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Global higher integrability of minimizers of variational problems with mixed boundary conditions

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Abstract

We consider functionals of the type

$$\mathcal{F}(u) := \int_\Omega F(x, u, \mathrm{D}u)\,\mathrm{d}x,$$

where $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain with mixed boundary conditions such that $\partial\Omega = \overline{\partial_D\Omega} \cup \overline{\partial_N\Omega}$. The aim of this paper is to prove that, under uniform estimates within certain classes of p-growth and coercivity assumptions on the density F, the minimizers u are of higher integrability order, meaning $u \in W^{1,p+\epsilon}(\Omega;\mathbb{R}^N)$ for a uniform $\epsilon > 0$. The results are applied to a model describing damage evolution in a nonlinear elastic body and to a model for shape memory alloys.

1 Introduction

This paper investigates integrability properties for vector-valued minimizers of integral functionals on nonsmooth domains with densities having p-growth and with mixed boundary conditions. The natural regularity for minimizers of such functionals is $W^{1,p}$-regularity.

However, in many applications further regularity properties for minimizers, or for solutions to PDEs are desirable. It is well-known that, in the case of solutions to PDEs, higher regularity properties of solutions allow, for instance, to predict convergence rates of numerical schemes for PDEs, or to derive first order necessary conditions in optimal control problems. Analogously, for minimizers of functionals of the type we are considering, it is fundamental in many applications to obtain higher integrability. In particular, higher integrability properties of minimizers are important in the discussion of elasticity problems which are coupled with further phenomena like phase separation or damage processes, and where the elasticity coefficients depend on the phase field or damage variables.

Since in typical applications the domain representing the model reference configuration has a nonsmooth boundary with corners and edges and since different types of boundary conditions may be imposed on different parts of the boundary, regularity results are needed that take into account all these peculiarities. We also highlight that in the applications to elasticity models with state-dependent coefficients, it is of great interest to study the robustness of the higher integrability properties with respect to classes of coefficients satisfying uniform bounds. Uniform bounds are required as well in general time-dependent models, in homogenization problems and in general problems where a further passage to the limit has to be performed.

The paper focuses on minimizers of integral functionals of the type

$$\mathcal{F}(u) = \int_\Omega F(x, u, \mathrm{D}u)\,\mathrm{d}x, \tag{1.1}$$

where $u : \Omega \to \mathbb{R}^N$ is a vector-valued function and Ω is the n-dimensional physical domain. The energy density F shall have p-growth and satisfy a suitable coercivity estimate. Convexity or differentiability of F in $\mathrm{D}u$ are not required. As main result, we derive the global higher integrability of minimizers and their gradients on nonsmooth domains, with mixed boundary conditions and with nonsmooth coefficients, i.e. we show that there exists $q > p$ such that the minimizers belong to $W^{1,q}(\Omega)$. Moreover, we provide results and estimates, which are uniform within certain classes of functionals.

We are aware that the higher integrability result itself is not "surprising". However, for systems with mixed boundary conditions on nonsmooth domains, no results are available in the literature, as we point out here below.

The investigation of higher integrability of solutions to elliptic PDEs and of minimizers of integral functionals has a long tradition. For scalar linear elliptic equations (i.e. $p = 2$) of second order with L^∞-coefficients and mixed boundary conditions a general integrability result was proved in [Grö89], stating that there exists a $q > 2$ such that the differential operator is an isomorphism between the Sobolev spaces $W_D^{1,q}(\Omega) = \{u \in W^{1,q}(\Omega) \,|\, u|_{\partial_D\Omega} = 0\}$ and $W^{-1,q}(\Omega)$ with suitable boundary conditions. This result was extended in [HMW11] to the system of linear elasticity and a closely related strongly monotone system for nonlinear elasticity ($p = 2$). The arguments used are localization principles, fixed point and reflection arguments and rely on a generalized Gårding inequality for the Laplace operator [Sim72].

A different approach was followed by Giaquinta and Giusti [GG82, Giu03] who in a first step showed, for the vector-valued case and general p, that minimizers of integral-functionals satisfy a Caccioppoli inequality which is a reverse Hölder inequality on increasing domains. Subsequently, a generalized Gehring-type lemma due to Giaquinta and Modica [GM79] allows to conclude the higher integrability of minimizers meaning that there exists $q > p$ such that minimizers belong to $W^{1,q}(\Omega)$. The results are proved for nonsmooth domains with pure Dirichlet boundary conditions for quite general functionals satisfying suitable upper p-growth estimates and a coercivity estimate. In [NW91] these arguments were extended to nonlinear elliptic systems of second order with mixed boundary conditions, with the Sobolev space $W^{1,2}(\Omega)$ as the basic space (i.e. $p = 2$). In [SW94], the elasticity system is studied on smooth domains with similar techniques. Further, refined estimates for smooth domains under strong convexity assumptions on the energy density can be found in [KM06]. To our knowledge, for the general vector-valued case with energy densities of general p-growth ($p \neq 2$) on nonsmooth domains with mixed boundary conditions no global integrability results are available in the literature.

In the proof of our main result, we follow the general lines of Naumann-Wolff's proof of higher integrability for systems of elliptic PDEs ([NW91]). More precisely, we use the localization techniques introduced in [Grö89] and we are interested in deriving Caccioppoli-like estimates for model problems on cubes with mixed boundary conditions. However, we do not deal with PDEs, but with minimizers of possibly non-differentiable functionals with general p-growth and more general coercivity assumptions, allowing to treat also elasticity models with symmetric gradients. For these reasons, the proof of the Caccioppoli-type inequality in our case is more delicate, compared to [NW91], and we need to adapt the tools presented in [Giu03] to the situation with mixed boundary condition. In view of the applications we have in mind, special attention is devoted to the uniformity of our estimates.

In the last part of the paper, we apply our Main Theorem 3.1 to time-dependent elastic models with internal parameters. In particular, we prove the higher integrability of the displacement field of a rate-independent damage model. This model is based on a quasi-convex elastic energy, a multiplicative coupling between the damage variable and the elastic fields and an inequality constraint on the damage variable preventing self-healing. We also use the higher integrability result to generalize the boundary condition considered in the models analyzed in [Fia10, FKS11] for phase transitions and damage evolution, respectively. The evolution considered there is written in terms of Young measures and to prove the stability condition, in particular, continuity properties along suitable sequences of minimizers are needed. This continuity is actually obtained thanks to the uniform higher integrability of the solutions to the discrete minimization problem. With similar arguments, the uniform global higher integrability finally is derived for the displacements and the internal variables of so-called stable states, cf. [FM06], occurring in the energetic formulation of rate-independent processes, see Section 4.5.

2 Setting of the problem and assumptions

Let $C_r(y) = y + (-r, r)^n \subset \mathbb{R}^n$ be the open cube with side length $2r$ centered in y.

Definition 2.1 (Regular domain [Grö89])**.** *A set $G \subset \mathbb{R}^n$ is called regular, if G is a bounded domain and if for every $x^0 \in \partial G$ there exist subsets $U_{x^0} \subset \mathbb{R}^n$ and a bi-Lipschitz transformation $T_{x^0} : U_{x^0} \to C_1(0)$ such that U_{x^0} is an open neighborhood of $x^0 \in \mathbb{R}^n$ and $T_{x^0}(U_{x^0}) = C_1(0)$. Furthermore, $T_{x^0}(x^0) = 0$ and the image $T_{x^0}(U_{x^0} \cap G)$ is one of the following sets:*

$$
\begin{aligned}
E_1 &:= \{y \in \mathbb{R}^n \,|\, |y|_\infty < 1, y_n > 0\},\\
E_2 &:= \{y \in \mathbb{R}^n \,|\, |y|_\infty < 1, y_n \geq 0\},\\
E_3 &:= \{y \in E_2 \,|\, y_n > 0 \text{ or } y_1 > 0\}.
\end{aligned}
$$

Here, $|\cdot|_\infty$ denotes the supremum norm so that $E_i, i = 1, 2, 3$, are n-dimensional cuboids. On regular domains in the sense of Definition 2.1, the Sobolev embedding theorems and therefore Poincaré type inequalities are valid, e.g. see Theorems 3.11, 3.12 and 3.13 in [Giu03]. In the proofs of the underlying Theorems 3.6 and 3.10 in [Giu03], one can easily check that the assumptions on the boundary therein are not more restrictive than in Definition 2.1. We set

$$
\begin{gathered}
\Omega = \overset{\circ}{G},\ \partial_N\Omega = (\overset{\circ}{G\backslash\Omega}) \text{ for the Neumann boundary and}\\
\partial_D\Omega = \partial\Omega\backslash\overline{\partial_N\Omega} \text{ for the Dirichlet boundary.}
\end{gathered} \tag{2.1}
$$

Definition 2.2 (Admissible test function [Grö89])**.** *For $p \in (1, \infty)$, we call φ an admissible test function on Ω, if $\varphi \in W^{1,p}_D(\Omega; \mathbb{R}^N)$ with*

$$
W^{1,p}_D(\Omega; \mathbb{R}^N) := \big\{\varphi \in W^{1,p}(\Omega; \mathbb{R}^N) \,\big|\, \varphi|_{\partial_D\Omega} = 0\big\}.
$$

The functional $\mathcal{F} : W_D^{1,p}(\Omega;\mathbb{R}^N) \to \mathbb{R}$ that will be subject to our upcoming investigations is defined as

$$\mathcal{F}(u) := \int_\Omega F(x, u, \mathrm{D}u)\,\mathrm{d}x,$$

with a density $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{n\times N} \to \mathbb{R}$, which will be specified later. The following definition of Q-minimizers is in the spirit of quasi-minimizers as in Definition 6.1 in [Giu03].

Definition 2.3 (Q-minimizer)**.** *We call $u \in W_D^{1,p}(\Omega;\mathbb{R}^N)$ a Q-minimizer of $\mathcal{F}$ for some $Q \geq 1$, if for all compact sets $K \subset \mathbb{R}^n$ and all $\varphi \in W_D^{1,p}(\Omega;\mathbb{R}^N)$ such that $\varphi|_{\Omega\setminus K} = 0$ there holds*

$$\int_{\Omega\cap K} F(x, u, Du)\, dx \leq Q \int_{\Omega\cap K} F(x, u+\varphi, D(u+\varphi))\, dx.$$

Obviously, every global minimizer of $\mathcal{F}$ with respect to $W_D^{1,p}(\Omega;\mathbb{R}^N)$ is a Q-minimizer with $Q = 1$. Throughout the paper the following assumptions should hold true.

Assumption on the domain Ω:

(A1) The set $G \subset \mathbb{R}^n$, $n \geq 2$ is regular as in Definition 2.1 and Ω, $\partial_D\Omega$, $\partial_N\Omega$ are given as in (2.1). Note that $\partial_D\Omega$ or $\partial_N\Omega$ may possibly be empty.

Assumptions on the integrand F:

(A2) The volume density $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{n\times N} \to \mathbb{R}$ is a Carathéodory function satisfying for $p \in (1,\infty)$, $c_0 > 0$ and almost all $x \in \Omega$ and all $u \in \mathbb{R}^N$, $A \in \mathbb{R}^{n\times N}$ the upper estimate

$$|F(x, u, A)| \leq c_0 \left(|A| + \vartheta(x, u)\right)^p,$$

where $\vartheta(x, u)^p = b_1(x)|u|^\gamma + b_2(x)$ for some

$$\gamma \in \begin{cases} (0, p^*) & \text{with } p^* = \frac{pn}{n-p}, \quad \text{if } p < n, \\ (0,\infty), & \text{if } p \geq n. \end{cases} \tag{2.2}$$

Here, the functions b_1 and b_2 satisfy $b_2 \in L^1(\Omega)$ and $b_1 \in L^\sigma(\Omega)$ with $b_1, b_2 \geq 0$ and

$$\sigma = \begin{cases} \frac{p^*}{p^*-\gamma}, & \text{if } p < n, \\ 1+\delta, & \text{if } p = n, \\ 1, & \text{if } p > n, \end{cases} \tag{2.3}$$

for some arbitrary $\delta > 0$.

We define for $p \leq n$:

$$p^* := \begin{cases} \frac{pn}{n-p}, & \text{if } p < n, \\ \frac{\gamma(1+\delta)}{\delta}, & \text{if } p = n, \end{cases} \tag{2.4}$$

with γ, δ as in (A2). Observe that $W^{1,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$ for $p \leq n$ and for $p > n$, we find $W^{1,p}(\Omega) \subset L^\infty(\Omega)$.

(A3) There exists a function $\widetilde{F} : \mathbb{R}^{n\times N} \to \mathbb{R}$ such that

(a) $\widetilde{F}(0) = 0$,

(b) there exist constants $\nu > 0$, $c_1 \geq 0$ such that for every $\varphi \in W^{1,p}_D(\Omega;\mathbb{R}^N)$ there holds

$$\int_\Omega \widetilde{F}(\mathrm{D}\varphi)\,\mathrm{d}x \geq \nu\|\mathrm{D}\varphi\|^p_{L^p(\Omega)} - c_1\|\varphi\|^p_{L^p(\Omega)},$$

(c) for almost all $x \in \Omega$ and all $u \in \mathbb{R}^N$, $z \in \mathbb{R}^{n\times N}$ the lower estimate

$$F(x, u, A) \geq \widetilde{F}(A) - \vartheta(x, u)^p$$

is valid.

Remark 2.1. *According to Theorem 4.3.1 in [Zie89], an element ℓ of the dual space $W^{1,p}_D(\Omega;\mathbb{R}^N)^*$ can be represented by a pair $(H_0, H_1) \in L^{p'}(\Omega;\mathbb{R}^N) \times L^{p'}(\Omega;\mathbb{R}^{n\times N})$ with $\frac{1}{p} + \frac{1}{p'} = 1$ such that*

$$\langle \ell, u\rangle = \int_\Omega H_0 \cdot u + H_1 : \nabla u \, dx \qquad \text{for all } u \in W^{1,p}_D(\Omega;\mathbb{R}^N).$$

Here, $A : B = tr(A^T B)$ for $A, B \in \mathbb{R}^{n\times N}$. If the assumptions (A2)-(A3) hold true for a function $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{n\times N} \to \mathbb{R}$, then for every (H_0, H_1), the function

$$\hat{F}(x, u, A) := F(x, u, A) - H_0 \cdot u - H_1 : A$$

satisfies (A2)-(A3), too, with $\hat{\gamma}, \hat{b}_1, \hat{b}_2$ and $\widetilde{\hat{F}}$ as given below:

$$\begin{aligned}
&\textit{Case } \gamma \in (0,1): && \hat{\gamma} = 1,\ \hat{\sigma} = \begin{cases} \frac{p^*}{p^*-1}, & \text{if } p < n, \\ 1 + \hat{\delta} \text{ for some } \hat{\delta} > 0, & \text{if } p = n, \\ 1, & \text{if } p > n, \end{cases} \\
& && \hat{c}_0 = c(c_0, p, \gamma) \text{ and } \hat{\vartheta}(x, u)^p = \hat{b}_1|u|^{\hat{\gamma}} + \hat{b}_2 \text{ with} \\
& && \hat{b}_1 = b_1^{(1-\sigma(1-\gamma))\gamma^{-1}} + |H_0| \in L^{\hat{\sigma}}(\Omega),\ \hat{b}_2 = b_2 + b_1^{\sigma} + |H_1|^{p'} \in L^1(\Omega). \\
&\textit{Case } \gamma \in [1, p^*): && \hat{\gamma} = \gamma,\ \hat{\sigma} = \sigma \\
& && \hat{c}_0 = c(c_0, p, \gamma) \text{ and } \hat{\vartheta}(x, u)^p = \hat{b}_1|u|^{\hat{\gamma}} + \hat{b}_2 \text{ with} \\
& && \hat{b}_1 = b_1 + |H_0|^{\frac{p-\gamma}{p-1}} \in L^{\hat{\sigma}}(\Omega),\ \hat{b}_2 = b_2 + |H_0|^{p'} + |H_1|^{p'} \in L^1(\Omega). \\
&\textit{In any case:} && \widetilde{\hat{F}}(A) = \widetilde{F}(A) - \tfrac{\nu}{2}|A|^p,\ \hat{\nu} = \tfrac{\nu}{2} \text{ and } \hat{c}_1 = c_1.
\end{aligned}$$

For the sake of Remark 2.1, we can absorb linear functionals of the type $u \mapsto \langle \ell, u\rangle$ into the density F.

