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On Fuzzy Upper and Lower Theta Star Semicontinuous Multifunctions

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Abstract

We introduce the concepts of fuzzy upper and lower theta star (respectively, theta)-semicontinuous multifunction on fuzzy topological spaces in the Šostak sense. In L-fuzzy topological spaces, the mutual relationships of these fuzzy upper (respectively, fuzzy lower) theta star (respectively, theta)-semicontinuous multifunctions are established along with several characterizations and properties. Later, researchers looked at the composition and union of these multifunctions.

Keywords: Fuzzy upper theta star (respectively, theta)-semicontinuous multifunction; Fuzzy lower theta star (respectively, theta)-semicontinuous multifunction

MSC 2010 No.: 54A40, 54C08, 54C60

1. Introduction

Chang (1968) and Goguen (1973) proposed the concept of (L)-fuzzy topological space as a generalisation of L-topological spaces, which were initially named (L)-fuzzy topological spaces by Kubiak (1985) and Šostak (1985). It's the degree to which a L-fuzzy set is open. Höhle (1980), Höhle and Šostak (1999), Kubiak (1985), Kubiak and Šostak (1997) and Šostak (1985) developed a generic method to the study of topological type structures on fuzzy power sets.

The multimapping function was introduced by Berge (1963). Following Chang (1968) introducing the notion of fuzzy topology, numerous authors have defined and researched continuity of multifunctions in fuzzy topological spaces from various perspectives (e.g., see Alimohammady et al. (2011), Mahmoud (2003), Mukherjee and Malakar (1991), Mughil et al. (2021), Papageorgiou (1985)). In Chang (1968) fuzzy topology, Tsiporkova et al. (1997) introduced the continuity of fuzzy multivalued mappings. The ideas of fuzzy upper and lower semi-continuous multifunctions, fuzzy upper and lower beta-continuous multifunctions in L-fuzzy topological spaces were later introduced by Abbas et al. (2014). In L-fuzzy topological spaces, Hebeshi and Taha (2015) proposed the ideas of fuzzy upper and lower alpha-continuous multifunctions.

The ideas of fuzzy upper and lower theta star (respectively, theta)-semicontinuous multifunction on fuzzy topological spaces are introduced in the Šostak sense in this study. In L-fuzzy topological spaces, several characterizations and features of these multifunctions are described, as well as their mutual interactions. Later, the composition and union of these multifunctions were investigated.

2. Preliminaries

The definition of a fuzzy multifunction (FM, for short), Normalized, a crisp, a image, composition are defined by Abbas et al. (2014). An L-fuzzy topological space (L-fts, in short) and their basic definitions are given by Höhle and Šostak (1999), Kubiak (1985), Liu and Luo (1997), Šostak (1985). An z-fuzzy θ-interior respective θ-closure, θ-open (respectively, θ-closed) (briefly, z-$f\theta o$ (respectively, z-$f\theta c$)), z-fuzzy θ-semiopen (respectively, z-fuzzy θ-semiclosed) (briefly, z-$f\theta\mathcal{S}o$ (respectively, z-$f\theta\mathcal{S}c$)) are defined by Vijayalakshmi et al. (2019). An z-fuzzy α-open (respectively, z-fuzzy semiopen and z-fuzzy γ-open) (briefly, z-$f\alpha o$ (respectively, z-$f\mathcal{S}o$ and z-$f\gamma o$)) are defined by Ramadan et al. (1992). An z-fuzzy θ^*-semiopen (respectively, z-fuzzy θ^*-semiclosed) (briefly z-$f\theta^*\mathcal{S}o$ (respectively, z-$f\theta^*\mathcal{S}c$)), z-$f\theta^*\mathcal{S}$ (respectively, z-$f\theta\mathcal{S}$) interior, z-$f\theta^*\mathcal{S}$ (respectively, z-$f\theta\mathcal{S}$) closure are defined by Mughil et al. (2021). Fuzzy upper semi (or Fuzzy upper) (in short, FUS (or FU)) continuous and Fuzzy lower semi (or Fuzzy lower) (in short, FLS (or FL)) continuous are defined by Abbas et al. (2014), and $FU\alpha$ continuous and $FL\alpha$ continuous are defined by Hebeshi and Taha (2015), $FU\theta$ continuous and $FL\theta$ continuous by Ibedou and Abbas (2019). The product fuzzy topology is defined by Alimohammady et al. (2011) and Wong (1974) and graph fuzzy multifunction by Alimohammady et al. (2011) and Mukherjee and Malakar (1991).

3. Fuzzy upper and lower θ^* (resp. θ)-semicontinuous multifunctions

Definition 3.1.

Let $f_m : S \multimap T$ be a FM between two L-fts's $(S,\ \chi),\ (T,\ \chi^*)$ and $z \in L_0$. Then f_m is called:

(i) Fuzzy upper θ^* (respectively, θ)-semicontinuous (in short, $FU\theta^*\mathcal{S}Cts$ (respectively, $FU\theta\mathcal{S}Cts$)) at any L-fp $u_t \in dom(f_m)$ if $u_t \in f_m^u(A)\ \forall\ A \in L^T$ and $\chi^*(A) \geq z$ there exists z-$f\theta^*\mathcal{S}o$ (respectively, z-$f\theta\mathcal{S}o$) set, $B \in L^S$ and $u_t \in B$ such that $B \wedge dom(f_m) \leq f_m^u(A)$.

(ii) Fuzzy lower θ^* (respectively, θ)-semicontinuous (in short, $FL\theta^*\mathcal{S}Cts$ (respectively, $FL\theta\mathcal{S}Cts$)) at any L-fp $u_t \in dom(f_m)$ if $u_t \in f_m^l(A)$ for each $A \in L^T$ and $\chi^*(A) \geq z$ there exists z-$f\theta^*\mathcal{S}o$ (respectively, z-$f\theta\mathcal{S}o$) set, $B \in L^S$ and $u_t \in B$ such that $B \leq f_m^l(A)$.

(iii) $FU\theta^*\mathcal{S}Cts$ (respectively, $FL\theta^*\mathcal{S}Cts$, $FU\theta\mathcal{S}Cts$ and $FL\theta\mathcal{S}Cts$) if it is $FU\theta^*\mathcal{S}Cts$ (respectively, $FL\theta^*\mathcal{S}Cts$, $FU\theta\mathcal{S}Cts$ and $FL\theta\mathcal{S}Cts$) at every $u_t \in dom(f_m)$.

Definition 3.2.

Let f_m be normalized, then f_m is $FU\theta^*\mathcal{S}Cts$ (respectively, $FU\theta\mathcal{S}Cts$) at an L-fp $u_t \in dom(f_m)$ if $u_t \in f_m^u(A)\ \forall\ A \in L^T$ and $\chi^*(A) \geq z$ there exists $B \in L^S$, B is z-$f\theta^*\mathcal{S}o$ (respectively, z-$f\theta\mathcal{S}o$) set and $u_t \in B$ such that $B \leq f_m^u(A)$.

Theorem 3.1.

Let $f_m : S \multimap T$ be a FM between two L-fts's $(S,\ \chi)$, $(T,\ \chi^*)$ & $A \in L^T$, then the following are interchangeable:

(i) f_m is $FL\theta^*\mathcal{S}Cts$.

(ii) $f_m^l(A)$ is z-$f\theta^*\mathcal{S}o$ set, for any $\chi^*(A) \geq z$.

(iii) $f_m^u(A)$ is z-$f\theta^*\mathcal{S}c$ set, for any $\chi^*(\overline{1} - A) \geq z$.

(iv) $\theta^*\mathcal{S}C_\chi(f_m^u(A), z) \leq f_m^u(C_{\chi^*}(A, z))$, for any $A \in L^T$.

(v) $C_\chi(I_\chi(C_\chi(f_m^u(A), z), z), z) \wedge I_\chi(\theta C_\chi(f_m^u(A), z), z) \leq f_m^u(C_{\chi^*}(A, z))$, for any $A \in L^T$.

