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Abstract

We introduce the new concept of maximal dissipative solutions for the Navier–Stokes and Euler equations and show that these solutions exist and the solution set is closed and convex. The concept of maximal dissipative solutions coincides with the concept of weak solutions as long as the weak solutions inherits enough regularity to be unique. A maximal dissipative solution is defined as the minimizer of a convex functional and we argue that this definition bears several advantages.

Contents

1 Introduction

Nonlinear partial differential equations require generalized solution concepts, mainly because smooth solutions do not exist in general (see [7, Sec. 11.3.2])

Leray introduced in his seminal work [17], the concept of weak solutions to the Navier–Stokes equations, which is nowadays widely accepted and used for numerous different problems. Often, they still lac uniqueness due to insufficient regularity properties. In two spatial dimension, the weak solutions are known to be unique. For higher space dimensions, this is not known. Probably the most well-known uniqueness result is due to Serrin [20] (see Remark 3.1).

Beside weak solutions, there is a plethora of different solutions concepts for different problems. They range from weak, measure-valued, statistical, over viscosity to different dissipative solution concepts. These solution concepts have different properties, advantages, and disadvantages, but so far do not allow to show existence and uniqueness for the Navier–Stokes and Euler equations.

We follow the line of our previous work on dissipative solutions [12] and define the concept of maximal dissipative solutions. As we will show, maximal dissipative solutions can be shown to exists in any space dimension. We did not succeed to show the uniqueness and continuous dependence on the given data for arbitrary times, but only within a possibly short time interval. This may be not too surprising, if turbulent flows in the Navier Stokes case or shock waves in the Euler case are considered.

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The idea behind a dissipative solution is that the equations do not have to be fulfilled in some distributional sense anymore, but the distance of the solution to smooth test functions fulfilling the equation only approximately is measured in terms of the relative energy and relative dissipation (to be made precise later on). The concept of dissipative solutions was first introduced by Pierre-Louis Lions in the context of the Euler equations [19, Sec. 4.4] with ideas originating from the Boltzmann equation [18]. It is also applied in the context of incompressible viscous electro-magneto-hydrodynamics (see Arsénio and Saint-Raymond [1]) and equations of viscoelastic diffusion in polymers [23]. For the more involved Ericksen–Leslie system, it was found that the dissipative solution concept, in comparison to measure-valued solutions, captures the quantity of interest (see [13] and [12]) and is also more amenable from the point of view of an Galerkin approximation (see [15]).

Since this concept proved worthwhile for more difficult systems, it may be also a good solution concept for simpler systems such as the Navier–Stokes equation. A problem arises, since dissipative solutions are not unique, even though they enjoy the weak-strong uniqueness property: They coincide with a local strong solution, as long as the latter exists. Thus, naturally the question arises, whether it is possible to design an additional criterion in order to choose a special solution from these many different dissipative solutions in order to may gain uniqueness of the solution.

We propose a step into this direction by introducing the concept of maximal dissipative solutions. Following Dafermos [4], we want to choose the solution dissipating the most energy. This is done by taking the supremum over all test functions and the minimum over all possible solutions in an altered formulation of the relative energy. The concept has several advantages in comparison to the concept of weak solutions. Firstly, we show that it coincides with the weak solutions exhibiting enough regularity to be unique.

As in the dissipative solution framework, maximal dissipative solutions are not known to fulfill the equation in distributional sense. But since all equations are modeled starting from energies and dissipation mechanisms, clinging to the equation may not simplify the analysis. Additionally, recent approaches showed that weak solution may not be physically relevant, if they exceed certain regularity assumption. For a given energy profile, it is known that there exist infinitely many weak solutions to the Euler equations [11] and to the Navier–Stokes equations [3]. Therefore, these solution concepts may not be the appropriate ones. Thus the time seems to be ready to consider alternative solution concepts. One key idea for the proposed solution concept is that the solutions are compared via the relative energy to test functions with enough regularity to be physically meaningful as a solution, *i.e.,* exhibit no non-physical non-uniqueness. The maximal dissipative solutions only coincide with weak solutions, as long as the weak solution is unique.

In his seminal paper, Leray [17] observed that a physically relevant solution to the Navier–Stokes equation only needs the energy and the dissipation to be bounded. The disadvantage of the concept of weak solution is that this does not suffice for the weak sequential compactness of the formulation. In contrast, this is the case for the proposed concept of maximal dissipative solutions, *i.e.*, it is weak sequentially stable with respect to the weak compactness properties read of the energy inequality. The solution concept of maximal dissipative solution has the additional advantage, that it is written as the minimizer of a convex functional. This allows to use standard methods from the calculus of variations for the existence proof (see the proof of Theorem 3.1 below) and minimizers of functionals often exhibit additional regularity, or are more amenable for standard elliptic regularity estimates (see [9]). As it is the case for the Ericksen–Leslie equations, we hope that the new concept of maximal dissipative solutions may also inspire stable numerical schemes for the Navier–Stokes equations.

Formulating the solution concept as a minimizing problem may helps to tailor selection criteria to guarantee uniqueness of the solution. This can for example be achieved by altering the cost functional in a way to make it strictly convex.

The proposed solution concept is very general and may be applied to various kinds of problems, we want to explain the concept here at the example of the Navier–Stokes equations for the sake of readability. But it can be applied in the sense of Definition 2.1 (below) to other systems featuring the relative energy inequality like systems in complex fluids like nematic liquid crystals [6], models in phase transition [16], or more generally GENERIC systems [10].

Plan of the paper: First, we introduce the concept of maximal dissipative solution and collect some preliminary material. Afterwards, we show the existence for maximal dissipative solutions for the Navier–Stokes equations, their weak-strong uniqueness, and that the solution set is convex. Finally, we show the same conclusions for the Euler equations.

