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Torsion-free covers of solvable minimax groups

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Abstract

We prove that every finitely generated solvable minimax group can be realized as a quotient of a torsion-free solvable minimax group. This result has an application to the investigation of random walks on finitely generated solvable minimax groups. Our methods also allow us to completely characterize the solvable minimax groups that are homomorphic images of torsion-free solvable minimax groups.

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1 Introduction

In this paper we establish a new property of finitely generated solvable groups of finite abelian section rank. This enables us to resolve a difficult issue that arises in Ch. Pittet and L. Saloff-Coste's study [**9**] of random walks on the Cayley graphs of such groups.

We are concerned here with *solvable minimax groups*; these are groups that possess a series of finite length in which each factor is either cyclic or quasicyclic. The set of primes associated with the quasicyclic factors in such a series is an invariant of the group, called its *spectrum*. Among finitely generated solvable groups, the minimax ones are of central importance because they comprise all the finitely generated solvable groups without any sections isomorphic to $C_p \wr C_\infty$ for any prime p (see [**5**]). In particular, any finitely generated solvable group of finite abelian section rank is minimax.

Our goal is to prove Theorem A below. In its statement, we write $\mathrm{Fitt}(G)$ for the *Fitting subgroup* of a group G, namely, the subgroup generated by all the nilpotent normal subgroups.

Theorem A. *If G is a finitely generated solvable minimax group, then there is a finitely generated torsion-free solvable minimax group G^* with the same spectrum as G and an epimorphism $\phi : G^* \to G$ such that $\phi(\mathrm{Fitt}(G^*)) = \mathrm{Fitt}(G)$.*

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The above theorem may be invoked to reduce certain questions about finitely generated solvable minimax groups to the case where the group is torsion-free. One such application, to random walks, was communicated to the authors by Lison Jacoboni. Immediately upon learning of our result, she pointed out that it could be employed to fill an apparent gap in an argument contained in [**9**]. We thank Lison Jacoboni for sharing this insight with us and giving permission for its inclusion here (see Theorem 5.1).

In proving Theorem A, we introduce a new method for studying solvable minimax groups, one that we hope will give rise to further advances in the theory of such groups. Our approach involves embedding G in a locally compact topological group and taking advantage of certain features of the structure of this new group. In this section, we describe the main aspects of the argument, assuming, for the sake of simplicity, that the maximal torsion normal subgroup P of G is a direct product of quasicyclic p-groups for a single prime p. The first step is to embed $N = \text{Fitt}(G)$ in a nilpotent, locally compact topological group N_p whose compact subgroups are all polycyclic pro-p groups. The group N_p is constructed by forming the direct limit of the pro-p completions of the finitely generated subgroups of N. In the abelian case, this is equivalent to tensoring N with the ring of p-adic integers.

Drawing on a technique originally due to Peter Hilton [**2**], we build next a group extension $1 \to N_p \to G_{(N,p)} \to Q \to 1$ that fits into a diagram of the form

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & N_p & \longrightarrow & G_{(N,p)} & \longrightarrow & Q & \longrightarrow & 1. \end{array}$$

The benefit of the group $G_{(N,p)}$ is that it contains a maximal radicable normal subgroup $R < N_p$ and a residually finite subgroup X such that $[G_{(N,p)} : RX] < \infty$. Writing $\Gamma_0 = R \rtimes X$, we have an obvious group epimorphism from Γ_0 to RX. Moreover, Γ_0 enjoys two cohomological properties that we can exploit to obtain the desired result. First, the fact that X is a complement to R in Γ_0 provides us with a surjective restriction map $H^2(\Gamma_0/P, A) \to H^2(X, A)$, where A is the maximal elementary abelian subgroup of P. In addition, the radicability of R and the semidirect product structure of Γ_0 enable us to deduce that the restriction map $H^2(\Gamma_0, A) \to H^2(P, A)^{\Gamma_0}$ is epic.

In the most important stage of the proof, we employ the surjectivity of the two maps in cohomology described above in order to construct an inverse system of groups $\cdots \xrightarrow{\phi_3} \Gamma_2 \xrightarrow{\phi_2} \Gamma_1 \xrightarrow{\phi_1} \Gamma_0$ with the following properties:

(i) ϕ_i is surjective and Ker $\phi_i \cong A$ for $i \geq 1$;

(ii) the inverse limit Γ^* of the system is a virtually torsion-free nilpotent-by-polycyclic group whose finitely generated subgroups are all minimax.

Now we take $\{h_1, \ldots, h_r\} \subset \Gamma^*$ to be the inverse image under the epimorphism $\Gamma^* \to RX$ of a generating set of $G \cap RX$ and let $H = \langle h_1, \ldots, h_r \rangle$. Then H is a finitely generated, virtually torsion-free solvable minimax group that maps homomorphically onto $G \cap RX$, a subgroup of finite index in G. In this way, we find ourselves within easy reach of our final objective.

The proof of Theorem A is not presented until §4. In the preceding two sections, we lay the groundwork for the proof: §3 investigates how to construct the groups N_p and $G_{(N,p)}$ discussed above, and §2 examines the class $\mathfrak{N}_p$ of nilpotent topological groups inhabited by

N_p. Of particular significance in our discussion of $\mathfrak{N}_p$ are the final two results in §2, which treat the properties of the maximal radicable subgroup of a group in that class. The second of these states that, if N is a group in $\mathfrak{N}_p$ that is a normal subgroup of a group G such that G/N is a finitely generated, virtually abelian group, then N can be expressed as a product of a closed radicable G-invariant subgroup and a G-invariant, polycyclic pro-p subgroup. This property is required for the construction of the subgroup X mentioned above in our outline of the proof of Theorem A.

After treating the application to random walks in §5, the article concludes with an appendix containing a generalization of Theorem A. This result specifies the precise extent to which the condition that the group is finitely generated may be weakened while still guaranteeing the existence of a torsion-free solvable minimax cover. The proof involves merely a slight elaboration of the argument employed for Theorem A.

Below we describe the notation and terminology that we will employ throughout the article. In addition, we state a proposition that lists some basic properties of solvable minimax groups; proofs of these may be found in [**7**].

Notation and terminology. If p is a prime, then $\mathbb{Z}_p$ is the ring of p-adic integers and $\mathbb{Q}_p$ the ring of p-adic rational numbers. In addition, $\mathbb{Z}_{p^\infty}$ is the quasicyclic p-group.

If π is a set of primes, then a *π-number* is an integer whose prime divisors are all in π. Moreover, $\mathbb{Z}_{(\pi)}$ denotes the subring of $\mathbb{Q}$ consisting of all fractions of the form m/n where $m \in \mathbb{Z}$ and n is a nonzero π-number.

Let G be a group. We define the *finite residual* of G, written $R(G)$, to be the intersection of all the subgroups of finite index in G.

If H is a subgroup of a group G and $g \in G$, then $H^g = g^{-1}Hg$.

Let π be a set of primes. A group G is said to be *π-radicable* if, for any π-number n and $g \in G$, there is an $x \in G$ such that $x^n = g$. A *radicable* group is a group that is π-radicable where π is the set of all primes.

If N is a nilpotent group, then nil N represents the nilpotency class of N.

If a group G acts nilpotently on a group N as a group of operators, then the nilpotency class of the action is denoted $\mathrm{nil}_G\, N$.

If π is set of primes, then a *π-minimax group* is a solvable minimax group whose spectrum is contained in π.

The term *module* will always refer to a left module. Moreover, if R is a ring and A an R-module, we will write the operation $R \times A \to A$ as $(r, a) \mapsto r \cdot a$.

A *section* of a module is a modular quotient of a submodule. Similarly, a *section* of a group is a quotient of a subgroup.

Assume that G is a topological group. A *topological quotient* of G is a quotient of G by a closed normal subgroup. A *topological section* of G is a topological quotient of a closed subgroup. If $H < G$, then $\overline{H}$ denotes the closure of H in G. We say that G is *topologically finitely generated* if there are elements $g_1, \ldots, g_r$ of G such that $G = \overline{\langle g_1, \ldots, g_r \rangle}$.

A *polycyclic pro-p group* is a pro-p group with a series of finite length consisting of closed subgroups such that each factor is a cyclic pro-p group, that is, either a finite cyclic p-group or a copy of $\mathbb{Z}_p$.

Proposition 1.1. *Let G be a solvable minimax group and $N = \mathrm{Fitt}(G)$. Then the following three statements hold.*

(i) N is nilpotent and G/N virtually abelian.

(ii) If G is virtually torsion-free, then G/N is finitely generated.

(iii) $R(G)$ is a direct product of finitely many quasicyclic groups. □

2 The class $\mathfrak{N}_p$

This section is concerned with the properties of topological groups that belong to the following class.

Definition. For any prime p, we define $\mathfrak{N}_p$ to be the class of all nilpotent topological groups N that have a series

$$1 = N_0 \lhd N_1 \lhd \cdots \lhd N_r = N$$

consisting of closed subgroups such that, for every $i \geq 1$, N_i/N_{i-1} is of one of the following two types with respect to the topology inherited from N:

(i) a cyclic pro-p group; or

(ii) a quasicyclic p-group with the discrete topology.

If N resides in the class $\mathfrak{N}_p$, we define the $\mathfrak{N}_p$*-rank* of N, denoted $n_p(N)$, to be the number of infinite factors in any series of the sort described above. By the version of Schreier's refinement theorem for topological groups, this number does not depend on the particular series chosen.

We can immediately make the three elementary observations below about groups in the class $\mathfrak{N}_p$.