Definition 2.4. *For a set $G \subset \mathbb{R}^n$ satisfying (A1) and a set of parameters and functions $(p, \nu, c_0, c_1, b_1, b_2, \gamma)$ with $p \in (1, \infty)$, $\nu, c_0 > 0$, $c_1 \geq 0$, $b_1 \in L^\sigma(\Omega)$, $b_2 \in L^1(\Omega)$ with σ and γ as in (2.3) and (2.2), respectively, we introduce the class of functionals*

$$\mathbb{F}(\Omega, p, \nu, c_0, c_1, b_1, b_2, \gamma) := \Big\{ \textit{functionals } \mathcal{F}(u) = \int_\Omega F(x, u, Du)\, dx \textit{ with densities } F \textit{ fulfilling}$$
$$\textit{assumptions (A2)-(A3) with the set of parameters and functions } (p, \nu, c_0, c_1, b_1, b_2, \gamma) \Big\}.$$

3 Main Theorem and proof

The main result of this paper is the following theorem, which states the uniform higher integrability for the gradient Du of Q-minimizers u.

Theorem 3.1 (Main Theorem)**.** *Assume (A1) holds true and let the set of parameters and functions $(p, \nu,\ c_0, c_1, b_1, b_2, \gamma)$ be chosen according to Definition 2.4. Further, let $Q \geq 1$, p^* as in (2.4), σ as in (2.3) and assume that,*

$$\textit{there exists } t > 1 \textit{ such that } b_1^\sigma, b_2 \in L^t(\Omega). \tag{3.1}$$

Let further $C_b \geq 0$. Then there exist constants $c > 0$ and $q > 1$ such that for all $\mathcal{F} \in \mathbb{F}(\Omega, p, \nu, c_0, c_1, b_1, b_2, \gamma)$ and all Q-minimizers $u \in W_D^{1,p}(\Omega; \mathbb{R}^N)$ of $\mathcal{F}$ satisfying

$$\|u\|_{W^{1,p}(\Omega)} \leq C_b, \tag{3.2}$$

it holds: $Du \in L^{pq}(\Omega; \mathbb{R}^{n\times N})$. Moreover, if $p \leq n$ there holds

$$\int_\Omega \left(|Du|^p + |u|^{p^*}\right)^q\, dx \leq c\left\{ \left(\int_\Omega |Du|^p + |u|^{p^*}\, dx \right)^q + \int_\Omega (b_1^\sigma + b_2 + 1)^q\, dx \right\}, \tag{3.3}$$

and if $p > n$

$$\int_\Omega |Du|^{pq}\, dx \leq c\left\{ \left(\int_\Omega |Du|^p\, dx \right)^q + \int_\Omega (b_1^\sigma + b_2 + 1)^q\, dx \right\}. \tag{3.4}$$

Remark 3.1. *In consideration of the upcoming Remarks 3.3 and A.1 in the Appendix, the constants c and q in Theorem 3.1 only depend on the parameters $Q, p, \nu, c_0, c_1, \gamma, C_b$ and the full norm $\|b_1^\sigma + b_2\|_{L^1(\Omega)}$, but not on local properties of b_1 and b_2.*

The Main Theorem provides uniform higher integrability estimates for all minimizers of functionals belonging to certain classes $\mathbb{F}(\Omega, p, \nu, c_0, c_1, b_1, b_2, \gamma)$ and admitting the same upper bound C_b. A sufficient condition leading to such uniform bounds is formulated in Lemma 3.1, here below.

Lemma 3.1. *Assume that Ω satisfies assumption (A1) and $\lambda^{n-1}(\partial_D\Omega) > 0$. Let the set of parameters and functions $(p, \nu, c_0, c_1, b_1, b_2, \gamma)$ be chosen as in Definition 2.4 with $\gamma < p$ and*

$c_1 = 0$. Let further $Q \geq 1$. Then there exists a constant $c > 0$ such that for all $\mathcal{F}$ from $\mathbb{F}(\Omega, p, \nu, c_0, c_1, b_1, b_2, \gamma)$ and all Q-minimizers $u \in W_D^{1,p}(\Omega; \mathbb{R}^N)$ of $\mathcal{F}$, we have

$$\|u\|^p_{W^{1,p}(\Omega)} \leq c \left(\|b_1\|^\alpha_{L^\sigma(\Omega)} + \|b_2\|_{L^1(\Omega)} \right),$$

where $\alpha = \frac{p}{p-\gamma} \in (1, \infty)$ and c only depend on the parameters p, ν, c_0, γ and Q.

Proof. Let $u \in W_D^{1,p}(\Omega; \mathbb{R}^N)$ be a Q-minimizer of $\mathcal{F}$ for an arbitrary $\mathcal{F}$ belonging to $\mathbb{F}(\Omega, p, \nu, c_0, c_1, b_1, b_2, \gamma)$. By choosing $\varphi = -u$ and K such that $\Omega \cap K = \Omega$ in Definition 2.3, we find

$$\mathcal{F}(u) \leq Q\mathcal{F}(0) \leq Q \int_\Omega c_0 \vartheta(x, 0)^p \, \mathrm{d}x = Qc(c_0)\|b_2\|_{L^1(\Omega)}. \tag{3.5}$$

Now we derive a lower estimate for $\mathcal{F}(u)$ by exploiting the assumption (A3) with $c_1 = 0$:

$$\mathcal{F}(u) \geq \int_\Omega \widetilde{F}(\mathrm{D}u) - \vartheta(x, u)^p \, \mathrm{d}x \geq \nu\|\mathrm{D}u\|^p_{L^p(\Omega)} - \int_\Omega b_1|u|^\gamma + b_2 \, \mathrm{d}x. \tag{3.6}$$

Combining (3.5)-(3.6), one obtains

$$\nu\|\mathrm{D}u\|^p_{L^p(\Omega)} \leq c(Q, c_0)\|b_2\|_{L^1(\Omega)} + \int_\Omega b_1|u|^\gamma \, \mathrm{d}x. \tag{3.7}$$

In the case $p \leq n$, applying Hölder's inequality with σ and $\sigma' = \frac{p^*}{\gamma}$ and taking into account the embedding $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ in combination with the Poincaré inequality gives

$$\nu\|\mathrm{D}u\|^p_{L^p(\Omega)} \leq c(Q, p, c_0) \left(\|b_2\|_{L^1(\Omega)} + \|b_1\|_{L^\sigma(\Omega)} \|\mathrm{D}u\|^\gamma_{L^p(\Omega)} \right). \tag{3.8}$$

In the case $p > n$, similar considerations based on the embedding $W^{1,p}(\Omega) \subset L^\infty(\Omega)$ yield (3.8), as well. We apply Young's inequality to the second term on the right-hand side in (3.8) with $\alpha = \frac{p}{p-\gamma} \in (1, \infty)$ and $\alpha' = \frac{p}{\gamma}$ so that we obtain for every $\epsilon > 0$

$$\nu\|\mathrm{D}u\|^p_{L^p(\Omega)} \leq c(Q, p, c_0) \left(\|b_2\|_{L^1(\Omega)} + C(\epsilon)\|b_1\|^\alpha_{L^\sigma(\Omega)} + \epsilon\|\mathrm{D}u\|^p_{L^p(\Omega)} \right). \tag{3.9}$$

If we now choose $\epsilon < \frac{\nu}{2c}$, we obtain by the classical Poincaré inequality

$$\|u\|^p_{W^{1,p}(\Omega)} \leq 2c(Q, p, \nu, c_0, \gamma) \left(\|b_2\|_{L^1(\Omega)} + \|b_1\|^\alpha_{L^\sigma(\Omega)} \right).$$

□

The proof of the Main Theorem 3.1 will now be given in several steps and follows the structure of [NW91]:

1. Transformation of the open sets $U_{x^0} \subset \mathbb{R}^n$ onto cubes $C_1(0)$ with a one-to-one Lipschitz mapping T_{x^0}.

2. Proof of a Caccioppoli-type inequality in the spirit of Theorem 6.5 in [Giu03] for a model problem on the half cube E_3 (Lemmata 3.2 and 3.3).

3. Extension of the estimates from half cubes to full cubes by reflection (Corollary 3.1).
4. Deriving from Caccioppoli's inequality the higher integrability of the gradient $\mathrm{D}u$ by applying a result from Giaquinta and Modica (Theorem A.3), which is based on Gehring's lemma.

3.1 The Transformation T

We recall that $C_1(0)$ is the unit cube with side length 2 centered in 0 and $C_1^+(0)$ is its upper half. For $x^0 \in \partial\Omega$, let T_{x^0} be the bi-Lipschitz transformation $T_{x^0} : U_{x^0} \cap \Omega \to C_1^+(0)$ with $T_{x^0}(x) = y$, existing after (A1). Since the domain $\overline{\Omega}$ is compact, there exists an open covering of $\overline{\Omega}$ of the form

$$\overline{\Omega} \subset \Omega_0 \cup \bigcup_{i=1}^{N} T_{x_i^0}^{-1}(C_{\frac{1}{8}}(0)), \tag{3.10}$$

for a finite number of $x_i^0 \in \partial\Omega$ and some set $\Omega_0 \subset\subset \Omega$. In the following, we focus on the boundary sets U_{x^0}. The higher integrability result for Ω_0 can be found in Definition 6.3 and Theorem 6.7 in [Giu03]. Further, there exist $0 < \lambda_0 \le \lambda^* < \infty$ such that for

$$\lambda_i(y) := |\det \mathrm{D}T_{x_i^0}^{-1}(y)|$$

we have $\lambda_i \in L^\infty(C_1(0))$ with $\lambda_0 \le \lambda_i(y) \le \lambda^*$ almost everywhere. For $x^0 \in \{x_i^0\}_{i=1}^N$ fixed, we set

$$v := u \circ T_{x^0}^{-1} \quad \text{and} \quad \hat{F}(y, v, A) := F(T_{x^0}^{-1}(y), v, A\mathrm{D}T_{x^0}|_{T_{x^0}^{-1}(y)})\lambda_i(y). \tag{3.11}$$

In the following, we suppress the dependency of the transformation T_{x^0} on x^0. The transformation formula yields for $T^{-1}(C_1^+(0)) = U_{x^0} \cap \Omega$:

$$\begin{aligned}
&\int_{U_{x^0}\cap\Omega} F(x, u(x), \mathrm{D}_x u(x))\,\mathrm{d}x \\
&\quad = \int_{C_1^+(0)} F(T^{-1}(y), u(T^{-1}(y)), \mathrm{D}_x u(T^{-1}(y))\mathrm{D}T|_{T^{-1}(y)})|\det \mathrm{D}T^{-1}(y)|\,\mathrm{d}y \\
&\quad = \int_{C_1^+(0)} \hat{F}(y, v(y), \mathrm{D}_y v(y))\,\mathrm{d}y.
\end{aligned}$$

Next we show that there exist constants and functions $\hat{\nu}, \hat{c}_0, \hat{c}_1, \hat{b}_1, \hat{b}_2$ with $\hat{\nu}, \hat{c}_0 > 0, \hat{c}_1 \ge 0$, $\hat{b}_1 \in L^\sigma(C_1^+(0)), \hat{b}_2 \in L^1(C_1^+(0))$ such that for all $\mathcal{F} \in \mathbb{F}(\Omega, p, \nu, c_0, c_1, b_1, b_2, \gamma)$ and all $x^0 \in \{x_i^0\}_{i=1}^N$ it holds

$$\hat{\mathcal{F}}(w) := \int_{C_1^+(0)} \hat{F}(y, w, \mathrm{D}w)\,\mathrm{d}y \in \mathbb{F}(C_1^+, p, \hat{\nu}, \hat{c}_0, \hat{c}_1, \hat{b}_1, \hat{b}_2, \gamma).$$

Indeed, let us set

$$\widetilde{\hat{F}}(y, A) := \widetilde{F}(A\,\mathrm{D}T|_{T^{-1}(y)})\lambda_i(y) \quad \text{and} \quad \hat{\vartheta}(y, w) := \vartheta(T^{-1}(y), w)\lambda_i(y)^{\frac{1}{p}}.$$

Then it obviously holds for almost all $y \in C_1^+(0)$ and all $v \in \mathbb{R}^N, A \in \mathbb{R}^{n\times N}$ that

(A2*) $|\hat{F}(y,w,A)| \leq c(p,c_0,\lambda^*,\rho^*)\left(|A|+\hat{\vartheta}(y,w)\right)^p$, where $\rho^* = \max_{x^0}\Big\{1, \epsilon^{\frac{p}{2}} \,|\, \epsilon$ is the biggest eigenvalue of the matrix $(\mathrm{D}T_{x^0})^T(\mathrm{D}T_{x^0})\Big\}$,

(A3*) $\hat{F}(y,w,A) \geq \widetilde{\hat{F}}(y,A) - \hat{\vartheta}(y,w)^p$. Moreover, given a function $\psi \in W^{1,p}(C_1^+(0);\mathbb{R}^N)$ with

$$\psi|_{\partial C_1^+(0)\setminus(E_i\cap\{y_n=0\})} = 0, \tag{3.12}$$

where $E_i = T(U_{x^0}\cap G)$, there holds for $\varphi(x) := \psi(T(x))$ with the extension $\varphi = 0$ on $\Omega\setminus U_{x^0}$:

$$\begin{aligned}
\int_{C_1^+(0)} \widetilde{\hat{F}}(y,\mathrm{D}_y\psi)\,\mathrm{d}y &= \int_{U_{x^0}\cap\Omega} \widetilde{F}(\mathrm{D}_x\varphi(x))\,\mathrm{d}x \\
&= \int_{\Omega} \widetilde{F}(\mathrm{D}_x\varphi(x))\,\mathrm{d}x \quad (\text{since } \widetilde{F}(0)=0) \\
&\underset{(A3)}{\geq} \nu\int_{U_{x^0}} |\mathrm{D}_x\varphi(x)|^p\,\mathrm{d}x - c_1\int_{U_{x^0}} |\varphi(x)|^p\,\mathrm{d}x \\
&= \nu\int_{C_1^+(0)} \left|\mathrm{D}\psi(y)\mathrm{D}T|_{x=T^{-1}(y)}\right|^p |\det \mathrm{D}T^{-1}(y)|\,\mathrm{d}y \\
&\quad -c_1\int_{C_1^+(0)} |\psi(y)|^p |\det \mathrm{D}T^{-1}(y)|\,\mathrm{d}y \\
&\geq \nu\lambda_0\rho_0\int_{C_1^+(0)} |\mathrm{D}\psi(y)|^p\,\mathrm{d}y - c_1\lambda^*\int_{C_1^+(0)} |\psi(y)|^p\,\mathrm{d}y,
\end{aligned}$$

where $\rho_0 = \min_{x^0}\Big\{\epsilon^{\frac{p}{2}} \,|\, \epsilon$ is the smallest eigenvalue of the matrix $(\mathrm{D}T_{x^0})^T(\mathrm{D}T_{x^0})\Big\}$.

