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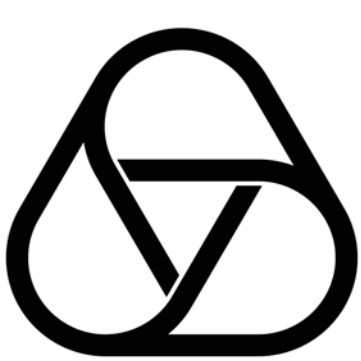

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KATHRIN BRINGMANN AND BERNHARD HEIM

Hecke Duality Relations of Jacobi Forms

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HECKE DUALITY RELATIONS OF JACOBI FORMS

KATHRIN BRINGMANN AND BERNHARD HEIM

ABSTRACT. In this paper we introduce a new subspace of Jacobi forms of higher degree via certain relations among Fourier coefficients. We prove that this space can also be characterized by duality properties of certain distinguished embedded Hecke operators. We then show that this space is Hecke invariant with respect to all good Hecke operators. As explicit examples we give Eisenstein series. Conversely we show the existence of forms that are not contained in this space.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper we prove the existence of a non-trivial Hecke invariant proper subspace of the space of Jacobi forms on $\mathbb{H}_2 \times \mathbb{C}^2$ which satisfies Hecke duality relations.

Thirty years ago Saito und Kurokawa [11] conjectured the existence of a distinguished subspace of M_k^2, the vector space of Siegel modular forms of degree 2 and weight $k \in \mathbb{N}$. From the degeneration of the spinor L-function of Hecke eigenforms they conjectured that there exists a correspondence to a space of elliptic modular forms. At the same time, Maass studied the Fourier coefficients of Eisenstein series of degree 2 and discovered interesting relations among them. He introduced the Spezialschar [12, 13, 14], which is a subspace of Siegel modular forms of degree 2 that is defined via certain relations among Fourier coefficients. A Siegel modular form $F \in M_k^2$ is in the *Maass Spezialschar* if for all positive definite half-integral 2×2 matrices T the Fourier coefficients $A(T)$ of F satisfy the relation

$$A\big([n,r,m]\big) = \sum_{d|(n,r,m)} d^{k-1} A\left(\left[\frac{nm}{d^2}, \frac{r}{d}, 1\right]\right), \tag{1.1}$$

where we identify $T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$ with the quadratic form $[n,r,m]$. Relation (1.1) is equivalent to the fact that $A(T)$ only depends on the discriminant of T and the greatest common divisor of its entries. Explicit examples for elements in the Spezialschar are given by Eisenstein series. Moreover the Maass Spezialschar is Hecke invariant and provides an example of forms that do not satisfy an analogue of the Ramanujan-Petersson conjecture in higher dimensions [19].

At the end of the last century Duke and Imamoglu generalized the Saito-Kurokawa conjecture to higher degree. Let M_k^n be the space of Siegel modular forms of weight k and degree $n \in \mathbb{N}$. Duke and Imamoglu conjectured that if $n + k \equiv 0 \pmod 2$, then there exists a Hecke invariant isomorphism between the space of the elliptic forms of weight $2k$ and a subspace of M_{k+n}^{2n}. Note that M_{k+n}^{2n} has even degree. In 2001 this conjecture was proven by Ikeda [8] and subsequently Kohnen and Kojima [10] gave a linearization of this lift. Recently the second author characterized these lift via Hecke operators [7].

It would be interesting to construct lifts and formulas also in the case of Siegel modular forms of odd degree. Here we make a first step towards this goal and investigate Jacobi forms of degree 2 since these arise for example as Fourier Jacobi coefficients of Siegel modular forms of degree 3. Let $\Phi \in J_{k,m}^2$ (the vector space of Jacobi forms of weight k, degree 2, and index m) with Fourier

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coefficients $C(N,R)$. We use the parametrization $N = \left(\begin{smallmatrix} n_{11} & \frac{n_{12}}{2} \\ \frac{n_{12}}{2} & n_{22} \end{smallmatrix}\right)$ and $R = (r_1, r_2)$, and define the invariants:

$$D_1 := -4 \cdot \det \begin{pmatrix} n_{11} & \frac{r_1}{2} \\ \frac{r_1}{2} & m \end{pmatrix}, \qquad D_2 := -4 \cdot \det \begin{pmatrix} n_{22} & \frac{r_2}{2} \\ \frac{r_2}{2} & m \end{pmatrix}, \qquad D := -4 \cdot \det \begin{pmatrix} \frac{n_{12}}{2} & \frac{r_1}{2} \\ \frac{r_2}{2} & m \end{pmatrix}. \tag{1.2}$$

It will turn out that $C(N,R)$ only depends upon D_1, D_2, D, and r_1 and r_2 modulo $2m$. Therefore the following is well-defined:

$$C_{r_1,r_2}(D_1, D_2, D) := C(N,R), \tag{1.3}$$

where r_1 and r_2 are defined modulo $2m$. Moreover if m is either 1 or a prime, then $C(N,R)$ only depends upon D_1, D_2, and D. In this case we set

$$C(D_1, D_2, D) := C(N,R). \tag{1.4}$$

Next define the space $\mathbb{E}_{k,m}$ of "distinguished" Jacobi forms.

Definition (Hecke duality relation). *Let $\Phi \in J^2_{k,m}$ with Fourier coefficients $C_{r_1,r_2}(D_1,D_2,D)$ with $D_1, D_2, D, r_1, r_2 \in \mathbb{Z}$. Let p be a prime with $(p, 2m) = 1$. Then Φ is an element of $\mathbb{E}^{(p)}_{k,m}$ if the following relation is satisfied:*

$$\begin{aligned} \left(\chi_{D_1}(p) - \chi_{D_2}(p)\right) C_{r_1,r_2}(D_1,D_2,D) \;\; = \\ p^{2-k}\left(C_{r_1,pr_2}\left(D_1, D_2p^2, Dp\right) - C_{pr_1,r_2}\left(D_1p^2, D_2, Dp\right)\right) \\ +p^{k-1}\left(C_{r_1,\bar{p}r_2}\left(D_1, \frac{D_2}{p^2}, \frac{D}{p}\right) - C_{\bar{p}r_1,r_2}\left(\frac{D_1}{p^2}, D_2, \frac{D}{p}\right)\right). \end{aligned} \tag{1.5}$$

Here $p\bar{p} \equiv 1 \pmod{2m}$ and $\chi_* := \left(\frac{*}{p}\right)$. Furthermore we define

$$\mathbb{E}_{k,m} := \bigcap_{\substack{p \text{ prime} \\ (p,2m)=1}} \mathbb{E}^{(p)}_{k,m}. \tag{1.6}$$

Since we show that this relation is equivalent to a property involving Hecke Jacobi operators, we refer to it as *Hecke duality relation*. For $m = 1$ it is known by work of Ibukiyama [9] that the space $J^2_{k,1}$ is isomorphic to a space of Siegel modular forms of degree 2 and half-integral weight. In this case he conjectured the existence of a certain distinguished subspace which seems to be different from the space considered here. It would be interesting to determine the connection between those.

Theorem 1.1. *Assume that $\Phi \in J^2_{k,m}$ and that p is a prime with $(p, 2m) = 1$. Then the following two conditions are equivalent:*

1. *The function Φ is an element of $\mathbb{E}^{(p)}_{k,m}$.*
2. *We have $\Phi | \left(T^J(p)^{\uparrow} - T^J(p)^{\downarrow}\right) = 0$.*

Here $T^J(p)^{\uparrow}$ and $T^J(p)^{\downarrow}$ are two canonical Hecke Jacobi operators obtained by embedding the classical Hecke Jacobi operator $T^J(p)$ in two different ways (see Section 3.2).

The spaces $\mathbb{E}^{(p)}_{k,m}$ are invariant with respect to all "good" Hecke operators.

Theorem 1.2. *Assume that p and q are distinct primes with $(pq, 2m) = 1$. Let $\mathcal{H}^{J,2}_p$ be the local Hecke Jacobi algebra of degree 2 and define $\mathcal{H}^{J,2} := \otimes_{(p,2mq)=1}\mathcal{H}^{J,2}_p$. If m is either 1 or a prime, then $\mathbb{E}^{(q)}_{k,m}$ is invariant with respect to $\mathcal{H}^{J,2}$.*

Next we prove that Jacobi Eisenstein series $E^{J,2}_{k,m}$ (see Section 6) are contained in $\mathbb{E}_{k,m}$. As a by-product we show a new decomposition of Eisenstein series.

Theorem 1.3. *Assume that m is square-free. Then we have for all primes p*

$$E^{J,2}_{k,m} \mid \left(T^J(p)^{\uparrow} - T^J(p)^{\downarrow}\right) = 0. \tag{1.7}$$

If m is arbitrary, then (1.7) is also true if $(p,m) = 1$.

To show that $\mathbb{E}_{k,m}$ is not in general equal to the whole space of Jacobi forms, we give in the last section explicit examples which do not satisfy the Hecke duality relations. In particular if $k \geq 10$ is even, then we have $\mathbb{E}_{k,1} \subsetneqq J^2_{k,1}$.

2. Basic facts about Automorphic forms

In this section we recall some basic facts about automorphic forms with respect to the symplectic and to the Jacobi group. Throughout we let R be a commutative ring. The symplectic group $Sp_n(R)$ acts on the Siegel upper half-space $\mathbb{H}_n$ by

$$g \circ \mathcal{T} := (A\mathcal{T} + B)(C\mathcal{T} + D)^{-1}.$$

If $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp_n(R)$, $k \in \mathbb{N}_0$, and Φ is a complex-valued function on $\mathbb{H}_n$, then define

$$F|_k g(\mathcal{T}) := (\det J(g,\mathcal{T}))^{-k} F(g \circ \mathcal{T}),$$

where $J(g,\mathcal{T}) := (C\mathcal{T} + D)$. We let M^n_k be the vector space of Siegel modular forms of weight k and degree n with respect to the Siegel modular group $\Gamma_n := Sp_n(\mathbb{Z})$, i.e., the space of holomorphic functions $F : \mathbb{H}_n \to \mathbb{C}$ that satisfy $F|_k g = F$ for all $g \in \Gamma_n$ and that have a Fourier expansion

$$F(\mathcal{T}) = \sum_S A(S)\ e^{2\pi i \mathrm{tr}\,(S\mathcal{T})},$$

where S runs through the set of half-integral semi-definite matrices.