Lemma 2.1. *Assume that p is a prime and N is a member of the class $\mathfrak{N}_p$. Then the following assertions are true.*

(i) N is locally compact and totally disconnected.

(ii) Every compact subgroup of N is a polycyclic pro-p group.

(iii) For each natural number m, N possesses only finitely many subgroups of index m.

Proof. According to [**11**, Theorem 6.15], both local compactness and total disconnectivity are preserved by extensions of topological groups. Hence statement (i) follows readily from the fact that cyclic pro-p groups and discrete groups are both locally compact and totally disconnected.

To show (ii), we observe that every compact subgroup is closed and hence a member of $\mathfrak{N}_p$. Thus, lacking any quasicyclic topological sections, it must be a polycyclic pro-p group.

Statement (iii) is an immediate consequence of the fact that the profinite completion of N is finitely generated as a topological group. The latter assertion can be easily established by employing induction on the length of a series of the sort described in the definition of $\mathfrak{N}_p$ and invoking the right-exactness of the profinite completion functor.

□

Abelian groups that reside in $\mathfrak{N}_p$ have the following properties.

Lemma 2.2. *Assume that p is a prime. If A is an abelian group in $\mathfrak{N}_p$, then A is a $\mathbb{Z}_p$-module. Furthermore, a subgroup B of A is closed if and only if it is a $\mathbb{Z}_p$-submodule.*

Proof. Any topologically cyclic group belonging to $\mathfrak{N}_p$ is either a finite p-group or a copy of $\mathbb{Z}_p$. Hence any abelian group A in $\mathfrak{N}_p$ must be a $\mathbb{Z}_p$-module. Let B be a subgroup of A.

If B is closed, then, for every $b \in B$, $\overline{\langle b \rangle} < B$, so that B is a $\mathbb{Z}_p$-submodule. To prove the converse, assume that B is a $\mathbb{Z}_p$-submodule, and let

$$0 = A_0 \lhd A_1 \lhd \cdots \lhd A_r = A$$

be a series of closed subgroups in which each factor is either a cyclic pro-p group or a quasicyclic p-group with the discrete topology. Then, for each $i \geq 1$, $A_i \cap B / A_{i-1} \cap B$ is a $\mathbb{Z}_p$-submodule of A_i/A_{i-1} and hence locally compact. Therefore, since local compactness is preserved by forming topological group extensions, we have that B is locally compact. Thus, in view of [**11**, Theorem 4.8], B is closed. □

In Lemma 2.4 below, we succeed in characterizing all the abelian groups that lie in the class $\mathfrak{N}_p$. For this purpose, we require the following simple calculation.

Lemma 2.3. *For any prime p, $\mathrm{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p, \mathbb{Z}_p) = 0$.*

Proof. The short exact sequence $0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Z}_{p^\infty} \to 0$ gives rise to an exact sequence

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_p) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p, \mathbb{Z}_p) \longrightarrow 0. \tag{2.1}$$

To investigate $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_p)$, we refer to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}_p \to \mathbb{Z}_p/\mathbb{Z} \to 0$. Since $\mathbb{Z}_p/\mathbb{Z}$ is both p-torsion-free and divisible, this short exact sequence yields $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_p)$.

From the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, we obtain $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Q}/\mathbb{Z})$. Thus $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Z}) \cong \mathbb{Z}_p$, so that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_p) \cong \mathbb{Z}_p$. Since $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p, \mathbb{Z}_p)$ is divisible, it follows from (2.1) that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p, \mathbb{Z}_p) = 0$. Therefore, $\mathrm{Ext}^1_{\mathbb{Z}_p}(\mathbb{Q}_p, \mathbb{Z}_p) = 0$. □

Lemma 2.4. *For any prime p, a topological abelian group A belongs to the class $\mathfrak{N}_p$ if and only if A is a direct sum of finitely many groups of the following four types:*
(i) a finite cyclic p-group;
(ii) a quasicyclic p-group;
(iii) a copy of $\mathbb{Z}_p$;
(iv) a copy of $\mathbb{Q}_p$.

Proof. The "if" direction is obvious. For the "only if" part, we proceed by induction on $n_p(A)$. If $n_p(A) = 0$, then A is finite, so that it is a direct sum of finitely many cyclic p-groups. Suppose $n_p(A) > 0$. Then A has a closed subgroup B such that A/B has the form (ii) or (iii). Moreover, by the inductive hypothesis, B is a direct sum of finitely many $\mathbb{Z}_p$-modules of types (i), (ii), (iii), and (iv). If $A/B \cong \mathbb{Z}_p$, then A splits over B, so that the conclusion of the lemma follows immediately. Suppose $A/B \cong \mathbb{Z}_{p^\infty}$. Let D be the sum of all the divisible summands in the decomposition of B and F the sum of those that are finite. Then $B \cong D \oplus F \oplus B_1 \oplus \cdots \oplus B_r$, where $B_i \cong \mathbb{Z}_p$ for $1 \leq i \leq r$. If A/F happens to split over $D \oplus B_1 \oplus \cdots \oplus B_r$, then we can deduce the conclusion very easily. Suppose, then, that this is not the case. This means that, for some $i \in \{1, \cdots, r\}$, the extension

$$0 \to B_i \to A/(D \oplus F \oplus B_1 \oplus \cdots \oplus B_{i-1} \oplus B_{i+1} \oplus \cdots \oplus B_r) \to A/B \to 0$$

must fail to split. Hence

$$A/(D \oplus F \oplus B_1 \oplus \cdots \oplus B_{i-1} \oplus B_{i+1} \oplus \cdots \oplus B_r) \cong \mathbb{Q}_p.$$

It follows, then, from Lemma 2.3 that A splits over $D \oplus F \oplus B_1 \oplus \cdots B_{i-1} \oplus B_{i+1} \oplus \cdots \oplus B_r$, yielding the conclusion.

□

In the next lemma, we list three properties possessed by all groups in the class $\mathfrak{N}_p$.

Lemma 2.5. *Let p be a prime. If N is a member of the class $\mathfrak{N}_p$, then the three properties below hold.*

(i) $\gamma_i N$ is a closed subgroup of N for each $i \geq 1$.

(ii) For each $i \geq 1$, there is a $\mathbb{Z}_p$-module epimorphism

$$\theta_i : \underbrace{N_{ab} \otimes_{\mathbb{Z}_p} \cdots \otimes_{\mathbb{Z}_p} N_{ab}}_{i} \to \gamma_i N/\gamma_{i+1} N.$$

(iii) N is a union of countably many open compact subgroups.

Proof. We will dispose of (i) and (ii) with a single argument. If $i \geq 1$, forming iterated commutators of weight i defines a continuous function

$$f_i : \underbrace{N/\overline{N'} \times \cdots \times N/\overline{N'}}_{i} \to \overline{\gamma_i N}/\overline{\gamma_{i+1} N}$$

between topological spaces. Moreover, the continuity of f_i renders it $\mathbb{Z}_p$-linear in each component. Thus f_i gives rise to a $\mathbb{Z}_p$-module homomorphism

$$\theta_i : \underbrace{N/\overline{N'} \otimes_{\mathbb{Z}_p} \cdots \otimes_{\mathbb{Z}_p} N/\overline{N'}}_{i} \to \overline{\gamma_i N}/\overline{\gamma_{i+1} N}.$$

The image of θ_i is $\gamma_i N/\overline{\gamma_{i+1} N}$. Since this image is a $\mathbb{Z}_p$-module, it must be closed in the abelian $\mathfrak{N}_p$-group $\overline{\gamma_i N}/\overline{\gamma_{i+1} N}$. Therefore, $\gamma_i N$ must be closed in N. The $\mathbb{Z}_p$-module homomorphism θ_i, then, can serve as the map required for (ii).

We establish statement (iii) by inducting on $n_p(N)$, the case $n_p(N) = 0$ being trivial. Assume $n_p(N) > 0$. If N has no quasicyclic topological sections, then N is itself compact, making the conclusion true. Assume that N has at least one quasicyclic topological section. It follows, then, from (ii) and Lemma 2.4 that N/N' must possess a topological quotient N/N_0 that is quasicyclic. Thus there is an infinite chain of open normal subgroups $N_0 < N_1 < \cdots$ such that $N = \bigcup_{i=0}^{\infty} N_i$ and N_i/N_{i-1} is finite and nontrivial for $i \geq 1$. Moreover, in view of the inductive hypothesis, each N_i is a union of countably many compact open subgroups. Therefore, the same is true for N.

□

We continue our investigation of the structure of an $\mathfrak{N}_p$-group by examining its finite residual.

Proposition 2.6. *Assume that p is a prime. If N is a topological group belonging to $\mathfrak{N}_p$, then the following three statements are true:*

(i) $R(N)$ is closed in N;

(ii) $N/R(N)$ is compact;
(iii) $R(N)$ is radicable.

Proof. In a topologically finitely generated pro-p group, every normal subgroup of finite index is open ([**12**, Theorem 4.3.5]). Hence it follows from Lemma 2.5(iii) that every normal subgroup of finite index in N is open and therefore closed. Thus $R(N)$ is closed. Let $Q = N/R(N)$ and, for each $i \geq 1$, set $Q_i = Z_i(Q)/Z_{i-1}(Q)$. Then, if $i \geq 1$, there is an injective $\mathbb{Z}_p$-module homomorphism

$$\xi_i : Q_{i+1} \to \mathrm{Hom}_{\mathbb{Z}_p}(Q_{ab}, Q_i)$$

such that, for each $z \in Z_{i+1}(Q)$, $\xi_i(zZ_i(Q))$ is the $\mathbb{Z}_p$-homomorphism $xQ' \mapsto [x, z]$ from Q_{ab} to Q_i. As a residually finite abelian group in $\mathfrak{N}_p$, Q_1 must be a direct sum of finitely many cyclic pro-p groups. Thus the monomorphisms ξ_i allow us to deduce that Q_i must also be a direct sum of finitely many cyclic pro-p groups for $i > 1$. Therefore, Q is a polycyclic pro-p group.