Remark 3.2. *Let $u \in W_D^{1,p}(\Omega;\mathbb{R}^N)$ be a Q-minimizer of $\mathcal{F}$. Then $v = (u|_{U_{x^0}\cap\Omega})\circ T^{-1} \in W^{1,p}(C_1^+(0);\mathbb{R}^N)$ is a Q-minimizer of the functional $\hat{\mathcal{F}}$ with*

$$\hat{\mathcal{F}}(w) := \int_{C_1^+(0)} \hat{F}(y,w,Dw)\,dy$$

with $\hat{F}$ as in (3.11), in the following sense: For every compact set $K \subset \mathbb{R}^n$ and every $\psi \in W^{1,p}(C_1^+(0);\mathbb{R}^N)$ satisfying (3.12) and with $\psi_{C_1^+(0)\setminus K} = 0$ it holds

$$\int_{C_1^+(0)\cap K} \hat{F}(y,v,Dv)\,dy \leq Q\int_{C_1^+(0)\cap K} \hat{F}(y,v+\psi,D(v+\psi))\,dy.$$

Indeed, let $\psi \in W^{1,p}(C_1^+(0);\mathbb{R}^N)$ satisfy (3.12) and $\psi|_{C_1^+(0)\setminus K} = 0$. Then the function $\varphi : \Omega \to \mathbb{R}^N$ defined by

$$\varphi(x) := \begin{cases} \psi(T(x)), & \text{if } x \in T^{-1}(C_1^+(0)), \\ 0, & \text{if } x \in \Omega\setminus T^{-1}(C_1^+(0)), \end{cases}$$

is an admissible test function for $\mathcal{F}$ in $W^{1,p}_D(\Omega;\mathbb{R}^N)$ with $\varphi|_{\Omega\setminus\overline{T^{-1}(C_1^+(0)\cap K)}}=0$ and since u is a Q-minimizer, it follows that

$$\int_{C_1^+(0)\cap K}\hat{F}(y,v,\mathrm{D}v)\,\mathrm{d}y=\int_{T^{-1}(C_1^+(0)\cap K)}F(x,u,\mathrm{D}u)\,\mathrm{d}x$$
$$\leq Q\int_{T^{-1}(C_1^+(0)\cap K)}F(x,u+\varphi,\mathrm{D}(u+\varphi))\,\mathrm{d}x=Q\int_{C_1^+(0)\cap K}\hat{F}(y,v+\psi,\mathrm{D}(v+\psi))\,\mathrm{d}y.$$

From now on, we denote $\hat{F}$ with f for shortness.

3.2 A Caccioppoli-type inequality on E_3

Depending on the choice of $x^0\in\overline{\partial\Omega}$, we are left with three different cases:

1. For $x^0\in\partial_D\Omega$, we have $T_{x^0}(U_{x^0}\cap G)=E_1$,
2. For $x^0\in\partial_N\Omega$, we have $T_{x^0}(U_{x^0}\cap G)=E_2$,
3. For $x^0\in\overline{\partial_D\Omega}\cap\overline{\partial_N\Omega}$, we have $T_{x^0}(U_{x^0}\cap G)=E_3$,

where we concentrate on the last case. The other cases can be seen as special cases of the third one. If $x^0\in\Omega_0$, see (3.10), the subsequent considerations can be adapted to this case in a straightforward manner. We will now study a model problem on E_3, defining

$$\Gamma_D:=\{y\in\mathbb{R}^n\,|\,y_1<0\text{ and }y_n=0\}\quad\text{and}\quad\Gamma_N:=\{y\in\mathbb{R}^n\,|\,y_1\geq 0\text{ and }y_n=0\}$$

as the images of the Dirichlet and Neumann boundary under the transformation with T_{x^0}. Further, we define the set of admissible test functions by

$$W^{1,p}_{\mathrm{ad}}(C_1^+(0);\mathbb{R}^N):=\big\{\psi\in W^{1,p}(C_1^+(0);\mathbb{R}^N)\,\big|\,\psi|_{\partial C_1^+(0)\setminus\Gamma_N}=0\big\}.$$

A Caccioppoli-type inequality will be derived for functions $v\in W^{1,p}_{\Gamma_D}(C_1^+(0);\mathbb{R}^N)$, where

$$W^{1,p}_{\Gamma_D}(C_1^+(0);\mathbb{R}^N):=\big\{\psi\in W^{1,p}(C_1^+(0);\mathbb{R}^N)\,\big|\,\psi|_{\Gamma_D}=0\big\},$$

satisfying for some $Q\geq 1$ and all compact sets $K\subset\mathbb{R}^n$ and all $\psi\in W^{1,p}_{\mathrm{ad}}(C_1^+(0);\mathbb{R}^N)$ with $\psi|_{C_1^+(0)\setminus K}=0$ the inequality

$$\int_{C_1^+(0)\cap K}f(y,v,\mathrm{D}v)\,\mathrm{d}y\leq Q\int_{C_1^+(0)\cap K}f(y,v+\psi,\mathrm{D}(v+\psi))\,\mathrm{d}y.\tag{3.13}$$

Let us introduce further notation of open cuboids by setting

$$C_r^+(y^0):=C_r(y^0)\cap E_1\quad\text{and}\quad C_r^-(y^0):=C_r(y^0)\setminus\overline{C_r^+(y^0)}.$$

Hence for $y^0\in C^+_{1/4}(0)$ and $0<r<\frac{1}{4}$, we have $C_{3r}(y^0)\subset C_1(0)$. In Lemmata 3.2 and 3.3, here below, we distinguish two cases for the test cuboid $C_r^+(y^0)$:

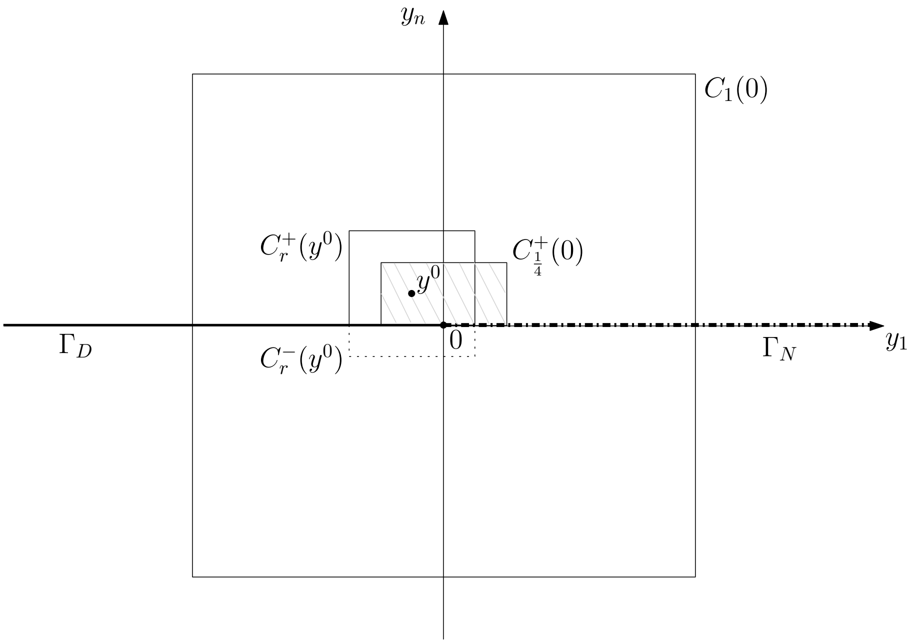

Figure 1: Position of the cube $C_r^+(y^0)$ in the Case (II).

(I) The test cuboid $C_r^+(y^0)$ has no Dirichlet boundary: $C_r(y^0) \cap \Gamma_D = \emptyset$.

(II) The test cuboid $C_r^+(y^0)$ has a Dirichlet boundary: $C_r(y^0) \cap \Gamma_D \neq \emptyset$.

Lemma 3.2. *Let $y^0 \in C_{1/4}^+(0)$ and $0 < r < \frac{1}{4}$. Assume (I) and let $\hat{p} = \frac{pn}{n+p}$. For every $Q \geq 1$ and $C_b \geq 0$, there exists a constant $c > 0$, independent of r and y^0 such that for all functions $v \in W^{1,p}_{\Gamma_D}(C_1^+(0); \mathbb{R}^N)$ with $\|v\|_{W^{1,p}(C_1^+)} \leq C_b$ satisfying (3.13) for some*

$$\mathcal{F}(w) = \int_{C_1^+} f(y, w, Dw)\, dy \in \mathbb{F}(C_1^+, p, \nu, c_0, c_1, b_1, b_2, \gamma)$$

it holds: If $p \leq n$ and p^* as in (2.4)

$$\int_{C^+_{\frac{r}{2}}(y^0)} |Dv|^p + |v|^{p^*}\, dy \leq c \Bigg\{ \frac{1}{r^p} \left(\int_{C_r^+(y^0)} \left(|Dv|^p + |v|^{p^*} \right)^{\frac{\hat{p}}{p}} dy \right)^{\frac{p}{\hat{p}}} + \int_{C_r^+(y^0)} b_1^\sigma + b_2 + 1\, dy \Bigg\}. \tag{3.14}$$

In the case $p > n$ it holds

$$\int_{C^+_{\frac{r}{2}}(y^0)} |Dv|^p\, dy \leq c \Bigg\{ \frac{1}{r^p} \left(\int_{C_r^+(y^0)} |Dv|^{\hat{p}}\, dy \right)^{\frac{p}{\hat{p}}} + \int_{C_r^+(y^0)} b_1 + b_2 + 1\, dy \Bigg\}. \tag{3.15}$$

Remark 3.3. *The constant c in Lemma 3.2 only depends on the parameters $Q, p, \nu, c_0, c_1, \gamma$ and the uniform bound C_b. Either C_b is given or derived depending on the given data as in Lemma 3.1. In any case, C_b is independent of local properties of the given data b_1 and b_2 and so is c.*

Observe that, for all $p \in (1, \infty)$, there holds $\hat{p} \le p$ and we find $\hat{p}^* = p$, i.e. $W^{1,\hat{p}}(\Omega) \subset L^p(\Omega)$.

Proof. Let $y^0 \in C^+_{1/4}(0)$ and $0 < r < \frac{1}{4}$ so that $C_{3r}(y^0) \subset C_1(0)$ and assume that $C_r(y^0) \cap \Gamma_D = \emptyset$. In the following we omit the variable y^0 and just write C_r instead of $C_r(y^0)$ and C_r^+ instead of $C_r^+(y^0)$. Assume $v \in W^{1,p}_{\Gamma_D}(C_1^+; \mathbb{R}^N)$ is a Q-minimizer satisfying (3.13) for some arbitrary $\mathcal{F} \in \mathbb{F}(C_1^+, p, \nu, c_0, c_1, b_1, b_2, \gamma)$. Let $\frac{r}{2} \le t < s \le r$. Moreover, let $\varsigma \in C^1(\mathbb{R}^n; [0,1])$ be a cut-off function such that

$$\varsigma|_{C_t} = 1, \quad \varsigma|_{\mathbb{R}^n \setminus C_s} = 0 \quad \text{and} \quad |\mathrm{D}\varsigma| \le \tfrac{\omega}{s-t}, \tag{3.16}$$

with $\omega > 0$, independent of t and s. We set $\psi := (v - v_C)\varsigma$, where $v_C = \fint_{C_s^+} v \, \mathrm{d}y$ is the mean value of v, and we observe that $\psi \in W^{1,p}_{\mathrm{ad}}(C_1^+; \mathbb{R}^N)$ is an admissible test function on C_s^+. The gradient for the difference $v - \psi = v_C + (1-\varsigma)(v - v_C)$ satisfies

$$\mathrm{D}(v - \psi) = (v_C - v)\mathrm{D}\varsigma + (1-\varsigma)\mathrm{D}v. \tag{3.17}$$

Further, we estimate the gradient $|\mathrm{D}v|^p$ with $\mathrm{D}v = \mathrm{D}\psi$ on C_t^+ by using (A3) as follows:

$$\begin{aligned} \int_{C_t^+} |\mathrm{D}v|^p \, \mathrm{d}y &\le \int_{C_s^+} |\mathrm{D}\psi|^p \, \mathrm{d}y \\ &\le c(\nu) \int_{C_s^+} f(y, v, \mathrm{D}\psi) + \vartheta(y,v)^p + c_1 |\psi|^p \, \mathrm{d}y \\ &= c(\nu) \int_{C_s^+} f(y, v, \mathrm{D}v) + f(y, v, \mathrm{D}\psi) - f(y, v, \mathrm{D}v) + \vartheta(y,v)^p + c_1 |\psi|^p \, \mathrm{d}y. \end{aligned} \tag{3.18}$$

The proof will now be given for the cases $p \le n$ and $p > n$, separately. We start with the first case following the argumentation of Theorem 6.5 in [Giu03] closely.

