Local Well-Posedness of Strong Solutions to the Three-Dimensional Compressible Primitive Equations

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Abstract

This work is devoted to establishing the local-in-time well-posedness of strong solutions to the three-dimensional compressible primitive equations of atmospheric dynamics. It is shown that strong solutions exist, are unique, and depend continuously on the initial data, for a short time in two cases: with gravity but without vacuum, and with vacuum but without gravity.

Contents

1. Introduction

1.1. The Compressible Primitive Equations

The general hydrodynamic and thermodynamic equations (see, e.g., [36]) with Coriolis force and gravity are used to model the motion and state of the atmosphere, which is a specific compressible fluid. However, such equations are extremely complicated and prohibitively expensive computationally. However, since the vertical scale of the atmosphere is significantly smaller than the planetary horizontal scale, the authors in [17] take advantage, as it is commonly done in planetary scale geophysical models, of the smallness of this aspect ratio between these two orthogonal directions to formally derive the compressible primitive equations (CPE) from the compressible Navier–Stokes equations. Specifically, in the CPE the vertical component of the momentum in the compressible Navier–Stokes equations is replaced by the hydrostatic balance equation $(1.1)_3$, below, which is also known as the quasi-static equilibrium equation. It turns out that the hydrostatic approximation equation is accurate enough for practical applications and has become a fundamental equation in atmospheric science. It is the starting point of many large scale models in the theoretical investigations and practical weather predictions (see, e.g., [35]). This has also been observed by meteorologists (see, e.g., [39,44]). In fact, such an approximation is reliable and useful in the sense that the balance of gravity and pressure dominates the dynamic in the vertical direction and that the vertical velocity is usually hard to observe in reality. In many simplified models, it is assumed that the atmosphere is under adiabatic process and therefore the entropy remains unchanged along the particle path. In particular, if the entropy is constant in the spatial variables initially, it remains so in later time. On the other hand, instead of the molecular viscosity, eddy viscosity is used to model the statistical effect of turbulent motion in the atmosphere. The observations above and more perceptions from the meteorological point of view can be found in [39, Chapter 4]. Therefore, under the above assumptions, one can write down the isentropic compressible primitive equations as in (1.1), below. Moreover, we also study the problem by further neglecting the gravity in (1.2), below. We remark here that, although it does not cause any additional difficulty, we have omitted the Coriolis force in this work for the convenience of presentation. That is, the local well-posedness theorems still work for systems (1.1) and (1.2) with the Coriolis force.

The first mathematical treatment of the compressible primitive equations (CPE) can be tracked back to Lions, Temam and Wang [35]. Actually, the authors formulated the compressible primitive equations in the pressure coordinates (p-coordinates) and show that in the new coordinate system, the equations are in the form of classical primitive equations (called primitive equations, or PE hereafter) with the incompressibility condition. In yet another work [34], the authors modeled the nearly incompressible ocean by the PE. It is formulated as the hydrostatic approximation of the Boussinesq equations. The authors show the existence of global weak solutions and therefore indirectly study the CPE (see, e.g., [32,33] for additional work by the authors). Notably, the PE have been the subject of intensive mathematical research. For instance, Guillén-González, Masmoudi and Rodríguez-Bellido in

[21] study the local existence of strong solutions and global existence of strong solutions for small initial data to the PE. In [47] the authors address the global existence of strong solutions to PE in a domain with small depth for restricted large initial data depending on the depth. In [38], the authors study the Sobolev and Gevrey regularity of the solutions to PE. The first breakthrough concerning the global well-posedness of PE is obtained by Cao and Titi in [8], in which the authors show the existence of unique global strong solutions (see, also, [9,22–25,27–29,46] and the references therein for related study). On the other hand, with partial anisotropic diffusion and viscosity, Cao, Li and Titi in [3–7,10] establish the global well-posedness of strong solutions to PE. For the inviscid primitive equations, or hydrostatic incompressible Euler equations, in [1,26,37], the authors show the short time existence of solutions in the analytic function space and in H^s space. More recently, the authors in [2,45] construct finite-time blowup for the inviscid PE in the absence of rotation. Also, in [20], the authors establish the Gevrey regularity of hydrostatic Navier–Stokes equations with only vertical viscosity.

Despite the fruitful study of the primitive equations, it still remains interesting to study the compressible equations. On the one hand, it is a more direct model to study the atmosphere and perform practical weather predictions. On the other hand, the former deviation of the PE from the CPE in the p-coordinates did not treat the corresponding derivation of the boundary conditions. In fact, due to the change of pressure on the boundary, the appropriate studying domain for the PE should be evolving together with the flows in order to recover the solutions to the CPE. Thus, even though the formulation of the PE significantly simplifies the equations of the CPE, the boundary conditions are more complicated than before in order to study the motion of the atmosphere. We believe that this might be one of the reasons that is responsible for the not-completely successful prediction of the weather by using the PE.

Recently, Gatapov, Kazhikhov, Ersoy, Ngom construct a global weak solution to some variant of two-dimentional compressible primitive equations in [16,19]. Meanwhile, Ersoy, Ngom, Sy, Tang, Gao study the stability of weak solutions to the CPE in [17,41] in the sense that a sequence of weak solutions satisfying some entropy conditions contains a subsequence converging to another weak solution. In recent work, we show the existence of such weak solutions in [31]. See also [43].

In this and subsequent works, we aim to address several problems concerning the compressible primitive equations. In this work, we start by studying the local well-posedness of strong solutions to the CPE. That is, we will establish the local strong solutions to (1.1) and (1.2), below, in the domain $\Omega = \Omega_h \times (0, 1)$, with $\Omega_h = \mathbb{T}^2 = [0, 1]^2 \subset \mathbb{R}^2$ being the fundamental periodic domain. In comparison with the compressible Navier–Stokes equations [18], the absence of evolutionary equations for the vertical velocity (vertical momentum) causes the main difficulty. This is the same difficulty as in the case of the PE. In fact, the procedure of recovering the vertical velocity is a classical one in the modeling of the atmosphere [39, Chapter 5]. This is done with the help of the hydrostatic equation, which causes the stratification of density profiles in the CPE. On the one hand, in (1.1), as one will see later, the hydrostatic equation implies that if there is vacuum in the physical domain Ω, the sound speed will be at most $1/2$-Hölder continuous. Thus the H^2 estimate

of the density is not available in the presence of vacuum. However in (1.2), such an obstacle no longer exists. For this reason, the local well-posedness established in this work doesn't allow vacuum in the presence of gravity, but vacuum is allowed in the case without gravity. On the other hand, the hydrostatic equation does have some benefits. Indeed, such a relation yields that the density admits a stratified profile along the vertical direction. This fact will help us recover the vertical velocity from the continuity equation (see (1.8) and (1.13), below).

In this work, we will first reformulate the compressible primitive equations (1.1), (1.2) by making use of the stratified density profile. Then we will study the local well-posedness of the reformulated systems under the assumption that there is no vacuum initially. This is done via a fixed point argument. Next, in order to obtain the existence of strong solutions to (1.2) with non-negative density, we establish some uniform estimates independent of the lower bound of the density. We point out that in comparison to the compressible Navier–Stokes equations (see, e.g., [12–15]), we will require H^2 estimate of $\rho^{1/2}$ in order to derive the above mentioned uniform estimates. Such estimates are not available in the case with gravity (1.1). To this end, continuity arguments are used to establish the solutions with vacuum. We also study the continuous dependence on the initial data and the uniqueness of the strong solutions.

Through out this work, we will use $\vec{x} := (x, y, z)^\top$, $\vec{x}_h := (x, y)^\top$ to represent the coordinates in Ω and Ω_h, respectively. In addition, we will use the following notations to denote the differential operators in the horizontal direction:

$$\nabla_h := (\partial_x, \partial_y)^\top, \ \partial_h \in \{\partial_x, \partial_y\},$$
$$\mathrm{div}_h := \nabla_h \cdot, \ \Delta_h := \mathrm{div}_h \nabla_h.$$

The isentropic compressible primitive equations with gravity are governed by the system

$$\begin{cases} \partial_t \rho + \mathrm{div}_h (\rho v) + \partial_z(\rho w) = 0 & \text{in } \Omega, \\ \partial_t(\rho v) + \mathrm{div}_h (\rho v \otimes v) + \partial_z(\rho w v) + \nabla_h P = \mu \Delta_h v + \mu \partial_{zz} v \\ \qquad + (\mu + \lambda)\nabla_h \mathrm{div}_h v & \text{in } \Omega, \\ \partial_z P - \rho g = 0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

with $P := \rho^\gamma$. We will study in this work only the case when $\gamma = 2$ in (1.1) for the sake of simplifying our presentation. For general $\gamma > 1$, we refer to Remark 1, below.

On the other hand, the isentropic compressible primitive equations without gravity are governed by the system

$$\begin{cases} \partial_t \rho + \mathrm{div}_h (\rho v) + \partial_z(\rho w) = 0 & \text{in } \Omega, \\ \partial_t(\rho v) + \mathrm{div}_h (\rho v \otimes v) + \partial_z(\rho w v) + \nabla_h P = \mu \Delta_h v + \mu \partial_{zz} v \\ \qquad + (\mu + \lambda)\nabla_h \mathrm{div}_h v & \text{in } \Omega, \\ \partial_z P = 0 & \text{in } \Omega, \end{cases} \quad (1.2)$$

with $P := \rho^\gamma$ and $\gamma > 1$.

In the above systems, (1.1) and (1.2), the viscosity coefficients μ, λ are assumed to be strictly positive. Also, (1.1) and (1.2) are supplemented with the following boundary conditions:

$$w = 0, \ \partial_z v = 0 \quad \text{on} \ \Omega_h \times \{0, 1\}. \tag{1.3}$$

The rest of this paper will be organized as follows: in section 1.2, we present a reformulation of (1.1) and (1.2) by making use of the stratified density profiles. Also, we present the formula for recovering the vertical velocity and the main theorems of this work. After listing some useful inequalities and notations, we study in section 2 the existence theory. Next, in section 3 we show the continuous dependence on the initial data and the uniqueness of strong solutions.

1.2. Reformulation, Analysis and Main Theorems

In this section, we will reformulate (1.1) and (1.2) and point out how to recover the vertical velocity in terms of the density and the horizontal velocity.

The Case with Gravity and $\gamma = 2$ We first consider (1.1). From $(1.1)_3$, one has

$$\rho^{\gamma-1}(\vec{x}, t) = \frac{\gamma - 1}{\gamma} gz + \rho^{\gamma-1}(\vec{x}_h, 0, t).$$

Denote by $\xi = \xi(\vec{x}_h, t) := \rho^{\gamma-1}(\vec{x}_h, 0, t)$. The continuity equation $(1.1)_1$ implies

$$\partial_t \xi + v \cdot \nabla_h \xi + (\gamma - 1) \left(\xi + \frac{\gamma - 1}{\gamma} gz \right) (\text{div}_h \, v + \partial_z w) + \frac{\gamma - 1}{\gamma} gw = 0. \tag{1.4}$$

In particular, since $\gamma = 2$, we have $\rho(\vec{x}, t) = \xi(\vec{x}_h, t) + \frac{1}{2} gz$ and
(1.1) can be written as

$$\begin{cases} \partial_t \xi + v \cdot \nabla_h \xi + \left(\xi + \frac{1}{2} gz \right) \text{div}_h \, v + \partial_z \left(\xi w + \frac{1}{2} gzw \right) = 0 & \text{in } \Omega, \\ \left(\xi + \frac{1}{2} gz \right) (\partial_t v + v \cdot \nabla_h v + w \partial_z v) + (2\xi + gz) \nabla_h \xi \\ \qquad = \mu \Delta_h v + \mu \partial_{zz} v + (\mu + \lambda) \nabla_h \text{div}_h \, v & \text{in } \Omega, \\ \partial_z \xi = 0 & \text{in } \Omega. \end{cases} \tag{1.5}$$

Hereafter, we denote, for any $f : \Omega \mapsto \mathbb{R}$,

$$\overline{f} := \int_0^1 f \, dz, \ \ \widetilde{f} := f - \overline{f}. \tag{1.6}$$

Then averaging over the vertical variable in $(1.5)_1$ yields, thanks to (1.3),

$$\partial_t \xi + \overline{v} \cdot \nabla_h \xi + \xi \overline{\text{div}_h \, v} + \frac{1}{2} g \overline{z \text{div}_h \, v} = 0. \tag{1.7}$$

Then comparing (1.7) with (1.5)$_1$ implies

$$\partial_z(\rho w) = \partial_z\left(\xi w + \frac{1}{2}gzw\right) = -\widetilde{v}\cdot\nabla_h\xi - \xi\operatorname{div}_h\widetilde{v} - \frac{g}{2}z\widetilde{\operatorname{div}_h v}\ \text{ in }\Omega.$$

Therefore, the vertical velocity w is determined, thanks to the boundary condition (1.3), by the relation

$$\rho w = (\xi + \frac{1}{2}gz)w = -\int_0^z \left(\operatorname{div}_h(\xi\widetilde{v}) + \frac{g}{2}z\widetilde{\operatorname{div}_h v}\right)\mathrm{d}z. \tag{1.8}$$

System (1.5) is complemented with the initial data

$$(\xi, v)|_{t=0} = (\xi_0, v_0), \tag{1.9}$$

with $\xi_0, v_0 \in H^2(\Omega)$. Also the following compatible conditions are imposed:

$$\begin{aligned}
&\rho_0 = \xi_0 + \frac{1}{2}gz \geqq \underline{\rho} > 0 \text{ in } \Omega, \text{ and } \partial_z v_0|_{z=0,1} = 0,\\
&\mu\Delta_h v_0 + \mu\partial_{zz}v_0 + (\mu+\lambda)\nabla_h\operatorname{div}_h v_0 - (2\xi_0+gz)\nabla_h\xi_0\\
&\quad - \rho_0 v_0\cdot\nabla_h v_0 - \rho_0 w_0\partial_z v_0 =: \rho_0 V_1, \quad \text{with } V_1 \in L^2(\Omega),\\
&\text{and } \rho_0 w_0 = -\int_0^z\left(\operatorname{div}_h(\xi_0\widetilde{v_0}) + \frac{g}{2}z\widetilde{\operatorname{div}_h v_0}\right)\mathrm{d}z.
\end{aligned} \tag{1.10}$$

Also, we will denote the bounds

$$\left\|\xi_0\right\|_{H^2}^2 \leqq B_{g,1}, \quad \left\|v_0\right\|_{H^2}^2 + \left\|V_1\right\|_{L^2}^2 \leqq B_{g,2}. \tag{1.11}$$

Theorem 1. *Suppose the initial data* $(\rho_0, v_0) = (\xi_0 + \frac{1}{2}gz, v_0)$ *satisfy* (1.11) *and the compatible conditions* (1.10). *Then there is a unique strong solution* (ρ, v) *to system* (1.1), *with the boundary condition* (1.3), *in* $\Omega \times (0, T)$, *for some positive constant* $T = T(B_{g,1}, B_{g,2}, \underline{\rho}) > 0$. *Also, the solution satisfies*

$$\begin{aligned}
&\rho \in L^\infty(0, T; H^2(\Omega)), \partial_t\rho \in L^\infty(0, T; H^1(\Omega)),\\
&v \in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)),\\
&\partial_t v \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).
\end{aligned}$$

Furthermore, for some positive constant $C(B_{g,1}, B_{g,2}, \underline{\rho})$,

$$\inf_{(\vec{x},t)\in\Omega\times(0,T)} \rho(\vec{x}, t) \geqq \frac{1}{2}\underline{\rho} > 0,$$

$$\sup_{0\leqq t\leqq T}\left(\left\|\rho(t)\right\|_{H^2}^2 + \left\|\partial_t\rho(t)\right\|_{H^1}^2 + \left\|v(t)\right\|_{H^2}^2 + \left\|\partial_t v(t)\right\|_{L^2}^2\right)$$

$$+ \int_0^T\left(\left\|v(t)\right\|_{H^3}^2 + \left\|\partial_t v(t)\right\|_{H^1}^2\right)dt \leqq C(B_{g,1}, B_{g,2}, \underline{\rho}).$$

Moreover, for any two solutions (ρ_i, v_i), $i = 1, 2$ *with initial data* $(\rho_{i,0}, v_{i,0})$, $i = 1, 2$ *satisfying the conditions mentioned above, we have the following inequality*

$$\|\rho_1 - \rho_2\|_{L^\infty(0,T;L^2(\Omega))} + \|v_1 - v_2\|_{L^\infty(0,T;L^2(\Omega))}$$
$$+ \|\nabla(v_1 - v_2)\|_{L^2(0,T;L^2(\Omega))} \leq C_{\mu,\lambda,B_{g,1},B_{g,2},\underline{\rho},T}$$
$$\times \left(\|\rho_{1,0} - \rho_{2,0}\|_{L^2(\Omega)} + \|v_{1,0} - v_{2,0}\|_{L^2(\Omega)}\right),$$

for some positive constant $C_{\mu,\lambda,B_{g,1},B_{g,2},\underline{\rho},T}$.