2 Definition and preliminaries

We start with a general definition of maximal dissipative solutions:

Definition 2.1. Let there be given two linear spaces $\mathscr{X}$, $\mathscr{Y}$, and forms $\mathscr{R}:\mathscr{X}\times\mathscr{X}\to\mathbb{R}_{\geq 0}$, $\mathscr{W}:\mathscr{X}\times\mathscr{X}\to\mathbb{R}_{\geq 0}$, $\mathscr{K}:\mathscr{Y}\to\mathbb{R}_{\geq 0}$, which we call relative energy, relative dissipation, and regularity measure, respectively. Additionally, let there be a solution operator $\mathscr{A}:\mathscr{Y}\to\mathscr{X}^*$. Then a function $\boldsymbol{u}$ is called maximal dissipative solution, if

$$\boldsymbol{u}=\arg\min_{\boldsymbol{v}\in\mathscr{X}}\sup_{\tilde{\boldsymbol{v}}\in\mathscr{Y}}\mathscr{F}(\boldsymbol{v}|\tilde{\boldsymbol{v}}),$$

where

$$\begin{aligned}\mathscr{F}(\boldsymbol{v}|\tilde{\boldsymbol{v}})=\sup_{\phi\in\tilde{\mathscr{C}}(0,T)}\Big(&-\int_0^T\phi'(t)\mathscr{R}(\boldsymbol{v}(t)|\tilde{\boldsymbol{v}}(t))e^{-\int_0^t\mathscr{K}(\tilde{\boldsymbol{v}}(s))\,\mathrm{d}s}\mathrm{d}t-\phi(0)\mathscr{R}(\boldsymbol{v}_0,\tilde{\boldsymbol{v}}(0))\\&+\int_0^T\phi(t)\left(\mathscr{W}(\boldsymbol{v}(t)|\tilde{\boldsymbol{v}}(t))+\langle\mathscr{A}(\tilde{\boldsymbol{v}}(t)),\boldsymbol{v}(t)-\tilde{\boldsymbol{v}}(t)\rangle\right)e^{-\int_0^t\mathscr{K}(\tilde{\boldsymbol{v}}(s))\,\mathrm{d}s}\mathrm{d}t\Big),\end{aligned}$$

where $\phi\in\tilde{\mathscr{C}}([0,T])$ as long as $\phi\in\mathscr{C}^1([0,T])$ with $\phi\geq 0$, and $\phi'\leq 0$ on $[0,T]$ as well as $\phi(0)=1$ and $\phi(T)=0$.

Remark 2.1. For every system, the forms have to be defined individually. Usually, for convex energy and dissipation, they are defined via the first-order Taylor approximation, *i.e.*,

$$\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}})=\mathscr{E}(\boldsymbol{v})-\mathscr{E}(\tilde{\boldsymbol{v}})-\langle\partial\mathscr{E}(\tilde{\boldsymbol{v}}),\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle\quad\text{and}\quad\mathscr{W}(\boldsymbol{v}|\tilde{\boldsymbol{v}})=\mathscr{D}(\boldsymbol{v})-\mathscr{D}(\tilde{\boldsymbol{v}})-\langle\partial\mathscr{D}(\tilde{\boldsymbol{v}}),\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle,$$

where $\mathscr{E}$ denotes the energy, $\mathscr{D}$ the dissipation of the system, and ∂ the directional (or Gâteaux) derivative (or sub-differential). But this may differ for different problems and often also involves some freedom of choice. Concerning the choice of $\mathscr{K}$, there is some freedom in this solution concept as well. The regularity usually has to be sufficient to provide uniqueness of solutions. For Navier–Stokes, we define it according to Serrin's uniqueness criterion. But it may also be chosen differently, such that the emerging maximal dissipative solution differs and depends on the choice of $\mathscr{K}$.

Note that the minimum of the functional is always zero, since we may observe

$$0\leq\sup_{\tilde{\boldsymbol{v}}\in\mathscr{Y}}\inf_{\boldsymbol{v}\in\mathscr{X}}\mathscr{F}(\boldsymbol{v}|\tilde{\boldsymbol{v}})\leq\inf_{\boldsymbol{v}\in\mathscr{X}}\sup_{\tilde{\boldsymbol{v}}\in\mathscr{Y}}\mathscr{F}(\boldsymbol{v}|\tilde{\boldsymbol{v}})\leq 0,$$

where the first equality follows by construction, the second one holds in general and the last one follows from the existence proof. This implies that all inequalities in the above chain are indeed equations.

For an introduction into $\min$–$\max$ problems, we refer to [5, Chapter VI]. It is also possible to derive optimality conditions for such problems [5, Proposition 1.6, Chapter VI].

Throughout this paper, let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. The space of smooth solenoidal functions with compact support is denoted by $\mathscr{C}^\infty_{c,\sigma}(\Omega;\mathbb{R}^d)$. By $\boldsymbol{L}^p_\sigma(\Omega)$, $\boldsymbol{H}^1_{0,\sigma}(\Omega)$, and $\boldsymbol{W}^{1,p}_{0,\sigma}(\Omega)$, we denote the closure of $\mathscr{C}^\infty_{c,\sigma}(\Omega;\mathbb{R}^d)$ with respect to the norm of $\boldsymbol{L}^p(\Omega)$, $\boldsymbol{H}^1(\Omega)$, and $\boldsymbol{W}^{1,p}(\Omega)$ respectively. Note that $\boldsymbol{L}^2_\sigma(\Omega)$ can be characterized by $\boldsymbol{L}^2_\sigma(\Omega) = \{\boldsymbol{v} \in L^2(\Omega) \,|\, \nabla\cdot \boldsymbol{v} = 0 \text{ in } \Omega, \boldsymbol{n}\cdot\boldsymbol{v} = 0 \text{ on } \partial\Omega\}$, where the first condition has to be understood in the distributional sense and the second condition in the sense of the trace in $H^{-1/2}(\partial\Omega)$. The dual space of a Banach space V is always denoted by V^* and equipped with the standard norm; the duality pairing is denoted by $\langle\cdot,\cdot\rangle$. We use the standard notation $(\boldsymbol{H}^1_0(\Omega))^* = \boldsymbol{H}^{-1}(\Omega)$.

