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Abstract. Sweeping processes are a class of evolution differential inclusions arising in elastoplasticity and were introduced by J.J. Moreau in the early seventies. The solution operator of the sweeping processes represents a relevant example of *rate independent operator*. As a particular case we get the so called *play operator*, which is a typical example of a hysteresis operator. The continuity properties of these operators were studied in several works. In this note we address the continuity with respect to the strict metric in the space of functions of bounded variation with values in the metric space of closed convex subsets of a Hilbert space. We provide counterexamples showing that for all BV-formulations of the sweeping process the corresponding solution operator is not continuous when its domain is endowed with the strict topology of BV and its codomain is endowed with the L^1-topology. This is at variance with the play operator which has a BV-extension that is continuous in this case.

1. Introduction

A sweeping process is an evolution problem arising in elastoplasticity that can be described in the following way. Let $\mathcal{H}$ be a real Hilbert space and let $\mathcal{C}(t) \subseteq \mathcal{H}$ be a moving closed convex set, i.e. a family of closed convex sets indexed by the time parameter $t \in [0, T]$, T being the final time of the evolution. One has to find a function $y : [0, T] \longrightarrow \mathcal{H}$ such that

$$y(t) \in \mathcal{C}(t) \qquad \forall t \in [0, T]\,, \tag{1.1}$$

$$-\,y'(t) \in N_{\mathcal{C}(t)}(y(t)) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0, T], \tag{1.2}$$

$$y(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0), \tag{1.3}$$

where $\mathcal{L}^1$ is the one dimensional Lebesgue measure, $y_0 \in \mathcal{H}$ is prescribed, $\mathrm{Proj}_{\mathcal{C}(0)}(y_0)$ is its projection on $\mathcal{C}(0)$, and $N_{\mathcal{C}(t)}(y(t))$ is the normal cone to $\mathcal{C}(t)$ at $y(t)$ (cf. the definition in formula (2.12) below: precise definitions will be given in the next Section 2). The sweeping process formulated as in (1.1)–(1.3) is well posed in the framework of Lipschitz continuous functions,

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indeed it can be shown that there exists a unique Lipschitz continuous function $y : [0, T] \longrightarrow \mathcal{H}$ satisfying (1.1)–(1.3), once $y_0 \in \mathcal{H}$ and the mapping $t \longmapsto \mathcal{C}(t)$ is Lipschitz continuous when the class of closed convex subsets of $\mathcal{H}$ is endowed with the Hausdorff metric (see (2.13) below). The proof of this fact can be found in [13]. For an overview on sweeping processes we refer the reader to [12].

If one wants to deal with more general movements of $\mathcal{C}(t)$, for instance when $t \longmapsto \mathcal{C}(t)$ is of bounded variation, then the above formulation has to be modified. In [14] the following generalized formulation is proposed. One has to look for a function $y : [0, T] \longrightarrow \mathcal{H}$ of bounded variation and a positive measure μ such that the distributional derivative $\mathrm{D}y$ of y satisfies the equality $\mathrm{D}y = w\mu$ for some $w \in L^1([0, T], \mu; \mathcal{H})$ and the condition

$$-w(t) \in N_{\mathcal{C}(t)}(y(t)) \qquad \text{for } \mu\text{-a.e. } t \tag{1.4}$$

holds together with (1.3). Such a function y is also called *a solution of the sweeping process in the sense of the differential measures.* We will call $\mathsf{S} : \mathcal{C} \longrightarrow y$ the solution operator of the sweeping process, associating with $\mathcal{C}(t)$ the solution $y(t)$.

The operator S is a very relevant example of a *rate independent operator*, i.e. an operator S such that

$$\mathsf{S}(\mathcal{C} \circ \phi)(t) = (\mathsf{S}(\mathcal{C}) \circ \phi)(t) \quad \forall t \in [0, T]. \tag{1.5}$$

whenever $\phi : [0, T] \longrightarrow [0, T]$ is an increasing surjective function. This fact was already observed by J.J. Moreau in (cf. [14, Proposition 2i]), even if he did not use the term “rate independence”.

A relevant particular case of sweeping processes is obtained when $\mathcal{C}(t) = u(t) - \mathcal{Z}$, $\mathcal{Z}$ being a fixed closed convex set in $\mathcal{H}$ and $u : [0, T] \longrightarrow \mathcal{H}$ a given function. In this case the solution operator P mapping u to the solution of the corresponding sweeping process can be equivalently defined on the space of $\mathcal{H}$-valued functions of bounded variation. The operator P is usually called *play operator* and plays an important role in elastoplasticity (cf. e.g. [7, 21, 4, 8, 11]).

The study of the continuity properties of the solution operators S and P has been addressed in several works. For instance in [14] it is shown that S is continuous with respect to the topology of the uniform convergence. Instead in [8] it is proved that P is continuous with respect to the BV strict topology when P is restricted to the space of continuous functions of bounded variation provided $\mathcal{Z}$ is a bounded convex closed set whose interior contains 0, for every boundary point x of $\mathcal{Z}$ there exists a unique outward normal $n(x)$, and the mapping $x \to n(x)$ is continuous. All these assumptions are dropped in [17], where $\mathcal{Z}$ is allowed to be an arbitrary closed convex set. Geometric conditions on $\mathcal{Z}$ are given in Section 3 in order to characterize when P is continuous from the space of left continuous functions of bounded variation into itself when the domain is endowed with the strict topology and the codomain is endowed with the L^1-topology.

The aim of the present note is to show that this continuity property does not hold for the general solution operator S. This is achieved by exhibiting a concrete example in the one dimensional case $\mathcal{H} = \mathbb{R}$.

Here is a brief plan of the paper. In the following Section 2 we present all the technical tools in order to deal with the sweeping processes: BV functions with values in a metric space and convex sets in a Hilbert space. In Section 3 we state the main known existence and continuity results about the sweeping process and in Section 4 we present our counterexample showing the BV-discontinuity of its solution operator. In the final section we make some remarks connecting the BV-discontinuity with the existence of multiple geodesics in the space of closed convex subsets of a Hilbert space.

