Random-matrix theory for the Lindblad master equation

Cite as: Chaos **31**, 023101 (2021); https://doi.org/10.1063/5.0033486
Submitted: 16 October 2020 • Accepted: 12 January 2021 • Published Online: 01 February 2021

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Submitted: 16 October 2020 · Accepted: 12 January 2021 ·
Published Online: 1 February 2021

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Note: This paper is part of the Focus Issue, Dissipative Quantum Chaos.
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ABSTRACT

Open quantum systems with Markovian dynamics can be described by the Lindblad equation. The quantity governing the dynamics is the Lindblad superoperator. We apply random-matrix theory to this superoperator to elucidate its spectral properties. The distribution of eigenvalues and the correlations of neighboring eigenvalues are obtained for the cases of purely unitary dynamics, pure dissipation, and the physically realistic combination of unitary and dissipative dynamics.

The theory of ensembles of random matrices has proved useful in understanding the energy spectra of complex closed quantum systems, such as heavy atomic nuclei and classically chaotic billiards. In these cases, the Hamiltonian describing the system is drawn from a suitable random-matrix ensemble. More recently, it has been realized that random-matrix theory can also shed light on open quantum systems. Their dynamics is not described by a Hamiltonian but by a so-called Lindblad generator. Using random-matrix ensembles suitable for the Lindblad generator, we study its spectral properties, which are important for the dynamics of open quantum systems.

I. INTRODUCTION

The theory of open quantum systems is central for the description and understanding of nature. Any real system is coupled to its environment and in many cases this coupling cannot be ignored. The coupling to the environment is particularly important for the fields of quantum information science and quantum chaos, as exemplified by this focus issue.

Statistical descriptions of open quantum systems are provided by quantum master equations of various flavors. A quantum master equation is an equation of motion for the reduced density operator of the open system. In principle, it can be derived from the von Neumann equation for the full density operator by tracing out the degrees of freedom of the environment and making suitable approximations.[1–8] While the dynamics of a closed system is unitary, this is generally not the case for an open system since the quantum master equation contains dissipative terms. In addition, the coupling to the environment typically leads to memory effects, which makes the quantum master equation nonlocal in time. If memory effects can be neglected the dynamics is called *Markovian* and the equation becomes local in time.

In the following, we assume the dynamics to be Markovian and homogeneous in time. Moreover, we take the dimension of the reduced Hilbert space of the open system to be finite and denote it by N. The form of the Markovian quantum master equation is strongly constrained by general principles: on the one hand, it must be a linear equation in the reduced density operator ρ since the underlying quantum mechanics is linear. This means that the equation can be written in the form

$$\frac{d}{dt}\rho = \mathscr{L}\rho, \tag{1}$$

where $\mathscr{L}$ is a superoperator acting on the space of potential density operators on the reduced Hilbert space. On the other hand,

the density operator must remain positive semidefinite and of unit trace for all times since its eigenvalues represent probabilities. These properties together with Markovianity are formalized by the theory of quantum dynamical semigroups,[9–11] which allows to show that the master equation can then be written in the Gorini–Kossakowski–Sudarshan–Lindblad form,[12,13]

$$\frac{d}{dt}\rho = \mathscr{L}\rho = -i[H,\rho] + \mathscr{L}_d\rho, \tag{2}$$

where the *generator* $\mathscr{L}$ of the semigroup consists of two terms. The first term describes unitary dynamics in terms of a Hermitian Hamiltonian H and the second one, $\mathscr{L}_d$, encodes dissipative processes. It acts as

$$\mathscr{L}_d\rho = \frac{1}{2}\sum_{\alpha,\beta=1}^{\tilde{N}} K_{\alpha\beta}\left(\left[L_\alpha, \rho L_\beta^\dagger\right] + \left[L_\alpha\rho, L_\beta^\dagger\right]\right), \tag{3}$$

where we define $\tilde{N} \equiv N^2 - 1$ in terms of the dimension $N > 1$ of the reduced Hilbert space, the *Lindblad operators* L_α are traceless operators that are orthonormal with respect to the Hilbert–Schmidt inner product, $\mathrm{Tr} L_\alpha^\dagger L_\beta = \delta_{\alpha\beta}$, and together with the identity operator form a basis of the space of all superoperators, and $K_{\alpha\beta} \in \mathbb{C}$ are the components of the Hermitian, positive semidefinite $\tilde{N} \times \tilde{N}$ *Kossakowski matrix* K. Equation (3) can easily be rewritten as

$$\mathscr{L}_d\rho = \sum_{\alpha,\beta=1}^{\tilde{N}} K_{\alpha\beta}\left(L_\alpha\rho L_\beta^\dagger - \frac{1}{2}\left\{L_\beta^\dagger L_\alpha, \rho\right\}\right). \tag{4}$$

For complex systems, it is impossible in practice to determine the large number of coefficients appearing in the Lindblad equation. This situation recalls the problem of the energy spectra of heavy nuclei, which one cannot obtain by diagonalizing a known Hamiltonian because of the large number of degrees of freedom and the strong interactions between the nucleons. Wigner[14–16] as well as Landau and Smorodinsky[17] realized that at least *statistical* properties of the spectra can be inferred by assuming the Hamiltonian to be a typical representative of an ensemble of Hamiltonians defined by Hermiticity and appropriate global symmetries. This idea was further fleshed out by Dyson[18–22] and was also successfully applied to other complicated quantum systems, such as classically chaotic billiards.[23,24] The motivation in terms of statistical properties naturally leads one to study the spectra of random matrices with certain properties and characterize them in the limit of large matrix size.

The situation for complex open quantum systems described by the Lindblad equation is similar, suggesting the application of random-matrix theory to this case. The Lindblad master equation and the Schrödinger equation have in common that they relate the time derivative of the quantity of interest (the density matrix and the state vector, respectively) to a linear function of this quantity, which can thus be characterized by a matrix of coefficients. The properties of the coefficient matrix and thus the relevant random-matrix ensembles are distinct, though.

As a first step, one of us has studied systems of linear rate equations within random-matrix theory.[25] Such rate equations, also called the Pauli master equation, can be understood as a master equation for only the diagonal components of a reduced density matrix, i.e., for the probabilities. They emerge from a full quantum master equation when the off-diagonal components, the coherences, can be neglected due to rapid decoherence.[8] However, it is in principle possible to derive a Pauli master equation directly without reference to the coherences.[3,4] The Pauli master equation has the general form

$$\frac{d}{dt}P_i = \sum_{j=1}^{N} A_{ij}P_j, \tag{5}$$

where P_i are probabilities of states of the open system enumerated by $i = 1, \ldots, N$ and A_{ij} are the components of a transition-rate matrix A. In Ref. 25, rigorous and numerical results for random-matrix ensembles of transition-rate matrices were studied. The off-diagonal elements A_{ij} denote transition rates from state j to state i. The interpretation as rates requires $A_{ij} \geq 0$ for $i \neq j$. Furthermore, the conservation of probability, $\sum_i P_i = 1$, requires that the column sums vanish, $\sum_i A_{ij} = 0$ for all j. These two conditions must be satisfied by all elements of the ensemble. By making an ansatz $P_i = p_i e^{\lambda t}$, one can see that the eigenvalues λ of A describe the dynamics of eigenmodes of the open system. Since the matrices A are real but generally not symmetric, their spectrum consists of real and pairs of complex conjugate eigenvalues. One can easily show that there is always a zero eigenvalue, which corresponds to a stationary solution. One can further show that the real parts of all eigenvalues are non-positive. The real eigenvalues λ describe exponentially decaying modes, whereas the complex conjugate pairs can be combined to real oscillating solutions with exponentially decaying envelope.

In Ref. 25, the off-diagonal components of the $N \times N$ transition-rate matrices A were taken to be independently identically distributed according to an exponential distribution with mean $\langle R \rangle$ [the exponential general rate-matrix ensemble (EGRE)]. In the limit of large N, the distribution of nonzero eigenvalues λ is then centered about $-N\langle R \rangle$, its width in both the real and imaginary directions scales with $\sqrt{N}\langle R \rangle$, and it approaches a distinctive shape for large N.[25] The distribution of distances between nearest-neighbor eigenvalues, defined by the absolute value of their difference, is linear in the distance for small distance and large N, i.e., the exponent of this distribution is $\beta = 1$.[25] This agrees with the Ginibre orthogonal ensemble (GinOE, also known as the real Ginibre ensemble) of real matrices with independently normally distributed components.[26,27]

It is then natural to extend the random-matrix analysis to the full Lindblad equation. This was done in the master's thesis[28] of the first author, supervised by the second. Since the field seemed somewhat esoteric at that time, we unwisely refrained from publishing the results. However, since then, there has been a significant increase in the interest in open quantum systems, in part driven by the highly active fields of quantum information and quantum chaos. Denisov *et al.*[29] have recently studied random generators for the Lindblad equation, unaware of Ref. 28, which was not even available online at that time, and obtaining results consistent with it. In the present paper, we summarize and clarify the main results of Ref. 28 and give a few additional results and simpler derivations. We also establish connections to recent developments.

II. SPECTRAL STATISTICS FOR THE LINDBLAD EQUATION

By writing the quantum master equation (1) in components with respect to an orthonormal basis, we obtain a set of N^2 coupled ordinary homogeneous linear differential equations of first order for the components ρ_{mn} of the reduced density operator. It is useful to interpret $\alpha = (m, n)$ as a compound index. Then, ρ is a vector with N^2 components and $\mathscr{L}$ is an $N^2 \times N^2$ matrix.

Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of the matrix $\mathscr{L}$ with algebraic multiplicities $n_1, \ldots, n_k$. Then,

$$\mathrm{Tr}\mathscr{L}^l = \sum_{i=1}^{k} n_i \lambda_i^l \tag{6}$$

holds for all $l \geq 1$. The theory of coupled ordinary differential equations shows that the general solution of the Lindblad equation is of the form

$$\rho(t) = \sum_{i=1}^{k} P_i(t)\, e^{\lambda_i t}, \tag{7}$$

where the vectors $P_i(t)$ are polynomials in t of degrees not larger than $n_i - 1$. The real parts of eigenvalues describe exponential decay for $\mathrm{Re}\lambda_i < 0$ or growth for $\mathrm{Re}\lambda_i > 0$, while the imaginary parts describe overlaid oscillations. For the Lindblad equation, like for the Pauli master equation, the eigenvalues are real or form complex conjugate pairs, there is always a zero eigenvalue and thus a stationary solution, and the real parts are non-positive.

We will be concerned with random-matrix ensembles, for which the multiplicities are $n_i = 1$ with probability one. We can thus write the general solution as

$$\rho(t) = \sum_{i=0}^{\tilde{N}} P_i\, e^{\lambda_i t}, \tag{8}$$

where $\lambda_0 = 0$, $\mathrm{Re}\lambda_i \leq 0$ for all i, and P_i are time-independent vectors. Insertion into the Lindblad equation yields

$$\mathscr{L} P_i = \lambda_i P_i \tag{9}$$

so that P_i are the right eigenvectors of $\mathscr{L}$ to the eigenvalues λ_i.

Our next goal is to construct random-matrix ensembles for the generator $\mathscr{L}$. The Lindblad equation contains a unitary part $-i[H, \rho]$ and a dissipative part $\mathscr{L}_d\rho$. In order to be physically meaningful, the ensemble should be invariant under basis transformations. In addition, we require that every allowed generator $\mathscr{L}$ occurs in the ensemble.

A. Random Lindblad generators: Unitary part

We first consider the unitary part, which is described by a random Hamiltonian H. Here, we can fall back on the theory of Hermitian matrix ensembles.[14–22,30] Quantum systems without any symmetries are modeled using the Gaussian unitary ensemble (GUE),[18,30] which is defined by the real diagonal components and the real and imaginary parts of the components of the strict upper triangular matrix being independently normally distributed with standard deviation σ. The strict lower triangular matrix is then fixed by Hermiticity. We will restrict ourselves to the GUE for the unitary part.

The von Neumann equation $d\rho/dt = -i[H, \rho]$ can be written in components as

$$\frac{d}{dt}\rho_{mn} = -i \sum_{p,q=1}^{N} (H_{mp}\delta_{qn} - \delta_{mp}H_{qn})\rho_{pq}. \tag{10}$$

Using compound indices α, β as above, this can be rewritten as

$$\frac{d}{dt}\rho_\alpha = \sum_{\beta=1}^{N^2} \mathscr{L}^0_{\alpha\beta}\rho_\beta, \tag{11}$$

where

$$\mathscr{L}^0_{\alpha\beta} \equiv \mathscr{L}^0_{mn,pq} \equiv -i\,(H_{mp}\delta_{qn} - \delta_{mp}H_{qn}) \tag{12}$$

are the components of the *Liouvillian* $\mathscr{L}_0$. The ensemble of Liouvillians $\mathscr{L}_0$ is independent of the choice of the Hilbert-space basis and covers all allowed operators because the GUE has these properties.