Lemma 2.2. Let $f \in L^1(0,T)$ and $g \in L^\infty(0,T)$ with $g \geq 0$ a.e. in $(0,T)$. Then the two inequalities

$$-\int_0^T \phi'(t)g(t)\,\mathrm{d}t - \phi(0)g(0) + \int_0^T \phi(t)f(t)\,\mathrm{d}t \leq 0$$

for all $\phi \in \mathscr{C}^\infty_c([0,T))$ with $\phi \geq 0$, and $\phi' \leq 0$ on $[0,T]$ and

$$g(t) - g(0) + \int_0^t f(s)\,\mathrm{d}s \leq 0 \quad \text{for a.e. } t \in (0,T) \tag{1}$$

are equivalent.

Proof. The proof of the first implication is a simple adaptation of the classical variational lemma (compare to [14]). It can be seen, by choosing a sequence $\{\phi_\varepsilon\}$ as a suitable (monotone decreasing) approximation of the indicator function on $[0,t]$, *i.e.*, $\chi_{[0,t]}$, with the condition $\phi_\varepsilon(0) = 1$.

The reverse implication can be seen, by testing (1) by $-\phi'$ and integrate-by-parts in the last two terms.

□

3 Existence for Navier–Stokes equations

First we recall the Navier–Stokes equations for the sake of completeness.

$$\begin{aligned} \partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu\Delta\boldsymbol{v} + \nabla p = 0, \qquad \nabla\cdot\boldsymbol{v} &= 0 \quad \text{in } \Omega\times(0,T), \\ \boldsymbol{v}(0) &= \boldsymbol{v}_0 \quad \text{in } \Omega, \\ \boldsymbol{v} &= 0 \quad \text{on } \partial\Omega\times(0,T). \end{aligned} \tag{2}$$

The underlying spaces in the Navier–Stokes case are given by $\mathscr{X} = L^\infty(0,T;\boldsymbol{L}^2_\sigma) \cap L^2(0,T;\boldsymbol{H}^1_{0,\sigma})$ and $\mathscr{Y} = C^1([0,T];\mathscr{C}^\infty_{c,\sigma}(\Omega;\mathbb{R}^d))$. We define the relative energy $\mathscr{R}$ by

$$\mathscr{R}(\boldsymbol{v}|\tilde{\boldsymbol{v}}) = \frac{1}{2}\|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}, \tag{3a}$$

the relative dissipation $\mathscr{W}$ by

$$\mathscr{W}(\boldsymbol{v}|\tilde{\boldsymbol{v}}) = \frac{\nu}{2}\|\nabla\boldsymbol{v} - \nabla\tilde{\boldsymbol{v}}\|^2_{L^2(\Omega)}, \tag{3b}$$

the regularity measure $\mathscr{K}$ by

$$\mathscr{K}(\tilde{\boldsymbol{v}}) = \mathscr{K}_{s,r}(\tilde{\boldsymbol{v}}) = c\|\tilde{\boldsymbol{v}}\|_{L^r(\Omega)}^s \quad \text{for } \frac{2}{s} + \frac{d}{r} \leq 1\,, \tag{3c}$$

and the solution operator $\mathscr{A}$ by

$$\mathscr{A}(\tilde{\boldsymbol{v}}) = \langle \partial_t \tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}} - \nu\Delta\tilde{\boldsymbol{v}} - \boldsymbol{f} + \nabla\tilde{p}, \cdot\rangle\,, \tag{3d}$$

which has to be understood in a weak sense, at least with respect to space. Note that the solution operator does not include boundary condition, since they are encoded in the underlying spaces. This may changes for different boundary conditions.

We may state now the main theorem of this article:

Theorem 3.1. Let $\nu > 0$, $\boldsymbol{v}_0 \in \boldsymbol{L}^2_\sigma$, and $\boldsymbol{f} \in L^2(0,T;H^{-1}(\Omega))$ be given. Let $\mathscr{R}$, $\mathscr{W}$, $\mathscr{K}$, and $\mathscr{A}$ be given as above. Then there exists a maximal dissipative solution $\boldsymbol{v} \in \mathscr{X}$ in the sense of Definition 2.1 and the solution set is convex. By construction maximal dissipative solutions enjoy the weak strong uniqueness property.

Remark 3.1 (Comparison to weak solutions). In the case that there exists a weak solution to the Navier–Stokes equation complying to Serrin's uniqueness criterion, we observe that it is a maximal dissipative solution. Indeed, let $\boldsymbol{v}$ be a weak solution enjoying the regularity

$$\boldsymbol{v} \in L^s(0,T;L^r(\Omega)) \quad \text{for } \frac{2}{s} + \frac{d}{r} \leq 1\,,$$

then the regularity measure $\mathscr{K}$ is bounded and we may use it as a test function $\tilde{\boldsymbol{v}}$ (or rather approximate it by test functions) in the formulation of Definition 2.1. Note that using density arguments, $\mathscr{Y}$ could be replaced by $\mathscr{X} \cap L^s(0,T;L^r(\Omega)) \cap W^{1,2}(0,T;\left(\boldsymbol{H}^1_{0,\sigma}\right)^*)$ with s and r fulfilling the above condition. We observe that $\mathscr{F}(\boldsymbol{v}|\boldsymbol{v}) = 0$, which is indeed the minimum since for every other function $\boldsymbol{u} \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;\boldsymbol{H}^1_{0,\sigma})$ emanating from the same initial datum, we observe that $\mathscr{F}(\boldsymbol{u}|\boldsymbol{v}) > 0$. Thus, maximal dissipative solutions coincide with weak solutions as long as the latter are unique.