2. Preliminaries

From now on T will be a fixed strictly positive number and $\mathbb{N}$ is the set of integers that are greater or equal than one. The family of Borel sets in $[0, T]$ will be denoted by $\mathscr{B}([0, T])$.

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2.1. Functions with bounded variation

We assume that

$$(\mathcal{X}, d) \text{ is a complete metric space.} \tag{2.1}$$

If $x \in \mathcal{X}$ and $S \subseteq \mathcal{X}$, $S \neq \varnothing$, we set $d(x, S) := \inf_{y \in S} d(x, y)$.

We will mainly deal with spaces of $\mathcal{X}$-valued functions defined on $[0, T]$. As usual the space of continuous functions is denoted by $C([0, T]\,; \mathcal{X})$. In the next definition we recall the most simple space containing discontinuous functions.

Definition 2.1. *Given a function $u : [0, T] \longrightarrow \mathcal{X}$ and a subinterval $J \subseteq [0, T]$, the* (pointwise) variation of u on J *is defined by*

$$\mathrm{V}(u, J) := \sup \left\{ \sum_{j=1}^{m} d(u(t_{j-1}), u(t_j)) \ :\ m \in \mathbb{N},\ t_j \in J\ \forall j,\ t_0 < \cdots < t_m \right\}. \tag{2.2}$$

If $\mathrm{V}(u, [0, T]) < \infty$ *we say that* u is of bounded variation on $[0, T]$ *and we set* $BV([0, T]\,; \mathcal{X}) := \{u : [0, T] \longrightarrow \mathcal{X}\ :\ \mathrm{V}(u, [0, T]) < \infty\}$.

It is well-known that every $u \in BV([0, T]\,; \mathcal{X})$ admits one sided limits $u(t-), u(t+)$ at every point $t \in [0, T]$, with the convention that $u(0-) := u(0)$ and $u(T+) := u(T)$. In this note we will limit ourselves to left continuous functions, i.e. we will deal with the space

$$BV_L([0, T]\,; \mathcal{X}) := \{u \in BV([0, T]\,; \mathcal{X})\ :\ u(t-) = u(t) \quad \forall t \in [0, T]\}. \tag{2.3}$$

When we consider left continuous functions we are essentially dealing with Lebesgue equivalence classes of functions with a special view on the initial point 0, allowing us to take into account Dirac masses at 0. In the next definition we introduce some natural metrics in $BV_L([0, T]\,; \mathcal{X})$.

Definition 2.2. *For every $u, v \in BV_L([0, T]\,; \mathcal{X})$ we set*

$$d_\infty(u, v) := \sup_{t \in [0, T]} d(u(t), v(t)), \tag{2.4}$$

$$d_s(u, v) := \int_0^T d(u(t), v(t))\,\mathrm{d}t + d(u(0), v(0)) + |\,\mathrm{V}(u, [0, T]) - \mathrm{V}(v, [0, T])|. \tag{2.5}$$

We call d_s strict metric *and we say that* $u_n \to u$ strictly on $[0, T]$ *if* $d_s(u_n, u) \to 0$ *as* $n \to \infty$. *The topology induced by* d_s *is called* strict topology.

Of course d_∞ is the distance inducing the topology of uniform convergence. Observe that if $u, v \in BV_L([0, T]\,; \mathcal{X})$ then $t \longmapsto d(u(t), v(t))$ is a measurable integrable function (cf. [6, Section 4.5.10, p. 505]), thus formula (2.5) makes sense.

The strict metric d_s is the natural metric in BV in the metric framework. It is also used when one deals with approximation procedures (see, e.g., [2]). In connection with hysteresis, it has been studied in [21, 4, 8, 16, 19]. Notice that the strict topology is different from the strong (or norm) BV-topology (see (2.9) below): the norm topology is usually too strong for applications (indeed it is often called the $W^{1,1}$-topology, that cannot be adapted to the metric framework).

Usually in the definition of strict metric the term $d(u(0), v(0))$ is missing. The reason why we insert it, is that we are considering left continuous functions on the closed interval $[0, T]$ and we want to take into account the value of these functions at the point $t = 0$. This is equivalent to artificially extend any function $u : [0, T] \longrightarrow \mathcal{X}$ from $[0, T]$ to $[-1, T]$ by setting $u(t) = u(0)$ for every $t < 0$. If we write down the classical notion of strict metric for these extended functions, we get exactly our d_s of Definition 2.2.

Let us recall that the space of Lipschitz continuous functions is defined by

$$Lip([0,T]\,;\mathcal{X}) := \left\{ u : [0,T] \longrightarrow \mathcal{X} \ : \ \sup_{t\neq s} \frac{d(u(s),u(t))}{|t-s|} < \infty \right\}. \tag{2.6}$$

It is clear that $Lip([0,T]\,;\mathcal{X}) \subseteq C([0,T]\,;\mathcal{X}) \cap BV([0,T]\,;\mathcal{X})$. In the following definition we recall the notion of geodesic in a metric space.

Definition 2.3. *If $x_0, y_0 \in \mathcal{X}$ and there is a curve $g \in Lip([0,1]\,;\mathcal{X})$ such that $g(0) = x_0$, $g(1) = y_0$ and $\mathrm{d}(x_0,y_0) = \mathrm{V}(g,[0,1])$, then g is called* a geodesic connecting x_0 and y_0.