If E_n, $n = 1, \ldots, N$ are the eigenvalues of H, then the eigenvalues of $\mathscr{L}_0$ are $-i\,(E_m - E_n)$, $m, n = 1, \ldots, N$. If H is taken from the GUE the eigenvalues E_n are distinct with probability one. Hence, with probability one, N of the N^2 eigenvalues of $\mathscr{L}_0$ are zero. For later, we recall that in the limit of large N, the distribution of GUE eigenvalues approaches the semicircle law[14,30,31] $p^\infty_{\mathrm{GUE}}(E) = (2/\pi r)\sqrt{1 - (E/r)^2}$ for $-r \leq E \leq r$ with $r \equiv \sqrt{8\sigma^2 N}$. Moreover, the eigenvalues repel each other in such a way that the distribution function of separations ΔE of nearest-neighbor eigenvalues scales as ΔE^2 for small ΔE.

Furthermore, Eq. (6) implies that $\mathrm{Tr}\mathscr{L}_0 = 0$ and

$$\mathrm{Tr}\mathscr{L}_0^2 = -2N^2\,\overline{(E - \overline{E})^2}, \tag{13}$$

where the overline denotes the average over all eigenvalues for a given Hamiltonian H, without ensemble averaging. The last expression is negative because the spectrum of $\mathscr{L}_0$ is lying on the imaginary axis. Taking the average over the GUE, we obtain $\langle E\rangle_{\mathrm{GUE}} = 0$ and $\langle (E - \langle E\rangle_{\mathrm{GUE}})^2\rangle_{\mathrm{GUE}} = \langle E^2\rangle_{\mathrm{GUE}} = 2N\sigma^2$ for all N. This implies that $\langle \mathrm{Tr}\mathscr{L}_0^2\rangle_{\mathrm{GUE}} = -4N^3\sigma^2$.

B. Random Lindblad generators: Dissipative part

Next, we turn to the dissipative part $\mathscr{L}_d\rho$, which, according to Eqs. (3) and (4), involves the Kossakowski matrix K and a complete orthonormal set of traceless Lindblad operators L_α. Since $\mathscr{L}_d$ is form invariant under unitary transformations of this set, an arbitrary but fixed complete orthonormal set can be used. We now construct a convenient one starting from an orthonormal Hilbert-space basis $\{|n\rangle \mid n = 1, \ldots, N\}$. The matrices $\tilde{L}_{mn} = |m\rangle\langle n|$ for $m \neq n$ and $\tilde{L}_{mm} = |m\rangle\langle m| - |m+1\rangle\langle m+1|$ for $m = n < N$ form a basis of the space of traceless matrices. However, they are not orthonormal with respect to the Hilbert–Schmidt inner product. We construct an orthonormal basis $\{L_{mn}\}$ by applying the Gram–Schmidt process to

$\{\tilde{L}_{mn}\}$. The resulting Lindblad operators $L_\alpha \equiv L_{mn}$ are

$$L_{mn} \equiv |m\rangle\langle n| \tag{14}$$

for $m \neq n$ and

$$L_{mm} \equiv \frac{|1\rangle\langle 1| + |2\rangle\langle 2| + \cdots + |m\rangle\langle m| - m\,|m+1\rangle\langle m+1|}{\sqrt{m(m+1)}} \tag{15}$$

for $m = n < N$. Note that these Lindblad operators satisfy $L^\dagger_{mn} = L_{nm}$. It will prove useful to also define

$$L_{N^2} \equiv \frac{1}{\sqrt{N}}\,\mathbb{1}, \tag{16}$$

where $\mathbb{1}$ is the $N \times N$ unit matrix (note that L_{N^2} does not appear in $\mathscr{L}_d$). L_{mn} for $m, n = 1, \ldots, N$ then form an orthonormal basis of the space of all $N \times N$ matrices. Equations (14)–(16) can be summarized as

$$L_{mn} = \sum_{p,q=1}^{N} \mathscr{U}_{mn,pq} |p\rangle\langle q|, \tag{17}$$

with $\mathscr{U}_{mn,mn} = 1$ for $m \neq n$, $\mathscr{U}_{mm,pp} = O_{mp}$, where

$$O = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ \frac{1}{\sqrt{N(N-1)}} & \frac{1}{\sqrt{N(N-1)}} & \frac{1}{\sqrt{N(N-1)}} & \cdots & -\frac{N-1}{\sqrt{N(N-1)}} \\ \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \cdots & \frac{1}{\sqrt{N}} \end{pmatrix}, \tag{18}$$

and all other components of $\mathscr{U}$ vanish.

In analogy to Eq. (12), the dissipative part of the generator can be written in components,

$$\mathscr{L}^d_{mn,pq} = \sum_{\alpha,\beta=1}^{\tilde{N}} K_{\alpha\beta} \left(L^\alpha_{mp} L^{\beta\dagger}_{qn} - \frac{1}{2} \sum_{k=1}^{N} L^{\beta\dagger}_{mk} L^\alpha_{kp} \delta_{qn} - \frac{1}{2} \sum_{k=1}^{N} \delta_{mp} L^{\beta\dagger}_{qk} L^\alpha_{kn} \right), \tag{19}$$

where $L^\alpha_{mp} = \langle m|L_\alpha|p\rangle$ are the matrix elements of L_α. Inserting the choice of Lindblad operators from Eqs. (14) and (15) leads to an explicit form for $\mathscr{L}^d_{mn,pq}$. The result is rather lengthy because of the m dependence of L_{mm} and is, therefore, relegated to Appendix A. A more convenient representation is obtained by including L_{N^2} into the sums in Eq. (4),

$$\mathscr{L}_d \rho = \sum_{\alpha,\beta=1}^{N^2} \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\beta} \left(L_\alpha \rho L^\dagger_\beta - \frac{1}{2} \left\{ L^\dagger_\beta L_\alpha, \rho \right\} \right). \tag{20}$$

It is then possible to transform the full set L_α, $\alpha = 1, \ldots, N^2$ back to the simple dyads $D_{mn} \equiv |m\rangle\langle n|$, which results in

$$\mathscr{L}_d \rho = \sum_{\alpha,\beta=1}^{N^2} \tilde{K}_{\alpha\beta} \left(D_\alpha \rho D^\dagger_\beta - \frac{1}{2} \left\{ D^\dagger_\beta D_\alpha, \rho \right\} \right), \tag{21}$$

with

$$\tilde{K}_{\alpha\beta} \equiv \sum_{\gamma,\delta=1}^{N^2} \mathscr{U}^T_{\alpha\gamma} \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}_{\gamma\delta} \mathscr{U}^*_{\delta\beta} = \sum_{\gamma,\delta=1}^{\tilde{N}} \mathscr{U}^T_{\alpha\gamma} K_{\gamma\delta} \mathscr{U}^*_{\delta\beta} \tag{22}$$

and $\mathscr{U}$ as defined in Eq. (17). This is now easy to write in components,

$$\mathscr{L}^d_{mn,pq} = \tilde{K}_{mp,nq} - \frac{\delta_{mp}}{2} \sum_{r=1}^{N} \tilde{K}_{rn,rq} - \frac{\delta_{nq}}{2} \sum_{r=1}^{N} \tilde{K}_{rp,rm}. \tag{23}$$

As Eq. (19) shows, an ensemble of random generators $\mathscr{L}_d$ can be obtained by using a convenient fixed set of Lindblad operators L_α and randomizing the Kossakowski matrix K. Clearly, the resulting ensemble should not depend on which fixed set of Lindblad operators is used. This can be formalized by requiring the random $\mathscr{L}_d$ ensemble to be invariant under unitary transformations of the set of Lindblad operators. It is then also invariant under unitary transformations of the basis $\{|n\rangle\}$ of the Hilbert space because the latter transformations form a subgroup of the former.[32] Since a unitary transformation of all L_α is equivalent to a corresponding unitary transformation of K, the requirement can be met by using an ensemble of random Kossakowski matrices that is invariant under unitary transformations. We now define such an ensemble.

Since K is a positive semidefinite $\tilde{N} \times \tilde{N}$ matrix, there exists a complex $\tilde{N} \times \tilde{N}$ matrix A so that $K = A^\dagger A$. Using this relationship, we construct random matrices K from random matrices A. The components $A_{\alpha\beta} \in \mathbb{C}$ of the matrix A are drawn in such a way that their real and imaginary parts are independently normally distributed with standard deviation $\tilde{\sigma}$. This defines the Ginibre unitary ensemble (GinUE, also known as the complex Ginibre ensemble).[26] The resulting ensemble of matrices $K = A^\dagger A$ is indeed invariant under basis changes: let U be a unitary $\tilde{N} \times \tilde{N}$ matrix that transforms one orthonormal set of Lindblad operators L_α into another. Under such a transformation, K is mapped onto $UKU^\dagger = UA^\dagger AU^\dagger = UA^\dagger U^\dagger UAU^\dagger$ and the invariance of the ensemble of matrices K follows from the corresponding property of the GinUE, which was shown by Mezzadri.[33] In addition, all positive semidefinite matrices are contained in the ensemble. The real version of this ensemble was studied by Wishart already in 1928.[34] For this reason, we call the ensemble the Wishart positive unitary ensemble (WPUE). The same ensemble has been used by Denisov *et al.*[29] independently. Furthermore, Sá *et al.*[35] have recently studied the generalized case where A is chosen as an $r \times \tilde{N}$ matrix with $r \leq \tilde{N}$. Here, r represents the number of independent system operators (jump operators), which couple the open system to its environment.

Finally, we obtain $\tilde{K}$ using the transformation in Eq. (22) with the known matrix $\mathscr{U}$ and then get the components of $\mathscr{L}_d$ from Eq. (23). We emphasize that while this procedure uses the specific Lindblad operators defined in Eqs. (14) and (15), this choice does not matter because of the unitary invariance of the ensemble.

As noted above, $\mathscr{L}_d$ always has an eigenvalue $\lambda_0 = 0$, which is nondegenerate with probability one, and the other $\tilde{N}$ eigenvalues are real or form complex conjugate pairs. It is important to realize that the spectrum of $\mathscr{L}_d$ is distinct from the spectrum of the Kossakowski matrix K, which consists of real, non-negative numbers. Further details can be found in Ref. 28.

III. RESULTS

In this section, we present our main results for the spectra of random Lindblad generators $\mathscr{L}$. We will first consider the

unitary and dissipative parts of the Lindblad equation separately and then address the full problem. In all cases, we are interested in the distributions of eigenvalues and of their nearest-neighbor separations.

We denote eigenvalues of $\mathscr{L}$ by λ with suitable subscripts. Ensemble averages are denoted by $\langle\cdots\rangle$. Since the Lindblad generator is guaranteed to have a zero eigenvalue, which corresponds to the stationary solution, we will often exclude this eigenvalue from distributions and averages, which we then denote by a prime, e.g., $\langle\cdots\rangle'$.

A. Purely unitary dynamics

The dynamics of a quantum system is purely unitary if it is decoupled from its environment. The Lindblad operator then consists only of the Liouvillian $\mathscr{L}_0$ [see Eq. (12)], and its eigenvalues are $-i(E_m - E_n)$, where E_m are the eigenvalues of the Hamiltonian H. Obviously, the eigenvalues form complex conjugate pairs lying on the imaginary axis, and thus the mean value of the spectrum is zero. Upon averaging over the GUE, the variance of the eigenvalues along the imaginary axis is

$$\sigma_\lambda^2 \equiv \left\langle \frac{\mathrm{Tr}(i\mathscr{L}_0)^2}{N^2} \right\rangle = -\frac{1}{N^2}\langle \mathrm{Tr}\mathscr{L}_0^2\rangle. \tag{24}$$

Using results from Sec. II A, we obtain

$$\sigma_\lambda = 2\sqrt{N}\sigma, \tag{25}$$

where σ is the standard deviation in the GUE for H.

The distribution function $p_\lambda(\mathrm{Im}\lambda)$ of imaginary parts of eigenvalues of $\mathscr{L}_0$ contains a δ distribution for the N zero eigenvalues and a smooth distribution for all others,

$$p_\lambda(\mathrm{Im}\lambda) = \frac{1}{N}\delta(\mathrm{Im}\lambda) + \left(1 - \frac{1}{N}\right) p'_\lambda(\mathrm{Im}\lambda). \tag{26}$$

The smooth part stems from unequal eigenvalues and can, in principle, be obtained from the known joint distribution function $p_{\mathrm{GUE}}(E_1, \ldots, E_N)$ of eigenvalues of GUE matrices,[36]

$$p'_\lambda(y) = \int dy \int dE_1 \cdots dE_N\, p_{\mathrm{GUE}}(E_1, \ldots, E_N)\delta(y + E_1 - E_2). \tag{27}$$

We have obtained $p'_\lambda(\mathrm{Im}\lambda)$ by means of Monte Carlo simulations for random matrices from the GUE, choosing the standard deviation of components of H to be $\sigma = 1/\sqrt{8N}$. This choice assures that the spectral radius of the distribution function of eigenvalues of H approaches $r = 1$ for large N. Equation (25) shows that the width σ_λ of p'_λ is then $1/\sqrt{2}$. The results are shown in Fig. 1.

