$, we arrive at the recurrence

$$B_{n+1}(x,y) = H(X, D+y)\, B_n(x,y). \tag{15}$$

The exponential generating function of polynomials $B_n(x,y)$ is defined by

$$B(x,y,\lambda) = \sum_{n=0}^{\infty} B_n(x,y)\frac{\lambda^n}{n!}. \tag{16}$$

Note that, similarly to Eq. (14), one has

$$B(x,y,\lambda) = e^{-xy} e^{\lambda H(X,D)} e^{xy}. \tag{17}$$

Differentiating $B(x,y,\lambda)$ with respect to λ yields

$$\begin{aligned} \partial_\lambda\, B(x,y,\lambda) &= \sum_{n=0}^{\infty} B_{n+1}(x,y)\frac{\lambda^n}{n!} \\ &= \sum_{n=0}^{\infty} H(X, D+y)\, B_n(x,y)\frac{\lambda^n}{n!}, \end{aligned}$$

which gives the following partial differential equation

$$\partial_\lambda\, B(x,y,\lambda) = H(X, D+y)\, B(x,y,\lambda), \tag{18}$$

with initial condition $B(x,y,0) = 1$.

In this way we have obtained formal resolution to the problem. Observe that the right hand sides of equations

$$(H(X,D))^n = B_n(X,D), \tag{19}$$
$$e^{\lambda H(X,D)} = B(X,D,\lambda) \tag{20}$$

are in the normally ordered form. We note that in many cases Eq. (18) can be solved providing explicit analytic formulas.

3.3 Enumeration of urn histories

Here we will show utility of ordering methods in calculating generating function of urn histories of Eqs. (3) and (4). Recall from Section 2.3 that for a given process $\mathcal{H}$ we have

$$H^n x^l = \sum_k G^{(n)}_{l\to k}\, x^k, \tag{21}$$

which if multiplied by $y^l/l!$ and summed over l yields generating functions of urn histories

$$G^{(n)}(x,y) = H^n \, e^{xy} \tag{22}$$

and

$$G(x,y,z) = e^{zH} \, e^{xy}. \tag{23}$$

For efficient use of Eqs. (22) and (23) one needs to know the action of operators $(H(X,D))^n$ and $e^{zH(X,D)}$ on functions (in this particular case on e^{xy}). It becomes trivial when the normally ordered form of operator is known. Hence, from Eqs. (19) and (20) we have

$$G^{(n)}(x,y) = B_n(x,y) \, e^{xy} \tag{24}$$

and

$$G(x,y,z) = B(x,y,z) \, e^{xy} \tag{25}$$

proving that calculation of both functions $G(x,y,z)$ and $B(x,y,z)$ is equivalent. In this way we have shown the one-to-one correspondence between the problems of enumeration of urn histories of iterated process $\mathcal{H}$ and normal ordering of powers of the associated operator $H(X,D)$. From one side ordering methods provide efficient tools capable of tracing of urn histories, and vice versa – the urn model illustrates the procedure itself.

For illustration consider urn process $\mathcal{H} = \mathcal{X}\mathcal{D} + g\,\mathcal{X} + g\,\mathcal{D}$. It corresponds to the Hamiltonian $H = a^\dagger a + g\,(a^\dagger + a)$ describing oscillator driven by external force – a simple model of a system coupled to the environment. Particularly interesting is the picture of dynamics generated by H in terms of urn histories. It translates into the process $\mathcal{H}$ in which at each step one ball can be either inspected, added or removed with relative probabilities 1, g and g respectively. Inspecting $\mathcal{X}\mathcal{D}$ means just drawing a ball, looking at and then putting it back into the urn – hence in no way affecting number of the balls in the urn – representing free (undisturbed) evolution of a system. The remaining two terms introduce disturbance into the scheme by random addition $\mathcal{X}$ or removal $\mathcal{D}$ of a ball, interpreted as the effect of external factor. Consequently, number of balls in the urn may change and urn histories proliferate as conveniently described by generating function of Eq. (4)

$$G(x,y,z) = e^{(x+g)(y+g)(e^z-1)} e^{-g^2 z} e^{xy}, \tag{26}$$

obtained from Eqs. (18) and (25). Statistical properties of the model are straightforward application of combinatorial analysis [6].

4 Outlook

We have pointed out that abstract Heisenberg-Weyl commutation relation have purely combinatorial underpinning. It allowed for construction of elementary urn model illustrating the operator normal ordering procedure which was proved equivalent to enumeration of urn histories. In this way, these both seemingly unrelated procedures gain new perspective and may draw on methods taken one from another. From one side combinatorial enumeration of histories is facilitated if the normally ordered form of involved operators is known. On the other hand abstract mathematical construction of operator ordering gains straightforward interpretation as enumeration of histories in simple combinatorial models. The latter particularly interesting in description of quantum systems, and pointing at efficiency of combinatorial methods based on generating functions approach.

Anticipating further analogies we observe that multi-mode systems can be modeled by urns containing balls of different type, *e.g.* each mode having different colour. Mixing terms in the Hamiltonian, introducing entanglement into quantum system, may be then interpreted as processes swapping between different types of balls in the urn. We expect that at least some of strange quantum phenomena can be made more intuitive if looked from this simple combinatorial perspective.

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