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Counting Curves on Toric Surfaces Tropical  
Geometry & the Fock Space

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# COUNTING CURVES ON TORIC SURFACES TROPICAL GEOMETRY & THE FOCK SPACE

RENZO CAVALIERI PAUL JOHNSON HANNAH MARKWIG DHRUV RANGANATHAN

ABSTRACT. We study the stationary descendant Gromov–Witten theory of toric surfaces by combining and extending a range of techniques – tropical curves, floor diagrams, and Fock spaces. A correspondence theorem is established between tropical curves and descendant invariants on toric surfaces using maximal toric degenerations. An intermediate degeneration is then shown to give rise to floor diagrams, giving a geometric interpretation of this well-known bookkeeping tool in tropical geometry. In the process, we extend floor diagram techniques to include descendants in arbitrary genus. These floor diagrams are then used to connect tropical curve counting to the algebra of operators on the bosonic Fock space, and are shown to coincide with the Feynman diagrams of appropriate operators. This extends work of a number of researchers, including Block–Göttsche, Cooper–Pandharipande, and Block–Gathmann–Markwig.

## 1. INTRODUCTION

1.1. **Overview.** The scope of this manuscript is to establish an equivalence between the following enumerative and combinatorial geometric theories of surfaces, studied by a number of researchers in the last decade:

- (1) decorated floor diagram counting;
- (2) logarithmic descendant Gromov–Witten theory of Hirzebruch surfaces;
- (3) tropical descendant Gromov–Witten theory of Hirzebruch surfaces;
- (4) matrix elements of operators on a bosonic Fock space.

**Floor diagrams** are loop free graphs on a linearly ordered set of vertices, further endowed with vertex, edge, and half-edge decorations as specified in Definition 4.1. Each diagram is counted with a weight which is the product of vertex and edge multiplicities. Floor diagrams capture the combinatorial essence of the other three theories, in the sense that the simplest way to exhibit the above equivalences is through a weight preserving bijection between floor diagrams and specific ways to organize the enumeration in the other theories.

**Logarithmic Gromov–Witten theory** studies the intersection theory on moduli spaces of maps from pointed curves to a target surface, with specified tangency conditions along certain boundary divisors, as in Definition 2.3. These moduli spaces admit a virtual fundamental class, and we construct zero dimensional cycles by capping with the virtual class two types of cycles: point conditions, corresponding to requiring a point on the curve to map to a specified point on the surface; and descendant insertions, which are Euler classes of certain tautological line bundles on the moduli space, associated to each marked point. The word *stationary* refers to the fact that descendant insertions are always coupled with point conditions. In this work, we specify special tangencies to the  $0$  and  $\infty$  sections of Hirzebruch surfaces, with transverse contact along the torus invariant fibers, taking inspiration from the geometry of double Hurwitz numbers. Such invariants may be computed by studying maps to a degeneration of the target surface consisting of a chain of

Hirzebruch surfaces glued to each other along the sections. In Theorem 4.9, the equivalence of the two theories is established, by identifying the decorated graphs that naturally organize the degeneration as floor diagrams.

**Tropical Gromov–Witten theory** of surfaces consists of the study of piecewise linear, balanced maps from tropical curves into  $\mathbb{R}^2$ , see Definition 3.3. One obtains a finite count by imposing point conditions (i.e. specifying the image of a contracted marked end on the plane), and tropical descendant conditions. The descendant conditions constrain the valency of the vertex adjacent to a marked end. Each map is counted with a weight that arises as a product of local vertex multiplicities and tropical intersection-theoretic factors coming from cycles and point evaluations. In good cases, the latter factors can again be spread out as a product over areas of dual polygons of non-marked vertices. The directions and multiplicities of the infinite ends define a Newton fan, which determines at the same time a toric surface, a curve class on it, and prescribed tangencies along the toric divisors, offering a natural candidate for a correspondence between the logarithmic and tropical theories.

The simplest way to establish this correspondence is via a combinatorial geometry argument; showing that the tropical curve count matches with the floor diagram count, see Theorem 4.11: if the point conditions are horizontally stretched, then all tropical curves contributing to the counting problem become *floor decomposed*, meaning that certain subgraphs of the tropical curves may be contracted to give rise to a floor diagram.

For completeness and its conceptual value, in Theorem 3.14 we also offer a direct proof of the correspondence between logarithmic and tropical invariants, as we feel it provides valuable insight into the geometric motivation for such a correspondence: each tropical curve contributing to a particular invariant identifies a subdivision of the polytope of the surface, and hence a degeneration of the target surface. A common refinement of these subdivisions gives a degeneration of the target surface such that each tropical curve arises as the dual graph of some map to this degeneration. The key new ingredient in this aspect of the work is the recently established *decomposition formula* for logarithmic Gromov–Witten invariants [2].

The **bosonic Fock space** is a countably infinite dimensional vector space with a basis indexed by ordered pairs of partitions consisting of non-negative integers. It has an action of a Heisenberg algebra of operators, generated by two families of operators  $a_s, b_s$  parameterized by the integers. The distinguished basis vectors can naturally be identified with tangency conditions along the 0 and  $\infty$  sections of a Hirzebruch surface. In Definition 5.1, we construct a family of linear operators  $M_1$  on the Fock space which are naturally associated to stationary descendant insertions. To each Gromov–Witten invariant then corresponds a *matrix element* for an operator obtained as an appropriate composition of the  $M_1$ 's above. The equality between a Gromov–Witten invariant and the corresponding matrix element goes through a comparison with the floor diagrams count: by Wick's theorem a matrix element can be naturally evaluated as a weighted sum over Feynman graphs (see Definition 5.4). In Theorem 5.3 we exhibit a weight preserving bijection between the Feynman graphs for a given matrix element, and the floor diagrams for the corresponding Gromov–Witten invariant.

**1.2. Context and Motivation.** This work provides an extension and unification of several previous lines of investigation on the subject. Correspondence theorems between tropical curve counts and primary Gromov–Witten invariants of surfaces – those with only point conditions and no descendant insertions – were established by Mikhalkin, Nishinou–Siebert, and Gathmann–Markwig

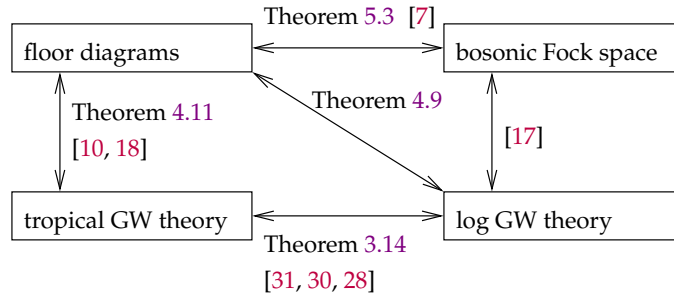


FIGURE 1. An overview of the content and background.

in [31, 33, 20]; the tropical descendant invariants in genus 0 was first investigated by Markwig–Rau [30], and correspondence theorems were established independently, using different techniques, by A. Gross [23] and by Mandel–Ruddat [28]. Tropical descendants have also arisen in aspects of the SYZ conjecture [24, 35].

Cooper and Pandharipande pioneered a Fock space approach to the Severi degrees of  $\mathbb{P}^1 \times \mathbb{P}^1$  and  $\mathbb{P}^2$  by using degeneration techniques [17]. Block and Göttsche generalized their work to a broader class of surfaces (the  $h$ -transverse surfaces, see for instance [4] and [10]), and to refined curve counts, via quantum commutators on the Fock space side [7]. Both cases deal only with primary invariants. Block and Göttsche assign an operator on the Fock space to point insertions, and observe the connection between floor diagrams and Feynman graphs. We generalize their operator to a family of operators, one for each descendant insertion, and notice that the operators can be written with summands naturally corresponding to the possible *sizes of the floor* (see Definition 5.1) that contains a particular descendant insertion. In the primary case, there were only floors of size 0 (elevators) or 1 (floors), and hence the operator had two terms.

Section 4 contains a brief summary of how floor diagrams came to be employed for these type of enumerative problems (Subsection 4.1). This discussion follows our definition of floor diagrams (Definition 4.1) to explain and motivate some of the minor combinatorial tweaks we made in order to adapt to the current geometric context.

It is at this point a well understood philosophy that correspondence theorems between classical and tropical enumerative invariants are based on the fact that tropical curves encode the combinatorics of possible degenerations of the classical objects. It is especially satisfactory when one can turn this philosophy into a precise mathematical statement, to understand the “seed” geometric data that one needs in order for tropical geometry to combinatorially reconstruct the classical theory. In this work, we appeal to logarithmic Gromov–Witten theory to achieve this. The ability to generalize the degeneration formula to targets with simple normal crossing boundaries (see the recent results of [2]) allows us to realize the tropical-classical curve count correspondence as an instance of the degeneration formula; the geometric inputs then appear in our formulas as the vertex multiplicities associated to tropical maps.

An appealing feature of the generality provided by the logarithmic setup is that it establishes a formula from which one can witness the collapsing of geometric inputs in different settings to give rise to a purely combinatorial theory. In genus 0, the descendant contributions collapse into closed combinatorial formulas. Conceptually, this is because the intersection theory of the space of genus 0 logarithmic maps is essentially captured by the intersection theory on a particular toric variety, see [36]. Without descendants but still in higher genus, there is a different collapsing – on

a surface, one can degenerate in such a way that all the algebraic inputs are 1 up to multiplicity – the multiplicity can be detected combinatorially, leading to Mikahlkin’s formula (3.15).

Restricting our attention to the study of invariants of **Hirzebruch surfaces** is a stylistic choice, as we strived to write a paper that communicates the various connections we explore, rather than making the most general statements possible. Results of Section 3 could as well be formulated for any toric surface, results of Sections 4 and 5 for any toric surface dual to a so-called h-transverse lattice polygon, see [7, Section 2.3].

This paper is a sequel to the authors’ work in [13], in which the relationship between tropical curves, Fock spaces, and degeneration techniques was studied for target curves, combining Okounkov and Pandharipande’s seminal work in [34], with the tropical perspective on the enumerative geometry of target curves [11, 12, 15, 14]. We refer the reader to [13] for a more detailed discussion of the history of the target curve case.

The paper is organized as follows. In Section 2 we present some basic facts about the geometry of Hirzebruch surfaces, and introduce logarithmic stationary descendant Gromov–Witten invariants. Section 3 introduces the tropical theory of descendant stationary invariants of Hirzebruch surfaces, and proves the correspondence theorem with the classical theory. In Section 4 we define our version of decorated floor diagrams, explain the connection with the previous notions in the literature, and then provide two correspondence theorems. First we compare floor diagram counts with the classical theory, as an application of the degeneration formula. Next we provide a correspondence theorem with the tropical theory. Section 5 provides a brief and hopefully friendly introduction to the Fock space, and then proves the correspondence theorem between floor diagram counts and matrix elements for specific operators in the Fock space.

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## 2. LOGARITHMIC DESCENDANTS OF HIRZEBRUCH SURFACES

For  $k \geq 0$ , the **Hirzebruch surface**  $\mathbb{F}_k$  is defined to be the ruled surface  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k))$ ; it is a smooth projective toric surface. The 1-skeleton of its fan  $\Sigma_k$  is given by the four vectors  $e_1, \pm e_2, -e_1 + ke_2$ . The 2-dimensional cones are spanned by the consecutive rays in the natural counterclockwise ordering. The zero section  $B$ , the infinity section  $E$ , and the fiber  $F$  have intersections

$$B^2 = k, \quad E^2 = -k, \quad BF = EF = 1, \quad \text{and} \quad F^2 = BE = 0.$$

The Picard group of  $\mathbb{F}_k$  is isomorphic to  $\mathbb{Z} \times \mathbb{Z}$  generated by the classes of  $B$  and  $F$ . In particular, we have  $E = B - kF$ . A curve in  $\mathbb{F}_k$  has **bidegree**  $(a, b)$  if its class is  $aB + bF$ . The polygon depicted in Figure 2 defines  $\mathbb{F}_k$  as a projective toric variety, equivariantly polarized by an  $(a, b)$  curve.

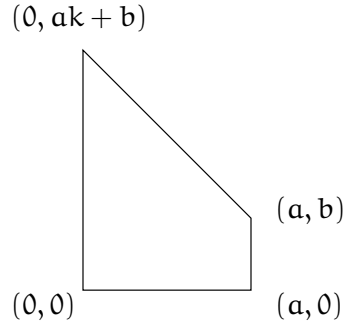


FIGURE 2. The polygon defining the Hirzebruch surface  $\mathbb{F}_k$  as a toric surface with hyperplane section the class of a curve of bidegree  $(a, b)$ . The vertical sides corresponds to the sections  $B$  (left) and  $E$  (right).

We study the virtual enumerative invariants of curves in Hirzebruch surfaces that have prescribed special contact orders with the zero and infinity sections, with generic intersection with the fibers. This numerical data is encoded in terms of the *Newton fan*.

**Definition 2.1.** A *Newton fan* is a sequence  $\delta = \{v_1, \dots, v_k\}$  of vectors  $v_i \in \mathbb{Z}^2$  satisfying

$$\sum_{i=1}^k v_i = 0.$$

If  $v_i = (v_{i1}, v_{i2})$ , then the positive integer  $w_i = \gcd(v_{i1}, v_{i2})$  (resp. the vector  $\frac{1}{w_i}v_i$ ) is called the **expansion factor** (resp. the **primitive direction**) of  $v_i$ . We use the notation

$$\delta = \{v_1^{m_1}, \dots, v_k^{m_k}\}$$

to indicate that the vector  $v_i$  appears  $m_i$  times in  $\delta$ .