2.2. Convex sets in Hilbert spaces

Let us assume that

$$\begin{cases} \mathcal{H} \text{ is a real Hilbert space with inner product } \langle\cdot,\cdot\rangle, \\ \|x\|_{\mathcal{H}} := \langle x,x\rangle^{1/2}. \end{cases} \tag{2.7}$$

If $\mu : \mathscr{B}([0,T]) \longrightarrow [0,+\infty]$ is a measure and $p \in [1,\infty]$, then the space of functions $u : [0,T] \longrightarrow \mathcal{H}$ such that $t \longmapsto \|u(t)\|_{\mathcal{H}}^p$ is integrable with respect to μ will be denoted by $L^p([0,T]\,,\mu;\mathcal{H})$ or by $L^p(\mu;\mathcal{H})$ if no confusion may arise. For the theory of integration of vector valued functions we refer to [3, Appendix]. When $\mu = \mathcal{L}^1$, the one dimensional Lebesgue measure, we will simply write $L^p(0,T;\mathcal{H}) := L^p([0,T],\mu;\mathcal{H})$. We warn the reader that we do not identify two functions which are equal $\mathcal{L}^1$-almost everywhere ($\mathcal{L}^1$-a.e.). For example, two functions u and v that are identical on $(0,T]$ but satisfy $u(0) \neq v(0)$ are not considered as two representation function of the same equivalence class. We recall that the Sobolev space $W^{1,p}([0,T]\,;\mathcal{H})$ is the space of functions $u \in L^p(0,T;\mathcal{H})$ such that there exists $v \in L^p(0,T;\mathcal{H})$ satisfying the equality $u(t) = u(0) + \int_0^t v(s)\,\mathrm{d}s$ for every $t \in [0,T]$ (cf., e.g., the Appendix of [3]). We also have that $Lip([0,T]\,;\mathcal{H}) = W^{1,\infty}([0,T]\,;\mathcal{H})$. Moreover if $u \in W^{1,1}([0,T]\,;\mathcal{H})$, then there exists the derivative $u'(t)$ for $\mathcal{L}^1$-a.e. $t \in [0,T]$, and we have the equality

$$\mathrm{V}(u,[0,T]) = \int_0^T \|u'(t)\|_{\mathcal{H}}\,\mathrm{d}t, \qquad \forall u \in W^{1,1}([0,T]\,;\mathcal{H}). \tag{2.8}$$

Since $\mathcal{H}$ has a linear structure we can also consider the so called strong BV-metric (or $W^{1,1}$-metric) in $BV_L([0,T]\,;\mathcal{H})$:

$$d_{BV}(u,v) := \int_0^T \|u(t)-v(t)\|_{\mathcal{H}}\,\mathrm{d}t + \|u(0)-v(0)\|_{\mathcal{H}} + \mathrm{V}(u-v,[0,T]), \qquad u,v \in BV_L([0,T]\,;\mathcal{H}), \tag{2.9}$$

In the regular case we obtain the topology of the Sobolev space $W^{1,1}([0,T]\,;\mathcal{H})$ induced by the norm

$$\|u\|_{W^{1,1}([0,T];\mathcal{H})} := \int_0^T \|u(t)\|_{\mathcal{H}}\,\mathrm{d}t + \int_0^T \|u'(t)\|_{\mathcal{H}}\,\mathrm{d}t, \qquad u \in W^{1,1}([0,T]\,;\mathcal{H}). \tag{2.10}$$

Now we set

$$\mathscr{C}_{\mathcal{H}} := \{\mathcal{K} \subseteq \mathcal{H} \ : \ \mathcal{K} \text{ nonempty, bounded, closed and convex}\}. \tag{2.11}$$

If $\mathcal{K} \in \mathscr{C}_{\mathcal{H}}$ and $x \in \mathcal{H}$, then the projection on $\mathcal{K}$ of x is denoted by $\mathrm{Proj}_{\mathcal{K}}(x)$. If $\mathcal{K} \in \mathscr{C}_{\mathcal{H}}$ and $x \in \mathcal{K}$, we recall that the *(exterior) normal cone to $\mathcal{K}$ at x* is defined by

$$N_{\mathcal{K}}(x) := \{y \in \mathcal{H} \ : \ \langle y, v-x\rangle \leq 0 \ \forall v \in \mathcal{K}\}. \tag{2.12}$$

We endow the set $\mathscr{C}_{\mathcal{H}}$ with the Hausdorff distance. Here is the definition.

Definition 2.4. *The* Hausdorff distance $d_{\mathscr{H}} : \mathscr{C}_{\mathcal{H}} \times \mathscr{C}_{\mathcal{H}} \longrightarrow [0, \infty[$ *is defined by*

$$d_{\mathscr{H}}(\mathcal{A}, \mathcal{B}) := \max\left\{ \sup_{a \in \mathcal{A}} d(a, \mathcal{B}), \sup_{b \in \mathcal{B}} d(b, \mathcal{A}) \right\}, \qquad \mathcal{A}, \mathcal{B} \subseteq \mathscr{C}_{\mathcal{H}}. \tag{2.13}$$

The distance $d_{\mathscr{H}}$ makes $\mathscr{C}_{\mathcal{H}}$ a complete metric space (see, e.g., [1, Theorem 3.85, Section 3.17, p. 116]).

For the sake of simplicity we assumed that the elements of $\mathscr{C}_{\mathcal{H}}$ are bounded. If this assumption is dropped, then $d_{\mathscr{H}}$ is a metric that may take on the value ∞, thus some supplementary technical details have to be added. However for our purposes this assumption is not restrictive and in order to prove the BV discontinuity of the sweeping process we can limit ourselves to the bounded case.

2.3. Differential measures

We recall that a *($\mathcal{H}$-valued Borel) vector measure* on $[0, T]$ is a map $\mu : \mathscr{B}([0, T]) \longrightarrow \mathcal{H}$ such that $\mu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n)$ whenever (B_n) is a sequence of mutually disjoint sets in $\mathscr{B}([0, T])$. Let us also recall that if $\mu : \mathscr{B}([0, T]) \longrightarrow \mathcal{H}$ is a vector measure, then $|\mu| : \mathscr{B}([0, T]) \longrightarrow [0, \infty]$ is defined by

$$|\mu|(B) := \sup\left\{ \sum_{n=1}^{\infty} \|\mu(B_n)\|_{\mathcal{H}} \ : \ B = \bigcup_{n=1}^{\infty} B_n, \ B_n \in \mathscr{B}([0, T]), \ B_h \cap B_k = \varnothing \text{ if } h \neq k \right\}.$$

The map $|\mu|$ is a positive measure which is called *total variation of* μ and the vector measure μ is said to be *with bounded variation* if $|\mu|([0, T]) < \infty$ (see, e.g., [5, Chapter I, Section 3.]).