We next consider Jacobi forms. The Heisenberg group of degree n

$$H_n(R) := \{(\lambda,\mu,\kappa) \mid \lambda,\mu \in R^n \text{ and } \kappa \in R\}$$

has the group law

$$(\lambda_1,\mu_1,\kappa_1) + (\lambda_2,\mu_2,\kappa_2) = \left(\lambda_1+\lambda_2, \mu_1+\mu_2, \kappa_1+\kappa_2+\lambda_1\mu_2^t - \mu_1\lambda_2^t\right).$$

Define the Jacobi group $G^J_n(R) := Sp_n(R) \ltimes H_n(R)$. This group can be viewed as a subgroup of $Sp_{n+1}(R)$ via the embedding $\widehat{\ }: G^J_n(R) \longrightarrow Sp_{n+1}(R)$, where $\gamma = (g,h) = \left(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), (\lambda,\mu,\kappa)\right)$ maps to

$$\widehat{\gamma} := \begin{pmatrix} A & 0 & B & \mu' \\ \lambda & 1 & \mu & \kappa \\ C & 0 & D & -\lambda' \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here λ', μ' are uniquely determined. Moreover we have $\widehat{\gamma} = \widehat{g}\cdot\widehat{h}$. The group $G^J_n(R)$ acts on $\mathbb{H}^J_n$ via

$$\gamma \circ (\mathcal{T}, Z) := \left(g \circ \mathcal{T}, (Z\ + \lambda\mathcal{T} + \mu)\, J^{-1}(g,\mathcal{T})\right).$$

We define the cocycle $J_{k,m}$ by

$$J_{k,m}\big((g,h),\mathcal{T},Z\big) := J_{k,m}\big(g, h\circ(\mathcal{T},Z)\big) \cdot J_{k,m}\big(h,(\mathcal{T},Z)\big),$$

where

$$\begin{aligned} J_{k,m}\big(g,(\mathcal{T},Z)\big) &:= \det\left(J(g,\mathcal{T})\ e\left(m\,\mathrm{tr}\left(\left(J^{-1}(g,\mathcal{T})C\right)[Z^t]\right)\right)\right), \\ J_{k,m}\left(h,(\mathcal{T},Z)\right) &:= e\left(-m\,\mathrm{tr}\left(\mathcal{T}[\lambda^t] + 2\lambda Z^t + \kappa + \mu\lambda^t\right)\right). \end{aligned}$$

Here $A[B] := B^t AB$ for matrices of suitable sizes and $e(x) := e^{2\pi i x}$ for $x \in \mathbb{C}$. For all $\gamma_2, \gamma_2 \in G_n^J(\mathbb{R})$, the cocyle $J_{k,m}$ has the property

$$J_{k,m}\big(\gamma_1\gamma_2, (\mathcal{T}, Z)\big) = J_{k,m}\big(\gamma_1, \gamma_2 \circ (\mathcal{T}, Z)\big)\ J_{k,m}\big(\gamma_2, (\mathcal{T}, Z)\big).$$

Define the Petersson slash operator $|_{k,m}$ for complex-valued functions F on $\mathbb{H}_n^J$ and $\gamma \in G_n^J(\mathbb{R})$ by

$$\Phi|_{k,m}\gamma(\mathcal{T}, Z) := J_{k,m}^{-1}(\gamma, (\mathcal{T}, Z))\ \ \Phi\left(\gamma \circ (\mathcal{T}, Z)\right).$$

For positive integers k, m, and n we let $J_{k,m}^n$ be the space of Jacobi forms of degree n, weight k, and index m, i.e., the space of complex-valued functions Φ on $\mathbb{H}_n^J$ that satisfy $\Phi|_{k,m}\gamma = \Phi$ for $\gamma \in \Gamma_n^J := G_n^J(\mathbb{Z})$ and that have a Fourier expansion of the form

$$\Phi\left(\mathcal{T}, Z\right) = \sum_{N,R} C(N,R)\ e^{2\pi i\left(\mathrm{tr}(N\mathcal{T}) + RZ^t\right)}.$$

Here the sum runs over all N and R such that $\left(\begin{smallmatrix} N & R^t \\ R & m \end{smallmatrix}\right)$ is a half-integral semi-definite matrix of size $n+1$. Examples of Jacobi forms are given by Fourier Jacobi coefficients of Siegel modular forms. If we write for $n > 1$ an element of $\mathbb{H}_n$ as $\left(\begin{smallmatrix} \mathcal{T} & Z^t \\ Z & \omega_0 \end{smallmatrix}\right)$ with $\omega_0 \in \mathbb{H}$, then $F \in M_k^n$ has a Fourier Jacobi expansion of the form

$$F\begin{pmatrix} \mathcal{T} & Z^t \\ Z & \omega_0 \end{pmatrix} = \sum_{m=0}^{\infty} \Phi_m^F\left(\mathcal{T}, Z\right)\ \ e(m\,\omega_0),$$

where $\Phi_m^F \in J_{k,m}^{n-1}$. We drop the index $n = 1$ to simplify notation. For $\Phi \in J_{k,m}^n$ we define

$$\widehat{\Phi}\begin{pmatrix} \mathcal{T} & Z^t \\ Z & \omega_0 \end{pmatrix} := \Phi\left(\mathcal{T}, Z\right)\ \ e(m\,\omega_0),$$

which has the property

$$\Phi|_{k,m}\gamma = e(-m\,\omega_0)\ \widehat{\Phi}|_k\widehat{\gamma}. \tag{2.1}$$

We next embed $Sp_n \times Sp_n$ into Sp_{2n} by

$$Sp_n \times Sp_n \to Sp_{2n}, \qquad \left(\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} A_2 & B_2 \\ C_2 & D_2 \end{smallmatrix}\right) \mapsto \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}$$

and identify this image with $Sp_n \times Sp_n$. Further we embed $G_n^J \times G_n^J$ into G_{2n}^J via

$$\Big((\lambda_1, \mu_1, \kappa_1), g_1\Big) \times \left((\lambda_2, \mu_2, \kappa_2), g_2\right) \mapsto \left(((\lambda_1, \lambda_2), (\mu_1, \mu_2), (\kappa_1 + \kappa_2), g_1 \times g_2)\right).$$

In the following we use the symbols $\gamma^{\uparrow}$ and $\widetilde{\gamma}^{\downarrow}$ to indicate the embeddings of $\gamma \times I_2$ and $I_2 \times \widetilde{\gamma}$.

3. Hecke Theory and the lifting operator

3.1. **The symplectic Hecke algebra.** Let us first consider the symplectic Hecke algebra $\mathcal{H}^n$ of the Hecke pair $(\Gamma_n, Sp_n(\mathbb{Q}))$ [1, 16] which decomposes as $\mathcal{H}^n = \otimes_p \mathcal{H}_p^n$. Here the local Hecke algebra $\mathcal{H}_p^n$ is generated by $\left(T_i^{(n)}(p)\right)_{(0 \le i \le n)}$, where

$$T_i^{(n)}(p) := p^{-1}\,\Gamma_n\,\mathrm{diag}\left(1, \ldots, 1, \underbrace{p, \ldots, p}_{i}; p^2, \ldots, p^2, \underbrace{p \ldots, p}_{i}\right)\Gamma_n.$$

We need an explicit left coset decomposition of the generators of the Hecke operators for $n = 1$ and $n = 2$. Since $T_n^{(n)}(p) = I_{2n}$, we can omit the case $i = n$.

If $n = 1$ we can choose as a Γ-left coset decomposition of $p\,T_0^{(1)}(p)$:

$$\Gamma\, N_1 + \sum_a \Gamma\, N_2(a) + \sum_b \Gamma\, N_3(b), \tag{3.1}$$

where a runs through $(\mathbb{Z}/p\mathbb{Z})^*$, b through $\left(\mathbb{Z}/p^2\mathbb{Z}\right)$, and where $N_1 := \left(\begin{smallmatrix} p^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $N_2(a) := \left(\begin{smallmatrix} p & a \\ 0 & p \end{smallmatrix}\right)$, and $N_3(b) := \left(\begin{smallmatrix} 1 & b \\ 0 & p^2 \end{smallmatrix}\right)$.