For $N = 2$, the distribution follows the result[19,30] for the separation of eigenvalues of the GUE because there is only a single positive $\mathrm{Im}\lambda$, which is the difference between the two eigenvalues of the random Hamiltonian H. For $N = 8$ and $N = 12$, the distribution shows a modulation close to the maximum, which can be understood as follows: for finite N, the positive $\mathrm{Im}\lambda$ are made up of differences between first ($k = 1$), second ($k = 2$), etc. neighbors of eigenvalues of H up to $k = N - 1$. Thus, the distribution function $p'_\lambda(\mathrm{Im}\lambda)$ can be written as a sum of contributions $p_\lambda^{(k)}$ from kth neighbors. The

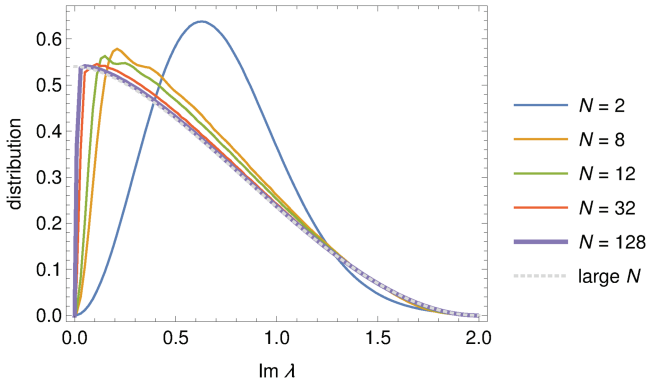

FIG. 1. Distribution function p'_λ of positive imaginary parts of eigenvalues of the Liouvillian $\mathscr{L}_0$ for Hamiltonians H taken from the GUE, for various matrix sizes $N \times N$. The full distribution is symmetric about $\mathrm{Im}\lambda = 0$. The standard deviation for the GUE has been chosen as $\sigma = 1/\sqrt{8N}$. We have performed Monte Carlo simulations using n_H random matrices with $n_H N^2 \geq 5 \times 10^7$ and collected all nonzero $\mathrm{Im}\lambda$ in histograms with 100 bins. The dotted gray curve is the asymptotic form in Eq. (29), which is based on the assumption that for $N \to \infty$ only the GUE eigenvalue distribution matters but not its eigenvalue correlations.

position of the maximum of $p_\lambda^{(k)}$ increases with k for fixed N. This leads to a modulation of $p'_\lambda(\mathrm{Im}\lambda)$ for not too large N but due to the overlap of the functions $p_\lambda^{(k)}$, one cannot generally resolve $N - 1$ separate peaks.

For large N, the separation between the maxima of the functions $p_\lambda^{(k)}$ approaches zero, the modulation vanishes, and the scaled distribution approaches a smooth limiting function. Only at small $\mathrm{Im}\lambda \lesssim 1/N$, the distribution is suppressed due to the repulsion between eigenvalues in the GUE (note that the typical separation between eigenvalues in the GUE scales with $r/N \cong 1/N$). For larger $\mathrm{Im}\lambda$, the repulsion matters less and we, therefore, conjecture that the limiting function can be obtained from the asymptotic semicircular distribution $p_{\mathrm{GUE}}^{\infty}$ of GUE eigenvalues[30] but ignoring eigenvalue correlations. The limiting function is then a convolution of the semicircular distribution function with itself,

$$p_\lambda'^{\infty}(y) = \frac{4}{\pi^2 r}\int_{-1+\frac{|y|}{2r}}^{1-\frac{|y|}{2r}} du \sqrt{1 - \left(u + \frac{y}{2r}\right)^2}\sqrt{1 - \left(u - \frac{y}{2r}\right)^2}, \tag{28}$$

where $r = \sqrt{8\sigma^2 N}$ is the spectral radius of the GUE. The integral evaluates to

$$p_\lambda'^{\infty}(y) = \frac{16}{3\pi^2 r}(1-a)\left[(1+a^2)\,E\left(\arcsin\frac{1-a}{1+a}\middle|\frac{(1+a)^2}{(1-a)^2}\right) + 2a\,F\left(\arcsin\frac{1-a}{1+a}\middle|\frac{(1+a)^2}{(1-a)^2}\right)\right], \tag{29}$$

where $a = |y|/2r$ and $E(\phi|m)$ and $F(\phi|m)$ are elliptic integrals. Figure 1 shows that the results for large N indeed approach $p_\lambda'^{\infty}(\mathrm{Im}\lambda)$.

Next, we turn to the distribution function $p'_{\Delta\lambda}$ of separations between neighboring $\mathrm{Im}\lambda$ for the nonzero eigenvalues. Here, higher-order correlations of GUE eigenvalues come into play since

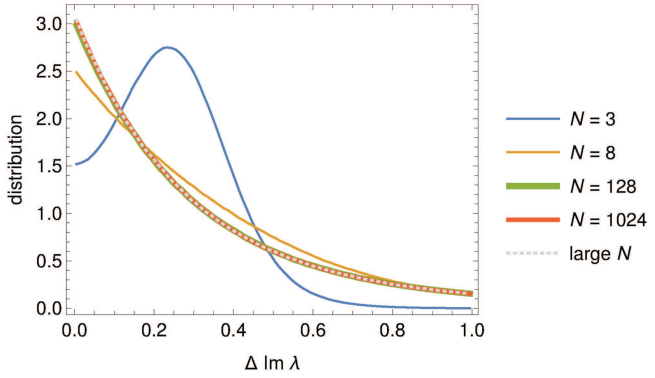

FIG. 2. Distribution function $p'_{\Delta\lambda}$ of separations $\Delta\mathrm{Im}\lambda$ of positive imaginary parts of eigenvalues of the Liouvillian $\mathscr{L}_0$ for Hamiltonians H taken from the GUE, for various matrix sizes $N \times N$. The other parameters are chosen as in Fig. 1. The dotted gray curve is obtained by drawing energies independently from the asymptotic semicircular form $p^{\infty}_{\mathrm{GUE}}$ of the GUE eigenvalue distribution, for $N = 1024$.

these separations are differences *of differences* of GUE eigenvalues. Figure 2 shows Monte Carlo results for various values of N. Since for $N = 2$ we obtain only a single positive imaginary part $|E_1 - E_2|$, the simplest nontrivial case is $N = 3$. Figure 2 shows that the distribution for $N = 3$ is strongly suppressed for small separations, compared to large N. This is easy to understand: H has three eigenvalues, which we order so that $E_1 < E_2 < E_3$. The Liouvillian $\mathscr{L}_0$ then has eigenvalues with the positive imaginary parts $y_{21} = E_2 - E_1$, $y_{32} = E_3 - E_2$, and $y_{31} = E_3 - E_1 = y_{21} + y_{32}$. One nearest-neighbor separation is $|y_{32} - y_{21}| = |E_3 - 2E_2 + E_1|$ and the other is $|y_{31} - y_{21}| = E_3 - E_2$ ($|y_{31} - y_{32}| = E_2 - E_1$) for $y_{21} > y_{32}$ ($y_{21} < y_{32}$). Hence, one of the two separations is the separation of two eigenvalues of H. Thus, for $N = 3$, half of the distribution is determined by the eigenvalue repulsion for the GUE.

For increasing N, this contribution from the eigenvalue repulsion decreases; the nearest-neighbor separations of $\mathscr{L}_0$ eigenvalues are increasingly due to eigenvalues of H that are far apart and thus weakly correlated. We expect that the limiting function is the one obtained from a semicircular distribution of uncorrelated energies E. To test this hypothesis, we have performed simulations by repeatedly drawing N energies independently from the semicircular distribution and then evaluating the differences $\Delta\lambda$ as above, for a large value of $N = 1024$. The result in Fig. 2 confirms that the distribution of nearest-neighbor separations of imaginary parts of eigenvalues of $\mathscr{L}_0$ indeed approaches this uncorrelated limit.

B. Pure dissipation

We now turn to the opposite limiting case where the generator $\mathscr{L}$ consists only of the dissipative part $\mathscr{L}_d$. This case is not easy to motivate physically. One might think that it corresponds to strong coupling between system and environment but this is not generally true since the coupling also contributes to the unitary dynamics. Our motivation for considering this case is rather its kinship with the Pauli master or rate equations discussed in Sec. I, which also do not involve a unitary part.

The master equation can be written in components as

$$\frac{d}{dt}\rho_{mn} = \sum_{p,q=1}^{N} \mathscr{L}^{d}_{mn,pq}\,\rho_{pq}, \tag{30}$$

where $\mathscr{L}^{d}_{mn,pq}$ is obtained in terms of random Kossakowski matrices from the WPUE as discussed in Sec. II B.

Equation (19) gives

$$\begin{aligned}\mathrm{Tr}\mathscr{L}_d &= \sum_{m,n=1}^{N} \mathscr{L}^{d}_{mn,mn} \\ &= \sum_{\alpha,\beta=1}^{\tilde{N}} K_{\alpha\beta}\left(\mathrm{Tr}L_\alpha \mathrm{Tr}L^{\dagger}_{\beta} - N\mathrm{Tr}L^{\dagger}_{\beta}L_\alpha\right) \\ &= -N\sum_{\alpha=1}^{\tilde{N}} K_{\alpha\alpha} = -N\,\mathrm{Tr}K\end{aligned} \tag{31}$$

and thus

$$\frac{\mathrm{Tr}\mathscr{L}_d}{\tilde{N}} = -\frac{N}{\tilde{N}}\mathrm{Tr}K. \tag{32}$$

This implies that for any given Kossakowski matrix K, the mean of the nonzero eigenvalues of the Lindblad generator $\mathscr{L}_d$ equals $-N$ times the mean eigenvalue of K. This holds even without ensemble averaging over the WPUE.

The ensemble average $\langle\lambda\rangle'$ of the nonzero eigenvalues for $\mathscr{L}_d$ is real and negative since the eigenvalues are real or form complex conjugate pairs and have non-positive real parts. With Eq. (32), we obtain

$$\begin{aligned}\langle\lambda\rangle' &= \frac{\langle\mathrm{Tr}\mathscr{L}_d\rangle_{\mathrm{WPUE}}}{\tilde{N}} \\ &= -\frac{N}{N^2-1}\langle\mathrm{Tr}K\rangle_{\mathrm{WPUE}} \equiv -N\langle k\rangle_{\mathrm{WPUE}}.\end{aligned} \tag{33}$$

As discussed above, we take $K = A^{\dagger}A$ with A distributed according to the GinUE. We thus find

$$\langle\lambda\rangle' = -\frac{N}{\tilde{N}}\langle\mathrm{Tr}A^{\dagger}A\rangle_{\mathrm{GinUE}} = -\frac{N}{\tilde{N}}\sum_{\alpha,\beta=1}^{\tilde{N}}\langle A^{*}_{\alpha\beta}A_{\alpha\beta}\rangle_{\mathrm{GinUE}}. \tag{34}$$

Since the components of A are equally normally distributed with standard deviation $\tilde{\sigma}$ this average can be evaluated as

$$\langle\lambda\rangle' = -2N\tilde{N}\tilde{\sigma}^2. \tag{35}$$

In the limit of large N, the distribution of singular values of GinUE matrices follows a quarter-circle law,[37] $p^{\infty}_{\mathrm{GinUE}}(a) = (4/\pi\tilde{r})\sqrt{1 - a^2/\tilde{r}^2}$ for $0 \le a \le \tilde{r}$. The spectral radius is $\tilde{r} = \sqrt{8\tilde{\sigma}^2\tilde{N}}$ in terms of the standard deviation $\tilde{\sigma}$. The eigenvalues of K are the squares of the singular values of A and thus the distribution

function of the eigenvalues of K is, in the limit of large N,

$$p^{\infty}_{\mathrm{WPUE}}(k) = \int_0^{\tilde{r}} da\, \delta(k-a^2)\, p^{\infty}_{\mathrm{GinUE}}(a) = \frac{2}{\pi\tilde{r}^2}\sqrt{\frac{\tilde{r}^2}{k}-1} \tag{36}$$

for $0 < k \le \tilde{r}^2 = 8\tilde{\sigma}^2\tilde{N}$. The asymptotic mean eigenvalue is

$$\langle k\rangle_{\infty} = \int_0^{\tilde{r}^2} dk\, k\, p^{\infty}_{\mathrm{WPUE}}(k) = \frac{\tilde{r}^2}{4} \tag{37}$$

and thus

$$\langle\lambda\rangle'_{\infty} = -N\langle k\rangle_{\infty} = -\frac{N}{4}\tilde{r}^2 = -2N\tilde{N}\tilde{\sigma}^2 \tag{38}$$

in the limit of large N, consistent with Eq. (35). The typical relaxation rate $-\langle\lambda\rangle' > 0$ is thus given by the dimension N of the Hilbert space of the open system, multiplied by its typical coupling strength to the environment, $\langle k\rangle_{\mathrm{WPUE}}$. It is clear that the relaxation rate must increase for stronger coupling. The factor N stems from the fact that for any state of the open system, there are of order N other states it can relax to. An analogous result is found for the limit of the Pauli master equation.[25]