Remark 1. For general $\gamma > 1$, after multiplying (1.4) with

$$\left(\frac{\xi + \frac{\gamma-1}{\gamma}gz}{\xi}\right)^{\frac{2-\gamma}{\gamma-1}}$$

and averaging the resultant in the z-variable, one obtains

$$\frac{\gamma}{g}\left((\xi + \frac{\gamma-1}{\gamma}g)^{\frac{1}{\gamma-1}}\xi^{\frac{\gamma-2}{\gamma-1}} - \xi\right)\partial_t \xi + \xi^{\frac{\gamma-2}{\gamma-1}}\overline{\left(\xi + \frac{\gamma-1}{\gamma}gz\right)^{\frac{2-\gamma}{\gamma-1}} v \cdot \nabla_h \xi}$$
$$+ (\gamma - 1)\xi^{\frac{\gamma-2}{\gamma-1}}\overline{\left(\xi + \frac{\gamma-1}{\gamma}gz\right)^{\frac{1}{\gamma-1}} \operatorname{div}_h v} = 0 \text{ in } \Omega_h.$$

Consequently, by eliminating $\partial_t \xi$ from the above equation and (1.4), it follows that

$$(\gamma - 1)\partial_z\left((\xi + \frac{\gamma-1}{\gamma}gz)^{\frac{1}{\gamma-1}}w\right) = -(\gamma - 1)\left(\xi + \frac{\gamma-1}{\gamma}gz\right)^{\frac{1}{\gamma-1}}\operatorname{div}_h v$$
$$- (\xi + \frac{\gamma-1}{\gamma})^{\frac{2-\gamma}{\gamma-1}} v \cdot \nabla_h \xi + \frac{g\xi^{\frac{\gamma-2}{\gamma-1}}\left(\xi + \frac{\gamma-1}{\gamma}gz\right)^{\frac{2-\gamma}{\gamma-1}}}{\gamma\left((\xi + \frac{\gamma-1}{\gamma}g)^{\frac{1}{\gamma-1}}\xi^{\frac{\gamma-2}{\gamma-1}} - \xi\right)}$$
$$\times \left(\overline{\left(\xi + \frac{\gamma-1}{\gamma}gz\right)^{\frac{2-\gamma}{\gamma-1}} v \cdot \nabla_h \xi} + (\gamma - 1)\overline{\left(\xi + \frac{\gamma-1}{\gamma}gz\right)^{\frac{1}{\gamma-1}}\operatorname{div}_h v}\right).$$

Therefore, from above, and as in the case when $\gamma = 2$, the vertical velocity w can be represented in the form

$$w = \int_0^z H(\operatorname{div}_h v, v, \nabla_h \xi, \xi)\, dz',$$

similarly to (1.8), for an explicit function $H(\cdot)$. Notably, the arguments and proofs, below, apply equally, and similar conclusion of Theorem 1 also holds for $\gamma > 1$.

The Case Without Gravity and $\gamma > 1$ Concerning system (1.2), since (1.2)$_3$ already yields the independence of the density of the vertical variable, after taking the vertical average of (1.2)$_1$, as before, one has

$$\partial_t \rho + \operatorname{div}_h (\rho \overline{v}) = 0. \tag{1.12}$$

Comparing (1.12) with (1.2)$_1$ yields, thanks to the boundary condition (1.3), that the vertical velocity w is determined by the relation

$$\rho w = -\int_0^z \operatorname{div}_h (\rho \widetilde{v}) \, dz. \tag{1.13}$$

In particular, by denoting $\sigma := \rho^{1/2}$, from (1.12) and (1.13), one has either $\sigma = 0$ or

$$\partial_t \sigma + \overline{v} \cdot \nabla_h \sigma + \frac{1}{2} \sigma \overline{\operatorname{div}_h v} = 0, \tag{1.14}$$

$$\sigma w = -\int_0^z \left(\sigma \widetilde{\operatorname{div}_h v} + 2\widetilde{v} \cdot \nabla_h \sigma \right) dz. \tag{1.15}$$

In fact, for (σ, v) regular enough, (1.14), (1.15) hold regardless of whether $\sigma = 0$ or not. See also the justification in the beginning of section 3.2.

System (1.2) is complemented with the initial data

$$(\rho, v)|_{t=0} = (\rho_0, v_0), \quad \text{or equivalently } (\sigma, v)|_{t=0} = (\sigma_0, v_0), \tag{1.16}$$

with $\sigma_0 = \rho_0^{1/2}$, $v_0 \in H^2(\Omega)$, and the initial total mass and physical energy satisfy

$$0 < \int_\Omega \rho_0 \, d\vec{x} = \int_\Omega \sigma_0^2 \, d\vec{x} = M < \infty,$$

$$0 < \int_\Omega \rho_0 |v_0|^2 \, d\vec{x} + \frac{1}{\gamma - 1} \int_\Omega \rho_0^\gamma \, d\vec{x} = \int_\Omega \sigma_0^2 |v_0|^2 \, d\vec{x} \tag{1.17}$$

$$+ \frac{1}{\gamma - 1} \int_\Omega \sigma_0^{2\gamma} \, d\vec{x} = E_0 < \infty.$$

Also the following compatible conditions are imposed:

$$\rho_0 \geqq 0, \quad \partial_z v_0|_{z=0,1} = 0,$$

$$\mu \Delta_h v_0 + \mu \partial_{zz} v_0 + (\mu + \lambda) \nabla_h \operatorname{div}_h v_0 - \nabla_h \rho_0^\gamma - \rho_0 v_0 \cdot \nabla_h v_0$$

$$- \rho_0 w_0 \partial_z v_0 =: \rho_0^{1/2} h_1, \quad \text{with } h_1 \in L^2(\Omega), \tag{1.18}$$

$$\text{and} \quad \rho_0 w_0 = -\int_0^z \operatorname{div}_h (\rho_0 \widetilde{v}_0) \, dz.$$

Also, we will denote the bounds

$$\|\sigma_0\|_{H^2}^2 = \|\rho_0^{1/2}\|_{H^2}^2 \leqq B_1, \quad \|v_0\|_{H^2}^2 + \|h_1\|_{L^2}^2 \leqq B_2. \tag{1.19}$$

Moreover, if $\rho = \sigma^2 > 0$, (1.2) can be written as

$$\begin{cases} \partial_t \sigma + v \cdot \nabla_h \sigma + w \partial_z \sigma + \dfrac{1}{2}\sigma(\text{div}_h\, v + \partial_z w) = 0 & \text{in } \Omega, \\ \sigma^2(\partial_t v + v \cdot \nabla_h v + w \partial_z v) + \nabla_h \sigma^{2\gamma} \\ \qquad = \mu \Delta_h v + \mu \partial_{zz} v + (\mu + \lambda)\nabla_h \text{div}_h\, v & \text{in } \Omega, \\ \partial_z \sigma = 0 & \text{in } \Omega. \end{cases} \tag{1.20}$$

Theorem 2. *Suppose the initial data $(\rho_0, v_0) = (\sigma_0^2, v_0)$ satisfy* (1.17)*,* (1.19) *and the compatible conditions* (1.18)*. Then there is a unique strong solution (ρ, v) to system* (1.2)*, with the boundary condition* (1.3)*, in $\Omega \times (0, T^*)$, for some positive constant $T^* = T^*(B_1, B_2) > 0$. Also, the solution satisfies*

$$\rho^{1/2} \in L^\infty(0, T^*; H^2(\Omega)), \quad \partial_t \rho^{1/2} \in L^\infty(0, T^*; H^1(\Omega)),$$
$$v \in L^\infty(0, T^*; H^2(\Omega)) \cap L^2(0, T^*; H^3(\Omega)), \quad \partial_t v \in L^2(0, T^*; H^1(\Omega))$$
$$\rho^{1/2}\partial_t v \in L^\infty(0, T^*; L^2(\Omega)).$$

Furthermore, for some positive constant $\mathcal{C}(B_1, B_2)$,

$$\inf_{(\vec{x},t)\in\Omega\times(0,T^*)} \rho(\vec{x}, t) \geqq 0,$$
$$\sup_{0\leqq t\leqq T^*} \left(\left\|\rho^{1/2}(t)\right\|_{H^2}^2 + \left\|\partial_t \rho^{1/2}(t)\right\|_{H^1}^2 + \left\|v(t)\right\|_{H^2}^2 + \left\|(\rho^{1/2}v_t)(t)\right\|_{L^2}^2 \right)$$
$$+ \int_0^{T^*} \left(\left\|v(t)\right\|_{H^3}^2 + \left\|v_t(t)\right\|_{H^1}^2 \right) dt \leqq \mathcal{C}(B_1, B_2).$$

Moreover, for any two strong solutions $(\rho_i, v_i), i = 1, 2$, with initial data $(\rho_{i,0}, v_{i,0}), i = 1, 2$, satisfying the conditions mentioned above, we have the inequality

$$\left\|\rho_1^{1/2} - \rho_2^{1/2}\right\|_{L^\infty(0,T^*;L^2(\Omega))} + \left\|\rho_1^{1/2}(v_1 - v_2)\right\|_{L^\infty(0,T^*;L^2(\Omega))}$$
$$+ \left\|\rho_2^{1/2}(v_1 - v_2)\right\|_{L^\infty(0,T^*;L^2(\Omega))} + \left\|v_1 - v_2\right\|_{L^2(0,T^*;L^2(\Omega))}$$
$$+ \left\|\nabla(v_1 - v_2)\right\|_{L^2(0,T^*;L^2(\Omega))}$$
$$\leqq C_{\mu,\lambda,B_1,B_2,T^*}\left(\left\|\rho_{1,0}^{1/2} - \rho_{2,0}^{1/2}\right\|_{L^2(\Omega)} + \left\|v_{1,0} - v_{2,0}\right\|_{L^2(\Omega)} \right)$$

for some positive constant $C_{\mu,\lambda,B_1,B_2,T^*}$.

1.3. Preliminaries

We will use $|\cdot|, \|\cdot\|$ to denote norms in $\Omega_h \subset \mathbb{R}^2$ and $\Omega \subset \mathbb{R}^3$, respectively. After applying Ladyzhenskaya's and Agmon's inequalities in Ω_h and Ω, directly we have

$$|f|_{L^4} \leqq C|f|_{L^2}^{1/2}|f|_{H^1}^{1/2}, \quad |f|_{L^\infty} \leqq C|f|_{L^2}^{1/2}|f|_{H^2}^{1/2},$$
$$\|f\|_{L^3} \leqq C\|f\|_{L^2}^{1/2}\|f\|_{H^1}^{1/2} \tag{1.21}$$

for any function f with bounded right-hand sides. Also, $\left|\overline{f}\right|_{L^p}, \left\|\widetilde{f}\right\|_{L^p} \leq C\|f\|_{L^p}$, for every $p \geq 1$. Considering any quantities A, B, we use the notation $A \lesssim B$ to denote $A \leq CB$ for some generic positive constant C, which may be different from line to line. In what follows $\delta, \omega > 0$ are arbitrary constants which will be chosen later in the relevant paragraphs to be adequately small. C_q represents a positive constant depending on the quantity q. We will also need the following classical inequality:

Lemma 1. *Let* $2 \leq p \leq 6$, *and* $\rho \geq 0$ *such that* $0 < \int_\Omega \rho\, d\vec{x} = M < \infty$, *and* $\int_\Omega \rho^\gamma\, d\vec{x} \leq E_0$, *for some* $\gamma \in (1, \infty)$. *Then one has*

$$\|f\|_{L^p} \leq C\|\nabla f\|_{L^2} + C\|\rho^{1/2} f\|_{L^2} \tag{1.22}$$

for some constant $C = C(M, E_0)$, *provided the right-hand side is finite.*

Proof. This is standard. See, e.g., [18, Lemma 3.2]. □

2. Associated Linear Systems and Existence Theory

In this section, we will establish the local existence theory of (1.1) and (1.2). To do this, we will first study the local existence of solutions to (1.5) and (1.20) via the Schauder–Tchonoff fixed point theorem capitalizing on some a priori estimates. In fact, under the assumption that

$$\rho_0 = \begin{cases} \xi_0 + \dfrac{1}{2}gz & \text{in the case with gravity} \\ (\sigma_0)^2 & \text{in the case without gravity} \end{cases} > \underline{\rho} > 0, \tag{2.1}$$

we will first introduce linear systems and the function spaces $\mathfrak{Y}$ associated with (1.5) and (1.20) with some given input states (ξ^o, v^o) and (σ^o, v^o), respectively, in section 2.1 and 2.2. Here, $\mathfrak{Y}$ are compactly embedded in some corresponding spaces $\mathfrak{V}$. Also, we will show that the maps $\mathcal{T} : \mathfrak{X} \mapsto \mathfrak{X}$, for some convex bounded subsets $\mathfrak{X}$ of $\mathfrak{Y}$, given by

$$(\xi^o, v^o) \rightsquigarrow (\xi, v) \quad \text{in the case with gravity, and}$$
$$(\sigma^o, v^o) \rightsquigarrow (\sigma, v) \quad \text{in the case without gravity,}$$

are well-defined; observing that $\mathfrak{X}$ are convex subsets of $\mathfrak{Y}$ and hence compact in $\mathfrak{V}$. We will use the same notations $\mathfrak{X}, \mathfrak{Y}, \mathfrak{V}, \mathcal{T}$ to denote the convex bounded sets, the compact function spaces, the embedded function spaces and the constructed maps in both cases. We summarize the relevant regularity estimates in section 2.3 and show that the Schauder-Tchonoff fixed point theorem will yield the existence of solutions to (1.5) and (1.20) in the corresponding set. Recall that the Schauder-Tchonoff fixed point theorem states that for a Banach space V with a convex compact subset $X \subset V$, if $F : X \mapsto X$ is continuous, then F has at least one fixed point in X. In our case, we will take $X = \mathfrak{X}$ and $V = \mathfrak{V} :=$ $\{(\xi, v)|\xi, v \in L^\infty(0, T; L^2(\Omega)), \nabla v \in L^2(0, T; L^2(\Omega))\}$ in the case with gravity,

or $V = \mathfrak{V} := \{(\sigma, v) | \sigma, v \in L^\infty(0, T; L^2(\Omega)), \nabla v \in L^2(0, T; L^2(\Omega))\}$ in the case without gravity, with the corresponding norms.

We will only sketch the key steps in this paper. For more detailed calculation, we refer to our preprint [30].