If $n = 2$ the generators of the Hecke algebra are given by:

$$T_0^{(2)}(p) = p^{-1}\Gamma_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix} \Gamma_2, \qquad T_1^{(2)}(p) = p^{-1}\Gamma_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \Gamma_2, \qquad T_2^{(2)}(p) = \Gamma_2.$$

Representatives can be choosen of the form $p^{-1}\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in Sp_2(\mathbb{Q})$, with $D = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$, $A = p^2(D^t)^{-1}$, and $D^t\,B = B^t\,D$ (see [1]). Then D runs through all Γ-left cosets of $\Gamma\backslash\left(\bigcup_{j=1}^6 \Gamma\mathcal{D}_j\Gamma\right)$ with

$$\mathcal{D}_1 = I_2, \quad \mathcal{D}_2 = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \quad \mathcal{D}_3 = pI_2, \quad \mathcal{D}_4 = \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}, \quad \mathcal{D}_5 = p\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \quad \mathcal{D}_6 = p^2 I_2.$$

Each of the double cosets related to $\mathcal{D}_1, \mathcal{D}_3$, and $\mathcal{D}_6$ decompose into one left coset. ¿From the decomposition of $\Gamma\backslash\Gamma\mathcal{D}_4\Gamma$ in (3.1) and the identity $p\mathcal{D}_2 = \mathcal{D}_5$, it is sufficient to consider

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma = \Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + \sum_{a \pmod p} \Gamma \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}. \tag{3.2}$$

Next we calculate the corresponding representatives $M \in Sp_2(\mathbb{Q})$. If a representative $D \in \mathrm{Mat}(2,\mathbb{Z})$ is fixed, then $B \in \mathrm{Mat}(2,\mathbb{Z})$ runs through a set of representatives modulo D, i.e., B satisfies $D^t\,B = B^t\,D$ and the congruence relation $\sim$. Here $B_1 \sim B_2$ if and only if $(B_1 - B_2)D^{-1} \in Mat(2,\mathbb{Z})$. Using the algorithm given in [18] we obtain the following representatives:

$$M_1(x,y,z) := \begin{pmatrix} p^{-1} & 0 & \frac{y}{p} & \frac{x}{p} \\ 0 & p^{-1} & \frac{x}{p} & \frac{z}{p} \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}_{x,y,z \,(\mathrm{mod}\, p^2)} \qquad M_2(s,x) := \begin{pmatrix} p^{-1} & 0 & \frac{s}{p} & \frac{x}{p} \\ 0 & 1 & x & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\substack{s \,(\mathrm{mod}\, p^2) \\ x \,(\mathrm{mod}\, p)}}$$

$$M_3 := \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p^{-1} & 0 \\ 0 & 0 & 0 & p^{-1} \end{pmatrix} \qquad M_4(a) := \begin{pmatrix} p^{-1} & 0 & \frac{a}{p} & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p^{-1} \end{pmatrix}_{a \,(\mathrm{mod}\, p^2)}$$

$$M_5 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p^{-1} \end{pmatrix} \qquad M_6(a,b) := \begin{pmatrix} p & 0 & 0 & 0 \\ -a & 1 & 0 & \frac{b}{p} \\ 0 & 0 & p^{-1} & \frac{a}{p} \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\substack{a \,(\mathrm{mod}\, p) \\ b \,(\mathrm{mod}\, p)*}}$$

$$M_7(a,b,c) := \begin{pmatrix} 1 & 0 & 0 & \frac{c}{p} \\ -\frac{a}{p} & 1 & \frac{c}{p} & \frac{b}{p} \\ 0 & 0 & 1 & \frac{a}{p} \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\substack{a,b,c \,(\mathrm{mod}\, p) \\ a \not\equiv 0 \,(mod\, p)}} \qquad M_8(a,x) := \begin{pmatrix} p & 0 & 0 & 0 \\ -\frac{a}{p} & \frac{1}{p} & 0 & \frac{x}{p} \\ 0 & 0 & p^{-1} & \frac{a}{p} \\ 0 & 0 & 0 & p \end{pmatrix}_{a,x \,(\mathrm{mod}\, p^2)}$$

$$M_9(a,x,z,\kappa) := \begin{pmatrix} 1 & 0 & \frac{x}{p} & z \\ -\frac{a}{p} & \frac{1}{p} & \frac{z}{p} & \frac{\kappa}{p} \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & p \end{pmatrix}_{\substack{a,z \,(\mathrm{mod}\, p) \\ \kappa \,(\mathrm{mod}\, p^2)}} \qquad M_{10} := I_4$$

$$M_{11}(a) := \begin{pmatrix} p & 0 & 0 & 0 \\ -a & 1 & 0 & 0 \\ 0 & 0 & p^{-1} & \frac{a}{p} \\ 0 & 0 & 0 & 1 \end{pmatrix}_{a \,(\mathrm{mod}\, p)} \qquad M_{12}(a,x,z,\kappa) := \begin{pmatrix} 1 & 0 & \frac{x}{p} & \frac{xa+zp}{p} \\ -\frac{a}{p} & \frac{1}{p} & \frac{z}{p} & \frac{\kappa}{p} \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & p \end{pmatrix}_{\substack{a,z \,(\mathrm{mod}\, p) \\ \kappa \,(\mathrm{mod}\, p^2) \\ x \,(\mathrm{mod}\, p)^*}}.$$

3.2. **Hecke Jacobi operators.** In the setting of Jacobi forms complications arise since the Jacobi group is not reductive. It is well known that the related Hecke Jacobi algebra in not commutative and does not decompose into local Hecke algebras. For our purpose it will be sufficient to consider double cosets attached to the symplectic part of the Jacobi group $G_n^J(\mathbb{Q})$ which includes the Hecke operators introduced in [4, 15].

For $l \in \mathbb{N}$ we define $\mathbb{X}(l) := \Gamma^J \left(\begin{smallmatrix} l & 0 \\ 0 & l^{-1} \end{smallmatrix}\right) \Gamma^J$. We have $\mathbb{X}(l_1 l_2) = \mathbb{X}(l_1) \cdot \mathbb{X}(l_2)$ for all positive coprime integers l_1 and l_2. Let $\mathcal{H}^J$ be the Hecke Jacobi algebra generated by $\{\mathbb{X}(l) | \, l \in \mathbb{N}\}$ over $\mathbb{Q}$ and $\mathcal{H}_p^J$ the local Hecke Jacobi algebra generated by $\{\mathbb{X}(p^n) | \, n \in \mathbb{N}_0\}$ over $\mathbb{Q}$. Then we have $\mathcal{H}^J = \otimes_p \mathcal{H}_p^J$. In the following we identify the Hecke Jacobi algebra and the related Hecke Jacobi operators.

For $\Phi \in J_{k,m}$ define the Hecke operator $T^J(\ell)$ as

$$\Phi | T^J(\ell) := \ell^{k-4} \sum_{\substack{M \in Mat(2,\mathbb{Z}) \\ \det(M) = \ell^2 \\ \gcd(M) = \square}} \sum_{(\lambda,\mu) \in \mathbb{Z}^2/\ell\mathbb{Z}^2} \Phi|_{k,m} \left(\frac{1}{\ell} M(\lambda,\mu) \right). \tag{3.3}$$

Then $T^J(p)$ is related to $\mathbb{X}(p)$ by $\mathbb{X}(p) = p^{3-k} T^J(p)$. Using (2.1), we can rewrite (3.3) as

$$e(-m\,\omega_0)\, \ell^{k-4} \sum_{\substack{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Mat(2,\mathbb{Z}) \\ \det(M) = \ell^2 \\ \gcd(M) = \square}} \sum_{(\lambda,\mu) \in \mathbb{Z}^2/\ell\mathbb{Z}^2} \hat{\Phi}|_k \left(\frac{1}{\ell} \left(\begin{smallmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \right).$$

For $\Phi \in J_{k,m}^2$ we introduce the operators $T^J(l)^\uparrow$ and $T^J(l)^\downarrow$:

$$\Phi | T^J(l)^\uparrow := \frac{e(-m\,\omega_0)}{\ell^{4-k}} \sum_{\substack{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Mat(2,\mathbb{Z}) \\ \det(M) = \ell^2 \\ \gcd(M) = \square}} \sum_{(\lambda,\mu) \in \mathbb{Z}^2/\ell\mathbb{Z}^2} \hat{\Phi}|_k \left(\frac{1}{\ell} \left(\begin{smallmatrix} a & 0 & 0 & b & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ c & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 & 0 & 0 & 0 & \mu \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 1 & \mu & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\lambda \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right) \right),$$

$$\Phi | T^J(l)^\downarrow := \frac{e(-m\,\omega_0)}{\ell^{4-k}} \sum_{\substack{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Mat(2.\mathbb{Z}) \\ \det(M) = \ell^2 \\ \gcd(M) = \square}} \sum_{(\lambda,\mu) \in \mathbb{Z}^2/\ell\mathbb{Z}^2} \hat{\Phi}|_k \left(\frac{1}{\ell} \left(\begin{smallmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & c & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \mu \\ 0 & \lambda & 1 & 0 & \mu & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right) \right).$$

For $S \in Sp_n(\mathbb{Q})$ and $\Phi \in J_{k,m}^n$, we define the Hecke Jacobi operators $T_n^J(S)$:

$$\Phi | T_n^J(S) := \sum_{g \in \Gamma_n \backslash \Gamma_n S \Gamma_n} \sum_{h \in M(l)} \Phi|_k gh,$$

where l is the smallest integer such that $lS \in Mat(2n,\mathbb{Z})$ and $M(l)$ is the set of all $M(\lambda,\mu)$ with $\lambda,\mu \in \mathbb{Z}^n/l\mathbb{Z}^n$. We note that $T_1^J\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and $T_1^J\begin{pmatrix}p&0\\0&1/p\end{pmatrix}$ generate $\mathcal{H}_p^J$. Let further $\mathcal{H}_p^{J,2}$ be the Hecke Jacobi algebra generated by $(T^J(S_i))_i$, where

$$S_1 := I_4, \qquad S_2 := \begin{pmatrix}1/p&0&0&0\\0&1/p&0&0\\0&0&p&0\\0&0&0&p\end{pmatrix}, \qquad S_3 := \begin{pmatrix}1/p&0&0&0\\0&1&0&0\\0&0&p&0\\0&0&0&1\end{pmatrix}. \tag{3.4}$$

4. Proof of Theorem 1.1

Throughout, we let $\Phi \in J_{k,m}^2$ with Fourier coefficients $C(N,R)$ and write $N = \begin{pmatrix}n_{11}&\frac{n_{12}}{2}\\\frac{n_{12}}{2}&n_{22}\end{pmatrix}$ and $R = (r_1,r_2)$. ¿From the transformation law of Φ one can conclude:

Lemma 4.1. *The Fourier coefficients $C(N,R)$ only depend upon D_1, D_2, D, and the values of r_1 and r_2 modulo $2m$. In particular if m is either 1 or a prime, then they only depend on D_1, D_2, and D.*

Two remarks.
1) The invariants D_1, D_2, and D are natural since in the case that m is 1 or a prime, Lemma 4.1 is equivalent to the fact that the coefficient $C(N,R)$ only depends on $(4N - R^t \cdot R)$.
2) For fixed D_1, D_2, r_1 and r_2 there exist only finitely many Fourier coefficients $C(N,R)$.