We next turn to the width of the distribution of eigenvalues. We here calculate the average

$$\langle\tilde{\lambda}^2\rangle' \equiv \langle(\lambda - \langle\lambda\rangle')^2\rangle', \tag{39}$$

which is related to the widths in the real and imaginary directions by

$$\langle\tilde{\lambda}^2\rangle' = \langle(\mathrm{Re}\tilde{\lambda} + i\mathrm{Im}\tilde{\lambda})^2\rangle' = \langle(\mathrm{Re}\tilde{\lambda})^2\rangle' - \langle(\mathrm{Im}\tilde{\lambda})^2\rangle'. \tag{40}$$

In the last step, we have use that for any eigenvalue λ, there exists also an eigenvalue λ^*. We find

$$\langle\tilde{\lambda}^2\rangle' = \langle\lambda^2\rangle' - (\langle\lambda\rangle')^2 = \frac{\langle\mathrm{Tr}\mathscr{L}_d^2\rangle}{\tilde{N}} - N^2\langle k\rangle^2_{\mathrm{WPUE}}, \tag{41}$$

where Eq. (23) can be used to obtain

$$\begin{aligned}\mathrm{Tr}\mathscr{L}_d^2 &= \sum_{m,n,p,q=1}^{N} \mathscr{L}^d_{mn,pq}\,\mathscr{L}^d_{pq,mn}\\ &= \sum_{m,n,p,q=1}^{N} \tilde{K}_{mp,nq}\,\tilde{K}_{pm,qn}\\ &\quad - \sum_{m,n,q,r=1}^{N} \tilde{K}_{mm,nq}\,\tilde{K}_{rq,rn} - \sum_{m,n,p,r=1}^{N} \tilde{K}_{mp,nn}\,\tilde{K}_{rm,rp}\\ &\quad + \frac{N}{2}\sum_{n,q,r,s=1}^{N} \tilde{K}_{rn,rq}\tilde{K}_{sq,sn} + \frac{1}{2}\sum_{n,q,r,s=1}^{N} \tilde{K}_{rn,rn}\tilde{K}_{sq,sq}.\end{aligned} \tag{42}$$

Now, note that

$$\begin{aligned}\sum_{m=1}^{N}\tilde{K}_{mm,nq} &= \sum_{\gamma,\delta=1}^{N^2}\sum_{m=1}^{N}\mathscr{U}_{\gamma,mm}\begin{pmatrix}K & 0\\ 0 & 0\end{pmatrix}_{\gamma\delta}\mathscr{U}^*_{\delta,nq}\\ &= \sum_{r=1}^{N}\sum_{\delta=1}^{N^2}\sum_{m=1}^{N} O_{rm}\begin{pmatrix}K & 0\\ 0 & 0\end{pmatrix}_{rr,\delta}\mathscr{U}^*_{\delta,nq} = 0\end{aligned} \tag{43}$$

since the row sums of O vanish for rows $r = 1,\ldots,N-1$ [see Eq. (18)], and the last row of the extended Kossakowski matrix contains only zeros. Analogously, we find $\sum_{n=1}^{N}\tilde{K}_{mp,nn} = 0$. Moreover, the last term in Eq. (42) is $1/2\,(\mathrm{Tr}\tilde{K})^2 = 1/2\,(\mathrm{Tr}K)^2$. We thus obtain

$$\mathrm{Tr}\mathscr{L}_d^2 = \sum_{m,n,p,q=1}^{N}\tilde{K}_{mp,nq}\,\tilde{K}_{pm,qn} + \frac{N}{2}\sum_{n,q,r,s=1}^{N}\tilde{K}_{rn,rq}\tilde{K}_{sq,sn} + \frac{1}{2}(\mathrm{Tr}K)^2. \tag{44}$$

Next, the ensemble average is taken. The calculations are somewhat tedious and details are given in Appendix B. The result is

$$\begin{aligned}\langle\mathrm{Tr}\mathscr{L}_d^2\rangle &= \tilde{N}^2(\tilde{N}+1)\,\langle|A_{**}|^2\rangle^2 + \frac{\tilde{N}^3(\tilde{N}+1)}{2}\,\langle|A_{**}|^2\rangle^2\\ &\quad + \frac{\tilde{N}^2(\tilde{N}^2+1)}{2}\,\langle|A_{**}|^2\rangle^2\\ &= \tilde{N}^2\left(\tilde{N}^2 + \frac{3}{2}\tilde{N} + \frac{3}{2}\right)\langle|A_{**}|^2\rangle^2,\end{aligned} \tag{45}$$

where A_{**} represents any element of the matrix A, which are identically distributed according to the GinUE. The average of $|A_{**}|^2$ can be expressed in terms of the standard deviation as $\langle|A_{**}|^2\rangle = 2\tilde{\sigma}^2$, giving

$$\langle\mathrm{Tr}\mathscr{L}_d^2\rangle = 2\tilde{N}^2\left(2\tilde{N}^2 + 3\tilde{N} + 3\right)\tilde{\sigma}^4. \tag{46}$$

Using Eqs. (33), (35), and (41), we obtain

$$\langle\tilde{\lambda}^2\rangle' = 2\tilde{N}\left(2\tilde{N}^2 + 3\tilde{N} + 3\right)\tilde{\sigma}^4 - 4N^2\tilde{N}^2\tilde{\sigma}^4 = 2\tilde{N}(\tilde{N}+3)\,\tilde{\sigma}^4. \tag{47}$$

This quantity is positive for all N, implying that the distribution of nonzero eigenvalues of $\mathscr{L}_d$ is always broader in the real direction than in the imaginary direction.

In the following, we choose $\tilde{\sigma} = 1/\sqrt{8\tilde{N}}$ so that the GinUE spectral radius $\tilde{r}$ approaches unity for large N. The corresponding limits then become $\langle k\rangle_{\infty} = 1/4$, $\langle\lambda\rangle'_{\infty} = -N/4$, and $\langle\tilde{\lambda}^2\rangle'_{\infty} = 1/32$. Physically, the assumption of N-independent $\langle k\rangle_{\infty}$ means that the typical coupling strength to the environment does not depend on the dimension of the Hilbert space of the open system. Note that the width also becomes independent of N for this choice.

We now present numerical results for the distribution of nonzero eigenvalues of $\mathscr{L}_d$. To get an idea on the structure of the distribution, we plot it for $N = 3$ in Fig. 3, where the center has been shifted to the asymptotic large-N average $\langle\lambda\rangle'_{\infty} = -N/4$. Figure 3 shows that the eigenvalues form two distinct families: there are strictly real eigenvalues and complex conjugate pairs, which are evidently repelled by the real axis. This is similar to the Ginibre orthogonal ensemble (GinOE).[26] At first glance, this is surprising since the GinOE consists of real matrices, whereas $\mathscr{L}_d$ is complex. However, Eq. (19) together with the Hermiticity of the Kossakowski matrix implies that

$$\mathscr{L}^{d*}_{mn,pq} = \mathscr{L}^{d}_{nm,qp}. \tag{48}$$

Using this generalized reality condition for the $N^2\times N^2$ matrix $\mathscr{L}_d$, one can show that $\mathscr{L}_d P = \lambda P$ is equivalent to $\mathscr{L}_d P^{\dagger} = \lambda^* P^{\dagger}$. Consequently, the eigenvalues of $\mathscr{L}_d$ are real or form complex conjugate pairs. Equation (48) also implies that there exists a unitary matrix Ω, which is independent of $\mathscr{L}_d$, so that $\Omega\,\mathscr{L}_d\,\Omega^{\dagger}$ is real. The proof

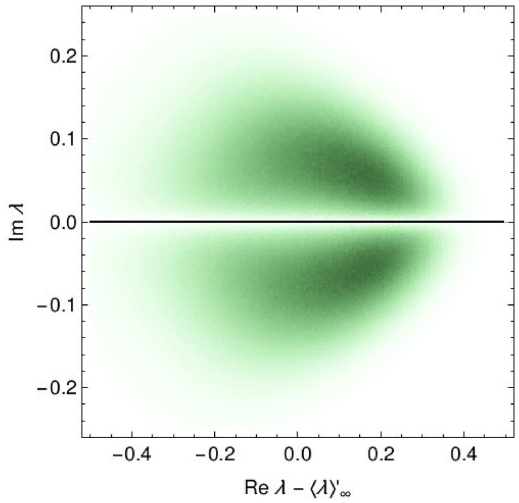

FIG. 3. Distribution function p'_λ of nonzero eigenvalues of the Lindblad generator $\mathscr{L}_d$ for the case of purely dissipative dynamics for $N = 3$. We have performed Monte Carlo simulations using $n_K = 2 \times 10^7$ random 8×8 Kossakowski matrices K taken from the WPUE and collected all nonzero eigenvalues λ in a histogram with 301×301 bins. The standard deviation for the WPUE has been chosen as $\tilde{\sigma} = 1/\sqrt{8\tilde{N}}$, which guarantees that the spectral radius of the Kossakowski matrices approaches unity for large N. The line along the real axis has an apparent nonzero thickness due to the finite bin size.

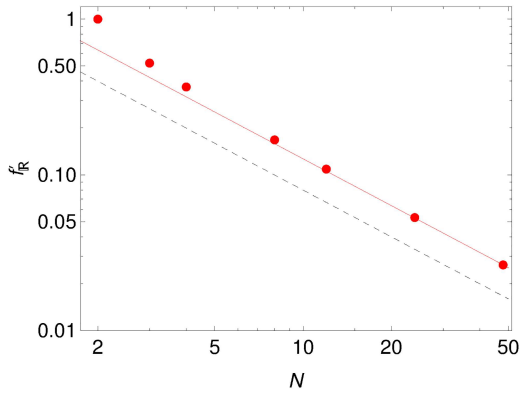

FIG. 4. Fraction $f'_{\mathbb{R}}$ of nonzero real eigenvalues of the dissipative Lindblad generator $\mathscr{L}_d$ that are real, for various dimensions N of the Hilbert space of the reduced system (red circles). We have performed Monte Carlo simulations using n_K random Kossakowski matrices so that $n_K N^2 \geq 10^8$. The solid red line shows the function const$/N$ fitted to the data point at $N = 48$. The dashed black line denotes the large-N asymptotic form of the corresponding fraction $f_{\mathbb{R},\,\mathrm{GinOE}} = \sqrt{2/\pi}/N$ for the GinOE.[38]

is relegated to Appendix C. Thus, the ensemble of generators $\mathscr{L}_d$ is unitarily equivalent to an ensemble of real $N^2 \times N^2$ matrices. Of course, this ensemble is not the GinOE because the components of $\mathscr{L}_d$ are not independently normally distributed but rather are determined by the random Kossakowski matrix. Analogous arguments apply to $\mathscr{L}_0$, and hence also to $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_d$.

In the following, we will treat the real and complex eigenvalues of $\mathscr{L}_d$ separately. The first question is what fraction $f'_{\mathbb{R}}(N)$ of nonzero eigenvalues is real, on average. For $N = 2$, there is only a single nonzero eigenvalue, which must be real. Thus, we find $f'_{\mathbb{R}}(2) = 1$. For larger N, the fraction has been obtained by Monte Carlo simulations and is plotted in Fig. 4. We see that the fraction is consistent with a scaling with $1/N$ for large N. As a function of the matrix size $\tilde{N}$ of the Kossakowski matrix K or $\tilde{N} + 1$ of the generator $\mathscr{L}_d$, this corresponds to $1/\sqrt{\tilde{N}}$ for large $\tilde{N}$. We have also found that the scaling does not change if we generate the random Kossakowski matrices such that their eigenvalues are independently identically distributed random variables with the distribution $p^{\infty}_{\mathrm{WPUE}}$[28] (not shown). The power law agrees with the fraction of real eigenvalues for the GinOE, which scales with the matrix size N as $1/\sqrt{N}$, as shown in Ref. 38. However, the fraction $f'_{\mathbb{R}}(N)$ for our case exceeds the GinOE asymptotics by a factor of about 1.59. It would be interesting to obtain the probability $p'_0(N)$ of having *no* real eigenvalue except for the always present $\lambda_0 = 0$. For the GinOE, the corresponding probability is a stretched exponential, $p_0(N) \cong C \exp(-\zeta(3/2)\sqrt{N/2\pi})$, for large N, where C is a constant and $\zeta(z)$ is the Riemann zeta function.[39]

Figure 5 shows the distribution function of nonzero real eigenvalues of the purely dissipative Lindblad generator $\mathscr{L}_d$ for various N, shifted by the mean $\langle\lambda\rangle'_\infty = -N/4$. The results suggest that the distribution becomes symmetric about zero and has an N-independent width for $N \to \infty$. Moreover, the tails for $|\lambda - \langle\lambda\rangle'_\infty| > 1/2$ are suppressed for large N, consistent with a compact support. This is analogous to the GUE[14,30,31] and the GinOE.[26,38] On the other hand, the distribution function is distinct from both the semicircle law for the GUE and the flat distribution for the GinOE. We have not tried to find a proof for a compact support nor to derive the analytical form of the asymptotic distribution.