The following proposition (cf. [5, Theorem 1, section III.17.2, p. 358]) provides a connection between functions with bounded variation and vector measures.

Theorem 2.1. *If $f \in BV_L([0, T]; \mathcal{H})$ then there exists a unique vector measure of bounded variation $\mu_f : \mathscr{B}([0, T]) \longrightarrow \mathcal{H}$ such that for every $c, d \in [0, T]$ with $c < d$ we have*

$$\mu_f(]c, d[) = f(d) - f(c+), \qquad \mu_f([c, d]) = f(d+) - f(c),$$
$$\mu_f([c, d[) = f(d) - f(c), \qquad \mu_f(]c, d]) = f(d+) - f(c+).$$

Vice versa if $\mu : \mathscr{B}([0, T]) \longrightarrow \mathcal{H}$ is a vector measure with bounded variation, then the map $f_\mu : [0, T] \longrightarrow \mathcal{H}$ defined by $f_\mu(t) := \mu([a, t[)$ is such that $f_\mu \in BV_L([0, T]; \mathcal{H})$ and $\mu_{f_\mu} = \mu$.

The measure μ_f is called *Lebesgue-Stieltjes measure* or *differential measure* of f. Like in the scalar case if $f \in BV([0, T]; \mathcal{H})$ then $\mu_f = \mathrm{D}f$, with $\mathrm{D}f$ being the distributional derivative of f, i.e.

$$-\int_0^T \varphi'(t) f(t) \, \mathrm{d}t = \int_0^T \varphi \, \mathrm{d}\mathrm{D}f \qquad \forall \varphi \in C_c^1(]0, T[)$$

(cf. [17, Section 2]).

3. The sweeping processes and the play operator

Now we can present the general existence result for the sweeping processes in BV_L. This is the main result in [14]: the existence theorem for right continuous data is [14, Propositions 2a and 3a] and the left continuous case can be deduced from [14, Section 2d].

Theorem 3.1. *Let $\mathcal{C} \in BV_L([0, T]; \mathscr{C}_{\mathcal{H}})$ and $y_0 \in \mathcal{H}$ be given. There exists a unique $y =: \mathsf{S}_{y_0}(\mathcal{C}) \in BV_L([0, T]; \mathcal{H})$ such that there exist a measure $\mu : \mathscr{B}([0, T]) \longrightarrow [0, \infty[$ and* *a function* $w \in L^1(\mu; \mathcal{H})$ *satisfying*

$$\mathrm{D}y = w\mu, \tag{3.1}$$

$$y(t) \in \mathcal{C}(t) \qquad \forall t \in [0,T]\,, \tag{3.2}$$

$$-\,w(t) \in N_{\mathcal{C}(t)}(y(t)) \qquad \text{for } \mu\text{-a.e. } t \in [0,T], \tag{3.3}$$

$$y(0) = \mathrm{Proj}_{\mathcal{C}(0)}(y_0), \tag{3.4}$$

where $w\mu$ *is the vector measure defined by* $w\mu(B) := \int_B w\,\mathrm{d}\mu$, $B \in \mathscr{B}([0,T])$. *If* $\mathcal{C} \in BV([0,T]\,;\mathscr{C}_{\mathcal{H}}) \cap C([0,T]\,;\mathscr{C}_{\mathcal{H}})$, *then* $\mathsf{S}_{y_0}(\mathcal{C}) \in BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H})$. *Finally if* $\mathcal{C} \in Lip([0,T]\,;\mathcal{H})$ *then* $\mathsf{S}_{y_0}(\mathcal{C}) \in Lip([0,T]\,;\mathcal{H})$ *and problem* (3.1)–(3.4) *reads as* (1.1)–(1.3).

The two last statements of the previous theorem can be easily deduced by the representation formula of [18, Theorem 3.1] for the continuous case. Thanks to the previous theorem the following solution operator is defined

$$\mathsf{S}_{y_0} : BV_L([0,T]\,;\mathscr{C}_{\mathcal{H}}) \longrightarrow BV_L([0,T]\,;\mathcal{H}).$$

If $\mathcal{Z} \in \mathscr{C}_{\mathcal{H}}$, $0 \in \mathcal{Z}$ and $z_0 \in \mathcal{Z}$, we consider the *play operator*

$$\mathsf{P}_{z_0} : BV_L([0,T]\,;\mathcal{H}) \longrightarrow BV_L([0,T]\,;\mathcal{H})$$

which is defined by

$$\mathsf{P}_{z_0}(u) := \mathsf{S}_{u(0)-z_0}(u - \mathcal{Z}), \qquad u \in BV_L([0,T]\,;\mathcal{H}).$$

The restrictions of S_{y_0} and P_{z_0} to the various spaces subspaces of BV will be denoted by the same symbols S_{y_0} and P_{z_0}. Using this notation, it holds that $\mathsf{P}_{z_0} : Lip([0,T]\,;\mathcal{H}) \longrightarrow Lip([0,T]\,;\mathcal{H})$ is the *classical play operator*, which is the operator associating with $u \in Lip([0,T]\,;\mathcal{H})$ the only function $y \in Lip([0,T]\,;\mathcal{H})$ satisfying (1.1)–(1.3) with $\mathcal{C}(t) = u(t) - \mathcal{Z}$ and $y_0 = u(0) - z_0$.

Let us observe that the operator P_{z_0} here defined coincides with the one introduced in [9]. Indeed, as observed in [9], the two operators coincide on the set of left continuous step functions (cf. formula (3.8)) and they are both continuous with respect to the d_∞-metric (cf. Theorem 3.2 (iii) below and [9, Theorem 2.3]).