Using Lemma 4.1, we can rewrite the Fourier expansion of Φ using the following theta decomposition

$$\Phi(\mathcal{T},Z) = \sum_{r_1,r_2 \pmod{2m}} \Theta_{r_1,r_2}(\mathcal{T},Z)\, g_{r_1,r_2}(\mathcal{T}), \tag{4.1}$$

where

$$\begin{aligned}\Theta_{r_1,r_2}(\mathcal{T},Z) &:= \sum_{\substack{\lambda_1,\lambda_2\in\mathbb{Z}\\ \lambda_i\equiv r_i \pmod{2m}}} e^{2\pi i\left(\frac{\lambda_1^2}{4m}\tau+\frac{\lambda_2^2}{4m}\zeta+\frac{\lambda_1\lambda_2}{2m}u+\lambda_1 z_1+\lambda_2 z_2\right)},\\ g_{r_1,r_2}(\mathcal{T}) &:= \sum_{D_1,D_2,D} C_{r_1,r_2}(D_1,D_2,D)\, e^{2\pi i\left(-\frac{D_1}{4m}\tau-\frac{D_2}{4m}\zeta-\frac{D}{2m}u\right)}.\end{aligned}$$

Throughout we write $\mathcal{T} = \left(\begin{smallmatrix}\tau&u\\u&\zeta\end{smallmatrix}\right)$ and $Z = (z_1,z_2)$. We note that the involved theta series are linear independent (see Section 3 of [20]).

Proof of Theorem 1.1. We start with the Γ-left coset decomposition of $\Gamma\begin{pmatrix}1&0\\0&p^2\end{pmatrix}\Gamma$ stated in (3.1). We first consider N_1 and define

$$\Phi_1 := \sum_{\lambda,\mu \pmod{p}} \Phi|_{k,m}\left(\frac{N_1}{p},(\lambda,\mu)\right)^{\uparrow}.$$

One computes that

$$\Phi_1(\mathcal{T},Z) = \sum_{\lambda,\mu \pmod{p}} \sum_{N,R} C(N,R)\, e^{2\pi i\left(\left(n_{11}p^2+r_1\lambda p+m\lambda^2\right)\tau+(n_{12}p+r_2\lambda)u+(pr_1+2m\lambda)z_1+r_2z_2+n_{22}\zeta\right)}.$$

We make the change of variables

$$n_{11}' := p^2n_{11}+p\lambda r_1+\lambda^2 m, \quad n_{12}' := pn_{12}+\lambda r_2, \quad n_{22}' := n_{22}, \quad r_1' := pr_1+2\lambda m, \quad r_2' := r_2.$$

Since $(p, 2m) = 1$ the condition $n_{11}, n_{12}, n_{22}, r_1, r_2 \in \mathbb{Z}$ is equivalent to $r_1' \equiv 2\lambda m \pmod{p}$, $D_1' \equiv 0 \pmod{p^2}$, and $D' \equiv 0 \pmod{p}$. Then we obtain the new invariants $D_1' = p^2 D_1$, $D_2' = D_2$, $D' = pD$, $r_1' = pr_1$, and $r_2' = r_2$. This yields independent of λ and μ

$$C_{r_1,r_2}(D_1, D_2, D) = C_{\bar{p}r_1',r_2'}\left(\frac{D_1'}{p^2}, D_2', \frac{D'}{p}\right).$$

Therefore we obtain

$$\begin{aligned}\Phi_1(\mathcal{T}, Z) &= p^{k+1} \sum_{\lambda \pmod{p}} \sum_{\substack{r_1,r_2,n_{11},n_{12},n_{22} \in \mathbb{Z} \\ r_1 \equiv 2\lambda m \pmod{p}}} C_{\bar{p}r_1,r_2}\left(\frac{D_1}{p^2}, D_2, \frac{D}{p}\right) e^{2\pi i\left(\operatorname{tr}(N\mathcal{T}) + RZ^t\right)} \\ &= p^{k+1} \sum_{r_1,r_2 \pmod{2m}} \Theta_{r_1,r_2}(\mathcal{T}, Z) \sum_{D_1,D,D_2} C_{\bar{p}r_1,r_2}\left(\frac{D_1}{p^2}, D_2, \frac{D}{p}\right) e^{2\pi i\left(-\frac{D_1}{4m}\tau - \frac{D_2}{4m}\zeta - \frac{D}{2m}u\right)}.\end{aligned}$$

We next consider $N_2(a)$, and define

$$\Phi_2 := \sum_{\substack{\lambda,\mu \pmod{p} \\ a \pmod{p}^*}} \Phi|_{k,m}\left(\frac{N_2(a)}{p}, (\lambda, \mu)\right)^{\uparrow}.$$

One can prove that

$$\Phi_2(\mathcal{T}, Z) = \sum_{\substack{\lambda,\mu \pmod{p} \\ a \pmod{p}^*}} \sum_{N,R} C(N, R)\, e^{\frac{2\pi i a n_{11}}{p}}\, e^{2\pi i\left(\left(n_{11} + \lambda r_1 + \lambda^2 m\right)\tau + (n_{12} + \lambda r_2)u + (r_1 + 2\lambda m)z_1 + r_2 z_2 + n_{22}\zeta\right)}.$$

We consider Φ_2^* which arises from Φ_2 by completing the sum over a into a sum over a modulo p. The new sum over a vanishes unless $p|n_{11}$ in which case it equals p. We make the change of variables

$$n_{11}' := n_{11} + \lambda r_1 + \lambda^2 m, \quad n_{12}' := n_{12} + \lambda r_2, \quad n_{22}' := n_{22}, \quad r_1' := r_1 + 2\lambda m, \quad r_2' := r_2.$$

The new invariants are $D_1' = D_1, D_2' = D_2, D' = D, r_1' = r_1$, and $r_2' = r_2$. The condition $r_1, r_2, n_{11}, n_{12}, n_{22} \in \mathbb{Z}$ is equivalent to $r_1', r_2', n_{11}', n_{12}', n_{22}' \in \mathbb{Z}$. Moreover the congruence $n_{11} \equiv 0 \pmod{p}$ is equivalent to the congruence

$$4m^2\left(\lambda - \overline{2m} r_1'\right)^2 \equiv D_1' \pmod{p}. \tag{4.2}$$

The number of solutions λ of the congruence (4.2) equals $1 + \chi_{D_1}(p)$. Hence

$$C_{r_1,r_2}(D_1, D_2, D) = C_{r_1',r_2'}\left(D_1', D_2', D'\right)$$

and

$$\Phi_2^*(\mathcal{T}, Z) = p^2 \sum_{r_1,r_2 \pmod{2m}} \Theta_{r_1,r_2}(\mathcal{T}, Z) \sum_{D_1,D,D_2} \chi_{D_1}(p) C_{r_1,r_2}\left(D_1, D_2, D\right) e^{2\pi i\left(-\frac{D_1}{4m}\tau - \frac{D_2}{4m}\zeta - \frac{D}{2m}u\right)}.$$

We next consider $N_3(b)$, and set

$$\Phi_3 := \sum_{\substack{\lambda,\mu \pmod{p} \\ b \pmod{p^2}}} \Phi|_{k,m}\left(\frac{N_3(b)}{p}, (\lambda, \mu)\right)^{\uparrow}.$$

One can show that Φ_3 equals

$$\sum_{\substack{\lambda,\mu \pmod p \\ b \pmod{p^2}}} \sum_{N,R} C(N,R)\, e^{2\pi i\left(\frac{bn_{11}}{p^2}+\frac{\mu r_1}{p}\right)} e^{2\pi i\left(\left(\frac{n_{11}}{p^2}+\frac{\lambda r_1}{p}+\lambda^2 m\right)\tau+\left(\frac{n_{12}}{p}+\lambda r_2\right)u+\left(\frac{r_1}{p}+2\lambda m\right)z_1+r_2z_2+n_{22}\zeta\right)}.$$

The sum over b vanishes unless $p^2|n_{11}$ in which case it equals p^2. Moreover the sum over μ vanishes unless $p|r_1$ in which case it equals p. We make the change of variables

$$n'_{11} := \frac{n_{11}}{p^2}+\frac{\lambda r_1}{p}+\lambda^2 m, \quad n'_{12} := \frac{n_{12}}{p}+\lambda r_2, \quad n_{22'} := n_{22}, \quad r'_1 := \frac{r_1}{p}+2\lambda m, \quad r'_2 := r_2.$$

The restrictions $p^2|n_{11}, p|r_1$, and $r_2, n_{12}, n_{22} \in \mathbb{Z}$ are equivalent to $n'_{11}, n'_{12}, n'_{22}, r'_1, r'_2 \in \mathbb{Z}$. In particular r'_1 runs through $\mathbb{Z}$ for each λ. We obtain the invariants $D'_1 = \frac{D_1}{p^2}$, $D'_2 = D_2$, $D' = \frac{D}{p}$, $r'_1 = \overline{p}r_1$, and $r'_2 = r_2$. Thus

$$C_{r_1,r_2}(D_1,D_2,D) = C_{pr'_1,r'_2}\left(p^2D'_1, D'_2, pD'\right).$$

This yields

$$\Phi_3(\mathcal{T},Z) = p^{4-k} \sum_{r_1,r_2 \pmod{2m}} \Theta_{r_1,r_2}(\mathcal{T},Z) \sum_{D_1,D,D_2} C_{pr_1,r_2}\left(p^2D_1,D_2,pD\right) e^{2\pi i\left(-\frac{D_1}{4m}\tau-\frac{D_2}{4m}\zeta-\frac{D}{2m}u\right)}.$$

In a similar manner we treat $\Phi|_{k,m}\left(T^J(p)\right)^{\downarrow}$. Now the claim of the theorem follows by comparing Fourier coefficients and by using the linear independence of the theta series Θ_{r_1,r_2}. □

5. Proof of Theorem 1.2

Throughout we let $\Phi \in \mathbb{E}^{(q)}_{k,m}$. We show that for all S_i as defined in (3.4) we have $\Phi|T^J_2(S_i) \in \mathbb{E}^{(q)}_{k,m}$. We actually show that each package of representatives M_i which correspond to one of the S_i already preserves the Hecke duality. As a by-product we explicitly determine this action on the Fourier coefficients of Φ.