Proof:

(i) $\Rightarrow$ (ii): Let $u_t \in dom(f_m),\ A \in L^T,\ \chi^*(A) \geq z$ and $u_t \in f_m^l(A)$. Then, there exist $B \in L^S$, B is z-$f\theta^*\mathcal{S}o$ set and $u_t \in B \ni B \leq f_m^l(A)$ and hence $u_t \in \theta^*\mathcal{S}I_\chi(f_m^l(A), z)$. Therefore, we obtain $f_m^l(A) \leq \theta^*\mathcal{S}I_\chi(f_m^l(A), z)$. Thus, $f_m^l(A)$ is z-$f\theta^*\mathcal{S}o$ set.

(ii) $\Rightarrow$ (iii): Let $A \in L^T$ and $\chi^*(\overline{1} - A) \geq z$. Hence, by (ii), $f_m^l(\overline{1} - A) = \overline{1} - f_m^u(A)$ is z-$f\theta^*\mathcal{S}o$. Then, $f_m^u(A)$ is z-$f\theta^*\mathcal{S}c$.

(iii) $\Rightarrow$ (iv): Let $A \in L^T$. Hence, by (iii), $f_m^u(C_{\chi^*}(A, z))$ is z-$f\theta^*\mathcal{S}c$. Then, we obtain

$\theta^*\mathcal{S}C_\chi(f_m^u(A),\ z) \leq f_m^u(C_{\chi^*}(A,\ z))$.

(iv) $\Rightarrow$ (v): Let $A \in L^T$. Hence, by (iv), we obtain $C_\chi(I_\chi(C_\chi(f_m^u(A), z), z), z) \wedge I_\chi(\theta C_\chi(f_m^u(A), z), z) \leq \theta^*\mathcal{S}C_\chi(f_m^u(A), z) \leq f_m^u(C_{\chi^*}(A,\ z))$.

(v) $\Rightarrow$ (ii): Let $A \in L^T$, $\chi^*(A) \geq z$. Hence, by (v), we have

$$\begin{aligned}\overline{1} - f_m^l(A) =& f_m^u(\overline{1} - A)\\ \geq& C_\chi(I_\chi(C_\chi(f_m^u(\overline{1} - A), z), z), z) \wedge I_\chi(\theta C_\chi(f_m^u(\overline{1} - A), z), z)\\ =& C_\chi(I_\chi(C_\chi(\overline{1} - f_m^l(A), z), z), z) \wedge I_\chi(\theta C_\chi(\overline{1} - f_m^l(A), z), z)\\ =& \overline{1} - [I_\chi(C_\chi(I_\chi(f_m^l(A), z), z), z) \wedge C_\chi(\theta I_\chi(f_m^l(A), z), z)]\\ f_m^l(A) \leq& I_\chi(C_\chi(I_\chi(f_m^l(A), z), z), z) \wedge C_\chi(\theta I_\chi(f_m^l(A), z), z).\end{aligned}$$

Hence, $f_m^l(A)$ is z-$f\theta^*\mathcal{S}o$.

(ii) $\Rightarrow$ (i): Let $u_t \in dom(f_m),\ A \in L^T,\ \chi^*(A) \geq z$, with $u_t \in f_m^l(A)$. We have by (ii), $f_m^l(A)$ is z-$f\theta^*\mathcal{S}o$ set. Let $f_m^l(A) = B$ (say), then there exists $B \in L^S$, B is z-$f\theta^*\mathcal{S}o$ set and $u_t \in B$ such that $B \leq f_m^l(A)$. Thus, f_m is $FL\theta^*\mathcal{S}Cts$. ■

Theorem 3.2.

Let $f_m : S \multimap T$ be a FM and normalized between two L-fts's $(S,\ \chi)$, $(T,\ \chi^*)$ & $A \in L^T$. Then, the following are interchangeable:

(i) f_m is $FU\theta^*\mathcal{S}Cts$.

(ii) $f_m^u(A)$ is z-$f\theta^*\mathcal{S}o$ set, for any $\chi^*(A) \geq z$.

(iii) $f_m^l(A)$ is z-$f\theta^*\mathcal{S}c$ set, for any $\chi^*(\overline{1} - A) \geq z$.

(iv) $\theta^*\mathcal{S}C_\chi(f_m^l(A), z) \leq f_m^l(C_{\chi^*}(A, z))$, for any $A \in L^T$.

(v) $C_\chi(I_\chi(C_\chi(f_m^l(A), z), z), z) \wedge I_\chi(\theta C_\chi(f_m^l(A), z), z) \leq f_m^l(C_{\chi^*}(A, z))$, for any $A \in L^T$.

Proof:

(i) $\Rightarrow$ (ii): Let $u_t \in dom(f_m),\ A \in L^T,\ \chi^*(A) \geq z$ and $u_t \in f_m^u(A)$. Then, there exist $B \in L^S$, B is z-$f\theta^*\mathcal{S}o$ set and $u_t \in B \ni B \leq f_m^u(A)$ and hence $u_t \in \theta^*\mathcal{S}I_\chi(f_m^u(A), z)$. Therefore, we obtain $f_m^u(A) \leq \theta^*\mathcal{S}I_\chi(f_m^u(A), z)$. Thus, $f_m^u(A)$ is z-$f\theta^*\mathcal{S}o$ set.

(ii) $\Rightarrow$ (iii): Let $A \in L^T$ and $\chi^*(\overline{1} - A) \geq z$. Hence, by (ii), $f_m^u(\overline{1} - A) = \overline{1} - f_m^l(A)$ is z-$f\theta^*\mathcal{S}o$. Then, $f_m^l(A)$ is z-$f\theta^*\mathcal{S}c$.

(iii) $\Rightarrow$ (iv): Let $A \in L^T$. Hence, by (iii), $f_m^l(C_{\chi^*}(A, z))$ is z-$f\theta^*\mathcal{S}c$. Then, we obtain $\theta^*\mathcal{S}C_\chi(f_m^l(A),\ z) \leq f_m^l(C_{\chi^*}(A,\ z))$.

(iv) $\Rightarrow$ (v): Let $A \in L^T$. Hence, by (iv), we obtain $C_\chi(I_\chi(C_\chi(f_m^l(A), z), z), z) \wedge I_\chi(\theta C_\chi(f_m^l(A), z), z) \leq \theta^*\mathcal{S}C_\chi(f_m^l(A), z) \leq f_m^l(C_{\chi^*}(A,\ z))$.

(v) $\Rightarrow$ (ii): Let $A \in L^T$, $\chi^*(A) \geq z$. Hence, by (v), we have

$$\begin{aligned}\overline{1} - f_m^u(A) =& f_m^l(\overline{1} - A)\\ \geq& C_\chi(I_\chi(C_\chi(f_m^l(\overline{1} - A), z), z), z) \wedge I_\chi(\theta C_\chi(f_m^l(\overline{1} - A), z), z)\\ =& C_\chi(I_\chi(C_\chi(\overline{1} - f_m^u(A), z), z), z) \wedge I_\chi(\theta C_\chi(\overline{1} - f_m^u(A), z), z)\\ =& \overline{1} - [I_\chi(C_\chi(I_\chi(f_m^u(A), z), z), z) \wedge C_\chi(\theta I_\chi(f_m^u(A), z), z)]\\ f_m^u(A) \leq& I_\chi(C_\chi(I_\chi(f_m^u(A), z), z), z) \wedge C_\chi(\theta I_\chi(f_m^u(A), z), z).\end{aligned}$$

Hence, $f_m^u(A)$ is z-$f\theta^*\mathcal{S}o$.

(ii) $\Rightarrow$ (i): Let $u_t \in dom(f_m),\ A \in L^T,\ \chi^*(A) \geq z$, with $u_t \in f_m^u(A)$. We have by (ii), $f_m^u(A)$ is z-$f\theta^*\mathcal{S}o$ set. Let $f_m^u(A) = B$ (say). Then, there exists $B \in L^S$, B is z-$f\theta^*\mathcal{S}o$ set and $u_t \in B$ such that $B \leq f_m^u(A)$. Thus, f_m is $FU\theta^*\mathcal{S}Cts$. ∎

Corollary 3.1.