For a Newton fan  $\delta$ , one can construct a polarized toric surface, identified by the **dual polygon**  $\Pi_\delta$  in  $\mathbb{R}^2$ , in the following way: for each primitive integer direction  $(\alpha, \beta)$  in  $\delta$ , we consider the vector  $w(-\beta, \alpha)$ , where  $w$  is the sum of the expansion factors of all vectors in  $\delta$  with primitive integer direction  $(\alpha, \beta)$ . Up to translation,  $\Pi_\delta$  is the unique (convex, positively oriented) polygon whose oriented edges are exactly the vectors  $w(-\beta, \alpha)$ .

**Notation 2.2.** (*Discrete data*) Fix a Hirzebruch surface  $\mathbb{F}_k$ . The following discrete conditions govern the enumerative geometric problems we study throughout the paper:

- A positive integer  $n$ ;
- Non-negative integers  $g, a, k_1, \dots, k_n, n_1, n_2$ ;
- A vector  $\underline{\phi} = (\varphi_1, \dots, \varphi_{n_1}) \in (\mathbb{Z} \setminus 0)^{n_1}$ ;
- A vector  $\underline{\mu} = (\mu_1, \dots, \mu_{n_2}) \in (\mathbb{Z} \setminus 0)^{n_2}$ .
- We assume that  $\underline{\phi}$  and  $\underline{\mu}$  are non-decreasing sequences.
- We denote by  $(\underline{\phi}^+, \underline{\mu}^+)$  the positive entries of  $(\underline{\phi}, \underline{\mu})$ , and by  $(\underline{\phi}^-, \underline{\mu}^-)$  the negative ones.

Further, the following two equations must be satisfied:

$$(1) \quad \sum_{i=1}^{n_1} \varphi_i + \sum_{i=1}^{n_2} \mu_i + ka = 0;$$

$$(2) \quad n_2 + 2a + g - 1 = n + \sum_{j=1}^n k_j.$$

The first enumerative geometric problem we introduce is stationary, descendant, logarithmic Gromov–Witten invariants of  $\mathbb{F}_k$ , which morally count curves in  $\mathbb{F}_k$  with prescribed tangency conditions along the boundary, and satisfying some further geometric constraints, called **descendant insertions** (see Section 2.1), at a number of fixed points in the interior of the surface. In this context,  $g$  is the arithmetic genus of the curves being counted,  $n$  is the number of ordinary marked points on the curves, and the  $k_i$  are the degrees of the descendant insertions at each point. The sequences  $(\underline{\phi}, \underline{\mu})$  identify a curve class in  $H_2(\mathbb{F}_k, \mathbb{Z})$ , as well as the required tangency with the toric boundary, as we now explain.

The tuple  $(\underline{\phi}, \underline{\mu})$  determines the curve class

$$(3) \quad \beta = aB + \left( \sum_{\varphi_i \in \underline{\phi}^+} \varphi_i + \sum_{\mu_i \in \underline{\mu}^+} \mu_i \right) F.$$

The compatibility condition (1) ensures that  $\beta$  is an effective, integral curve class in  $H_2(\mathbb{F}_k, \mathbb{Z})$ .

The Newton fan

$$(4) \quad \delta_{(\underline{\phi}, \underline{\mu})} := \{(0, -1)^a, (k, 1)^a, \varphi_1 \cdot (1, 0), \dots, \varphi_{n_1} \cdot (1, 0), \mu_1 \cdot (1, 0), \dots, \mu_{n_2} \cdot (1, 0)\}.$$

encodes contact orders a curve may have with the toric boundary of  $\mathbb{F}_k$ . Such a curve is necessarily of class  $\beta$ .

We count curves with contact orders  $|\varphi_i|$  for  $\varphi_i < 0$  (resp.  $\varphi_i > 0$ ) with the zero (resp. infinity) section at fixed points, and contact orders  $|\mu_i|$  for  $\mu_i < 0$  (resp.  $\mu_i > 0$ ) with the zero (resp. infinity) section at arbitrary points.

**2.1. Virtual enumerative invariants.** Logarithmic stable maps and logarithmic Gromov–Witten invariants were developed in the articles [1, 16, 25]. Consider the moduli space

$$\overline{M}_{g, n+n_1+n_2}(\mathbb{F}_k, \delta_{(\underline{\phi}, \underline{\mu})}),$$

parameterizing families of minimal logarithmic stable maps

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathbb{F}_k \\ \downarrow & & \\ S, & & \end{array}$$



where  $\mathcal{C}$  is a family of connected marked genus  $g$  nodal curves and  $f$  is a map of logarithmic schemes, whose underlying map is stable in the usual sense. The minimality condition is a technical condition on the logarithmic structure on  $S$ . Since it only plays a background role here, we refer the reader to the literature for a discussion of the concept.

Concerning the discrete data, contact orders with the toric boundary are specified by the Newton fan  $\delta_{(\underline{\phi}, \underline{\mu})}$ . We choose to mark the points of contact with the zero and infinity sections, and to not mark the points of contact with the torus-invariant fibers, where the behavior requested is generic.

This moduli space is a Deligne-Mumford stack of virtual dimension  $(g-1) + 2a + n + n_1 + n_2$ , and it carries a virtual fundamental class denoted by  $[1]^{\text{vir}}$ . For each of the first  $n$  marked points, which carry trivial contact orders, there are evaluation morphisms

$$ev_i : \overline{M}_{g, n+n_1+n_2}(\mathbb{F}_k, \delta_{(\underline{\phi}, \underline{\mu})}) \rightarrow \mathbb{F}_k$$

The points marking the contact points with the zero and infinity sections give rise to evaluation morphisms

$$\widehat{ev}_i : \overline{M}_{g, n+n_1+n_2}(\mathbb{F}_k, \delta_{(\underline{\phi}, \underline{\mu})}) \rightarrow \mathbb{P}^1.$$

Here, the target  $\mathbb{P}^1$  is the the zero section  $B$  for negative entries of  $\underline{\phi}$  or  $\underline{\mu}$ , and the infinity section  $E$  for positive entries.

For each of the first  $n$  marks (with trivial contact order) there is a cotangent line bundle, whose first Chern class is denoted  $\psi_i$ .

**Definition 2.3.** Fix a Hirzebruch surface  $\mathbb{F}_k$  and discrete data as in Notation 2.2.

The *stationary descendant log Gromov–Witten invariant* is defined as the following intersection number on  $\overline{M}_{g, n+n_1+n_2}(\mathbb{F}_k, \delta_{(\underline{\phi}, \underline{\mu})})$ :

$$(5) \quad \langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}) \dots \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g = \int_{[1]^{\text{vir}}} \prod_{j=1}^n \psi_j^{k_j} ev_j^*([\text{pt}]) \prod_{i=n+1}^{n+n_1} \widehat{ev}_i^*([\text{pt}])$$

Condition (2) comes from equating the expected dimension of the moduli space with the codimension of the intersection cycle, and hence it is a necessary condition for Equation (5) to be non-zero.

We also define the *disconnected* descendant log Gromov–Witten invariant

$$\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}) \dots \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^\bullet$$

to be the intersection number obtained via an analogous moduli space of log stable maps where the source curves are not required to be connected.

Since they appear as local vertex multiplicities of tropical stable maps contributing to tropical stationary descendant Gromov–Witten invariants, we also define marked descendant Gromov–Witten invariants for curves in other toric surfaces.

**Definition 2.4.** Let  $\delta = \delta_\phi \cup \delta_\mu$  be a two part partition of a Newton fan. Denoting  $S_\delta$  the polarized toric surface identified by  $\delta$ , for each entry in  $\delta_\phi$ , fix a point in the corresponding torus invariant line of  $S_\delta$ . We consider log-stable maps intersecting the toric boundary as prescribed by  $\delta$ , and in addition meeting the fixed point in the toric boundary for each entry of  $\delta_\phi$ . We define the intersection number on  $\overline{M}_{g, n+\ell(\delta)}(S_\delta, \delta)$ :

$$(6) \quad \langle \tau_{k_1}(\text{pt}) \dots \tau_{k_n}(\text{pt}) \rangle_{\delta_\phi \cup \delta_\mu, g} = \int_{[1]^{\text{vir}}} \prod_{j=1}^n \psi_j^{k_j} ev_j^*([\text{pt}]) \prod_{i=n+1}^{n+\ell(\delta_\phi)} \widehat{ev}_i^*([\text{pt}]).$$

In Equation (6), the quantity  $\ell(\delta)$  denotes the number of vectors in the Newton fan  $\delta$ , and for each vector in  $\delta$  we have an evaluation morphism to the corresponding torus invariant divisor.

### 3. TROPICAL DESCENDANTS OF HIRZEBRUCH SURFACES

**3.1. Tropical preliminaries.** An **(abstract) tropical curve** is a connected metric graph  $\Gamma$  with unbounded rays or “ends” and a genus function  $g : \Gamma \rightarrow \mathbb{N}$  which is nonzero only at finitely many points. Locally around a point  $p$ ,  $\Gamma$  is homeomorphic to a star with  $r$  half-rays. The number  $r$  is called the **valence** of the point  $p$  and denoted by  $\text{val}(p)$ . We require that there are only finitely many points with  $\text{val}(p) \neq 2$ . The points of nonzero genus, valence larger than 2 and a finite subset of the bivalent points will be called **vertices**. By abuse of notation, the underlying graph with this vertex set is also denoted by  $\Gamma$ . Correspondingly, we can speak about **edges** and **flags** of  $\Gamma$ . A flag is a tuple  $(V, e)$  of a vertex  $V$  and an edge  $e$  with  $V \in \partial e$ . It can be thought of as an element in the tangent space of  $\Gamma$  at  $V$ , i.e. as a germ of an edge leaving  $V$ , or as a half-edge (the half of  $e$  that is attached to  $V$ ). Edges which are not ends have a finite length and are called **bounded edges**.

A **marked tropical curve** is a tropical curve such that some of its ends are labeled. An isomorphism of a tropical curve is a homeomorphism respecting the metric, the markings of ends, and the genus function. The **genus** of a tropical curve is the first Betti number  $b^1(\Gamma)$  plus the genera of all vertices. A curve of genus 0 is called **rational**.

The **combinatorial type** of a tropical curve is obtained by dropping the information on the metric.

Let  $\Sigma$  be a complete fan in  $\mathbb{R}^2$ .

**Definition 3.1.** A **tropical stable map to  $\Sigma$**  is a tuple  $(\Gamma, f)$  where  $\Gamma$  is a marked abstract tropical curve and  $f : \Gamma \rightarrow \Sigma$  is a piecewise integer-affine map of polyhedral complexes satisfying:

- On each edge  $e$  of  $\Gamma$ ,  $f$  is of the form

$$t \mapsto \mathbf{a} + t \cdot \mathbf{v} \text{ with } \mathbf{v} \in \mathbb{Z}^2,$$

where we parametrize  $e$  as an interval of size the length  $\ell(e)$  of  $e$ . The vector  $\mathbf{v}$ , called the **direction**, arising in this equation is defined up to sign, depending on the starting vertex of the parametrization of the edge. We will sometimes speak of the direction of a flag  $\mathbf{v}(V, e)$ . If  $e$  is an end we use the notation  $\mathbf{v}(e)$  for the direction of its unique flag.

- The **balancing condition** holds at every vertex, i.e.

$$\sum_{e \in \partial V} \mathbf{v}(V, e) = 0.$$

- The **stability condition** holds, i.e. for every 2-valent vertex  $v$  of  $\Gamma$ , the star of  $v$  is not contained in the relative interior of any single cone of  $\Sigma$ .

For an edge with direction  $\mathbf{v} = (v_1, v_2) \in \mathbb{Z}^2$ , we call  $w = \gcd(v_1, v_2)$  the **expansion factor** and  $\frac{1}{w} \cdot \mathbf{v}$  the **primitive direction** of  $e$ .

An isomorphism of tropical stable maps is an isomorphism of the underlying tropical curves respecting the map. The **degree** of a tropical stable map is the Newton fan given as the multiset of directions of its ends. The **combinatorial type** of a tropical stable map is the data obtained when dropping the metric of the underlying graph. More explicitly, it consists of the data of a finite graph  $\Gamma$ , and (1) for each vertex  $v$  of  $\Gamma$ , the cone  $\sigma_v$  of  $\Sigma$  to which this vertex maps, and (2) for each edge  $e$  of  $\Gamma$ , the expansion factor and primitive direction of  $e$ .

**Convention 3.2.** We consider tropical stable maps to Hirzebruch surfaces, i.e. the degree is a Newton fan dual to the polygons of Figure 2. Furthermore, we require the vertical and diagonal ends to be non-marked and of expansion factor 1. The horizontal ends can have any expansion factor, and are marked.

In what follows, we fix conditions for tropical stable maps — the degree, the genus, point conditions, high valency ( $\psi$ -power) conditions, and end conditions — and then count tropical stable maps satisfying the conditions, with multiplicity. We consider degrees containing integer multiples of  $(1, 0)$ . An end whose direction vector is a multiple of  $(1, 0)$  is mapped to a line segment of the form  $\{(a, b) + t \cdot (\pm 1, 0)\}$ , where  $(a, b) \in \mathbb{R}^2$ . The unique  $b$  appearing here is the **y-coordinate** of the respective end. Our end conditions fix some of the y-coordinates of ends.