2.1. The Case with Gravity and $\gamma = 2$

2.1.1. Associated Linear Inhomogeneous System Consider a finite positive time T, which will be determined later. Let $\mathfrak{Y} = \mathfrak{Y}_T$ be the function space defined by

$$\begin{aligned}
\mathfrak{Y} = \mathfrak{Y}_T :=\{(\xi, v) | \xi &\in L^\infty(0, T; H^2(\Omega)), \partial_t \xi \in L^\infty(0, T; H^1(\Omega)), \\
v &\in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)), \\
\partial_t v &\in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))\},
\end{aligned} \tag{2.2}$$

with the norm

$$\begin{aligned}
\|(\xi, v)\|_{\mathfrak{Y}} :=&\|\xi\|_{L^\infty(0,T;H^2(\Omega))} + \|\partial_t \xi\|_{L^\infty(0,T;H^1(\Omega))} + \|v\|_{L^\infty(0,T;H^2(\Omega))} \\
&+ \|v\|_{L^2(0,T;H^3(\Omega))} + \|\partial_t v\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t v\|_{L^2(0,T;H^1(\Omega))}.
\end{aligned}$$

Notice that, thanks to the Aubin compactness theorem (see, e.g., [42, Theorem 2.1] and [11,40]), every bounded subset of $\mathfrak{Y}$ is a compact subset of the space

$$\mathfrak{V} = \mathfrak{V}_T := \{(\xi, v) | \xi, v \in L^\infty(0, T; L^2(\Omega)), \nabla v \in L^2(0, T; L^2(\Omega))\}, \tag{2.3}$$

with the norm

$$\begin{aligned}
\|(\xi, v)\|_{\mathfrak{V}} :=&\|\xi\|_{L^\infty(0,T;L^2(\Omega))} + \|v\|_{L^\infty(0,T;L^2(\Omega))} \\
&+ \|v\|_{L^2(0,T;H^1(\Omega))}.
\end{aligned} \tag{2.4}$$

Let $\mathfrak{X} = \mathfrak{X}_T$ be a bounded subset of $\mathfrak{Y}$ defined by

$$\begin{aligned}
\mathfrak{X} =\mathfrak{X}_T := \Big\{&(\xi, v) \in \mathfrak{Y} | (\xi, v)|_{t=0} = (\xi_0, v_0), \partial_z v|_{z=0,1} = 0, \partial_z \xi = 0, \\
&\xi + \frac{1}{2}gz \geq \frac{1}{2}\rho > 0, \sup_{0 \leq t \leq T} \|\xi(t)\|_{H^2}^2 \leq 2M_0, \sup_{0 \leq t \leq T} \|\partial_t \xi(t)\|_{H^1}^2 \leq C_2, \\
&\sup_{0 \leq t \leq T} \{\|v(t)\|_{H^2}^2 + \|\partial_t v(t)\|_{L^2}^2\} + \int_0^T \Big(\|v(t)\|_{H^3}^2 \\
&+ \|\partial_t v(t)\|_{H^1}^2\Big) dt \leq C_1 M_1\Big\},
\end{aligned} \tag{2.5}$$

where M_0, M_1 are the bounds of initial data in (2.9) and $C_1 = C_1(M_0, \mu, \lambda, \rho)$, $C_2 = C_2(M_0, C_1 M_1)$ are given below in (2.34), (2.17), respectively. Notice, for $(\xi, v) \in \mathfrak{X}$,

$$\int_0^1 \Big(\text{div}_h(\xi\widetilde{v}) + \frac{g}{2}z\widetilde{\text{div}_h v}\Big) dz = 0.$$

Let $(\xi^o, v^o) \in \mathfrak{X}$. The following inhomogeneous linear system is inferred from (1.5) using (ξ^o, v^o) as an input:

$$
\begin{cases}
\partial_t \xi + \overline{v^o} \cdot \nabla_h \xi + \xi \overline{\mathrm{div}_h \, v^o} + \dfrac{g}{2} \overline{z \mathrm{div}_h \, v^o} = 0 & \text{in } \Omega, \\[2mm]
(\xi^o + \dfrac{1}{2} gz)(\partial_t v + v^o \cdot \nabla_h v^o + w^o \partial_z v^o) + (2\xi^o + gz)\nabla_h \xi^o & \\[1mm]
\qquad = \mu \Delta_h v + \mu \partial_{zz} v + (\mu + \lambda)\nabla_h \mathrm{div}_h \, v & \text{in } \Omega, \\[2mm]
\partial_z \xi = 0 & \text{in } \Omega.
\end{cases}
\tag{2.6}
$$

Here w^o is given by (1.8) with (ξ^o, v^o) instead of (ξ, v), i.e.,

$$
\rho^o w^o = (\xi^o + \frac{1}{2} gz) w^o := - \int_0^z \left(\mathrm{div}_h \, \widetilde{(\xi^o \widetilde{v^o})} + \frac{g}{2} z \widetilde{\mathrm{div}_h \, v^o} \right) \mathrm{d}z.
\tag{2.7}
$$

Notice that $(2.6)_1$ is inferred from (1.7). For details, see the deviation from (1.5) to (1.8). Hereafter, denote by $\rho^o := \xi^o + \dfrac{1}{2} gz$. The initial and boundary conditions for the linear system (2.6) are given by

$$
(\xi, v)|_{t=0} = (\xi_0, v_0), \ \partial_z v|_{z=0,1} = 0.
\tag{2.8}
$$

The compatible conditions in (1.10) are still imposed and we require

$$
\left\| \xi_0 \right\|_{H^2}^2 \leqq M_0, \ \left\| v_0 \right\|_{H^2}^2 + \left\| V_1 \right\|_{L^2}^2 \leqq M_1.
\tag{2.9}
$$

Recall that V_1 is given in (1.10), essentially, $V_1 = v_t|_{t=0}$.

Then the map $\mathcal{T}$, in this case, is defined as

$$
\mathcal{T} : (\xi^o, v^o) \rightsquigarrow (\xi, v),
\tag{2.10}
$$

where (ξ, v) is the unique solution to the linear system (2.6) with $(\xi^o, v^o) \in \mathfrak{X}$. We claim that $\mathcal{T}$ is a well defined map from $\mathfrak{X}$ to $\mathfrak{X}$, which is the consequence of the following two propositions:

Proposition 1. *For given $(\xi^o, v^o) \in C^\infty(\overline{\Omega} \times [0, T]) \cap \mathfrak{X}_T$, there is a unique strong solution $(\xi, v) \in \mathfrak{Y}_T$ of system (2.6) with the initial and boundary conditions (2.8).*

Suppose, in addition, that $\xi_0, v_0 \in H^3(\Omega)$. One will have the following regularity of the unique solution (ξ, v) of system (2.6):

$$
\begin{aligned}
&\xi \in L^\infty(0, T; H^3(\Omega)), \ \partial_t \xi \in L^\infty(0, T; H^2(\Omega)), \\
&v \in L^\infty(0, T; H^3(\Omega)) \cap L^2(0, T; H^4(\Omega)), \\
&\partial_t v \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)).
\end{aligned}
\tag{2.11}
$$

Proposition 2. *Consider the initial data with the bounds M_0, M_1 in (2.9) and $(\xi^o, v^o) \in \mathfrak{X} = \mathfrak{X}_T$. There is a $T_g = T_g(M_0, M_1, \mu, \lambda, \underline{\rho}) > 0$ sufficiently small such that for any $T \in (0, T_g)$, there exists a unique solution to (2.6). Moreover, the solution belongs to $\mathfrak{X} = \mathfrak{X}_T$. Therefore, for any such T, the map $\mathcal{T}$ in (2.10) is a well defined map from $\mathfrak{X}$ into $\mathfrak{X}$.*

We omit the proof of Proposition 1, and refer to [30] for the details. The existence of solutions in Proposition 2 follows from Proposition 1 and a standard approximating argument. We only show the required *a priori* estimates in the rest of this subsection, which are sufficient to establish Proposition 2. See Propositions 3 and 4, below.

2.1.2. A Priori Estimates for the Inhomogeneous Linear System Hereafter, we assume that the solution (ξ, v) to the linear system (2.6) is smooth enough so that the following estimates are rigorous.

We start by establishing some estimates for the solutions of $(2.6)_1$. In particular, we will establish the following:

Proposition 3. *There exists a $T' = T'(M_0, C_1 M_1, \rho) > 0$ sufficiently small such that for any $T \in (0, T']$, the solution ξ to $(2.6)_1$ satisfies that*

$$\xi + \frac{1}{2}gz \geq \frac{1}{2}\underline{\rho}; \ \sup_{0 \leq t \leq T} \left\|\xi(t)\right\|_{H^2}^2 \leq 2M_0; \ \sup_{0 \leq t \leq T} \left\|\partial_t\xi(t)\right\|_{H^1}^2 \leq C_2,$$

where M_0 is as in (2.9).

The lower bound for ξ

In order to derive the lower bound of ξ, we employ the following Stampaccia-like argument. Let $M = M(t) > 0$ be a nonnegative integrable function to be determined later. Consider $\eta = \eta(x, y, t) := \xi - \underline{\rho} + \int_0^t M(s)\,ds$. Then according to $(2.6)_1$, η satisfies the equation

$$\partial_t\eta + \overline{v^o} \cdot \nabla_h\eta + \eta\overline{\mathrm{div}_h\,v^o} = -(\underline{\rho} - \int_0^t M(s)\,ds)\overline{\mathrm{div}_h\,v^o}$$

$$- \frac{g}{2}\overline{z\mathrm{div}_h\,v^o} + M(t).$$

Let

$$\mathbb{1}_{\{\eta<0\}} = \begin{cases} 1 & \text{whenever } \{\eta < 0\}, \\ 0 & \text{otherwise,} \end{cases}$$

and denote by $\eta_- := -\eta\mathbb{1}_{\{\eta<0\}} \geq 0$. Observe that since $\xi \in H^1(\Omega \times [0, T])$, so is η_-. Thus, one has

$$\frac{d}{dt}\int_{\Omega_h} \eta_-\,d\vec{x}_h = \int_{\{\eta<0\}} \left((\underline{\rho} - \int_0^t M(s)\,ds)\overline{\mathrm{div}_h\,v^o} + \frac{g}{2}\overline{z\mathrm{div}_h\,v^o} - M(t)\right)d\vec{x}_h.$$

Now, let $0 < M(t) := C\max\{\left|\overline{\mathrm{div}_h\,v^o}\right|_{L^\infty}, \left|\overline{z\mathrm{div}_h\,v^o}\right|_{L^\infty}\} \leq C\|v^o\|_{H^3} < \infty$, *a.e.,,* for some constant $C > 0$. Then the integrand on the right-hand side of the above equation satisfies

$$\left(\underline{\rho} - \int_0^t M(s)\,ds\right)\overline{\mathrm{div}_h\,v^o} + \frac{g}{2}\overline{z\mathrm{div}_h\,v^o} - M(t)$$

$$\leq \frac{1}{C}\left(\underline{\rho} + C\int_0^T \|v^o\|_{H^3}(s)\,ds + \frac{g}{2}\right)M(t) - M(t) < 0,$$

provided C is large enough and T is small enough such that

$$\frac{1}{C}\left(\underline{\rho} + 1 + \frac{g}{2}\right) < 1, \text{ and}$$

$$C\int_0^T \|v^o\|_{H^3}(s)\,ds \leq CT^{1/2}\left(\int_0^T \|v^o\|_{H^3}^2(s)\,ds\right)^{1/2} < 1.$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_h} \eta_- \, \mathrm{d}\vec{x}_h \leqq 0 \quad a.e.,$$

which, after integrating over $[0, t_0]$ for any $t_0 \in [0, T]$, thanks to the fact $\eta_-(0) \equiv 0$, yields

$$\int_{\Omega_h} \eta_-(t_0) \, \mathrm{d}\vec{x}_h \leqq 0. \tag{2.12}$$

Hence $\eta_- = 0$ in $\Omega_h \times [0, T]$. That is, $\eta(t) = \xi(t) - \underline{\rho} + \int_0^t M(s) \, \mathrm{d}s \geqq 0$ and

$$\begin{aligned} \xi(t) + \frac{1}{2} gz > \xi(t) &\geqq \underline{\rho} - C \int_0^T \|v^o\|_{H^3}(s) \, \mathrm{d}s \\ &\geqq \underline{\rho} - CC_1^{1/2} M_1^{1/2} T^{1/2} \geqq \frac{1}{2} \underline{\rho} \end{aligned} \tag{2.13}$$

for $t \in [0, T]$, $T \leqq T_1$, with $T_1 = T_1(C_1 M_1, \underline{\rho})$ sufficiently small.

The $H^2(\Omega)$ norm for ξ

Applying the standard H^2 estimate of linear transport equations to $(2.6)_1$ yields, since $|\xi|_{H^2} = \|\xi\|_{H^2}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\xi\|_{H^2}^2 \leq C \|v^o\|_{H^3} \|\xi\|_{H^2}^2 + C \|v^o\|_{H^3} \|\xi\|_{H^2}.$$

Thanks to the Grönwall and the Hölder inequalities, one has

$$\begin{aligned} \sup_{0 \leqq t \leqq T} \|\xi(t)\|_{H^2}^2 &\leq \frac{1}{4} \sup_{0 \leqq t \leqq T} \|\xi\|_{H^2}^2 + C^2 C_1 M_1 e^{2CC_1^{1/2} M_1^{1/2} T^{1/2}} T \\ &+ e^{CC_1^{1/2} M_1^{1/2} T^{1/2}} M_0. \end{aligned} \tag{2.14}$$

Then for $T \in (0, T_2]$ with $T_2(M_0, C_1 M_1)$ sufficiently small, (2.14) yields

$$\sup_{0 \leqq t \leqq T} \|\xi(t)\|_{H^2}^2 \leqq 2M_0. \tag{2.15}$$

The $H^1(\Omega)$ norm for $\partial_t \xi$ Using equation $(2.6)_1$, $\partial_t \xi$ can be represented in terms of the spatial derivatives of ξ, v^o. Then applying the Hölder and the Sobolev embedding inequalities implies

$$\|\partial_t \xi\|_{H^1}^2 \leq C \|v^o\|_{H^2}^2 \|\xi\|_{H^2}^2 + C \|v^o\|_{H^2}^2 \leq C_2, \tag{2.16}$$

where $C_2 = C_2(M_0, C_1 M_1)$ is given by

$$C(1 + 2M_0) C_1 M_1 =: C_2. \tag{2.17}$$

Proof of Proposition 3. By choosing $T' = \min\{T_1, T_2\}$, the proof of Proposition 3 follows from (2.13), (2.15) and (2.16). $\square$

Next, we show

Proposition 4. *There exists a $T'' = T''(M_0, M_1, C_1, C_2, \rho) \in (0, \infty)$, sufficiently small, such that for every $T \in (0, T'']$, the solution v to $(2.6)_2$ satisfies*

$$\sup_{0 \leq t \leq T} (\|v(t)\|_{H^2}^2 + \|v_t(t)\|_{L^2}^2) + \int_0^T \left(\|v(t)\|_{H^3}^2 + \|v_t(t)\|_{H^1}^2 \right) dt \leq C_1 M_1.$$

Horizontal spatial derivative estimates for v

Applying ∂_{hh} to $(2.6)_2$ will yield

$$\rho^o \partial_t \partial_{hh} v - \mu \Delta_h \partial_{hh} v - \mu \partial_{zz} \partial_{hh} v - (\mu + \lambda) \nabla_h \text{div}_h \, \partial_{hh} v$$
$$= -2\partial_h \rho^o \partial_t \partial_h v - \partial_{hh} \rho^o \partial_t v - \partial_{hh} (\rho^o v^o \cdot \nabla_h v^o) - \partial_{hh} (\rho^o w^o \partial_z v^o)$$
$$- \partial_{hh} ((2\xi^o + gz) \nabla_h \xi^o),$$

$$(2.18)$$

where $\rho^o = \xi^o + \frac{1}{2} gz$. After taking the inner product of (2.18) with $\partial_{hh} v$ and integrating by parts, one has

$$\frac{d}{dt} \left\{ \frac{1}{2} \int_\Omega \rho^o |\partial_{hh} v|^2 \, d\vec{x} \right\} + \int_\Omega \left(\mu |\nabla_h \partial_{hh} v|^2 + \mu |\partial_{hhz} v|^2 + (\mu + \lambda) \right.$$
$$\left. \times |\text{div}_h \, \partial_{hh} v|^2 \right) d\vec{x} = - \int_\Omega \left(2\partial_h \rho^o \partial_t \partial_h v \cdot \partial_{hh} v + \partial_{hh} \rho^o \partial_t v \cdot \partial_{hh} v \right) d\vec{x}$$
$$+ \frac{1}{2} \int_\Omega \partial_t \rho^o |\partial_{hh} v^o|^2 \, d\vec{x} + \int_\Omega \partial_h (\rho^o v^o \cdot \nabla_h v^o) \cdot \partial_{hhh} v \, d\vec{x}$$
$$+ \int_\Omega \partial_h (\rho^o w^o \partial_z v^o) \cdot \partial_{hhh} v \, d\vec{x} + \int_\Omega \partial_h ((2\xi^o + gz) \nabla_h \xi^o) \cdot \partial_{hhh} v \, d\vec{x}$$
$$=: \sum_{i=1}^5 I_i. \qquad (2.19)$$

While we only will omit the detailed estimates, which are standard, we list below the estimates for the I_i terms (see [30] for details). We will use the fact $\|\rho^o\|_{H^2}^2 \leq C\|\xi^o\|_{H^2}^2 + Cg^2 \leq CM_0 + C$, $\|\partial_t \rho^o\|_{L^2}^2 = \|\partial_t \xi^o\|_{L^2}^2 \leq C_2$. Also, hereafter the estimates hold for every $\delta, \omega > 0$ which will be chosen later to be adequately small. Correspondingly, $C_\delta, C_\omega, C_{\delta,\omega}$ are some positive constants depending on δ, ω:

$$I_1 \lesssim \delta \|\partial_{hh} v\|_{H^1}^2 + \omega \|v_t\|_{H^1}^2 + C_{\delta,\omega} (M_0^2 + 1)(\|v\|_{H^2}^2 + \|\partial_t v\|_{L^2}^2).$$
$$I_2 \lesssim \omega \|v^o\|_{H^3}^2 + C_\omega C_2^2 C_1 M_1.$$
$$I_3 \lesssim \delta \|\partial_{hh} v\|_{H^1}^2 + C_\delta (M_0 + 1) C_1^2 M_1^2.$$
$$I_5 \lesssim \delta \|\partial_{hh} v\|_{H^1}^2 + C_\delta (M_0^2 + 1).$$