Let $f_m : S \multimap T$ be a FM between two fts's $(S,\ \chi)$, $(T,\ \chi^*)$ & $A \in L^T$. Following that,

(i) If f_m is normalized, then f_m is $FU\theta^*\mathcal{S}Cts$ at u_t if and only if $u_t \in z$-$f\theta^*\mathcal{S}o$ set of $f_m^u(A)$, $\forall$ $\chi^*(A) \geq z$ and $u_t \in f_m^u(A)$.

(ii) f_m is $FL\theta^*\mathcal{S}Cts$ at u_t if and only if $u_t \in z$-$f\theta^*\mathcal{S}o$ set of $f_m^l(A)$, $\forall$ $\chi^*(A) \geq z$ and $u_t \in f_m^l(A)$.

Proof:

(i) Let $u_t \in dom(f_m),\ A \in L^T,\ \chi^*(A) \geq z$ and $u_t \in f_m^u(A)$. Then, there exist $B \in L^S$, B is z-$f\theta^*\mathcal{S}o$ set and $u_t \in B \ni B \leq f_m^u(A)$ and hence $u_t \in \theta^*\mathcal{S}I_\chi(f_m^u(A), z)$. Therefore, we obtain $f_m^u(A) \leq \theta^*\mathcal{S}I_\chi(f_m^u(A), z)$. Thus, $f_m^u(A)$ is z-$f\theta^*\mathcal{S}o$ set.

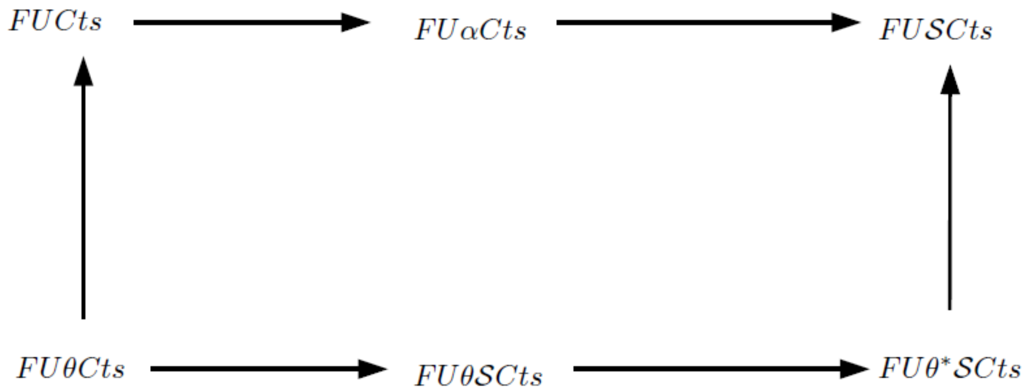
Conversely, let $u_t \in dom(f_m),\ A \in L^T,\ \chi^*(A) \geq z$, with $u_t \in f_m^u(A)$. We have by (ii), $f_m^u(A)$ is z-$f\theta^*\mathcal{S}o$ set. Let $f_m^u(A) = B$ (say). Then, there exists $B \in L^S$, B is z-$f\theta^*\mathcal{S}o$ set and $u_t \in B$ such that $B \leq f_m^u(A)$. Thus, f_m is $FU\theta^*\mathcal{S}Cts$.

(ii) Let $u_t \in dom(f_m),\ A \in L^T,\ \chi^*(A) \geq z$ and $u_t \in f_m^l(A)$. Then, there exist $B \in L^S$, B is z-$f\theta^*\mathcal{S}o$ set and $u_t \in B \ni B \leq f_m^l(A)$ and hence $u_t \in \theta^*\mathcal{S}I_\chi(f_m^l(A), z)$. Therefore, we obtain $f_m^l(A) \leq \theta^*\mathcal{S}I_\chi(f_m^l(A), z)$. Thus, $f_m^l(A)$ is z-$f\theta^*\mathcal{S}o$ set.

Conversely, let $u_t \in dom(f_m),\ A \in L^T,\ \chi^*(A) \geq z$, with $u_t \in f_m^l(A)$ we have by (ii), $f_m^l(A)$ is z-$f\theta^*\mathcal{S}o$ set. Let $f_m^l(A) = B$ (say). Then, there exists $B \in L^S$, B is z-$f\theta^*\mathcal{S}o$ set and $u_t \in B$ such that $B \leq f_m^l(A)$. Thus, f_m is $FL\theta^*\mathcal{S}Cts$. ∎

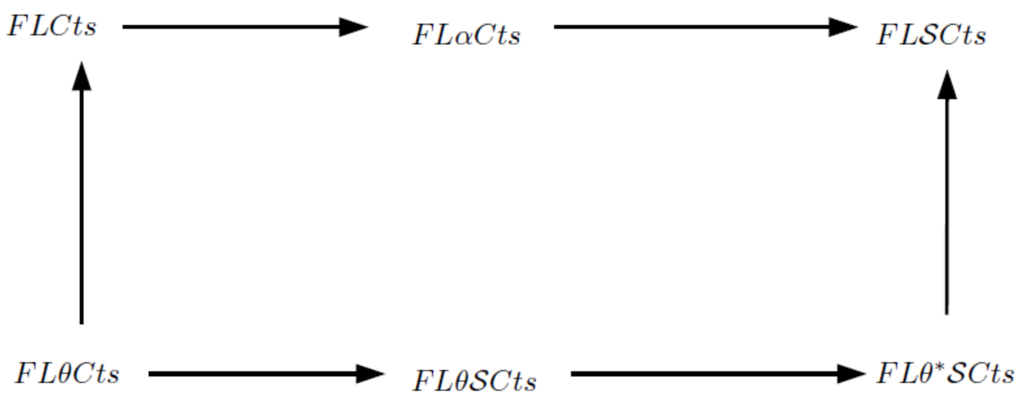
Remark 3.1.

The following implications can be deduced from the above definitions.



Remark 3.2.

The following implications can be deduced from the above definitions.



The converses of these implications are not true, as shown in the following examples.

Example 3.1.

Let $S = \{u_{11},\ u_{22}\}$, $Y = \{v_1,\ v_2,\ v_3\}$ and $f_m : S \multimap T$ be a FM defined by $G_{f_m}(u_{11},\ v_1) = 0.88$, $G_{f_m}(u_{11},\ v_2) = 0.99$, $G_{f_m}(u_{11},\ v_3) = 0.88$, $G_{f_m}(u_{22},\ v_1) = \overline{1}$, $G_{f_m}(u_{22},\ v_2) = 0.77$, and $G_{f_m}(u_{22},\ v_3) = 0.33$. Let B_1, B_2, B_3, B_4 and B_5 be a fuzzy subsets of S be defined as $B_1(u_{11}) = 0.33$, $B_1(u_{22}) = 0.11$; $B_2(u_{11}) = 0.11$, $B_2(u_{22}) = 0.33$; $B_3(u_{11}) = 0.33$, $B_3(u_{22}) = 0.33$; $B_4(u_{11}) = 0.11$, $B_4(u_{22}) = 0.11$ and $B_5(u_{11}) = 0.22$, $B_5(u_{22}) = 0.33$ and A be a fuzzy subset of Y defined as $A(v_1) = 0.33$, $A(v_2) = 0.11$, $A(v_3) = 0.22$. We make the assumption that $\overline{1} = 1$ and $\overline{0} = 0$. Define L-ft's $\chi : L^S \to L$ and $\chi^* : L^T \to L$ as:

$$\chi(B) = \begin{cases} 1, & \text{if } B = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } B = B_1\ , B_2,\ B_3,\ B_4, \\ 0, & \text{otherwise}, \end{cases} \qquad \chi^*(A) = \begin{cases} 1, & \text{if } A = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } A = A, \\ 0, & \text{otherwise}, \end{cases}$$