Case $p \le n$: We recall that from the definition of p^* in (2.4) and the Sobolev embedding theorems, it follows that $W^{1,p}(C_1^+; \mathbb{R}^N)$ is continuously embedded in $L^{p^*}(C_1^+; \mathbb{R}^N)$. Next we add the quantity $\mu \int_{C_t^+} |v|^{p^*} \, \mathrm{d}y$ to both sides of (3.18) and choose the constant $\mu > 0$ later on so that we have for now

$$\begin{aligned} \int_{C_t^+} |\mathrm{D}v|^p + \mu |v|^{p^*} \, \mathrm{d}y \le c(\nu) \int_{C_s^+} & \underbrace{f(y, v, \mathrm{D}v) + \mu |v|^{p^*}}_{\text{Step 1}} \\ & + \underbrace{f(y, v, \mathrm{D}\psi) - f(y, v, \mathrm{D}v) + \vartheta(y,v)^p + c_1 |\psi|^p}_{\text{Step 2}} \, \mathrm{d}y. \end{aligned} \tag{3.19}$$

We continue to estimate the right-hand side of (3.19) from above in three steps.
Step 1. By exploiting (3.13) with $K = \overline{C_s^+}$, we obtain

$$\int_{C_s^+} f(y, v, \mathrm{D}v) \, \mathrm{d}y \le Q \int_{C_s^+} f(y, v - \psi, \mathrm{D}(v - \psi)) \, \mathrm{d}y. \tag{3.20}$$

Reinserting the term $\mu \int_{C_s^+} |v|^{p^*} \mathrm{d}y$ in (3.20) yields

$$\begin{aligned}
\int_{C_s^+} f(y,v,\mathrm{D}v) + \mu|v|^{p^*}\,\mathrm{d}y &\leq c(Q)\left\{\int_{C_s^+} |f(y,v-\psi,\mathrm{D}(v-\psi))| + \mu|v|^{p^*}\,\mathrm{d}y\right\} \\
&\underset{(A2)}{\leq} c(Q,c_0)\left\{\int_{C_s^+} |\mathrm{D}(v-\psi)|^p + \vartheta(y,v-\psi)^p + \mu|v|^{p^*}\,\mathrm{d}y\right\} \\
&\underset{(3.17)}{\leq} c(Q,p,c_0)\left\{\int_{C_s^+} |(v_C-v)\mathrm{D}\varsigma|^p + |(1-\varsigma)\mathrm{D}v|^p + \mu|v|^{p^*} + b_1|v-\psi|^\gamma + b_2\,\mathrm{d}y\right\}. \qquad (3.21)
\end{aligned}$$

By applying Young's inequality with $\epsilon_1 > 0$, the estimation of the product $b_1|v-\psi|^\gamma$ yields (with $\sigma > 1$ and $\sigma' = \frac{p^*}{\gamma}$)

$$b_1|v|^\gamma \leq C(\epsilon_1)b_1^\sigma + \tfrac{\gamma}{p^*}\epsilon_1^{\frac{p^*}{\gamma}}|v|^{p^*}. \qquad (3.22)$$

Introducing the triangle inequalities $|v-\psi| \leq |v-v_C| + |v_C|$ as well as $|v| \leq |v-v_C| + |v_C|$ and remembering the assumptions on the cut-off function ς, inequality (3.21) takes the form

$$\begin{aligned}
\int_{C_s^+} f(y,v,\mathrm{D}v) + \mu|v|^{p^*}\,\mathrm{d}y \leq c(Q,p,\nu,c_0,\gamma)\bigg\{ \int_{C_s^+} \frac{1}{(s-t)^p}|v-v_C|^p \\
+ c(\mu,\epsilon_1)\left(|v-v_C|^{p^*} + |v_C|^{p^*}\right) + C(\epsilon_1)b_1^\sigma + b_2\,\mathrm{d}y + \int_{C_s^+\setminus C_t^+} |\mathrm{D}v|^p\,\mathrm{d}y\bigg\}, \qquad (3.23)
\end{aligned}$$

where $c(\mu,\epsilon_1) = \left(\mu + \epsilon_1^{\frac{p^*}{\gamma}}\right)$. The factor $\frac{\gamma}{p^*}$ is absorbed by the first constant $c(Q,p,\nu,c_0,\gamma)$. We are now treating the difference $|v-v_C|^{p^*}$ by applying Lemma A.1 with a constant $C_{e,P} > 0$ independent of s so that we obtain

$$\begin{aligned}
\int_{C_s^+} |v-v_C|^{p^*}\mathrm{d}y &\leq C_{e,P}^{p^*}\left(\int_{C_s^+} |\mathrm{D}v|^p\,\mathrm{d}y\right)^{\frac{p^*}{p}} \\
&= C_{e,P}^{p^*}\left(\int_{C_s^+} |\mathrm{D}v|^p\,\mathrm{d}y\right)^{\frac{p^*-p}{p}}\left(\int_{C_s^+} |\mathrm{D}v|^p\,\mathrm{d}y\right) \\
&\leq C_{e,P}^{p^*}C_b^{\frac{p^*-p}{p}}\int_{C_s^+} |\mathrm{D}v|^p\,\mathrm{d}y, \qquad (3.24)
\end{aligned}$$

where C_b is the uniform bound for v assumed in Lemma 3.2. Inserting (3.24) in (3.23) and splitting the integral $\int_{C_s^+} |\mathrm{D}v|^p\,\mathrm{d}y$ in one part with $C_s^+\setminus C_t^+$ and another with C_t^+, we arrive at

$$\begin{aligned}
\int_{C_s^+} f(y,v,\mathrm{D}v) + \mu|v|^{p^*}\,\mathrm{d}y \leq c(Q,p,\nu,c_0,\gamma,C_b,C_{e,P})\bigg\{ \int_{C_s^+} \frac{1}{(s-t)^p}|v-v_C|^p + c(\mu,\epsilon_1)|v_C|^{p^*} \\
+ C(\epsilon_1)b_1^\sigma + b_2\,\mathrm{d}y + (1+c(\mu,\epsilon_1))\int_{C_s^+\setminus C_t^+} |\mathrm{D}v|^p\,\mathrm{d}y + c(\mu,\epsilon_1)\int_{C_t^+} |\mathrm{D}v|^p\,\mathrm{d}y\bigg\}. \qquad (3.25)
\end{aligned}$$

Step 2. We return to inequality (3.19) and, using assumption (A2) and $\mathrm{D}\psi = \mathrm{D}v$ on C_t^+, we continue to estimate as follows

$$\begin{aligned}
&\int_{C_s^+} f(y,v,\mathrm{D}\psi) - f(y,v,\mathrm{D}v) + \vartheta(y,v)^p + c_1|\psi|^p \,\mathrm{d}y \\
&\quad = \int_{C_s^+\setminus C_t^+} f(y,v,\mathrm{D}\psi) - f(y,v,\mathrm{D}v)\,\mathrm{d}y + \int_{C_s^+} \vartheta(y,v)^p + c_1|\psi|^p\,\mathrm{d}y \\
&\quad \le c(c_0)\left\{\int_{C_s^+\setminus C_t^+} |\mathrm{D}\psi|^p + |\mathrm{D}v|^p\,\mathrm{d}y + \int_{C_s^+} \vartheta(y,v)^p + c_1|\psi|^p\,\mathrm{d}y\right\} \\
&\quad \le c(p,c_0)\left\{\int_{C_s^+} \frac{1}{(s-t)^p}|v-v_C|^p\,\mathrm{d}y + \int_{C_s^+\setminus C_t^+} |\mathrm{D}v|^p\,\mathrm{d}y + \int_{C_s^+} \vartheta(y,v)^p + c_1|\psi|^p\,\mathrm{d}y\right\}.
\end{aligned} \tag{3.26}$$

Applying once more Young's inequality with $\epsilon_2 > 0$ and $\frac{p^*}{p} \in (1,\infty)$ yields analogously to (3.22)

$$\begin{aligned}
\int_{C_s^+} |\psi|^p\,\mathrm{d}y &\le \int_{C_s^+} 1\cdot|v-v_C|^p\,\mathrm{d}y \\
&\le c(\epsilon_2)\int_{C_s^+} |v-v_C|^{p^*}\,\mathrm{d}y + C(\epsilon_2)\int_{C_s^+} 1\,\mathrm{d}y,
\end{aligned} \tag{3.27}$$

where $c(\epsilon_2) = \frac{p}{p^*}\epsilon_2^{\frac{p^*}{p}}$. The first term in (3.27) can be treated analogously to (3.24). The term including ϑ in (3.26) can now be estimated analogously to (3.22) and (3.24) so that (3.26) can be written as

$$\begin{aligned}
&\int_{C_s^+} f(y,v,\mathrm{D}\psi) - f(y,v,\mathrm{D}v) + \vartheta(y,v)^p + c_1|\psi|^p\,\mathrm{d}y \\
&\le c(Q,p,\nu,c_0,c_1,\gamma,C_b)\bigg\{\int_{C_s^+} \frac{1}{(s-t)^p}|v-v_C|^p + c(\epsilon_2)|v_C|^{p^*} + C(\epsilon_1,\epsilon_2)\,(b_1^\sigma+1) + b_2\,\mathrm{d}y \\
&\quad + (1+c(\mu,\epsilon_1)+c(\epsilon_2))\int_{C_s^+\setminus C_t^+} |\mathrm{D}v|^p\,\mathrm{d}y + (c(\mu,\epsilon_1)+c(\epsilon_2))\int_{C_t^+} |\mathrm{D}v|^p\,\mathrm{d}y\bigg\}.
\end{aligned} \tag{3.28}$$

We are now in the position to return to inequality (3.19) by combining the results (3.25) and (3.28). Choosing μ, ϵ_1 and ϵ_2 such that $c(Q,p,\nu,c_0,c_1,\gamma,C_b)(c(\mu,\epsilon_1)+c(\epsilon_2)) = \frac{1}{2}$, we can subtract the term $\frac{1}{2}\int_{C_t^+}|\mathrm{D}v|^p\mathrm{d}y$ from the right-hand side such that we obtain

$$\begin{aligned}
\int_{C_t^+} |\mathrm{D}v|^p + |v|^{p^*}\,\mathrm{d}y \le c\bigg\{&\int_{C_s^+} \frac{1}{(s-t)^p}|v-v_C|^p + b_1^\sigma + b_2 + 1\,\mathrm{d}y + |C_s^+||v_C|^{p^*} \\
&+ \int_{C_s^+\setminus C_t^+} |\mathrm{D}v|^p\,\mathrm{d}y\bigg\},
\end{aligned} \tag{3.29}$$

where $c = c(Q,p,\nu,c_0,c_1,\gamma,C_b,C_{e,P})$.

Step 3. To conclude (3.14) from (3.29), we aim to apply Lemma A.3. Since $\frac{r}{2} < s \leq r$, we estimate, by applying Hölder's inequality with $p^* \frac{\hat{p}}{p} \in (1, \infty)$,

$$
\begin{aligned}
|C_s^+||v_C|^{p^*} &\leq 2^{np^*}|C_r^+| \left(\fint_{C_r^+} |v| \,\mathrm{d}y \right)^{p*} \\
&\leq 2^{np^*}|C_r^+| \left(\fint_{C_r^+} |v|^{p^* \frac{\hat{p}}{p}} \,\mathrm{d}y \right)^{\frac{p}{\hat{p}}} \\
&\leq 2^{n(p^*+1)} \underbrace{r^n \left(r^{-n}\right)^{\frac{n+p}{n}}}_{r^{-p}} \left(\int_{C_r^+} |v|^{p^* \frac{\hat{p}}{p}} \,\mathrm{d}y \right)^{\frac{p}{\hat{p}}}.
\end{aligned} \tag{3.30}
$$

Further, applying Poincaré's inequality (Lemma A.1) with $\hat{p}^* = p$, we find

$$
\int_{C_s^+} |v - v_C|^p \,\mathrm{d}y \leq \int_{C_r^+} |v - v_C|^p \,\mathrm{d}y \leq C_{e,P}^p \left(\int_{C_r^+} |\mathrm{D}v|^{\hat{p}} \,\mathrm{d}y \right)^{\frac{p}{\hat{p}}}. \tag{3.31}
$$

Enlarging the right-hand side term $\int_{C_s^+ \setminus C_t^+} |\mathrm{D}v|^p \mathrm{d}y$ in (3.29) to $\int_{C_s^+ \setminus C_t^+} |\mathrm{D}v|^p + |v|^{p^*} \mathrm{d}y$, adding on both sides $c \int_{C_t^+} |\mathrm{D}v|^p + |v|^{p^*} \mathrm{d}y$ and then dividing by $c+1$, yields with (3.30) and (3.31)

$$
\begin{aligned}
\int_{C_t^+} |\mathrm{D}v|^p + |v|^{p^*} \,\mathrm{d}y \leq \frac{c}{c+1} \Bigg\{ &\frac{1}{(s-t)^p} \left(\int_{C_r^+} |\mathrm{D}v|^{\hat{p}} \,\mathrm{d}y \right)^{\frac{p}{\hat{p}}} + \frac{1}{r^p} \left(\int_{C_r^+} |v|^{p^* \frac{\hat{p}}{p}} \,\mathrm{d}y \right)^{\frac{p}{\hat{p}}} \\
&+ \int_{C_r^+} b_1^\sigma + b_2 + 1 \,\mathrm{d}y \Bigg\} + \frac{c}{c+1} \int_{C_s^+} |\mathrm{D}v|^p + |v|^{p^*} \,\mathrm{d}y.
\end{aligned} \tag{3.32}
$$

Finally, observing $0 < \frac{c}{c+1} < 1$ and setting $\beta := 1 < p$, $Z(t) := \int_{C_t^+} |\mathrm{D}v|^p + |v|^{p^*} \,\mathrm{d}y$ as well as

$$
A := \left(\int_{C_r^+} |\mathrm{D}v|^{\hat{p}} \,\mathrm{d}y \right)^{\frac{p}{\hat{p}}}, \; B := 0 \text{ and } C := \frac{1}{r^p} \left(\int_{C_r^+} |v|^{p^* \frac{\hat{p}}{p}} \,\mathrm{d}y \right)^{\frac{p}{\hat{p}}} + \int_{C_r^+} b_1^\sigma + b_2 + 1 \,\mathrm{d}y,
$$

we find using Lemma A.3 that

$$
\int_{C_{\frac{r}{2}}^+} |\mathrm{D}v|^p + |v|^{p^*} \,\mathrm{d}y \leq c \Bigg\{ \frac{1}{r^p} \left(\int_{C_r^+} |\mathrm{D}v|^{\hat{p}} \,\mathrm{d}y \right)^{\frac{p}{\hat{p}}} + \frac{1}{r^p} \left(\int_{C_r^+} |v|^{p^* \frac{\hat{p}}{p}} \,\mathrm{d}y \right)^{\frac{p}{\hat{p}}} + \int_{C_r^+} b_1^\sigma + b_2 + 1 \,\mathrm{d}y \Bigg\}. \tag{3.33}
$$

From (3.33) follows (3.14) directly, where the constant c only depends on the parameters Q, p, ν, c_0, c_1, γ and the uniform bound C_b, which finishes the proof of the case $p \leq n$.

Case $p > n$: Reviewing (3.18), we start our estimations from the inequality

$$
\int_{C_t^+} |\mathrm{D}v|^p \,\mathrm{d}y \leq c(\nu) \int_{C_s^+} \underbrace{f(y, v, \mathrm{D}v)}_{\text{Step 1}} + \underbrace{f(y, v, \mathrm{D}\psi) - f(y, v, \mathrm{D}v) + \vartheta(y, v)^p + c_1|\psi|^p}_{\text{Step 2}} \,\mathrm{d}y, \tag{3.34}
$$

and we will, as in the case $p \leq n$, continue in three steps.