In Fig. 6, we plot the distribution function of complex eigenvalues with positive imaginary part, shifted by the mean $\langle\lambda\rangle'_\infty = -N/4$. We note that since the fraction $f'_{\mathbb{R}}(N)$ of real nonzero

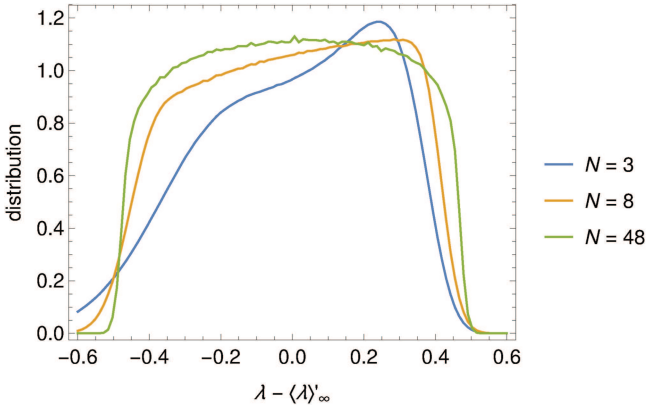

FIG. 5. Distribution function $p'_{\lambda\in\mathbb{R}}$ of nonzero real eigenvalues of the dissipative Lindblad generator $\mathscr{L}_d$, for various N. 100 bins have been used. The other parameters are chosen as in Fig. 3. We have performed Monte Carlo simulations using n_K random Kossakowski matrices so that $n_K N^2 \geq 10^8$.

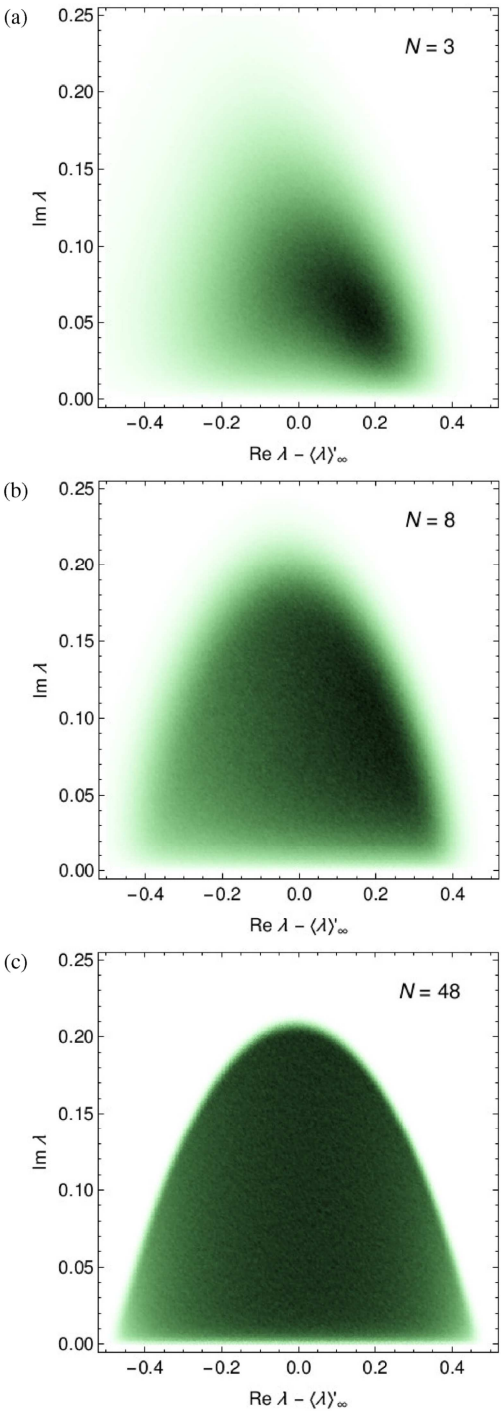

FIG. 6. Distribution function of complex eigenvalues with positive imaginary part of the Lindblad generator $\mathscr{L}_d$ for the case of purely dissipative dynamics, for (a) $N = 3$, (b) $N = 8$, and (c) $N = 48$. The distribution is symmetric with respect to the real axis. The parameters of the Monte Carlo simulations are as for the previous figures. We have used 301 × 301 bins.

eigenvalues and the relative weight of the single zero eigenvalue approach zero for $N \to \infty$, the distribution of complex eigenvalues asymptotically gives the complete distribution. Like for the real eigenvalues, the skewness decreases for increasing N and the distribution appears to approach a compact support. Moreover, the width of the distribution in both the real and imaginary directions approaches a constant in the large-N limit, consistent with our result $\langle(\mathrm{Re}\tilde{\lambda})^2\rangle'_\infty - \langle(\mathrm{Im}\tilde{\lambda})^2\rangle'_\infty = 1/32$. This figure corresponds to Fig. 2 in Ref. 29. Using the quaternionic extension of the theory of free probability,[40–44] Denisov *et al.*[29] have found the boundary of the asymptotic support in terms of the solution of an algebraic equation involving elliptic integrals. They call this a *lemon* shape. Sá *et al.*[35] find the same in the limit of many jump operators.

For comparison, the complex eigenvalues of the GinOE also have a compact support for $N \to \infty$ but it is here given by a circle.[45,46] For the Pauli master equation,[25] the distribution also appears to have a compact support for $N \to \infty$, which is distinct from both the GinOE and the ensemble of dissipative Lindblad generators.

Figure 6 shows a suppression of the distribution function close to the real axis. To analyze this further, we plot in Fig. 7 the distribution function of the (positive) imaginary parts Imλ of complex eigenvalues. We find that the distribution increases linearly for small Imλ. The same behavior is found for the GinOE.[27]

Next, we turn to the correlations between eigenvalues. Figures 6 and 7 show that the complex eigenvalues are repelled by the real axis. We now discuss the real and complex eigenvalues separately. We have found above that the width of the distribution of the real eigenvalues approaches a constant for large N if the WPUE is defined in such a way that the spectral radius of the Kossakowski matrices approaches unity. Moreover, the fraction of real eigenvalues scales as $1/N$ and thus the number of real eigenvalues scales as $N^2/N = N$. Ignoring correlations, the mean separation between real eigenvalues then scales as $1/N$. It is thus convenient to rescale separations between eigenvalues with a factor of N. Figure 8 shows the distribution function $p'_{N\Delta\lambda,\mathbb{R}}(\Delta\lambda)$ of $N\Delta\lambda$, where $\Delta\lambda > 0$ is the separation

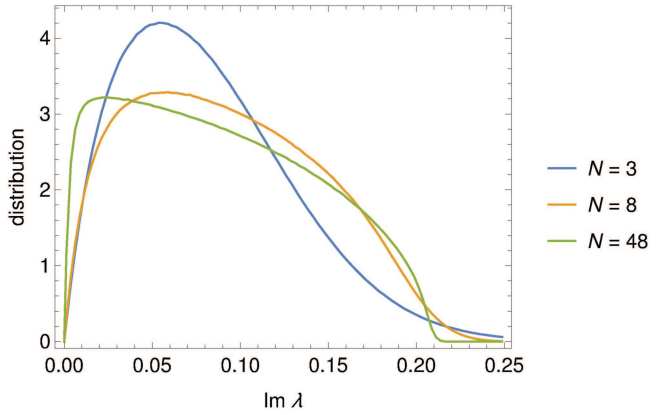

FIG. 7. Distribution function of the imaginary parts of the complex eigenvalues shown in Fig. 6. The data have been collected in 100 bins.

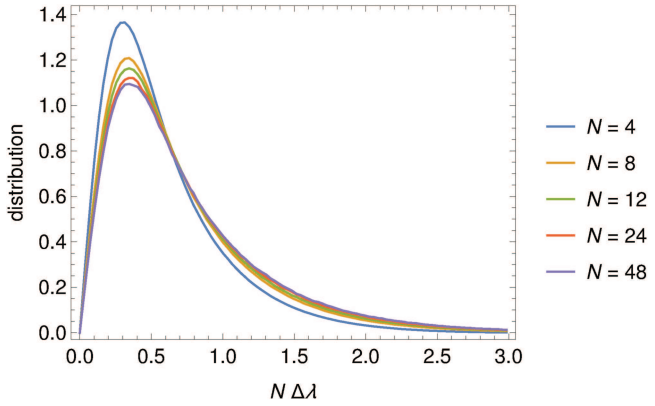

FIG. 8. Distribution function of separations $\Delta\lambda > 0$ of neighboring nonzero real eigenvalues of the Lindblad generator $\mathscr{L}_d$ for the case of purely dissipative dynamics and various values of N. The parameters of the Monte Carlo simulations are as for the previous figures. We have used 100 bins.

between neighboring real eigenvalues, for various N. The rescaled distribution approaches an N-independent limit for large N. The most relevant result is the *linear* dependence on $\Delta\lambda$ for small separations. This is the same linear suppression as for the GinOE[27] and for the Pauli master equation.[25]

The distribution of complex eigenvalues also assumes constant widths in both real and imaginary directions for large N if the WPUE is chosen as above. Figure 9 shows the distribution function $p'_{\Delta\lambda,\mathbb{C}}(\Delta\lambda)$ of complex differences of neighboring eigenvalues with positive imaginary part, rescaled with N, for $N = 4$ and $N = 48$. By "neighboring eigenvalues," we mean that for any eigenvalue λ with $\mathrm{Im}\lambda > 0$, we have found the eigenvalue $\lambda' \neq \lambda$ with $\mathrm{Im}\lambda' > 0$ that minimizes $|\lambda' - \lambda|$ and then added $\Delta\lambda = \lambda' - \lambda$ to a histogram. For small N, the distribution is rather asymmetric since the two eigenvalues forming a nearest-neighbor pair typically lie far apart and are thus affected by the overall distribution of eigenvalues. For large N, the distribution of separations appears to become isotropic, which is reasonable since here the separation is small compared to the width of the eigenvalue distribution and the plot is only sensitive to local correlations. The plots look very similar to the case of the Pauli master equation.[25]

Because of the asymptotic isotropy of $p'_{\Delta\lambda,\mathbb{C}}(\Delta\lambda)$ in the complex plane, it is of interest to plot the distribution of $|\Delta\lambda|$, which is shown in Fig. 10. We observe that the distribution is suppressed like $|\Delta\lambda|^3$ for small $|\Delta\lambda|$. This result also agrees with the GinOE[27] and the case of the Pauli master equation.[25]

To conclude this part, the correlations between neighboring eigenvalues are essentially the same for the WPUE, the GinOE, and the EGRE for the Pauli master equation, whereas the distributions of the eigenvalues themselves remain distinct for all N. The WPUE consists of complex matrices $\mathscr{L}_d$, whereas the GinOE and the EGRE are ensembles of real matrices. We conjecture that the correlations in the WPUE agrees with the real ensembles because of the generalized reality condition (48), which implies that the eigenvalues λ of $\mathscr{L}_d$ are real or form complex conjugate pairs. Moreover, neighboring

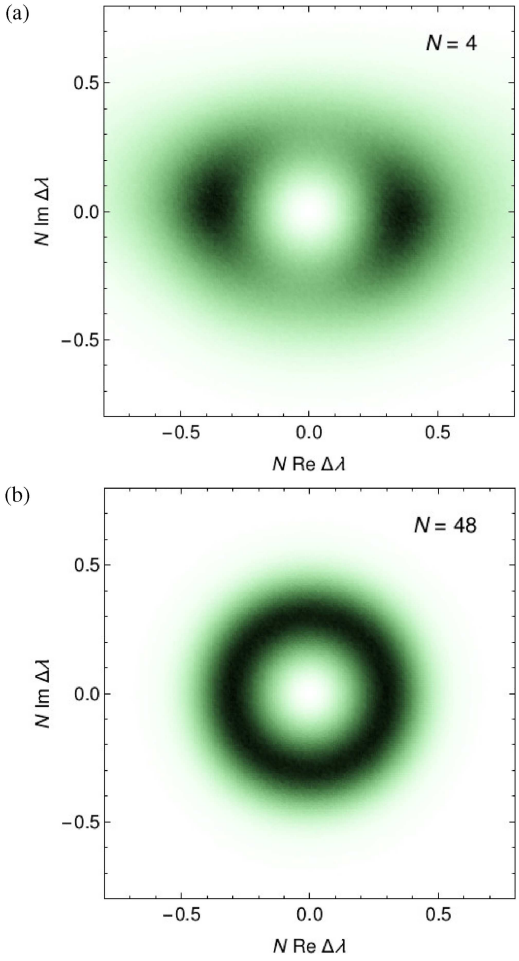

FIG. 9. Distribution function of complex separations $\Delta\lambda$ of neighboring (see text) eigenvalues with positive imaginary part of the Lindblad generator $\mathscr{L}_d$ for the case of purely dissipative dynamics, for (a) $N = 4$ and (b) $N = 48$. The parameters of the Monte Carlo simulations are as for the previous figures. We have used 301×301 bins.

eigenvalues are increasingly close to each other for large N so that the distribution of nearest-neighbor separation becomes insensitive to the overall shape of the eigenvalue distribution.

C. Full Lindblad generator

For general open systems, the Lindblad generator $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_d$ is the sum of a unitary part $\mathscr{L}_0$ and a dissipative part $\mathscr{L}_d$. To define a random-matrix ensemble for $\mathscr{L}$, we use a statistically independent combination of the ensembles for $\mathscr{L}_0$ and $\mathscr{L}_d$ studied above. Then, like for pure dissipation, $\mathscr{L}$ has an eigenvalue $\lambda_0 = 0$, which is nondegenerate with probability one. The mean of the nonzero eigenvalues is only determined by the dissipative part since $\mathrm{Tr}\mathscr{L}_0 = 0$ and thus equals $\langle\lambda\rangle' = -N\langle k\rangle_{\mathrm{WPUE}}$.