**Definition 3.3.** Fix discrete invariants as in Notation 2.2. Let

$$\Delta = \delta_{(\phi, \underline{\mu})} \cup \{0^n\}$$

identify a degree for tropical stable maps. Fix  $n$  points  $p_1, \dots, p_n \in \mathbb{R}^2$  in general position, and two sets  $E_0$  and  $E_\infty$  of pairwise different real numbers together with bijections  $E_0 \rightarrow \{\varphi_i | \varphi_i < 0\}$  (resp.  $E_\infty \rightarrow \{\varphi_i | \varphi_i > 0\}$ ).

*The tropical descendant Gromov–Witten invariant*

$$\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(p_1) \dots \tau_{k_n}(p_n) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{trop}}$$

is the weighted number of marked tropical stable maps  $(\Gamma, f)$  of degree  $\Delta$  and genus  $g$  satisfying:

- For  $j = 1, \dots, n$ , the marked end  $j$  is contracted to the point  $p_j \in \mathbb{R}^2$ .
- The end  $j$  is adjacent to a vertex  $V$  in  $\Gamma$  of valence  $\text{val}(V) = k_j + 3 - g(V)$ .
- $E_0$  and  $E_\infty$  are the y-coordinates of ends marked by the set  $\phi$ .

Each such tropical stable map is counted with **multiplicity**  $\text{mult}(\Gamma, f)$ , to be defined in Definition 3.7.

**3.2. Tropical moduli, evaluations, and rigid curves.** Fixing a combinatorial type of tropical stable maps, the set of tropical stable maps of this type can be parametrized by a polyhedral cone in a real vector space see. In the present generality and notation, a proof is recorded in [37], though the main ideas can be found in [21, 22, 32]. The **expected dimension** of the cone associated to the type of a map  $(\Gamma, f)$  is

$$\#\{\text{ends}\} + b^1(\Gamma) - 1 - \sum k_i - \sum_V (\text{val}(V) - 3) = \#\{\text{bounded edges}\} - 2b^1(\Gamma),$$

see for instance [37, Section 2.2]. When a combinatorial type has this expected dimension, it is said to be **non-superabundant**. In the non-superabundant cases, a tropical curve meeting general point constraints will always be rigid. In superabundant cases, there may be nontrivial families even when the expected dimension is zero, so we require the above notion of rigidity to reduce to a finite combinatorial count.

The following result follows from a simple adaptation of the proof of [31, Lemma 4.20].

**Lemma 3.4.** Let  $(\Gamma, f)$  be a rigid stable map satisfying the conditions of Definition 3.3. Then every connected component of  $\Gamma$  minus the marked ends is rational and contains exactly one end with a non-fixed end.

Ranging over combinatorial types, these cones may be glued to form a (generalized) cone complex  $T_{g, n+n_1+n_2}(\Sigma_k, \delta_{(\phi, \underline{\mu})})$ , constructed in [37, Section 2]. As in the algebraic case, there exist

evaluation morphisms

$$\text{ev}_i : T_{g, n+n_1+n_2}(\Sigma_k, \delta_{(\underline{\phi}, \underline{\mu})}) \rightarrow \Sigma_k,$$

and

$$\widehat{\text{ev}}_i : T_{g, n+n_1+n_2}(\Sigma_k, \delta_{(\underline{\phi}, \underline{\mu})}) \rightarrow \mathbb{R},$$

to the surface and boundary strata respectively.

We will use the following definition to identify tropical curves that have been “fixed” by the stationary constraints.

**Definition 3.5.** *Choose general points  $p_1, \dots, p_n \in \mathbb{R}^2$ , a degree, genus, incidence, and descendant constraints defining a tropical descendant Gromov–Witten invariant. Let  $(\Gamma, f)$  be a tropical stable map satisfying these chosen constraints. The map  $(\Gamma, f)$  is said to be **rigid** if  $(\Gamma, f)$  is not contained in any nontrivial family of tropical curves having the same combinatorial type.*

It should be noted that the notion of rigid that we use here is in line with the virtual decomposition theorem [2], but is weaker than the notion of rigidity used in [28].

**3.3. The multiplicity of a tropical curve.** In this section, we record the multiplicity of rigid tropical curves that will be used to establish the correspondence theorem. The main content of the section is to separately extract the various combinatorial and algebro-geometric multiplicities from the logarithmic decomposition formula. In Section 4, we exploit the specific combinatorics of our situation to give a vastly simplified method to compute these invariants using floor diagram calculus.

Let  $(\Gamma, f)$  be a tropical stable map of combinatorial type  $\Theta$ . There is a cone  $\sigma_\Theta$  parametrizing tropical stable maps of type  $\Theta$ , see for instance [37, Proposition 2.2.6]. We require a numerical invariant from this construction – the **saturation index** – defined as follows. To describe a particular map  $[f]$  of type  $\Theta$  we must assign to each vertex  $v_i$  a point  $f(v_i)$  in the associated cone  $\sigma_{v_i}$  dictated by  $\Theta$ . Similarly, we must assign to each edge  $e_j$ , a length  $\ell_{e_j}$ . Since these assignments must define a continuous balanced map to  $\Delta$ , we need that for each edge  $e$  between vertices  $v$  and  $v'$

$$f(v) - f(v') = \ell_e w_e,$$

where, as in previous notation,  $w_e$  is the vector slope given by  $\Theta$ . In other words,

$$\sigma_\Theta = \left\{ ((f(v)_v, (\ell_e)_e) \in \prod_{v \in V} \sigma_v \times \prod_{e \in E} \mathbb{R}_{\geq 0} : \text{For all } e = vv', f(v) - f(v') = \ell_e w_e \right\}.$$

Since we work with toric surfaces, all the cones  $\sigma_v$  appearing above are orthants. Thus,  $\sigma_\Theta$  is cut out of an orthant  $\mathbb{R}_{\geq 0}^L$  by a collection of linear equations, thus defining a cone.

**Definition 3.6.** *In the notation of the paragraph above, the **saturation index**  $m_\Theta$  of the type  $\Theta$  is the index of the lattice  $\sigma_\Theta \cap \mathbb{Z}_{\geq 0}^L$  inside the integral points of  $\sigma_\Theta$ . Given a tropical stable map of type  $\Theta$ , define  $m_{(\Gamma, f)}$  to be  $m_\Theta$ .*

**Definition 3.7.** *Let  $(\Gamma, f)$  be a tropical stable map meeting the constraints of a tropical descendant invariant  $\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(p_1) \dots \tau_{k_n}(p_n) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{trop}}$  as in Definition 3.3.*

*If  $(\Gamma, f)$  is not rigid, set  $\text{mult}(\Gamma, f) = 0$ . Otherwise, orient the edges of  $\Gamma$  minus the marked ends in each component towards the unique end with a non-fixed end.*

**From oriented edges to boundary incidence conditions.** *Locally around each vertex  $V$  of  $\Gamma$ , the directions of the adjacent flags define a Newton fan  $\delta_V$ . If for a vertex, all incoming ends have distinct directions,*

we let  $\delta_\phi$  be the subset given by all entries of  $\delta_V$  corresponding to edges which are oriented towards  $V$ , and  $\delta_\mu$  consist of the vectors in  $\delta_V$  oriented away from  $V$ . If a vertex has multiple incoming ends with the same direction, then choose one of these to be in  $\delta_\phi$  and the remaining in  $\delta_\mu$ .

Define the **local multiplicity at  $V$**  to be

$$\text{mult}_V(\Gamma, f) = \langle \tau_{k_i}(\mathbf{pt}) \rangle_{\delta_\phi \cup \delta_\mu, g_V}$$

if the marked end  $i$  is adjacent to  $V$  and

$$\text{mult}_V(\Gamma, f) = \langle \rangle_{\delta_\phi \cup \delta_\mu, g_V}$$

otherwise. Here  $g_V$  denotes the genus of  $\Gamma$  at  $V$ . Since we require the marked ends to meet distinct points, there cannot be more than one end adjacent to a vertex  $V$ .

Given a rigid tropical stable map  $(\Gamma, f)$  of type  $\Theta$ , let

$$\text{ev}_\Theta : \sigma_\Theta \rightarrow (\mathbb{R}^2)^n \times (\mathbb{R})^{\#\Phi},$$

be the product of all evaluation morphisms (both interior and boundary). The **multiplicity of  $(\Gamma, f)$**  is defined to be

$$\text{mult}(\Gamma, f) := \frac{1}{|\text{Aut}(f)|} \cdot \mathfrak{m}_{(\Gamma, f)} \cdot \prod_V \text{mult}_V(\Gamma, f) \cdot \det(\text{ev}_\Theta)$$

where the product runs over all vertices  $V$  of  $\Gamma$ .

**A note on the orientation.** Given a vertex  $v$  of a rigid tropical stable map  $(\Gamma, f)$ , the multiplicity of  $v$  is a combination of local multiplicities and logarithmic Gromov–Witten invariants. The discrete invariants of this Gromov–Witten invariant are determined by the local picture (i.e. the star) of  $v$  in  $\Gamma$ . In order to make this determination, it is necessary to know whether any given outgoing direction at  $v$  meets the boundary of the corresponding toric surface at a fixed point or a moving point. The need for this can be understood as follows. If  $v$  and  $v'$  are adjacent via an edge  $e$ , let  $t$  and  $t'$  be the flags along  $e$  based at  $t$  and  $t'$  respectively. In order to glue a map dual to  $v$  and a map dual to  $v'$ , one of the nodes dual to the flags  $t$  and  $t'$  must be constrained (i.e. incident to a fixed marked point) while the other is unconstrained. Further, the dimension count for the virtual dimension of the moduli spaces of logarithmic stable maps shows that for a vertex containing a marked point, all boundary conditions must be unconstrained, while for a vertex that does not host a marking exactly one boundary condition must be moving, as we show in Remark 3.8. The orientation above determines the unique way to make a consistent choice of boundary conditions, whereby incoming flags correspond to fixed conditions while outgoing flags correspond to moving conditions.

**Remark 3.8.** The only possibly non-vanishing local vertex multiplicities happen when the virtual dimension of the moduli space of logarithmic stable maps equals 0 in the case of an unmarked vertex, and  $k_i + 2$  for a vertex adjacent to the  $i$ -th mark. Let  $v$  denote a vertex whose star gives the Newton fan  $\delta$ . Let  $\delta_\phi \cup \delta_\mu = \delta$  be an arbitrary two-part partition of  $\delta$ , and let  $M_v$  the moduli space of logarithmic stable maps identified by this data. The virtual dimension is:

$$(7) \quad \text{virdim}(M_v) = g - 1 + \text{val}(v) - \ell(\phi)$$

If  $v$  is an unmarked vertex, using  $\text{val}(v) = \ell(\phi) + \ell(\mu)$ , it follows that for the virtual dimension of  $M_v$  to equal 0,

$$\ell(\mu) = 1 - g.$$





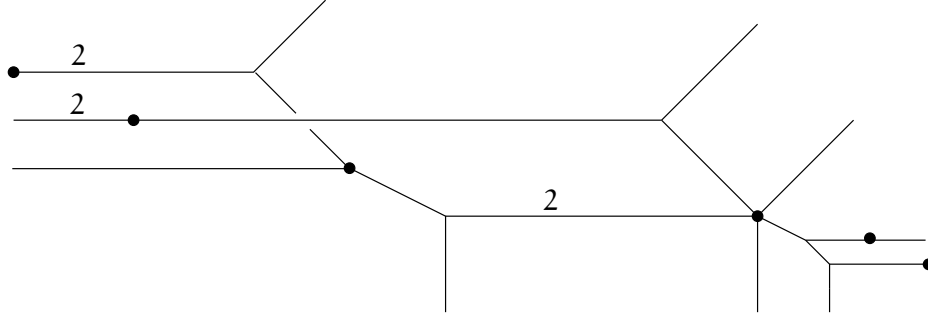


FIGURE 4. A tropical stable map to  $\mathbb{F}_1$  contributing to the invariant  $\langle((-2), (-2, -1)) | \tau_0(p_1) \tau_1(p_2) \tau_3(p_3) \tau_0(p_4) | ((1), (1)) \rangle_0^{\text{trop}}$  as in Example 3.11.

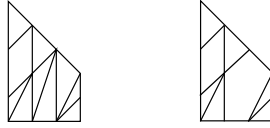


FIGURE 5. The dual subdivisions of the images of the stable maps of Example 3.11.

toric surface with hyperplane section the class of a curve of bidegree  $(a, \sum_{i|\varphi_i > 0} \varphi_i + \sum_{i|\mu_i > 0} \mu_i)$ . Figure 5 shows the dual Newton subdivisions of the images of the stable maps of Example 3.11.

**Remark 3.13.** Assume a tropical stable map  $(\Gamma, f)$  meets the constraints of a tropical descendant invariant  $\langle(\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(p_1) \dots \tau_{k_n}(p_n) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{trop}}$  as in Definition 3.3. The multiplicity with which it contributes contains the factors  $m_{(\Gamma, f)} \cdot \det(\text{ev}_\Theta)$  according to Definition 3.7. Assume away from marked points  $\Gamma$  has only trivalent vertices. Then the factor above equals the product of all (normalized) areas of triangles dual to the trivalent non-marked vertices in the dual subdivision, divided by the weights of fixed ends [22, 30, 29].