We first consider the action of the Heisenberg group on $\Phi|_{k,m}g$ with $g \in \Gamma_2\backslash\Gamma_2 S\Gamma_2$. For this we define $M := \sum_{\lambda,\mu \pmod p} M(\lambda,\mu)$, where $\lambda := (\lambda_1,\lambda_2), \mu := (\mu_1,\mu_2)$, and where

$$M(\lambda,\mu) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \mu_1 \\ 0 & 1 & 0 & 0 & 0 & \mu_2 \\ \lambda_1 & \lambda_2 & 1 & \mu_1 & \mu_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\lambda_1 \\ 0 & 0 & 0 & 0 & 1 & -\lambda_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $M(\lambda,\mu) \circ \begin{pmatrix} \mathcal{T} & Z^t \\ Z & w_0 \end{pmatrix}$ equals

$$\begin{pmatrix} \tau & u & \lambda_1\tau+\lambda_2u+z_1 \\ u & \zeta & \lambda_1u+\lambda_2\zeta+z_2 \\ \lambda_1\tau+\lambda_2u+z_1 & \lambda_1u+\lambda_2\zeta+z_2 & \lambda_1^2\tau+2\lambda_1\lambda_2u+2\lambda_1z_1+2\lambda_2z_2+\lambda_2^2\zeta+w_0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \mu_1 \\ 0 & 0 & \mu_2 \\ \mu_1 & \mu_2 & \lambda_1\mu_1+\lambda_2\mu_2 \end{pmatrix}.$$

It will turn out that $\Phi|_{k,m}g$ has a Fourier expansion with $n_{11}, n_{12}, n_{22} \in \mathbb{Z}$ and $r_1, r_2 \in \frac{1}{p}\mathbb{Z}$. The sum over μ vanishes unless $p|r_1, r_2$ in which case it equals p^2. We make the change of variables

$$n'_{11} := n_{11} + \lambda_1 r_1 + \lambda_1^2 m, \qquad n'_{12} := n_{12} + \lambda_2 r_1 + \lambda_1 r_2 + 2\lambda_1\lambda_2 m,$$
$$n'_{22} := n_{22} + \lambda_2 r_2 + \lambda_2^2 m, \qquad r'_1 := r_1 + 2\lambda_1 m, \qquad r'_2 := r_2 + 2\lambda_2 m,$$

which doesn't change the associated invariants.

We next consider the action of the matrices M_i $(1 \le i \le 12)$. We start with $M_1 := \sum_{x,y,z \pmod{p^2}} M_1(x,y,z)$. Then

$$M_1(x,y,z) \circ \begin{pmatrix} \mathcal{T} & Z^t \\ Z & w_0 \end{pmatrix} = \begin{pmatrix} \frac{\tau}{p^2} & \frac{u}{p^2} & \frac{z_1}{p} \\ \frac{u}{p^2} & \frac{\zeta}{p^2} & \frac{z_2}{p} \\ \frac{z_1}{p} & \frac{z_2}{p} & w_0 \end{pmatrix} + \begin{pmatrix} \frac{y}{p^2} & \frac{x}{p^2} & 0 \\ \frac{x}{p^2} & \frac{z}{p^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The sum over x, y, and z vanishes unless $p^2|n_{11}, n_{12}$, and n_{22} in which case it equals p^6. We make the change of variables $n'_{11} := \frac{n_{11}}{p^2}$, $n'_{12} := \frac{n_{12}}{p^2}$, $n'_{22} := \frac{n_{22}}{p^2}$, $r'_1 := \frac{r_1}{p}$, and $r'_2 := \frac{r_2}{p}$. Observe that n'_{11}, n'_{12}, and $n'_{22} \in \mathbb{Z}$ and $r'_1, r'_2 \in \frac{1}{p}\mathbb{Z}$. We obtain the invariants $D'_1 = \frac{D_1}{p^2}$, $D'_2 = \frac{D_2}{p^2}$, $D' = \frac{D}{p^2}$ which yields

$$C(D_1, D_2, D) = C\left(p^2 D'_1, p^2 D'_2, p^2 D'\right).$$

¿From the above considerations we see that applying the Heisenberg part reduces the summation to r_1 and $r_2 \in \mathbb{Z}$, multiplies the sum by p^2, and leaves the invariants unchanged. Thus $\Phi|_{k,m} M_1 M(\mathcal{T}, Z)$ equals

$$p^{10-2k} \sum_{\substack{D_1, D, D_2 \\ r_1, r_2}} C\left(p^2 D_1, p^2 D_2, p^2 D\right) e^{2\pi i\left(-\frac{D_1}{4m}\tau - \frac{D_2}{4m}\zeta - \frac{D}{2m}u\right)} e^{2\pi i\left(\frac{r_1^2}{4m}\tau + \frac{r_2^2}{4m}\zeta + \frac{r_1 r_2}{2m}u + r_1 z_1 + r_2 z_2\right)},$$

where here in the following we have as before $D_1 = r_1^2 - 4n_{11}m$, $D_2 = r_2^2 - 4n_{22}m$, and $D = r_1 r_2 - 2n_{12}^2$. Let

$$A(D_1, D_2, D) := C\left(p^2 D_1, p^2 D_2, p^2 D\right).$$

We show that this function satisfies (1.5) using that Φ satisfies (1.5) and that $\chi_{D_1}(q) = \chi_{p^2 D_1}(q)$. This yields that $\left(\chi_{D_1}(q) - \chi_{D_2}(q)\right)\ A\left(D_1, D_2, D\right)$ equals

$$\begin{aligned}
&\left(\chi_{D_1 p^2}(q) - \chi_{D_2 p^2}(q)\right)\ C\left(p^2 D_1, p^2 D_2, p^2 D\right) \\
&\quad = q^{2-k}\ \left(C\left(p^2 D_1, p^2 q^2 D_2, p^2 q D\right) - C\left(q^2 p^2 D_1, p^2 D_2, p^2 q D\right)\right) \\
&\quad + q^{k-1}\left(C\left(p^2 D_1, \frac{p^2 D_2}{q^2}, \frac{p^2 D}{q}\right) - C\left(\frac{p^2 D_1}{q^2}, p^2 D_2, \frac{p^2 D}{q}\right)\right) \\
&\qquad = q^{2-k}\ \left(A\left(D_1, q^2 D_2, q D\right) - A\left(q^2 D_1, D_2, q D\right)\right) \\
&\qquad\qquad + q^{k-1}\left(A\left(D_1, \frac{D_2}{q^2}, \frac{D}{q}\right) - A\left(\frac{D_1}{q^2}, D_2, \frac{D}{q}\right)\right)
\end{aligned}$$

as claimed. The matrices M_2, M_3, M_4, and M_5 are treated in a similar way. We next consider $M_6 := \sum_{\substack{a \pmod p \\ b \pmod p^*}} M_6(a,b)$. We have

$$M_6(a,b) \circ \begin{pmatrix} \mathcal{T} & Z^t \\ Z & w_0 \end{pmatrix} = \begin{pmatrix} p^2\tau & -ap\tau + pu & pz_1 \\ -ap\tau + pu & a^2\tau - 2au + \zeta & -az_1 + z_2 \\ pz_1 & -az_1 + z_2 & w_0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{b}{p} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since we will see later that M_{11} preserves (1.5) we may complete the sum over b into a sum over all b modulo p which we denote by M_6^*. The sum over b vanishes unless $p|n_{22}$ in which case it equals p. We make the change of variables

$$n_{11}' := p^2 n_{11} - apn_{12} + a^2 n_{22}, \quad n_{12}' := pn_{12} - 2an_{22}, \quad n_{22}' := n_{22}, \quad r_1' := pr_1 - ar_2, \quad r_2' := r_2.$$

We compute the invariants

$$D_1 = \frac{1}{p^2}\left(D_1' + 2aD' + a^2 D_2'\right), \quad D_2 = D_2', \quad D = \frac{1}{p}\left(D' + aD_2'\right), \quad r_1 = \frac{1}{p}\left(r_1' + ar_2'\right), \quad r_2 = r_2'.$$

Applying the Heisenberg transformation preserves those invariants. We denote the new variables with tildes. We have the following equivalent conditions:

$$p|n_{22} \quad \Leftrightarrow \quad p|n_{22}' \quad \Leftrightarrow \quad \widetilde{n}_{22} - \lambda_2 \widetilde{r}_2 + \lambda_2^2 m \equiv 0 \pmod p \quad \Leftrightarrow \quad 4m^2\left(\lambda - \overline{2m}\widetilde{r}_2\right)^2 \equiv \widetilde{D}_2 \pmod p.$$