are ft's on S and Y. For $z = \frac{1}{2}$, then

(i) f_m is $FUCts$ but not $FU\theta Cts$ because $\chi^*(A) \geq \frac{1}{2}$ in $(T,\ \chi^*)$, $f_m^u(A) = B_2$ and $f_m^u(A) = B_2$ is not $\frac{1}{2}$-$f\theta o$ in $(S,\ \chi)$.
(ii) f_m is $FU\theta^\star\ SCts$ but not $FU\theta\ SCts$ because $\chi^*(A) \geq \frac{1}{2}$ in $(T,\ \chi^*)$, $f_m^u(A) = B_2$ and $f_m^u(A) = B_2$ is not $\frac{1}{2}$-$f\theta\ So$ in $(S,\ \chi)$.
(iii) f_m is $FL\alpha Cts$ but not $FLCts$ because $\chi^*(A) \geq \frac{1}{2}$ in $(T,\ \chi^*)$, $f_m^l(A) = B_5$ and $f_m^l(A) = B_5$ is not $\frac{1}{2}$-fo in $(S,\ \chi)$.
(iv) f_m is $FL\theta^\star\ SCts$ but not $FL\theta\ SCts$ because $\chi^*(A) \geq \frac{1}{2}$ in $(T,\ \chi^*)$, $f_m^l(A) = B_5$ and $f_m^l(A) = B_5$ is not $\frac{1}{2}$-$f\theta\ So$ in $(S,\ \chi)$.

Example 3.2.

Let $S = \{u_{11},\ u_{22}\}$, $Y = \{v_1,\ v_2,\ v_3\}$ and $f_m : S \multimap T$ be a FM defined by $G_{f_m}(u_{11},\ v_1) = 0.88$, $G_{f_m}(u_{11},\ v_2) = 0.99$, $G_{f_m}(u_{11},\ v_3) = 0.88$, $G_{f_m}(u_{22},\ v_1) = \overline{1}$, $G_{f_m}(u_{22},\ v_2) = 0.77$, and $G_{f_m}(u_{22},\ v_3) = 0.33$. Let B_1, B_2 and B_3 be a fuzzy subsets of S be defined as $B_1(u_{11}) = 0.11$, $B_1(u_{22}) = 0.33$; $B_2(u_{11}) = 0.11$, $B_2(u_{22}) = 0.11$ and $B_3(u_{11}) = 0.22$, $B_3(u_{22}) = 0.33$ and A be a fuzzy subset of Y defined as $A(v_1) = 0.33$, $A(v_2) = 0.11$, $A(v_3) = 0.22$. We make the assumption that $\overline{1} = 1$ and $\overline{0} = 0$. Define L-ft's $\chi : L^S \to L$ and $\chi^* : L^T \to L$ as

$$\chi(B) = \begin{cases} 1, & \text{if } B = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } B = B_1\ , B_2, \\ 0, & \text{otherwise,} \end{cases} \qquad \chi^*(A) = \begin{cases} 1, & \text{if } A = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } A = A, \\ 0, & \text{otherwise,} \end{cases}$$

are ft's on S and Y. For $z = \frac{1}{2}$, then
(i) f_m is $FL\ SCts$ but not $FL\alpha Cts$ because $\chi^*(A) \geq \frac{1}{2}$ in $(T,\ \chi^*)$, $f_m^l(A) = B_3$ and $f_m^l(A) = B_3$ is not $\frac{1}{2}$-$f\alpha o$ in $(S,\ \chi)$.
(ii) f_m is $FL\ SCts$ but not $FL\theta^\star\ SCts$ because $\chi^*(A) \geq \frac{1}{2}$ in $(T,\ \chi^*)$, $f_m^l(A) = B_3$ and $f_m^l(A) = B_3$ is not $\frac{1}{2}$-$f\theta^\star\ So$ in $(S,\ \chi)$.

Example 3.3.

Let $S = \{u_{11},\ u_{22}\}$, $Y = \{v_1,\ v_2,\ v_3\}$ and $f_m : S \multimap T$ be a FM defined by $G_{f_m}(u_{11},\ v_1) = 0.88$, $G_{f_m}(u_{11},\ v_2) = 0.99$, $G_{f_m}(u_{11},\ v_3) = 0.88$, $G_{f_m}(u_{22},\ v_1) = \overline{1}$, $G_{f_m}(u_{22},\ v_2) = 0.77$, and $G_{f_m}(u_{22},\ v_3) = 0.33$. Let B_1, B_2 and B_3 be a fuzzy subsets of S be defined as $B_1(u_{11}) = 0.99$, $B_1(u_{22}) = 0.77$; $B_2(u_{11}) = 0.99$, $B_2(u_{22}) = 0.99$ and $B_3(u_{11}) = 0.22$, $B_3(u_{22}) = 0.33$ and A be a fuzzy subset of Y defined as $A(v_1) = 0.33$, $A(v_2) = 0.11$, $A(v_3) = 0.22$. We make the assumption that $\overline{1} = 1$ and $\overline{0} = 0$. Define L-ft's $\chi : L^S \to L$ & $\chi^* : L^T \to L$ as

$$\chi(B) = \begin{cases} 1, & \text{if } B = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } B = B_1\ , B_2,\ B_3, \\ 0, & \text{otherwise,} \end{cases} \qquad \chi^*(A) = \begin{cases} 1, & \text{if } A = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } A = A, \\ 0, & \text{otherwise,} \end{cases}$$

are ft's on S and Y. For $z = \frac{1}{2}$, then
(i) f_m is $FLCts$ but not $FL\theta Cts$ because $\chi^*(A) \geq \frac{1}{2}$ in $(T,\ \chi^*)$, $f_m^l(A) = B_3$ and $f_m^l(A) = B_3$ is not $\frac{1}{2}$-$f\theta o$ in $(S,\ \chi)$.
(ii) f_m is $FL\theta\ SCts$ but not $FL\theta Cts$ because $\chi^*(A) \geq \frac{1}{2}$ in $(T,\ \chi^*)$, $f_m^l(A) = B_3$ and

$f^l_m(A) = B_3$ is not $\frac{1}{2}$-$f\theta o$ in $(S,\ \chi)$.

Example 3.4.

Let $S = \{u_{11},\ u_{22}\}$, $Y = \{v_1,\ v_2,\ v_3\}$ and $f_m : S \multimap T$ be a FM defined by $G_{f_m}(u_{11},\ v_1) = 0.88$, $G_{f_m}(u_{11},\ v_2) = 0.99$, $G_{f_m}(u_{11},\ v_3) = 0.88$, $G_{f_m}(u_{22},\ v_1) = \overline{1}$, $G_{f_m}(u_{22},\ v_2) = 0.77$, and $G_{f_m}(u_{22},\ v_3) = 0.33$. Let B_1, B_2, B_3, B_4 and B_5 be a fuzzy subsets of S be defined as $B_1(u_{11}) = 0.33$, $B_1(u_{22}) = 0.11$; $B_2(u_{11}) = 0.11$, $B_2(u_{22}) = 0.33$; $B_3(u_{11}) = 0.33$, $B_3(u_{22}) = 0.33$; $B_4(u_{11}) = 0.11$, $B_4(u_{22}) = 0.11$ and $B_5(u_{11}) = 0.77$, $B_5(u_{22}) = 0.77$ and A be a fuzzy subset of Y defined as $A(v_1) = 0.77$, $A(v_2) = 0.99$, $A(v_3) = 0.88$. We make the assumption that $\overline{1} = 1$ & $\overline{0} = 0$. Define L-ft's $\chi : L^S \to L$ & $\chi^* : L^T \to L$ as

$$\chi(B) = \begin{cases} 1, & \text{if } B = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } B = B_1\ , B_2,\ B_3,\ B_4, \\ 0, & \text{otherwise,} \end{cases} \qquad \chi^*(A) = \begin{cases} 1, & \text{if } A = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } A = A, \\ 0, & \text{otherwise,} \end{cases}$$

are ft's on S and Y. For $z = \frac{1}{2}$, then

(i) f_m is $FU\alpha Cts$ but not $FUCts$ because $\chi^*(A) \geq \frac{1}{2}$ in $(T,\ \chi^*)$, $f^u_m(A) = B_5$ and $f^u_m(A) = B_5$ is not $\frac{1}{2}$-fo in $(S,\ \chi)$.