**Theorem 3.14** (Correspondence theorem). *Fix a Hirzebruch surface  $\mathbb{F}_k$  and discrete data as in Notation 2.2. The tropical stationary descendant log Gromov–Witten invariant coincides with its algebro-geometric counterparts, i.e. we have*

$$\langle(\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(pt) \dots \tau_{k_n}(pt) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g = \langle(\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(pt) \dots \tau_{k_n}(pt) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{trop}}$$

*Proof.* As mentioned in the introduction, this result can in principle be deduced from the results of the following sections of the paper – since the Gromov–Witten invariants are unchanged by variation of the point constraints, we may choose the constraints in such a way that the contributing tropical curves become floor decomposed. By a combinatorial argument to follow this is equal to the corresponding floor diagram count, via a degeneration to be implemented in the next section.

We sketch a direct proof, without specializing conditions, implementing the recent decomposition formula for virtual classes in Gromov–Witten theory, due to Abramovich, Chen, Gross, and Siebert [2]. We explain the geometric setup, and how to deduce the multiplicity above from the formulation in loc. cit. We assume that there are no fixed boundary conditions to lower the burden of the notation; the general case is no more complicated.

Consider the moduli space  $\overline{M}_{g, n+n_1+n_2}(\mathbb{F}_k, \delta_{(\underline{\phi}, \underline{\mu})})$ , and on it, the descendant cycle class given by  $\psi_1^{k_1} \dots \psi_n^{k_n}$ . We compute the invariant by degenerating the point conditions, and cutting down

the class

$$\psi_1^{k_1} \cdots \psi_n^{k_n} \cap [1]^{\text{vir}}$$

by evaluation morphisms. Working over  $\text{Spec}(\mathbb{C}((t)))$ , choose points  $p_1, \dots, p_n \in T \subset \mathbb{F}_k$ , whose tropicalizations  $p_1^{\text{trop}}, \dots, p_n^{\text{trop}}$  are in general position in  $\mathbb{R}^2$ . Since the tropical moduli space with the prescribed discrete data has only finitely many cones, it follows that there are finitely many rigid tropical stable maps meeting the stationary constraints. Suppose  $(\Gamma, f)$  is a tropical stable map with an end  $p_i$  incident to a vertex  $V$ . Since a point  $p_i$  must support the descendant class  $\psi_i^{k_i}$ , a dimension argument forces that the valency of  $V$  is  $k_j + 3 - g(V)$ . In other words, the tropical curves contributing to the count are precisely the ones outlined in Definition 3.3.

Enumerate the finitely many tropical stable maps  $(\Gamma_1, f_1), \dots, (\Gamma_r, f_r)$  contributing to the invariant

$$\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(p_1^{\text{trop}}) \cdots \tau_{k_n}(p_n^{\text{trop}}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{trop}}.$$

Choose a polyhedral decomposition  $\mathcal{P}$  of  $\mathbb{R}^2$  such that every tropical stable map  $f_i^{\text{trop}}$  factors through the one-skeleton of  $\mathcal{P}$  and that the fan of unbounded directions of  $\mathcal{P}$  (i.e. the recession fan) is the fan  $\Sigma_k$ . Note that  $\mathcal{P}$  can always be chosen to be a common refinement of the images of  $f_i^{\text{trop}}$ . The contact order conditions on the tropical maps ensure that the recession fan is  $\Sigma_k$ .

The polyhedral decomposition  $\mathcal{P}$  determines a toric degeneration  $\mathcal{X}$  of  $\mathbb{F}_k$ , over  $\text{Spec}(\mathbb{C}[[t]])$ , see [26]. By the deformation invariance property of logarithmic Gromov–Witten invariants, we may compute  $\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(pt) \cdots \tau_{k_n}(pt) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g$  on the central fiber of this degeneration, as

$$ev^*(p) \cap \psi_1^{k_1} \cdots \psi_n^{k_n} \cap [1]^{\text{vir}},$$

where

$$ev : \overline{M}_{g, n+n_1+n_2}(\mathcal{X}, \delta_{(\underline{\phi}, \underline{\mu})}) \rightarrow \mathcal{X}^n,$$

is the product of all evaluation morphisms, and  $p$  is the specialization of the point  $(p_1, \dots, p_n)$  chosen above.

By applying the decomposition formula for logarithmic Gromov–Witten invariants for point conditions [2, Theorem 6.3.9], this invariant can be written as a sum of the invariants associated to each tropical curve. In order to calculate these, we now use the special choice of degeneration  $\mathcal{X}$ . Fix a rigid tropical stable map  $(\Gamma, f)$  and let  $v$  be a vertex of  $\Gamma$ . Consider the tropical map

$$\text{Star}(v, \Gamma) \rightarrow \text{Star}(f(v), \mathcal{P}).$$

This determines the discrete data of a logarithmic stable map space to the toric variety  $X_v$  dual to  $\text{Star}(f(v), \mathcal{P})$ . Denote this moduli space by  $\mathcal{L}(v)$ . Given logarithmic stable maps  $C_v \rightarrow X_v$  for each vertex  $v$ , the underlying maps can be glued. We claim that every such glued map can be enhanced to a logarithmic map. To see this, observe that after possibly performing a logarithmic modification of  $\mathcal{X}$ , we can arrange that in any such glued map

$$\varphi : \cup_v C_v \rightarrow \cup_v X_v,$$

each irreducible component is torically transverse to the strata of  $\mathcal{X}_0$  and the nodes of  $C$  are pre-deformable. That is, for any point  $p \in C$  that is a node mapping to a double curve  $D$  of the special fiber  $\mathcal{X}_0$ , the intersection multiplicity of the two branches of  $C$  with  $D$  are equal. That such maps lift to logarithmic morphisms is now immediate from [33, Proposition 7.1] or [2, Theorem 5.4.1]. Note that by virtual birational invariance [3], the logarithmic modifications made in the process of ensuring toric transversality have no effect on the argument. By our choice of degeneration, given any such map  $C_v \rightarrow X_v$ ,  $C_v$  is torically transverse to the codimension 2 strata of the degeneration.



Gluing the curves  $C_v \rightarrow X_v$  ranging over all  $v$ , the transversality implies that there exist logarithmic lifts of all such stable maps. To count the number of such logarithmic lifts we may apply [2, Theorem 5.5.1]. The fact that this number is equal to the product

$$m_{(\Gamma, f)} \cdot \det(\text{ev}_\Theta)$$

is a well-known but tedious calculation in tropical geometry. A careful proof is recorded in [33, Proposition 8.8] and [19, Section 5]. Since we will provide a more practical formula in a later section, we leave these details to the reader. Taking into account the automorphism factors and summing over all tropical curves, the result follows.  $\square$

**Remark 3.15.** The multiplicity in Definition 3.3 and the correspondence principle in Theorem 3.14 collapses substantially in special cases to the following two tropical curve counts studied in [30, 31].

- (1) **If all  $\psi$ -powers are 0, i.e.  $k_1 = \dots = k_n = 0$ :** the valency condition implies that the vertex adjacent to end  $i$  is trivalent and of genus 0. Since the end  $i$  is contracted, the image of a neighbourhood of this vertex just looks like an edge passing through  $p_i$ . We thus count plane tropical curves passing through the points (and possibly with some fixed  $y$ -coordinates for the ends). An example can be found in Example 3.11, see Figure 3. They are counted with multiplicity equal to the product of the normalized areas of the triangles in the dual subdivision (notice that all vertices are trivalent and of genus 0 for dimension reasons). In case of fixed  $y$ -coordinates, the product above has to be multiplied in addition with  $\prod_e \frac{1}{w(e)}$ , where the product goes over all fixed ends  $e$  and  $w(e)$  denotes their expansion factor [20]. This equals the multiplicity defined in Definition 3.7 because of Remark 3.13. That all local Gromov–Witten invariants are 1 follows from the correspondence theorem proved in [33, 36].
- (2) **If the genus  $g = 0$ :** just as in [30], we count stable maps satisfying point (and end) conditions, and higher valency conditions according to the  $\psi$ -powers. In [30], they are counted with multiplicity equal to the product of the normalized areas of the triangles (dual to non-marked vertices) in the dual subdivision as above, with a factor of  $\prod_e \frac{1}{w(e)}$  for fixed ends, see [6]. The correspondence theorem for such invariants is proved in the papers [23, 36] and [28].

#### 4. FLOOR DIAGRAMS

Floor diagrams provide a combinatorial connection between the classical and tropical computations of descendant Gromov–Witten invariants of Hirzebruch surfaces. On the tropical side, the images of tropical stable maps become floor decomposed by choosing horizontally stretched point conditions. On the classical side, floor diagrams naturally organize the computation of a Gromov–Witten invariant via degeneration formula. We make here a definition of floor diagrams which is especially adapted to our context and needs, and follow it by a brief discussion of its connection to previous work on the subject (Section 4.1). We prove correspondences of floor diagram weighted counts with classical descendant Gromov–Witten invariants in Section 4.2, and with tropical maps counts in Section 4.3.

**Definition 4.1.** Let  $D$  be a loop-free connected graph on a linearly ordered vertex set.  $D$  has two types of edges: compact edges, composed of two flags (or half-edges), adjacent to different vertices, and unbounded edges, also called ends, with only one flag.  $D$  is called a **floor diagram** for  $\mathbb{F}_k$  of degree  $(\underline{\phi}, \underline{\mu})$  if:

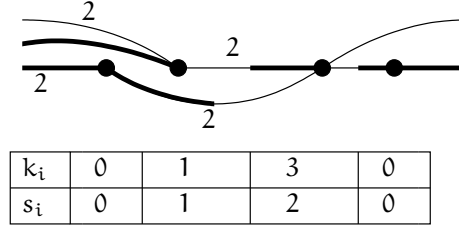


FIGURE 6. An example of a floor diagram. The genus at all vertices is 0.

- (1) Three non-negative integers are assigned to each vertex  $V$ :  $g_V$  (called the **genus** of  $V$ ),  $s_V$  (called the **size** of  $V$ ) and  $k_V$  (called the  **$\psi$ -power** of  $V$ ).
- (2) Each flag may be decorated with a thickening. We require that for each compact edge precisely one of its two half-edges is thickened.
- (3) At each vertex  $V$ ,  $k_V + 2 - 2s_V - g_V$  adjacent half-edges are thickened.
- (4) Each edge  $e$  comes with an **expansion factor**  $w(e) \in \mathbb{N}_{>0}$ .
- (5) At each vertex  $V$ , the signed sum of expansion factors of the adjacent edges (where we use negative signs for edges pointing to the left and positive signs for edges pointing to the right) equals  $-k \cdot s_V$ .
- (6) The sequence of expansion factors of non-thick ends (where we use negative signs for the ends pointing to the left and positive signs for the ends pointing to the right) is  $(\underline{\phi})$ , and the sequence of expansion factors of thickened ends (with the analogous sign convention) is  $(\underline{\mu})$ .
- (7) The ends of the graph are marked by the parts of  $(\underline{\phi}, \underline{\mu})$ .

The **genus** of a floor diagram is defined to be the first Betti number of the graph plus the sum of the genera at all vertices.

**Example 4.2.** Figure 6 shows a floor diagram for  $\mathbb{F}_1$  of degree  $((-2, 1), (-2, -1, 1))$  and genus 0.

**Definition 4.3.** Given a floor diagram for  $\mathbb{F}_k$ , let  $V$  be a vertex of genus  $g_V$ , size  $s_V$  and with  $\psi$ -power  $k_V$ . Let  $(\underline{\phi}_V, \underline{\mu}_V)$  denote the expansion factors of the flags adjacent to  $V$ ; the first sequence encodes the normal half edges, the second the thickened ones. We define the multiplicity  $\text{mult}(V)$  of  $V$  to be the one-point stationary descendant invariant

$$\text{mult}(V) = \langle (\underline{\phi}_V^-, \underline{\mu}_V^-) | \tau_{k_V}(\text{pt}) | (\underline{\phi}_V^+, \underline{\mu}_V^+) \rangle_{g_V}.$$

**Definition 4.4.** Fix discrete data as in Notation 2.2. We define:

$$\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}) \dots \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{floor}}$$

to be the weighted count of floor diagrams  $D$  for  $\mathbb{F}_k$  of degree  $(\underline{\phi}, \underline{\mu})$  and genus  $g$ , with  $n$  vertices with  $\psi$ -powers  $k_1, \dots, k_n$ , such that  $\mathbf{a}$  equals the sum of all sizes of vertices,  $\mathbf{a} = \sum_{V=1}^n s_V$ .

Each floor diagram is counted with multiplicity

$$\text{mult}(D) = \prod_{e \in \text{C.E.}} w(e) \cdot \prod_V \text{mult}(V),$$

where the second product is over the set C.E. of compact edges and  $w(e)$  denotes their expansion factors; the third product ranges over all vertices  $V$  and  $\text{mult}(V)$  denotes their multiplicities as in Definition 4.3.

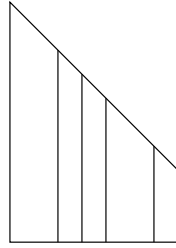


FIGURE 7. The subdivision of a floor decomposed tropical curve refines the sliced Newton polygon. Each strip corresponds to a floor, and the integral width of the strip is called the size of the floor.