Remark 3.2 (Reintroduction of the pressure). In this work, we only consider the velocity field for simplicity. Due to the fact that no equation is fulfilled in the maximal dissipative solution concept, we do not have to worry about choosing the pressure in such a way that the full Navier–Stokes equation is fulfilled in a distributional sense (see [21]). We propose to calculate the pressure by solving the usual elliptic Neumann boundary value problem

$$\begin{aligned} -\Delta p &= \operatorname{tr}\left(\nabla \boldsymbol{v}^2\right) - \nabla\cdot \boldsymbol{f}\,, && \text{in } \Omega \\ \boldsymbol{n}\cdot\nabla p &= \boldsymbol{n}\cdot\boldsymbol{f} - \boldsymbol{n}\cdot((\boldsymbol{v}\cdot\nabla)\boldsymbol{v})\,, && \text{on } \partial\Omega \end{aligned}$$

in a very weak sense with $p \in L^2(\Omega)$ and the additional normalization $\int_\Omega p(t)\,\mathrm{d}\boldsymbol{x} = 0$ a.e. in $(0,T)$. The previous formulation for the pressure especially makes sense, if one considers a suitable approximation of the Navier–Stokes equation, *i.e.*, by a Galerkin approximation with a Galerkin space spanned by eigenfunctions of the Stokes operator.

Remark 3.3 (other boundary conditions). In order to incorporate different boundary conditions it is sufficient to adapt the function space for the solution, *i.e.*, $\mathscr{X}$, the test functions, *i.e.*, $\mathscr{Y}$, and the formulation of the operator $\mathscr{A}$.

Remark 3.4 (Enhance the concept by another selection criterion to get uniqueness of solutions). The problem when trying to prove uniqueness of maximal dissipative solutions is that the functional $\sup_{\tilde{v}\in\mathscr{Y}}\mathscr{F}(\cdot|\tilde{v})$ is only convex and not strictly convex. There are different possibilities to get strict convexity, but we only found ways, which alter the assertion of Remark 3.1. For example, one could restrict $\mathscr{Y}$ to be the space $L^s(0,T;L^r(\Omega))$ with $2/s+d/r\le 1$ such that $\|\tilde{v}\|_{L^s(0,T;L^r(\Omega))}\le C$ for some possibly big constant C. On the one hand, unique weak solutions would only coincide with the maximal solutions, as long as $\|v\|_{L^s(0,T;L^r(\Omega))}\le C$ holds for the weak solution. On the other hand, under this additional restriction the functional $\sup_{\tilde{v}\in\mathscr{Y}}\mathscr{F}(\cdot|\tilde{v})$ is strictly convex and the maximal dissipative solution would be thus unique.

Another possibility would by to add a term to $\mathscr{F}$ to guarantee its strict convexity, for example $\tilde{\mathscr{F}}(v|\tilde{v})=\mathscr{F}(v|\tilde{v})+\varepsilon\|v\|^2_{L^2(\Omega\times(0,T))}$ for a possible small constant $\varepsilon>0$. This adaptation coincides with a standard Tikhonov regularization of the functional and guarantee uniqueness. Choosing this regularization would coincide with the selection criterion of picking the solution with the smallest energy. But also this regularization may violate the assertion of Remark 3.1.

Finally, we want to remark that the solution set is convex and closed, which may makes it easy to design certain uniqueness criteria. A possibility would be to choose the Hilbert-space projection (possibly for the Hilbert-space $L^2(\Omega\times(0,T))$) of 0 onto the closed convex solution set. Since the projection onto a closed convex set is unique (see [2, Theorem 5.2]), the associated solution would be unique.

Additionally, we prove a conditional result on the continuous dependence of maximal dissipative solutions. Since the result is only conditional, we fail to show well posedness in the sense of Hadamard. From a physical view point, one maybe wouldn't even expect continuous dependence in a turbulent regime, but rather fast changes from the given value.

Proposition 3.2. Assume that there exists a weak solution $v\in\mathscr{X}$ with $v\in L^s(0,T;L^r(\Omega))$ with $2/s+d/r\le 1$ to given right-hand side $f\in L^2(0,T;H^{-1}(\Omega))$ and initial datum $v_0\in L^2_\sigma$. For any $f^1\in L^2(0,T;H^{-1}(\Omega))$ and $v_0^1\in L^2_\sigma$ the associated maximal dissipative solution v^1 fulfills the estimate

$$\mathscr{R}(v^1(t)|v(t))+\int_0^t\mathscr{W}(v^1,v)e^{\int_s^t\mathscr{K}(v)\,\mathrm{d}\tau}\,\mathrm{d}s\le\mathscr{R}(v_0^1|v_0)e^{\int_0^t\mathscr{K}(v)\,\mathrm{d}s}+\frac{1}{2\nu}\int_0^t\|f-f^1\|^2_{H^{-1}(\Omega)}e^{\int_s^t\mathscr{K}(v)\,\mathrm{d}\tau}\,\mathrm{d}s$$

for a.e. $t\in(0,T)$.