(ii) f_m is $FU\theta\ SCts$ but not $FU\theta Cts$ because $\chi^*(A) \geq \frac{1}{2}$ in $(T,\ \chi^*)$, $f^u_m(A) = B_5$ and $f^u_m(A) = B_5$ is not $\frac{1}{2}$-$f\theta o$ in $(S,\ \chi)$.

(i) f_m is $FU\ SCts$ but not $FU\theta^\star\ SCts$ because $\chi^*(A) \geq \frac{1}{2}$ in $(T,\ \chi^*)$, $f^u_m(A) = B_5$ and $f^u_m(A) = B_5$ is not $\frac{1}{2}$-$f\theta^\star\ So$ in $(S,\ \chi)$.

Example 3.5.

Let $S = \{u_{11},\ u_{22}\}$, $Y = \{v_1,\ v_2,\ v_3\}$ and $f_m : S \multimap T$ be a FM defined by $G_{f_m}(u_{11},\ v_1) = 0.88$, $G_{f_m}(u_{11},\ v_2) = 0.99$, $G_{f_m}(u_{11},\ v_3) = 0.88$, $G_{f_m}(u_{22},\ v_1) = \overline{1}$, $G_{f_m}(u_{22},\ v_2) = 0.77$, and $G_{f_m}(u_{22},\ v_3) = 0.33$. Let B_1, B_2 and B_3 be a fuzzy subsets of S be defined as $B_1(u_{11}) = 0.99$, $B_1(u_{22}) = 0.77$; $B_2(u_{11}) = 0.99$, $B_2(u_{22}) = 0.99$; $B_3(u_{11}) = 0.22$, $B_3(u_{22}) = 0.33$ and $B_4(u_{11}) = 0.77$, $B_4(u_{22}) = 0.77$ and A be a fuzzy subset of Y defined as $A(v_1) = 0.77$, $A(v_2) = 0.99$, $A(v_3) = 0.88$. We make the assumption that $\overline{1} = 1$ and $\overline{0} = 0$. Define L-ft's $\chi : L^S \to L$ and $\chi^* : L^T \to L$ as

$$\chi(B) = \begin{cases} 1, & \text{if } B = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } B = B_1\ , B_2,\ B_3, \\ 0, & \text{otherwise,} \end{cases} \qquad \chi^*(A) = \begin{cases} 1, & \text{if } A = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } A = A, \\ 0, & \text{otherwise,} \end{cases}$$

are ft's on S and Y. For $z = \frac{1}{2}$, then

(i) f_m is $FU\ SCts$ but not $FU\alpha Cts$ because $\chi^*(A) \geq \frac{1}{2}$ in $(T,\ \chi^*)$, $f^l_m(A) = B_4$ and $f^l_m(A) = B_4$ is not $\frac{1}{2}$-$f\alpha o$ in $(S,\ \chi)$.

Remark 3.3.

Although from the above examples, the following theorems are provided to be some applications of their obtained results.

Theorem 3.3.

Let $\{f_m{}^o\}_{o\in\Gamma}$ be a family of $FL\theta^*\mathcal{S}Cts$ between two fts's $(S,\ \chi)$ and $(T,\ \chi^*)$. Then $\bigcup\limits_{o\in\Gamma} f_m{}^o$ is $FL\theta^*\mathcal{S}Cts$.

Proof:

Let $A \in L^T$, then $(\bigcup\limits_{o\in\Gamma} f_m{}^o)^l(A) = \bigvee\limits_{o\in\Gamma}(f_m{}^{ol}(A))$ by Theorem 2.11 (2) in Abbas et al. (2014). Since $\{f_m{}^o\}_{o\in\Gamma}$ is a family of $FL\theta^*\mathcal{S}Cts$ between two fts's $(S,\ \chi)$ and $(T,\ \chi^*)$, then $f_m{}^{ol}(A)$ is z-$f\theta^*\mathcal{S}o$, for any $\chi^*(A) \geq z$. Then, we have $(\bigcup\limits_{o\in\Gamma} f_m{}^o)^l(A) = \bigvee_{o\in\Gamma}(f_m{}^{ol}(A))$ is z-$f\theta^*\mathcal{S}o$ set for any $\chi^*(A) \geq z$. Hence, $\bigcup\limits_{o\in\Gamma} f_m{}^o$ is $FL\theta^*\mathcal{S}Cts$. ■

Theorem 3.4.

Let $\{f_m{}^o\}_{o\in\Gamma}$ be a family of normalized $FU\theta^*\mathcal{S}Cts$ between two fts's $(S,\ \chi)$ and $(T,\ \chi^*)$. Then $f_m{}^1 \bigcup f_m{}^2$ is $FU\theta^*\mathcal{S}Cts$.

Proof:

Let $A \in L^T$, then $(f_m{}^1 \cup f_m{}^2)^u(A) = f_m{}^{1u}(A) \wedge f_m{}^{2u}(A)$ by Theorem 2.11 (3) in Abbas et al. (2014). Since $\{f_m{}^o\}_{o\in\Gamma}$ is a family of normalized $FU\theta^*\mathcal{S}Cts$ between two fts's $(S,\ \chi)$ and $(T,\ \chi^*)$, then $(f_m{}^{ou}(A))$ if z-$f\theta^*\mathcal{S}o$, for any $\chi^*(A) \geq z$ for each $o \in \{1,2\}$. Then, for each $A \in L^T$, we have $(f_m{}^1 \cup f_m{}^2)^u(A) = f_m{}^{1u}(A) \wedge f_m{}^{2u}(A)$ is z-$f\theta^*\mathcal{S}o$ set for any $\chi^*(A) \geq z$. Hence, $f_m{}^1 \cup f_m{}^2$ is $FU\theta^*\mathcal{S}Cts$. ■

Definition 3.3.

A fuzzy set B in a fts $(S,\ \chi)$ is called fuzzy $\theta^*\mathcal{S}$ (respectively, fuzzy $\theta\mathcal{S}$)-compact (briefly, $f\theta^*\mathcal{S}com$ (respectively, $f\theta\mathcal{S}com$)) if every family in $\{A\ :\ A$ is z-$f\theta^*\mathcal{S}o$ (respectively, z-$f\theta\mathcal{S}o$), $A \in L^S$ and $z \in L\}$ covering B has a finite subcover.

Definition 3.4.

Let $F : X \multimap Y$ be a FM between two fts's (S,χ), (T,χ^*) and $z \in L_0$. Then, f_m is called fuzzy $\theta^*\mathcal{S}$ (respectively, $\theta\mathcal{S}$)-compact valued if $f_m(u_t)$ is $f\theta^*\mathcal{S}com$ (respectively, $f\theta\mathcal{S}com$) for each $u_t \in dom(f_m)$.

Theorem 3.5.

Let $f_m : S \multimap T$ be a crisp $FU\theta^*\mathcal{S}Cts$ and $f\theta^*\mathcal{S}com$ valued between two fts's $(S,\ \chi)$ and $(T,\ \chi^*)$. Then, the direct image of a $f\theta^*\mathcal{S}com$ in S under f_m is also $f\theta^*\mathcal{S}com$.