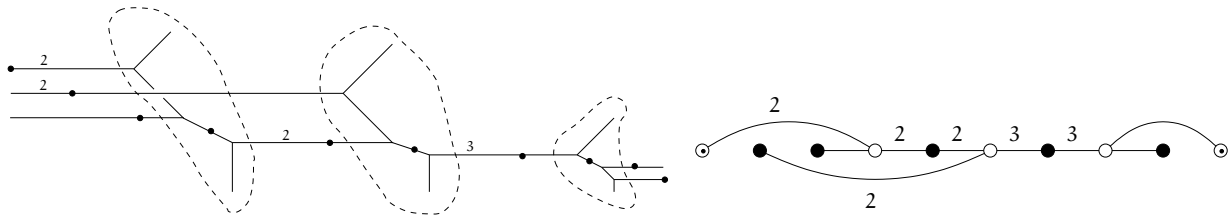


FIGURE 8. The floors in the tropical stable map of Example 3.11(1), and the corresponding floor diagram.

**4.1. Motivation and relation to other work.** Floor diagrams were introduced for counts of curves in  $\mathbb{P}^2$  by Brugallé-Mikhalkin [9], and further investigated by Fomin-Mikhalkin [18], leading to new results about node polynomials. The results were generalized to other toric surfaces, including Hirzebruch surfaces, in [4].

The main observation is that by picking *horizontally stretched point conditions*, the images of tropical stable maps contributing to a Gromov–Witten invariant become floor decomposed: this means that the dual subdivision of the Newton polygon is *sliced* (i.e. a refinement of a subdivision of the trapezoid by parallel vertical lines — see Figure 7). Floor diagrams are then obtained by shrinking each **floor** (i.e. a part of the plane tropical curve which is dual to a (Minkowski summand of a) slice in the Newton polygon) to a white vertex. Each floor contains precisely one marked point. Further marked points lie on horizontal edges which connect floors, the so-called *elevators*<sup>1</sup>, and are represented with black vertices. Fixed horizontal ends are given a (double circled) vertex, while other horizontal ends are shrunk so that the diagram has no unbounded edges.

**Example 4.5.** In Figure 8 we revisit the tropical stable map observed in the first part of Example 3.11. The floors are circled by dashed lines. On the right-hand side we have the corresponding floor diagram. Following the convention in [6], fixed ends terminate with a double circle, and other ends are contracted to the corresponding black vertex.

For rational stationary descendant Gromov–Witten invariants, the floor diagram technique was studied by Block, Gathmann and the third author [6] (the diagrams are called  $\psi$ -**floor diagrams**). There are two main difference with respect to the primary case:

- descendant insertions force us to consider floor decomposed curves with floors of size larger than one. The size of a floor thus becomes part of the data of a floor diagram: each

<sup>1</sup>The bizarre nomenclature makes intuitive sense if everything is rotated by 90°.

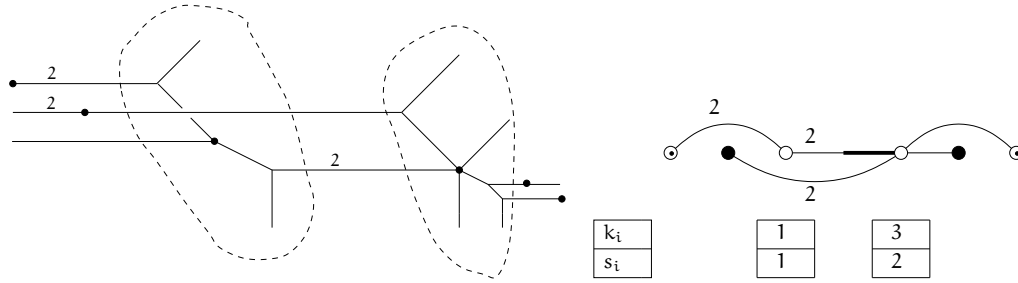


FIGURE 9. The floors in the tropical stable map of Example 3.11(2), and the corresponding floor diagram. The numbers below the white vertices indicate the  $\psi$ -power  $k_i$  of the marked point in the floor, and the size  $s_i$  of the floor.

floor vertex comes with two numbers, the  $\psi$ -power  $k_i$  of the corresponding marked point, and the size of the floor;

- marked points may now be supported at a vertex of the tropical stable map, and horizontal edges incident to such vertex are fixed by the point condition. This condition is encoded by thickening the corresponding half-edges in the floor diagram.

**Example 4.6.** Figure 9 illustrates the second part of Example 3.11. Some half-edges are thickened, indicating that the corresponding edge in the tropical curve leading to this diagram is adjacent to the marked point in the floor.

The language in [6] was modeled after the work by Fomin–Mikhalkin [18], which was motivated by a computational approach aiming at new results about node polynomials. Our current motivation to study floor diagrams comes from their connections to degeneration techniques and Fock space formalisms to enumerative geometry. Hence our definitions introduce the following modifications with respect to [18, 6]:

- (1) The distinction between floors and marked points on elevators is not needed anymore (to the contrary, it only complicates the combinatorics and clouds the connection to the Fock space). We do away with bi-colored vertices by considering marked points on elevators as floors of size zero. The adjacent half-edges have to be thickened, since they are adjacent to the marked point.
- (2) We thicken ends that correspond to a tangency at a non-fixed point, have unthickened ends for tangency to a fixed point (rather than marking the end with a double circle) and remove the vertices at the end of these edges.
- (3) We draw all elevator edges adjacent to marked points, as that allows us to record the complete tangency data for the invariant we are trying to compute (in the convention of [18, 6], obvious continuations of edges in the tropical curve are dropped in the floor diagram).

As an example, the floor diagram of Figure 9 becomes with our conventions the diagram in Figure 6.

**4.2. Floor Diagrams and Degeneration.** While the classical/tropical correspondence theorem relies on maximal degenerations and the logarithmic degeneration formula, the correspondence to floor diagrams follows from the simpler “accordion” degeneration, as originally observed and discussed in [8, 5]. We note here a choice is present in which enumerative problem to study. One may study the geometry  $\mathbb{F}_k$  relative to the toric boundary, fixing transverse contact orders along 0 and

$\infty$  fibers, and use the decomposition formula in [2]. Alternatively, one may study the geometry of  $\mathbb{F}_k$  relative only to the 0 and  $\infty$  sections, rather than the full toric boundary. In the latter case, since the relative conditions are at smooth divisors, the simpler degeneration formula due to Jun Li [27] is sufficient. We record a proof in the latter case, noting that the proof in the former case follows mutatis mutandis. We recall Li's theorem, stated in the specific geometric context that is of interest to us.

**Theorem 4.7.** *Let  $\mathcal{X}_t$  be a flat family of surfaces such that the general fiber is a smooth Hirzebruch surface  $\mathbb{F}_k$  and the central fiber is the union of two surfaces  $S_1 \cup_D S_2$  both isomorphic to  $\mathbb{F}_k$ , meeting transversely along the divisor  $D = E_{S_1} = B_{S_2}$ . Fix a two part partition of the set  $[n + m]$ : without loss of generality we may choose  $\{1, \dots, n\} \cup \{n + 1, \dots, n + m\}$ . Set discrete invariants  $g, n, k_1, \dots, k_n, (\underline{\phi}, \underline{\mu})$  as in Notation 2.2.*

Then:

$$(8) \quad \langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}), \dots, \tau_{k_{n+m}}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^\bullet = \sum \frac{\prod \lambda_i \eta_j}{|\text{Aut}(\underline{\lambda})| |\text{Aut}(\underline{\eta})|} \\ \langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}), \dots, \tau_{k_n}(\text{pt}) | (\underline{\lambda}, \underline{\eta}) \rangle_{g_1}^\bullet \langle (-\underline{\eta}, -\underline{\lambda}) | \tau_{k_{n+1}}(\text{pt}), \dots, \tau_{k_{n+m}}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_{g_2}^\bullet,$$

where  $\underline{\lambda} = \lambda_1, \dots, \lambda_r, \underline{\eta} = \eta_1, \dots, \eta_s$  are an  $r$ -tuple and an  $s$ -tuple of positive integers and the sum is over all discrete data  $(g_1, g_2, (\underline{\eta}, \underline{\lambda}))$  such that:

- (1)  $((\underline{\phi}^-, \underline{\lambda}), (\underline{\mu}^-, \underline{\eta}))$  (resp.  $((-\underline{\eta}, \underline{\phi}^+), (-\underline{\lambda}, \underline{\mu}^+))$ ) determines an effective curve class  $a_1 B_{S_1} + b_1 F_{S_1}$  (resp.  $a_2 B_{S_2} + b_2 F_{S_2}$ ) in  $H_2(\mathbb{F}_k, \mathbb{Z})$  with  $a_1, a_2 \geq 0, a_1 + a_2 = a, b_1 = a_2 k + b, b_2 = b$ ;
- (2)  $g = g_1 + g_2 + r + s - 1$ .

**Remark 4.8.** The following details are important in parsing Equation (8):

- (1) The formula is organized as a sum over the gluing data  $(\underline{\lambda}, \underline{\eta})$ . Each term in the summand is however weighted by a factor of  $\frac{1}{|\text{Aut}(\underline{\lambda})| |\text{Aut}(\underline{\eta})|}$ , which corrects the overcounting coming from different labelings of points that give rise to the same gluing. More geometrically, one may think that in Equation (8) the sum is over the distinct topological types of maps (where the points that get glued are unlabeled), and the multiplicity of each summand omits the above factor.
- (2) The switching of the roles of  $\underline{\lambda}$  and  $\underline{\eta}$  on the two sides of the product comes from the Kunneth decomposition of the class of the diagonal in  $\mathbb{P}^1 \cong D$ .

To realize the hypotheses of the theorem, one may start from a trivial family  $\mathbb{F}_k \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$  together with  $n + m$  non intersecting sections  $s_v$ , the first  $n$  staying away from  $E$ , the last  $m$  meeting but not tangent to  $E$  at  $t = 0$ ; one obtains  $\mathcal{X}_t$  by blowing up  $E \times \{0\}$  and considering the proper transforms of the sections. This construction may be iterated a finite number of times, and Theorem 4.7 applies with the appropriate bookkeeping. This is what gives rise to the correspondence with the floor diagram count, as we make explicit in the next theorem.

**Theorem 4.9.** *Fix a Hirzebruch surface  $\mathbb{F}_k$  and discrete data as in Notation 2.2. The descendant log Gromov–Witten invariant coincides with the weighted count of floor diagrams from Definition 4.4*

$$(9) \quad \langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}), \dots, \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\bullet, \text{floor}} = \langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}), \dots, \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^\bullet.$$

*Proof.* As with many proofs based on iterated applications of the degeneration formula, a completely explicit and accurate bookkeeping would be extremely cumbersome and cloud the actual

simplicity of the argument. We choose therefore to carefully outline the construction, and omit the bookkeeping.

Iterate the construction from the previous paragraph  $n - 1$  times, each time separating exactly one section from all others. In the end one obtains a family  $\mathcal{X}_t$  such that the general fiber is a smooth Hirzebruch surface  $\mathbb{F}_k$  and the central fiber is the union of  $n$  surfaces  $X_0 = S_1 \cup_{D_1} S_2 \cup_{D_2} \dots \cup_{D_{n-1}} S_n$ , where all surfaces  $S_i$  are isomorphic to  $\mathbb{F}_k$ , and  $S_i$  and  $S_{i+1}$  meet transversely along the divisor  $D_i = E_{S_i} = B_{S_{i+1}}$ . For  $V = 1, \dots, n$  the section  $s_V$  is obtained as the proper transform of the original section.

Applying the appropriately iterated version of Theorem 4.7, the stationary descendant invariant is expressed as a sum over the topological types of maps from nodal curves to the central fiber, weighted by the appropriate (disconnected) relative Gromov–Witten invariants. Since each  $S_i$  contains exactly one marked point, the disconnected maps to  $S_i$  have one connected component hosting the marked point; by dimension reasons, the other components consist of rational curves mapping with degree  $dF$  (multiple of the class of a fiber), and in fact mapping as a  $d$ -fold cover of a fiber, fully ramified at the points of contact with  $D_{i-1}$  and  $D_i$ . Further, the relative conditions at the boundary must have one fixed point on one side, and a moving point on the other. The contribution of any such component to the disconnected invariant is  $1/d$ .

For every summand in the degeneration formula, consider the dual graph of the source curve, label each edge with the ramification order of the corresponding point, and thicken half edges corresponding to moving boundary point conditions. For every two-valent vertex adjacent to two flags of opposite thickening, contract the vertex and the two neighboring flags. We claim (and leave the verification to the patient reader) that the object thus obtained is a floor diagram for the stationary descendant invariant we are trying to compute, and further that this construction establishes a bijection between the summands in the degeneration formula and the floor diagrams described in Definition 4.4.

The proof is concluded by showing that each floor diagram is counted with the same multiplicity. The degeneration formula assigns the same vertex and compact edge multiplicities to the dual graphs of maps as the floor diagram enumerative count. The proof is then concluded by noticing that the operation of removing a two-valent vertex (which contributes with multiplicity  $1/d$ ) and its two adjacent flags does not alter the multiplicity of the graph: for each such vertex removed we lose a compact edge of weight  $d$ , which contributes a factor of  $d$  to the multiplicity of the graph.  $\square$

Since the proof of the correspondence is based on a bijection between dual graphs of maps and floor diagrams that preserves connectedness, one immediately obtains the following corollary.

**Corollary 4.10.** *The version of Theorem 4.9 for connected invariants also holds:*

$$(10) \quad \langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}), \dots, \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{floor}} = \langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}), \dots, \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g.$$

**4.3. Floor diagrams and tropical curves.** In this section, we present a direct weighted bijection between the counts of tropical stable maps and floor diagrams (relying on Theorem 3.14). Even though such equality is a corollary of Theorems 3.14 and 4.9, the direct proof provides valuable intuition on the connection between tropical stable maps and floor diagrams.