Remark 3.5. The previous result gives no assertion on the continuous dependence on the data in general, but only conditionally, if a unique weak solution exists. This can only be proven to be the case locally in time (see [22]).

Proof of Theorem 3.1. The proof mainly relies on the direct method of the calculus of variations. Indeed, since the problem was formulated as a minimization problem, the proof relies on standard variational techniques. The existence proof mainly focuses on showing that the convex functional is not infinity everywhere. We show that there exists a candidate for which the functional is less or equal to zero for every test function $\tilde{u}$. This candidate is actually the weak solution with energy inequality. To show existence, we have to perform a calculation similar to the one in Serrin's conditional uniqueness proof (or a weak-strong uniqueness proof).

Let v be a weak solution to the Navier–Stokes equation with energy inequality. Then it fulfills the weak formulation

$$-\int_0^T\int_\Omega v\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t+\nu\int_0^T\int_\Omega\nabla v:\nabla\varphi+(v\cdot\nabla)v\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\langle f,\varphi\rangle\,\mathrm{d}t+\int_\Omega v\cdot\varphi(0)\,\mathrm{d}x \quad (4)$$

for $\boldsymbol{\varphi} \in \mathscr{C}_c^1([0,T);\mathscr{C}_{c,\sigma}^\infty(\Omega;\mathbb{R}^3))$and the energy inequality

$$\frac{1}{2}\|\boldsymbol{v}(t)\|_{L^2(\Omega)}^2 + \nu\int_0^t \|\nabla\boldsymbol{v}\|_{L^2(\Omega)}^2\,\mathrm{d}s \le \frac{1}{2}\|\boldsymbol{v}_0\|_{L^2(\Omega)}^2 + \int_0^t \langle \boldsymbol{f},\boldsymbol{v}\rangle\,\mathrm{d}s \quad \text{for a.e. } t\in(0,T)\,. \tag{5}$$

For a test function $\tilde{\boldsymbol{v}} \in \mathscr{Y}$, we find by testing the solution operator $\mathscr{A}(\tilde{\boldsymbol{v}})$ by $\phi\tilde{\boldsymbol{v}}$ with $\phi\in\mathscr{C}_c^1([0,T))$ and standard calculations that

$$\int_0^T \phi\,\langle \mathscr{A}(\tilde{\boldsymbol{v}})\tilde{\boldsymbol{v}}\rangle\,\mathrm{d}t = \\ -\int_0^T \phi'\frac{1}{2}\|\tilde{\boldsymbol{v}}(t)\|_{L^2(\Omega)}^2\,\mathrm{d}t + \int_0^T \phi\left(\nu\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}^2 - \langle \boldsymbol{f},\tilde{\boldsymbol{v}}\rangle\right)\mathrm{d}t - \phi(0)\frac{1}{2}\|\tilde{\boldsymbol{v}}(0)\|_{L^2(\Omega)}^2 \tag{6}$$

Testing again the solution operator $\mathscr{A}(\tilde{\boldsymbol{v}})$ by $\phi\boldsymbol{v}$ and (4) by $\phi\tilde{\boldsymbol{v}}$ with $\phi\in\mathscr{C}_c^1([0,T))$, we find

$$-\int_0^T \phi'\int_\Omega \boldsymbol{v}\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^T \phi\int_\Omega 2\nu\nabla\boldsymbol{v}:\nabla\tilde{\boldsymbol{v}} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ = \int_0^T \phi\,\langle \mathscr{A}(\tilde{\boldsymbol{v}}),\boldsymbol{v}\rangle\,\mathrm{d}t + \phi(0)\int_\Omega \boldsymbol{v}_0\cdot\tilde{\boldsymbol{v}}(0)\,\mathrm{d}\boldsymbol{x} + \int_0^T \phi\langle f,\tilde{\boldsymbol{v}}+\boldsymbol{v}\rangle\,\mathrm{d}t\,. \tag{7}$$

Reformulating (5) by Lemma 2.2, adding (6), and subtracting (7), let us deduce that

$$-\int_0^T \phi'\frac{1}{2}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2\,\mathrm{d}t + \nu\int_0^T \phi\|\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2\,\mathrm{d}t - \phi(0)\frac{1}{2}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 \\ \le \int_0^T \phi\int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^T \phi\,\langle \mathscr{A}(\tilde{\boldsymbol{v}}),\tilde{\boldsymbol{v}}-\boldsymbol{v}\rangle\,\mathrm{d}t \tag{8}$$

for all $\phi\in\tilde{C}([0,T])$. In the following, we estimate the convective terms as in the proof of Serrin's result. Therefore, we use some standard manipulations using the skew-symmetry of the convective term in the last two arguments and the fact that $\tilde{\boldsymbol{v}}$ is divergence free, to find

$$\int_\Omega (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)\tilde{\boldsymbol{v}}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega ((\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\tilde{\boldsymbol{v}} + (\tilde{\boldsymbol{v}}\cdot\nabla)(\tilde{\boldsymbol{v}}-\boldsymbol{v})\cdot(\boldsymbol{v}-\tilde{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x} \\ = \int_\Omega ((\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}\,.$$