It remains to establish statement (iii). We will accomplish this by demonstrating that $R(N)$ fails to have any proper subgroups of finite index. This will then imply that $(R(N))_{ab}$ must be a divisible abelian group, which will allow us to conclude that $R(N)$ is radicable by applying the epimorphisms in Lemma 2.5(ii). Assume that H is a subgroup of $R(N)$ with finite index. Then, in view of Lemma 2.1(ii), H contains subgroup $K \lhd N$ with $[R(N) : K] < \infty$. Since N/K is a polycyclic pro-p group, it is residually finite. Thus N contains a subgroup L of finite index such that $R(N) \cap L < K$. But $R(N) < L$, so that $R(N) = H$. Therefore, $R(N)$ has no proper subgroups of finite index.

□

Fundamental to our reasoning in the proof of Theorem A will be the following property of an $\mathfrak{N}_p$-group that sits inside a larger group as a normal subgroup with an associated quotient group that is finitely generated and virtually abelian. Notice that the hypotheses of the lemma are satisfied in particular if $M = R(N)$.

Proposition 2.7. *Assume that G is a group and p a prime. Suppose that M and N are normal nilpotent subgroups of G satisfying the following three properties:*
(i) $Q = G/N$ is finitely generated and virtually abelian;
(ii) N belongs to the class $\mathfrak{N}_p$;
(iii) M is a closed subgroup of N such that N/M is compact.
Then N possesses a G-invariant compact subgroup X such that $N = MX$.

The proof of Proposition 2.7 relies upon the following result from [**8**].

Proposition 2.8. (Kropholler, Lorensen, and Robinson [**8**, Proposition 2.1]) *Let G be an abelian group and R a principal ideal domain such that R/Ra is finite for every nonzero element a of R. Assume that A and B are RG-modules that are R-torsion-free and have finite R-rank. Suppose further that A fails to contain a nonzero RG-submodule that is isomorphic to a submodule of B. Then there is a positive integer m such that $m \cdot \mathrm{Ext}^n_{RG}(A, B) = 0$ for all $n \geq 0$.* □

Proof of Proposition 2.7. We preface the proof by observing that we fail to lose any real generality if we assume that Q is abelian. To demonstrate this, we suppose that the proposition is true if Q is abelian. Then there is a normal subgroup $G_0 > N$ of finite index in G and a compact G_0-invariant subgroup X_0 of N such that $N = MX_0$. If $\{g_1, \dots, g_r\}$ is a complete set of coset representatives of G_0 in G, then $X = \langle X_0^{g_1}, \dots, X_0^{g_r}\rangle$ is a compact G-invariant subgroup of N and $N = MX$. Hence we may safely assume that Q is abelian.

First we treat the case where M enjoys the following two properties:

(1) M is abelian;

(2) for every G-invariant closed subgroup P of M, either P or M/P is finite.

The statement is obviously true if N is compact; hence we suppose that it is not. Notice that the epimorphisms from Lemma 2.5(ii) imply that N/N', too, fails to be compact. Therefore, $M/N' \cap M$ must be infinite, forcing $N' \cap M$ to be finite. As a result, we will not incur any significant loss of generality if we suppose $N' \cap M = 1$. Furthermore, under this assumption, N' is compact, which means that it suffices to consider the case $N' = 1$. Since N is then abelian, we elect to view N as an additive group and hence a $\mathbb{Z}_pQ$-module.

Suppose that M is not torsion as an abelian group. This means that M has a finite torsion subgroup. The conclusion of the proposition being obviously true otherwise, we will assume that M is not compact. Then, by Proposition 2.8, $\mathrm{Ext}^1_{\mathbb{Z}_pQ}(N/M, M)$ is torsion. Hence, applying [**8**, Proposition 2.2], we obtain a $\mathbb{Z}_pQ$-submodule U of N such that $U \cap M$ is finite and $[N : M + U] < \infty$. Notice further that U is compact.

Now let S be a finite set of right coset representatives of $M+U$ in N, and take V to be the $\mathbb{Z}_pQ$-submodule of N generated by S. Writing $R = R(V)$, we have $R < M$. Our intention is to show that V contains a compact $\mathbb{Z}_pQ$-submodule W such that $V = R + W$. Since $\mathbb{Z}_pQ$ is a Noetherian ring, V must be a Noetherian $\mathbb{Z}_pQ$-module. Hence R has a maximal compact submodule C. Set $A = V/R$ and $B = R/C$. Because A is compact and B fails to contain any nontrivial compact $\mathbb{Z}_pQ$-submodules, it follows from Proposition 2.8 that $\mathrm{Ext}^1_{\mathbb{Z}_pQ}(A, B)$ is torsion. But then the fact that B is a $\mathbb{Q}$-vector space forces $\mathrm{Ext}^1_{\mathbb{Z}_pQ}(A, B)$ to be trivial. Therefore, V possesses a $\mathbb{Z}_pQ$-submodule W with the desired properties. Finally, if we let $X = U + W$, then X fulfills our requirements.

Next we assume that M is torsion as an abelian group. Since $\mathbb{Z}_pQ$ is a Noetherian ring, N/M has type FP_∞ as a $\mathbb{Z}_pQ$-module. Thus $\mathrm{Ext}^1_{\mathbb{Z}_pQ}(N/M, M)$ is torsion. A repetition of the argument above, then, yields a compact $\mathbb{Z}_pQ$-submodule X such that $N = M + X$. This completes the proof for the case where M satisfies properties (1) and (2) above.

The general case of the proposition will be established by inducting on $n_p(M)$, the statement being trivially true if $n_p(M) = 0$. Suppose $n_p(M) > 0$, and let K be a G-invariant closed subgroup of M such that M/K is abelian and $n_p(K)$ is as large as possible while still remaining less than $n_p(M)$. For every G-invariant closed subgroup L/K of M/K, either L/K or M/L is finite. Therefore, by the case established above, N/K must contain a G-invariant compact subgroup Y/K such that $N = MY$ is finite. Applying the inductive hypothesis, we obtain a G-invariant compact subgroup $X < Y$ with $Y = KX$. The subgroup X, then, can serve as the desired subgroup.

□

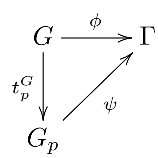
3 Tensor p-completion

In this section we construct, for each prime p, a functor from the category of countable groups to the category of topological groups whose topologically finitely generated subgroups are all pro-p groups. This functor will turn out to be an exact functor on the subcategory of countable nilpotent groups; moreover, when applied to an abelian group, it will be equivalent to tensoring by $\mathbb{Z}_p$. As a result, the functor will assign a topological group in $\mathfrak{N}_p$ to any nilpotent minimax group. At the end of the section, we will discuss how a solvable minimax group can be embedded in a locally compact topological group by applying the functor we have constructed to the Fitting subgroup of the minimax group. This embedding will play a crucial role on the proof of Theorem A.

Assume that p is a prime and G an arbitrary countable group. Let $\mathcal{U}$ be the set of all finitely generated subgroups of G. For any $H \in \mathcal{U}$, we denote the pro-p completion of H by $\hat{H}_p$. For any pair $H, K \in \mathcal{U}$ with $H < K$, define $\iota^{HK} : H \to K$ to be the inclusion map, and let $\hat{\iota}_p^{HK} : H_p \to K_p$ be the continuous homomorphism induced by ι^{HK}. The *tensor p-completion* of G, denoted G_p, is defined to be the inductive limit of the groups $\hat{H}_p$ for $H \in \mathcal{U}$. We endow G_p with a topology by making $U \subset G_p$ open if and only if its inverse image in $\hat{H}_p$ is open for every $H \in \mathcal{U}$. According to [**3**, Theorem 2.7], the countability of $\mathcal{U}$ and the compactness of the groups $\hat{H}_p$ ensure that G_p is a topological group under the topology we have selected. In the case that G is finitely generated, G_p is canonically isomorphic to the pro-p completion of G.

Another important observation concerning G_p is that the canonical maps $H \to \hat{H}_p$ for $H \in \mathcal{U}$ give rise to a map $t_p^G : G \to G_p$ that enjoys the following universal property.

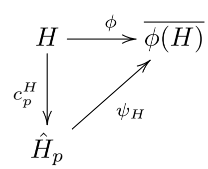
Proposition 3.1. *Let G be a countable group and p a prime. Suppose that $\phi : G \to \Gamma$ is a group homomorphism such that Γ is a topological group whose topologically finitely generated subgroups are all pro-p groups. Then there exists a unique continuous homomorphism ψ : $G_p \to \Gamma$ such that the diagram*



(3.1)

commutes.

Proof. For each finitely generated subgroup H of G, $\overline{\phi(H)}$ is a pro-p group. Hence there is a unique continuous map $\psi_H : \hat{H}_p \to \overline{\phi(H)}$ such that the diagram



commutes, where c_p^H denotes the pro-p completion map. The universal properties of direct limits for abstract groups and topological spaces yield, then, a unique continuous homomorphism $\psi : G_p \to \Gamma$ that makes diagram (3.1) commute.