Step 1. Analogously to (3.21), we find from (3.20) that it holds

$$\begin{aligned}\int_{C_t^+} f(y,v,\mathrm{D}v)\,\mathrm{d}y &\leq c(Q,p,c_0)\left\{\int_{C_s^+} |(v_C-v)\mathrm{D}\varsigma|^p + |(1-\varsigma)\mathrm{D}v|^p + b_1|v-\psi|^\gamma + b_2\,\mathrm{d}y\right\}\\ &\leq c(Q,p,c_0,\gamma)\left\{\int_{C_s^+} \frac{1}{(s-t)^p}|v-v_C|^p + b_1\left(|v-v_C|^\gamma + |v_C|^\gamma\right) + b_2\,\mathrm{d}y\right.\\ &\qquad\left. + \int_{C_s^+\setminus C_t^+} |\mathrm{D}v|^p\,\mathrm{d}y\right\}. \end{aligned}\tag{3.35}$$

At first, we discuss the product $b_1\left(|v-v_C|^\gamma + |v_C|^\gamma\right)$ in two steps. Exploiting the embedding $W^{1,p}(C_1^+;\mathbb{R}^N)\subset L^\infty(C_1^+;\mathbb{R}^N)$ with constant $C_e>0$, we obtain

$$\begin{aligned}\int_{C_s^+} b_1|v_C|^\gamma\,\mathrm{d}y = |v_C|^\gamma\|b_1\|_{L^1(C_s^+)} &\leq \|v\|^\gamma_{L^\infty(C_1^+)}\|b_1\|_{L^1(C_s^+)}\\ &\leq C_e\|v\|^\gamma_{W^{1,p}(C_1^+)}\|b_1\|_{L^1(C_s^+)} \leq C_eC_b^\gamma\|b_1\|_{L^1(C_s^+)},\end{aligned}$$

where C_b is the uniform bound assumed in Lemma 3.2. The other term including b_1 can be estimated analogously:

$$\int_{C_s^+} b_1|v-v_C|^\gamma\,\mathrm{d}y \leq \|v-v_C\|^\gamma_{L^\infty(C_1^+)}\|b_1\|_{L^1(C_s^+)} \leq 2C_eC_b^\gamma\|b_1\|_{L^1(C_s^+)}.\tag{3.36}$$

Thus, we can finish Step 1 by stating the inequality

$$\begin{aligned}\int_{C_t^+} f(y,v,\mathrm{D}v)\,\mathrm{d}y \leq c(Q,p,\nu,c_0,c_1,\gamma,C_b)&\left\{\int_{C_s^+} \frac{1}{(s-t)^p}|v-v_C|^p + b_1 + b_2\,\mathrm{d}y\right.\\ &\left.+ \int_{C_s^+\setminus C_t^+} |\mathrm{D}v|^p\,\mathrm{d}y\right\}.\end{aligned}\tag{3.37}$$

Step 2 and Step 3 follow completely analogously to the case $p\leq n$ by neglecting the term $|v|^{p^*}$ and estimating

$$\int_{C_s^+} |\psi|^p\,\mathrm{d}y \leq \int_{C_s^+} |v-v_C|^p\,\mathrm{d}y$$

as in (3.36) (with $b_1=1$). Thus, Lemma 3.2 is proven for all $p\in(1,\infty)$. $\square$

Lemma 3.3. *Let $y^0\in C^+_{1/4}(0)$ and $0<r<\frac{1}{4}$. Assume (II) and let $\hat{p}=\frac{pn}{n+p}$. For every $Q\geq 1$ and $C_b\geq 0$, there exists a constant $c>0$ independent of r and y^0 such that for all functions $v\in W^{1,p}_{\Gamma_D}(C_1^+(0);\mathbb{R}^N)$ with $\|v\|_{W^{1,p}(C_1^+)}\leq C_b$ satisfying (3.13) for some $\mathcal{F}\in\mathbb{F}(C_1^+,p,\nu,c_0,c_1,b_1,b_2,\gamma)$ it holds: If $p\leq n$ and p^* as in (2.4), then*

$$\int_{C^+_{\frac{r}{2}}(y^0)} |Dv|^p + |v|^{p^*}\,dy \leq c\left\{\frac{1}{r^p}\left(\int_{C^+_{3r}(y^0)}\left(|Dv|^p+|v|^{p^*}\right)^{\frac{\hat{p}}{p}}\,dy\right)^{\frac{p}{\hat{p}}} + \int_{C^+_{3r}(y^0)} b_1^\sigma + b_2 + 1\,dy\right\}.\tag{3.38}$$

For $p > n$ it holds

$$\int_{C^+_{\frac{r}{2}}(y^0)} |Dv|^p \, dy \le c \left\{ \frac{1}{r^p} \left(\int_{C^+_{3r}(y^0)} |Dv|^{\hat{p}} \, dy \right)^{\frac{p}{\hat{p}}} + \int_{C^+_{3r}(y^0)} b_1^\sigma + b_2 + 1 \, dy \right\}. \tag{3.39}$$

Proof. The structure of the proof is the same as in the proof for Lemma 3.2 so that we will only outline the modifications here. Let $\frac{r}{2} \le t < s \le r$ and let further denote ς the cut-off function satisfying (3.16). Assume $v \in W^{1,p}_{\Gamma_D}(C_1^+(0);\mathbb{R}^N)$ is a Q-minimizer according to (3.13) and we choose $\psi := v\varsigma \in W^{1,p}_{\mathrm{ad}}(C_1^+(0);\mathbb{R}^N)$ as an admissible test function on C_s^+.

Since $C_r \cap \Gamma_D \neq \emptyset$ in case (II) and $v|_{C_r \cap \Gamma_D} = 0$, we now apply Theorem A.2 and Lemma A.2, instead of Theorem A.1 and Lemma A.1 used in Case (I). In order to obtain a uniform bound for the constants involved, the estimates will be done for the cubes C^+_{3s} instead of C^+_s:
Let $\frac{r}{2} < s \le r$. Then we find a constant $c(n) > 0$ such that for all s and y^0

$$\frac{\lambda^{n-1}(C_{3s} \cap \Gamma_D)}{\lambda^{n-1}(\partial C_{3s})} \ge \frac{\frac{r}{2}(3r)^{n-2}}{c(n)(6r)^{n-1}} = \frac{2^{2-n}}{12c(n)}$$

is a uniform lower bound for the part of the Dirichlet boundary $C_{3s} \cap \Gamma_D$ with respect to ∂C^+_{3s}. Thus, by Lemma A.2, with $\frac{p^*}{\gamma}$ being the conjugate exponent to σ from (A2), there exists a constant $C_{e,P} > 0$ such that for all $s \in \left(\frac{r}{2}, r\right]$ it holds for $p \le n$

$$\int_{C^+_{3s}} |v|^{p^*} \, \mathrm{d}y \le C_{e,P} C_b^{\frac{p^*-p}{p}} \int_{C^+_{3s}} |\mathrm{D}v|^p \, \mathrm{d}y \quad \text{and} \quad \int_{C^+_{3s}} |v|^p \, \mathrm{d}y \le C_P \left(\int_{C^+_{3s}} |\mathrm{D}v|^{\hat{p}} \, \mathrm{d}y \right)^{\frac{p}{\hat{p}}}.$$

Similar calculations can be carried out also for the case $p > n$. Having this in mind, Lemma 3.3 can now be derived in the same way as Lemma 3.2. $\square$

3.3 Reflection

We are now going to extend the estimates from Lemmata 3.2 and 3.3 to full cubes $C_{3r}(y^0)$. For this purpose, we extend v from $C_1^+(0)$ onto $C_1^-(0)$ by reflection at the hyperplane
$\{y \in \mathbb{R}^n \mid y_n = 0\} = \overline{\Gamma}_D \cup \overline{\Gamma}_N$: Defining for almost all $y \in C_1(0)$

$$\widetilde{v}(y) := \begin{cases} v(y_1, \ldots, y_n), & \text{if } y \in C_1^+(0), \\ v(y_1, \ldots, y_{n-1}, -y_n), & \text{if } y \in C_1^-(0), \end{cases}$$

we obtain $\widetilde{v} \in W^{1,p}(C_1(0);\mathbb{R}^N)$, by Lemma 3.4 in [Giu03]. The functions b_1 and b_2 need to be extended as well, but since their extensions $\widetilde{b}_1$ and $\widetilde{b}_2$ satisfy under reflection the same properties assumed in (A2)-(A3) for b_1 and b_2, we will not distinguish them in notation.

It is an immediate observation that (3.14)-(3.15) hold true as well with $C^+_{3r}(y^0)$ instead of $C^+_r(y^0)$ on the right-hand side as in (3.38)-(3.39). We are now merging the Cases (I) and (II), considered in the Lemmata 3.2 and 3.3, in Corollary 3.1, here below.

Corollary 3.1. *Let the inequalities (3.38)-(3.39) hold true for $y^0 \in C^+_{1/4}(0)$, $0 < r < \frac{1}{4}$ and $p \le n, p > n$, respectively. Then we have for $y^0 \in C_{1/4}(0)$, $0 < r < \frac{1}{4}$ and $p \le n$*

$$\int_{C_{\frac{r}{2}}(y^0)} |D\widetilde{v}|^p + |\widetilde{v}|^{p^*}\, dy \le c\left\{\frac{1}{r^p}\left(\int_{C_{3r}(y^0)} \left(|D\widetilde{v}| + |\widetilde{v}|^{p^*}\right)^{\frac{\hat{p}}{p}} dy\right)^{\frac{p}{\hat{p}}} + \int_{C_{3r}(y^0)} b_1^\sigma + b_2 + 1\, dy\right\}, \tag{3.40}$$

and for $p > n$

$$\int_{C_{\frac{r}{2}}(y^0)} |D\widetilde{v}|^p\, dy \le c\left\{\frac{1}{r^p}\left(\int_{C_{3r}(y^0)} |D\widetilde{v}|^{\hat{p}}\, dy\right)^{\frac{p}{\hat{p}}} + \int_{C_{3r}(y^0)} b_1^\sigma + b_2 + 1\, dy\right\}, \tag{3.41}$$

where c is a positive constant depending on the same parameters as in the Lemmata 3.2 and 3.3 (independent of r and y^0).

Proof. We will distinguish again two cases.
Case 1.
The cube $C_r(y^0)$ lies entirely in $C_1^+(0)$ or $C_1^-(0)$ such that $C_r(y^0) \cap \{y \in \mathbb{R}^n \,|\, y_n = 0\} = \emptyset$.
a) Let $y_n^0 > 0$. Then we have $C_r^+(y^0) = C_r(y^0)$ and $\widetilde{v} = v$ so that Corollary 3.1 follows directly from Lemma 3.2.
b) Let $y_n^0 < 0$. We define $\hat{y}^0 := (y_1^0, \dots, y_{n-1}^0, -y_n^0)$ so that we obtain $C_r(\hat{y}^0) \subset \{y \in \mathbb{R}^n \,|\, y_n > 0\}$ and

$$\int_{C_r(y^0)} |\mathrm{D}\widetilde{v}|^p\, \mathrm{d}y = \int_{C_r(\hat{y}^0)} |\mathrm{D}\widetilde{v}|^p\, \mathrm{d}y = \int_{C_r(\hat{y}^0)} |\mathrm{D}v|^p\, \mathrm{d}y. \tag{3.42}$$

Again Corollary 3.1 follows directly from Lemma 3.2.
Case 2.
The cube $C_r(y^0)$ crosses the hyperplane $\overline{\Gamma}_D \cup \overline{\Gamma}_N$ such that $C_r(y^0) \cap \{y \in \mathbb{R}^n \,|\, y_n = 0\} \neq \emptyset$.
a) Let $y_n^0 \ge 0$. In this case we have

$$\int_{C_{\frac{r}{2}}(y^0)} |\mathrm{D}\widetilde{v}|^p\, \mathrm{d}y \le 2\int_{C^+_{\frac{r}{2}}(y^0)} |\mathrm{D}\widetilde{v}|^p\, \mathrm{d}y = 2\int_{C^+_{\frac{r}{2}}(y^0)} |\mathrm{D}v|^p\, \mathrm{d}y \tag{3.43}$$

as well as

$$\int_{C^+_{3r}(y^0)} |\mathrm{D}v|^{\hat{p}}\, \mathrm{d}y \le \int_{C_{3r}(y^0)} |\mathrm{D}\widetilde{v}|^{\hat{p}}\, \mathrm{d}y. \tag{3.44}$$

Now we apply Lemma 3.2 or Lemma 3.3 to the right-hand side in (3.43) and then apply (3.44) in order to obtain (3.40)-(3.41).
b) Let $y_n^0 < 0$. We define $\hat{y}^0$ as in Case 1b). With the help of (3.42), we are reduced to the Case 2a) so that we have

$$\int_{C_{\frac{r}{2}}(y^0)} |\mathrm{D}\widetilde{v}|^p\, \mathrm{d}y \underset{(3.42)}{=} \int_{C_{\frac{r}{2}}(\hat{y}^0)} |\mathrm{D}\widetilde{v}|^p\, \mathrm{d}y \underset{(3.43)}{\le} 2\int_{C^+_{\frac{r}{2}}(\hat{y}^0)} |\mathrm{D}v|^p\, \mathrm{d}y$$

as well as

$$\int_{C_r^+(\hat{y}^0)} |\mathrm{D}v|^p \,\mathrm{d}y \underset{(3.44)}{\leq} \int_{C_{3r}(\hat{y}^0)} |\mathrm{D}\widetilde{v}|^p \,\mathrm{d}y \underset{(3.42)}{=} \int_{C_{3r}(y^0)} |\mathrm{D}\widetilde{v}|^p \,\mathrm{d}y.$$

The inequalities (3.40)-(3.41) follow analogously to the Case 2a) and Corollary 3.1 is proved. □

3.4 Deriving the higher integrability of the gradient

We wish to apply the Giaquinta-Modica Theorem to Corollary 3.1 in order to derive the higher integrability of the gradient and inequalities (3.3)-(3.4), which finishes the proof of Theorem 3.1.