Hölder's, Gagliardo–Nirenberg's, and Young's inequality provide the estimate

$$\int_\Omega ((\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\nabla)(\boldsymbol{v}-\tilde{\boldsymbol{v}})\cdot\tilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x} \le \|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|_{L^p(\Omega)}\|\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}\|\tilde{\boldsymbol{v}}\|_{L^{2p/(p-2)}(\Omega)} \\ \le c_p\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^{(1-\alpha)}\|\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^{(1+\alpha)}\|\tilde{\boldsymbol{v}}\|_{L^{2p/(p-2)}(\Omega)} \\ \le \frac{\nu}{2}\|\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 + c\|\tilde{\boldsymbol{v}}\|_{L^{2p/(p-2)}(\Omega)}^{2/(1-\alpha)}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2\,,$$

where α is chosen according to Gagliardo–Nirenberg's inequality by

$$\alpha = d(p-2)/2p \quad \text{for} \quad d \le 2p/(p-2)\,.$$

Inserting this into (8) and replace ϕ by $\varphi e^{-\int_0^t \mathscr{K}(\tilde{\boldsymbol{v}})\,\mathrm{d}s}$ (or approximate it appropriately), we get

$$-\int_0^T \varphi'\frac{1}{2}\|\boldsymbol{v}(t)-\tilde{\boldsymbol{v}}(t)\|_{L^2(\Omega)}^2 e^{-\int_0^t c\|\tilde{\boldsymbol{v}}\|_{L^{2p/(p-2)}(\Omega)}^{2/(1-\alpha)}\,\mathrm{d}s}\,\mathrm{d}t - \frac{1}{2}\|\boldsymbol{v}_0-\tilde{\boldsymbol{v}}(0)\|_{L^2(\Omega)}^2 \\ + \int_0^T \varphi\left(\frac{\nu}{2}\|\nabla\boldsymbol{v}-\nabla\tilde{\boldsymbol{v}}\|_{L^2(\Omega)}^2 + \langle \mathscr{A}(\tilde{\boldsymbol{v}}),\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle\right) e^{-\int_0^t c\|\tilde{\boldsymbol{v}}\|_{L^{2p/(p-2)}(\Omega)}^{2/(1-\alpha)}\,\mathrm{d}t}\,\mathrm{d}s \le 0$$

for every smooth function $\tilde{\boldsymbol{v}} \in \mathscr{Y}$ and all $\varphi \in \tilde{\mathscr{C}}([0,T])$ .Thus the supremum is also bounded from above by zero. This shows that the functional $\sup_{\tilde{\boldsymbol{v}}\in\mathscr{Y}} \mathscr{F}(\cdot|\tilde{\boldsymbol{v}})$ is not always infinity, where $\mathscr{F}$ is given according to Definition 2.1. Hence we may conclude that there exists a minimizing sequence. By construction with the supremum taken over the test functions, the sequence is known to be bounded from below (Insert any smooth enough function (for example zero) to find a lower bound).

Let $\{\boldsymbol{v}_n\}$ be such a minimizing sequence, since $\sup_{\tilde{\boldsymbol{v}}\in\mathscr{Y}} \mathscr{F}(\boldsymbol{v}|\tilde{\boldsymbol{v}})$ is bounded, we also know that $\mathscr{F}(\boldsymbol{v}|0)$ is bounded, which immediately helps us together with Lemma 2.2 to read of *a priori* estimates and deduce the standard weak convergences

$$\boldsymbol{v}_n \overset{*}{\rightharpoonup} \boldsymbol{v}, \quad \text{in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\boldsymbol{H}^1_{0,\sigma}). \tag{9}$$

Considering the functional, we observe

$$\liminf_{n\to\infty} \sup_{\tilde{\boldsymbol{v}}\in\mathscr{Y}} \mathscr{F}(\boldsymbol{v}_n|\tilde{\boldsymbol{v}}) \geq \sup_{\tilde{\boldsymbol{v}}\in\mathscr{Y}} \liminf_{n\to\infty} \mathscr{F}(\boldsymbol{v}_n|\tilde{\boldsymbol{v}}) \geq \sup_{\tilde{\boldsymbol{v}}\in\mathscr{Y}} \mathscr{F}(\boldsymbol{v}|\tilde{\boldsymbol{v}}).$$

The convexity of the solution set follows directly from the fact that $\sup_{\tilde{\boldsymbol{v}}\in\mathscr{Y}} \mathscr{F}(\boldsymbol{v}|\tilde{\boldsymbol{v}})$ is a convex function in $\boldsymbol{v}$. Indeed, let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ be two solutions in the sense of Definition 2.1. Since they are both minimizers of the functional $\sup_{\tilde{\boldsymbol{v}}} \mathscr{F}(\cdot|\tilde{\boldsymbol{v}})$, we may consider a convex combination of both

$$\begin{aligned}\lambda \sup_{\tilde{\boldsymbol{v}}_1\in\mathscr{Y}} \mathscr{F}(\boldsymbol{v}_1|\tilde{\boldsymbol{v}}_1) + (1-\lambda) \sup_{\tilde{\boldsymbol{v}}_2\in\mathscr{Y}} \mathscr{F}(\boldsymbol{v}_2|\tilde{\boldsymbol{v}}_2) &\geq \sup_{\tilde{\boldsymbol{v}}\in\mathscr{Y}} (\lambda \mathscr{F}(\boldsymbol{v}_1|\tilde{\boldsymbol{v}}) + (1-\lambda)\mathscr{F}(\boldsymbol{v}_2|\tilde{\boldsymbol{v}})) \\ &\geq \sup_{\tilde{\boldsymbol{v}}\in\mathscr{Y}} \mathscr{F}(\lambda \boldsymbol{v}_1 + (1-\lambda)\boldsymbol{v}_2|\tilde{\boldsymbol{v}})\end{aligned}$$

for all $\lambda \in [0,1]$. This implies that $\lambda\boldsymbol{v}_1 + (1-\lambda)\boldsymbol{v}_2$ is also a maximal dissipative solution. By the same arguments as in the existence proof, we may observe that the solutions set is also closed. □