**Theorem 4.11.** *Fixing all discrete invariants as in Notation 2.2, the weighted count of floor diagrams equals the tropical descendant log Gromov–Witten invariant, i.e. we have*

$$(11) \quad \langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}) \dots \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{floor}} = \langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}) \dots \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{trop}}.$$



*Proof.* The proof of this theorem is in two parts. Construction 4.12 associates a floor diagram to a floor decomposed tropical stable map contributing to  $\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}) \dots \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{trop}}$ . By Proposition 4.20, the weighted number of tropical stable maps yielding the same floor diagram  $D$  under this procedure equals the multiplicity  $\text{mult}(D)$  from Definition 4.4.  $\square$

Let  $(\Gamma, f)$  be a rigid floor decomposed tropical stable map contributing to

$$\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}) \dots \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{trop}}.$$

Because of the horizontally stretched point conditions, each marked point is either on a horizontal edge (resp. **elevator edge**) of  $f(\Gamma)$  (i.e. an edge of primitive direction  $(1, 0)$ ) or on a part dual to a slice in the Newton subdivision. On each part dual to a slice, there is exactly one marked point. Consider the preimage in  $\Gamma$  under  $f$  of a part dual to a slice, this is a subgraph that we call  $\Gamma'$ . Assume the slice in the Newton polygon has width  $s > 0$  (i.e. in plane coordinates, it is a slice between the lines  $\{x = i\}$  and  $\{x = i + s\}$  for some  $i$ ). Since the image of  $\Gamma'$  is fixed by exactly one point (and conditions on the coordinates of its horizontal edges),  $\Gamma'$  consists of only rational connected components. Furthermore, all but one of these components is just one edge which is mapped horizontally. This connected component (which contains  $s$  ends of direction  $(0, -1)$  and  $s$  ends of direction  $(k, 1)$ ) is called a **floor of size**  $s$ . We refer to other connected components as horizontal edges passing through the floor. For an example, see Figures 8 and 9.

**Construction 4.12.** Let  $(\Gamma, f)$  be a (non-superabundant) floor decomposed tropical stable map contributing to  $\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}) \dots \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{trop}}$ .

We associate a floor diagram  $D$  contributing to  $\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}) \dots \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{floor}}$  to  $(\Gamma, f)$  by contracting each floor to a vertex; also marked points adjacent to only horizontal edges are considered vertices. The vertices are equipped with:

- the  $\psi$ -power  $k_i$  of the adjacent marked point  $i$ ,
- the size  $s_i$  (i.e. the width) of the dual slice of the Newton polygon for vertices corresponding to a floor;  $s_i = 0$  for marked points on elevators,
- the genus  $g_i$  of the vertex adjacent to the marked end  $i$  in the tropical curve.

We thicken flags if they come from half-edges of  $f(\Gamma)$  which are adjacent to a marked point.

*Proof.* We show that Construction 4.12 yields a floor diagram of the right degree and genus. Because of the horizontally stretched point conditions, we obtain a graph  $D$  on a linearly ordered vertex set.

The balancing condition satisfied by  $(\Gamma, f)$  implies that the signed sum of expansion factors of edges adjacent to vertex  $i$  of the floor diagram equals  $-k \cdot s_i$ .

By Lemma 3.4, removing from the subgraph underlying a floor of size  $s_i$  the marked end  $i$  together with its end vertex yields connected components each containing at most one of the  $2s_i$  ends of direction  $(0, -1)$  resp.  $(k, 1)$ . It follows that the valence of the vertex adjacent to the  $i$ -th mark is  $2s_i$  plus one (for the marked end itself) plus the number of adjacent horizontal edges. The latter correspond to the thick flags in the floor diagram  $D$ . Thus at vertex  $i$  of  $D$ ,  $(k_i - g_i + 3) - 1 - 2s_i$  edges are thickened, as required. Furthermore, each horizontal edge of  $\Gamma$  must be fixed, either by a condition on the  $y$ -coordinates of ends, or by a marked point. It cannot be fixed more than once because of the genericity of the conditions. It follows that every edge of the associated floor diagram  $D$  has precisely one thickened flag, as required. Since all floors of  $(\Gamma, f)$  are rational, the

genus of  $D$  is  $g$ . Obviously, the degree of  $D$  is  $(\underline{\phi}, \underline{\mu})$ . Thus  $D$  is a floor diagram contributing to  $\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(pt) \dots \tau_{k_n}(pt) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{floor}}$ . □

**Remark 4.13.** If we consider the case  $k_i = 0$  for all  $i$ , then the tropical descendant invariant considered above is nothing but a count of tropical plane curves satisfying point conditions. For such a count, it is well-known, see for instance [31], that all tropical stable maps  $(\Gamma, f)$  that contribute with nonzero multiplicity have  $g$  cycles which are visible in the image  $f(\Gamma)$  and have only trivalent vertices. It then follows that their spaces of deformation have the expected dimension, because the  $g$  visible cycles impose  $2g$  linearly independent conditions in the orthant parametrizing all lengths on bounded edges. Hence superabundance is no issue for tropical plane curve counts.

When descendant insertions are allowed, this is no longer true: even if all cycles are visible in the image, they do not need to impose linearly independent conditions. The existence of superabundant tropical stable maps satisfying the conditions implies the existence of rigid tropical stable maps with “additional overvalency”, e.g. a 4-valent vertex which is not adjacent to a marked point (see Example 3.10 in [21]).

In the following, we exclude such behavior for the case of floor-decomposed tropical stable maps. This is needed when considering the equality of the count of floor diagrams to the corresponding tropical descendant Gromov–Witten invariant.

**Lemma 4.14.** *Let  $(\Gamma, f)$  be a rigid floor decomposed tropical stable map, satisfying horizontally stretched conditions. Assume  $g'$  independent cycles are visible in the image  $f(\Gamma) \subset \mathbb{R}^2$ , then these cycles impose  $2g'$  linearly independent conditions, i.e. the space of deformations of  $(\Gamma, f)$  is of codimension  $2g'$  in the orthant parametrizing all lengths on bounded edges of  $\Gamma$ .*

*Proof.* Since  $(\Gamma, f)$  is rigid, there cannot be cycles contained in a floor (each floor is fixed by only one point condition). The  $g'$  cycles thus have to be between two floors each, and thus each involve at least two elevator edges. We can put an arbitrary order on the pairs of floors, and then on the cycles involving the same two floors as imposed by the maximal  $y$ -coordinates of the elevator edges. In this way, we can produce an upper triangular matrix cutting out the space of deformations of  $(\Gamma, f)$  from the orthant parametrizing all lengths of bounded edges, which consequently is of codimension  $2g'$  as expected. □

**Lemma 4.15.** *If a floor decomposed  $(\Gamma, f)$  contributes with non-zero multiplicity to a descendant Gromov–Witten invariant, then all cycles of  $\Gamma$  are visible in  $f(\Gamma)$  (i.e. no cycles are mapped to a line segment or contracted to a point).*

*Proof.* Since we assume that  $(\Gamma, f)$  contributes with non-zero multiplicity, it has to be rigid. If a cycle of  $\Gamma$  was contracted to a point, then  $(\Gamma, f)$  would not be rigid because the lengths of edges of the contracted cycle can be varied without changing the image  $f(\Gamma)$ . We could vary  $(\Gamma, f)$  in an at least one-dimensional family still meeting the point and  $y$ -coordinate conditions.

If a cycle of  $\Gamma$  is mapped to a line segment  $S$ , it has to be an elevator (floors are rational for rigid stable maps, they are fixed by just one point). Let  $V$  be a vertex of the cycle mapping to an endpoint of  $S$ . We may assume that no marking is incident to the vertex  $V$ : by the genericity of the incidence conditions at most one cycle vertex may be marked, and since we ruled out completely contracted cycles, the segment  $S$  has two distinct endpoints.

Now we argue that  $\Gamma$  does not contribute to the descendant Gromov–Witten invariant. If  $V$  is trivalent, it follows by the balancing condition that all three edges are mapped to the same line.



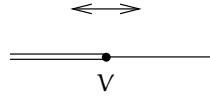


FIGURE 10. Tropical maps with a trivalent vertex at the end of a hidden cycle are not rigid.

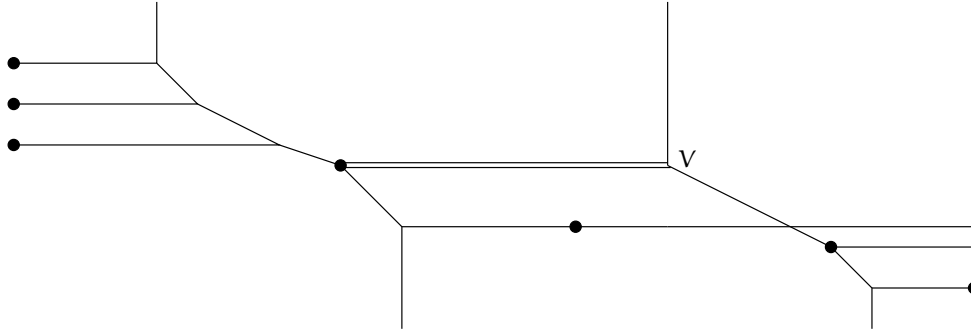


FIGURE 11. A rigid superabundant tropical stable map of multiplicity zero.

The lengths of the three edges may be varied in such a way that the image  $f(V)$  moves along the line, but  $f(\Gamma)$  remains unchanged (see Figure 10). The resulting tropical stable maps would still satisfy all the conditions. Thus  $(\Gamma, f)$  is not rigid.

If  $\text{val}(V) \geq 4$ , the local multiplicity at  $V$  is 0 by the dimension count in Remark 3.8, since there are at least two legs giving unconstrained boundary conditions. Hence for  $(\Gamma, f)$  to contribute with a non-zero multiplicity, it cannot have a cycle which is mapped to a line segment: all cycles of  $\Gamma$  are visible in  $f(\Gamma)$ .

□

**Example 4.16.** Figure 11 shows an example of a rigid superabundant tropical stable map which has multiplicity zero. This is the case because the multiplicity of the vertex  $V$  is zero for dimension reasons. The Figure is supposed to reflect both the image of the stable map and the parametrizing abstract graph — we draw two edges close together if their images in  $\mathbb{R}^2$  coincide.

Combining Lemmas 4.14 and 4.15, we deduce the following non-trivial fact:

**Corollary 4.17.** *For horizontally stretched point conditions leading to floor-decomposed tropical stable maps, any  $(\Gamma, f)$  that contributes with non-zero multiplicity to a tropical descendant Gromov–Witten invariant is not superabundant.*

*Proof.* From Lemma 4.15 we can conclude that all cycles of  $\Gamma$  are visible in the image  $f(\Gamma)$ . From Lemma 4.14 we can conclude that they form independent conditions. It follows that the space of deformations of  $(\Gamma, f)$  is of the expected dimension, and hence  $(\Gamma, f)$  is not superabundant. □

**Remark 4.18.** For tropical stable maps to  $\mathbb{R}^n$  with  $n \geq 3$ , there is no analogous statement known, i.e. it is not known whether there is a configuration of points such that all tropical stable maps (of non-zero multiplicity) satisfying the conditions are not superabundant, or even if there is a configuration of points forcing all cycles to be visible, which is a much weaker condition. In fact, analogous geometries for the Gromov–Witten theory of  $\mathbb{P}^3$  suggest that no such configurations exist.

**Lemma 4.19.** *Let  $(\Gamma, f)$  be a floor-decomposed tropical stable map contributing to a tropical descendant Gromov–Witten invariant with non-zero multiplicity. Then every vertex of  $\Gamma$  which is not adjacent to a marked end is trivalent.*

*Proof.* Let  $(\Gamma, f)$  be a floor-decomposed tropical stable map with non-zero multiplicity. Assume the marked points with  $\psi$ -conditions  $k_1, \dots, k_n$  are at vertices of genus  $g_1, \dots, g_n$ , and accordingly, of valence  $k_i + 3 - g_i$ .

By Corollary 4.17,  $(\Gamma, f)$  is not superabundant. The number of edges in the graph  $\Gamma$  is

$$n + n_1 + n_2 + 2a - 3 + 3(g - g_1 - \dots - g_n) - \sum_V (\text{val}(V) - 3),$$

which follows from an Euler characteristic computation. The space of deformations of  $(\Gamma, f)$  has dimension:

$$\begin{aligned} & 2 + \#\{\text{edges}\} - 2 \cdot \#\{\text{visible cycles}\} \\ &= n + n_1 + n_2 + 2a - 1 + (g - g_1 - \dots - g_n) - \sum_V (\text{val}(V) - 3) \\ &= n + n_1 + n + \sum_i (k_i - g_i) - \sum_V (\text{val}(V) - 3) \end{aligned}$$

by the requirement on the conditions. Since  $\sum_V (\text{val}(V) - 3) = \sum_i (k_i - g_i) + \sum_{V'} (\text{val}(V') - 3)$  (where now the sum goes over all vertices  $V'$  which are not adjacent to one of the marked ends  $i$ ) by the valency conditions, and since the  $y$ -coordinates of  $n_1$  ends are fixed and  $n$  generic point conditions are satisfied, the dimension has to be at least  $n_1 + 2n$ , which can only be satisfied if any vertex besides the ones adjacent to the marked ends, is trivalent.  $\square$

**Proposition 4.20.** *Let  $D$  be a floor diagram contributing to*

$$\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}) \dots \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{floor}}.$$

*The weighted number of tropical stable maps contributing to*

$$\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}) \dots \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{trop}}$$

*that yield  $D$  under the procedure described in Construction 4.12 equals  $\text{mult}(D)$ .*

*Proof.* Let  $(\Gamma, f)$  be a tropical stable map that yields  $D$  using Construction 4.12.