Let us start with the case $p \leq n$. Dividing both sides of inequality (3.40) by $|C_{r/2}| = r^n$, we arrive with $\frac{p}{\hat{p}} = \frac{n+p}{n}$ and $\frac{(r^n)^{(n+p)/n}}{r^n r^p} = 1$ at

$$\fint_{C_{\frac{r}{2}}(y^0)} \left(|\mathrm{D}\widetilde{v}|^p + |\widetilde{v}|^{p^*}\right) \mathrm{d}y \leq c\left\{\left(\fint_{C_{3r}(y^0)} \left(|\mathrm{D}\widetilde{v}|^p + |\widetilde{v}|^{p^*}\right)^{\frac{\hat{p}}{p}} \mathrm{d}y\right)^{\frac{p}{\hat{p}}} + \fint_{C_{3r}(y^0)} b_1^\sigma + b_2 + 1 \,\mathrm{d}y\right\}.$$

Now we can apply a variant of Theorem A.3 with pairs of cubes $Q, \widetilde{Q}$ as in Remark A.1. Thus let $Q = C_{r/2}(y^0) \subset \widetilde{Q} = C_{3r}(y^0) \subset\subset C_{1/4}(0)$, $y^0 \in C_{1/4}(0)$ and $r \in (0, \frac{1}{4})$. Let further $m = \frac{\hat{p}}{p}$, $g = |\mathrm{D}\widetilde{v}|^p + |\widetilde{v}|^{p^*}$ and $h = b_1^\sigma + b_2 + 1$. Recalling assumption (3.1), there exist, by Theorem A.3, constants $c > 0$ and $q > 1$ such that we have

$$\fint_{C_{\frac{1}{8}}(0)} \left(|\mathrm{D}\widetilde{v}|^p + |\widetilde{v}|^{p^*}\right)^q \mathrm{d}y \leq c\left\{\left(\fint_{C_{\frac{1}{4}}(0)} |\mathrm{D}\widetilde{v}|^p + |\widetilde{v}|^{p^*} \mathrm{d}y\right)^q + \fint_{C_{\frac{1}{4}}(0)} (b_1^\sigma + b_2 + 1)^q \,\mathrm{d}y\right\}. \quad (3.45)$$

Multiplying with $|C_{1/8}| = (\frac{1}{4})^n$ and using $|C_{1/4}| = (\frac{1}{2})^n$, we deduce from (3.45)

$$\int_{C_{\frac{1}{8}}(0)} \left(|\mathrm{D}\widetilde{v}|^p + |\widetilde{v}|^{p^*}\right)^q \mathrm{d}y \leq c\left\{\left(\int_{C_{\frac{1}{4}}(0)} |\mathrm{D}\widetilde{v}|^p + |\widetilde{v}|^{p^*} \mathrm{d}y\right)^q + \int_{C_{\frac{1}{4}}(0)} (b_1^\sigma + b_2 + 1)^q \,\mathrm{d}y\right\}.$$

Restriction to upper half cubes and a back transformation with $T_{x^0}^{-1}$ from Section 3.1 finally yields

$$\begin{aligned}\int_{T_{x^0}^{-1}(C_{\frac{1}{8}})\cap\Omega} \left(|\mathrm{D}u|^p + |u|^{p^*}\right)^q \mathrm{d}y \leq c&\left\{\left(\int_{T_{x^0}^{-1}(C_{\frac{1}{4}})\cap\Omega} |\mathrm{D}u|^p + |u|^{p^*} \mathrm{d}y\right)^q\right.\\ &\left.+ \int_{T_{x^0}^{-1}(C_{\frac{1}{4}})\cap\Omega} (b_1^\sigma + b_2 + 1)^q \,\mathrm{d}y\right\},\end{aligned}$$

and therefore (3.3) follows, since there exists a finite number of sets $T_{x^0}^{-1}(C_{1/8})$, which cover Ω. An analog argument can be used in the case $p > n$ in order to derive (3.4) from (3.41). Thus, Theorem 3.1 is proved.

4 Discussion and applications

4.1 Discussion of the assumptions

Let us first comment on assumption (A1) on the domain Ω. In the case of pure Dirichlet boundary conditions, i.e. $\partial\Omega = \partial_D\Omega$, higher integrability results are derived for more general domains than those described by our assumption (A1). Following e.g. Section 6.5 in [Giu03], in the case of pure Dirichlet conditions it is sufficient to consider domains with the property

$$\lambda^n(C_r(x^0)\backslash\Omega) \geq \alpha_0 r^n \tag{4.1}$$

for all $x^0 \in \partial\Omega$ and cubes $C_r(x^0)$. This implies e.g. that the domain has no interior cusps, but exterior cusps are not excluded. Condition (4.1) moreover guarantees uniform constants in the Poincaré inequality on sets $\Omega \cap C_r(x^0)$.

Our assumption (A1) is mainly a regularity assumption on the hypersurface that separates the Dirichlet boundary from the Neumann boundary: It means roughly speaking that the separating set is a Lipschitzian hypersurface in $\partial\Omega$, see Remark 1 in [Grö89]. The assumption (A1) implies that the constants in the Poincaré inequality are uniform with respect to the sets $\Omega \cap C_r(x^0)$ for $0 < r < R$ and $x^0 \in \overline{\Gamma}_D \cap \overline{\Gamma}_N$. Domains Ω that have a Lipschitz continuous boundary $\partial\Omega$ in the sense of graphs satisfy in particular (A1). Domains satisfying (A1) are Lipschitz domains in the sense of bi-Lipschitz maps. Let us note that also more general assumptions on the boundary between $\partial_D\Omega$ and $\partial_N\Omega$ give uniform constants in the Poincaré inequality: For example, "interior hyper-cusps" with respect to the Dirichlet boundary still give uniform constants, while for "exterior hyper-cusps" the constants degenerate at balls centered in the tip of the cusp, see Figure 2. In [HDKR], an isomorphism result in the spirit of [Grö89] will be derived for this more general class of domains.

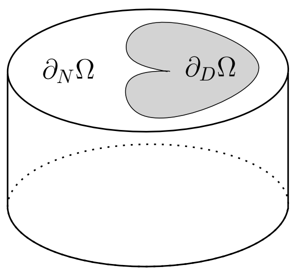

Figure 2: An admissible "interior hyper-cusp" on the surface of a cylinder.

The assumption (A2) on the integrand F can be slightly weakened by assuming that F is a normal integrand instead of a Carathéodory function, because it is only required that the integral $\int_\Omega F(x, u, \mathrm{D}u)\,\mathrm{d}x$ is well defined, see Chapter VIII 1.1-1.3 in [ET99].

The very recent paper by Wachsmuth et al. [HMW11] provides a result on the higher integrability of solutions to nonlinear, monotone elasticity systems for $p = 2$ with mixed boundary conditions and it states invertibility properties of the corresponding differential operators in $W^{1,p}$-spaces. We provide an exemplary energy density that satisfies the assumptions (A2)-(A3) in our paper for $p = 2$, but is not included in the considerations in [HMW11].

Let $p = n = N = 2$ and assume pure Dirichlet boundary conditions, i.e. $\partial\Omega = \partial_D\Omega$. Let the energy density $W : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be defined as

$$W(A) := \tfrac{1}{2}|A|^2 + g(\det A),$$

where $g \in C^2(\mathbb{R}; [0, \infty))$ with $g(1) = 0$ and $\sup_{a\in\mathbb{R}} |g'(a)| < \infty$. Furthermore, g is convex, nonlinear and satisfies for $C > 0$ and all $a \in \mathbb{R}$ the growth condition

$$g(a) \le C(1 + |a|).$$

The first summand of W, $A \mapsto \frac{1}{2}|A|^2$, is differentiable and strictly convex, whereas the second summand $A \mapsto g(\det A)$ is differentiable and quasi-convex, but no longer convex. There exists $\tilde{c} > 0$ such that it holds for all $A \in \mathbb{R}^{2\times 2}$

$$\tfrac{1}{2}|A|^2 \le W(A) \le \tfrac{1}{2}|A|^2 + C(1 + |\det A|) \le \tilde{c}(1 + |A|)^2,$$

whatfrom (A2)-(A3) follow.

Let now $u \in H_0^1(\Omega)$ be a minimizer of the functional $\mathcal{F}$, where

$$\mathcal{F}(u) = \int_\Omega W(\nabla u)\,\mathrm{d}x.$$

Then there holds $\mathrm{D}\mathcal{F}(u)[v] = 0$ for all $v \in H_0^1(\Omega)$. The derivative of the energy density W is given by

$$\mathrm{D}W(A) = \begin{cases} A + g'(\det A)\operatorname{cof} A, & \text{if } \det(A) \neq 0, \\ A, & \text{if } \det(A) = 0, \end{cases}$$

where $\operatorname{cof} A = \det A \cdot A^{-T}$ is the cofactor matrix. Hence, there holds for all $v \in H_0^1(\Omega)$

$$0 = \mathrm{D}\mathcal{F}(u)[v] = \int_\Omega \mathrm{D}W(\nabla u) : \nabla v\,\mathrm{d}x.$$

However, we now show that g can be chosen in such a way that $\mathrm{D}W : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is not monotone and hence the analysis from [HMW11] cannot be applied here. Our Main Theorem is applicable to $\mathcal{F}$. Indeed, chose $A_1 = \mathrm{Id}$ and $A_2 = \left(\begin{smallmatrix} 0 & -\frac{1}{2} \\ 1 & 0 \end{smallmatrix}\right)$. Then $\operatorname{cof} A_1 = \mathrm{Id}$, $\det A_1 = 1$, $\operatorname{cof} A_2 = \left(\begin{smallmatrix} 0 & -1 \\ \frac{1}{2} & 0 \end{smallmatrix}\right)$, $\det A_2 = \frac{1}{2}$ and we obtain with $g'(1) = 0$ that

$$\begin{aligned}(\mathrm{D}W(A_1) - \mathrm{D}W(A_2)) : (A_1 - A_2) &= \left(\mathrm{Id} + g'(1)\,\mathrm{Id} - \left(\begin{smallmatrix} 0 & -\frac{1}{2} \\ 1 & 0 \end{smallmatrix}\right) - g'(\tfrac{1}{2})\left(\begin{smallmatrix} 0 & -1 \\ \frac{1}{2} & 0 \end{smallmatrix}\right)\right) : \left(\begin{smallmatrix} 1 & \frac{1}{2} \\ -1 & 1 \end{smallmatrix}\right) \\ &= \tfrac{13}{4} + g'(\tfrac{1}{2}).\end{aligned}$$

Now, for functions g as described above with $g'(\frac{1}{2}) < -\frac{13}{4}$, the monotonicity condition from [HMW11] is violated.

4.2 Damage of nonlinear elastic materials at small strains

In this section, we will show an application of the Main Theorem to a quasistatic evolution model describing damage accumulation in an elastic body. In particular, we will prove on the basis of Theorem 3.1 higher integrability of the deformation gradient ∇u, for the damage of nonlinear elastic materials, presented in [TM10].

At first, we briefly recall the main aspects of the model: Let $\Omega \subset \mathbb{R}^n$ and $\partial_D\Omega \subset \partial\Omega$ satisfy assumption (A1) from Section 2 with $\lambda^{n-1}(\partial_D\Omega) > 0$. The state space $\mathcal{Q} := \mathcal{U} \times \mathcal{Z}$ is defined for $p, r \in (1,\infty)$ by

$$\mathcal{U} := W^{1,p}_D(\Omega;\mathbb{R}^n) \quad \text{and} \quad \mathcal{Z} := W^{1,r}(\Omega).$$

We define the energy functional $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R}$ by

$$\mathcal{E}(t,u,z) := \int_\Omega W(x,\nabla u,z) + \tfrac{\kappa}{r}|\nabla z|^r + \chi_{[0,1]}(z)\,\mathrm{d}x - \langle \ell(t), u\rangle, \tag{4.2}$$

where $\kappa > 0$ is a material constant and $\ell \in C^1([0,T];W^{-1,p'}(\Omega;\mathbb{R}^n))$, $\frac{1}{p}+\frac{1}{p'} = 1$, represents the volume and surface forces. The dissipation distance $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_\infty$ is defined as

$$\mathcal{D}(z_1,z_2) := \begin{cases} \int_\Omega \rho(x)(z_1(x)-z_2(x))\,\mathrm{d}x, & \text{if } z_2(x) \le z_1(x) \text{ a.e.,} \\ \infty, & \text{else.} \end{cases} \tag{4.3}$$

There, $\rho \in L^\infty(\Omega)$ with $0 < \rho_0 \le \rho$ almost everywhere is again a material dependent function and can be interpreted as a kind of fracture toughness.

We aim to consider the constraint $z \in [0,1]$, whereby the value $z = 1$ corresponds to intact material and $z = 0$ represents maximal damage. Due to the asymmetric definition of the dissipation distance, the damage variable z is monotonically decreasing in time, as will be clear from the evolution model (S) & (E), here below.

Hypotheses on the energy functional $\mathcal{E}$:

(H1) **Carathéodory function:** $W(x,\cdot,\cdot) \in C^0(\mathbb{R}^{n\times n} \times \mathbb{R})$ for almost every $x \in \Omega$ and $W(\cdot,A,z)$ is measurable in Ω for all $(A,z) \in \mathbb{R}^{n\times n} \times \mathbb{R}$.

(H2) **Quasi-convexity:** For almost every $x \in \Omega$ and every $z \in \mathbb{R}$ the function $A \mapsto W(x,A,z)$ is quasi-convex.

(H3) **p-growth and coercivity:** For almost all $x \in \Omega$ and all $z \in \mathbb{R}$ we have $W(x,\cdot,z) \in C^1(\mathbb{R}^{n\times n})$ and there exists a constant $C > 0$ such that for almost every $x \in \Omega$ and every $(A,z) \in \mathbb{R}^{n\times n} \times \mathbb{R}$ we have $\quad W(x,A,z) \le C(1+|A|^p)$.

Moreover, there exist $\widetilde{W} : \mathbb{R}^{n\times n} \to \mathbb{R}$ with $\widetilde{W}(0) = 0$ and a constant $\nu > 0$ such that for every element $u \in \mathcal{U}$ there holds

$$\int_\Omega \widetilde{W}(\nabla u)\,\mathrm{d}x \ge \nu \int_\Omega |\nabla u|^p\,\mathrm{d}x$$

and for almost all $x \in \Omega$ and all $(A,z) \in \mathbb{R}^{n\times n} \times \mathbb{R}$ it holds $W(x,A,z) \ge \widetilde{W}(A)$.

(H4) **Monotonicity:** There exists constants $k_1 > 0$, $k_2 \leq 0$ so that for almost every $x \in \Omega$ and all $(A, z), (A, \tilde{z}) \in \mathbb{R}^{d\times d}_{\text{sym}} \times [0,1]$ with $z \leq \tilde{z}$, we have

$$W(x, A, z) \leq W(x, A, \tilde{z}) \leq k_1(W(x, A, z) + k_2).$$

Example for an admissible energy: For some $\delta \in (0,1)$ let $g_\delta : \mathbb{R} \to \mathbb{R}$ be defined as

$$g_\delta(z) := \begin{cases} \delta, & \text{if } z < 0, \\ \delta + (1-\delta)z, & \text{if } 0 \leq z \leq 1, \\ 1, & \text{if } z > 1. \end{cases}$$

Moreover, let $\mathbb{C} \in L^\infty(\Omega; \mathrm{Lin}(\mathbb{R}^{n\times n}_{\text{sym}}; \mathbb{R}^{n\times n}_{\text{sym}}))$ denote the elasticity tensor satisfying for some constant $\nu > 0$ and all $e_1, e_2 \in \mathbb{R}^{n\times n}_{\text{sym}}$

$$\mathbb{C}e_1 : e_2 = \mathbb{C}e_2 : e_1 \quad \text{and} \quad \mathbb{C}e_1 : e_1 \geq \nu |e_1|^2. \tag{4.4}$$

An exemplary energy density is then given by $W(x, A, z) := \frac{1}{2} g_\delta(z)\mathbb{C}(x)A_{\text{sym}} : A_{\text{sym}}$, where $A_{\text{sym}} = \frac{1}{2}(A + A^T)$ denotes the symmetric part. Then the energy

$$\mathcal{E}(t, u, z) = \int_\Omega \tfrac{1}{2} g_\delta(z)\mathbb{C}e(u) : e(u) + \tfrac{\kappa}{r}|\nabla z|^r + \chi_{[0,1]}(z)\,\mathrm{d}x - \langle \ell(t), u\rangle,$$

where $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the linearized strain tensor, satisfies (H1)-(H4) with $p = 2$ and $\tilde{F}(A) = |A_{\text{sym}}|^2$.