Proof of Proposition 3.2. First note that for every solution $\boldsymbol{v}^1$, we have by construction of dissipative solutions

$$0 \geq \sup_{\tilde{\boldsymbol{v}}\in\mathscr{Y}} \mathscr{F}(\boldsymbol{v}^1|\tilde{\boldsymbol{v}}).$$

Choosing $\tilde{\boldsymbol{v}}$ to be the weak regular solution $\boldsymbol{v}$ (or approximate it appropriately), we find

$$\begin{aligned}0 \geq \sup_{\phi\in\tilde{\mathscr{C}}([0,T])} \Big(&-\int_0^T \phi'(t)\mathscr{R}(\boldsymbol{v}^1(t)|\boldsymbol{v}(t))e^{-\int_0^t \mathscr{K}(\boldsymbol{v}(s))\,\mathrm{d}s}\,\mathrm{d}t - \phi(0)\mathscr{R}(\boldsymbol{v}^1_0,\boldsymbol{v}_0) \\ &+ \int_0^T \phi(t)\left(\mathscr{W}(\boldsymbol{v}^1(t)|\boldsymbol{v}(t)) + \left\langle \mathscr{A}_{\boldsymbol{f}^1}(\boldsymbol{v}(t)), \boldsymbol{v}^1(t)-\boldsymbol{v}(t)\right\rangle\right) e^{-\int_0^t \mathscr{K}(\boldsymbol{v}(s))\,\mathrm{d}s}\,\mathrm{d}t\Big),\end{aligned} \tag{10}$$

where $\mathscr{A}_{f^1}$ denotes the solution operator (3d) with $\boldsymbol{f}$ replaced by $\boldsymbol{f}^1$. Since $\boldsymbol{v}$ is a solution for the right-hand side $\boldsymbol{f}$, we may estimate

$$\begin{aligned}\left\langle \mathscr{A}_{\boldsymbol{f}^1}(\boldsymbol{v}(t)), \boldsymbol{v}^1(t)-\boldsymbol{v}(t)\right\rangle &= \langle \mathscr{A}_{\boldsymbol{f}}(\boldsymbol{v}(t)) + \boldsymbol{f}(t) - \boldsymbol{f}^1(t), \boldsymbol{v}^1(t)-\boldsymbol{v}(t)\rangle \\ &\geq -\frac{1}{2\nu}\|\nabla\boldsymbol{v}^1(t)-\nabla\boldsymbol{v}(t)\|^2_{L^2(\Omega)} - \frac{1}{2\nu}\|\boldsymbol{f}(t)-\boldsymbol{f}^1(t)\|^2_{H^{-1}(\Omega)}.\end{aligned}$$

Reinserting this estimate into (10), applying Lemma 2.2 and multiplying by $e^{\int_0^t \mathscr{K}(\boldsymbol{v}(\tau))\,\mathrm{d}s}$, we find the assertion of Proposition 3.2.

□

4 Existence for Euler equations

A simple adaptation leads to the existence of Euler's equation. First we recall the Euler equations for the sake of completeness.

$$\begin{aligned} \partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p = \boldsymbol{f}, \qquad \nabla\cdot\boldsymbol{v} &= 0 \qquad \text{in } \Omega\times(0,T)\,,\\ \boldsymbol{v}(0) &= \boldsymbol{v}_0 \qquad \text{in } \Omega\,,\\ \boldsymbol{v}\cdot\boldsymbol{n} &= 0 \qquad \text{on } \partial\Omega\times(0,T)\,. \end{aligned}$$

For the Euler equation, the underlying spaces change to $\mathscr{X} := L^\infty(0,T;L^2_\sigma(\Omega))$ for the solutions and $\mathscr{Y} := \mathscr{C}^1([0,T];\mathscr{C}^\infty_{c,\sigma}(\Omega;\mathbb{R}^3))$ for the test functions. The definitions of the relative energy and the relative dissipation, as well as the solution operator are given as in (3) with $\nu = 0$. The regularity measure changes to $\mathscr{K}(\tilde{\boldsymbol{v}}) = \|(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym},-}\|_{L^\infty(\Omega)}$, where $(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym},-}$ denotes the negative part of the symmetrized gradient of $\tilde{\boldsymbol{v}}$, *i.e.*,

$$\|(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym},-}\|_{L^\infty(\Omega)} = \left\| \left(\sup_{|\boldsymbol{a}|=1} - \left(\boldsymbol{a}^T\cdot(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym}}\boldsymbol{a}\right)\right)\right\|_{L^\infty(\Omega)} .$$

We recall an existence result on dissipative solutions for the Euler equation by Pierre-Louis Lions [19, Sec 4.4]:

Theorem 4.1 (Existence of dissipative solutions)**.** Let $\boldsymbol{v}_0 \in L^2_\sigma(\Omega)$, and $\boldsymbol{f}\in L^2(0,T;L^2(\Omega))$ be given. Define $\mathscr{X} := L^\infty(0,T;L^2_\sigma(\Omega))$ and $\mathscr{Y} := \mathscr{C}^1([0,T];\mathscr{C}^\infty_{c,\sigma}(\Omega;\mathbb{R}^3))$ and let $\mathscr{R}$, $\mathscr{W}$, and $\mathscr{A}$ be given as in (3) with $\nu = 0$ and let $\mathscr{K}$ be given by $\mathscr{K}(\tilde{\boldsymbol{v}}) = \|(\nabla\tilde{\boldsymbol{v}})_{\mathsf{sym},-}\|_{L^\infty(\Omega)}$. Then there exists at least one function $\boldsymbol{v}\in L^\infty(0,T;L^2_\sigma(\Omega))$ such that

$$\mathscr{R}(\boldsymbol{v}(t)|\tilde{\boldsymbol{v}}(t)) \le \mathscr{R}(\boldsymbol{v}_0,\tilde{\boldsymbol{v}}(0))e^{\int_0^t \mathscr{K}(\tilde{\boldsymbol{v}})\,\mathrm{d}s} + \int_0^t \langle \mathscr{A}(\tilde{\boldsymbol{v}}),\tilde{\boldsymbol{v}}-\boldsymbol{v}\rangle\, e^{\int_s^t \mathscr{K}(\tilde{\boldsymbol{v}})\,\mathrm{d}\tau} \quad \text{for all } \tilde{\boldsymbol{v}}\in\mathscr{Y}\,. \tag{11}$$