Proof:

Let B be $f\theta^*\mathcal{S}com$ set in S and $\{\gamma_o :\ \gamma_o$ is r-$f\theta^*\mathcal{S}o$ set in T, $o \in \Gamma\}$ be a family of covering of

$f_m(B)$. i.e. $f_m(B) \le \bigvee_{o\in\Gamma} \gamma_o$. Since $B = \bigvee_{u_t\in B} u_t$, we have

$$f_m(B) = f_m(\bigvee_{u_t\in B} u_t) = \bigvee_{u_t\in B} f_m(u_t) \le \bigvee_{o\in\Gamma} \gamma_o.$$

It follows that for each $u_t \in B,\ f_m(u_t) \le \bigvee_{o\in\Gamma} \gamma_o$. Since f_m is $f\theta^*\mathcal{S}com$ valued, then there exists finite subset Γ_{u_t} of Γ such that $f_m(u_t) \le \bigvee_{n\in\Gamma_{u_t}} \gamma_n = \gamma_{u_t}$. By Theorem 2.10 (5) in Abbas et al. (2014), we have

$$u_t \le f_m^u(f_m(u_t)) \le f_m^u(\gamma_{u_t}) \text{ and } B = \bigvee_{u_t\in B} u_t = \bigvee_{u_t\in B} f_m^u(\gamma_{u_t}).$$

Since, $\chi^*(\gamma_{u_t}) \ge z$, then from Theorem 3.2, we have $f_m^u(\gamma_{u_t})$ is z-$f\theta^*\mathcal{S}o$ set. Hence, $\{f_m^u(\gamma_{u_t}) : f_m^u(\gamma_{u_t})$ is z-$f\theta^*\mathcal{S}o$ set, $u_t \in B\}$ is a family covering the set B. Since B is $f\theta^*\mathcal{S}com$, then there exists finite index set $N \ni B \le \bigvee_{n\in N} f_m^u(\gamma_{u_{t_n}})$. From Theorem 2.10 (4) in Abbas et al. (2014), we have

$$f_m(B) \le f_m(\bigvee_{n\in N} f_m^u(\gamma_{u_{t_n}})) = \bigvee_{n\in N} f_m(f_m^u(\gamma_{u_{t_n}})) \le \bigvee_{n\in N} \gamma_{u_{t_n}}.$$

Then, $f_m(B)$ is $f\theta^*\mathcal{S}com$. ■

Theorem 3.6.

Let $f_m : S \multimap T$ and $h_m : T \multimap W$ be two FM's and let $(S,\ \chi)$, $(T,\ \chi^*)$ and $(W,\ \delta)$ be three fts's. Following that, we have the following:

(i) If f_m and h_m are normalized, $FU\theta^*\mathcal{S}Cts$, then $h_m \circ f_m$ is $FU\theta^*\mathcal{S}Cts$.

(ii) If f_m and h_m are $FL\theta^*\mathcal{S}Cts$, then $h_m \circ f_m$ is $FL\theta^*\mathcal{S}Cts$.

Proof:

(i) Let f_m and h_m be normalized, $FU\theta^*\mathcal{S}Cts$ and $\nu \in L^W$. Then, from Theorem 2.17 in Abbas et al. (2014), we have $(h_m \circ f_m)^u(\nu) = f_m^u(h_m^u(\nu))$ is $f\theta^*\mathcal{S}o$ with $\nu(h_m^u(\nu)) \ge \delta(\nu)$. Thus, $h_m \circ f_m$ is $FU\theta^*\mathcal{S}Cts$.

(ii) Let f_m and h_m be $FL\theta^*\mathcal{S}Cts$ and $\nu \in L^W$. Then, from Theorem 2.17 in Abbas et al. (2014), we have $(h_m \circ f_m)^l(\nu) = f_m^l(h_m^l(\nu))$ is $f\theta^*\mathcal{S}o$ with $\nu(h_m^l(\nu)) \ge \delta(\nu)$. Thus, $h_m \circ f_m$ is $FL\theta^*\mathcal{S}Cts$. ■

Theorem 3.7.

Let $f_m : S \multimap T$ and $h_m : T \multimap W$ be two FM's and let $(S,\ \chi)$, $(T,\ \chi^*)$ and $(W,\ \delta)$ be three L-fts's. If f_m is $FL\theta^*\mathcal{S}Cts$ and h_m is $FLCts$, then $h_m \circ f_m$ is $FL\theta^*\mathcal{S}Cts$.

Proof:

Let $\nu \in L^W,\ \delta(\nu) \ge z$. Since h_m is $FLCts$, then by Theorem 3.5 in Abbas et al. (2014), $h_m^l(\nu)$ is z-fo set in T. Also, f_m is $FL\theta^*\mathcal{S}Cts$ implies $f_m^l(h_m^l(\nu))$ is $f\theta^*\mathcal{S}o$ set in S. Hence, we have $(h_m \circ f_m)^l(\nu) = f_m^l(h_m^l(\nu))$ is z-$f\theta^*\mathcal{S}o$. Thus, $h_m \circ f_m$ is $FL\theta^*\mathcal{S}Cts$. ■

Theorem 3.8.

Let $f_m : S \multimap T$ and $h_m : T \multimap W$ be two FM's and let $(S,\ \chi)$, $(T,\ \chi^*)$ and $(W,\ \delta)$ be three L-fts's. If f_m and h_m are normalized, f_m is $FU\theta^*\mathcal{S}Cts$ and h_m is $FUCts$, then $h_m \circ f_m$ is $FU\theta^*\mathcal{S}Cts$.

Proof:

Let $\nu \in L^W,\ \delta(\nu) \geq z$. Since h_m is $FUCts$, then by Theorem 3.5 in Abbas et al. (2014), $h_m^u(\nu)$ is z-fo set in T. Also, f_m is $FU\theta^*\mathcal{S}Cts$ implies $f_m^u(h_m^u(\nu))$ is $f\theta^*\mathcal{S}o$ set in S. Hence, we have $(h_m \circ f_m)^u(\nu) = f_m^u(h_m^u(\nu))$ is z-$f\theta^*\mathcal{S}o$. Thus, $h_m \circ f_m$ is $FU\theta^*\mathcal{S}Cts$. ■

Example 3.6.

Let $S = \{u_{11},\ u_{22}\}$, $Y = \{v_1,\ v_2,\ v_3\}$ and $f_m : S \multimap T$ and $h_m : T \multimap T$ be a FM defined by $G_{f_m}(u_{11},\ v_1) = 0.88$, $G_{f_m}(u_{11},\ v_2) = 0.99$, $G_{f_m}(u_{11},\ v_3) = 0.88$, $G_{f_m}(u_{22},\ v_1) = \overline{1}$, $G_{f_m}(u_{22},\ v_2) = 0.77$, and $G_{f_m}(u_{22},\ v_3) = 0.33$ and $G_{h_m}(v_1,\ v_1) = 0.88$, $G_{h_m}(v_2,\ v_2) = 0.99$ and $G_{h_m}(v_3,\ v_3) = 0.33$. Let B_1, B_2, B_3, B_4 and B_5 be a fuzzy subsets of S be defined as $B_1(u_{11}) = 0.33$, $B_1(u_{22}) = 0.11$; $B_2(u_{11}) = 0.11$, $B_2(u_{22}) = 0.33$; $B_3(u_{11}) = 0.33$, $B_3(u_{22}) = 0.33$; $B_4(u_{11}) = 0.11$, $B_4(u_{22}) = 0.11$ and $B_5(u_{11}) = 0.22$, $B_5(u_{22}) = 0.33$ and A be a fuzzy subset of Y defined as $A(v_1) = 0.33$, $A(v_2) = 0.11$, $A(v_3) = 0.22$. We make the assumption that $\overline{1} = 1$ and $\overline{0} = 0$. Define L-ft's $\chi : L^S \to L$, $\chi^* : L^T \to L$ and $\chi^{**} : L^T \to L^T$ as:

$$\chi(B) = \begin{cases} 1, & \text{if } B = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } B = B_1\ , B_2,\ B_3,\ B_4, \\ 0, & \text{otherwise,} \end{cases} \qquad \chi^*(A) = \begin{cases} 1, & \text{if } A = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } A = A, \\ 0, & \text{otherwise,} \end{cases}$$

$$\chi^{**}(A) = \begin{cases} 1, & \text{if } A = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } A = A, \\ 0, & \text{otherwise,} \end{cases}$$

are ft's on S and Y. For $z = \frac{1}{2}$, f_m is $FU\theta^*\mathcal{S}Cts$ and h_m is $FUCts$, but $h_m \circ f_m$ is not $FU\theta^*\mathcal{S}Cts$.