Then $\Phi|_{k,m} M_6^* M(\mathcal{T}, Z)$ equals

$$p^{4+k} \sum_{\substack{D_1, D, D_2 \\ r_1, r_2}} \sum_{\substack{\lambda_2, a \pmod p \\ 4m^2\left(\lambda_2 - \overline{2m}r_2\right)^2 \equiv D_2 \pmod p}} C\left(\frac{1}{p^2}(D_1 + 2aD + a^2 D_2), D_2, \frac{1}{p}(D + aD_2)\right)$$
$$e^{2\pi i\left(-\frac{D_1}{4m}\tau - \frac{D_2}{4m}\zeta - \frac{D}{2m}u\right)} e^{2\pi i\left(\frac{r_1^2}{4m}\tau + \frac{r_2^2}{4m}\zeta + \frac{r_1 r_2}{2m}u + r_1 z_1 + r_2 z_2\right)} e^{2\pi i\left(\frac{r_1^2}{4m}\tau + \frac{r_2^2}{4m}\zeta + \frac{r_1 r_2}{2m}u + r_1 z_1 + r_2 z_2\right)}.$$

We let

$$A(D_1, D_2, D) := \sum_{\substack{\lambda, a \pmod p \\ 4m^2\left(\lambda - \overline{2m}r_2\right)^2 \equiv D_2 \pmod p}} C\left(\frac{1}{p^2}(D_1 + 2aD + a^2 D_2), D_2, \frac{1}{p}(D + aD_2)\right) \tag{5.1}$$

and show that this function satisfies (1.5). First observe that in the sum over a in (5.1) we may choose as a set of representatives elements that are divisible by q. Using this we have

$$\chi_{D_1}(q) = \chi_{\frac{1}{p^2}(D_1 + 2aD + a^2 D_2)}(q).$$

This yields that $\left(\chi_{D_1}(q) - \chi_{D_2}(q)\right) A(D_1, D_2, D)$ equals

$$\left(\chi_{D_1}(q) - \chi_{D_2}(q)\right) \sum_{\substack{\lambda, a \pmod p \\ 4m^2\left(\lambda - \overline{2m}r_2\right)^2 \equiv D_2 \pmod p}} C\left(\frac{1}{p^2}(D_1 + 2aD + a^2 D_2), D_2, \frac{1}{p}(D + aD_2)\right)$$
$$= \sum_{\substack{\lambda, a \pmod p \\ 4m^2\left(\lambda - \overline{2m}r_2\right)^2 \equiv D_2 \pmod p \\ q|a}} \left(\chi_{\frac{1}{p^2}(D_1 + 2aD + a^2 D_2)}(q) - \chi_{D_2}(q)\right)$$
$$C\left(\frac{1}{p^2}(D_1 + 2aD + a^2 D_2), D_2, \frac{1}{p}(D + aD_2)\right).$$

Using (1.5) gives that this equals

$$q^{2-k} \sum_{\substack{\lambda,a \pmod p \\ 4m^2\left(\lambda-\overline{2m}r_2\right)^2\equiv D_2 \pmod p \\ q|a}} C\left(\frac{1}{p^2}(D_1+2aD+a^2D_2), q^2D_2, \frac{q}{p}(D+aD_2)\right)$$

$$- q^{2-k} \sum_{\substack{\lambda,a \pmod p \\ 4m^2\left(\lambda-\overline{2m}r_2\right)^2\equiv D_2 \pmod p \\ q|a}} C\left(\frac{q^2}{p^2}(D_1+2aD+a^2D_2), D_2, \frac{q}{p}(D+aD_2)\right)$$

$$+ q^{k-1} \sum_{\substack{\lambda,a \pmod p \\ 4m^2\left(\lambda-\overline{2m}r_2\right)^2\equiv D_2 \pmod p \\ q|a}} C\left(\frac{1}{p^2}(D_1+2aD+a^2D_2), \frac{D_2}{q^2}, \frac{1}{pq}(D+aD_2)\right)$$

$$- q^{k-1} \sum_{\substack{\lambda,a \pmod p \\ 4m^2\left(\lambda-\overline{2m}r_2\right)^2\equiv D_2 \pmod p \\ q|a}} C\left(\frac{1}{p^2q^2}(D_1+2aD+a^2D_2), D_2, \frac{1}{pq}(D+aD_2)\right).$$

We rewrite the occuring summands on the right hand side. Since $(p,q)=1$, we may change in the first sum a into aq and in the second a into aq and λ into $\lambda\bar{q}$. The other summands are treated similarly. This gives that $\left(\chi_{D_1}(q)-\chi_{D_2}(q)\right) A(D_1,D_2,D)$ equals

$$q^{2-k}\left(A\left(D_1,D_2q^2,Dq\right)-A\left(D_1q^2,D_2,Dq\right)\right)+q^{k-1}\left(A\left(D_1,\frac{D_2}{q^2},\frac{D}{q}\right)-A\left(\frac{D_1}{q^2},D_2,\frac{D}{q}\right)\right)$$

as claimed.

We next deal with the action of $M_7 := \sum_{\substack{b,c \pmod p \\ a \pmod p^*}} M_7(a,b,c)$. We have

$$M_7(a,b,c)\circ\begin{pmatrix}\mathcal{T} & Z^t\\ Z & w_0\end{pmatrix} = \begin{pmatrix}\tau & u-\frac{a}{p}\tau & z_1\\ u-\frac{a}{p}\tau & \frac{a^2}{p^2}\tau-2\frac{a}{p}u+\zeta & z_2-\frac{a}{p}z_1\\ z_1 & z_2-\frac{a}{p}z_1 & w_0\end{pmatrix}+\begin{pmatrix}0 & \frac{c}{p} & 0\\ \frac{c}{p} & \frac{bp-ca}{p^2} & 0\\ 0 & 0 & 0\end{pmatrix}.$$

The sum over b vanishes unless $n_{22}\equiv 0 \pmod p$. Moreover the sum over c vanishes unless $n_{12}-\frac{an_{22}}{p}\equiv 0 \pmod p$. We make the change of variables

$$n'_{11} := n_{11}-\frac{a}{p}n_{12}+\frac{a^2}{p^2}n_{22},\quad n'_{12} := n_{12}-2\frac{a}{p}n_{22},\quad n'_{22} := n_{22},\quad r'_1 := r_1-\frac{a}{p}r_2,\quad r'_2 := r_2.$$

Then

$$D_1 = D'_1+\frac{2a}{p}D'+\frac{a^2}{p^2}D'_2,\qquad D_2 = D'_2,\qquad D = D'+\frac{a}{p}D'_2.$$

We again denote the variables after the Heisenberg transformation with tildes. As before we see that the condition $n_{22}\equiv 0 \pmod p$ is equivalent to

$$4m^2\left(\lambda_2-\overline{2m}\widetilde{r}_2\right)^2\equiv \widetilde{D}_2 \pmod p. \tag{5.2}$$

Similarly we have the equivalence

$$n_{12}-\frac{an_{22}}{p}\equiv 0 \pmod p \quad\Leftrightarrow\quad n'_{12}+\frac{a}{p}n'_{22}\equiv 0 \pmod p,$$

which is equivalent to

$$\widetilde{n}_{12} - \lambda_2\widetilde{r}_1 - \lambda_1\widetilde{r}_2 + 2\lambda_1\lambda_2 m + \frac{a}{p}\left(\widetilde{n}_{22} - \lambda_2\widetilde{r}_2 + \lambda_2^2 m\right) \equiv 0 \pmod{p}. \tag{5.3}$$

This yields that $\Phi|_{k,m}M_7 M(\mathcal{T}, Z)$ equals

$$p^2 \sum_{\substack{D_1,D,D_2 \\ r_1,r_2}} \sum_{\substack{\lambda_1,\lambda_2 \pmod p \\ a \pmod p^*}}{}' C\left(D_1 + \frac{2aD}{p} + \frac{a^2 D_2}{p^2}, D_2, D + \frac{aD_2}{p}\right) e^{2\pi i\left(-\frac{D_1}{4m}\tau - \frac{D_2}{4m}\zeta - \frac{D}{2m}u\right)} e^{2\pi i\left(\frac{r_1^2}{4m}\tau + \frac{r_2^2}{4m}\zeta + \frac{r_1 r_2}{2m}u}$$

where in $\sum'$ the sum runs over those λ_1, λ_2 and a that satisfy (5.2) and (5.3). Now we can argue as before. The case of M_8 is proven similarly.

We next consider the action of $M_9 := \sum_{\substack{a,z \pmod p \\ x \pmod p^* \\ \kappa \pmod{p^2}}} M_9(a,x,z,\kappa)$. We compute

$$M_9(a,x,x,\kappa) \circ \begin{pmatrix} \mathcal{T} & Z^t \\ Z & w_0 \end{pmatrix} = \begin{pmatrix} \tau & -\frac{a\tau}{p} + \frac{u}{p} & z_1 \\ -\frac{a\tau}{p} + \frac{u}{p} & \frac{a^2\tau}{p^2} - \frac{2au}{p^2} + \frac{\zeta}{p^2} & -\frac{az_1}{p} + \frac{z_2}{p} \\ z_1 & -\frac{az_1}{p} + \frac{z_2}{p} & w_0 \end{pmatrix} + \begin{pmatrix} \frac{x}{p} & \frac{z}{p} & 0 \\ \frac{z}{p} & \frac{\kappa}{p^2} - \frac{az}{p^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since it turns out that M_{12} preserves (1.5) we may complete the sum over x into a sum over all x modulo p. The sum over x, z, and κ vanishes unless $p|n_{11}, n_{12}$ and $p^2|n_{22}$ in which case it equals p^4. We make the change of variables

$$n'_{11} := n_{11} - \frac{an_{12}}{p} + \frac{a^2}{p^2}n_{22}, \quad n'_{12} := \frac{n_{12}}{p} - 2\frac{a}{p^2}n_{22}, \quad n'_{22} := \frac{n_{22}}{p^2}, \quad r'_1 := r_1 - \frac{a}{p}r_2, \quad r'_2 := \frac{r_2}{p}$$

and obtain

$$D_1 = D'_1 + 2aD' + a^2 D'_2, \qquad D_2 = p^2 D'_2, \qquad D = p(D' + aD'_2).$$

This yields

$$\Phi|_{k,m}M_9 M(\mathcal{T}, Z)p^{8-k} \sum_{\substack{D_1,D,D_2 \\ r_1,r_2}} \sum_{a \pmod p^*} C\left(D_1 + 2aD + a^2 D_2, p^2 D_2, p(D + aD_2)\right)$$
$$e^{2\pi i\left(-\frac{D_1}{4m}\tau - \frac{D_2}{4m}\zeta - \frac{D}{2m}u\right)} e^{2\pi i\left(\frac{r_1^2}{4m}\tau + \frac{r_2^2}{4m}\zeta + \frac{r_1 r_2}{2m}u + r_1 z_1 + r_2 z_2\right)}.$$

Now we can argue as before. The matrices M_{10}, M_{11}, and M_{12} can be considered similarly.