□

As a consequence of the preceding result, any map $\phi : G \to H$ between countable groups induces a unique continuous map $\phi_p : G_p \to H_p$ that renders the diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ {\scriptstyle t_p^G}\downarrow & & \downarrow{\scriptstyle t_p^H} \\ G_p & \xrightarrow[\phi_p]{} & H_p \end{array}$$

commutative. In this way, tensor p-completion defines a functor from the category of countable groups to the category of topological groups whose topologically finitely generated subgroups are all pro-p groups.

When restricted to nilpotent groups, tensor p-completion enjoys several convenient properties, listed below.

Lemma 3.2. *Let N be a countable nilpotent group and p a prime.*
(i) If N is abelian, then $N_p \cong N \otimes \mathbb{Z}_p$.
(ii) The kernel of $t_p^N : N \to N_p$ is the p'-torsion subgroup of N.
(iii) For any $i \geq 1$, $(\gamma_i(N))_p$ is canonically isomorphic to $\gamma_i(N_p)$.
(iv) If $1 \to M \xrightarrow{\iota} N \xrightarrow{\epsilon} Q \to 1$ is exact, then so is $1 \to M_p \xrightarrow{\iota_p} N_p \xrightarrow{\epsilon_p} Q_p \to 1$.
(v) If N is minimax, then N_p lies in the class $\mathfrak{N}_p$.

Proof. (i) If N is finitely generated, then $N \otimes \mathbb{Z}_p$ is isomorphic to the pro-p completion of N, which coincides with N_p. The general case follows, therefore, from the fact that tensor products commute with direct limits.

(ii) The case where N is finitely generated follows from the properties of the pro-p completion of a nilpotent group. The general case is then an immediate consequence.

(iii) Let $\mathcal{U}$ be the set of all finitely generated subgroups of N. Since $N \cong \varinjlim_{H \in \mathcal{U}} H$, we have $\gamma_i(N) \cong \varinjlim_{H \in \mathcal{U}} \gamma_i(H)$. Being left-adjoint to the embedding functor, tensor p-completion commutes with direct limits. As a result, $(\gamma_i(N))_p \cong \varinjlim_{H \in \mathcal{U}} (\gamma_i(H))_p$. For finitely generated nilpotent groups, the pro-p completion functor commutes with the operators γ_i. Hence $(\gamma_i(H))_p \cong \gamma_i(H_p)$ for any $H \in \mathcal{U}$. Moreover, because $N_p = \varinjlim_{H \in \mathcal{U}} H_p$, we have $\gamma_i(N_p) = \varinjlim_{H \in \mathcal{U}} \gamma_i(H_p)$. Therefore, $(\gamma_i(N))_p \cong \gamma_i(N_p)$.

(iv) Forming inductive limits always preserves short exact sequences. In addition, the pro-p completion functor preserves short exact sequences of finitely generated nilpotent groups. Combining these two assertions, then, allows us to deduce the conclusion.

(v) This statement is proved by taking a series of finite length whose factors are cyclic or quasicyclic and then applying (i) and (iv).

□

We conclude this section by discussing how the tensor p-completion functor can be applied just to a normal subgroup of a group, thereby mapping the larger group into an extension of a topological group by a discrete group. This process is inspired by Hilton's notion of a *relative p-localization* from [**2**]. Essential to the construction is the following result of his concerning group extensions.

Proposition 3.3. (Hilton [**2**]) *Let $1 \to K \to G \to Q \to 1$ be a group extension and $\phi : K \to L$ a group homomorphism. Suppose further that there is a left action of G on L that satisfies the following two properties:*

(i) $k \cdot l = \phi(k) l \phi(k^{-1})$ for all $l \in L$ and $k \in K$;

(ii) $g \cdot \phi(k) = \phi(gkg^{-1})$ for all $g \in G$ and $k \in K$.

In addition, let $\bar{K} = \{(\phi(k), k^{-1}) \mid k \in K\} \subset L \rtimes G$. Then $\bar{K} \lhd L \rtimes G$. Furthermore, if $G^* = (K \rtimes G)/\bar{K}$, the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \| & & \\ 1 & \longrightarrow & L & \longrightarrow & G^* & \longrightarrow & Q & \longrightarrow & 1. \end{array}$$

commutes, where $\psi : G \to G^*$ is the map $g \mapsto (1, g)\bar{K}$. □

Now assume that G is a group with a countable normal subgroup K, and suppose that p is a prime. Let $\omega : G \to \mathrm{Aut}(K)$ be the homomorphism arising from conjugation. Further, define the map $\omega_p : G \to \mathrm{Aut}(K_p)$ by $\omega_p(g) = (\omega(g))_p$ for all $g \in G$, and allow G to act upon K_p via ω_p. This action, then, fulfills the following two conditions:

(i) $k \cdot x = t_p(k) x t_p(k^{-1})$ for all $x \in K_p$ and $k \in K$;

(ii) $g \cdot t_p(k) = t_p(gkg^{-1})$ for all $g \in G$ and $k \in K$.

Hence, by Hilton's proposition, $\bar{K} = \{(t_p(k), k^{-1}) \mid k \in K\}$ is a normal subgroup of $K_p \rtimes G$. We define the *partial tensor p-completion of G with respect to K*, denoted $G_{(K,p)}$, by $G_{(K,p)} = (K_p \rtimes G)/\bar{K}$. Then $G_{(K,p)}$ fits into a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle t_p} & & \downarrow & & \| & & \\ 1 & \longrightarrow & K_p & \longrightarrow & G_{(K,p)} & \longrightarrow & Q & \longrightarrow & 1. \end{array}$$

Our focus here will be on the partial tensor p-completion of a finitely generated solvable minimax group G with respect to its Fitting subgroup N. In this situation, the group $G_{(N,p)}$ is an extension of a group in $\mathfrak{N}_p$ by a finitely generated virtually abelian group. This will allow us to apply Proposition 2.7 to gain insights into the structure of $G_{(N,p)}$ that will assist us in the proof of Theorem A.

4 Proof of Theorem A

Before embarking on the proof of Theorem A, we need to establish a series of lemmas. We begin with a property which we require for the part of Theorem A pertaining to the Fitting subgroups.

Lemma 4.1. *Assume that $1 \to K \to G \to Q \to 1$ and $1 \to K \to H \to R \to 1$ are group extensions that fit into a diagram of the form*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \\ 1 & \longrightarrow & K & \longrightarrow & H & \underset{\epsilon}{\longrightarrow} & R & \longrightarrow & 1, \end{array}$$

in which ψ is surjective. If $\psi(\mathrm{Fitt}(Q)) = \mathrm{Fitt}(R)$, then $\phi(\mathrm{Fitt}(G)) = \mathrm{Fitt}(H)$.

Proof. We may identify G with $H \times_R Q$, so that $\phi(h, q) = h$ for all $(h, q) \in G$. It is straightforward to deduce $\mathrm{Fitt}(G) \cong \mathrm{Fitt}(H) \times_R \mathrm{Fitt}(Q)$. Therefore,

$$\phi(\mathrm{Fitt}(G)) = \mathrm{Fitt}(H) \cap \epsilon^{-1}(\psi(\mathrm{Fitt}(Q))) = \mathrm{Fitt}(H) \cap \epsilon^{-1}(\mathrm{Fitt}(R)) = \mathrm{Fitt}(H).$$

. □

Our next lemma establishes the special case of Theorem A for G virtually torsion-free.

Lemma 4.2. *Assume that π is a set of primes and G a virtually torsion-free π-minimax group. Then there is a torsion-free π-minimax group G^* and an epimorphism $\phi : G^* \to G$ such that $\phi(\mathrm{Fitt}(G^*)) = \mathrm{Fitt}(G)$.*

Proof. Let K be a torsion-free normal subgroup of G such that G/K is finite. Assume that F is a finitely generated free group with a normal subgroup R such that $F/R \cong G/K$. Then R_{ab} is a finitely generated free abelian group. Moreover, as shown in the proof of [**1**, Theorem 3], F/R' is torsion-free.

Let $M/K = \mathrm{Fitt}(G/K)$ and $c = \mathrm{nil}\ M/K$. Suppose further that N is the free nilpotent group of class c on the set of elements of M/K. Denote by θ the epimorphism $N \to M/K$ that maps the free generators of N to themselves. The group G/K acts on the set of elements of M/K by conjugation, and this action extends to an action of G/K on the group N. In this way, we can regard N as an F/R'-operator group via the epimorphism $\epsilon : F/R' \to G/K$. Writing $Q = N \rtimes F/R'$, we may extend the epimorphisms θ and ϵ to an epimorphism $\psi : Q \to G/K$ such that $\psi(N) = M/K$. Now we form a group extension $1 \to K \to G^* \to Q \to 1$ that fits into a commutative diagram of the form

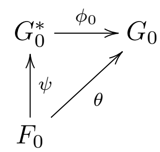
$$\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & G^* & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & \\
1 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & G/K & \longrightarrow & 1.
\end{array}$$

Then G^* is a torsion-free π-minimax group. Also, by Lemma 4.1, $\phi(\mathrm{Fitt}(G^*)) = \mathrm{Fitt}(G)$. □

Lemma 4.2 allows us to deduce the property below, an indispensible aid in the proof of Theorem A.