Notice first that if  $(\Gamma, f)$  contributes with multiplicity 0 (as e.g. the one in Figure 11), then also  $D$  contributes with multiplicity 0: the floor containing the vertex of multiplicity 0 also has multiplicity 0. Vice versa, if  $D$  has a floor of multiplicity 0, any tropical stable map producing  $D$  must have a vertex of multiplicity 0.

So we can assume now that  $(\Gamma, f)$  is of non-zero multiplicity. In particular it is rigid, and not superabundant, and all its non-marked vertices are trivalent by the above. Using our convention of marking horizontal ends, it follows also that  $(\Gamma, f)$  has no nontrivial automorphisms, since it cannot have edges which are not distinguishable.

Following Definition 3.7 and using Remark 3.13,  $(\Gamma, f)$  contributes a product of

- (1) areas of triangles dual to non-marked vertices and factors  $\frac{1}{w}$  for the weights of fixed ends, and
- (2) local vertex multiplicities  $\text{mult}_V(\Gamma, f)$ .

Every compact edge  $e$  of  $D$  of weight  $w(e)$  comes from a bounded edge  $e'$  of  $\Gamma$  of weight  $w(e)$ . Since  $e$  has precisely one non-thickened flag,  $e'$  is adjacent to precisely one trivalent vertex  $V$  not adjacent to a marked point (see Lemma 4.19). Denote by  $e''$  an edge in the floor which is adjacent to  $V$ . Every non-horizontal edge in a floor is of direction  $(0, 1) + c \cdot (1, 0)$  for some  $c$  (by the balancing condition, the fact that the floor is rational, and since we can connect every edge to an end of direction  $(0, -1)$ ), and so the area of the triangle dual to  $V$  (formed by the duals of  $e'$  and  $e''$ ) is  $w(e)$ .

A non-fixed end of  $\Gamma$  has to be adjacent to a marked point by rigidity, so it is not adjacent to a trivalent vertex as above. A fixed end of  $\Gamma$  is adjacent to a trivalent vertex whose dual triangle has area  $w(e)$  by the above.

Altogether we can see that the first item above — the product over all areas of triangles dual to non-marked vertices in the dual subdivision of  $(\Gamma, f)$  divided by factors  $w$  for fixed ends — equals the product of weights of the compact edges of  $D$ .

We cut  $(\Gamma, f)$  into floors. Each floor  $(\Gamma', f')$  can be viewed as a tropical stable map contributing to the count

$$\langle (\underline{\phi}_V^-, \underline{\mu}_V^-) | \tau_{k_V}(\text{pt}) | (\underline{\phi}_V^+, \underline{\mu}_V^+) \rangle_{g_V}$$

which gives the multiplicity of the floor viewed as a vertex  $V$  of  $D$ . As such, the floor contributes its tropical multiplicity, which is again a product as above.

Let  $v$  be a vertex of  $D$ . By Theorem 3.14,  $\text{mult}(v)$  equals the weighted sum of all floors  $(\Gamma', f')$  of some  $(\Gamma, f)$  that map to  $v$  under Construction 4.12. In this weighted count, each summand contributes with its tropical multiplicity as above. Since every end of  $\Gamma'$  which is not adjacent to the marked point in  $\Gamma'$  has to be fixed by rigidity, the only contribution we have for the whole floor is the local vertex multiplicity  $\text{mult}_V(\Gamma', f')$  of the vertex  $V$  of  $\Gamma'$  adjacent to the marked point. Thus,  $\text{mult}(v)$  equals the weighted sum over all floors that can possibly be inserted, each counted with the factor  $\text{mult}_V(\Gamma', f')$  where  $V$  is the vertex adjacent to the marked point.

Since we can freely combine floors by gluing them to elevator edges as imposed by  $D$ ,  $\text{mult}(D)$  equals the weighted count of all tropical stable maps contributing to the invariant

$$\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}) \dots \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^{\text{trop}}$$

and yielding  $D$  under the procedure described in Construction 4.12, where by Definition 4.4, each tropical stable map is counted with a product of weights for the compact edges of  $D$  times  $\text{mult}_V(\Gamma', f')$  where  $V$  is the vertex adjacent to the marked point. We have seen above that the product of weights for the compact edges of  $D$  equals the product of the areas of triangles dual to non-marked edges, divided by the weights of the fixed ends. Thus  $\text{mult}(D)$  equals the weighted count of all tropical stable maps yielding  $D$ , each weighted with its tropical multiplicity. The statement follows. □

## 5. THE FOCK SPACE APPROACH

In this section we build on work of Cooper and Pandharipande [17] and Block and Göttsche [7] and express relative descendant Gromov–Witten invariants of Hirzebruch surfaces as matrix elements for an operator on a Fock space. We begin the section by reviewing the formalism of Fock spaces in our context.

Let  $\mathcal{H}$  denote the algebra presented with generators  $a_n, b_n$  for  $n \in \mathbb{Z}$  satisfying the commutator relations

$$(12) \quad [a_n, a_m] = 0, \quad [b_n, b_m] = 0, \quad [a_n, b_m] = n \cdot \delta_{n, -m},$$

where  $\delta_{n, -m}$  is the Kronecker symbol. We let  $a_0 = b_0 = 0$ .

The Fock space  $F$  is the vector space generated by letting the generators  $a_n, b_n$  for  $n < 0$  act freely (as linear operators) on the so-called vacuum vector  $v_\emptyset$ . We define  $a_n \cdot v_\emptyset = b_n \cdot v_\emptyset = 0$  for  $n > 0$ . For a pair of partitions  $\underline{\phi} = (\varphi_1, \dots, \varphi_{n_1})$  and  $\underline{\mu} = (\mu_1, \dots, \mu_{n_2})$ , we denote

$$(13) \quad v_{\underline{\phi}, \underline{\mu}} = \frac{1}{|\text{Aut}(\underline{\phi})| \cdot |\text{Aut}(\underline{\mu})|} a_{-\varphi_1} \cdots a_{-\varphi_{n_1}} \cdot b_{-\mu_1} \cdots b_{-\mu_{n_2}} \cdot v_\emptyset.$$

The vectors  $\{v_{\underline{\phi}, \underline{\mu}}\}$  indexed by pairs of partitions  $\underline{\phi}, \underline{\mu}$  form a basis for  $F$ . We define an inner product on  $F$  by declaring  $\langle v_\emptyset | v_\emptyset \rangle = 1$  and  $a_n$  to be the adjoint of  $a_{-n}$ ,  $b_n$  of  $b_{-n}$ . The structure constants for the inner product in the two-partition basis are:

$$(14) \quad \langle v_{\underline{\phi}, \underline{\mu}} | v_{\underline{\phi}', \underline{\mu}'} \rangle = \prod \varphi_i \cdot \prod \mu_i \cdot \frac{1}{|\text{Aut}(\underline{\phi})|} \cdot \frac{1}{|\text{Aut}(\underline{\mu})|} \cdot \delta_{\underline{\phi}, \underline{\phi}'} \cdot \delta_{\underline{\mu}, \underline{\mu}'}$$

Following standard conventions, for  $\alpha, \beta \in F$  and an operator  $A \in \mathcal{H}$ , we write  $\langle \alpha | A | \beta \rangle$  for  $\langle \alpha | A \beta \rangle$ . Such expressions are referred to as **matrix elements**. We write  $\langle A \rangle$  for  $\langle v_\emptyset | A | v_\emptyset \rangle$ ; such a value is called a **vacuum expectation**.

We also introduce **normal ordering** of operators in  $\mathcal{H}$ . If  $c_i, i = 1, \dots, n$  are operators in  $\mathcal{H}$ , then the normally ordered product  $:\prod_{i=1}^n c_i$  reorders the  $c_i$  so that any  $c_i$  with  $i > 0$  occurs after the  $c_j$  with  $j < 0$ . For example, we have  $:a_2 b_{-2} a_2 a_{-1} := b_{-2} a_{-1} a_2 a_2$ .

As before, we fix  $k \in \mathbb{N}$  to identify a Hirzebruch surface  $\mathbb{F}_k$ .

**Definition 5.1.** Let  $m \in \mathbb{N}_{>0}$ ,  $l, s$  and  $g \in \mathbb{N}$  be given. Let  $\mathbf{z} \in (\mathbb{Z} \setminus \{0\})^m$  satisfy  $\sum_{i=1}^m z_i = -k \cdot s$ . Denote  $\underline{\mu} = (z_1, \dots, z_{l+2-2s-g})$  and  $\underline{\phi} = (z_{l+2-2s-g+1}, \dots, z_m)$ , and let superscripts  $\pm$  denote the subsets of positive (resp. negative) entries.

Define

$$\hat{a}_n = \begin{cases} u a_n & \text{if } n < 0 \\ a_n & \text{if } n > 0 \end{cases} \quad \text{and} \quad \hat{b}_n = \begin{cases} u b_n & \text{if } n < 0 \\ b_n & \text{if } n > 0 \end{cases}.$$

We define the following series of operators in  $\mathcal{H}[t, u]$ , indexed by  $l \in \mathbb{N}$ :

$$M_l = \sum_{g \in \mathbb{N}} u^{g-1} \sum_{s \in \mathbb{N}} t^s \sum_{m \in \mathbb{N}_{>0}} \sum_{\mathbf{z} \in \mathbb{Z}^m} \langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_l(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g \cdot \\ : \hat{b}_{z_1} \cdots \hat{b}_{z_{l+2-2s-g}} \cdot \hat{a}_{z_{l+2-2s-g+1}} \cdots \hat{a}_{z_m} :$$

where the fourth sum is taken over all  $\mathbf{z}$  satisfying  $\sum_i z_i = -k \cdot s$  (where  $s$  is the index of the second sum), and where the log descendant one-point Gromov–Witten invariant  $\langle \tau_l(\text{pt}) \rangle_{(\underline{\phi}, \underline{\mu}), g}$  depends on the indices  $l, g$  and  $\mathbf{z}$  as above.

**Remark 5.2.** Consider the operator  $M_0$ . It has only two summands for  $s$ ,  $s = 0$  and  $s = 1$ , since  $2 - 2s - g < 0$  for  $s > 1$ . If  $s = 0$ , the curve class in the Gromov–Witten invariant  $\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_0(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g$  is a multiple of the class of a fiber. This implies that the moduli space of maps is non-empty only if  $g = 0$  and  $m = 2$ . The invariant  $\langle \underline{\mu}^- | \tau_0(\text{pt}) | \underline{\mu}^+ \rangle_0$ , for  $\underline{\mu} = (d, -d)$  is readily seen to be 1: there is a unique map of degree  $d$  from a rational curve to the fiber identified

by the point condition, fully ramified at 0 and  $\infty$  (the intersections of the sections with the given fiber). Such a map has no automorphisms because we have marked one point on the rational curve.

If  $s = 1$ , we must have  $g = 0$  and no  $b$  factors. The invariants  $\langle \underline{\phi}^- | \tau_0(\text{pt}) | \underline{\phi}^+ \rangle_0$  are all 1 by the genus 0 correspondence theorem and a tropical computation, see [33, 36].

So we have

$$M_0 = \sum_{z_1+z_2=0} b_{z_1} \cdot b_{z_2} + \sum_{\Phi \in (\mathbb{Z} \setminus \{0\})^m} t \cdot u^{\#\{\Phi^-\}-1} a_{z_1} \cdot \dots \cdot a_{z_m},$$

where the second sum goes over all  $\mathbf{z} \in (\mathbb{Z} \setminus \{0\})^m$  satisfying  $\sum z_i = -k$ . Here the normal ordering is unnecessary since the  $a_i$  commute amongst themselves, as do the  $b_j$ . Since the genus can be computed from the Euler characteristic of the underlying Feynman graphs, the variable  $u$  is superficial in this scenario. Setting  $u = 1$ , we obtain the operator  $H_k(t)$  defined in [7], Theorem 1.1. Our family of operators  $M_l$  generalizes the operator of Block-Göttsche to one operator for each power of descendant insertions.

**Theorem 5.3.** *With discrete data fixed and denoted as in Notation 2.2, the disconnected log descendant Gromov–Witten invariant  $\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}) \dots \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^\bullet$  equals the matrix element*

$$(15) \quad \langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_1}(\text{pt}) \dots \tau_{k_n}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_g^\bullet \\ = \frac{|\text{Aut}(\underline{\mu})| |\text{Aut}(\underline{\Phi})|}{\prod |\mu_i| \prod |\phi_i|} \left\langle v_{\underline{\mu}^-, \underline{\phi}^-} \left| \text{Coeff}_{t^a u^h} \left( \prod_{i=1}^n M_{k_i} \right) \right| v_{\underline{\mu}^+, \underline{\phi}^+} \right\rangle,$$

where the operators  $M_{k_i}$  are as defined in Definition 5.1, and for a series of operators  $M \in \mathcal{H}[t, u]$   $\text{Coeff}_{t^a u^h}(M) \in \mathcal{H}$  denotes the  $t^a u^h$ -coefficient.