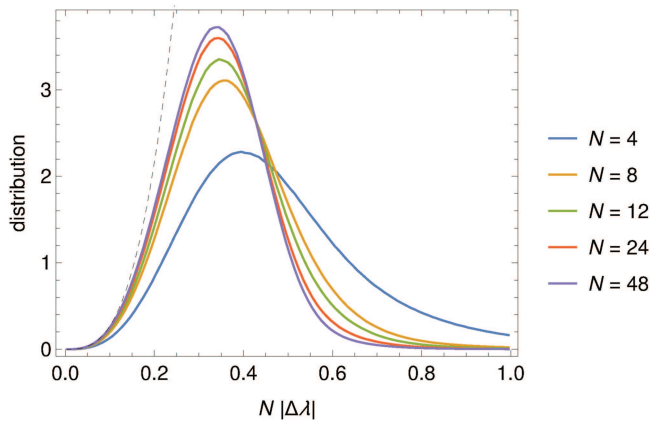

FIG. 10. Distribution function of the absolute values $|\Delta\lambda|$ for the separations between neighboring complex eigenvalues with positive imaginary part for various N. The parameters of the Monte Carlo simulations are as for the previous figures. We have used 100 bins. The black dashed curve shows the function $c\,N^3|\Delta\lambda|^3$, where c is fitted to the data for $N = 48$ at small $|\Delta\lambda|$.

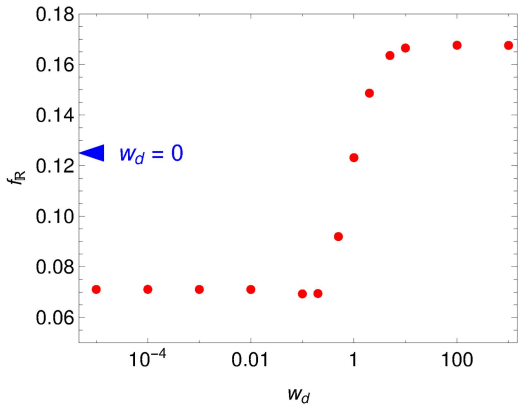

FIG. 11. Fraction $f_{\mathbb{R}}(N)$ of real eigenvalues of the full Lindblad generator $\mathscr{L}$ that are real as a function of the relative strength of dissipation, w_d, for dimension $N = 8$ of the Hilbert space of the reduced system. The blue triangle points to the exact result $f_{\mathbb{R}}(N) = 1/N$ for purely unitary dynamics ($w_d = 0$). We have performed Monte Carlo simulations using n_K random Kossakowski matrices and random Hamiltonians so that $n_K N^2 \geq 10^8$.

Generically, the inverse time (or energy) scales of $\mathscr{L}_0$ and $\mathscr{L}_d$ are unrelated. The relative scale interpolates between the previously studied cases of unitary dynamics and pure dissipation. In the following, we keep the distribution of $\mathscr{L}_0$ fixed and vary the scale of $\mathscr{L}_d$, which corresponds to varying the strength of dissipation. As noted above, the width of the distribution of the imaginary eigenvalues of $\mathscr{L}_0$ is given by $\sigma_\lambda = 2\sqrt{N}\sigma$, where σ is the standard deviation of the GUE for the Hamiltonian H. We choose $\sigma = 1/\sqrt{8N}$ like in Sec. III A.

Following Sec. III B, we take $(\langle\tilde{\lambda}^2\rangle'_\infty)^{1/2} = \sqrt{2}\,\tilde{N}\tilde{\sigma}^2$ as a measure of the width of the distribution of the nonzero eigenvalues of $\mathscr{L}_d$. For this quantity to be equal to the width of eigenvalues of $\mathscr{L}_0$, we would need to set $\tilde{\sigma} = 1/\sqrt{2\tilde{N}}$ since then

$$\langle\tilde{\lambda}^2\rangle'_\infty = \frac{1}{2} = \sigma_\lambda^2 \tag{49}$$

in the limit of large N. We will use

$$w_d \equiv \frac{\sqrt{\langle\tilde{\lambda}^2\rangle'_\infty}}{\sigma_\lambda} \tag{50}$$

as a measure for the strength of dissipation relative to the typical energy scale of the Hamiltonian H. For the numerics, this implies the choice $\tilde{\sigma}^2 = (\sqrt{2N}/\tilde{N})\,w_d\sigma = w_d/(2\tilde{N})$ and incidentally means that the mean eigenvalue of the Kossakowski matrix $\langle k\rangle_{\mathrm{WPUE}}$ approaches w_d for large N.

The distribution of eigenvalues λ of $\mathscr{L}$ is similar to the case of pure dissipation. In particular, there are distinct populations of real eigenvalues and of complex conjugate pairs. Figure 11 shows the average fraction of real eigenvalues as a function of w_d for $N = 8$. Evidently, this fraction approaches constant values in both the limits of weak and strong dissipation and it is largest for strong dissipation. This limiting value agrees with the one found for pure dissipation, plotted in Fig. 4. More interestingly, for weak dissipation this fraction is not continuous—the limit for $w_d \to 0$ is smaller than the exact value $f_{\mathbb{R}}(N) = 1/N$ for purely unitary dynamics. This is because infinitesimal dissipation splits the N-fold degeneracy of the zero eigenvalue of $\mathscr{L}_0$, typically generating some complex conjugate pairs and thereby reducing the fraction of real eigenvalues. The value of $f_{\mathbb{R}}(N)$ for infinitesimal nonzero dissipation is currently not understood.

The distribution of real eigenvalues for $N = 8$ and various values of w_d is shown in Fig. 12. The distribution gets broader for stronger dissipation, i.e., larger w_d. This is plausible since for purely unitary dynamics the width in the real direction is zero. We have rescaled the deviation $\lambda - \langle\lambda\rangle'_\infty$ by $1/w_d$ in Fig. 12. We observe a crossover of the *rescaled* distribution with an increase of its width for increasing w_d, in the same range where the fraction $f_{\mathbb{R}}$ increases.

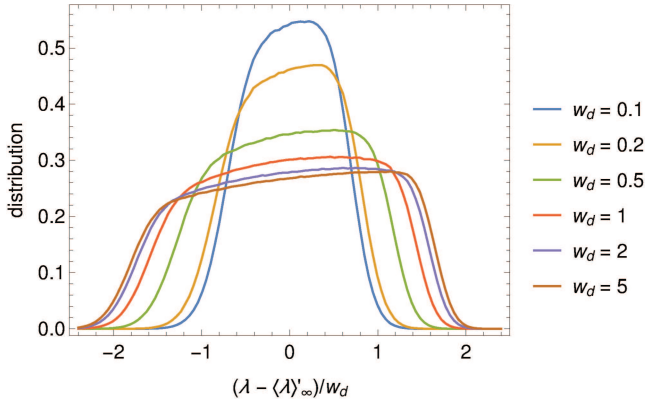

FIG. 12. Distribution function $p'_{\lambda\in\mathbb{R}}$ of nonzero real eigenvalues of the full Lindblad generator $\mathscr{L}$, for $N = 8$ and various values of w_d. 100 bins have been used. We have performed Monte Carlo simulations using n_K random Kossakowski matrices and random Hamiltonians so that $n_K N^2 \geq 10^8$.

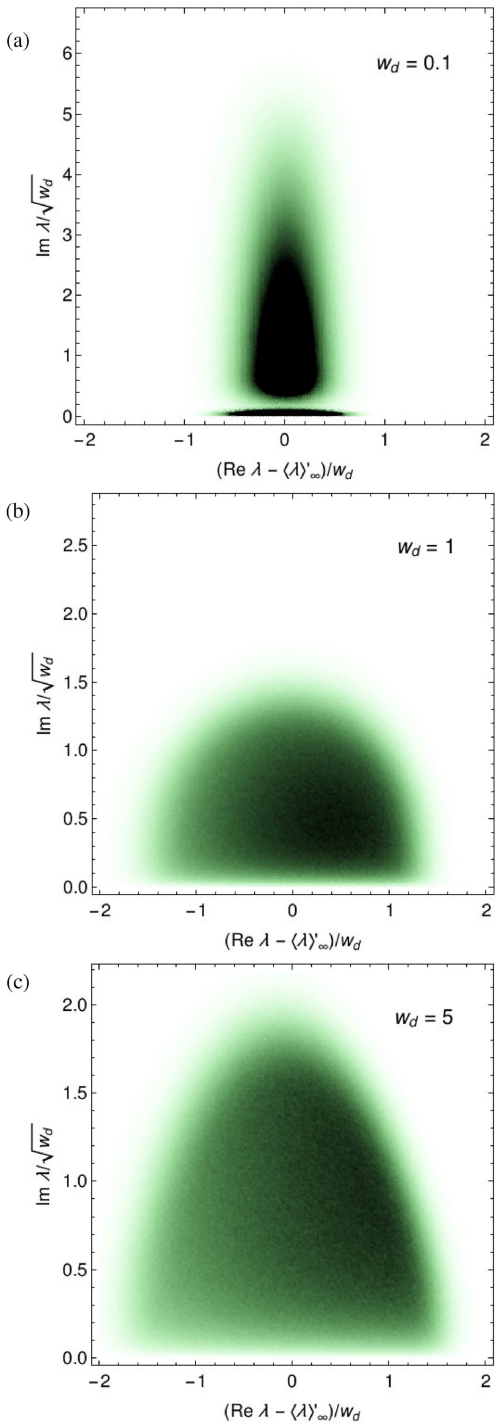

FIG. 13. Distribution function of complex eigenvalues with positive imaginary part of the full Lindblad generator $\mathscr{L}$ for $N = 8$ and the strength of dissipation (a) $w_d = 0.1$, (b) $w_d = 1$, and (c) $w_d = 5$. The distribution is symmetric with respect to the real axis. The parameters of the Monte Carlo simulations are as for the previous figures. We have used 301 × 301 bins.

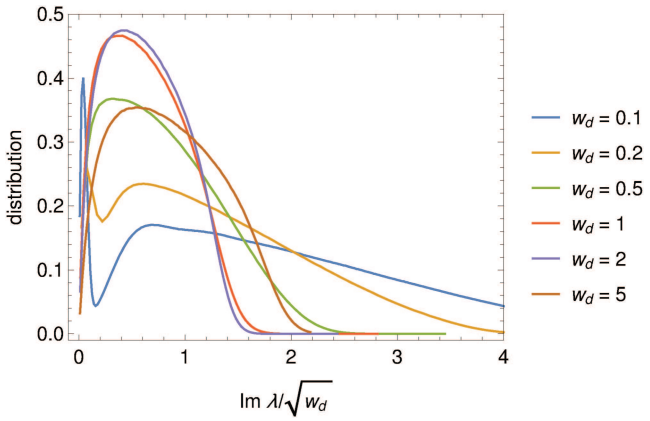

FIG. 14. Distribution function of the imaginary parts of the complex eigenvalues with positive imaginary part of the full Lindblad generator $\mathscr{L}$ for $N = 8$ and various values of w_d. We have performed Monte Carlo simulations using n_K random Kossakowski matrices and random Hamiltonians so that $n_K N^2 \geq 10^8$ and used 100 bins, except for $w_d = 0.1$, where $n_K N^2 = 5.12 \times 10^8$ and 400 bins have been used.

The distribution of complex eigenvalues with positive imaginary part is plotted in Fig. 13 for $N = 8$ and three values of w_d. The real parts have been rescaled with a factor of $1/w_d$ and the imaginary parts with $1/\sqrt{w_d}$ since this turns out to give weakly w_d-dependent widths. For strong dissipation, i.e., large w_d, the distribution is similar to the case of pure dissipation [compare Fig. 13(c) with Fig. 6(b)]. For smaller w_d, the lemon shape morphs into an ellipsoidal one, as also found by Denisov *et al.*[29] For $w_d < 0.5$, the distribution develops two maxima, one of which forms a band close to the real axis. This is also seen in Fig. 14, which shows the distribution of the imaginary part of the complex eigenvalues. What happens is that a fraction of $1/N$ of the complex eigenvalues splits off from the rest and approaches the real axis for $w_d \to 0$ to form the N-fold degenerate zero eigenvalue for purely unitary dynamics. These eigenvalues

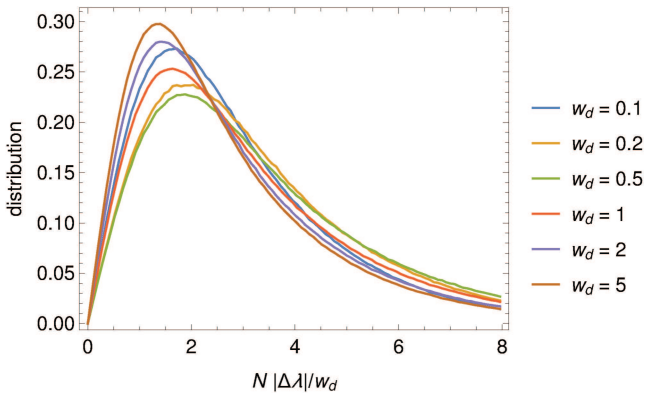

FIG. 15. Distribution function of separations $\Delta\lambda > 0$ of neighboring nonzero real eigenvalues of the full Lindblad generator $\mathscr{L}$ for $N = 8$ and various values of w_d. The parameters of the Monte Carlo simulations are as for the previous figures. We have used 100 bins.