Another possible extension of the play operator on $Lip([0,T]\,;\mathcal{H})$ to the space of functions of bounded variation is provided in [17], where an operator $\overline{\mathsf{P}}_{z_0} : BV_L([0,T]\,;\mathcal{H}) \longrightarrow BV_L([0,T]\,;\mathcal{H})$ is obtained as the continuous extension of the classical play operator $\mathsf{P}_{z_0} : Lip([0,T]\,;\mathcal{H}) \longrightarrow Lip([0,T]\,;\mathcal{H})$ with respect to the d_s-topology in the domain and the L^1-topology in the codomain. As shown in [17], the two operators coincide on $BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H})$, but in general they are different on $BV_L([0,T]\,;\mathcal{H})$.

A main feature of S_{y_0} and P_{z_0} is *rate independence*, i.e. if $\phi : [0,T] \longrightarrow [0,T]$ is an increasing surjective function, then

$$\mathsf{S}_{y_0}(\mathcal{C} \circ \phi) = \mathsf{S}_{y_0}(\mathcal{C}) \circ \phi, \tag{3.5}$$

$$\mathsf{P}_{z_0}(u \circ \phi) = \mathsf{P}_{z_0}(u) \circ \phi, \tag{3.6}$$

whenever $\mathcal{C} \in BV_L([0,T]\,;\mathscr{C}_{\mathcal{H}})$ and $u \in BV_L([0,T]\,;\mathcal{H})$ (cf. [14, Proposition 2i]). Now let $m \in \mathbb{N}$, $t_0 = 0 < t_1 < \cdots < t_m = T$, and $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_m \in \mathscr{C}_{\mathcal{H}}$. If $\mathcal{C} \in BV_L([0,T]\,;\mathscr{C}_{\mathcal{H}})$ is the step function defined by

$$\mathcal{C}(t) := \begin{cases} \mathcal{C}_0 & \text{if } t = 0, \\ \mathcal{C}_k & \text{if } t \in \,]t_{k-1}, t_k]\,, k \in 1, \ldots, m, \end{cases} \tag{3.7}$$

then for every $y_0 \in \mathcal{H}$ it turns out that (cf. [14, Formulas (1.13)-(1.14)])

$$\mathsf{S}_{y_0}(\mathcal{C})(t) = \begin{cases} \mathrm{Proj}_{\mathcal{C}_0}(y_0) & \text{if } t = 0, \\ \mathrm{Proj}_{\mathcal{C}_k}(y_{k-1}) & \text{if } t \in \left]t_{k-1}, t_k\right], k \in 1, \ldots, m. \end{cases} \tag{3.8}$$

Of course the same kind of result holds for the play operator P_{z_0}. Here we list some of the main continuity properties of S_{y_0}, of P_{z_0} and of the restrictions of these operators to various spaces subspaces of BV. We recall that these restrictions are denoted by the same symbols S_{y_0} and P_{z_0}.

Theorem 3.2. *Assume that $y_0 \in \mathcal{H}$, $\mathcal{Z} \in \mathscr{C}_{\mathcal{H}}$, and $0, z_0 \in \mathcal{Z}$. The following statements holds true.*

(i) $\mathsf{S}_{y_0} : BV_L([0,T]\,;\mathscr{C}_{\mathcal{H}}) \longrightarrow BV_L([0,T]\,;\mathcal{H})$ *is continuous with respect to the* d_∞*-topology.*

(ii) $\mathsf{S}_{y_0} : BV([0,T]\,;\mathscr{C}_{\mathcal{H}}) \cap C([0,T]\,;\mathscr{C}_{\mathcal{H}}) \longrightarrow BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H})$ *is continuous if its domain is endowed with the* d_s*-topology and its codomain is endowed with the* d_∞*-topology.*

(iii) $\mathsf{P}_{z_0} : BV_L([0,T]\,;\mathcal{H}) \longrightarrow BV_L([0,T]\,;\mathcal{H})$ *is continuous with respect to the* d_∞*-topology.*

(iv) $\mathsf{P}_{z_0} : BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H}) \longrightarrow BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H})$ *is continuous with respect to the* d_s*-topology.*

(v) $\overline{\mathsf{P}}_{z_0} : BV_L([0,T]\,;\mathcal{H}) \longrightarrow BV_L([0,T]\,;\mathcal{H})$ *is continuous if the domain is endowed with the strict topology and the codomain with the* L^1*-topology.*

(vi) *If $\mathcal{H}$ is of finite dimension and $\mathcal{Z}$ is a* non-obtuse polyhedron, *i.e.* $\mathcal{Z} = \{x \in \mathcal{H} : \langle n_j, x\rangle \le c_j,\ j = 1, \ldots, p\}$ *for some* $p \in \mathbb{N}$, $c_j \ge 0$ *and* $n_j \in \mathcal{H}$ *with* $\|n_j\|_{\mathcal{H}} = 1$ *and* $\langle n_j, n_k\rangle \le 0$ *whenever* $1 \le i < j \le p$, *then it follows that* $\mathsf{P}_{z_0} : BV_L([0,T]\,;\mathcal{H}) \longrightarrow BV_L([0,T]\,;\mathcal{H})$ *is continuous if its domain is endowed with the* d_s*-topology and its codomain is endowed with the* L^1*-topology.*