In order to estimate I_4, we shall plug in (2.7). One has

$$
\begin{aligned}
I_4 &= \int_\Omega \partial_h(\rho^o w^o)\partial_z v^o \cdot \partial_{hhh} v \, d\vec{x} + \int_\Omega \rho^o w^o \partial_{hz} v^o \cdot \partial_{hhh} v \, d\vec{x} \\
&= -\int_0^1 \int_{\Omega_h} \left[\int_0^z \left[\partial_h \mathrm{div}_h\,(\xi^o \widetilde{v^o}) + \frac{g}{2} z \widetilde{\mathrm{div}_h \partial_h v^o} \right] dz' \partial_z v^o \cdot \partial_{hhh} v \right] d\vec{x}_h \, dz \\
&\quad - \int_0^1 \int_{\Omega_h} \left[\int_0^z \left[\mathrm{div}_h\,(\xi^o \widetilde{v^o}) + \frac{g}{2} z \widetilde{\mathrm{div}_h v^o} \right] dz' \partial_{hz} v^o \cdot \partial_{hhh} v \right] d\vec{x}_h \, dz \\
&=: I_4' + I_4''.
\end{aligned}
$$

Then applying the Minkowski and the Sobolev embedding inequalities yields

$$
\begin{aligned}
I_4'' &= -\int_0^1 \int_0^z \int_{\Omega_h} \left[\mathrm{div}_h\,(\xi^o \widetilde{v^o}) + \frac{g}{2} z \widetilde{\mathrm{div}_h v^o} \right](\vec{x}_h, z', t) \\
&\quad \times \left[\partial_{hz} v^o \cdot \partial_{hhh} v \right](\vec{x}_h, z, t) \, d\vec{x}_h \, dz' \, dz \\
&\lesssim \int_0^1 \left(|\nabla_h \xi^o|_{L^4} |\widetilde{v^o}|_{L^\infty} + |\xi^o|_{L^\infty} |\widetilde{\nabla_h v^o}|_{L^4} + |\widetilde{\nabla_h v^o}|_{L^4} \right) dz' \\
&\quad \times \int_0^1 |\partial_{hz} v^o|_{L^4} |\partial_{hhh} v|_{L^2} \, dz \lesssim (\|\xi^o\|_{H^2} + 1) \|v^o\|_{H^2}^{3/2} \|v^o\|_{H^3}^{1/2} \\
&\quad \times \|\partial_{hh} v\|_{H^1} \lesssim \delta \|\partial_{hh} v\|_{H^1}^2 + \omega \|v^o\|_{H^3}^2 + C_{\delta,\omega}(M_0^2 + 1)C_1^3 M_1^3,
\end{aligned}
$$

and, similarly,

$$
I_4' \lesssim \delta \|\partial_{hh} v\|_{H^1}^2 + \omega \|v^o\|_{H^3}^2 + C_{\delta,\omega}(M_0^2 + 1)C_1^3 M_1^3,
$$

where we have employed (1.21). Summing up the above inequalities, with δ small enough, yields the following estimate

$$
\begin{aligned}
\frac{d}{dt} \|\sqrt{\rho^o}\partial_{hh} v\|_{L^2}^2 + c_{\mu,\lambda} \|\partial_{hh} v\|_{H^1}^2 &\lesssim \omega(\|\partial_t v\|_{H^1}^2 + \|v^o\|_{H^3}^2) \\
&\quad + C_\omega \mathcal{H}(M_0, C_1 M_1, C_2)(\|v\|_{H^2}^2 + \|\partial_t v\|_{L^2}^2 + 1).
\end{aligned}
$$

Hereafter, $\mathcal{H}$ will be used to denote a polynomial quantity of its arguments (i.e., the norms of the initial data and ξ^o, v^o) which may be different from line to line. Also $c_{\mu,\lambda}$, C_ω denote positive constants depending on μ, λ and ω, respectively. Similar arguments also hold for the lower order derivatives. Then after suitable choice of ω, one has

$$
\begin{aligned}
\frac{d}{dt}\big(\|\sqrt{\rho^o}v\|_{L^2}^2 + \|\sqrt{\rho^o}\nabla_h v\|_{L^2}^2 + \|\sqrt{\rho^o}\nabla_h^2 v\|_{L^2}^2\big) + c_{\mu,\lambda}\big(\|v\|_{H^1}^2 \\
+ \|\nabla_h v\|_{H^1}^2 + \|\nabla_h^2 v\|_{H^1}^2\big) \le \omega\big(\|\partial_t v\|_{H^1}^2 + \|v^o\|_{H^3}^2\big) \\
+ C_\omega \mathcal{H}(M_0, C_1 M_1, C_2)\big(\|v\|_{H^2}^2 + \|\partial_t v\|_{L^2}^2 + 1\big).
\end{aligned}
\tag{2.20}
$$

Time derivative estimates for v

Applying ∂_t to (2.6)$_2$ yields

$$\rho^o \partial_t v_t - \mu \Delta_h v_t - \mu \partial_{zz} v_t - (\mu + \lambda) \nabla_h \text{div}_h \, v_t = -\partial_t \rho^o \partial_t v \\ - \partial_t (\rho^o v^o \cdot \nabla_h v^o) - \partial_t (\rho^o w^o \partial_z v^o) - \partial_t ((2\xi^o + gz) \nabla_h \xi^o). \tag{2.21}$$

Consequently, one has

$$\frac{\text{d}}{\text{d}t} \frac{1}{2} \int_\Omega \rho^o |\partial_t v|^2 \, \text{d}\vec{x} + \int_\Omega \left(\mu |\nabla_h v_t|^2 + \mu |\partial_z v_t|^2 + (\mu + \lambda) |\text{div}_h \, v_t|^2 \right) \text{d}\vec{x}$$

$$= -\frac{1}{2} \int_\Omega \partial_t \rho^o |v_t|^2 \, \text{d}\vec{x} - \int_\Omega \partial_t (\rho^o v^o \cdot \nabla_h v^o) \cdot \partial_t v \, \text{d}\vec{x}$$

$$- \int_\Omega \partial_t (\rho^o w^o \partial_z v^o) \cdot \partial_t v \, \text{d}\vec{x} - \int_\Omega \partial_t ((2\xi^o + gz) \nabla_h \xi^o) \cdot \partial_t v \, \text{d}\vec{x}$$

$$=: \sum_{i=6}^{9} I_i. \tag{2.22}$$

Then applying similar estimates as before to the right-hand side of (2.22) (see [30] for details) and making use of the fact that w^o is given by (2.7), one has

$$\frac{\text{d}}{\text{d}t} \|\sqrt{\rho^o} v_t\|_{L^2}^2 + c_{\mu,\lambda} \|v_t\|_{H^1}^2 \leqq \omega \big(\|v^o\|_{H^3}^2 + \|\partial_t v^o\|_{H^1}^2 \big) \\ + C_\omega \mathcal{H}(M_0, C_1 M_1, C_2) \big(\|\partial_t v\|_{L^2}^2 + 1 \big). \tag{2.23}$$

Vertical derivative estimates for v

Taking the inner product of (2.6)$_2$ with v_t and applying similar estimates as before to the resultant implies

$$\frac{\text{d}}{\text{d}t} \big(\mu \|\nabla_h v\|_{L^2}^2 + \mu \|\partial_z v\|_{L^2}^2 + (\mu + \lambda) \|\text{div}_h \, v\|_{L^2}^2 \big) + \|\sqrt{\rho^o} v_t\|_{L^2}^2 \\ \leqq \mathcal{H}(M_0, C_1 M_1, C_2, \underline{\rho}). \tag{2.24}$$

On the other hand, (2.6)$_2$ can be written as

$$\mu \partial_{zz} v - \rho^o \partial_t v = -\mu \Delta_h v - (\mu + \lambda) \nabla_h \text{div}_h \, v + \rho^o v^o \cdot \nabla_h v^o \\ + \rho^o w^o \partial_z v^o + (2\xi^o + gz) \nabla_h \xi^o. \tag{2.25}$$

Taking the inner product of (2.25) with $\partial_t \partial_{zz} v$ and applying similar estimates as before will yield,

$$\frac{\text{d}}{\text{d}t} \big(\mu \|\nabla_h \partial_z v\|_{L^2}^2 + \mu \|\partial_{zz} v\|_{L^2}^2 + (\mu + \lambda) \|\text{div}_h \, \partial_z v\|_{L^2}^2 \big) \\ + c_{\mu,\lambda} \|\sqrt{\rho^o} \partial_z v_t\|_{L^2}^2 \leqq \omega \|v^o\|_{H^3}^2 \\ + C_\omega \mathcal{H}(M_0, C_1 M_1, C_2, \underline{\rho}) \big(\|v_t\|_{L^2}^2 + 1 \big). \tag{2.26}$$

Next, applying $\partial \in \{\partial_x, \partial_y, \partial_z\}$ to (2.25) yields

$$\mu \partial \partial_{zz} v - \rho^o \partial v_t = \partial \rho^o \partial_t v - \mu \Delta_h \partial v - (\mu + \lambda) \nabla_h \text{div}_h \, \partial v \\ + \partial (\rho^o v^o \cdot \nabla_h v^o) + \partial (\rho^o w^o \partial_z v^o) + \partial ((2\xi^o + gz) \nabla_h \xi^o). \tag{2.27}$$

This implies

$$\mu \left\| \partial \partial_{zz} v \right\|_{L^2} \lesssim \left\| \nabla_h^2 v \right\|_{H^1} + (M_0^{1/2} + 1)(\left\| v_t \right\|_{H^1} + M_0^{1/2} + C_1 M_1)$$
$$+ \left\| \partial (\rho^o w^o) \partial_z v^o \right\|_{L^2} + \left\| \rho^o w^o \partial \partial_z v^o \right\|_{L^2}.$$

Also, by employing the Minkowski's inequality, one obtains

$$\left\| \partial_h (\rho^o w^o) \partial_z v^o \right\|_{L^2}^2 \lesssim \int_0^1 |\partial_h (\rho^o w^o)|_{L^2}^2 |\partial_z v^o|_{L^\infty}^2 \, \mathrm{d}z$$
$$\lesssim \left(\int_0^1 (|\nabla_h^2 (\xi^o \widetilde{v^o})|_{L^2} + |\widetilde{\nabla_h^2 v^o}|_{L^2}) \, \mathrm{d}z' \right)^2 \times \int_0^1 |\partial_z v^o|_{H^1} |\partial_z v^o|_{H^2} \, \mathrm{d}z$$
$$\lesssim (\left\| \xi^o \right\|_{H^2}^2 + 1) \left\| v^o \right\|_{H^2}^3 \left\| v^o \right\|_{H^3} \lesssim \omega \left\| v^o \right\|_{H^3}^2 + C_\omega (M_0^2 + 1) C_1^3 M_1^3,$$

and, similarly,

$$\left\| \rho^o w^o \partial \partial_z v^o \right\|_{L^2}^2 \lesssim \omega \left\| v^o \right\|_{H^3}^2 + C_\omega (M_0^2 + 1) C_1^3 M_1^3,$$
$$\left\| \partial_z (\rho^o w^o) \partial_z v^o \right\|_{L^2}^2 \lesssim (M_0 + 1) C_1^2 M_1^2.$$

Therefore, we have

$$\left\| \partial_{zz} v \right\|_{H^1}^2 \leqq C \left\| \nabla_h^2 v \right\|_{H^1}^2 + C(M_0 + 1) \left\| v_t \right\|_{H^1}^2 + \omega \left\| v^o \right\|_{H^3}^2$$
$$+ C_\omega \mathcal{H}(M_0, C_1 M_1, C_2). \tag{2.28}$$

Proof of Proposition 4. From (2.20), (2.23), (2.24) and (2.26), there is a constant $c_{\mu,\lambda,\underline{\rho}}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_g(t) + c_{\mu,\lambda,\underline{\rho}} \left(\left\| v \right\|_{H^1}^2 + \left\| \nabla_h v \right\|_{H^1}^2 + \left\| \nabla_h^2 v \right\|_{H^1}^2 + \left\| v_t \right\|_{H^1}^2 \right)$$
$$\leqq \omega \left\| v_t \right\|_{H^1}^2 + \omega \left(\left\| v^o \right\|_{H^3}^2 + \left\| v_t^o \right\|_{H^1}^2 \right)$$
$$+ C_\omega \mathcal{H}(M_0, C_1 M_1, C_2, \underline{\rho})(\mathcal{E}_g(t) + 1), \tag{2.29}$$

where

$$\mathcal{E}_g(t) := \left\| \sqrt{\rho^o} v \right\|_{L^2}^2 + \left\| \sqrt{\rho^o} \nabla_h v \right\|_{L^2}^2 + \left\| \sqrt{\rho^o} \nabla_h^2 v \right\|_{L^2}^2$$
$$+ \left\| \sqrt{\rho^o} v_t \right\|_{L^2}^2 + \mu \left\| \nabla v \right\|_{L^2}^2 + (\mu + \lambda) \left\| \mathrm{div}_h v \right\|_{L^2}^2 \tag{2.30}$$
$$+ \mu \left\| \nabla \partial_z v \right\|_{L^2}^2 + (\mu + \lambda) \left\| \mathrm{div}_h \partial_z v \right\|_{L^2}^2.$$

Notice that for some positive constants $C_{i,\mu,\lambda,\underline{\rho},M_0}, i = 1, 2$, depending on $\mu, \lambda, \underline{\rho}, M_0$, we have

$$C_{1,\mu,\lambda,\underline{\rho},M_0}(\left\| v \right\|_{H^2}^2 + \left\| v_t \right\|_{L^2}^2) \leqq \mathcal{E}_g(t) \leqq C_{2,\mu,\lambda,\underline{\rho},M_0}(\left\| v \right\|_{H^2}^2 + \left\| v_t \right\|_{L^2}^2). \tag{2.31}$$

For $0 < \omega \leqq \frac{c_{\mu,\lambda,\rho}}{2}$, one infers from (2.29), that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_g(t) \leqq \omega(\|v^o\|_{H^3}^2 + \|v_t^o\|_{H^1}^2) + C_\omega\mathcal{H}(M_0, C_1M_1, C_2, \underline{\rho})(\mathcal{E}_g(t) + 1).$$

Therefore, applying the Grönwall's inequality yields

$$\sup_{0\leqq t\leqq T}\mathcal{E}_g(t) \leqq e^{C_\omega\mathcal{H}(M_0,C_1M_1,C_2,\underline{\rho})T}\left(\mathcal{E}_g(0) + \omega\int_0^T(\|v^o\|_{H^3}^2 + \|v_t^o\|_{H^1}^2)\,\mathrm{d}t\right.$$

$$\left. + \int_0^T C_\omega\mathcal{H}(M_0, C_1M_1, C_2, \underline{\rho})\,\mathrm{d}t\right) \leqq e^{C_\omega\mathcal{H}(M_0,C_1M_1,C_2,\underline{\rho})T}\left(C_{2,\mu,\lambda,\underline{\rho},M_0}M_1\right.$$

$$\left. + \omega C_1M_1 + C_\omega\mathcal{H}(M_0, C_1M_1, C_2, \underline{\rho})T\right), \quad \text{where }\omega\text{ is as above.}$$

Now, we integrate with respect to the time variable inequality (2.29). It follows, since $0 < \omega < \frac{c_{\mu,\lambda,\rho}}{2}$, that

$$\frac{c_{\mu,\lambda,\rho}}{2}\int_0^T\left(\|v\|_{H^1}^2 + \|\nabla_h v\|_{H^1}^2 + \|\nabla_h^2 v\|_{H^1}^2 + \|v_t\|_{H^1}^2\right)\mathrm{d}t \leqq \mathcal{E}_g(0) + \mathcal{E}_g(t)$$

$$+ \omega\int_0^T\left(\|v^o\|_{H^3}^2 + \|v_t^o\|_{H^1}^2\right)\mathrm{d}t + C_\omega\mathcal{H}(M_0, C_1M_1, C_2, \underline{\rho})\int_0^T\left(\mathcal{E}_g(t) + 1\right)\mathrm{d}t$$

$$\leqq \left(2 + TC_\omega\mathcal{H}(M_0, C_1M_1, C_2, \underline{\rho})\right)e^{C_\omega\mathcal{H}(M_0,C_1M_1,C_2,\underline{\rho})T}$$

$$\times \left(C_{2,\mu,\lambda,\underline{\rho},M_0}M_1 + \omega C_1M_1 + C_\omega\mathcal{H}(M_0, C_1M_1, C_2, \underline{\rho})T\right).$$