Lemma 4.3. *Assume that G is a solvable group and π a set of primes. Let G_0 be a subgroup of finite index in G that contains $\mathrm{Fitt}(G)$. Suppose that there is a virtually torsion-free π-minimax group G_0^* and an epimorphism $\phi_0 : G_0^* \to G_0$ such that $\phi_0^{-1}(\mathrm{Fitt}(G))$ is nilpotent. Then there is a torsion-free π-minimax group G^* and an epimorphism $\phi : G^* \to G$ with $\phi(\mathrm{Fitt}(G^*)) = \mathrm{Fitt}(G)$.*

Proof. It suffices to treat the case where $G_0 \lhd G$. Let θ be an epimorphism from a free group F to G, and set $F_0 = \phi^{-1}(G_0)$. Since F_0 is free, there is a homomorphism $\psi : F_0 \to G_0^*$ that makes the diagram



commute. Writing $K_\theta = \text{Ker } \theta$ and $K_\psi = \text{Ker } \psi$, we have $K_\psi < K_\theta$. Now let $f_1, \cdots, f_r$ be a complete list of coset representatives of F_0 in F. Put $K = \bigcap_{i=1}^{r} K_\psi^{f_i}$. Then $K \lhd F$ and $K < K_\theta$. Furthermore, for each $i = 1, \ldots, r$, $F_0/K_\psi^{f_i}$ is π-minimax and virtually torsion-free; hence the same holds for F_0/K.

We now have an epimorphism from the virtually torsion-free π-minimax group F/K to $G \cong F/K_\theta$. We claim that this epimorphism maps the Fitting subgroup of F/K onto the Fitting subgroup of G. In other words, if N/K_θ is the Fitting subgroup of F/K_θ, then N/K is nilpotent. To verify this, observe that our hypothesis concerning ϕ_0 implies that N/K_ψ is nilpotent. Thus N/K, too, is nilpotent, thereby proving our claim. All that remains, then, is to apply Lemma 4.2.

□

Our argument for Theorem A rests upon the following elementary observation about endomorphisms of abelian π-minimax groups.

Lemma 4.4. *Assume that π is a set of primes. Let A be a virtually torsion-free π-minimax abelian group. Then any endomorphism of A satisfies a polynomial in $\mathbb{Z}[t]$ in which the coefficient of the nonzero term of highest degree is a π-number.*

Proof. Suppose $\phi \in \text{End}(A)$. Then ϕ induces an endomorphism $\phi_{(\pi)}$ of $A \otimes \mathbb{Z}_{(\pi)}$. Moreover, $\phi_{(\pi)}$ satisfies a monic polynomial in $\mathbb{Z}_{(\pi)}[t]$. Multiplying by a large enough integer yields a polynomial $h(t) \in \mathbb{Z}[t]$ satisfied by $\phi_{(\pi)}$ whose nonzero term of highest degree has a coefficient that is a π-number. Hence $h(\phi)(A) < T$, where T is the torsion subgroup of A. But T is finite, implying that $\text{End}(h(\phi)(A))$ is finite. As a consequence, $(h(\phi))^r = (h(\phi))^s$ for a pair of distinct natural numbers r, s. Now let $f(t) = (h(t))^r - (h(t))^s$. Then $f(t) \in \mathbb{Z}[t]$ and $f(\phi) = 0$. Furthermore, the coefficient of the highest degree term in $f(t)$ is a π-number. □

In the proof of Theorem A, Lemma 4.4 will be applied in tandem with Proposition 4.5 below.

Proposition 4.5. *Assume that π is a finite set of primes. Let G be a polycyclic group and A a finitely generated $\mathbb{Z}G$-module whose underlying abelian group has no nonzero π-torsion. Suppose that, for each $g \in G$, there are integers $\alpha_0, \ldots, \alpha_m$ such that α_m is a nonzero π-number and $(\alpha_0 + \alpha_1 g + \cdots + \alpha_m g^m) \cdot a = 0$ for all $a \in A$. Then the underlying additive group of A is π-minimax.*

Proof. Let $B = A \otimes \mathbb{Z}_{(\pi)}$. Then B is a finitely generated $\mathbb{Z}_{(\pi)}G$-module. We will show that B is π-minimax as an abelian group, which will imply that the same holds for A. Our strategy is to induct on the length r of a series $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_r = G$ in which each factor G_i/G_{i-1} is cyclic. If $r = 0$, then B is a finitely generated $\mathbb{Z}_{(\pi)}$-module and therefore π-minimax. Suppose $r > 0$, and let S be a finite subset of B that generates B as a $\mathbb{Z}_{(\pi)}$-module. Also, take $g \in G$ such that gG_{r-1} is a generator of G/G_{r-1}. In addition, let

V be the $\mathbb{Z}_{(\pi)}G_{r-1}$-submodule of B generated by S. Notice that, in view of the inductive hypothesis, the underlying abelian group of V is π-minimax.

Put $V^+ = V + g \cdot V + g^2 \cdot V + \cdots$ and $V^- = V + g^{-1} \cdot V + g^{-2} \cdot V + \cdots$. We will show that both V^+ and V^- are π-minimax; since $B = V^+ + V^-$, this will imply that B is π-minimax. For each $k \geq 0$, write $V_k = V + g \cdot V + \cdots + g^k \cdot V$. The hypothesis implies that $V_k = V_{m-1}$ for every $k \geq m$. Hence $V^+ = V_{m-1}$, and so V^+ is π-minimax. Moreover, a similar argument can be adduced to establish that V^- is π-minimax.

□

The last of our preliminary results is the following cohomological property, a consequence of the Lyndon-Hochschild-Serre spectral sequence.

Proposition 4.6. (J-P. Serre and G. Hochschild [**3**, Theorem 2]) *Suppose that* $1 \to K \to G \to Q \to 1$ *is a group extension and* A *a* $\mathbb{Z}G$*-module. Let* $m \geq 1$*, and assume* $H^i(K, A) = 0$ *for* $0 < i < m$*. Then there is an exact sequence*

$$0 \longrightarrow H^m(Q, A^K) \xrightarrow{inf} H^m(G, A) \xrightarrow{res} H^m(K, A)^Q \longrightarrow H^{m+1}(Q, A^K) \xrightarrow{inf} H^{m+1}(G, A).$$

□

Proof of Theorem A. Denote the spectrum of G by π. Furthermore, let $N = \mathrm{Fitt}(G)$ and $Q = G/N$. Then N is nilpotent and Q virtually abelian. We prove the theorem by induction on the number of primes for which G contains a quasicyclic subgroup. If this number is zero then the conclusion follows by Lemma 4.2. Suppose that G has a quasicyclic subgroup for some prime p, and let P be the p-torsion part of $R(G)$. In addition, assume that A is the subgroup of P consisting of all the elements of order dividing p. Notice that, in view of the inductive hypothesis and Lemma 4.1, we fail to lose any real generality if we assume that G/P is torsion-free.

Set $R = R(N_p)$. Our first objective is to show that $G_{(N,p)}$ contains subgroup X such that X is an extension of a nilpotent, polycyclic pro-p group by a discrete polycyclic group and $[G_{(N,p)} : RX] < \infty$. By [**6**, Corollary 3.4], we can find a polycyclic subgroup U of G such that $[G : NU]$ is finite. Furthermore, Proposition 2.7 provides us with a compact $G_{(N,p)}$-invariant subgroup M of N_p with $N_p = RM$. Now let V be the image of U in $G_{(N,p)}$ and put $X = MV$. Then X fulfills our requirements.

Our next step is to form the semidirect product $\Gamma_0 = R \rtimes X$ and define the epimorphism ϵ from Γ_0 to the subgroup RX of $G_{(N,p)}$ by $\epsilon(r, x) = rx$ for all $r \in R$ and $x \in X$. The strategy we adopt for proving the theorem is to show that Γ_0 can be expressed as a homomorphic image of a virtually torsion-free solvable group whose finitely generated subgroups are all π-minimax, and that the inverse image of $\epsilon^{-1}(N_p)$ under this homomorphism is nilpotent. Since G has a subgroup of finite index that is isomorphic to a subgroup of RX, the conclusion of the theorem will then follow from Lemma 4.3.

In order to arrive at a group of the desired sort that maps onto Γ_0, we first construct a sequence of group epimorphisms

$$\cdots \xrightarrow{\phi_{i+1}} \Gamma_i \xrightarrow{\phi_i} \Gamma_{i-1} \xrightarrow{\phi_{i-1}} \cdots \xrightarrow{\phi_2} \Gamma_1 \xrightarrow{\phi_1} \Gamma_0 \qquad (4.1)$$

such that the following five properties hold for each $i \geq 0$.

(1) The group Γ_i is a semidirect product of a radicable nilpotent normal subgroup R_i and a subgroup X_i such that $R_0 = R$ and $X_0 = X$; moreover, if $i \geq 1$, $\phi_i(R_i) = R_{i-1}$ and $\phi_i(X_i) = X_{i-1}$.

(2) The subgroup R_i contains a copy of P which is normal in Γ_i. If $i \geq 1$, the copy of P in Γ_i is mapped by ϕ_i onto the copy of P that lies in Γ_{i-1}, and ϕ_i induces the epimorphism $x \mapsto x^p$ from P to P.

(3) If $i \geq 1$, then Ker $\phi_i = A$, so that ϕ_i induces an isomorphism $X_i \to X_{i-1}$.

(4) If $i \geq 1$, the automorphism of P arising from conjugation inside Γ_i by an arbitrary element g coincides with the automorphism of P that stems from conjugating the copy of P inside Γ_{i-1} by $\phi_i(g)$.