Proof. Part (i) of the previous theorem is proved in [14] while part (ii) is proved in [18]. Concerning the play operator P_{z_0}, part (iii) follows directly from part (i), since $d_\infty(u_n - Z, u - Z) \to 0$ whenever $u_n \to u$ uniformly on $[0,T]$. Part (iv) is instead proved in [17], where our extra-term $d(u(0), v(0))$ of the strict metric in (2.5) can be handled by means of the reduction method presented in [17, Section 4.4] taking into account the Lipschitz continuity of the projection. In order to prove (v) we recall that in [15] it is proved that $\mathsf{P}_{z_0} : BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H}) \longrightarrow BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H})$ admits a continuous extension to the above mentioned operator $\overline{\mathsf{P}}_{z_0} : BV_L([0,T]\,;\mathcal{H}) \longrightarrow BV_L([0,T]\,;\mathcal{H})$ if the domain is endowed with the strict topology and the codomain with the L^1-topology. This extension $\overline{\mathsf{P}}_{z_0}$ is however not necessarily equal to the sweeping process $\mathsf{P}_{z_0} = \mathsf{S}_{u(0)-z_0}(u - \mathcal{Z})$ on discontinuous functions considered in (vi): in [10] it is shown for $\mathcal{H}$ being finite dimensional that $\overline{\mathsf{P}}_{z_0} = \mathsf{P}_{z_0}$ on the whole $BV_L([0,T]\,;\mathcal{H})$ if and only if $\mathcal{Z}$ is a non-obtuse polyhedron. $\square$

In the one dimensional case the play operator P_{z_0} is continuous from $BV_L([0,T]\,;\mathbb{R})$ into itself endowed with strict topology, this is proved in [15]. In the vector case this property is in general false: this can be deduced from [17, Theorem 3.7] where it is shown that the restriction $\mathsf{P}_{z_0} : BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H}) \longrightarrow BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H})$ can be d_s-continuously extended to the whole $BV_L([0,T]\,;\mathcal{H})$ if and only if $\mathcal{Z}$ is a closed vector subspace or $\mathcal{Z} = \{x \in \mathcal{H} : -\alpha \le \langle f, x\rangle \le \beta\}$ for some $\alpha, \beta \in [0,\infty]$ and some $f \in \mathcal{H} \smallsetminus \{0\}$, hence for bounded $\mathcal{Z}$ the operator P_{z_0} is never continuous from the whole BV into itself, both domain and codomain endowed with the strict metric.

4. Metric BV discontinuity

From Theorem 3.2 we can infer that if $\mathcal{Z}$ is a non-obtuse polyhedron and $\mathcal{H}$ has finite dimension, then $\mathsf{P}_{z_0} : BV_L([0,T]\,;\mathcal{H}) \longrightarrow BV_L([0,T]\,;\mathcal{H})$ is continuous if the domain is endowed with the strict topology and the codomain with the L^1-topology, while the operator $\overline{\mathsf{P}}_{z_0} : BV_L([0,T]\,;\mathcal{H}) \longrightarrow BV_L([0,T]\,;\mathcal{H})$ is continuous in this sense even if these assumptions on $\mathcal{Z}$ and $\mathcal{H}$ are not satisfied.

Now we provide the counterexample showing that this last continuity property stated is in general not true for the solution operator S_{y_0} of the sweeping process.

Let us consider $\mathcal{H} = \mathbb{R}$ so that

$$\mathscr{C}_{\mathbb{R}} = \{I \subseteq \mathbb{R} \ :\ I \text{ is a bounded closed interval, } I \neq \varnothing\}. \tag{4.1}$$

If $a,b,c,d \in \mathbb{R}$ and $a \le b$, $c \le d$, then

$$d_{\mathscr{H}}([a,b]\,,[c,d]) = \max\{|a-c|, |b-d|\}. \tag{4.2}$$

Let us set

$$\mathcal{K}_0 := [0,2]\,, \qquad \mathcal{K}_1 := [1,4] \tag{4.3}$$

and fix $t_0 \in\,]0,T[$. So $\mathcal{K}_0, \mathcal{K}_1 \in \mathscr{C}_{\mathbb{R}}$ and we can define $\mathcal{C} : [0,T] \longrightarrow \mathscr{C}_{\mathbb{R}}$ by setting

$$\mathcal{C}(t) := \begin{cases} \mathcal{K}_0 & \text{if} \quad 0 \le t \le t_0, \\ \mathcal{K}_1 & \text{if} \quad t_0 < t \le T, \end{cases} \qquad t \in [0,T]\,. \tag{4.4}$$

Hence $\mathcal{C} \in BV([0,T]\,;\mathscr{C}_{\mathbb{R}})$ and $\mathrm{V}(\mathcal{C},[0,T]) = 2$. Let us now define $\mathcal{B} : [0,1] \longrightarrow \mathscr{C}_{\mathbb{R}}$ by

$$\mathcal{B}(t) := \begin{cases} [2t, 2+2t] & \text{if} \quad 0 \le t \le 2/3, \\ [2-t, 2+2t] & \text{if} \quad 2/3 < t \le 1. \end{cases} \tag{4.5}$$

Observe that $\mathcal{B} \in Lip([0,1]\,;\mathscr{C}_{\mathbb{R}})$ and thanks to [18, Proposition 6.1] we have that, up to reparametrization, the restriction of $\mathcal{B}$ to $[0,2/3]$ is a geodesic connecting $B(0) = \mathcal{K}_0$ to $\mathcal{B}(2/3) = [4/3, 10/3]$. On the other hand, up to reparametrization, the restriction of $\mathcal{B}$ to $[2/3,1]$ is a geodesic connecting $\mathcal{B}(2/3) = [4/3,10/3]$ and $B(1) = \mathcal{K}_1$. Hence

$$\mathrm{V}(\mathcal{B},[0,1]) = \mathrm{V}(\mathcal{B},[0,2/3]) + \mathrm{V}(\mathcal{B},[2/3,1]) = 4/3 + 2/3 = 2, \tag{4.6}$$

thus $\mathcal{B}$ is a geodesic connecting $\mathcal{K}_0$ and $\mathcal{K}_1$. Let $\mathcal{B}_n \in Lip([0,T]\,;\mathscr{C}_{\mathbb{R}})$ be the sequence defined (for n large enough) by

$$\mathcal{B}_n(t) := \begin{cases} \mathcal{K}_0 & \text{if } 0 \le t \le t_0, \\ \mathcal{B}(n(t-t_0)) & \text{if } t_0 < t \le t_0 + 1/n, \\ \mathcal{K}_1 & \text{if } t_0 + 1/n < t \le T. \end{cases}$$