Additionally, from (2.28), we have

$$\int_0^T\|\partial_{zz}v\|_{H^1}^2\,\mathrm{d}t \leqq C\int_0^T\left(\|\nabla_h^2 v\|_{H^1}^2 + (M_0 + 1)\|v_t\|_{H^1}^2\right)\mathrm{d}t$$

$$+ \omega C_1M_1 + C_\omega\mathcal{H}(M_0, C_1M_1, C_2)T.$$

Therefore, we conclude that

$$\sup_{0\leqq t\leqq T}(\|v(t)\|_{H^2}^2 + \|v_t(t)\|_{L^2}^2) + \int_0^T\left(\|v\|_{H^3}^2 + \|v_t\|_{H^1}^2\right)\mathrm{d}t$$

$$\leqq C_{1,\mu\lambda,\underline{\rho},M_0}^{-1}\sup_{0\leqq t\leqq T}\mathcal{E}_g(t) + \int_0^T\left(\|v\|_{H^1}^2 + \|\nabla_h v\|_{H^1}^2 + \|\nabla_h^2 v\|_{H^1}^2\right.$$

$$\left. + \|v_t\|_{H^1}^2\right)\mathrm{d}t + \int_0^T\|\partial_{zz}v\|_{H^1}^2 \leqq (M_0 + 1)\left(C_{3,\mu,\lambda,\underline{\rho},M_0}\right. \tag{2.32}$$

$$\left. + C_\omega\mathcal{H}(M_0, C_1M_1, C_2, \underline{\rho})T\right) \times e^{C_\omega\mathcal{H}(M_0,C_1M_1,C_2,\underline{\rho})T}\left(C_{2,\mu,\lambda,\underline{\rho},M_0}M_1\right.$$

$$\left. + \omega C_1M_1 + C_\omega\mathcal{H}(M_0, C_1M_1, C_2, \underline{\rho})T\right) + \omega C_1M_1$$

$$+ C_\omega\mathcal{H}(M_0, C_1M_1, C_2, \underline{\rho})T,$$

for some positive constant $C_{3,\mu,\lambda,\underline{\rho},M_0}$, depending on μ, λ, $\underline{\rho}$, M_0. Now fix $\omega = \frac{1}{2}\min\{\frac{c_{\mu,\lambda,\underline{\rho}}}{2}, \frac{1}{C_1}\}$ and let $T \in (0, T'']$, where $T'' = T''(M_0, M_1, C_1, C_2, \underline{\rho})$ is small enough and satisfying

$$C_\omega \mathcal{H}(M_0, C_1 M_1, C_2, \underline{\rho})T'' \leqq \min\{1, M_1\}.$$

Then (2.32) yields

$$\sup_{0 \leqq t \leqq T} (\|v(t)\|_{H^2}^2 + \|v_t(t)\|_{L^2}^2) + \int_0^T \left(\|v\|_{H^3}^2 + \|v_t\|_{H^1}^2 \right) dt \leq C_1 M_1,$$

(2.33)

where C_1 is given by

$$C_1 := (M_0 + 1)(C_{3,\mu,\lambda,\underline{\rho},M_0}e + e)(C_{2,\mu,\lambda,\underline{\rho},M_0} + 2) + 2.$$

(2.34)

This concludes the proof. □

2.2. The Case Without Gravity and $\gamma > 1$

Consider a finite positive time T, which will be determined later. Let $\mathfrak{Y} = \mathfrak{Y}_T$ be the function space defined by

$$\begin{aligned}
\mathfrak{Y} = \mathfrak{Y}_T := \{(\sigma, v) | &\sigma \in L^\infty(0, T; H^2(\Omega)), \partial_t \sigma \in L^\infty(0, T; H^1(\Omega)), \\
&v \in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)), \\
&\partial_t v \in L^\infty(0, T; L^2(\Omega)) L^2(0, T; H^1(\Omega))\}.
\end{aligned}$$

(2.35)

Notice that, thanks to Aubin compactness theorem (see [42, Theorem 2.1] and [11,40]), every bounded subset of $\mathfrak{Y}$ is a compact subset of the space

$$\mathfrak{V} = \mathfrak{V}_T := \{(\sigma, v) | \sigma, v \in L^\infty(0, T; L^2(\Omega)), \nabla v \in L^2(0, T; L^2(\Omega))\}.$$

(2.36)

Let $\mathfrak{X} = \mathfrak{X}_T$ be a bounded subset of $\mathfrak{Y}$ defined by

$$\begin{aligned}
\mathfrak{X} = \mathfrak{X}_T := \Big\{ &(\sigma, v) \in \mathfrak{Y} | (\sigma, v)|_{t=0} = (\sigma_0, v_0), \partial_z v|_{z=0,1} = 0, \partial_z \sigma = 0, \\
&\sigma^2 \geqq \frac{1}{2}\underline{\rho} > 0, \sup_{0 \leqq t \leqq T} \|\sigma(t)\|_{H^2}^2 \leqq 2M_0, \sup_{0 \leqq t \leqq T} \|\partial_t \sigma(t)\|_{H^1}^2 \leqq C_2, \\
&\sup_{0 \leqq t \leqq T} \left(\|v(t)\|_{H^2}^2 + \|v_t(t)\|_{L^2}^2 \right) + \int_0^T \left(\|v\|_{H^3}^2 + \|\partial_t v\|_{H^1}^2 \right) dt \\
&\leqq C_1 M_1 \Big\},
\end{aligned}$$

(2.37)

where $\underline{\rho}$ is the positive lower bound of initial density profile as in (2.1), for some positive constants $C_1 = C_1(M_0, \mu, \lambda, \underline{\rho})$, $C_2 = C_2(M_0, C_1 M_1)$. Notice, for $(\sigma, v) \in \mathfrak{X}$, $\int_0^1 \mathrm{div}_h (\sigma^2 \widetilde{v}) \, \mathrm{d}z = 0$. Let $(\sigma^o, v^o) \in \mathfrak{X}$. The linear system inspired by (1.20) with (σ^o, v^o) as an input is given by

$$
\begin{cases}
\partial_t \sigma + \overline{v^o} \cdot \nabla_h \sigma + \dfrac{1}{2} \sigma \mathrm{div}_h \, \overline{v^o} = 0 & \text{in } \Omega, \\
\rho^o \partial_t v + \rho^o v^o \cdot \nabla_h v^o + \sigma^o \sigma^o w^o \partial_z v^o + 2\gamma (\sigma^o)^{2\gamma - 1} \nabla_h \sigma^o & \\
\quad = \mu \Delta_h v + \mu \partial_{zz} v + (\mu + \lambda) \nabla_h \mathrm{div}_h \, v & \text{in } \Omega, \\
\partial_z \sigma = 0 & \text{in } \Omega,
\end{cases}
\tag{2.38}
$$

where $\rho^o = (\sigma^o)^2$ and w^o is determined, as in (1.15), by

$$
\sigma^o w^o := -\int_0^z \left(\sigma^o \mathrm{div}_h \, \widetilde{v^o} + 2\widetilde{v^o} \cdot \nabla_h \sigma^o \right) \mathrm{d}z.
\tag{2.39}
$$

The initial and boundary conditions for system (2.38) are given by

$$
(\sigma, v)|_{t=0} = (\sigma_0, v_0) = (\rho_0^{1/2}, v_0), \quad \partial_z v|_{z=0,1} = 0.
\tag{2.40}
$$

Here, in addition to the compatible conditions in (1.18), we require $\rho_0 \geqq \underline{\rho} > 0$, for some positive constant $\underline{\rho}$ as in (2.1). Also, we denote by $V_1 := h_1 / \rho_0^{1/2}$. Recall that h_1 is given in (1.18). Then $V_1 \in L^2(\Omega)$ and we require $\left\| \sigma_0 \right\|_{H^2}^2 \leqq M_0$, $\left\| v_0 \right\|_{H^2}^2 + \left\| V_1 \right\|_{L^2}^2 \leqq M_1$. Essentially $V_1 = v_t|_{t=0}$.

Then the map $\mathcal{T}$, in the case without gravity, is defined as

$$
\mathcal{T} : (\sigma^o, v^o) \rightsquigarrow (\sigma, v),
\tag{2.41}
$$

where (σ, v) is the unique solution to (2.38) for given $(\sigma^o, v^o) \in \mathfrak{X}$.

Proposition 5. *There is a $T_v = T_v(M_0, M_1, \mu, \lambda, \underline{\rho}) > 0$, sufficiently small, such that for every $T \in (0, T_v]$, there is a unique solution (σ, v) to (2.38) in the set $\mathfrak{X} = \mathfrak{X}_T$. Therefore, for such T, the map $\mathcal{T}$ defined in (2.41) is a well defined map from $\mathfrak{X}$ into $\mathfrak{X}$.*

The proof is similar as Proposition 1 and Proposition 2 in section 2.1 and therefore is omitted.

2.3. Existence Theory

In this subsection, we will establish the existence of solutions for (1.1) and (1.2) for given corresponding initial data and boundary conditions.

2.3.1. The Case With Gravity and $\gamma = 2$, but Without Vacuum   We will apply the Schauder-Tchonoff fixed point theorem to establish the existence of strong solutions to (1.1). With Proposition 2, it is sufficient to verify that $\mathcal{T}$, defined by (2.10), is continuous in $\mathfrak{V} = \mathfrak{V}_T$ given in (2.3), where the norm is given by

$$
\begin{aligned}
\left\| (\xi, v) \right\|_{\mathfrak{V}} := & \left\| \xi \right\|_{L^\infty(0,T;L^2(\Omega))} + \left\| v \right\|_{L^\infty(0,T;L^2(\Omega))} \\
& + \left\| v \right\|_{L^2(0,T;H^1(\Omega))}.
\end{aligned}
\tag{2.4}
$$

In order to show this, let $M_0 = B_{g,1}, M_1 = B_{g,2}$ in $\mathfrak{X}_T$ and $T \in (0, T_g]$, with T_g given in Proposition 2. Here $B_{g,1}, B_{g,2}$ are given in (1.11). We denote $(\xi_1^o, v_1^o), (\xi_2^o, v_2^o) \in \mathfrak{X}_T$ and

$$
(\xi_1, v_1) = \mathcal{T}(\xi_1^o, v_1^o), (\xi_2, v_2) = \mathcal{T}(\xi_1^o, v_1^o).
$$

Denote by $\xi_{12} := \xi_1 - \xi_2, v_{12} := v_1 - v_2, \xi_{12}^o := \xi_1^o - \xi_2^o, v_{12}^o := v_1^o - v_2^o$. Then $(\xi_{12}, v_{12})|_{t=0} = 0$. By taking the differences of the equations satisfied by $(\xi_i, v_i), i = 1, 2$, we have

$$
\begin{cases}
\partial_t \xi_{12} + \overline{v_1^o} \cdot \nabla_h \xi_{12} + \xi_{12}\overline{\mathrm{div}_h\, v_1^o} + \overline{v_{12}^o} \cdot \nabla_h \xi_2 + \xi_2 \overline{\mathrm{div}_h\, v_{12}^o} \\
\quad + \dfrac{g}{2}\overline{z\mathrm{div}_h\, v_{12}^o} = 0, \\
\rho_1^o \partial_t v_{12} - \mu \Delta_h v_{12} - \mu \partial_{zz} v_{12} - (\mu + \lambda)\nabla_h \mathrm{div}_h\, v_{12} = -\xi_{12}^o \partial_t v_2 \\
\quad - \nabla_h(\xi_{12}^o(\rho_1^o + \rho_2^o)) - \xi_{12}^o v_1^o \cdot \nabla_h v_1^o - \rho_2^o v_{12}^o \cdot \nabla_h v_1^o \\
\quad - \rho_2^o v_2^o \cdot \nabla_h v_{12}^o - (\rho_1^o w_1^o - \rho_2^o w_2^o)\partial_z v_1^o - \rho_2^o w_2^o \partial_z v_{12}^o.
\end{cases}
\tag{2.42}
$$

Now we perform standard L^2 estimates for (2.42). Take the L^2-inner product of (2.42)$_1$ with $2\xi_{12}$ and take the L^2-inner product of (2.42)$_2$ with $2v_{12}$. Then applying similar estimates as before yields, together with the Grönwall inequality, that

$$
\begin{aligned}
& \sup_{0 \leqq t \leqq T} \left(\left\| \xi_{12}(t) \right\|_{L^2}^2 + \left\| v_{12}(t) \right\|_{L^2}^2 \right) + \int_0^T \left\| \nabla v_{12}(t) \right\|_{L^2}^2 \, dt \\
& \leqq C_{M_0, C_1 M_1, C_2, \rho} \left(\sup_{0 < t < T} \left(\left\| \xi_{12}^o \right\|_{L^2}^2 + \left\| v_{12}^o \right\|_{L^2}^2 \right) + \int_0^T \left\| \nabla v_{12}^o \right\|_{L^2}^2 \, dt \right).
\end{aligned}
\tag{2.43}
$$

This implies the continuity of $\mathcal{T}$ in $\mathfrak{V}$. Therefore, after applying the fixed point theorem mentioned before, we have the following:

Proposition 6. *Consider*

$$
(\rho_0, v_0) = (\xi_0 + \frac{1}{2}gz, v_0),
$$

given in (1.9) *satisfying* (1.10) *and* (1.11). *There is a positive constant T depending on the initial data such that there is a strong solution* $(\rho, v) = (\xi + \frac{1}{2}gz, v)$ *to* (1.1) *(or equivalently* (1.5)*) with the boundary conditions* (1.3) *and with* $(\xi, v) \in \mathfrak{X}_T$.

2.3.2. The Case With Vacuum and $\gamma > 1$, but Without Gravity   When $\rho_0 \geqq \underline{\rho} > 0$, the existence of strong solutions to (1.2) follows from the estimates in section 2.2 and similar arguments to those in section 2.3.1. In fact, taking $M_0 = B_1$, $M_1 = B_2 + \underline{\rho}^{-1} B_2$, we have the following:

Proposition 7. *Suppose that* (1.17), (1.18), (1.19) *hold for the given initial data* (1.16) *with $\rho_0 \geqq \underline{\rho} > 0$. Then there is a positive constant T, depending on the initial data and $\underline{\rho}$, such that there exists a strong solution $(\rho, v) = (\sigma^2, v)$ to* (1.2) *(or equivalently* (1.20)*) satisfying the boundary conditions* (1.3) *and that $(\sigma, v) \in \mathfrak{X}_T$.*

In the following, we shall present some estimates independent of $\underline{\rho}$ and show that for a given non-negative initial density $\rho_0 \geqq 0$, there are strong solutions to equations (1.2). We will use here the notation $\sigma^2 = \rho$ and the alternative form of equations (1.20), as well as (1.2). Meanwhile, let us assume that

$$\begin{aligned}
\|\sigma_0\|_{H^2} &= \left\|\rho_0^{1/2}\right\|_{H^2} \leqq K_1, \\
\|v_0\|_{H^2} + \|h_1\|_{L^2} &\leqq K_2,
\end{aligned} \tag{2.44}$$

for given $K_1, K_2 > 0$. Recall essentially $h_1 = (\sigma v_t)|_{t=0}$ from (1.18). Also, taking inner product of (1.2)$_2$ with v yields, the conservation of physical energy

$$\begin{aligned}
\frac{1}{2} \int_\Omega \rho |v|^2 \, d\vec{x} &+ \frac{1}{\gamma - 1} \int_\Omega \rho^\gamma \, d\vec{x} + \int_0^T \int_\Omega \left(\mu |\nabla v|^2 + (\mu + \lambda) |\text{div}_h \, v|^2 \right) d\vec{x} \\
&= \frac{1}{2} \int_\Omega \rho_0 |v_0|^2 \, d\vec{x} + \frac{1}{\gamma - 1} \int_\Omega \rho_0^\gamma \, d\vec{x} < \infty,
\end{aligned} \tag{2.45}$$

where (1.2)$_1$ is also applied. Also, integrating (1.2)$_1$ over $\Omega \times (0, T)$ yields the conservation of total mass

$$0 < \int_\Omega \rho \, d\vec{x} = \int_\Omega \rho_0 \, d\vec{x} = M < \infty. \tag{2.46}$$

These facts are important when applying (1.22) in what follows.

A priori **assumptions**

Let (σ, v) be the solution to (1.20) given in Proposition 7. We assume first, for some constants $C_d \geqq K_2^2$, T_d (may depend on $\underline{\rho}$),

$$\sup_{0 \leqq t \leqq T_d} \left(\|v(t)\|_{H^2}^2 + \|\sigma v_t(t)\|_{L^2}^2 \right) + \int_0^{T_d} \left(\|v\|_{H^3}^2 + \|v_t\|_{H^1}^2 \right) dt < C_d. \tag{2.47}$$

In what follows, we will derive some *a priori* estimates independent of $\underline{\rho}$. Also, we set $T \in (0, T_d]$ to be determined later. We emphasize that the smallness of T in what follows is independent of $\underline{\rho}$.