(5) The map ϕ_i induces an isomorphism $\Gamma_i/P \to \Gamma_{i-1}/P$ for $i \geq 1$.

The above sequence will be defined inductively. We already know that all the relevant conditions from our list are satisfied for $i = 0$. Suppose now that, for an arbitrary $i \geq 1$, the group Γ_{i-1} and the various subgroups have been defined so that conditions (1)-(5) hold for Γ_{i-1}. The group Γ_i will be obtained by employing cohomology with coefficients in the $\mathbb{Z}\Gamma_{i-1}$-module A. We start by showing $H^n(\Gamma_{i-1}/R_{i-1}, A) \cong H^n(\Gamma_{i-1}/P, A)$ for $n \geq 0$. This will follow from the Lyndon-Hochschild-Serre spectral sequence for the group extension $1 \to R_{i-1}/P \to \Gamma_{i-1}/P \to \Gamma_{i-1}/R_{i-1} \to 1$ if we can establish that $H^n(R_{i-1}/P, A) = 0$ for $n \geq 1$. To verify this, we apply the universal coefficient theorem, acquiring the exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(R_{i-1}/P), A) \longrightarrow H^n(R_{i-1}/P, A) \longrightarrow \mathrm{Hom}(H_n(R_{i-1}/P), A) \longrightarrow 0 \quad (4.2)$$

for $n \geq 1$. Since R_{i-1}/P is torsion-free, nilpotent, and radicable, $H_n(R_{i-1}/P)$ is torsion-free and divisible for $n \geq 1$. As a result, the second and fourth groups in the sequence (4.2) are both trivial. Thus $H^n(R_{i-1}/P, A) = 0$ for $n \geq 1$.

Because Γ_{i-1} splits over R_{i-1}, the inflation map $H^n(\Gamma_{i-1}/R_{i-1}, A) \to H^n(\Gamma_{i-1}, A)$ must be injective for every n. Combining this observation with the cohomology isomorphism established in the preceding paragraph yields that the inflation homomorphism $H^n(\Gamma_{i-1}/P, A) \to H^n(\Gamma_{i-1}, A)$ is also injective. Moreover, by Proposition 4.6, there is an exact sequence

$$H^2(\Gamma_{i-1}/P, A) \xrightarrow{inf} H^2(\Gamma_{i-1}, A) \xrightarrow{res} H^2(P, A)^{\Gamma_{i-1}} \longrightarrow H^3(\Gamma_{i-1}/P, A) \xrightarrow{inf} H^3(\Gamma_{i-1}, A).$$

It follows, then, that the restriction map $H^2(\Gamma_{i-1}, A) \to H^2(P, A)^{\Gamma_{i-1}}$ must be surjective.

Consider now the short exact sequence $1 \to A \to P \xrightarrow{\rho} P \to 1$, where $\rho(x) = x^p$ for all $x \in P$. Let $\xi \in H^2(P, A)$ be the cohomology class of this extension. Employing the action of Γ_{i-1} on $H^2(P, A)$, we maintain that $g \cdot \xi = \xi$ for all $g \in \Gamma_{i-1}$. To show this, suppose $g \in \Gamma_{i-1}$. Let E be the subgroup of $P \times P$ consisting of all ordered pairs (x, y) such that $x^p = g^{-1}yg$. Define the epimorphism $\chi : E \to P$ by $\chi(x, y) = y$ and the monomorphism $\iota : A \to E$ by $\iota(a) = (a, 1)$. In addition, let $\psi : A \to A$ be the automorphism $a \mapsto g \cdot a$. Then $g \cdot \xi$ is the image of the cohomology class of the extension $1 \to A \xrightarrow{\iota} E \xrightarrow{\chi} P \to 1$ under the map $H^2(P, A) \to H^2(P, A)$ induced by ψ. However, if we define $\kappa : E \to P$ by $\kappa(x, y) = gxg^{-1}$, then the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \xrightarrow{\iota} & E & \xrightarrow{\chi} & P & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\kappa} & & \| & & \\
1 & \longrightarrow & A & \longrightarrow & P & \xrightarrow{\rho} & P & \longrightarrow & 1.
\end{array}$$

commutes. Hence $g \cdot \xi = \xi$; that is, $\xi \in H^2(P, A)^{\Gamma_{i-1}}$.

In this paragraph, we show that there is a $\zeta \in H^2(\Gamma_{i-1}, A)$ with the following two properties:

(i) ζ is mapped to ξ by the restriction map $H^2(\Gamma_{i-1}, A) \to H^2(P, A)^{\Gamma_{i-1}}$;

(ii) the image of ζ under the restriction map $H^2(\Gamma_{i-1}, A) \to H^2(X_{i-1}, A)$ is trivial.

Since the restriction map $H^2(\Gamma_{i-1}, A) \to H^2(P, A)^{\Gamma_{i-1}}$ is surjective, we can find an element ζ_0 of $H^2(\Gamma_{i-1}, A)$ whose image is ξ. Regarding X_{i-1} as a subgroup of Γ_{i-1}/P, we have a restriction map $H^2(\Gamma_{i-1}/P, A) \to H^2(X_{i-1}, A)$. Because X_{i-1} is a complement to a normal subgroup in Γ_{i-1}/P, this restriction map is surjective. Furthermore, the diagram

$$\begin{array}{ccc}
H^n(\Gamma_{i-1}/P, A) & \xrightarrow{inf} & H^n(\Gamma_{i-1}, A) \\
 & \searrow^{res} & \downarrow{\scriptstyle res} \\
 & & H^n(X_{i-1}, A)
\end{array}$$

commutes. Denote the image of ζ_0 in $H^2(X_{i-1}, A)$ by λ. Let $\nu \in H^2(\Gamma_{i-1}/P, A)$ be an inverse image of λ. Finally, take ζ_1 to be the image of ν in $H^2(\Gamma_{i-1}, A)$. Then $\zeta = \zeta_0 - \zeta_1$ fulfills conditions (i) and (ii).

Now we take $1 \to A \to \Gamma_i \xrightarrow{\phi_i} \Gamma_{i-1} \to 1$ to be a central group extension whose cohomology class is ζ. We will show that Γ_i enjoys properties (1)-(5) on our list. If $R_i = \phi_i^{-1}(R_{i-1})$, then properties (2) and (5) follow from condition (i) on ζ. Also, as an extension of P by R_{i-1}/P, R_i is radicable, disposing of a small portion of (1). Moreover, in view of condition (ii), the extension $1 \to A \to \phi_i^{-1}(X_{i-1}) \to X_{i-1} \to 1$ splits. Thus, if we take X_i to be a complement to A in $\phi_i^{-1}(X_{i-1})$, then the other aspects of (1), as well as (3), become true. In addition, the triviality of $H^1(P, A)$ ensures that, if ψ_1 and ψ_2 are automorphisms of P that make the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \longrightarrow & P & \xrightarrow{\rho} & P & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle\psi_2} & & \downarrow{\scriptstyle\psi_1} & & \\
1 & \longrightarrow & A & \longrightarrow & P & \xrightarrow{\rho} & P & \longrightarrow & 1
\end{array}$$

commute, then $\psi_2 = \psi_1$. As a consequence, property (4) holds. This concludes our argument concerning the construction of the sequence (4.1).

Let $N_0 = R \rtimes (X \cap N_p)$, so that $N_0 = \epsilon^{-1}(N_p)$ and N_0 is nilpotent. Also, for $i \geq 1$, put $N_i = \phi_i^{-1}(N_{i-1})$. Properties (4) and (5) imply that each N_i is nilpotent with nil $N_i \leq$ nil $(N_0/P) + \text{nil}_{N_0} P$. Define the group Γ^* to be the inverse limit of the system (4.1), letting P^*, R^*, N^*, and X^* be the inverse images in Γ^* of P, R, N_0, and X, respectively. Then $\Gamma^* = R^* \rtimes X^*$ and $X^* \cong X$. Moreover, N^* is nilpotent of class $\leq$ nil $(N_0/P) + \text{nil}_{N_0} P$, and Γ^*/N^* is polycyclic.

For each i, the group N_i can be endowed with the smallest topology that renders the map $N_i \to N_0$ continuous. This makes each N_i into a topological group situated in the class $\mathfrak{N}_p$. Moreover, since $N^* = \varprojlim N_i$, we can also make N^* into a topological group by equipping it with the topology induced by the product topology. With this topology, the subgroup P^* is closed and isomorphic, as a topological group, to a direct sum of finitely many copies of $\mathbb{Q}_p$. Therefore, N^*, too, belongs to $\mathfrak{N}_p$. Also, Γ^* must be virtually torsion-free.

We will show now that every finitely generated subgroup of Γ^* is π-minimax; as observed in the third paragraph of the proof, this will imply the conclusion of the theorem. The principal step toward establishing this property is to prove the assertion printed in italics below.

For each $g \in \Gamma^*$, there are integers $\alpha_0, \cdots, \alpha_m$ such that α_m is a π-number and

$$(\alpha_0 + \alpha_1 g + \cdots + \alpha_m g^m)^{2c} \cdot a = 0$$

for all $a \in N^*_{ab}$, where $c = \text{nil } N$.