We have that

$$\mathrm{V}(\mathcal{B}_n,[0,T]) = \mathrm{V}(\mathcal{C},[0,T]) = d_{\mathscr{H}}(\mathcal{K}_0,\mathcal{K}_1) \quad \forall n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} d_{\mathscr{H}}(\mathcal{B}_n(t), \mathcal{C}(t)) = 0 \qquad \forall t \in [0,T]\,.$$

Hence, as $d_{\mathscr{H}}(\mathcal{B}_n(t), \mathcal{C}(t)) \le d_{\mathscr{H}}(\mathcal{K}_0,\mathcal{K}_1)$, by the dominated convergence theorem we have that

$$\lim_{n\to\infty} \int_0^T d_{\mathscr{H}}(\mathcal{B}_n(t), \mathcal{C}(t))\,\mathrm{d}t = 0.$$

Hence $\mathcal{B}_n \to \mathcal{C}$ as $n \to \infty$ in the strict topology of $BV_L([0,T]\,;\mathscr{C}_{\mathbb{R}})$. If $y_0 := 0$, then $y_0 \in \mathcal{C}(0) = \mathcal{B}(0) = \mathcal{K}_0$ and, thanks to (3.7)–(3.8) we have

$$\mathsf{S}_0(\mathcal{C})(t) = \begin{cases} 0 & \text{if } 0 \le t \le t_0, \\ \mathrm{Proj}_{\mathcal{K}_1}(0) = 1 & \text{if } t_0 < t \le T. \end{cases} \tag{4.7}$$

It is also easy to check that

$$\mathsf{S}_0(\mathcal{B})(t) = \begin{cases} 2t & \text{if } 0 \le t \le 2/3, \\ 4/3 & \text{if } 2/3 < t \le 1, \end{cases}$$

therefore, using for instance rate independence, it follows that

$$\mathsf{S}_0(\mathcal{B}_n)(t) = \begin{cases} 0 & \text{if } 0 \le t \le t_0, \\ 2n(t-t_0) & \text{if } t_0 < t \le t_0 + 2/3n, \\ 4/3 & \text{if } t_0 + 2/3n < t \le T. \end{cases}$$

Hence, if $z : [0,T] \longrightarrow \mathbb{R}$ is defined by

$$z(t) = \begin{cases} 0 & \text{if } 0 \le t \le t_0, \\ 4/3 & \text{if } t_0 < t \le T, \end{cases} \tag{4.8}$$

we have that $\lim_{n\to\infty} \mathsf{S}_0(\mathcal{B}_n)(t) = z(t)$ for every $t \in [0,T]$, thus, thanks to the dominated convergence theorem, we infer that

$$\mathsf{S}_0(\mathcal{B}_n) \to z \qquad \text{in } L^1(0,T;\mathcal{H}). \tag{4.9}$$

Therefore, as $\mathsf{S}_0(\mathcal{C}) \neq z$, the operator S_0 is not continuous when its domain $BV_L([0,T]\,;\mathcal{H})$ is endowed with the topology induced by the strict convergence and its codomain is endowed with any reasonable topology weaker than or equal to the $L^1(0,T;\mathcal{H})$ topology.

5. Multiple geodesics

The lack of metric BV continuity of the solution operator of the sweeping process in the one dimensional case is somehow connected to the existence of more than one geodesic connecting points (sets) in $\mathscr{C}_{\mathbb{R}}$. Indeed let us consider the curve $\mathcal{A} : [0,1] \longrightarrow \mathscr{C}_{\mathbb{R}}$ defined by

$$\mathcal{A}(t) := (1-t)\mathcal{K}_0 + t\mathcal{K}_1, \qquad t \in [0,1]\,. \tag{5.1}$$

Observe that $\mathcal{A} \in Lip([0,1]\,;\mathscr{C}_{\mathbb{R}})$ and that $\mathcal{A} \neq \mathcal{B}$. In the case of the one dimensional play operator with characteristic $\mathcal{Z}$, the curve $\mathcal{A}$ corresponds to the input $\mathcal{A}(t) = a(t) - \mathcal{Z}$, $\mathcal{K}_j = \{k_j\}$ for some $k_j \in \mathcal{H}$, $j = 1,2$, with $a(t) = (1-t)k_1 + tk_2$, $t \in [0,1]$. Thanks to [18, Proposition 6.1] or [20, Prop. 1]: we have that $\mathcal{A}$ is a geodesic connecting $\mathcal{K}_0$ and $\mathcal{K}_1$. Let $\mathcal{A}_n \in Lip([0,T]\,;\mathscr{C}_{\mathbb{R}})$ be the sequence defined (for n large enough) by

$$\mathcal{A}_n(t) := \begin{cases} \mathcal{K}_0 & \text{if } 0 \le t \le t_0, \\ \mathcal{A}(n(t-t_0)) & \text{if } t_0 < t \le t_0 + 1/n, \\ \mathcal{K}_1 & \text{if } t_0 + 1/n < t \le T. \end{cases}$$

We have that

$$\mathrm{V}(\mathcal{A}_n, [0,T]) = \mathrm{V}(\mathcal{C}, [0,T]) = d_{\mathscr{H}}(\mathcal{C}_0, \mathcal{C}_1) \quad \forall n \in \mathbb{N},$$