Theorem 3.9.

Let $f_m : S \multimap T$ be a FM between two fts's $(S,\ \chi)$ and $(T,\ \chi^*)$. If G_f is $FL\theta^*\mathcal{S}Cts$, then f_m is $FL\theta^*\mathcal{S}Cts$.

Proof:

For the fuzzy sets $\rho \in L^S$, $\chi(\rho) \geq z$, $\nu \in L^T$ and $\chi^*(\nu) \geq z$, we take,

$$(\rho \times \nu)(m,\ s) = \begin{cases} 0, & \text{if } m \notin \rho, \\ \nu(s), & \text{if } m \in \rho. \end{cases}$$

Let $u_t \in dom(f_m)$, $A \in L^T$ and $\chi^*(A) \geq z$ with $u_t \in f_m^l(A)$. Then, we have $u_t \in G_f^l(S \times A)$ and $\chi^*(S \times A) \geq z$. Since G_f is $FL\theta^*\mathcal{S}Cts$, it follows that there exists $B \in L^S$, B is $f\theta^*\mathcal{S}o$

and $u_t \in B$ such that $B \leq G^l_f(S \times A)$. From here, we obtain that $B \leq f^l_m(A)$. Thus, f_m is $FL\theta^*\mathcal{S}Cts$. ■

Example 3.7.

Let $S = \{u_{11},\ u_{22}\}$, $Y = \{v_1,\ v_2,\ v_3\}$ and $f_m : S \multimap T$ be a FM defined by $G_{f_m}(u_{11},\ v_1) = 0.88$, $G_{f_m}(u_{11},\ v_2) = 0.99$, $G_{f_m}(u_{11},\ v_3) = 0.88$, $G_{f_m}(u_{22},\ v_1) = \overline{1}$, $G_{f_m}(u_{22},\ v_2) = 0.77$, and $G_{f_m}(u_{22},\ v_3) = 0.33$. Let B_1, B_2, B_3, B_4 and B_5 be a fuzzy subsets of S be defined as $B_1(u_{11}) = 0.33$, $B_1(u_{22}) = 0.11$; $B_2(u_{11}) = 0.11$, $B_2(u_{22}) = 0.33$; $B_3(u_{11}) = 0.33$, $B_3(u_{22}) = 0.33$; $B_4(u_{11}) = 0.11$, $B_4(u_{22}) = 0.11$ and $B_5(u_{11}) = 0.22$, $B_5(u_{22}) = 0.33$ and A be a fuzzy subset of Y defined as $A(v_1) = 0.33$, $A(v_2) = 0.11$, $A(v_3) = 0.22$. We make the assumption that $\overline{1} = 1$ and $\overline{0} = 0$. Define L-ft's $\chi : L^S \to L$ and $\chi^* : L^T \to L$ as:

$$\chi(B) = \begin{cases} 1, & \text{if } B = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } B = B_1\ , B_2,\ B_3,\ B_4, \\ 0, & \text{otherwise,} \end{cases} \qquad \chi^*(A) = \begin{cases} 1, & \text{if } A = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } A = A, \\ 0, & \text{otherwise,} \end{cases}$$

are ft's on S and Y. For $z = \frac{1}{2}$, G_f is $FL\theta^*\mathcal{S}Cts$, then f_m is $FL\theta^*\mathcal{S}Cts$ because $\chi^*(A) \geq \frac{1}{2}$ in $(T,\ \chi^*)$, $f^l_m(A) = B_5$ and $f^l_m(A) = B_5$ is $\frac{1}{2}$-$f\theta^*\mathcal{S}o$ in $(S,\ \chi)$.

Theorem 3.10.

Let $f_m : S \multimap T$ be a FM between two fts's $(S,\ \chi)$ and $(T,\ \chi^*)$. If G_f is $FU\theta^*\mathcal{S}Cts$, then f_m is $FU\theta^*\mathcal{S}Cts$.

Proof:

For the fuzzy sets $\rho \in L^S$, $\chi(\rho) \geq z$, $\nu \in L^T$ and $\chi^*(\nu) \geq z$, we take,

$$(\rho \times \nu)(m,\ s) = \begin{cases} 0, & \text{if } m \notin \rho, \\ \nu(s), & \text{if } m \in \rho. \end{cases}$$

Let $u_t \in dom(f_m)$, $A \in L^T$ and $\chi^*(A) \geq z$ with $u_t \in f^u_m(A)$, then we have $u_t \in G^u_f(S \times A)$ and $\chi^*(S \times A) \geq z$. Since G_f is $FU\theta^*\mathcal{S}Cts$, it follows that there exists $B \in L^S$, B is $f\theta^*\mathcal{S}o$ and $u_t \in B$ such that $B \leq G^u_f(S \times A)$. From here, we obtain that $B \leq f^u_m(A)$. Thus, f_m is $FU\theta^*\mathcal{S}Cts$. ■

Example 3.8.

Let $S = \{u_{11},\ u_{22}\}$, $Y = \{v_1,\ v_2,\ v_3\}$ and $f_m : S \multimap T$ be a FM defined by $G_{f_m}(u_{11},\ v_1) = 0.88$, $G_{f_m}(u_{11},\ v_2) = 0.99$, $G_{f_m}(u_{11},\ v_3) = 0.88$, $G_{f_m}(u_{22},\ v_1) = \overline{1}$, $G_{f_m}(u_{22},\ v_2) = 0.77$, and $G_{f_m}(u_{22},\ v_3) = 0.33$. Let B_1, B_2, B_3, B_4 and B_5 be a fuzzy subsets of S be defined as $B_1(u_{11}) = 0.33$, $B_1(u_{22}) = 0.11$; $B_2(u_{11}) = 0.11$, $B_2(u_{22}) = 0.33$; $B_3(u_{11}) = 0.33$, $B_3(u_{22}) = 0.33$; $B_4(u_{11}) = 0.11$, $B_4(u_{22}) = 0.11$ and $B_5(u_{11}) = 0.22$, $B_5(u_{22}) = 0.33$ and A be a fuzzy subset of Y defined as $A(v_1) = 0.33$, $A(v_2) = 0.11$, $A(v_3) = 0.22$. We make the assumption that $\overline{1} = 1$ and

$\overline{0} = 0$. Define L-ft's $\chi : L^S \to L$ and $\chi^* : L^T \to L$ as:

$$\chi(B) = \begin{cases} 1, & \text{if } B = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } B = B_1\ , B_2,\ B_3,\ B_4, \\ 0, & \text{otherwise,} \end{cases} \qquad \chi^*(A) = \begin{cases} 1, & \text{if } A = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } A = A, \\ 0, & \text{otherwise,} \end{cases}$$

are ft's on S and Y. For $z = \frac{1}{2}$, G_f is $FU\theta^*\mathcal{S}Cts$, then f_m is $FU\theta^*\mathcal{S}Cts$ because $\chi^*(A) \geq \frac{1}{2}$ in $(T,\ \chi^*)$, $f_m^u(A) = B_2$ and $f_m^u(A) = B_2$ is $\frac{1}{2}$-$f\theta^*So$ in $(S,\ \chi)$.

Theorem 3.11.

Let $(S,\ \chi)$ and $(S_o,\ \chi_o)$ be L-fts's $(o \in I)$. If a FM $f_m : S \multimap \Pi_{o\in I}S_o$ is $FL\theta^*\mathcal{S}Cts$ (where $\Pi_{o\in I}S_o$ is the product space), then $P_o \circ f_m$ is $FL\theta^*\mathcal{S}Cts\ \forall\ o \in I$, where $P_o : \Pi_{o\in I}S_o \multimap S_o$ is the projection multifunction which is defined by $P_o(u_o) = \{u_o\}\ \forall\ o \in I$.