Definition 4.1 (Energetic solution, Definition 2.1 in [TM10])**.** *A pair $(u, z) : [0, T] \to \mathcal{Q}$ is called energetic solution for the rate-independent process $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$, if $t \mapsto \partial_t \mathcal{E}(t, u(t), z(t)) \in L^1(0, T)$ and if for all $t \in [0, T]$ we have $\mathcal{E}(t, u(t), z(t)) < \infty$, stability (S) and energy balance (E):*

(S) *for all $(\tilde{u}, \tilde{z}) \in \mathcal{Q}$ holds: $\mathcal{E}(t, u(t), z(t)) \leq \mathcal{E}(t, \tilde{u}(t), \tilde{z}(t)) + \mathcal{D}(z(t), \tilde{z})$,*

(E) $\mathcal{E}(t, u(t), z(t)) + \mathrm{Diss}_\mathcal{D}(z, [0, t]) = \mathcal{E}(0, u(0), z(0)) + \int_0^t \partial_t \mathcal{E}(\tau, u(\tau), z(\tau))\, d\tau,$

where $\mathrm{Diss}_\mathcal{D}(z, [0, t]) := \sup \sum_{j=1}^M \mathcal{D}(z(\tau_{j-1}), z(\tau_j))$ and the supremum is taken over all partitions of the interval $[0, t]$.

An element $(t, u, z) \in [0, T] \times \mathcal{Q}$ such that (u, z) satisfies (S) in t is called *stable*. We will say that (u, z) is a *stable state at time t*.

From the energy balance follows directly that there holds $\mathrm{Diss}_\mathcal{D}(z, [0, t]) < \infty$ for all $t \in [0, T]$. Hence, the asymmetric definition of the dissipation distance $\mathcal{D}$ implies that the damage variable z is monotonically decreasing. In particular, there holds $\mathrm{Diss}_\mathcal{D}(z; [0, t]) = \int_\Omega \rho(z_0 - z(t))\,\mathrm{d}x$ for all monotonically decreasing functions z. Observe that, by setting $\tilde{z} = z(t)$ in the stability condition (S), we deduce

$$u(t) \in U(t, z(t)) := \operatorname*{Argmin}_{v\in\mathcal{U}} \mathcal{E}(t, v, z(t)) = \operatorname*{Argmin}_{v\in\mathcal{U}} \int_\Omega W(x, \nabla v, z(t))\,\mathrm{d}x - \langle \ell(t), v\rangle \tag{4.5}$$

for all $t \in [0,T]$. Since the term $\frac{\kappa}{r}|\nabla z(t)|^r$ is constant in v, we can neglect it while minimizing with respect to v. Our damage model is a particular case of the model introduced in [TM10] in the following sense: The hypothesis (H3) here is stronger than coercivity $(\mathrm{H3})_{\mathrm{TM}}$ and stress control $(\mathrm{H4})_{\mathrm{TM}}$ in [TM10]: By (H2)-(H3), we find $c > 0$ such that there holds for almost all $x \in \Omega$ and all $(A, z) \in \mathbb{R}^{n\times n} \times \mathbb{R}$ that $|\partial_A W(x, A, z)| \le c(1 + |A|^{p-1})$, which implies $(\mathrm{H4})_{\mathrm{TM}}$ in [TM10]. In [TM10, Theorem 3.1], the following existence result is shown:

Proposition 4.1 (Existence of energetic solutions)**.** *Let $\mathcal{Q} = \mathcal{U} \times \mathcal{Z}$ be defined as before, $\mathcal{E}$ as in (4.2) with (H1)-(H4) and $\mathcal{D}$ as in (4.3). Then, for every stable initial state $(u_0, z_0) \in \mathcal{Q}$, there exists an energetic solution for the rate-independent process defined by $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$.*

As for the irreversibility of the damage process, it is clear that $z(t) \le z_0 \le 1$ for all $t \in [0,T]$. We will now proceed to prove the higher integrability of the deformation gradient ∇u.

Theorem 4.1 (Higher integrability for u)**.** *Assume that $\Omega \subset \mathbb{R}^n$ satisfies (A1) and that the external forces are of the form $\langle \ell(t), v\rangle = \int_\Omega H_0(t)\cdot v + H_1(t) : \nabla v\, dx$ with $H_i \in C^0([0,T]; L^r(\Omega))$ for some $r > p$. Then there exist constants $q_1 > p$ and $c > 0$ such that for all $(t, z) \in [0,T] \times \mathcal{Z}$ with $0 \le z \le 1$ and for all $u \in \mathcal{U}$ satisfying*

$$u \in U(t,z) = \operatorname*{Argmin}_{v\in\mathcal{U}} \int_\Omega W(x, \nabla v, z)\, dx - \langle \ell(t), v\rangle,$$

it holds $u \in W^{1,q_1}(\Omega; \mathbb{R}^n)$ and $\|u\|_{W^{1,q_1}(\Omega)} \le c$.

Proof. We deduce the higher integrability for u by applying Theorem 3.1. Let $(t, z) \in [0,T] \times \mathcal{Z}$ be arbitrarily fixed with $0 \le z \le 1$, then $u \in U(t, z)$ is a global minimizer of the functional $\mathcal{F}_t(u) := \int_\Omega W(x, \nabla u, z)\, \mathrm{d}x - \langle \ell(t), u\rangle$. The domain Ω satisfies (A1) by assumption. Due to (H1) and (H3), the assumptions (A2)-(A3) are obviously satisfied for the energy density W. Due to the assumptions on ℓ, we can find uniform bounds, independent of t so that (A2)-(A3) hold independently of t and z and $\mathcal{F}_t \in \mathbb{F}(\Omega, p, \hat{\nu}, \hat{c}_0, \hat{c}_1, \hat{b}_1, \hat{b}_2, \gamma)$ as given in Remark 2.1.

Since the Dirichlet boundary $\partial_D\Omega$ has positive measure, $1 = \gamma < p$ and $c_1 = 0$, we obtain from Lemma 3.1 a uniform bound $C_b \ge 0$ such that $\|u\|_{W^{1,p}(\Omega)} \le C_b$ for all t and z. The right-hand side function $b_2 \equiv 1 \in L^\infty(\Omega)$ is obviously higher integrable and therefore, Theorem 3.1 yields $u \in W^{1,q_1}(\Omega; \mathbb{R}^n)$ for some $q_1 > p$. □

This means in particular that for any energetic solution (u, z) the displacement satisfies $u(t) \in W^{1,q_1}(\Omega; \mathbb{R}^n)$ for all $t \in [0,T]$ and some $q_1 > p$, independently of t and z.

4.3 Damage model without gradient of the damage variable

A further investigation of the damage model presented in [TM10] has been proposed in [FKS11], where no nonlocal damage effects are present, and consequently no compactifying terms depending on the gradient of the damage variable appear in the energy functional. In this case, hard technical difficulties prevent to obtain an existence result for the evolution in usual function spaces. Instead, a Young measure evolution notion is presented, satisfying a weaker version of the stability condition and a complete energy balance. We do not want to enter in the details of this definition here, and we refer the interested reader to [FKS11, Definition 4.1]. We just recall that, in order to perform the passage to the limit which provides the stability condition and to obtain the lower energy estimate, the higher integrability of approximate strains is crucial, and the related estimate is needed to be uniform with respect to the time-step chosen in the approximation (see [FKS11, estimate (5.7)]). Whereas the uniform higher integrability is easily obtained with the argument by Giaquinta and Giusti ([Giu03]) in the case where a fully Dirichlet boundary condition is imposed and no external forces are present, the more general case of mixed boundary conditions and external loads requires the higher integrability result proven in this paper.

4.4 Phase transitions

It is also possible to apply Theorem 3.1 to the phase-transition model presented in [Fia10]. In this paper a crystal material with finitely many phases and an elastic energy with quadratic growth ($p = 2$) is considered. As in the damage case, we deal with the deformation gradient ∇v and an internal variable z, playing the role of a phase indicator. Since we consider a multiphase material, a multiwell potential energy is to be expected. In [Fia10] no regularizing term depending on the gradient of the internal variable z is considered, therefore the lack of convexity of the energy functional is responsible for hard technical difficulties, which can be overcome by considering a suitable notion of Young measure quasi-static evolution (see [Fia10, Definition 6.2]).

In order to prove the convergence of the approximate solutions to a Young measure quasistatic evolution, a suitable higher integrability property for the approximate deformation gradients is needed (see [Fia10, Lemma 7.3]). This further regularity is proved in [Fia10] for a fully Dirichlet boundary condition and no external forces, by applying the results by Giaquinta and Giusti. Our higher integrability result allows us to consider the more general case of a Dirichlet boundary condition imposed just on a part $\partial_D\Omega$ of the boundary, and a nonzero external load $\ell \in C^1([0,T]; W_D^{1,1}(\Omega)^*)$.

The proof of the desired higher integrability properties for the approximate deformation gradients follows closely the argument in [Fia10, Lemma 7.2]. The reason why we need a more regular external load than in the model analyzed in Section 4.2 is related to the fact that, in the phase-transition case, Young measures need to be introduced from the very beginning in the proof of the existence result, in order to construct approximate solutions. This makes the application of the higher integrability result more delicate. In particular we need lower semicontinuity of the functional $u \to \langle \ell(t), u\rangle$ with respect to the strong topology of $W_D^{1,1}(\Omega)$, for every $t \in [0,T]$, to apply Ekeland's principle to the energy functional.

4.5 Higher integrability for a model describing shape memory alloys

As a further application of Theorem 3.1 we prove the uniform higher integrability of the stable states related to a rate-independent model describing shape memory alloys. Let $\Omega \subset \mathbb{R}^n$ satisfy (A1) with $\lambda^{n-1}(\partial_D\Omega) > 0$. Within the Souza-Auricchio model (see [SMZ98, AP02]) the state of a shape memory material occupying the domain Ω is completely characterized by the displacement field $u : [0,T]\times\Omega \to \mathbb{R}^n$ and the internal variable $z : [0,T]\times\Omega \to \mathbb{R}^{n\times n}_{\mathrm{dev}}$ describing the mesoscopic transformation strain. Here, $\mathbb{R}^{n\times n}_{\mathrm{dev}}$ is the set of symmetric $n \times n$ tensors with vanishing trace. Let the state space be given by $\mathcal{Q} = \mathcal{U} \times \mathcal{Z}$ with $\mathcal{U} = W^{1,2}_D(\Omega;\mathbb{R}^n)$ and $\mathcal{Z} = W^{1,2}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{dev}})$. For given time dependent loading $\ell \in C^0([0,T];\mathcal{U}^*)$ and $(v,\xi) \in \mathcal{U} \times \mathcal{Z}$ the stored energy is defined as

$$\mathcal{E}(t,v,\xi) = \int_\Omega \tfrac{1}{2}\mathbb{C}(e(v)-\xi) : (e(v)-\xi)\mathrm{d}x - \langle \ell(t), v\rangle \\ + \int_\Omega \frac{g_0(x)}{2}\,|\nabla z|^2 + g_1(x)\,|z| + g_2(x)\,|z|^2 + \chi(z)\mathrm{d}x, \quad (4.6)$$

where $\mathbb{C} \in L^\infty(\Omega;\mathbb{R}^{(n\times n)\times(n\times n)})$ is the elasticity tensor satisfying the symmetry and positivity properties from (4.4). Moreover, $e(v) = \frac{1}{2}(\nabla v + \nabla v^\top)$ denotes the linearized strain tensor. The transformation strains z take their values in the compact, convex set $\mathbf{Z} = \{z \in \mathbb{R}^{n\times n}_{\mathrm{dev}} \mid |z| \le \sigma_0\}$, where σ_0 is a positive constant. This constraint enters into the energy functional through the corresponding indicator function $\chi : \mathbb{R}^{n\times n}_{\mathrm{dev}} \to \{0,\infty\}$ with $\chi(\xi) = 0$ if $\xi \in \mathbf{Z}$ and $\chi(\xi) = \infty$ otherwise. The energy that is dissipated when switching between different transformation strains is taken into account via the dissipation functional $\mathcal{R}(\xi) = \int_\Omega \rho(x)\,|\xi(x)|\,\mathrm{d}x$ for $\xi \in \mathcal{Z}$ and fixed $\rho \in L^\infty(\Omega)$ with $\rho(x) \ge \rho_0 > 0$ a.e. Analogously to the damage model discussed in Section 4.2, the pair $(u,z) : [0,T] \to \mathcal{Q}$ is called an energetic solution to the rate-independent process defined by $(\mathcal{Q},\mathcal{E},\mathcal{R})$ if for all t it satisfies the stability condition (S) and the energy balance (E), specified in Definition 4.1. The existence and uniqueness of energetic solutions is investigated in [AMS08, MP07]. Here, we study the uniform higher integrability of the stable states. We recall that a pair $(u,z) \in \mathcal{U} \times \mathcal{Z}$ is called a stable state at time t if $(u,z) \in S(t)$, where the set of stable states at time t is defined as

$$S(t) = \{(u,z) \in \mathcal{Q} \,|\, \mathcal{E}(t,u,z) \le \mathcal{E}(t,v,\xi) + \mathcal{R}(z-\xi) \quad \text{for all } (v,\xi) \in \mathcal{Q}\}.$$

Theorem 4.2. *Assume that $\Omega \subset \mathbb{R}^n$ satisfies (A1) and that the functions g_i in (4.6) belong to $L^\infty(\Omega)$ with $g_i(x) \ge \alpha > 0$ for a.e. x and $0 \le i \le 2$. Assume furthermore that the external forces are of the form $\langle \ell(t), v\rangle = \int_\Omega H_0(t)\cdot v + H_1(t) : \nabla v dx$ with $H_i \in C^0([0,T];L^r(\Omega))$ for some $r > 2$.*

Then there exist $q_1, q_2 > 2$ and a constant $c > 0$ such that for all $t \in [0,T]$ and all stable states $(u,z) \in S(t)$ it holds $u \in W^{1,q_1}(\Omega;\mathbb{R}^n)$, $z \in W^{1,q_2}(\Omega;\mathbb{R}^{n\times n}_{dev})$ and $\|u\|_{W^{1,q_1}(\Omega)} + \|z\|_{W^{1,q_2}(\Omega)} \le c$.

In particular this theorem implies that energetic solutions of the Souza-Auricchio model are higher integrable in space, uniformly in time.