$\underline{\rho}$**-independent lower bound: non-negativity of ρ**

We will use the same Stampaccia-like argument as before to derive the lower bound of ρ. Consider

$$\eta = \eta(x, y, t) := \frac{\rho}{\inf_{\vec{x}_h \in \Omega_h} \rho_0(\vec{x}_h)} - 1 + \int_0^t 2 \big| \text{div}_h \, \overline{v}(s) \big|_{L^\infty} \, ds, \text{ for } t \in [0, T_d].$$

Due to (1.12), η satisfies

$$\partial_t \eta + \overline{v} \cdot \nabla_h \eta + \eta \mathrm{div}_h \, \overline{v} = \Big(\int_0^t 2 \big| \mathrm{div}_h \, \overline{v}(s) \big|_{L^\infty} \, \mathrm{d}s - 1 \Big) \times \mathrm{div}_h \, \overline{v}$$
$$+ 2 \big| \mathrm{div}_h \, \overline{v}(s) \big|_{L^\infty} \geqq -2 \big| \mathrm{div}_h \, \overline{v} \big| + 2 \big| \mathrm{div}_h \, \overline{v} \big|_{L^\infty} \geqq 0,$$

for every $t \in [0, T]$ with $T \in (0, T_1]$, and T_1 sufficiently small such that

$$2 \int_0^t \big| \mathrm{div}_h \, \overline{v}(s) \big|_{L^\infty} \, \mathrm{d}s \leqq 2C \int_0^t \big\| v(s) \big\|_{H^3} \, \mathrm{d}s$$
$$\leqq 2C T^{1/2} \Big(\int_0^t \big\| v(s) \big\|_{H^3}^2 \, \mathrm{d}s \Big)^{1/2} \leqq 2C C_d^{1/2} T_1^{1/2} \leqq \frac{1}{2}.$$

Denote by $\eta_- := -\eta \mathbb{1}_{\{\eta < 0\}} \geqq 0$. Then multiplying the above equation with $-\mathbb{1}_{\{\eta < 0\}}$ and integrating the resultant in the spatial variable yield

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_h} \eta_- \, \mathrm{d}\vec{x}_h \leqq 0.$$

Hence, $\eta_- = 0$ in $\Omega_h \times (0, T]$, since $\eta_-(0) \equiv 0$. Therefore, $\eta \geqq 0$ and

$$\rho = \inf_{\vec{x}_h \in \Omega_h} \rho_0(\vec{x}_h) \times \Big(\eta + 1 - \int_0^t 2 \big| \mathrm{div}_h \, \overline{v}(s) \big|_{L^\infty} \, \mathrm{d}s \Big)$$
$$\geqq \inf_{\vec{x}_h \in \Omega_h} \rho_0(\vec{x}_h) \times \Big(0 + 1 - \frac{1}{2} \Big) = \frac{1}{2} \inf_{\vec{x}_h \in \Omega_h} \rho_0(\vec{x}_h). \tag{2.48}$$

$\underline{\rho}$-independent estimate: $H^2(\Omega)$ for $\sigma = \rho^{1/2}$

After performing standard H^2 estimate of (1.14) and applying the Grönwall inequality to the result, one has

$$\sup_{0 \leqq t \leqq T} \big\| \sigma(t) \big\|_{H^2}^2 \leqq e^{C \int_0^T \| v \|_{H^3} \, \mathrm{d}t} \big\| \sigma_0 \big\|_{H^2}^2 \leqq e^{C C_d^{1/2} T^{1/2}} K_1^2 < 2 K_1^2, \tag{2.49}$$

for all $T \in (0, T_2]$, provided T_2 is sufficiently small.

$\underline{\rho}$-independent estimate: $H^1(\Omega)$ for σ_t

It directly follows from (1.14) that

$$\big\| \partial_t \sigma(t) \big\|_{H^1} \leqq C \big\| v(t) \big\|_{H^2} \big\| \sigma(t) \big\|_{H^2} \leqq \sqrt{2} C C_d^{1/2} K_1 =: K_3', \tag{2.50}$$

for all $t \in [0, T]$.

$\underline{\rho}$-independent estimate: $L^2(\Omega)$ for v_t

Taking the time derivative of $(1.20)_2$ yields

$$\sigma^2 \partial_t v_t - \mu \Delta_h v_t - \mu \partial_{zz} v_t - (\mu + \lambda) \nabla_h \mathrm{div}_h \, v_t = -2\sigma \partial_t \sigma \, \partial_t v$$
$$- \partial_t(\sigma^2 v \cdot \nabla_h v) - \partial_t(\sigma^2 w \partial_z v) - \partial_t \nabla_h \sigma^{2\gamma}. \tag{2.51}$$

Taking the L^2-inner product of (2.51) with $\partial_t v$ gives

$$\frac{1}{2}\frac{d}{dt}\|\sigma v_t\|_{L^2}^2 + \mu\|\nabla_h v_t\|_{L^2}^2 + \mu\|\partial_z v_t\|_{L^2}^2 + (\mu+\lambda)\|\mathrm{div}_h\, v_t\|_{L^2}^2$$

$$= -\int_\Omega \sigma\,\partial_t\sigma\,|v_t|^2\,d\vec{x} - \int_\Omega \partial_t(\sigma^2 v\cdot\nabla_h v)\cdot v_t\,d\vec{x}$$

$$-\int_\Omega \partial_t(\sigma^2 w\partial_z v)\cdot v_t\,d\vec{x} + \int_\Omega \partial_t\sigma^{2\gamma}\,\mathrm{div}_h\, v_t\,d\vec{x} =: \sum_{i=1}^4 L_i. \qquad (2.52)$$

Then one has the following estimates to the terms in the right-hand side of (2.52) (see [30] for details):

$$L_1 \lesssim \|\partial_t\sigma\|_{L^2}\|\sigma v_t\|_{L^3}\|v_t\|_{L^6} \lesssim \|\partial_t\sigma\|_{L^2}\|\sigma v_t\|_{L^2}^{1/2}\|\sigma\|_{L^\infty}^{1/2}\|v_t\|_{L^6}^{3/2}$$

$$\lesssim \delta\|\nabla v_t\|_{L^2}^2 + C_\delta C_d(K_1^2 K_3'^4 + 1).$$

$$L_2 \lesssim \|\partial_t\sigma\|_{L^2}\|v\|_{L^\infty}\|\nabla_h v\|_{L^6}\|\sigma v_t\|_{L^3} + \|\sigma v_t\|_{L^3}\|\nabla_h v\|_{L^6}\|\sigma v_t\|_{L^2}$$

$$+ \|\sigma\|_{L^\infty}\|v\|_{L^\infty}\|\nabla_h v_t\|_{L^2}\|\sigma v_t\|_{L^2} \lesssim \delta\|\nabla v_t\|_{L^2}^2$$

$$+ C_\delta(C_d + 1)(K_1^2 C_d + K_1 K_3' C_d^2 + 1).$$

$$L_4 \lesssim \|\sigma\|_{L^\infty}^{2\gamma-1}\|\partial_t\sigma\|_{L^2}\|\nabla v_t\|_{L^2} \lesssim \delta\|\nabla v_t\|_{L^2}^2 + C_\delta K_1^{4\gamma-2} K_3'^2.$$

We have applied above the Hölder inequality and (1.22). Notice that $\sigma = \rho^{1/2}$ and that the conservations of energy and mass (2.45), (2.46) hold. In order to estimate L_3 term, we substitute (1.13) and integrate by parts. Then

$$L_3 = -\int_\Omega \partial_t(\rho w)\partial_z v\cdot v_t\,d\vec{x} - \int_\Omega \rho w\partial_z v_t\cdot v_t\,d\vec{x}$$

$$= -\int_0^1\int_{\Omega_h}\left[\left(\int_0^z (\sigma^2\tilde{v})_t\,dz'\right)\cdot\nabla_h(\partial_z v\cdot v_t)\right]d\vec{x}_h\,dz$$

$$+ \int_0^1\int_{\Omega_h}\left[\left(\int_0^z \mathrm{div}_h\,(\sigma^2\tilde{v})\,dz'\right)\times\left(\partial_z v_t\cdot v_t\right)\right]d\vec{x}_h\,dz =: L_3' + L_3''.$$

Now we use (1.21), the Minkowski and Hölder inequalities,

$$L_3' = -\int_0^1\int_{\Omega_h}\left[\left(\int_0^z (\sigma\tilde{v}_t + 2\sigma_t\tilde{v})\,dz'\right)\cdot(\nabla_h\partial_z v\cdot\sigma v_t + \sigma\nabla_h v_t\cdot\partial_z v)\right]d\vec{x}_h\,dz$$

$$\lesssim \int_0^1\left(|\sigma\tilde{v}_t|_{L^2} + |\sigma_t|_{L^2}|\tilde{v}|_{L^\infty}\right)dz' \times \int_0^1\left(|\sigma|_{L^\infty}|\nabla_h\partial_z v|_{L^4}|v_t|_{L^4}\right.$$

$$+ |\sigma|_{L^\infty}|\nabla_h v_t|_{L^2}|\partial_z v|_{L^\infty}\Big)dz \lesssim \int_0^1\left(|\sigma\tilde{v}_t|_{L^2} + |\sigma_t|_{L^2}|\tilde{v}|_{H^2}\right)dz'$$

$$\times \int_0^1 |\sigma|_{H^2}|\partial_z v|_{H^1}^{1/2}|\partial_z v|_{H^2}^{1/2}|v_t|_{H^1}\,dz \lesssim \|\sigma\|_{H^2}(\|\sigma v_t\|_{L^2}$$

$$+ \|\sigma_t\|_{L^2}\|v\|_{H^2})\|v\|_{H^2}^{1/2}\|v\|_{H^3}^{1/2}(\|\sigma v_t\|_{L^2} + \|\nabla v_t\|_{L^2})$$

$$\lesssim \delta\|\nabla v_t\|_{L^2}^2 + \omega\|v\|_{H^3}^2 + C_{\delta,\omega}C_d(K_1^4(K_3'^4 + 1)C_d^2 + 1),$$

and, similarly,

$$L_3'' \lesssim \delta \|\nabla v_t\|_{L^2}^2 + C_\delta (K_1^{10} C_d^4 + K_1^{10/3} C_d^2).$$

After summing the above inequalities, (2.52) implies

$$\frac{d}{dt} \|\sigma v_t\|_{L^2}^2 + c_{\mu,\lambda} \|\nabla v_t\|_{L^2}^2 \leqq \omega \|v\|_{H^3}^2 + C_\omega \mathcal{H}(K_1, K_3', C_d), \qquad (2.53)$$

where, as before, $\mathcal{H}$ denotes a polynomial quantity of its arguments.

$\underline{\rho}$-**independent estimate: spatial derivatives of** v

Now we are able to derive the estimates on the spatial derivatives of v. Standard L^2 estimate of $(1.20)_2$ yields that

$$\frac{d}{dt} \|\sigma v\|_{L^2}^2 + c_{\mu,\lambda} \|\nabla v\|_{L^2}^2 \leqq C \|\sigma^{2\gamma}\|_{L^2}^2 \leqq C\mathcal{H}(K_1). \qquad (2.54)$$

Furthermore, taking the L^2-inner product of $(1.20)_2$ with v_t yields, after applying similar estimates as before,

$$\frac{d}{dt} (\mu \|\nabla_h v\|_{L^2}^2 + \mu \|\partial_z v\|_{L^2}^2 + (\mu + \lambda) \|\mathrm{div}_h v\|_{L^2}^2) + \|\sigma v_t\|_{L^2}^2 \qquad (2.55)$$
$$\leqq \mathcal{H}(K_1, C_d).$$

Next we estimate the second order spatial derivatives. Taking the L^2-inner product of $(1.20)_2$ with $\partial_{zz} v_t$ yields

$$\frac{1}{2} \frac{d}{dt} (\mu \|\nabla_h \partial_z v\|_{L^2}^2 + \mu \|\partial_{zz} v\|_{L^2}^2 + (\mu + \lambda) \|\mathrm{div}_h \partial_z v\|_{L^2}^2) + \|\sigma \partial_z v_t\|_{L^2}^2$$
$$= -\int_\Omega \partial_z (\sigma^2 v \cdot \nabla_h v) \cdot \partial_z v_t \, d\vec{x} - \int_\Omega \partial_z (\sigma^2 w \partial_z v) \cdot \partial_z v_t \, d\vec{x} =: L_8 + L_9. \qquad (2.56)$$

At the same time, taking the L^2-inner product of $(1.20)_2$ with $\Delta_h v_t$ yields

$$\frac{1}{2} \frac{d}{dt} (\mu \|\nabla_h^2 v\|_{L^2}^2 + \mu \|\nabla_h \partial_z v\|_{L^2}^2 + (\mu + \lambda) \|\nabla_h \mathrm{div}_h v\|_{L^2}^2)$$
$$+ \|\sigma \nabla_h v_t\|_{L^2}^2 = -2 \int_\Omega (\sigma \nabla_h \sigma \cdot \nabla_h v_t) \cdot v_t \, d\vec{x}$$
$$- \int_\Omega \nabla_h (\sigma^2 v \cdot \nabla_h v) : \nabla_h v_t \, d\vec{x} - \int_\Omega \nabla_h (\sigma^2 w \partial_z v) : \nabla_h v_t \, d\vec{x}$$
$$- \int_\Omega \nabla_h^2 \sigma^{2\gamma} : \nabla_h v_t \, d\vec{x} =: \sum_{i=10}^{13} L_i. \qquad (2.57)$$

Applying similar estimates as before to the right-hand sides of (2.56) and (2.57) yields, after summing up the results,

$$\frac{d}{dt} (\mu \|\nabla_h^2 v\|_{L^2}^2 + 2\mu \|\nabla_h \partial_z v\|_{L^2}^2 + \mu \|\partial_{zz} v\|_{L^2}^2$$
$$+ (\mu + \lambda) \|\nabla \mathrm{div}_h v\|_{L^2}^2) + \|\sigma \nabla v_t\|_{L^2}^2 \qquad (2.58)$$
$$\leqq \omega (\|\nabla v_t\|_{L^2}^2 + \|v\|_{H^3}^2) + C_\omega \mathcal{H}(K_1, C_d).$$

Finally, we provide estimates for the third spatial derivative of v. Applying $\partial \in \{\partial_x, \partial_y, \partial_z\}$ to $(1.20)_2$ yields

$$\mu \Delta_h \partial v + \mu \partial_{zz} \partial v + (\mu + \lambda)\nabla_h \mathrm{div}_h \, \partial v = \partial(\sigma^2 v_t) + \partial(\sigma^2 v \cdot \nabla_h v) \tag{2.59}$$
$$+ \partial(\sigma^2 w \partial_z v) + \partial \nabla_h \sigma^{2\gamma}.$$

Taking the L^2-inner product of (2.59) with $\Delta_h \partial v$, for $\partial \in \{\partial_h, \partial_z\}$, and integrating by parts imply, after applying the Cauchy-Schwarz inequality,

$$\left\| \nabla^3 v \right\|_{L^2}^2 \lesssim \left\| \nabla(\sigma^2 v_t) \right\|_{L^2}^2 + \left\| \nabla(\sigma^2 v \cdot \nabla_h v) \right\|_{L^2}^2 + \left\| \nabla(\sigma^2 w \partial_z v) \right\|_{L^2}^2$$
$$+ \left\| \nabla_h^2 \sigma^{2\gamma} \right\|_{L^2}^2. \tag{2.60}$$

Then, similarly to (2.28), noticing the fact that w are given by (1.15), (2.60) yields

$$\left\| v \right\|_{H^3}^2 \leqq \left\| \nabla^3 v \right\|_{L^2}^2 + C_d \leqq C K_1^4 \left\| \nabla v_t \right\|_{L^2}^2 + \mathcal{H}(K_1, C_d), \tag{2.61}$$

where C_d is as in (2.47)

We summarize the estimates obtained, so far, in this section in the following:

Proposition 8. *Consider the solution $(\sigma, v) = (\rho^{1/2}, v)$ to (1.2) with the bound (2.47) and initial data satisfying (1.17), (2.44). There is a positive constant $T^* = T^*(C_d, K_1, K_2)$, sufficiently small, such that (σ, v) admits the following bounds, for $T = \min\{T^*, T_d\}$,*

$$\inf_{(\vec{x},t) \in \Omega \times [0,T]} \rho(\vec{x}, t) \geqq \frac{1}{2} \inf_{\vec{x} \in \Omega} \rho_0 > 0, \quad \sup_{0 \leqq t \leqq T} \left\| \sigma(t) \right\|_{H^2} \leqq 2K_1,$$

$$\sup_{0 \leqq t \leqq T} \left\| \partial_t \sigma(t) \right\|_{H^1} \leqq K_3,$$

$$\sup_{0 \leqq t \leqq T} \left(\left\| v(t) \right\|_{H^2}^2 + \left\| (\sigma v_t)(t) \right\|_{L^2}^2 \right) + \int_0^T \left(\left\| v(t) \right\|_{H^3}^2 + \left\| v_t(t) \right\|_{H^1}^2 \right) dt \leqq K_4^2,$$

where $K_4 = \sqrt{2C_{\mu,\lambda}} K_2$, $K_3 = C K_1 K_4$ are given in (2.63) and (2.64). Notably, the bounds in these estimates depend only on the initial bounds K_1, K_2 and do not depend on the lower bound of density. Also, the smallness of T^* does not depend on $\underline{\rho}$, even though T_d may depend on $\underline{\rho}$.