**Important detail.** Notice the order of the partitions is switched on the two sides of Equation (15), thus the  $\mu_i$  entries are associated to  $a$  variables and vice-versa.

Before we start a formal proof of Theorem 5.3, we make a relevant definition and recall an important tool for the proof.

After translating the matrix element in Equation (15) to a vacuum expectation, we compute it as the weighted sum over **Feynman graphs** associated to each monomial contributing to the expectation. This can be viewed as a variant of Wick's theorem [39] and is proved in Proposition 5.2 of [7]. Generalizing the situation in [7], the Feynman graphs in question are essentially floor diagrams and Theorem 5.3 follows because of a natural weighted bijection of Feynman graphs and floor diagrams.

**Definition 5.4.** *Let  $P = m_+ \cdot m_1 \cdot \dots \cdot m_n \cdot m_-$  be a product of monomials in the variables  $a_s$  or  $b_s$ , such that:*

- for each monomial, all operators with negative indices stand left of all operators with positive indices;
- $m_+$  contains only positive factors (with  $s > 0$ );
- $m_-$  contains only negative factors (with  $s < 0$ ).

We associate graphs to  $P$  called **Feynman graphs** for  $P$ , via the following algorithm.

**Step 1: local pieces.** *To any monomial  $m_i$ , associate a star graph with vertex denoted  $v_i$ : for each factor  $a_s$  appearing in  $m_i$ , draw a (non-thickened) edge germ of weight  $|s|$  which is directed to the left if  $s < 0$  and to the right if  $s > 0$ . For each factor  $b_s$ , draw a thickened edge germ of weight  $|s|$  which is directed to the left if  $s < 0$  and to the right if  $s > 0$ .*

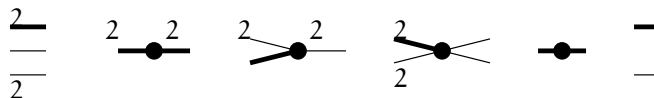


FIGURE 12. The weighted, directed, possibly thickened edge germs corresponding to the product  $P$  in Example 5.5. (We drop the marking of edge germs in the picture.)

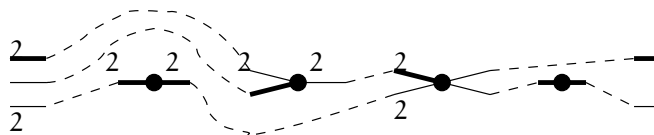


FIGURE 13. A Feynman graph completing the edge germs associated to the product  $P$  in Example 5.5.

To the special monomials  $m_+$ ,  $m_-$  associate a collection of disconnected, marked edge germs of weight equal to the absolute value of the index of each operator appearing in the monomials. Thicken the germs corresponding to the operators  $b_s$ .

**Step 2: Feynman fragment.** We call the Feynman fragment associated to  $P$  the disconnected graph obtained by linearly ordering the union of all the local pieces: first come the edge germs relative to  $m_+$ , then vertices  $v_i$  (ordered according to their index  $i$ , and finally the edge germs corresponding to  $m_-$ .

**Step 3: filling the gaps.** A Feynman graph completing the Feynman fragment is any (marked, weighted, ordered) graph obtained by promoting edge germs to half edges, and gluing pairs of half edges until there is none left. A pair of half edges may be glued if:

- one is directed to the right and the other to the left, and the vertex adjacent to the germ directed to the right is smaller than the one adjacent to the germ directed to the left,
- the two edge germs have the same weight, and
- one edge germ is thickened and one is not.

**Example 5.5.** Let  $P$  be the product

$$P = (b_2 \cdot a_1 \cdot a_2) \cdot (b_{-2} \cdot b_2) \cdot (a_{-2} \cdot b_{-1} \cdot a_2) \cdot (b_{-2} \cdot a_{-2} \cdot a_1 \cdot a_1) \cdot (b_{-1} \cdot b_1) \cdot (b_{-1} \cdot a_{-1}),$$

where the factors  $m_i$  are separated by parentheses. Following Definition 5.4, a Feynman graph for  $P$  is any graph completing the Feynman fragment depicted in Figure 12. In Figure 13, the dotted lines suggest a way to complete the fragment to a Feynman graph for  $P$ . After removing all external half edges, we recognize the floor diagram depicted in Figure 6.

**Proposition 5.6** (Wick's Theorem, see Proposition 5.2 of [7]). *The vacuum expectation  $\langle P \rangle$  for a product  $P$  as in Definition 5.4 equals the weighted sum of all Feynman graphs for  $P$ , where each Feynman graph is weighted by the product of weights of all edges (interior edges and ends).*

*Proof.* A detailed proof of this proposition may be found in [7]. Here we provide an intuitive and informal description of the mechanism that underlies the proof, as we feel this will be more beneficial to a reader who is not already an expert on these techniques.

In the product  $P$ , we take the right most factor  $a_i$  or  $b_i$  with  $i > 0$ , and try to move it to the right. To simplify notations, let us assume that this right most factor is  $a_i$  for some  $i > 0$ . If this factor  $a_i$  reaches the very right in a contribution we produce in this way (i.e. ends up being the right most

factor of a contributing term), then we obtain zero since by definition  $a_i \cdot v_\emptyset = b_i \cdot v_\emptyset = 0$  for  $i > 0$ . The commutator relations produce several contributing terms for  $\langle P \rangle$  when moving  $a_i$  to the right. We can make  $a_i$  jump over any  $a_j$ , or  $b_k$  with  $k \neq -i$ . If  $a_i$  is the left neighbour of  $b_{-i}$  however, the commutator relation replaces  $a_i b_{-i}$  by  $b_{-i} a_i + i$ . That is, we get two summands, one in which we manage to move  $a_i$  further to the right, and one where we cancel this factor together with its neighbour  $b_{-i}$ .

With both summands, we continue moving the right most factor with positive index right. For the summand in which we cancel  $a_i$  together with a factor of  $b_{-i}$  appearing right of  $a_i$  in  $P$ , we add to the Feynman fragment of  $P$  by drawing an edge connecting the germ corresponding to  $a_i$  and the germ corresponding to  $b_{-i}$ .

By following this procedure we draw all Feynman graphs completing the Feynman fragment for  $P$ . Each Feynman graph corresponds to a way to group the factors of  $P$  in pairs  $\{a_i, b_{-i}\}$  corresponding to edges completing the corresponding marked edge germs. Each such pair produces a contribution of  $i$  because of the commutator relations, so altogether each Feynman graph should be counted with weight equal to the product of its edge weights to produce  $\langle P \rangle$ .  $\square$

*Proof of Theorem 5.3:* First we express the matrix element in Equation (15) as a vacuum expectation:

$$(16) \quad \frac{|\text{Aut}(\underline{\mu})| |\text{Aut}(\underline{\phi})|}{\prod |\mu_i| \prod |\varphi_i|} \langle v_{\underline{\mu}^-, \underline{\phi}^-} | M | v_{\underline{\mu}^+, \underline{\phi}^+} \rangle = \frac{|\text{Aut}(\underline{\mu})| |\text{Aut}(\underline{\phi})|}{\prod |\mu_i| \prod |\varphi_i|} \frac{1}{|\text{Aut}(\underline{\phi}^+)| |\text{Aut}(\underline{\mu}^+)|} \frac{1}{|\text{Aut}(\underline{\phi}^-)| |\text{Aut}(\underline{\mu}^-)|} \langle v_\emptyset \left| \prod_{\mu_i \in \underline{\mu}^-} a_{|\mu_i|} \prod_{\varphi_i \in \underline{\phi}^-} b_{|\varphi_i|} M \prod_{\mu_i \in \underline{\mu}^+} a_{-\mu_i} \prod_{\varphi_i \in \underline{\phi}^+} b_{-\varphi_i} \right| v_\emptyset \rangle = \frac{1}{\prod |\varphi_i| \prod |\mu_i|} \langle \prod_{\mu_i \in \underline{\mu}^-} a_{|\mu_i|} \prod_{\varphi_i \in \underline{\phi}^-} b_{|\varphi_i|} M \prod_{\mu_i \in \underline{\mu}^+} a_{-\mu_i} \prod_{\varphi_i \in \underline{\phi}^+} b_{-\varphi_i} \rangle$$

By Theorem 4.9 (resp. Theorems 3.14 and 4.11), the left-hand side in Equation (16) equals an appropriate count of floor diagrams. By Proposition 5.6, each term contributing to the right-hand side can be expressed in terms of a weighted count of suitable Feynman diagrams. We show that the floor diagrams contributing to the left-hand side are essentially equal to the Feynman graphs contributing to the right, and that they are counted with the same weight on both sides.

Expand the left-hand side so that it becomes a sum of vacuum expectations, where each summand is of the form  $w_P \cdot P$  such that  $w_P$  is a number and  $P = m_+ \cdot \dots \cdot m_-$  a monomial as described in Definition 5.4. For each summand,

$$m_+ = \prod_{\mu_i \in \underline{\mu}^-} a_{|\mu_i|} \cdot \prod_{\varphi_i \in \underline{\phi}^-} b_{|\varphi_i|} \quad \text{and} \quad m_- = \prod_{\mu_i \in \underline{\mu}^+} a_{-\mu_i} \cdot \prod_{\varphi_i \in \underline{\phi}^+} b_{-\varphi_i}.$$

A factor  $m_i$  for  $i = 1, \dots, n$  comes from a summand of  $M_{k_i}$ , i.e. is of the form

$$\langle (\underline{\phi}^-, \underline{\mu}^-) | \tau_{k_i}(\text{pt}) | (\underline{\phi}^+, \underline{\mu}^+) \rangle_{g_i} \cdot \hat{b}_{z_1} \cdot \dots \cdot \hat{b}_{z_{k_i+2-2s_i-g}} \cdot \hat{a}_{z_{k_i+2-2s_i-g+1}} \cdot \dots \cdot \hat{a}_{z_m} \cdot,$$

where  $s_i$  is encoded in the power of  $t$  and  $g_i$  in the power of  $u$ .

Enrich the Feynman fragment for  $P$  by adding three numbers to each vertex  $i$ , namely the  $\psi$ -power  $k_i$  (imposed by the operator  $M_{k_i}$  of which the factor corresponding to vertex  $i$  is taken),



the size  $s_i$  (imposed by the power of  $t$ ) and the genus  $g_i$  (imposed by the power of  $u$ ). Any Feynman diagram completing this Feynman fragment is by definition a weighted loop-free graph with ends on the linearly ordered vertex set  $v_1, \dots, v_n$ . After removing all external half edges, the conditions (1), (2) and (3) we impose in the definition of a floor diagram (Definition 4.1) are satisfied. By definition of the operator  $M_l$  (see Definition 5.1), the signed sum of weights of edges adjacent to a vertex equals  $-k \cdot s_i$ , so condition (5) is satisfied. By definition of the operator  $M_l$ , in each factor  $m_i$ , exactly  $k_i + 2 - 2s_i - g_i$  factors are  $b$ -operators and thus correspond to thickened edge germs, so condition (4) is satisfied.

Since we take the  $t^a$  coefficient of the product  $M_{k_1} \cdot \dots \cdot M_{k_n}$  for the operator in Equation (15), we obtain floor diagrams satisfying  $a = \sum s_i$ . The degree  $(\phi, \underline{\mu})$  is determined by the boundary conditions. To see that the floor diagram is of the right genus, notice that the variable  $u$  is in charge of genus. Let us build a Feynman graph from the left to the right, starting with the left ends, and adding in vertex after vertex from 1 to  $n$ , taking the change in genus into account in each step. The genus of the graph consisting of  $\ell(\underline{\phi}^-) + \ell(\underline{\mu}^-)$  left ends (at first disconnected) has genus  $-\ell(\underline{\phi}^-) - \ell(\underline{\mu}^-) + 1$ . For the vertex  $i$  of local genus  $g_i$ , by definition of the operator  $M_l$ , we get a contribution of  $u^{g_i-1}$ , and we get as many additional factors of  $u$  as the vertex has incoming edges (by the  $\hat{a}_i$  resp.  $\hat{b}_i$  convention). Since  $h$  incoming edges potentially close up  $h - 1$  cycles, the vertex  $i$  increases the genus by  $g_i + h_i - 1$ , where  $h_i$  denotes the number of incoming edges. By taking the  $u^{g + \ell(\underline{\phi}^-) + \ell(\underline{\mu}^-) - 1}$  coefficient in total, we thus obtain floor diagrams of genus  $g$ .

Each Feynman graph for  $P$  can thus be viewed (after removing external half edges) as a floor diagram contributing to the left-right-handhand side, and vice versa, each floor diagram gives a Feynman graph.

It remains to show that a Feynman graph and the corresponding floor diagram contribute to Equation (15) with the same multiplicity. For the right-hand side, note that a Feynman graph contributes with the product of the weight of all of its edges times the coefficient  $w_P$  of the product  $P$  in the expansion of the product of the  $M_l$ -operators. Dividing by the factor  $\frac{1}{|\underline{\mu}|} \frac{1}{|\underline{\phi}|}$  (see the right-hand side of Equation (16)), we see that we are giving the Feynman graph weight equal to the product of the weights of its internal edges times the factor  $w_P = \prod_{v=1}^n \langle (\underline{\phi}_v^-, \underline{\mu}_v^-) | \tau_{k_v}(\text{pt}) | (\underline{\phi}_v^+, \underline{\mu}_v^+) \rangle_{g_v}$ . This is precisely the weight of the corresponding floor diagram in Equation (4.4).

□

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