Remark 4.1. Pierre Louis Lions also showed that $\boldsymbol{v}$ enjoys the regularity $\boldsymbol{v}\in\mathscr{C}_w([0,T];L^2(\Omega))$. We omit this here, since the regularity is not stable under the convergence with respect to $\mathscr{X}$.

We are now ready to state the existence result for the Euler equations.

Theorem 4.2. Let the assumptions of Theorem 4.1 be fulfilled. Then there exists a maximal dissipative solution $\boldsymbol{v}\in\mathscr{X}$ in the sense of Definition 2.1, the solution set is convex.

Additionally, we provide a conditional continuous dependence result similar to Proposition 3.2.

Proposition 4.3. Assume that there exists a unique weak solution $\boldsymbol{v}\in\mathscr{X}$ to the Euler equations with $\boldsymbol{v}\in L^1(0,T;W^{1,\infty}(\Omega))$ to given right-hand side $\boldsymbol{f}\in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$ and initial datum $\boldsymbol{v}_0\in\boldsymbol{L}^2_\sigma$. For any $\boldsymbol{f}^1\in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$ and $\boldsymbol{v}_0^1\in\boldsymbol{L}^2_\sigma$ the associated maximal dissipative solution $\boldsymbol{v}^1$ fulfills the estimate

$$\mathscr{R}(\boldsymbol{v}^1(t)|\boldsymbol{v}(t)) \le \mathscr{R}(\boldsymbol{v}_0^1|\boldsymbol{v}_0)e^{\int_0^t(\mathscr{K}(\boldsymbol{v})+1)\,\mathrm{d}s} + \frac{1}{2\nu}\int_0^t \|\boldsymbol{f}-\boldsymbol{f}^1\|^2_{\boldsymbol{L}^2(\Omega)} e^{\int_s^t(\mathscr{K}(\boldsymbol{v})+1)\,\mathrm{d}\tau}\,\mathrm{d}s$$

for a.e. $t\in(0,T)$.

Proof of Theorem 4.2. Multiplying (11) by $e^{-\int_0^t \mathscr{K}(\tilde{\boldsymbol{v}})\,\mathrm{d}s}$ and applying Lemma 2.2, we immediately observe that the function $\sup_{\tilde{\boldsymbol{v}}\in\mathscr{Y}} \mathscr{F}(\cdot|\tilde{\boldsymbol{v}})$ is not infinity everywhere, where $\mathscr{F}$ is given according to Definition 2.1. Thus, we may extract a minimizing sequence for the functional $\sup_{\tilde{\boldsymbol{v}}\in\mathscr{Y}} \mathscr{F}(\cdot|\tilde{\boldsymbol{v}})$. The rest of the proof coincides with the one for the Navier–Stokes case, if (9) is replaced by

$$\boldsymbol{v}_n \overset{*}{\rightharpoonup} \boldsymbol{v}\,, \quad \text{in } L^\infty(0,T;L^2(\Omega))\,.$$

□

Proof of Proposition 4.3. As in the proof of Proposition 3.2, we get (10). We continue by estimating

$$\begin{aligned}\left\langle \mathscr{A}_{\boldsymbol{f}^1}(\boldsymbol{v}(t)),\boldsymbol{v}^1(t)-\boldsymbol{v}(t)\right\rangle &= \left\langle \mathscr{A}_{\boldsymbol{f}}(\boldsymbol{v}(t))+\boldsymbol{f}(t)-\boldsymbol{f}^1(t),\boldsymbol{v}^1(t)-\boldsymbol{v}(t)\right\rangle \\ &\geq -\frac{1}{2}\|\boldsymbol{v}^1(t)-\boldsymbol{v}(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\boldsymbol{f}(t)-\boldsymbol{f}^1(t)\|^2_{L^2(\Omega)}\,.\end{aligned}$$

Inserting this into (10) for the Euler equations and choosing $\phi = \varphi e^{-t}$, we find

$$\begin{aligned}0 \geq \Big(&-\int_0^T \varphi'(t)\mathscr{R}(\boldsymbol{v}^1(t)|\boldsymbol{v}(t))e^{-\int_0^t(\mathscr{K}(\boldsymbol{v}(s))+1)\,\mathrm{d}s}\,\mathrm{d}t - \varphi(0)\mathscr{R}(\boldsymbol{v}_0^1,\boldsymbol{v}_0) \\ &+\int_0^T \varphi(t)(\left\langle \mathscr{A}_{\boldsymbol{f}^1}(\boldsymbol{v}(t)),\boldsymbol{v}^1(t)-\boldsymbol{v}(t)\right\rangle e^{-\int_0^t(\mathscr{K}(\boldsymbol{v}(s))+1)\,\mathrm{d}s}\,\mathrm{d}t\Big)\,.\end{aligned}$$

Applying Lemma 2.2 and multiplying by $e^{-\int_0^t \mathscr{K}(\boldsymbol{v}(s)+1)\,\mathrm{d}s}$, implies the assertion. □

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