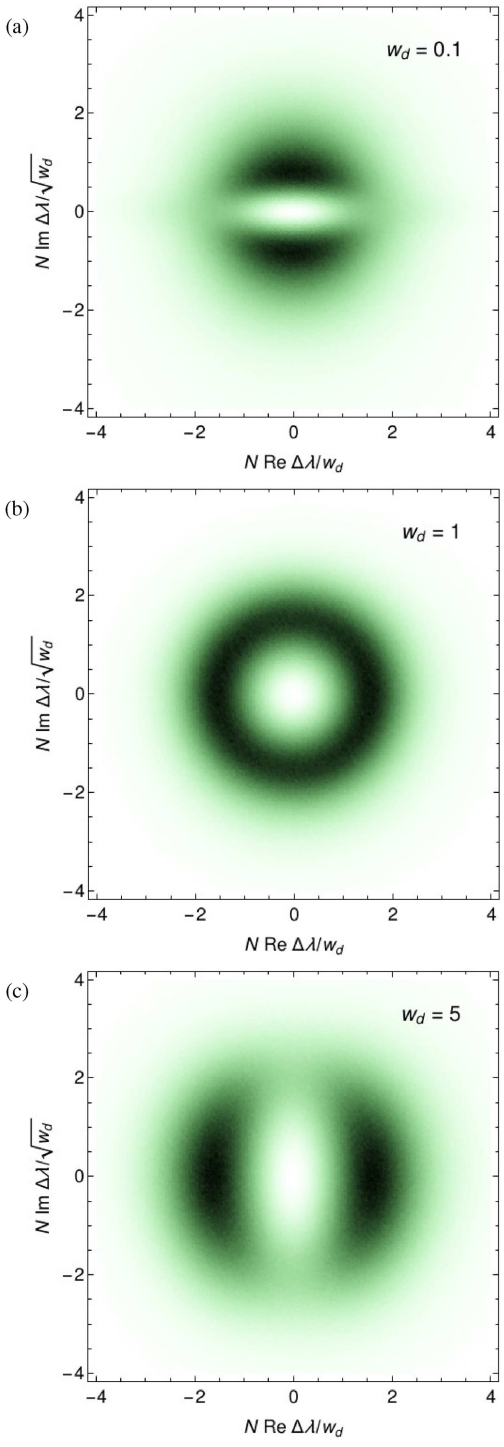

FIG. 16. Distribution function of complex separations $\Delta\lambda$ of neighboring eigenvalues with positive imaginary part of the full Lindblad generator $\mathscr{L}$ for $N = 8$ and (a) $w_d = 0.1$, (b) $w_d = 1$, and (c) $w_d = 5$. The parameters of the Monte Carlo simulations are as for the previous figures. We have used 301×301 bins.

thus make up the difference between $f_{\mathbb{R}}$ for small nonzero w_d and the exact $f_{\mathbb{R}} = 1/N$ for $w_d = 0$ in Fig. 11. We have also studied the eigenvalue distribution for larger N in Ref. 28. The results echo what we have found for purely dissipative dynamics and are not shown here: the distributions develop sharp boundaries and the fraction of real eigenvalues shows scaling consistent with the GinOE for large N.

Finally, we turn to the eigenvalue correlations for the full Lindblad generator, where we go beyond Ref. 28 by studying their dependence on the strength w_d of dissipation. Figures 13 and 14 show that the real eigenvalues repel the complex ones with a characteristic exponent of unity, like for the case of pure dissipation. For the real eigenvalues, we plot in Fig. 15 the distribution function $p'_{\Delta\lambda,\mathbb{R}}(\Delta\lambda)$ of the conveniently rescaled separations between neighboring real eigenvalues for $N = 8$ and various w_d. We see that while the detailed shape changes somewhat with w_d, the repulsion remains linear for call cases. Figure 16 shows the distribution function $p'_{\Delta\lambda,\mathbb{C}}(\Delta\lambda)$ of complex differences of neighboring eigenvalues with positive imaginary part for $N = 8$ and three values of w_d. Note the different scalings used for the real and imaginary axes. For strong dissipation, i.e., large w_d, there is an accumulation of weight in the real directions. This agrees with the purely dissipative limit for small N, see Fig. 9(a). For weak dissipation, we instead find an accumulation in the imaginary direction, which is plausible since the spectrum approaches the imaginary axis for purely unitary dynamics. For $w_d = 1$, we find a nearly circular distribution of $\Delta\lambda$. This should be understood as an accidental near cancellation of the effects of small N and small w_d. The behavior for small $|\Delta\lambda|$ is cubic in all cases (not shown).

IV. SUMMARY AND CONCLUSIONS

To summarize, we have studied the spectra of Lindblad generators governing open Markovian quantum systems using random-matrix theory. The Hamiltonian describing the unitary time evolution was taken from the GUE. The generator for the dissipative part was constructed from the Kossakowski matrix of coefficients in the Lindblad equation. The Kossakowski matrix was taken from the WPUE distribution first studied by Wishart.[34] The same choices have been made independently by Denisov *et al.*[29] We have studied the distribution of eigenvalues as well as the correlations of neighboring eigenvalues for the purely unitary limit, i.e., a closed system, the limit of pure dissipation and the combination of both, parametrized by the relative strength of the dissipative part.

For purely unitary dynamics, we give conjectures for the distributions of eigenvalues and eigenvalue separations in the limit of large dimension N of the Hilbert space. For pure dissipation, the lemon-shaped distribution of complex eigenvalues agrees with numerical and analytical results of Ref. 29. We also obtain the fraction of real eigenvalues, which shows scaling consistent with the GinOE, and the distribution of the real eigenvalues. The distribution of nearest-neighbor separations scales linearly with the separation for two real eigenvalues and for one real and one complex eigenvalue. It scales with the third power for two complex eigenvalues. This universal behavior also agrees with the GinOE. We conjecture that this agreement with the GinOE—in spite of the Lindblad generator not being real—results from the fact that the ensemble of random Lindblad generators is unitarily equivalent to an ensemble of real matrices, as proven in Appendix C.

For combined unitary and dissipative dynamics, clearly the physically most relevant case, the power laws describing the repulsion between eigenvalues are the same as for purely dissipative dynamics. By varying the relative strength of dissipation, we show how the full dynamics interpolates smoothly between the unitary and purely dissipative limits, except for the fraction $f_{\mathbb{R}}$ of real eigenvalues. In the limit of weak dissipation, this fraction approaches a value that is significantly lower than the exact fraction $1/N$ for purely unitary dynamics. This is a consequence of the existence of a population of eigenvalues with nonzero imaginary parts for arbitrarily weak but nonzero dissipation, which approaches but never quite reaches zero as the coupling between the system and its environment is reduced. This spectral discrepancy between the Liouvillian and the full Lindblad generator reflects the fundamental difference between the dynamics of closed and open quantum systems. While a closed quantum system with an N-dimensional Hilbert space has N steady states—the eigenstates of the Hamiltonian—an open quantum system typically only has a single steady state—the state in which it is in equilibrium with its environment—no matter how weak the coupling between system and environment. Weaker coupling (smaller $\langle k \rangle$) only means slower decoherence (smaller $|\langle\lambda\rangle'|$).

Additional results, including more detailed mathematical derivations and treatments of other random-matrix ensembles, are given in Ref. 28. We note that Sá *et al.*[47] have recently considered the complex ratio between next-nearest-neighbor and nearest-neighbor eigenvalues for the GUE and the GinUE as well as for their circular versions. By evaluating the distribution of this complex ratio for driven, dissipative spin chains, they demonstrate that it can be used to distinguish between regular and chaotic dynamics. To our knowledge, a systematic study of the complex ratio vs the Hilbert-space dimension and the relative strength of dissipation has not been done so far but would be a promising focus of future work.

ACKNOWLEDGMENTS

We thank K. Życzkowski, S. Denisov, D. Chruściński, N. Mertig, and W. Timmermann for enlightening discussions. C.T. acknowledges financial support by the Deutsche Forschungsgemeinschaft, in part through the Cluster of Excellence on Complexity and Topology in Quantum Matter ct.qmat (EXC 2147) and also, when this work was initiated, through Research Unit Towards Molecular Spintronics (FOR 1154).

APPENDIX A: EXPLICIT LINDBLAD GENERATOR

In this appendix, we present the explicit form of the dissipative part $\mathscr{L}_d$ of the Lindblad generator. The derivation is cumbersome but trivial and we omit intermediate steps. Insertion of Eqs. (14) and (15) into Eq. (19) gives

$$\mathscr{L}^d_{mn,pq} = \mathscr{A}_{mn,pq} + \mathscr{B}_{mn,pq} + \mathscr{C}_{mn,pq} + \mathscr{D}_{mn,pq}, \tag{A1}$$

where the four terms stem from different combinations of off-diagonal and diagonal Lindblad operators L_{mn}. They are

$$\mathscr{A}_{mn,pq} = \bar{\delta}_{mp}\bar{\delta}_{nq}K_{mp,nq} - \frac{\delta_{mp}}{2}\sum_{k=1,\,k\neq n,q}^{N} K_{kn,kq} - \frac{\delta_{nq}}{2}\sum_{k=1,\,k\neq m,p}^{N} K_{kp,km}, \tag{A2}$$

$$\begin{aligned}\mathscr{B}_{mn,pq} = {}& \bar{\delta}_{mp}\delta_{nq}\left(-\sqrt{\frac{n^-}{n}}\,K_{mp,n^-n^-} + \sum_{k=n}^{N-1}\frac{K_{mp,kk}}{\sqrt{k(k+1)}}\right) - \frac{\delta_{mp}\bar{\delta}_{nq}}{2}\left(-\sqrt{\frac{q^-}{q}}\,K_{qn,q^-q^-} + \sum_{k=q}^{N-1}\frac{K_{qn,kk}}{\sqrt{k(k+1)}}\right)\\ & - \frac{\bar{\delta}_{mp}\delta_{nq}}{2}\left(-\sqrt{\frac{m^-}{m}}\,K_{mp,m^-m^-} + \sum_{k=m}^{N-1}\frac{K_{mp,kk}}{\sqrt{k(k+1)}}\right),\end{aligned} \tag{A3}$$

$$\begin{aligned}\mathscr{C}_{mn,pq} = {}& \delta_{mp}\bar{\delta}_{nq}\left(-\sqrt{\frac{m^-}{m}}\,K_{m^-m^-,nq} + \sum_{k=m}^{N-1}\frac{K_{kk,nq}}{\sqrt{k(k+1)}}\right) - \frac{\bar{\delta}_{mp}\delta_{nq}}{2}\left(-\sqrt{\frac{p^-}{p}}\,K_{p^-p^-,pm} + \sum_{k=p}^{N-1}\frac{K_{kk,pm}}{\sqrt{k(k+1)}}\right)\\ & - \frac{\delta_{mp}\bar{\delta}_{nq}}{2}\left(-\sqrt{\frac{n^-}{n}}\,K_{n^-n^-,nq} + \sum_{k=n}^{N-1}\frac{K_{kk,nq}}{\sqrt{k(k+1)}}\right),\end{aligned} \tag{A4}$$

$$\begin{aligned}\mathscr{D}_{mn,pq} = {}& \delta_{mp}\delta_{nq}\left[\sqrt{\frac{m^-n^-}{mn}}\,K_{m^-m^-,n^-n^-} - \sqrt{\frac{m^-}{m}}\sum_{l=n}^{N-1}\frac{K_{m^-m^-,ll}}{\sqrt{l(l+1)}} - \sqrt{\frac{n^-}{n}}\sum_{k=m}^{N-1}\frac{K_{kk,n^-n^-}}{\sqrt{k(k+1)}} + \sum_{k=m}^{N-1}\sum_{l=n}^{N-1}\frac{K_{kk,ll}}{\sqrt{k(k+1)l(l+1)}}\right.\\ & - \frac{1}{2}\left(\frac{m^-}{m}\,K_{m^-m^-,m^-m^-} - \sqrt{\frac{m^-}{m}}\sum_{k=m}^{N-1}\frac{K_{m^-m^-,kk}+K_{kk,m^-m^-}}{\sqrt{k(k+1)}} + \sum_{k,l=m}^{N-1}\frac{K_{kk,ll}}{\sqrt{k(k+1)l(l+1)}}\right)\\ & \left. - \frac{1}{2}\left(\frac{n^-}{n}\,K_{n^-n^-,n^-n^-} - \sqrt{\frac{n^-}{n}}\sum_{k=n}^{N-1}\frac{K_{n^-n^-,kk}+K_{kk,n^-n^-}}{\sqrt{k(k+1)}} + \sum_{k,l=n}^{N-1}\frac{K_{kk,ll}}{\sqrt{k(k+1)l(l+1)}}\right)\right],\end{aligned} \tag{A5}$$

where $\bar{\delta}_{mn} \equiv 1 - \delta_{mn}$ and $m^- \equiv m - 1$.