Proof. Denote by

$$\mathcal{E}(t) := \left\| \sigma v_t \right\|_{L^2}^2 + \left\| \sigma v \right\|_{L^2}^2 + \mu \left\| \nabla v \right\|_{L^2}^2 + (\mu + \lambda) \left\| \mathrm{div}_h \, v \right\|_{L^2}^2$$
$$+ \mu \left\| \nabla_h^2 v \right\|_{L^2}^2 + 2\mu \left\| \nabla_h \partial_z v \right\|_{L^2}^2 + \mu \left\| \partial_{zz} v \right\|_{L^2}^2 \tag{2.62}$$
$$+ (\mu + \lambda) \left\| \nabla \mathrm{div}_h \, v \right\|_{L^2}^2.$$

Then from (2.53), (2.54), (2.55) and (2.58), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(t) + c_{\mu,\lambda} \left(\left\| \nabla v_t \right\|_{L^2}^2 + \left\| \nabla v \right\|_{L^2}^2 + \left\| \sigma v_t \right\|_{L^2}^2 + \left\| \sigma \nabla v_t \right\|_{L^2}^2 \right)$$
$$\leqq \omega \left(\left\| v \right\|_{H^3}^2 + \left\| \nabla v_t \right\|_{L^2}^2 \right) + C_\omega \mathcal{H}(K_1, K_3', C_d).$$

Then integrating the above inequality yields, for $T \in (0, T_d]$, where T_d is as in (2.47),

$$
\sup_{0 \leq t \leq T} \mathcal{E}(t) + c_{\mu,\lambda} \int_0^T \left(\left\| \nabla v_t \right\|_{L^2}^2 + \left\| \nabla v \right\|_{L^2}^2 + \left\| \sigma v_t \right\|_{L^2}^2 \right)
$$

$$
+ \left\| \sigma \nabla v_t \right\|_{L^2}^2 \right) \mathrm{d}t \leq \mathcal{E}(0) + \omega C_d + C_\omega T \mathcal{H}(K_1, K_3', C_d).
$$

Then together with (2.61), we have, after choosing ω small enough and then T sufficiently small,

$$
\sup_{0 \leq t \leq T} \left(\left\| v(t) \right\|_{H^2}^2 + \left\| \sigma v_t(t) \right\|_{L^2}^2 \right) + \int_0^T \left(\left\| v \right\|_{H^3}^2 + \left\| v_t \right\|_{H^1}^2 \right) \mathrm{d}t
$$
$$
\leq C_{\mu,\lambda} K_2^2 + \omega C_d C_{\mu,\lambda}(K_1^4 + 1) + C_\omega T \mathcal{H}(K_1, K_3', C_d) \tag{2.63}
$$
$$
\leq 2 C_{\mu,\lambda} K_2^2 =: K_4^2,
$$

where we have employed inequality (1.22) and the fact for some positive constant $C_{\mu,\lambda} > 0$ we have

$$
C_{\mu,\lambda}^{-1}(\left\| v \right\|_{H^2}^2 + \left\| \sigma v_t \right\|_{L^2}^2) \leq \mathcal{E}(t) \leq C_{\mu,\lambda}(\left\| v \right\|_{H^2}^2 + \left\| \sigma v_t \right\|_{L^2}^2).
$$

Then plugging in (2.63) back into (2.50) implies

$$
\left\| \partial_t \sigma \right\|_{H^1} \leq C K_4 K_1 =: K_3. \tag{2.64}
$$

Thus the conclusion is drawn from (2.48), (2.49), (2.63) and (2.64). □

Existence of strong solutions with vacuum but no gravity and $\gamma > 1$

Now we are in the place to remove the strict positivity (of the initial density profile) assumption in Proposition 7. In order to do so, we introduce a sequence of approximating initial data $(\rho_{0,n}, v_{0,n})$ satisfying, in addition to (1.17), (1.18), (1.19),

$$
\rho_{0,n} \geq \frac{1}{n} > 0,
$$

such that

$$
\rho_{0,n}^{1/2} \to \rho_0^{1/2}, \, v_{0,n} \to v_0
$$

in $H^2(\Omega)$, as $n \to \infty$, where (ρ_0, v_0) (or equivalently (σ_0, v_0)) is given in (1.16) satisfying (1.18) and (1.19).

We require that the initial physical energy and total mass given in (1.17) with ρ_0, v_0 replaced by $\rho_{0,n}$, $v_{0,n}$ satisfy

$$
0 < \int_\Omega \rho_{0,n} \, \mathrm{d}\vec{x} = M < \infty,
$$
$$
0 < \int_\Omega \rho_{0,n} \left| v_{0,n} \right|^2 \mathrm{d}\vec{x} + \frac{1}{\gamma - 1} \int_\Omega \rho_{0,n}^\gamma \, \mathrm{d}\vec{x} \leq E_0 + 1 < \infty,
$$

uniformly in n, so that when we apply inequality (1.22), the constant in the inequality is independent of n.

Now we apply Proposition 7 with the initial data $(\rho_{0,n}, v_{0,n})$. Indeed, consider $M_0 = B_1$, $M_1 = B_2 + nB_2$. Then Proposition 7 guarantees that there is a $T_1 = T_1(n, B_1, B_2)$ such that (1.2) admits a strong solution $(\rho_n, v_n) = (\sigma_n^2, v_n)$ satisfying

$$\sup_{0 \le t \le T_1} \left\| \sigma_n(t) \right\|_{H^2}^2 \le 2B_1, \quad \sup_{0 \le t \le T_1} \left\| \partial_t \sigma_n(t) \right\|_{H^1}^2 \le \mathcal{C}_1(B_1, B_2, n),$$

$$\sup_{0 \le t \le T_1} \left(\left\| v_n(t) \right\|_{H^2}^2 + \left\| \partial_t v_n(t) \right\|_{L^2}^2 \right) + \int_0^{T_1} \left(\left\| v_n(t) \right\|_{H^3}^2 + \left\| \partial_t v_n(t) \right\|_{H^1}^2 \right) dt$$

$$\le \mathcal{C}_2(B_1, B_2, n), \quad \text{and } \rho_n = \sigma_n^2 \ge \frac{1}{2n}.$$

Sequently, we apply Proposition 8 with $K_1 = B_1^{1/2}$, $K_2 = 2B_2^{1/2}$, $C_d = (1 + 2n)\mathcal{C}_2(B_1, B_2, n)$ and $T_d = T_1$. It yields that there is a $T_2 = T_2(B_1, B_2, n) \le T_1$ such that the following bounds are satisfied

$$\sup_{0 \le t \le T_2} \left\| \sigma_n(t) \right\|_{H^2} \le 2B_1^{1/2}, \quad \sup_{0 \le t \le T_2} \left\| \partial_t \sigma_n(t) \right\|_{H^1} \le \mathcal{C}_3(B_1, B_2),$$

$$\sup_{0 \le t \le T_2} \left(\left\| v_n(t) \right\|_{H^2}^2 + \left\| (\sigma_n v_{n,t})(t) \right\|_{L^2}^2 \right)$$

$$\tag{2.65}$$

$$+ \int_0^{T_2} \left(\left\| v_n(t) \right\|_{H^3}^2 + \left\| v_{n,t}(t) \right\|_{H^1}^2 \right) dt \le \mathcal{C}_4(B_1, B_2),$$

$$\text{and} \inf_{(\vec{x},t) \in \Omega \times [0,T_2]} \rho_n \ge \frac{1}{2n}.$$

Next, let $(\sigma_n, v_n)|_{t=T_2}$ as a new initial data for (1.2). The same arguments as above yield the bound (2.65) with lower bound of ρ_n replaced by $\frac{1}{4n}$, B_1 replaced by $4B_1$ and B_2 replaced by $\mathcal{C}_4(B_1, B_2)$. That is, for some $\delta T = \delta T(B_1, B_2, n) > 0$,

$$\inf_{(\vec{x},t) \in \Omega \times (T_2, T_2+\delta T)} \rho_n \ge \frac{1}{4n}, \quad \sup_{T_2 < t < T_2+\delta T} \left\| \sigma_n(t) \right\|_{H^2} \le 4B_1^{1/2},$$

$$\sup_{T_2 < t < T_2+\delta T} \left\| \partial_t \sigma_n(t) \right\|_{H^1} \le \mathcal{C}_3(4B_1, \mathcal{C}_4(B_1, B_2)),$$

$$\sup_{T_2 < t < T_2+\delta T} \left(\left\| v_n(t) \right\|_{H^2}^2 + \left\| \sigma_n v_{n,t}(t) \right\|_{L^2}^2 \right)$$

$$+ \int_{T_2}^{T_2+\delta T} \left(\left\| v_n(t) \right\|_{H^3}^2 + \left\| v_{n,t}(t) \right\|_{H^1}^2 \right) dt$$

$$\le \mathcal{C}_4(4B_1, \mathcal{C}_4(B_1, B_2)).$$

Now we apply Proposition 8 with $T_d = T_2 + \delta T$ and $C_d = \mathcal{C}_4(4B_1, \mathcal{C}_4(B_1, B_2))$ in the time interval $(0, T_d)$. This will yield that there is a $T^* = T^*(B_1, B_2)$ and $T_3 := \min\{T^*, T_2 + \delta T\}$, the bounds in (2.65) hold with T_2 replaced by T_3.

If $T_3 = T^*$, we have got an existence time independent of n and this finishes the job. Otherwise, let $(\sigma_n, v_n)|_{t=T_3}$ as a new initial data and repeat the arguments

above to get the bounds in (2.65) with T_2 replaced by $T_4 := \min\{T^*, T_3 + \delta T\} = \min\{T^*, T_2 + 2\delta T\}$. Keep repeating this process, one will eventually get that there is a $m \in \mathbb{Z}^+$ sufficiently large that $T_m := \min\{T^*, T_2 + (m - 2)\delta T\} = T^*$.

Therefore, we have got a sequence of approximating solutions $(\rho_n, v_n) = (\sigma_n^2, v_n)$ with a uniform existence time T^* independent of n for the approximating initial data $(\rho_{0,n}, v_{0,n})$ constructed above. In particular, (σ_n, v_n) satisfies the bounds in (2.65) with T_2 replaced by T^*. Thus by taking $n \to \infty$, it is straightforward to check that we have got a strong solution $(\rho, v) = (\sigma^2, v)$ to (1.2). In fact, we have the following:

Proposition 9. *Consider the initial data (ρ_0, v_0) (or equivalently (σ_0, v_0)) given in (1.16) satisfying (1.17), (1.18) and (1.19). There is a constant $T^* > 0$ such that there exists a solution $(\rho, v) = (\sigma^2, v)$ to equation (1.2) satisfying*

$$
\begin{aligned}
&\sigma \in L^\infty(0, T^*; H^2(\Omega)), \quad \partial_t \sigma \in L^\infty(0, T^*; H^1(\Omega)), \\
&v \in L^\infty(0, T^*; H^2(\Omega)) \cap L^2(0, T^*; H^3(\Omega)), \quad \partial_t v \in L^2(0, T^*; H^1(\Omega)) \\
&\sigma \partial_t v \in L^\infty(0, T^*; L^2(\Omega)),
\end{aligned}
\tag{2.66}
$$

and

$$
\begin{aligned}
&\sup_{0 \leqq t \leqq T^*} \|\sigma(t)\|_{H^2} \leqq 2B_1^{1/2}, \quad \sup_{0 \leqq t \leqq T^*} \|\partial_t \sigma(t)\|_{H^1} \leqq \mathcal{C}_3(B_1, B_2), \\
&\sup_{0 \leqq t \leqq T^*} \left(\|v(t)\|_{H^2}^2 + \|(\sigma v_t)(t)\|_{L^2}^2 \right) + \int_0^{T^*} \left(\|v(t)\|_{H^3}^2 + \|v_t(t)\|_{H^1}^2 \right) dt \\
&\leqq \mathcal{C}_4(B_1, B_2), \quad and \quad \inf_{(\vec{x},t) \in \Omega \times [0,T^*]} \rho \geqq 0,
\end{aligned}
\tag{2.67}
$$

for some constant $\mathcal{C}_3 = \mathcal{C}_3(B_1, B_2), \mathcal{C}_4 = \mathcal{C}_4(B_1, B_2)$.

3. Continuous Dependence on Initial Data and Uniqueness

In this section, we will show the continuous dependence of the solutions of (1.1) and (1.2) on the initial data. This will also imply the uniqueness of strong solutions constructed in Proposition 6 and Proposition 9.

3.1. The Case With Gravity and $\gamma = 2$, but Without Vacuum

Consider two sets of initial data $(\rho_{i,0}, v_{i,0}) = (\xi_{i,0} + \frac{1}{2}gz, v_{i,0}), i = 1, 2$, in (1.9) for (1.1) satisfying (1.10), (1.11). Denote $(\rho_i, v_i) = (\xi_i + \frac{1}{2}gz, v_i), i = 1, 2$, as the corresponding strong solutions constructed in Proposition 6 in the interval $[0, T]$ for some $T > 0$. Then we have $(\xi_i, v_i) \in \mathfrak{X}_T, i = 1, 2$. Throughout this

section we will denote the constant $C > 0$ which may be different from line to line and depends on μ, λ, $B_{g,1}$, $B_{g,2}$, $\underline{\rho}$, T. Also, we will use the notations

$$\xi_{12} := \xi_1 - \xi_2, \ v_{12} := v_1 - v_2,$$
$$\xi_{12,0} := \xi_{1,0} - \xi_{2,0}, \ v_{12,0} := v_{1,0} - v_{2,0}.$$

Taking the difference of the equations satisfied by (ξ_i, v_i), $i = 1, 2$, as in (2.42), then (ξ_{12}, v_{12}) satisfies

$$\begin{cases} \partial_t \xi_{12} + \overline{v_1} \cdot \nabla_h \xi_{12} + \xi_{12} \overline{\mathrm{div}_h \, v_1} + \overline{v_{12}} \cdot \nabla_h \xi_2 + \xi_2 \overline{\mathrm{div}_h \, v_{12}} \\ \quad + \dfrac{g}{2} z \overline{\mathrm{div}_h \, v_{12}} = 0, \\ \rho_1 \partial_t v_{12} - \mu \Delta_h v_{12} - \mu \partial_{zz} v_{12} - (\mu + \lambda) \nabla_h \mathrm{div}_h \, v_{12} = -\xi_{12} \partial_t v_2 \\ \quad - \nabla_h (\xi_{12} (\rho_1 + \rho_2)) - \xi_{12} v_1 \cdot \nabla_h v_1 - \rho_2 v_{12} \cdot \nabla_h v_1 \\ \quad - \rho_2 v_2 \cdot \nabla_h v_{12} - (\rho_1 w_1 - \rho_2 w_2) \partial_z v_1 - \rho_2 w_2 \partial_z v_{12}. \end{cases}$$

Then after applying standard L^2 estimates to the above system and applying the Grönwall's inequality to the resultant, one can show the following:

Proposition 10. *Given two sets of initial data* $(\rho_{i,0}, v_{i,0}) = (\xi_{i,0} + \frac{1}{2} gz, v_{i,0})$, $i = 1, 2$, *satisfying* (1.10) *and* (1.11)*, the corresponding strong solutions* $(\rho_i, v_i) = (\xi_i + \frac{1}{2} gz, v_i)$, $i = 1, 2$, *of* (1.1) *constructed in Proposition 6 in the interval* $[0, T]$, *for some* $T > 0$, *satisfy*

$$\begin{aligned} \| \rho_1 &- \rho_2 \|_{L^\infty (0,T; L^2(\Omega))} + \| v_1 - v_2 \|_{L^\infty (0,T; L^2(\Omega))} \\ &+ \| \nabla (v_1 - v_2) \|_{L^2 (0,T; L^2(\Omega))} \leqq C_{\mu, \lambda, B_{g,1}, B_{g,2}, \underline{\rho}, T} \\ &\times (\| \rho_{1,0} - \rho_{2,0} \|_{L^2(\Omega)} + \| v_{1,0} - v_{2,0} \|_{L^2(\Omega)}). \end{aligned}$$

In particular, if $\rho_{1,0} = \rho_{2,0}$, $v_{1,0} = v_{2,0}$, *we have* $\rho_1 = \rho_2$, $v_1 = v_2$ *in* $[0, T]$.