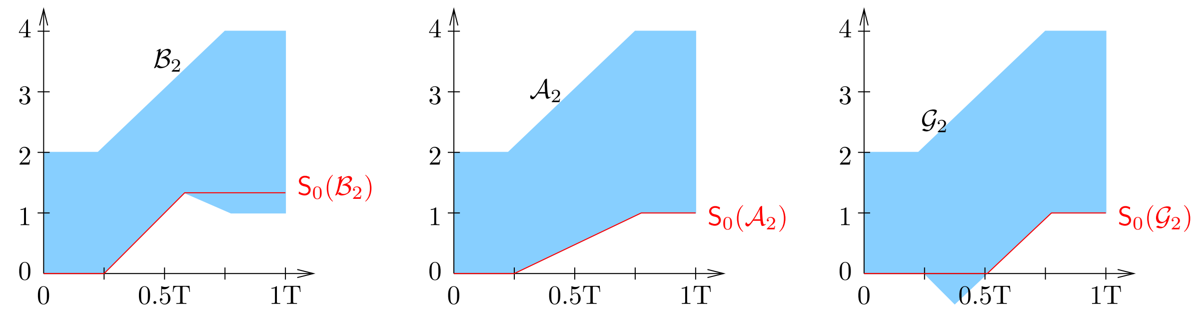

Figure 1. The approximations $\mathcal{B}_2$ (left) and $\mathcal{A}_2$ (center) for $\mathcal{C}$ and the corresponding output of the sweeping process S_0 for $T = 1$ and $t_0 = 0.25$. On the right, the approximation $\mathcal{G}_2$, being analogously derived by using the geodesic $\mathcal{G}$, and the corresponding output of the sweeping process are shown.

and

$$\lim_{n\to\infty} d_{\mathscr{H}}(\mathcal{A}_n(t), \mathcal{C}(t)) = 0 \qquad \forall t \in [0, T]\,.$$

Hence, as $d_{\mathscr{H}}(\mathcal{A}_n(t), \mathcal{C}(t)) \le d_{\mathscr{H}}(\mathcal{K}_0, \mathcal{K}_1)$, by the dominated convergence theorem we have that

$$\lim_{n\to\infty} \int_0^T d_{\mathscr{H}}(\mathcal{A}_n(t), \mathcal{C}(t))\,\mathrm{d}t = 0.$$

Hence $\mathcal{A}_n \to \mathcal{C}$ as $n \to \infty$ in the strict topology of $BV_L([0,T]\,;\mathscr{C}_{\mathbb{R}})$. Taking again $y_0 := 0$ we find $y_0 \in \mathcal{C}(0) = \mathcal{A}(0) = \mathcal{K}_0$ and

$$\mathsf{S}_0(\mathcal{A})(t) = t,$$

therefore

$$\mathsf{S}_0(\mathcal{A}_n)(t) = \begin{cases} 0 & \text{if } 0 \le t \le t_0, \\ n(t - t_0) & \text{if } t_0 < t \le t_0 + 1/n, \\ 1 & \text{if } t_0 + 1/n < t \le T. \end{cases}$$

Thus

$$\mathsf{S}_0(\mathcal{A}_n) \to \mathsf{S}_0(\mathcal{C}) \qquad \text{in } L^1(0, T; \mathbb{R}). \tag{5.2}$$

Hence, the sequence $\mathcal{A}_n$ approximating $\mathcal{C}$ by using the geodesic $\mathcal{A}$ does not allow to prove that S_0 is not continuous. [1] But, using this sequences, it follows for **every** extension $\mathsf{E}_{\mathsf{S},0} : BV_L([0,T]\,;\mathscr{C}_{\mathbb{R}}) \to BV_L([0,T]\,;\mathcal{H})$ of the restriction of S_0 to $Lip([0,1]\,;\mathscr{C}_{\mathbb{R}})$ that $\mathsf{E}_{\mathsf{S},0}$ is **not** continuous if the domain is endowed with the strict topology and the codomain with the L^1-topology: If $\mathsf{E}_{\mathsf{S},0}$ were continuous as requested it would follow by the computations for the sequence $\mathcal{A}_n$ that $\mathsf{E}_{\mathsf{S},0}(\mathcal{C}) = \mathsf{S}_0(\mathcal{C})$, while the computations for $\mathcal{B}_n$ would yield that $\mathsf{E}_{\mathsf{S},0}(\mathcal{C}) = z \ne \mathsf{S}_0(\mathcal{C})$.

[1] The same holds for the geodesic $\mathcal{G}$ connecting $\mathcal{K}_0$ and $\mathcal{K}_1$ that is defined as in [20, Theorem 1]:

$$\mathcal{G}(t) := \delta_{t\rho}(\mathcal{K}_1) \cap \delta_{(1-t)\rho}(\mathcal{K}_2)$$

with $\rho := d_{\mathscr{H}}(\mathcal{K}_0, \mathcal{K}_1) = 2$ and $\delta_\lambda(K) := \bigcup_{x\in X} \{y \in X \ : \ d(x,y) \le \lambda\}$ for $\lambda > 0$ and $K \in \mathscr{C}_{\mathbb{R}}$. We have

$$\mathcal{G}(t) = [-2t, 2+2t] \cap [1-2(1-t), 4+2(1-t)] = \begin{cases} [-2t, 2t+2] & \text{if } t \le 1/4, \\ [1-2(1-t), 2t+2] & \text{if } t > 1/4. \end{cases}$$

Hence, we see that considering different geodesics connecting two elements in $\mathscr{C}_{\mathbb{R}}$ yields that there is no extension of the restriction of the sweeping process with the requested continuity properties.

6. Conclusion

We have provided a one dimensional counterexample showing that the solution operator of the sweeping process is not continuous when its domain is endowed with the strict topology of BV and its codomain is endowed with the L^1-topology. For a general Hilbert space $\mathcal{H}$ one obtains a corresponding discontinuity for any BV-formulation of the sweeping process by introducing for any $e \in \mathcal{H}$ with $e \neq 0_H$ the mapping $x \longmapsto \langle x, e\rangle$ and considering the pre-images of the intervals used above intersected with some appropriate bounded convex set. This is at variance with the case of the play operator on $Lip([0,T]\,;\mathcal{H})$ and its continuous BV-extension $\overline{\mathsf{P}}_{z_0}$ (cf. Theorem 3.2 (v)).

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