APPENDIX B: DERIVATION OF THE ENSEMBLE AVERAGE

In order to derive Eq. (45) by performing the WPUE average of Eq. (44), it is useful to consider the average $\langle K_{\alpha\beta}K_{\gamma\delta}\rangle$, where $\alpha,\beta,\gamma,\delta = 1,\ldots,\tilde{N}$. We start with

$$\begin{aligned}\langle K_{\alpha\beta}K_{\gamma\delta}\rangle &= \sum_{\mu,\nu=1}^{\tilde{N}} \left\langle A^{\dagger}_{\alpha\mu}A_{\mu\beta}A^{\dagger}_{\gamma\nu}A_{\nu\delta}\right\rangle \\ &= \sum_{\mu,\nu=1}^{\tilde{N}} \left\langle A^{*}_{\mu\alpha}A_{\mu\beta}A^{*}_{\nu\gamma}A_{\nu\delta}\right\rangle. \end{aligned} \tag{B1}$$

Since $\langle\cdots\rangle$ is a Gaussian average, we can apply Wick's theorem,

$$\begin{aligned}\cdots &= \sum_{\mu,\nu=1}^{\tilde{N}} \left(\left\langle A^{*}_{\mu\alpha}A_{\mu\beta}\right\rangle\left\langle A^{*}_{\nu\gamma}A_{\nu\delta}\right\rangle + \left\langle A^{*}_{\mu\alpha}A^{*}_{\nu\gamma}\right\rangle\left\langle A_{\mu\beta}A_{\nu\delta}\right\rangle \right.\\ &\quad \left. + \left\langle A^{*}_{\mu\alpha}A_{\nu\delta}\right\rangle\left\langle A_{\mu\beta}A^{*}_{\nu\gamma}\right\rangle\right) \\ &= \sum_{\mu,\nu=1}^{\tilde{N}} \left(\delta_{\alpha\beta}\delta_{\gamma\delta}\,\langle|A_{**}|^2\rangle^2 + 0 + \delta_{\mu\nu}\delta_{\alpha\delta}\delta_{\beta\gamma}\,\langle|A_{**}|^2\rangle^2\right) \\ &= \delta_{\alpha\beta}\delta_{\gamma\delta}\,\tilde{N}^2\,\langle|A_{**}|^2\rangle^2 + \delta_{\alpha\delta}\delta_{\beta\gamma}\,\tilde{N}\,\langle|A_{**}|^2\rangle^2, \end{aligned} \tag{B2}$$

where A_{**} is any element of A, which have identical averages.

The WPUE average of the first term in Eq. (44) now becomes

$$\begin{aligned}\left\langle \sum_{m,n,p,q=1}^{N} \tilde{K}_{mp,nq}\tilde{K}_{pm,qn}\right\rangle &= \sum_{m,n,p,q=1}^{N}\sum_{\alpha,\beta,\gamma,\delta=1}^{\tilde{N}} \left\langle \mathcal{U}^{T}_{mp,\alpha}K_{\alpha\beta}\mathcal{U}^{*}_{\beta,nq}\mathcal{U}^{T}_{pm,\gamma}K_{\gamma\delta}\mathcal{U}^{*}_{\delta,qn}\right\rangle = \sum_{m,n,p,q=1}^{N}\sum_{\alpha,\beta,\gamma,\delta=1}^{\tilde{N}} \mathcal{U}^{T}_{mp,\alpha}\mathcal{U}^{*}_{\beta,nq}\mathcal{U}^{T}_{pm,\gamma}\mathcal{U}^{*}_{\delta,qn}\,\langle K_{\alpha\beta}K_{\gamma\delta}\rangle \\ &= \sum_{m,n,p,q=1}^{N}\sum_{\alpha,\beta,\gamma,\delta=1}^{\tilde{N}} \mathcal{U}^{T}_{mp,\alpha}\mathcal{U}^{*}_{\beta,nq}\mathcal{U}^{T}_{pm,\gamma}\mathcal{U}^{*}_{\delta,qn}\left(\delta_{\alpha\beta}\delta_{\gamma\delta}\,\tilde{N}^2\,\langle|A_{**}|^2\rangle^2 + \delta_{\alpha\delta}\delta_{\beta\gamma}\,\tilde{N}\,\langle|A_{**}|^2\rangle^2\right) \\ &= \sum_{m,n,p,q=1}^{N}\left(\sum_{\alpha,\gamma=1}^{\tilde{N}} \mathcal{U}^{T}_{mp,\alpha}\mathcal{U}^{*}_{\alpha,nq}\mathcal{U}^{T}_{pm,\gamma}\mathcal{U}^{*}_{\gamma,qn}\,\tilde{N}^2\,\langle|A_{**}|^2\rangle^2 + \sum_{\alpha,\beta=1}^{\tilde{N}} \mathcal{U}^{T}_{mp,\alpha}\mathcal{U}^{*}_{\beta,nq}\mathcal{U}^{T}_{pm,\beta}\mathcal{U}^{*}_{\alpha,qn}\,\tilde{N}\,\langle|A_{**}|^2\rangle^2\right). \end{aligned} \tag{B3}$$

The matrix $\mathcal{U}$ satisfies $\mathcal{U}^{T}\mathcal{U}^{*} = \mathbb{1}$. However, we have to be careful since the sum over the common index of $\mathcal{U}^{T}$ and $\mathcal{U}^{*}$ only runs from 1 to $\tilde{N} = N^2 - 1$, whereas the unitarity relation requires the sum up to N^2. We thus get

$$\sum_{\alpha=1}^{\tilde{N}} \mathcal{U}^{T}_{mp,\alpha}\mathcal{U}^{*}_{\alpha,nq} = \delta_{mn}\delta_{pq} - \mathcal{U}^{T}_{mp,N^2}\mathcal{U}_{N^2,nq} = \delta_{mn}\delta_{pq} - \delta_{mp}\delta_{nq}\,\frac{1}{N} \tag{B4}$$

and analogously for the other sums. Inserting this into Eq. (B3), we obtain

$$\begin{aligned}\cdots &= \sum_{m,n,p,q=1}^{N}\left[\left(\delta_{mn}\delta_{pq} - \frac{\delta_{mp}\delta_{nq}}{N}\right)\left(\delta_{pq}\delta_{mn} - \frac{\delta_{pm}\delta_{qn}}{N}\right)\tilde{N}^2\,\langle|A_{**}|^2\rangle^2 + \left(\delta_{mq}\delta_{pn} - \frac{\delta_{mp}\delta_{qn}}{N}\right)\left(\delta_{pn}\delta_{mq} - \frac{\delta_{pm}\delta_{nq}}{N}\right)\tilde{N}\,\langle|A_{**}|^2\rangle^2\right] \\ &= (N^2-1)\,\tilde{N}^2\,\langle|A_{**}|^2\rangle^2 + (N^2-1)\,\tilde{N}\,\langle|A_{**}|^2\rangle^2 \\ &= \tilde{N}^2(\tilde{N}+1)\,\langle|A_{**}|^2\rangle^2. \end{aligned} \tag{B5}$$

Similarly, the average of the second term in Eq. (44) contains

$$\begin{aligned}\left\langle \sum_{n,q,r,s=1}^{N} \tilde{K}_{rn,rq}\tilde{K}_{sq,sn}\right\rangle &= \sum_{n,q,r,s=1}^{N}\sum_{\alpha,\beta,\gamma,\delta=1}^{\tilde{N}} \mathcal{U}^{T}_{rn,\alpha}\mathcal{U}^{*}_{\beta,rq}\mathcal{U}^{T}_{sq,\gamma}\mathcal{U}^{*}_{\delta,sn}\,\langle K_{\alpha\beta}K_{\gamma\delta}\rangle \\ &= \sum_{n,q,r,s=1}^{N}\sum_{\alpha,\beta,\gamma,\delta=1}^{\tilde{N}} \mathcal{U}^{T}_{rn,\alpha}\mathcal{U}^{*}_{\beta,rq}\mathcal{U}^{T}_{sq,\gamma}\mathcal{U}^{*}_{\delta,sn}\left(\delta_{\alpha\beta}\delta_{\gamma\delta}\,\tilde{N}^2\,\langle|A_{**}|^2\rangle^2 + \delta_{\alpha\delta}\delta_{\beta\gamma}\,\tilde{N}\,\langle|A_{**}|^2\rangle^2\right) \\ &= \sum_{n,q,r,s=1}^{N}\left(\sum_{\alpha,\gamma=1}^{\tilde{N}} \mathcal{U}^{T}_{rn,\alpha}\mathcal{U}^{*}_{\alpha,rq}\mathcal{U}^{T}_{sq,\gamma}\mathcal{U}^{*}_{\gamma,sn}\,\tilde{N}^2\,\langle|A_{**}|^2\rangle^2 + \sum_{\alpha,\beta=1}^{\tilde{N}} \mathcal{U}^{T}_{rn,\alpha}\mathcal{U}^{*}_{\beta,rq}\mathcal{U}^{T}_{sq,\beta}\mathcal{U}^{*}_{\alpha,sn}\,\tilde{N}\,\langle|A_{**}|^2\rangle^2\right) \end{aligned}$$

$$= \sum_{n,q,r,s=1}^{N} \left[\left(\delta_{nq} - \frac{\delta_{rn}\delta_{rq}}{N} \right) \left(\delta_{qn} - \frac{\delta_{sq}\delta_{sn}}{N} \right) \tilde{N}^2 \langle |A_{**}|^2 \rangle^2 + \left(\delta_{rs} - \frac{\delta_{rn}\delta_{sn}}{N} \right) \left(\delta_{sr} - \frac{\delta_{sq}\delta_{rq}}{N} \right) \tilde{N} \langle |A_{**}|^2 \rangle^2 \right]$$

$$= \frac{(N^2-1)^2}{N} \tilde{N}^2 \langle |A_{**}|^2 \rangle^2 + \frac{(N^2-1)^2}{N} \tilde{N} \langle |A_{**}|^2 \rangle^2$$

$$= \frac{\tilde{N}^3(\tilde{N}+1)}{N} \langle |A_{**}|^2 \rangle^2. \tag{B6}$$

The proof of the latter result was missing from Ref. 28. The essential new step is using Wick's theorem.

Finally, we require

$$\langle (\mathrm{Tr}K)^2 \rangle = \sum_{\alpha,\gamma=1}^{\tilde{N}} \langle K_{\alpha\alpha} K_{\gamma\gamma} \rangle. \tag{B7}$$

With Eqs. (B1) and (B2) we obtain

$$\langle (\mathrm{Tr}K)^2 \rangle = \sum_{\alpha,\gamma=1}^{\tilde{N}} \left(\tilde{N}^2 \langle |A_{**}|^2 \rangle^2 + \delta_{\alpha\gamma} \tilde{N} \langle |A_{**}|^2 \rangle^2 \right)$$

$$= \tilde{N}^2(\tilde{N}^2+1) \langle |A_{**}|^2 \rangle^2. \tag{B8}$$

This completes the derivation of Eq. (45).

APPENDIX C: REAL FORM OF THE LINDBLAD GENERATOR

In this appendix, we show that there exists a unitary Ω, which is independent of $\mathscr{L}_d$, so that $\Omega \mathscr{L}_d \Omega^\dagger$ is real. To that end, we rewrite Eq. (48) as

$$\mathscr{L}_d^* = \mathscr{V} \mathscr{L}_d \mathscr{V}^\dagger, \tag{C1}$$

where $\mathscr{V}$ has the components

$$\mathscr{V}_{mn,pq} = \delta_{mq}\delta_{np}. \tag{C2}$$

It is easy to see that $\mathscr{V}$ is unitary.

Moreover, $\mathscr{V}$ is symmetric. Hence, there exists a unitary matrix Ω such that

$$\mathscr{D} = \Omega \mathscr{V} \Omega^T \tag{C3}$$

(note the transpose) is a real diagonal matrix with non-negative entries (Autonne–Takagi factorization). Since $\mathscr{V}$ is unitary, $\mathscr{D}$ is also unitary, which implies that $\mathscr{D} = \mathbb{1}$. Using this and $\mathscr{V} = \mathscr{V}^\dagger$, we obtain

$$\Omega^\dagger = \mathscr{V} \Omega^T = \mathscr{V}^\dagger \Omega^T, \tag{C4}$$

$$\Omega = \left(\mathscr{V}^\dagger \Omega^T \right)^\dagger = \Omega^* \mathscr{V} \tag{C5}$$

and finally arrive at

$$\left(\Omega \mathscr{L}_d \Omega^\dagger \right)^* = \left(\Omega^* \mathscr{V} \mathscr{L}_d \mathscr{V}^\dagger \Omega^T \right)^* = \left(\Omega^* \mathscr{L}_d^* \Omega^T \right)^*$$

$$= \Omega \mathscr{L}_d \Omega^\dagger. \tag{C6}$$

The matrix Ω evidently only depends on $\mathscr{V}$, not on $\mathscr{L}_d$.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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