6. Proof of Theorem 1.3

As explicit examples of Jacobi forms of degree 2 which are elements of $\mathbb{E}_{k,m}$, we define Jacobi Eisenstein series of Siegel type

$$E^{J,n}_{k,m}(\mathcal{T}, Z) := \sum_{\gamma \in \Gamma^J_\infty \backslash \Gamma^J_n} J^{-1}_{k,m}(\gamma, (\mathcal{T}, Z)).$$

Here Γ^J_∞ is the stabilizer group of the function $J^{-1}_{k,m}$. The series $E^{J,n}_{k,m}$ is absolutely convergent for $k > n + 2$ and defines a non-trivial Jacobi form. Since related functions occur in work of Arakawa [2], we choose for the readers convenience to use his parametrization of the Jacobi group in terms of matrices to simplify consulting this paper for related calculations. We note that the Eisenstein series viewed as functions are the same. Only formally the sets of representatives of cosets defining the Eisenstein series have a different parametrization which can be related to each other by conjugation.

We have that $\Gamma_\infty^J = \left\{\gamma \in \Gamma_n^J |\, \lambda = 0,\, g \in \Gamma_\infty\right\}$, where Γ_∞ is the subgroup of Γ_n with $C = 0$. It follows from the definition of the Eisenstein series that

$$E_{k,m}^{J,n}(\mathcal{T}, Z) = \sum_{\lambda \in \mathbb{Z}^n} \sum_{g \in \Gamma_\infty \backslash \Gamma_n} J_{k,m}^{-1}\Big(\left((\lambda, 0, 0)\,, g\right), (\mathcal{T}, Z)\Big).$$

In the following we restrict to $n = 2$ and analyze the decomposition of $\Gamma_\infty\backslash\Gamma_2$ with respect to $\Gamma \times \Gamma$. Work of Garrett [5] implies that

$$\Gamma_2 = \Gamma_\infty(\Gamma \times \Gamma) \cup \bigcup_{d=1}^{\infty} \Gamma_\infty\, h_d'\, \Gamma \times \Gamma,$$

where $h_d' := \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & d & 1 & 0 \\ d & 0 & 0 & 1 \end{smallmatrix}\right)$. A straightforward calculation gives that $\Gamma_\infty\backslash\Gamma_2$ can be written as the union of $(\Gamma_\infty\backslash\Gamma) \times (\Gamma_\infty\backslash\Gamma)$ and $\bigcup_{d=1}^\infty h_d' \left((\Gamma(d)\backslash\Gamma) \times \Gamma\right)$, where $\Gamma(d) := \left(\begin{smallmatrix} d^{-1} & 0 \\ 0 & d \end{smallmatrix}\right) \Gamma \left(\begin{smallmatrix} d & 0 \\ 0 & d^{-1} \end{smallmatrix}\right) \cap \Gamma$. Since $I_2 \times \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is an element of $I_2 \times \Gamma$, we can replace h_d' by

$$h_d := h_d' \left(I_2 \times \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -d \\ d & 1 & 0 & 0 \end{smallmatrix}\right).$$

We denote the subseries corresponding to $(\Gamma_\infty\backslash\Gamma) \times (\Gamma_\infty\backslash\Gamma)$ and $\bigcup_{d=1}^\infty h_d'\big((\Gamma(d)\backslash\Gamma) \times \Gamma\big)$ by E_I and E_{II}, respectively. One computes that

$$E_I(\mathcal{T}, Z) = \sum_{\substack{\lambda \in \mathbb{Z}^2 \\ g,h \in \Gamma_\infty\backslash\Gamma}} J_{k,m}^{-1}\left((\lambda, 0, 0)g \times h), (\mathcal{T}, Z)\right).$$

The conjugation law of the Heisenberg group implies that

$$h_d^{-1}\big((x, y), (0, 0), 0\big)\, h_d = (x, 0, 0) \times (0, -y, 0).$$

This yields

$$E_{II}(\mathcal{T}, Z) = \sum_{d=1}^{\infty} \sum_{\gamma_1, \gamma_2} J_{k,m}^{-1}\left(h_d \cdot (\gamma_1 \times \gamma_2), (\mathcal{T}, Z)\right),$$

where $\gamma_1 \in \big((\mathbb{Z}, 0, 0), \big(\Gamma(d)\backslash\Gamma\big) \times I_2\big)$ and $\gamma_2 \in \big((0, \mathbb{Z}, 0), I_2 \times \Gamma\big)$. We treat the subseries E_I and E_{II} separately.

6.1. The subseries E_I.

Proposition 6.1. *The series $E_I(\mathcal{T}, Z)$ satisfies (1.5) for all primes p with $(p, 2m) = 1$. If m is square-free, then this is true for all primes p.*

Proof. The cocyle relation of $J_{k,m}$ yields

(6.1)
$$E_I(\mathcal{T}, Z) = \sum_{\gamma, \widetilde{\gamma} \in \Gamma_\infty^J \backslash \Gamma^J} J_{k,m}^{-1}\left(\gamma \times \widetilde{\gamma}, (\mathcal{T}, Z)\right) = \sum_{\widetilde{\gamma} \in \Gamma_\infty^J \backslash \Gamma^J} E_{k,m}^J\left(\left(\widetilde{\gamma}^{\downarrow} \circ (\mathcal{T}, Z)\right)^*\right) J_{k,m}^{-1}\left(\widetilde{\gamma}^{\downarrow}, (\mathcal{T}, Z)\right),$$

where $(\mathcal{T}, Z)^* := (\tau, z_1) \in \mathbb{H}^J$. Without loss of generality we may assume that $T^J(p) = \sum_j \Gamma^J \eta_j$, where p is a prime and $\eta_j \in G^J(\mathbb{Q})$. Using (6.1) gives that $\left(E_I | T^J(p)^{\uparrow}\right)(\mathcal{T}, Z)$ equals

$$\sum_{\widetilde{\gamma} \in \Gamma_\infty^J \backslash \Gamma^J} \sum_j E_{k,m}^J\left(\left(\widetilde{\gamma}^{\downarrow} \eta_j^{\uparrow} \circ (\mathcal{T}, Z)\right)^*\right) J_{k,m}^{-1}\left(\widetilde{\gamma}^{\downarrow}, \eta_j^{\uparrow} \circ (\mathcal{T}, Z)\right)\; J_{k,m}^{-1}\left(\eta_j^{\uparrow}, (\mathcal{T}, Z)\right)$$

$$= \sum_j \sum_{\widetilde{\gamma} \in \Gamma_\infty^J \backslash \Gamma^J} E_{k,m}^J\Big(\eta_j \circ \left(\widetilde{\gamma}^{\downarrow} \circ (\mathcal{T}, Z)\right)^*\Big)\; J_{k,m}^{-1}\left(\eta_j, \left(\widetilde{\gamma}^{\downarrow} \circ (\mathcal{T}, Z)\right)^*\right)\; J_{k,m}^{-1}\left(\widetilde{\gamma}^{\downarrow}, (\mathcal{T}, Z)\right).$$

This yields

$$\left(E_I|T^J(p)^{\uparrow}\right)(\mathcal{T},Z)=\sum_{\widetilde{\gamma}\in\Gamma^J_\infty\backslash\Gamma^J}\left(E^J_{k,m}|T^J(p)\right)\left(\left(\widetilde{\gamma}^{\downarrow}\circ(\mathcal{T},Z)\right)^*\right)\ J^{-1}_{k,m}\left(\widetilde{\gamma}^{\downarrow},(\mathcal{T},Z)\right).$$

By formula (13) in [4] it follows that for $(p,m)=1$ the Jacobi Eisenstein series of degree 1 and index m is a Hecke Jacobi eigenform. If m is square-free it follows from formula (51) in the table on page 224 of [6] that this also true if $p|m$. Thus the claim of the proposition follows. □

6.2. **The subseries E_{II}.** In this section we prove an explicit formula for E_{II} as an infinite sum of certain functions $P^J_{k,m}(\mathcal{T},Z)$ of Poincaré type. For this purpose define the function

$$\psi_{k,m}(\mathcal{T},Z):=(\tau+2u+\zeta)^k\,e\left(\frac{m(z_1+z_2)^2}{\tau+2u+\zeta}\right),$$

which is related to the Jacobi Eisenstein series. A direct calculation leads to

$$J_{k,m}(h_d,(\mathcal{T},Z))=\psi_{k,m}(\mathbb{M}_d\circ(\mathcal{T},Z)). \tag{6.2}$$

Here $\mathbb{M}_d:=\left(\begin{smallmatrix}d&0\\0&d^{-1}\end{smallmatrix}\right)\times I_2$. Next define the unique involution $\#: G^J(\mathbb{R})\to G^J(\mathbb{R})$ on the generators of the Jacobi group by $(\lambda,0,0)^{\#}:=(-\lambda,0,0)$, $(0,\mu,\kappa)^{\#}:=(0,\mu,\kappa)$, $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)^{\#}:=\left(\begin{smallmatrix}d&b\\c&a\end{smallmatrix}\right)$ that satisfies $(\gamma_1\gamma_2)^{\#}:=\gamma_2^{\#}\gamma_1^{\#}$ for all $\lambda,\mu,\kappa\in\mathbb{R}$ and $\gamma_1,\gamma_2\in G^J(\mathbb{R})$. This involution preserves double cosets in the Hecke Jacobi algebra. A straightforward but lengthy calculation gives.