3.2. The Case With Vacuum and $\gamma > 1$, but Without Gravity

First, we claim that any solution $(\rho, v) = (\sigma^2, v)$ to (1.2) satisfying (2.66) with the bounds in (2.67) will also satisfy the following equations:

$$\begin{cases} \partial_t \sigma + \overline{v} \cdot \nabla_h \sigma + \dfrac{1}{2} \sigma \overline{\mathrm{div}_h \, v} = 0 & \text{in } \Omega, \\ \sigma w = - \int_0^z \widetilde{\sigma \mathrm{div}_h \, v} + 2 \widetilde{v} \cdot \nabla_h \sigma \, dz & \text{in } \Omega, \\ \sigma^2 \partial_t v + \sigma^2 v \cdot \nabla_h v + \sigma \sigma w \partial_z v + \nabla_h \sigma^{2\gamma} & \\ \quad = \mu \Delta_h v + \mu \partial_{zz} v + (\mu + \lambda) \nabla_h \mathrm{div}_h \, v & \text{in } \Omega, \\ \partial_z \sigma = 0 & \text{in } \Omega. \end{cases} \quad (1.2')$$

To show this claim, we first consider the non-degenerate variable $\rho + \varepsilon = \sigma^2 + \varepsilon$, for some constant $\varepsilon > 0$. From (1.12), one has

$$\partial_t (\rho + \varepsilon) + \overline{v} \cdot \nabla_h (\rho + \varepsilon) + (\rho + \varepsilon) \overline{\mathrm{div}_h \, v} - \varepsilon \overline{\mathrm{div}_h \, v} = 0.$$

Then after dividing $(\rho + \varepsilon)^{1/2}$, one has

$$2\partial_t(\rho + \varepsilon)^{1/2} + 2\overline{v} \cdot \nabla_h(\rho + \varepsilon)^{1/2} + (\rho + \varepsilon)^{1/2}\overline{\text{div}_h\, v}$$
$$- \frac{\varepsilon}{(\rho + \varepsilon)^{1/2}}\overline{\text{div}_h\, v} = 0. \tag{3.1}$$

Now it is easy to verify that (3.1) will converge to $(1.2')_1$ in the sense of distribution, as $\varepsilon \to 0$. On the other hand, from (1.13), one has

$$\sigma^2 w = -\sigma \int_0^z \left(\widetilde{\sigma\, \text{div}_h\, v} + 2\widetilde{v} \cdot \nabla_h\sigma\right) dz.$$

We define

$$\sigma w_\sigma := -\int_0^z \widetilde{\sigma\, \text{div}_h\, v} + 2\widetilde{v} \cdot \nabla_h\sigma\, dz.$$

Then $\sigma\sigma w_\sigma = \rho w$ and we will use hereafter the notation $\sigma w = \sigma w_\sigma$. As before it is easy to verify that $(1.2')_3$ is equivalent to $(1.2')_2$ in the sense of distribution. Summing up the facts above, we have shown that the solutions to (1.2) satisfying the (2.66) regularity with the bounds in (2.67) are also solutions to $(1.2')$.

Consider two sets of initial data $(\rho_{i,0}, v_{i,0}) = (\sigma_{i,0}^2, v_{i,0})$, $i = 1, 2$, in (1.16) for (1.2) satisfying (1.18) and (1.19). Denote $(\rho_i, v_i) = (\sigma_i^2, v_i)$, $i = 1, 2$, as the corresponding strong solutions constructed in Proposition 9 in the interval $[0, T^*]$, for some $T^* > 0$. Then we have (σ_i, v_i), $i = 1, 2$, satisfying the bounds in (2.67). Also (σ_i, v_i), $i = 1, 2$, are solutions to $(1.2')$. Throughout this section, we will denote the constant $C > 0$ which may be different from line to line and depends on $\mu, \lambda, B_1, B_2, T^*$. Also, we will use the notations

$$\sigma_{12} := \sigma_1 - \sigma_2, \; v_{12} := v_1 - v_2,$$
$$\sigma_{12,0} := \sigma_{1,0} - \sigma_{2,0}, \; v_{12,0} := v_{1,0} - v_{2,0}.$$

Taking the difference of the equations satisfied by (σ_i, v_i), $i = 1, 2$, we have

$$\begin{cases} \partial_t\sigma_{12} + \overline{v_1} \cdot \nabla_h\sigma_{12} + \dfrac{1}{2}\sigma_{12}\overline{\text{div}_h\, v_1} + \overline{v_{12}} \cdot \nabla_h\sigma_2 \\ \quad + \dfrac{1}{2}\sigma_2\overline{\text{div}_h\, v_{12}} = 0, \\ \sigma_1^2\partial_t v_{12} - \mu\Delta_h v_{12} - \mu\partial_{zz}v_{12} - (\mu + \lambda)\nabla_h\text{div}_h\, v_{12} \\ \quad = -\sigma_{12}(\sigma_1 + \sigma_2)\partial_t v_2 - \nabla_h\left(\sigma_{12}\dfrac{\sigma_1^{2\gamma} - \sigma_2^{2\gamma}}{\sigma_1 - \sigma_2}\right) \\ \quad - \sigma_{12}(\sigma_1 + \sigma_2)v_2 \cdot \nabla_h v_2 - \sigma_1^2 v_{12} \cdot \nabla_h v_2 \\ \quad - \sigma_1^2 v_1 \cdot \nabla_h v_{12} - \sigma_{12}\sigma_2 w_2\partial_z v_2 - \sigma_1(\sigma_1 w_1 - \sigma_2 w_2)\partial_z v_2 \\ \quad - \sigma_1\sigma_1 w_1\partial_z v_{12}, \\ \sigma_i w_i = -\int_0^z \left(\widetilde{\sigma_i\, \text{div}_h\, v_i} + 2\widetilde{v_i} \cdot \nabla_h\sigma_i\right) dz, \quad i = 1, 2. \end{cases} \tag{3.2}$$

Then as before, one can derive

$$\sup_{0 \leqq t \leqq T^*} \left(\left\| \sigma_{12}(t) \right\|_{L^2}^2 + C'_{\mu,\lambda} \left\| (\sigma_1 v_{12})(t) \right\|_{L^2}^2 \right) + \int_0^{T^*} \left\| \nabla v_{12} \right\|_{L^2}^2 \, \mathrm{d}t$$

$$\leqq C \left(\left\| \sigma_{12,0} \right\|_{L^2}^2 + C'_{\mu,\lambda} \left\| \sigma_{1,0} v_{12,0} \right\|_{L^2}^2 \right) \leqq C \left(\left\| \sigma_{12,0} \right\|_{L^2}^2 + \left\| v_{12,0} \right\|_{L^2}^2 \right).$$

Therefore, after employing (1.22) and noticing the fact that we can interchange (σ_1, v_1), (σ_2, v_2) in the previous arguments, we will have the following:

Proposition 11. *Given two sets of initial data* $(\rho_{i,0}, v_{i,0}) = (\sigma_{i,0}^2, v_{i,0})$, $i = 1, 2$, *for* (1.2) *satisfying* (1.17), (1.18) *and* (1.19), *the corresponding strong solutions* $(\rho_i, v_i) = (\sigma_i^2, v_i)$, $i = 1, 2$, *constructed in Proposition 9 in the interval* $[0, T^*]$, *for some* $T^* > 0$, *satisfy*

$$\left\| \sigma_1 - \sigma_2 \right\|_{L^\infty(0,T^*;L^2(\Omega))} + \left\| \sigma_1(v_1 - v_2) \right\|_{L^\infty(0,T^*;L^2(\Omega))}$$

$$+ \left\| \sigma_2(v_1 - v_2) \right\|_{L^\infty(0,T^*;L^2(\Omega))} + \left\| v_1 - v_2 \right\|_{L^2(0,T^*;L^2(\Omega))}$$

$$+ \left\| \nabla(v_1 - v_2) \right\|_{L^2(0,T^*;L^2(\Omega))}$$

$$\leqq C_{\mu,\lambda,B_1,B_2,T^*} \left(\left\| \sigma_{1,0} - \sigma_{2,0} \right\|_{L^2(\Omega)} + \left\| v_{1,0} - v_{2,0} \right\|_{L^2(\Omega)} \right).$$

In particular, if $\rho_{1,0} = \rho_{2,0}$, $v_{1,0} = v_{2,0}$, *we have* $\rho_1 = \rho_2$, $v_1 = v_2$ *in* $[0, T^*]$.

3.3. Proofs of the Main Theorems

Theorem 1 follows from Proposition 6 and Proposition 10. Theorem 2 follows from Proposition 9 and Proposition 11.

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References

1. Brenier, Y.: Homogeneous hydrostatic flows with convex velocity profiles. *Nonlinearity* **12**(3), 495–512, 1999
2. Cao, C., Ibrahim, S., Nakanishi, K., Titi, E.S.: Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics. *Commun. Math. Phys.* **337**(2), 473–482, 2015
3. Cao, C., Li, J., Titi, E.S.: Global well-posedness of strong solutions to the 3D primitive equations with horizontal eddy diffusivity. *J. Differ. Equ.* **257**(11), 4108–4132, 2014
4. Cao, C., Titi, E.S.: Local and global well-posedness of strong solutions to the 3D primitive equations with vertical eddy diffusivity. *Arch. Ration. Mech. Anal.* **214**(1), 35–76, 2014
5. Cao, C., Li, J., Titi, E.S.: Global well-posedness of the three-dimensional primitive equations with only horizontal viscosity and diffusion. *Commun. Pure Appl. Math.* **69**(8), 1492–1531, 2016
6. Cao, C., Li, J., Titi, E.S.: Strong solutions to the 3D primitive equations with only horizontal dissipation: near H^1 initial data. *J. Funct. Anal.* **272**(11), 4606–4641, 2017
7. Cao, C., Li, J., Titi, E.S.: Global well-posedness of the 3D primitive equations with horizontal viscosity and vertical diffusivity. *Phys. D: Nonlinear Phenomena* **412**, 132606, 2020
8. Cao, C., Titi, E.S.: Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics. *Ann. Math.* **166**(1), 245–267, 2007
9. Cao, C., Titi, E.S.: Global well-posedness and finite-dimensional global attractor for a 3-D planetary geostrophic viscous model. *Commun. Pure Appl. Math.* **56**(2), 198–233, 2003
10. Cao, C., Titi, E.S.: Global well-posedness of the 3D primitive equations with partial vertical turbulence mixing heat diffusion. *Commun. Math. Phys.* **310**(2), 537–568, 2012
11. Chen, X., Liu, J.-G.: Two nonlinear compactness theorems in $L^p(0, T; B)$. *Appl. Math. Lett.* **25**(12), 2252–2257, 2012
12. Cho, Y., Choe, H.J., Kim, H.: Unique solvability of the initial boundary value problems for compressible viscous fluids. *J. Math. Pures Appl.* **83**(2), 243–275, 2004
13. Cho, Y., Kim, H.: Existence results for viscous polytropic fluids with vacuum. *J. Differ. Equ.* **228**(2), 377–411, 2006
14. Cho, Y., Kim, H.: On classical solutions of the compressible Navier–Stokes equations with nonnegative initial densities. *Manuscr. Math.* **120**(1), 91–129, 2006
15. Choe, H.J., Kim, H.: Strong solutions of the Navier–Stokes equations for isentropic compressible fluids. *J. Differ. Equ.* **190**(2), 504–523, 2003
16. Ersoy, M., Ngom, T.: Existence of a global weak solution to compressible primitive equations. *C. R. Acad. Sci. Paris Ser. I* **350**(7–8), 379–382, 2012
17. Ersoy, M., Ngom, T., Sy, M.: Compressible primitive equations: formal derivation and stability of weak solutions. *Nonlinearity* **24**(1), 79–96, 2011
18. Feireisl, E.: Dynamics of Viscous Compressible Fluids. Oxford Lecture Series in Mathematics and Its Applications, vol. 26. Oxford University Press (2004).
19. Gatapov, B.V., Kazhikhov, A.V.: Existence of a global solution to one model problem of atmosphere dynamics. *Sib. Math. J.* **46**(5), 805–812, 2005
20. Gerard-Varet, D., Masmoudi, N., Vicol, V.: Well-posedness of the hydrostatic Navier–Stokes equations. *Anal. PDE* **13**(5), 1417–1455, 2020
21. Guillén-González, F.M., Masmoudi, N., Bellido, M.Á.R.: Anisotropic estimates and strong solutions of the primitive equations. *Differ. Integr. Equ.* **14**(11), 1381–1408, 2001
22. Hieber, M., Kashiwabara, T.: Global strong well-posedness of the three dimensional primitive equations in L^p-Spaces. *Arch. Ration. Mech. Anal.* **221**(3), 1077–1115, 2016

23. IGNATOVA, M., KUKAVICA, I., ZIANE, M.: Local existence of solutions to the free boundary value problem for the primitive equations of the ocean. *J. Math. Phys*. **53**(10), 103101, 2012

24. KOBELKOV, G.M.: Existence of a solution 'in the large' for the 3D large-scale ocean dynamics equations. *C. R. Acad. Sci. Paris Ser. I* **343**(4), 283–286, 2006

25. KUKAVICA, I., PEI, Y., RUSIN, W., ZIANE, M.: Primitive equations with continuous initial data. *Nonlinearity* **27**(6), 1135–1155, 2014

26. KUKAVICA, I., TEMAM, R., VICOL, V.C., ZIANE, M.: Local existence and uniqueness for the hydrostatic Euler equations on a bounded domain. *J. Differ. Equ*. **250**(3), 1719–1746, 2011

27. KUKAVICA, I., ZIANE, M.: On the regularity of the primitive equations of the ocean. *Nonlinearity* **20**(12), 2739–2753, 2007

28. KUKAVICA, I., ZIANE, M.: The regularity of solutions of the primitive equations of the ocean in space dimension three. *C. R. Acad. Sci. Paris Ser. I* **345**(5), 257–260, 2007

29. LI, J., TITI, E.S.: Existence and uniqueness of weak solutions to viscous primitive equations for a certain class of discontinuous initial data. *SIAM J. Math. Anal*. **49**(1), 1–28, 2017

30. LIU, X., TITI, E.S.: Local well-posedness of strong solutions to the three-dimensional compressible primitive equations. arXiv:1806.09868 (2018)

31. LIU, X., TITI, E.S.: Global existence of weak solutions to the compressible primitive equations of atmospheric dynamics with degenerate viscosities. *SIAM J. Math. Anal*. **51**(3), 1913–1964, 2019

32. LIONS, J.-L., TEMAM, R., WANG, S.: On mathematical problems for the primitive equations of the ocean: the mesoscale midlatitude case. *Nonlinear Anal. Theory Methods Appl*. **40**(1–8), 439–482, 2000

33. LIONS, J.-L., TEMAM, R., WANG, S.: Geostrophic asymptotics of the primitive equations of the atmosphere. *Topol. Methods Nonlinear Anal*. **4**(2), 253–287, 1994

34. LIONS, J.-L., TEMAM, R., WANG, S.: On the equations of the large-scale ocean. *Nonlinearity* **5**(5), 1007–1053, 1992

35. LIONS, J.-L., TEMAM, R., WANG, S.: New formulations of the primitive equations of atmosphere and applications. *Nonlinearity* **5**(2), 237–288, 1992

36. LIONS, P.-L.: *Mathematical Topics in Fluid Mechanics, Incompressible Models. Oxford Lecture Series in Mathematics and Its Applications*, vol. 1. Oxford University Press, 1996

37. MASMOUDI, N., WONG, T.K.: On the H^s theory of hydrostatic Euler equations. *Arch. Ration. Mech. Anal*. **204**(1), 231–271, 2012

38. PETCU, M., WIROSOETISNO, D.: Sobolev and Gevrey regularity results for the primitive equations in three space dimensions. *Appl. Anal*. **84**(8), 769–788, 2005

39. RICHARDSON, L.F.: *Weather Prediction by Numerical Process*. Cambridge University Press, 2007

40. SIMON, J.: Compact sets in the space $L^p(0, T; B)$. *J. Annali di Matematica pura ed applicata* **146**(1), 65–96, 1986

41. TANG, T., GAO, H.: On the stability of weak solution for compressible primitive equations. *Acta Appl. Math*. **140**(1), 133–145, 2015

42. TEMAM, R.: *Navier–Stokes Equations: Theory and Numerical Analysis. Studies in Mathematics and Its Applications*. Elsevier Science, 2016

43. WANG, F., DOU, C., JIU, Q.: Global weak solutions to 3D compressible Primitive equations with density-dependent viscosity. *J. Math. Phys*. **61**, 021507, 2020

44. WASHINGTON, W.M., PARKINSON, C.L.: *University Science Books. An Introduction to Three-Dimensional Climate Modeling*, 2005

45. WONG, T.K.: Blowup of solutions of the hydrostatic Euler equations. *Proc. Am. Math. Soc*. **143**(3), 1119–1125, 2014

46. ZELATI, M.C., HUANG, A., KUKAVICA, I., TEMAM, R., ZIANE, M.: The primitive equations of the atmosphere in presence of vapour saturation. *Nonlinearity* **28**(3), 625–668, 2015

47. Zɪᴀɴᴇ , M., Tᴇᴍᴀᴍ , R., Cʜᴀɴɢʙɪɴɢ , H.: The primitive equations on the large scale
     ocean under the small depth hypothesis. *Discrete Contin. Dyn. Syst*. **9**(1), 97–131, 2002

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