To prove the assertion, we let q be the image of g in $G_{(N,p)}/N \cong Q$ under the composition of maps $\Gamma^* \to \Gamma_0 \to G_{(N,p)} \to G_{(N,p)}/N$. Since G is finitely generated and Q finitely presented, N_{ab} is a finitely generated $\mathbb{Z}Q$-module. As a consequence, N_{ab} is a Noetherian $\mathbb{Z}Q$-module, for $\mathbb{Z}Q$ is a Noetherian ring. Hence N_{ab} is virtually torsion-free as an abelian group, so that Lemma 4.4 implies that there are integers $\alpha_0, \ldots, \alpha_m$ such that α_m is a π-number and $(\alpha_0 + \alpha_1 q + \cdots + \alpha_m q^m) \cdot a = 0$ for all a in the $\mathbb{Z}Q$-module N_{ab}. It follows, then, that $\alpha_0 + \alpha_1 q + \cdots + \alpha_m q^m$ annihilates the $\mathbb{Z}Q$-module $(N_p)_{ab}$. Thus, for any $y \in G_{(N,p)}$ whose image in Q is q,

$$(\alpha_0 + \alpha_1 y + \cdots + \alpha_m y^m) \cdot a = 0 \tag{4.3}$$

for all $a \in (N_p)_{ab}$.

We claim further that, for any $y \in \Gamma_0$ whose image in Q is q, $(\alpha_0 + \alpha_1 y + \cdots + \alpha_m y^m)^c$ destroys the $\mathbb{Z}\Gamma_0$-modules $(N_0)_{ab}$ and P. To show this, write $y = (r, x)$, where $r \in R$ and $x \in X$. It follows from (4.3) and Lemma 4.7 below that $(\alpha_0 + \alpha_1 x + \cdots + \alpha_m x^m)^c$ annihilates P. Moreover, since there is a $\mathbb{Z}X$-module epimorphism $R_{ab} \oplus (X \cap N_p)_{ab} \to (N_0)_{ab}$, $(\alpha_0 + \alpha_1 x + \cdots + \alpha_m x^m)^c$ also annihilates $(N_0)_{ab}$. Thus, since R acts trivially on $(N_0)_{ab}$ and P, $(\alpha_0 + \alpha_1 y + \cdots + \alpha_m y^m)^c$ must obliterate both modules.

Next we turn our attention to the group Γ_i and its nilpotent normal subgroup N_i for an arbitrary i. Observe that $(N_i)_{ab}$ fits into an exact sequence $P \to (N_i)_{ab} \to (N_0/P)_{ab} \to 0$ of $\mathbb{Z}\Gamma_i$-modules. From the statement established in the preceding paragraph, as well as property (4) of Γ_i, we conclude that, if y is an element of Γ_i that is mapped to q, $(\alpha_0 + \alpha_1 y + \cdots + \alpha_m y^m)^{2c} \cdot a = 0$ for all $a \in (N_i)_{ab}$.

Finally, we examine the action of g on N^*_{ab}. Observe first that the universal property of inverse limits of topological groups yields a continuous abelian group epimorphism $\theta : N^*_{ab} \to \varprojlim (N_i)_{ab}$. Notice further that θ is a $\mathbb{Z}\Gamma^*$-module epimorphism. Also, if $n \in N^*$ such that the image of n in N_i lies in N_i' for each i, then the fact that $(N^*)'$ is closed in N^* ensures that $n \in (N^*)'$. In other words, θ is an isomorphism of $\mathbb{Z}\Gamma^*$-modules. However, by what was shown in the previous paragraph, the element $(\alpha_0 + \alpha_1 g + \cdots + \alpha_m g^m)^{2c}$ of $\mathbb{Z}\Gamma^*$ annihilates $(N_i)_{ab}$ for all i. Therefore, this element must also decimate N^*_{ab}, as desired.

To finish the proof of the theorem, assume that H is a finitely generated subgroup of Γ^*. Set $K = N^* \cap H$ and $L = H/K$. Then L is polycyclic, and K_{ab} is a finitely generated $\mathbb{Z}L$-module. Combining the italicized assertion above with Lemma 4.7 below allows us to deduce from Proposition 4.5 that K_{ab} is an abelian group extension of a π-torsion group by a π-minimax one. The epimorphism from the ith tensor power of K_{ab} to $\gamma_i K/\gamma_{i+1}K$, then, yields that $\gamma_i K/\gamma_{i+1}K$ is an extension of the same form. Therefore, appealing to Lemma 4.8 below, we can argue by induction on j that $\gamma_{d-j}K$ is π-minimax for all $j \geq 0$, where $d = \text{nil } K$. In particular, K is π-minimax, and so H is π-minimax. The proof of Theorem A is now complete. □

It remains to establish the following two lemmas that were invoked in the proof of Theorem A.

Lemma 4.7. *Let G be a group. Assume that N is a nilpotent G-operator group with* nil $N = c$. *Suppose $r \in \mathbb{Z}G$ such that $r \cdot a = 0$ for all $a \in N_{ab}$. Then, for any G-invariant subgroup M of N, $r^c \cdot b = 0$ for all $b \in M_{ab}$.*

Proof. The $\mathbb{Z}G$-module M_{ab} has a series $0 = M_{c+1} \subset M_c \subset \cdots \subset M_1 = M_{ab}$ of submodules such that, for $1 \leq i \leq c$, M_i/M_{i+1} is a $\mathbb{Z}G$-module section of $\gamma_i N/\gamma_{i+1}N$. Moreover, the $\mathbb{Z}G$-module epimorphism from the ith tensor power of N_{ab} to $\gamma_i N/\gamma_{i+1}N$ allows us to conclude that r annihilates $\gamma_i N/\gamma_{i+1}N$ for $1 \leq i \leq c$. Hence r also obliterates M_i/M_{i+1} for $1 \leq i \leq c$, thereby yielding the conclusion of the lemma. □

Lemma 4.8. *Let π be a finite set of primes. Assume that N is a nilpotent group without any nontrivial π-torsion. Suppose further that N contains a normal π-minimax subgroup M such that N/M is π-torsion. Then N is π-minimax.*

Proof. We induct on nil N. Assume nil $N = 1$. Then $N \otimes \mathbb{Z}_{(\pi)} \cong M \otimes \mathbb{Z}_{(\pi)}$, so that $N \otimes \mathbb{Z}_{(\pi)}$ is π-minimax. Hence N is π-minimax. Suppose nil $N > 1$, and let $Z = Z(N)$. Then N/Z and Z are π-minimax by virtue of the inductive hypothesis. Thus N is π-minimax. □

The hypothesis that the group is finitely generated cannot be dropped from Theorem A. The simplest reason for its necessity is that every torsion-free solvable minimax group has a finitely generated Fitting quotient whereas that does not hold for all solvable minimax groups. Of course, this immediately leads one to ask whether the condition of finite generation can perhaps be replaced by the weaker one that merely the Fitting quotient is finitely generated. The example below, however, illustrates that the latter hypothesis alone would not suffice to guarantee the conclusion of the theorem.

Example 4.9. Assume that p is a prime, and choose λ to be an element of the multiplicative group of the ring $\mathbb{Z}_p$ that is transcendental over $\mathbb{Z}$. Suppose that $Q = \langle u \rangle$ is an infinite cyclic group, and endow $\mathbb{Z}_{p^\infty}$ with a Q-action by defining $u \cdot x = \lambda \cdot x$ for $x \in \mathbb{Z}_{p^\infty}$. Let A be the $\mathbb{Z}Q$-module obtained in this manner, and set $G = A \rtimes Q$. Then $\text{Fitt}(G) = A$, so that $G/\text{Fitt}(G)$ is cyclic. We claim that, nevertheless, G cannot be expressed as a quotient of a torsion-free solvable minimax group. To show this, suppose that there is a torsion-free solvable minimax group G^* and an epimorphism $\phi : G^* \to G$. Put $N^* = \text{Fitt}(G^*)$. Since G^*/N^* is a finitely generated virtually abelian group, $\phi(N^*) = A$. Let u^* be an element of G^* such that $\phi(u^*) = u$, and take $\psi \in \text{Aut}(N^*_{ab})$ to be the automorphism induced by conjugation by u^*. According to Lemma 4.5, there is a nonzero polynomial $f(t) \in \mathbb{Z}[t]$ such that $f(\psi) = 0$. This means that λ is also a root of $f(t)$, thus contradicting our choice of λ.

In §6 we will show that a transcendental action on the abelianization of the Fitting subgroup, like the one described above, is a necessary attribute for any such example.

5 An application to random walks on Cayley graphs

In this brief section we describe the application of Theorem A that was shown to us by Lison Jacoboni. For the statement of the result, we require some additional notation. First, if f and g are functions from the positive real numbers to the nonnegative real numbers, we write $f(t) \gtrsim g(t)$ whenever there are positive constants a and b such that $f(t) \geq ag(bt)$. Also, if $f(t) \gtrsim g(t)$ and $g(t) \gtrsim f(t)$, then we write $f(t) \sim g(t)$.

Suppose that G is a group with a finite generating set S that is closed under inversion. Consider the simple random walk on the Cayley graph of G with respect to S. For any positive integer n, let $P_{(G,S)}(2n)$ be the probability of returning to one's starting position after $2n$ steps. It is shown in [**10**] that, for any other generating set T closed under inversion, $P_{(G,T)}(2n) \sim P_{(G,S)}(2n)$.

In [**9**] the following theorem is enunciated.

Theorem 5.1. (Pittet and Saloff-Coste [**9**, Theorem 1.1]) *Assume that G is a solvable minimax group with a finite generating set S that is closed under inversion. Then*

$$P_{(G,S)}(2n) \gtrsim \exp(-n^{\frac{1}{3}}).$$

The proof of Theorem 5.1 offered in [**9**] relies on [**9**, Proposition 4.1], which erroneously asserts that the torsion subgroup of a nilpotent group of finite rank is finite. For this reason, the proof appears to handle the virtually torsion-free case as claimed, but some further argument seems necessary in the general case. This shortcoming was pointed out to us by Lison Jacoboni, who also noticed that our Theorem A is tailor-made to complete the proof. We present her reasoning below.