Proof. First, we prove the higher integrability for the displacement field as in Section 4.2 and subsequently for the internal variable. Observe that due to the stability condition (S) there exists a constant $C_b > 0$ such that the uniform estimate

$$\sup_{t\in[0,T],(u,z)\in S(t)} \|u\|_{W^{1,2}(\Omega)} + \|z\|_{W^{1,2}(\Omega)} \leq C_b \tag{4.7}$$

is valid. Now in the same way as in the proof of Theorem 4.1 the uniform higher integrability can be deduced for the displacement field u.

As for the higher integrability of z, we show that z can be interpreted as a Q-minimizer of a suitable functional, and then apply again Theorem 3.1. For this we proceed in the spirit of Example 6.4 in [Giu03]. Let $(u,z) \in S(t)$ and let the functional $\mathcal{F}_{u,z} : W^{1,2}(\Omega; \mathbb{R}^{n\times n}_{\text{dev}}) \to \mathbb{R}$ be defined as

$$\begin{aligned}\mathcal{F}_{u,z}(\xi) &= \int_\Omega \tfrac{1}{2}\mathbb{C}(e(u)-\xi) : (e(u)-\xi) + g_1\,|\xi| + g_2\,|\xi|^2 + \frac{g_0}{2}\,|\nabla\xi|^2 \,\mathrm{d}x + \int_\Omega \rho|z-\xi|\,\mathrm{d}x\\ &= \int_\Omega F_{u,z}(x,\xi) + \frac{g_0(x)}{2}\,|\nabla\xi|^2\,\mathrm{d}x,\end{aligned}$$

where $F_{u,z}(x,\xi) = \frac{1}{2}\mathbb{C}(e(u(x))-\xi) : (e(u(x))-\xi) + g_1\,|\xi| + g_2\,|\xi|^2 + \rho|z(x)-\xi|$. Since $(u,z) \in S(t)$, it holds

$$z \in \operatorname{Argmin}\{\mathcal{F}_{u,z}(\xi)\,|\,\xi \in W^{1,2}(\Omega;\mathbb{R}^{n\times n}_{\text{dev}}),\ \xi(x) \in \mathbf{Z}\}. \tag{4.8}$$

Let now $K \subset \mathbb{R}^n$ be compact and $\eta \in W^{1,2}(\Omega;\mathbb{R}^{n\times n}_{\text{dev}})$ with $\eta|_{\Omega\setminus K} = z$. We define $\tilde\eta := P_{\mathbf{Z}}(\eta(x))$, where $P_{\mathbf{Z}} : \mathbb{R}^{n\times n}_{\text{dev}} \to \mathbf{Z}$ is the projection onto the convex and closed set $\mathbf{Z}$. Observe that $\tilde\eta \in W^{1,2}(\Omega;\mathbb{R}^{n\times n}_{\text{dev}})$ with $\tilde\eta|_{\Omega\setminus K} = z$ and hence it is admissible for the minimization problem (4.8). Therefore, with $M := \{x \in \Omega\,|\,\eta(x) \in \mathbf{Z}\}$ the following estimate is valid:

$$\begin{aligned}\int_{\Omega\cap K} F_{u,z}(x,z) + \frac{g_0}{2}|\nabla z|^2\,\mathrm{d}x &\leq \int_{\Omega\cap K} F_{u,z}(x,\tilde\eta) + \frac{g_0}{2}|\nabla\tilde\eta|^2\,\mathrm{d}x\\ &= \int_{\Omega\cap K\cap M} F_{u,z}(x,\eta) + \frac{g_0}{2}|\nabla\eta|^2\,\mathrm{d}x + \int_{(\Omega\cap K)\setminus M} F_{u,z}(x,\tilde\eta) + \frac{g_0}{2}|\nabla\tilde\eta|^2\,\mathrm{d}x.\end{aligned}$$

From the uniform bound (4.7) for u, the boundedness of the set $\mathbf{Z}$ and the Lipschitz continuity of the projection $P_{\mathbf{Z}}$, it follows that there exist constants $\kappa, C_1 > 0$, which are independent of η, t, z and K such that

$$\int_{(\Omega\cap K)\setminus M} F_{u,z}(x,P_{\mathbf{Z}}(\eta)) + \frac{g_0}{2}|\nabla P_{\mathbf{Z}}(\eta)|^2\mathrm{d}x \leq \kappa \int_{(\Omega\cap K)\setminus M} C_1(1+|\nabla u|^2) + \frac{g_0}{2}|\nabla\eta|^2\mathrm{d}x.$$

Altogether it follows that

$$\int_{\Omega\cap K} F_{u,z}(x,z) + \frac{g_0}{2}|\nabla z|^2\mathrm{d}x \leq (1+\kappa)\int_{\Omega\cap K} F_{u,z}(x,\eta) + \frac{g_0}{2}|\nabla\eta|^2 + C_1(1+|\nabla u|^2)\mathrm{d}x.$$

For $x \in \Omega$ and $\xi \in \mathbb{R}^{n\times n}_{\mathrm{dev}}$, we define $F^0_{u,z}(x,\xi) := F_{u,z}(x,\xi) + C_1(1+|\nabla u(x)|^2)$. The above calculations show that z is a Q-minimizer of (the functional)

$$\mathcal{F}^0_{u,z}(\eta) := \int_\Omega F^0_{u,z}(x,\eta) + \frac{g_0}{2}|\nabla\eta|^2 \mathrm{d}x$$

with respect to $W^{1,2}(\Omega;\mathbb{R}^{n\times n}_{\mathrm{dev}})$ and for $Q = 2+\kappa$. It can easily be checked that $\mathcal{F}^0_{u,z}$ satisfies the assumptions of Theorem 3.1, which finishes the proof of Theorem 4.2. □

Remark 4.1. *The considerations can immediately be extended to quasiconvex energy densities and with general compact, convex constraints* $\mathbf{Z} \subset \mathbb{R}^m$*, covering in this way the rate-independent models studied in [FM06, Section 4].*

A Appendix

Theorem A.1 (Poincaré type inequality)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with Lipschitz boundary* $\partial\Omega$, $\hat{p} = \frac{pn}{n+p}$ *and* u_Ω *the mean value defined by*

$$u_\Omega := \frac{1}{|\Omega|}\int_\Omega u\,dx = \fint_Q u\,dx.$$

Then there exists a constant $C_P > 0$*, only depending on* n, p *and* Ω *such that there holds for all* $u \in W^{1,p}(\Omega;\mathbb{R}^N)$

$$\|u - u_\Omega\|_{L^p(\Omega)} \le C_P \|Du\|_{L^{\hat{p}}(\Omega)}.$$

Proof. See Theorem 3.15 in [Giu03]. □

Lemma A.1. *Let* $p \le n$*, we define* $p^* = \frac{pn}{n-p}$*, if* $p < n$ *and* $p^* = \frac{\gamma(1+\delta)}{\delta}$ *with* $\gamma, \delta > 0$ *from (A2), if* $p = n$ *. There exists a constant* $C_{e,P} > 0$ *such that for all* $s \in (0, \frac{1}{2}]$, $y^0 \in C^+_{1/2}(0)$ *(notation from Section 3.1) it holds for all* $u \in W^{1,p}(C^+_s(y^0);\mathbb{R}^N)$

$$\|u - u_{C^+_s(y^0)}\|_{L^{p^*}(C^+_s(y^0))} \le C_{e,P}\|Du\|_{L^p(C^+_s(y^0))}.$$

Proof. The proof relies on a scaling argument. For $s \in (0, \frac{1}{2}]$ and $y^0 \in C^+_{1/2}(0)$, we define the affine transformation $T_{s,y^0} : C^+_1 \to C^+_s(y^0)$ by

$$T_{s,y^0}(x) := A_{s,y^0}x + b_{s,y^0},$$

$$\text{where } A_{s,y^0} := \left(\begin{array}{ccc|c} s & & & 0 \\ & \ddots & & \vdots \\ & & s & 0 \\ \hline 0 & \cdots & 0 & s + \min\{y^0_n, s\} \end{array}\right) \in \mathbb{R}^{n\times n}$$

and $b_{s,y^0} \in \mathbb{R}^n$ is a suitable translation. Observe that it holds $s^n \le |\det \mathrm{D}T_{s,y^0}| \le 2s^n$. Since we have

$$\frac{|\det \mathrm{D}T_{s,y^0}|}{|C^+_s(y^0)|} = \frac{s^{n-1}(s+\min\{y^0_n, s\})}{(2s)^{n-1}(s+\min\{y^0_n,s\})} = \frac{1}{2^{n-1}} = \frac{1}{|C^+_1|},$$

the transformation formula reveals for the mean values that $u_{C_s^+(y^0)} = (u \circ T_{s,y^0})_{C_1^+}$. By exploiting the transformation formula, the embedding $W^{1,p}(\Omega;\mathbb{R}^N) \to L^{p^*}(\Omega;\mathbb{R}^N)$ and the classical Poincaré inequality, calculating the norm for $p < n$ yields

$$
\begin{aligned}
\|u - u_\Omega\|^{p^*}_{L^{p^*}(C_s^+)} &= \int_{C_1^+} |u(T_{s,y^0}(x)) - (u \circ T_{s,y^0})_{C_1^+}|^{p^*} \cdot |\det \mathrm{D}T_{s,y^0}| \,\mathrm{d}x \\
&\le 2C_e|s|^n \left(\int_{C_1^+} |u(T_{s,y^0}(x)) - (u \circ T_{s,y^0})_{C_1^+}|^p + |\mathrm{D}_x u(T_{s,y^0}(x))|^p \,\mathrm{d}x \right)^{\frac{p^*}{p}} \\
&\le 2C_{e,P}|s|^n \left(\int_{C_1^+} |\mathrm{D}_x u(T_{s,y^0}(x))|^p \,\mathrm{d}x \right)^{\frac{p^*}{p}} \\
&\le 4C_{e,P}|s|^{n+p^*} \left(\int_{C_s^+} |\mathrm{D}_y u(y)|^p \cdot |\det \mathrm{D}T^{-1}_{s,y^0}| \,\mathrm{d}y \right)^{\frac{p^*}{p}} \\
&\le 4C_{e,P}\|\mathrm{D}u\|^{p^*}_{L^p(C_s^+)},
\end{aligned}
$$

where $n + p^* - n\frac{p^*}{p} = 0$. For $p = n$ the last line reads $C_{e,P}|s|^n\|\mathrm{D}u\|^{p^*}_{L^p(C_s^+)} \le 4C_{e,P}\|\mathrm{D}u\|^{p^*}_{L^p(C_s^+)}$, since $n + p^* - n\frac{p^*}{p} = n$ and $s \le 1$. ☐

Theorem A.2 (Poincaré-Friedrichs type inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and $\hat{p} = \frac{pn}{n+p}$. Then, for every $u \in W^{1,p}(\Omega)$ taking the value zero in a set $A \subset \partial\Omega$ of positive measure, we have $\|u\|_{L^p(\Omega)} \le C_P \|Du\|_{L^{\hat{p}}(\Omega)}$, where C_P is a positive constant only depending on n, p, A and Ω.*

Proof. See Theorem 5.4.3 in [ABM06]. ☐

Lemma A.2. *Let Γ_D be as introduced in Section 3 and $p \le n$. Then there exists a constant $C_{e,P} > 0$ such that for all $s \in (0, \frac{1}{2}]$, $y^0 \in C^+_{1/2}(0)$ with $\frac{\lambda^{n-1}(C_s(y^0)\cap\Gamma_D)}{\lambda^{n-1}(\partial C_s(y^0))} \ge \kappa_0 > 0$ it holds for all $u \in W^{1,p}(C_s^+(y^0);\mathbb{R}^N)$ with $u|_{\Gamma_D} = 0$*

$$\|u\|_{L^{p^*}(C_s^+(y^0))} \le C_{e,P}\|Du\|_{L^p(C_s^+(y^0))}.$$

Proof. According to the assumptions, we have

$$\frac{\lambda^{n-1}(C_s(y^0) \cap \Gamma_D)}{\lambda^{n-1}(\partial C_s(y^0))} = \frac{(2s)^{n-2}h_{s,y^0}}{c(n)(2s)^{n-1}} \ge \kappa_0,$$

where $c(n)$ is a dimension dependent constant and h_{s,y^0} is the length of the Dirichlet boundary projected on the y_1-axis so that we arrive at $h_{s,y^0} \ge 2\kappa_0 c(n)$. Thus, we find

$$A := \{y \in \Gamma_D \,|\, y_1 \le -1 + \min\{1, 2\kappa_0 c(n)\}\} \subset T^{-1}_{s,y^0}\left(C_s(y^0) \cap \Gamma_D\right) \subset \partial C_1^+(0),$$

with $\lambda^{n-1}(A) \ge \min\{1, 2\kappa_0 c(n)\} > 0$ independently of s and y^0. The proof is now completely analog to the one of Lemma A.1, if one applies Theorem A.2 instead of Theorem A.1. ☐

Lemma A.3 (Lemma 6.1, [Giu03]). *Let $Z(t)$ be a bounded non-negative function in the interval $[\rho, R]$. Assume that for $\rho \leq t < s \leq R$, we have*

$$Z(t) \leq \left[A(s-t)^{-\alpha} + B(s-t)^{-\beta} + C\right] + \vartheta Z(s)$$

with $A, B, C \geq 0$, $\alpha > \beta > 0$ and $0 \leq \vartheta < 1$. Then

$$Z(\rho) \leq c(\alpha, \vartheta) \left[A(R-\rho)^{-\alpha} + B(R-\rho)^{-\beta} + C\right].$$

Theorem A.3 (Giaquinta and Modica, Theorem 6.6 in [Giu03]). *Let $g, h \in L^1(Q_R)$ with $g, h \geq 0$ almost everywhere and assume that for every pair of concentric cubes $Q \subset \widetilde{Q} \subset\subset Q_R$, where $\widetilde{Q}$ has the double diameter of Q, we have for some constant $B > 0$*

$$\fint_Q g \, dx \leq B \left\{ \left(\fint_{\widetilde{Q}} g^m \, dx \right)^{\frac{1}{m}} + \fint_{\widetilde{Q}} h \, dx \right\}, \tag{A.1}$$

with $0 < m < 1$. Assume the function h belongs to $L^s(Q_R)$ for some $s > 1$. Then there exist constants $c > 0$ and $q > 1$ such that $g \in L^q(Q_{R/2})$ and

$$\fint_{Q_{\frac{R}{2}}} g^q \, dx \leq c \left\{ \left(\fint_{Q_R} g \, dx \right)^q + \fint_{Q_R} h^q \, dx \right\}. \tag{A.2}$$

Remark A.1. *As remarked in [NW91], a close inspection of the proof of Theorem 6.6 in [Giu03] shows that the result remains valid if $\widetilde{Q}$ has six times the diameter of Q instead of two times. The constant B in (A.1) is proportional to the Ciaccoppoli constant c from Corollary 3.1 with $B = B(c, n) \sim 4^n c$ and the constants c and q in (A.2) only depend on the parameters m and B and not on local properties of the datum h.*

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