Lemma 6.2. *For $\gamma\in G^J(\mathbb{R})$ we have*

$$\psi^{-1}_{k,m}|\gamma^{\uparrow}=\psi^{-1}_{k,m}|\ \gamma^{\#\downarrow}.$$

We next define for $D\in Sp_1(\mathbb{Q})$ the holomorphic function

$$P^J_{k,m}((\mathcal{T},Z),D):=\sum_{\gamma\in Z(\Gamma^J)\backslash\Gamma^J\ D\,\Gamma^J}\left(\psi^{-1}_{k,m}|\gamma^{\uparrow}\right)(\mathcal{T},Z),$$

where $Z(\Gamma^J)$ is the center of Γ^J. This series is absolutely and locally uniformly convergent. Its restriction to $\mathbb{H}^J\times\mathbb{H}^J$ yields the Poincaré series given in [2]. Moreover there exist a unique $d\in\mathbb{N}$ with $D=\left(\begin{smallmatrix}d&0\\0&d^{-1}\end{smallmatrix}\right)$. Lemma 6.2 and the invariance of double cosets with respect to the involution $\#$ gives that

$$P^J_{k,m}((\mathcal{T},Z),D)=\sum_{\gamma\in Z(\Gamma^J)\backslash\Gamma^J\ D\,\Gamma^J}\left(\psi^{-1}_{k,m}|\gamma^{\downarrow}\right)(\mathcal{T},Z).$$

Moreover for $U\in SL_2(\mathbb{Z})$, we define $P^J_{k,m}(\mathcal{T},Z):=P^J_{k,m}((\mathcal{T},Z),U)$. Then we have

$$\left(P^J_{k,m}|\mathbb{X}(d)^{\uparrow}\right)(\mathcal{T},Z)=P^J_{k,m}((\mathcal{T},Z),D). \tag{6.3}$$

Lemma 6.3. *Assume that m is square-free. Then we have*

$$\left(P^J_{k,m}|\mathbb{X}(l)^{\uparrow}\right)\left((\mathcal{T},Z),\left(\begin{smallmatrix}d&0\\0&d^{-1}\end{smallmatrix}\right)\right)=\left(P^J_{k,m}|\mathbb{X}(l)^{\downarrow}\right)\left((\mathcal{T},Z),\left(\begin{smallmatrix}d&0\\0&d^{-1}\end{smallmatrix}\right)\right). \tag{6.4}$$

If m is arbitary (6.4) is also satisfied if $(l,d)=1$ or $(ld,m)=1$.

Proof. The conditions that m is square-free and that $(l,d)=1$ or $(ld,m)=1$ otherwise, imply that on $J_{k,m}$ we have

$$\mathbb{X}(l)\,\mathbb{X}(d)=\mathbb{X}(d)\ \mathbb{X}(l).$$

Finally we apply formula (6.3) to get the lemma. □

Proposition 6.4. *We have*

$$E_{II}(\mathcal{T},Z)=\sum_{d=1}^{\infty}\left(P_{k,m}^{J}|\mathbb{X}(d)^{\uparrow}\right)(\mathcal{T},Z)\;d^{-k}. \tag{6.5}$$

Proof. Denote by E_{II}^{d} the subseries of E_{II} corresponding to $d\in\mathbb{N}$. Then we have by (6.2)

$$E_{II}^{d}(\mathcal{T},Z)=\sum_{\gamma_1,\gamma_2}\psi_{k,m}^{-1}\left(\mathbb{M}_d\left(\gamma_1\times\gamma_2\right)\circ(\mathcal{T},Z)\right)\;J_{k,m}^{-1}\left(\gamma_1\times\gamma_2,(\mathcal{T},Z)\right).$$

Here $\gamma_1\in\Big((\mathbb{Z},0,0),\Gamma(d)\backslash\Gamma\times I_2\Big)$ and $\gamma_2\in\Big((0,\mathbb{Z},0),I_2\times\Gamma\Big)$. Using Lemma 6.2, a straightforward, but lengthy calculation gives

$$E_{II}^{d}(\mathcal{T},Z)=d^{-k}\sum_{\gamma_1,\gamma_2}\psi_{k,m}^{-1}\left(\left(\gamma_2^{\#}\left(\begin{smallmatrix}d&0\\0&d^{-1}\end{smallmatrix}\right)\gamma_1\right)^{\uparrow}\circ(\mathcal{T},Z)\right)J_{k,m}^{-1}\left(\left(\gamma_2^{\#}\left(\begin{smallmatrix}d&0\\0&d^{-1}\end{smallmatrix}\right)\gamma_1\right)^{\uparrow},(\mathcal{T},Z)\right).$$

To complete the proof it remains to analyze the set

$$\left\{\gamma_2^{\#}\left(\begin{smallmatrix}d&0\\0&d^{-1}\end{smallmatrix}\right)\gamma_1\;\middle|\;\gamma_2=((0,\mu,0)g)\,,\;\gamma_1=((\lambda,0,0)h)\;\text{ with } g\in\Gamma,\;h\in\Gamma(d)\backslash\Gamma\text{ and }\lambda,\mu\in\mathbb{Z}\right\}. \tag{6.6}$$

First we note that $\Big((0,\mathbb{Z},0)\,\Gamma\Big)^{\#}=\Gamma\,(0,\mathbb{Z},0)$. Hence the set in (6.6) is equal to $\Gamma\;(0,\mathbb{Z},0)\;\left(\begin{smallmatrix}d&0\\0&d^{-1}\end{smallmatrix}\right)(\mathbb{Z},0,0)\;\Gamma(d)\backslash\Gamma$, which equals

$$\Gamma\,(0,\mathbb{Z},0)\;(\mathbb{Z},0,0)\left(\begin{smallmatrix}d&0\\0&d^{-1}\end{smallmatrix}\right)(d\mathbb{Z}\backslash\mathbb{Z},0,0)\Gamma(d)\backslash\Gamma,$$

since $\left(\begin{smallmatrix}d^{-1}&0\\0&d\end{smallmatrix}\right)(\lambda',0,0)\left(\begin{smallmatrix}d&0\\0&d^{-1}\end{smallmatrix}\right)=(d\lambda',0,0)$. Here λ' runs modulo d. Hence the set (6.6) is equal to $Z(\Gamma^J)\backslash\Gamma^J\left(\begin{smallmatrix}d&0\\0&d^{-1}\end{smallmatrix}\right)\Gamma^J$. Thus

$$E_{II}^{d}(\mathcal{T},Z)=d^{-k}\sum_{\gamma\in Z(\Gamma^J)\backslash\Gamma^J\left(\begin{smallmatrix}d&0\\0&d^{-1}\end{smallmatrix}\right)\Gamma^J}\psi_{k,m}^{-1}\left(\gamma^{\uparrow}\circ(\mathcal{T},Z)\right)\;J_{k,m}^{-1}\left(\gamma^{\uparrow},(\mathcal{T},Z)\right), \tag{6.7}$$

which leads to the proof of the proposition. □

7. Eisenstein series of Klingen type

In this final section we show that for even $k>4$, the space $\mathbb{E}_k$ is a proper subspace of $J_{k,m}^{2}$. Let $\Phi\in J_{k,m}^{\text{cusps}}$, the space of Jacobi cusp forms, a non-trivial Hecke-Jacobi eigenform for all $T^J(l)$ with $(l,m)=1$. Denote by $E_{k,m}^{\mathrm{Kl}}\left(\Phi,(\mathcal{T},Z)\right)$ the Jacobi Klingen Eisenstein series of degree 2 associated to Φ. Then $E_{k,m}^{\mathrm{Kl}}\left(\Phi\right)$ is a non-trivial element of $J_{k,m}^{2}$.

Proposition 7.1. *We have*

$$E_{k,m}^{Kl}\left(\Phi\right)\notin\mathbb{E}_{k,m}.$$

Proof. We prove the existence of at least one prime p with $(p,m)=1$, such that

$$E_{k,m}^{\mathrm{Kl}}\left(\Phi\right)|_{k,m}\left(T^J(p)^{\uparrow}-T^J(p)^{\downarrow}\right)\neq 0.$$

This is in particular satisfied if its restriction to $\mathbb{H}^J\times\mathbb{H}^J$ has this property. This function equals

$$\left(T^J(p)\otimes\mathrm{id}-\mathrm{id}\otimes T^J(p)\right)\left(E_{k,m}^{\mathrm{Kl}}\left(\Phi\right)\Big|_{\mathbb{H}^J\times\mathbb{H}^J}\right). \tag{7.1}$$

In [3] Arakawa and the second author have shown that

$$E_{k,m}^{\mathrm{Kl}}\left(\Phi\right)|_{\mathbb{H}^J\times\mathbb{H}^J}=E_{k,m}^{J}\otimes\Phi+\Phi\otimes E_{k,m}^{J}+G,$$

where $G \in \mathrm{Sym}^2 J_{k,m}^{\mathrm{cusp}}$. Denoting by λ_E and λ_Φ the eigenvalues of $E_{k,m}^J$ and Φ with respect to the Hecke Jacobi operator $T^J(p)$ gives that (7.1) equals

$$(\lambda_E - \lambda_\Phi)\, E_{k,m}^J \otimes \Phi + (\lambda_\Phi - \lambda_E)\, \Phi \otimes E_{k,m}^J + G',$$

where $G' \in \mathrm{Sym}^2 J_{k,m}^{\mathrm{cusp}}$. There exists at least one prime p such that λ_E is different from λ_Φ, since these eigenvalues correspond to Eisenstein series and cusp forms of weight $2k-2$ (see [17]). □

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