Proof. According to Theorem A, there is a torsion-free solvable minimax group G^* and an epimorphism $\phi : G^* \to G$. Moreover, we can select G^* so that it has a finite generating set S^* such that S^* is closed under inversion and $\phi(S^*) = S$. Consider random walks on the Cayley graphs of both groups with respect to these generating sets, with both walks commencing at the identity element. Then $P_{(G,S)}(2n)$ is equal to the probability that the random walk on G^* will take us to an element of Ker ϕ after $2n$ steps. As a consequence, $P_{(G,S)}(2n) \geq P_{(G^*,S^*)}(2n)$. Moreover, by what Pittet and Saloff-Coste prove in [**9**], we have $P_{(G^*,S^*)}(2n) \gtrsim \exp(-n^{\frac{1}{3}})$. Hence $P_{(G,S)}(2n) \gtrsim \exp(-n^{\frac{1}{3}})$. □

6 Appendix: Non-finitely-generated groups

While delivering a lecture about the contents of this paper in the Penn State Topology Seminar, the second author was asked by his colleague Michael Weiner whether the method used to prove Theorem A would allow us to identify exactly which π-minimax groups can be realized as quotients of torsion-free π-minimax groups. In this appendix, we provide such a characterization by proving Theorem 6.1 below. The authors are grateful to Michael Weiner for posing the question that led to this generalization of Theorem A.

Theorem 6.1. *Let π be a set of primes and G a π-minimax group with Fitting subgroup N. Then G can be expressed as a homomorphic image of a torsion-free π-minimax group if and only if the following two properties hold.*

(i) The quotient G/N is finitely generated.

(ii) For every $g \in G$, there exist integers $\alpha_0, \ldots, \alpha_m$ such that α_m is a π-number and

$$(\alpha_0 + \alpha_1 g + \cdots + \alpha_m g^m) \cdot a = 0$$

for all $a \in N_{ab}$.
Moreover, if the above conditions are satisfied, then we can choose the torsion-free π-minimax group G^* and epimorphism $\phi : G^* \to G$ so that $\phi(\mathrm{Fitt}(G^*)) = \mathrm{Fitt}(G)$.

In additon to the machinery involved in the proof of Theorem A, the argument for Theorem 6.1 makes use of the following elementary fact about nilpotent groups.

Lemma 6.2. *Let π be a set of primes. Assume that $1 \to K \to N \xrightarrow{\epsilon} Q \to 1$ is a nilpotent group extension in which N is π-torsion-free, K π-radicable, and Q π-minimax. Then N contains a π-minimax subgroup X such that $\epsilon(X) = Q$.*

Proof. We induct on $\mathrm{nil}_N\ K$. First suppose that K is central in N. Let $\bar{H}$ be a finitely generated subgroup of Q with the same Hirsch length as Q. Now take H to be a finitely generated subgroup of N such that $\epsilon(H) = \bar{H}$. Assume that X is the π-isolator of H in N. Then Lemma 4.8 implies that X is π-minimax. In addition, the π-radicability of K ensures that $\epsilon(X) = Q$.

Finally, we treat the case where $\mathrm{nil}_N\ K > 1$. Set $M = K \cap Z(N)$. By the inductive hypothesis, N contains a subgroup Y such that $M < Y$, Y/M is π-minimax, and $\epsilon(Y) = Q$. The base case yields, then, a π-minimax subgroup X of Y such that $\epsilon(X) = Q$.

□

We also rely on the more general version of Proposition 4.5 below. This can be proved in exactly the same fashion as its predecessor.

Proposition 6.3. *Assume that π is a finite set of primes and G a polycyclic group. Let A be a $\mathbb{Z}G$-module whose underlying abelian group has no nonzero π-torsion, and which is generated as a $\mathbb{Z}G$-module by a π-minimax additive subgroup. Suppose further that, for each $g \in G$, there are integers $\alpha_0, \ldots, \alpha_m$ such that α_m is a nonzero π-number and $(\alpha_0 + \alpha_1 g + \cdots + \alpha_m g^m) \cdot a = 0$ for all $a \in A$. Then the underlying additive group of A is π-minimax.* □

Proof of Theorem 6.1. It is straightforward to show the "only if " part. Hence we concern ourselves solely with proving the "if " statement. We proceed exactly as in the proof of Theorem A, creating the group Γ^* with a nilpotent subgroup N^* satisfying the italicized statement on page 17. This time, however, our focus will be on another group that we will form using Γ^*. Let ϕ_0^* be the epimorphism from Γ^* to Γ_0 arising from the definition of Γ^*. Employing the map $\epsilon : \Gamma_0 \to RX$ from the proof of Theorem A, put $\epsilon^* = \epsilon\phi_0^*$ and $\Lambda = \mathrm{Ker}\ \epsilon^* \cap N^*$. Next let $\Lambda^{\mathbb{Q}}$ be the Mal'cev completion of the quotient of Λ by its torsion subgroup and denote the canonical map $\Lambda \to \Lambda^{\mathbb{Q}}$ by c.

Employing the action of Γ^* on $\Lambda^{\mathbb{Q}}$ induced by the action on Λ, we define

$$\Gamma^\dagger = (\Lambda^{\mathbb{Q}} \rtimes \Gamma^*)/\{(c(x), x^{-1}) : x \in \Lambda\},$$

so that $\Gamma^\dagger$ fits into a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Lambda & \longrightarrow & \Gamma^* & \xrightarrow{\eta^*} & \Gamma^*/\Lambda & \longrightarrow & 1 \\
 & & {\scriptstyle c}\downarrow & & {\scriptstyle \psi}\downarrow & & \| & & \\
1 & \longrightarrow & \Lambda^{\mathbb{Q}} & \longrightarrow & \Gamma^\dagger & \xrightarrow{\eta^\dagger} & \Gamma^*/\Lambda & \longrightarrow & 1.
\end{array}$$

Composing $\eta^\dagger$ with the map $\Gamma^*/\Lambda \to RX$ induced by ϵ^* yields an epimorphism $\epsilon^\dagger : \Gamma^\dagger \to RX$. Set $N^\dagger = (\epsilon^\dagger)^{-1}(N_p)$. Then $N^\dagger = \Lambda^{\mathbb{Q}}\psi(N^*)$. Hence $N^\dagger$ is nilpotent, and there is a $\mathbb{Z}\Gamma^*$-module epimorphism $\kappa : (\Lambda_{ab} \otimes \mathbb{Q}) \oplus N^*_{ab} \to N^\dagger_{ab}$. Letting $c^* = \text{nil } N^*$, we claim that, for any $g^\dagger \in \Gamma^\dagger$, there are integers $\alpha_0, \ldots, \alpha_m$ such that α_m is a π-number and $(\alpha_0 + \alpha_1 g^\dagger + \cdots + \alpha_m (g^\dagger)^m)^{c^*} \cdot a = 0$ for all $a \in N^\dagger_{ab}$. To show this, suppose $g^\dagger \in \Gamma^\dagger$. Then $g^\dagger = \lambda\psi(g)$, where $g \in \Gamma^*$ and $\lambda \in \Lambda^{\mathbb{Q}}$. Also, there are integers $\alpha_0, \ldots, \alpha_m$ such that α_m is a π-number and $\alpha_0 + \alpha_1 g + \cdots + \alpha_m g^m$ annihilates N^*_{ab}. It follows from the existence of the map κ that $(\alpha_0 + \alpha_1 g + \cdots + \alpha_m g^m)^{c^*}$ obliterates $N^\dagger_{ab}$, and so $(\alpha_0 + \alpha_1 g^\dagger + \cdots + \alpha_m (g^\dagger)^m)^{c^*}$ does as well.

By Lemma 6.2, $N^\dagger$ has a π-minimax subgroup Y such that $\epsilon^\dagger(Y) = N$. Recalling the definition of the subgroup V of RX, we let $V^\dagger$ be a polycyclic subgroup of $\Gamma^\dagger$ with $\epsilon^\dagger(V^\dagger) = V$. Assume that K is the $V^\dagger$-invariant subgroup of $N^\dagger$ generated by Y. Next put $H = KV^\dagger$. Then $\epsilon^\dagger(H) = G \cap RX$, so that $\epsilon^\dagger(H)$ has finite index in G. We will establish that H is π-minimax, thereby proving the theorem. Writing $L = H/K$, we have that L is polycyclic and K is generated by Y as an L-operator group. Proposition 6.3 implies, then, that K_{ab} is an extension of a π-torsion group by one that is π-minimax. Thus the argument advanced in the last paragraph of the proof of Theorem A allows us to deduce that H is π-minimax. □

Remark. It follows from Theorem 6.1 that the homomorphic images of torsion-free π-minimax groups are precisely the π-minimax groups that belong to the class $\mathcal{U}$ from [**8**]. In that paper the second author shows that the groups in this class enjoy two cohomological properties that are not manifested by all π-minimax groups.

We conclude the article by pointing out the following immediate consequence of Theorem 6.1.

Corollary 6.4. *Let π be a set of primes and N a nilpotent π-minimax group. Then there is a torsion-free nilpotent π-minimax group N^* admitting an epimorphism $N^* \to N$.* □

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