Proof:

Let $A_{o_0} \in L^{X_{o_0}}$ and $\chi_o(A_{o_0}) \geq z$. Then,

$$(P_{o_0} \circ F)^l(A_{o_0}) = f_m^l(P_{o_0}^l(A_{o_0})) = f_m^l(A_{o_0} \times \Pi_{o\neq o_0}S_o).$$

Since f_m is $FL\theta^*\mathcal{S}Cts$ and $\chi_o(A_{o_0} \times \Pi_{o\neq o_0}S_o) \geq z$, it follows that $f_m^l(A_{o_0} \times \Pi_{o\neq o_0}S_o)$ is $f\theta^*\mathcal{S}o$ set. Then, $P_o \circ f_m$ is an $FL\theta^*\mathcal{S}Cts$. ■

Theorem 3.12.

Let $(S,\ \chi)$ and $(S_o,\ \chi_o)$ be L-fts's $(o \in I)$. If a FM $f_m : S \multimap \Pi_{o\in I}S_o$ is $FU\theta^*\mathcal{S}Cts$ (where $\Pi_{o\in I}S_o$ is the product space), then $P_o \circ f_m$ is $FU\theta^*\mathcal{S}Cts\ \forall\ o \in I$, where $P_o : \Pi_{o\in I}S_o \multimap S_o$ is the projection multifunction which is defined by $P_o(u_o) = \{u_o\}\ \forall\ o \in I$.

Proof:

Let $A_{o_0} \in L^{X_{o_0}}$ and $\chi_o(A_{o_0}) \geq z$. Then, $(P_{o_0} \circ F)^u(A_{o_0}) = f_m^u(P_{o_0}^u(A_{o_0})) = f_m^u(A_{o_0} \times \Pi_{o\neq o_0}S_o)$. Since f_m is $FU\theta^*\mathcal{S}Cts$ and $\chi_o(A_{o_0} \times \Pi_{o\neq o_0}S_o) \geq z$, it follows that $f_m^u(A_{o_0} \times \Pi_{o\neq o_0}S_o)$ is $f\theta^*\mathcal{S}o$ set. Then, $P_o \circ f_m$ is an $FU\theta^*\mathcal{S}Cts$. ■

Theorem 3.13.

Let $(S_o,\ \chi_o)$ and $(T_o,\ \chi_o^*)$ be L-fts's and ${f_m}^o : S_o \multimap T_o$ be a FM $\forall\ o \in I$. Suppose that $F : \Pi_{o\in I}S_o \multimap \Pi_{o\in I}T_o$ is defined by $f_m(u_o) = \Pi_{o\in I}{f_m}^o(u_o)$. If f_m is $FL\theta^*\mathcal{S}Cts$, then ${f_m}^o$ is $FL\theta^*\mathcal{S}Cts\ \forall\ o \in I$.

Proof:

Let $A_o \in L^{T_o}$ and $\chi_o^*(A_o) \geq z$. Then, $\chi_o^*(A_o \times \Pi_{o\neq j}T_j) \geq z$. Since f_m is $FL\theta^*\mathcal{S}Cts$, it follows that $f_m^l(A_o \times \Pi_{o\neq j}T_j) = f_m^l(A_o) \times \Pi_{o\neq j}S_j$ is $f\theta^*\mathcal{S}o$. Consequently, we obtain that $f_m^l(A_o)$ is z-$f\theta^*\mathcal{S}o$ $\forall\ o \in I$. Thus, ${f_m}^o$ is $FL\theta^*\mathcal{S}Cts$. ■

Theorem 3.14.

Let $(S_o,\ \chi_o)$ and $(T_o,\ \chi_o^*)$ be L-fts's and ${f_m}^o : S_o \multimap T_o$ be a FM $\forall\ o \in I$. Suppose that

$f_m : \Pi_{o\in I}S_o \multimap \Pi_{o\in I}T_o$ is defined by $f_m(u_o) = \Pi_{o\in I}{f_m}^o(u_o)$. If f_m is $FU\theta^*\mathcal{S}Cts$, then ${f_m}^o$ is $FU\theta^*\mathcal{S}Cts\,\forall\, o \in I$.

Proof:

Let $A_o \in L^{T_o}$ and $\chi^*_o(A_o) \geq z$. Then, $\chi^*_o(A_o \times \Pi_{o\neq j}T_j) \geq z$. Since f_m is $FU\theta^*\mathcal{S}Cts$, it follows that $f^u_m(A_o \times \Pi_{o\neq j}T_j) = f^u_m(A_o) \times \Pi_{o\neq j}S_j$ is $f\theta^*\mathcal{S}o$. Consequently, we obtain that $f^u_m(A_o)$ is z-$f\theta^*\mathcal{S}o\,\forall\, o \in I$. Thus, ${f_m}^o$ is $FU\theta^*\mathcal{S}Cts$. ■

Remark 3.4.

The Theorems 3.1 to 3.14 are also true for z-$f\theta\mathcal{S}o$ sets.

Conclusion

In this paper, the concepts of fuzzy upper and lower theta star (respectively, theta) semicontinuous multifunction on fuzzy topological spaces in the Šostak sense are introduced. Also, in L-fuzzy topological spaces, the mutual relationships of these fuzzy upper (respectively, fuzzy lower) theta star (respectively, theta) semicontinuous multifunctions are established, as well as several characterizations and properties and we expect that the findings in this paper will aid researchers in improving and promoting additional research on fuzzy multifunctions in order to develop a broad framework for their practical applications.

REFERENCES

Abbas, S.E., Hebeshi, M.A. and Taha, I.M. (2014). On fuzzy upper and lower semi-continuous multifunctions, Journal of Fuzzy Mathematics, Vol. 22, No. 4, pp. 951–962.

Alimohammady, M., Ekici, E., Jafari, S. and Roohi, M. (2011). On fuzzy upper and lower contra continuous multifunctions, Iranian Journal of Fuzzy Systems, Vol. 8, No. 3, pp. 149-158.

Berge, C. (1963). *Topological Spaces Including a Treatment of Multi-valued Functions*, Vector Spaces and Convexity, Oliver, Boyd London.

Chang, C.L. (1968). Fuzzy topological spaces, J. Math. Anal. Appl., Vol. 24, pp. 182–189.

Goguen, J.A. (1973). The fuzzy Tychonoff Theorem, J. Math. Anal. Appl., Vol. 43, No. 3, pp. 734–742.

Höhle, U. (1980). Upper semicontinuous fuzzy sets and applications, J. Math. Anal. Appl., Vol. 78, pp. 659–673.

Höhle, U. and Šostak, A.P. (1999). Axiomatic Foundations of Fixed-Basis fuzzy topology, in *The Handbooks of Fuzzy Sets Series*, Kluwer Academic Publishers, Vol. 3, pp. 123–272.

Hebeshi, M.A. and Taha, I.M. (2015). On upper and lower α-continuous fuzzy multifunctions, Journal of Intelligent and Fuzzy Systems, Vol. 28, No. 6, pp. 2537-2546.

Ibedou, I. and Abbas, S. (2019). Generalized forms of upper and lower continuous fuzzy multifunctions, Journal of New Theory, Vol. 26, pp. 1-12.

Kubiak, T. (1985). *On Fuzzy Topologies*, Ph.D. Thesis, A. Mickiewicz, Poznan.

Kubiak, T. and Šostak, A.P. (1997). Lower set valued fuzzy topologies, Questions Math., Vol. 20, No. 3, pp. 423-429.

Liu, Y. and Luo, M. (1997). Fuzzy topology, World Scientific Publishing Singapore, pp. 229-236.

Mahmoud, R.A. (2003). An application of continuous fuzzy multifunctions, Chaos, Solitons and Fractals, Vol. 17, pp. 833-841.

Mukherjee, M.N. and Malakar, S. (1991). On almost continuous and weakly continuous fuzzy multifunctions, Fuzzy Sets and Systems, Vol. 41, pp. 113–125.

Mughil, A., Vadivel, A., Uma Maheswari, O. and Saravanakumar, G. (2021). Continuous maps via l-fuzzy θ^*-semiopen sets in Sostak's fuzzy topological spaces, Journal of Physics: Conference Series, 2070, 012024.

Papageorgiou, N.S. (1985). Fuzzy topolgy and fuzzy multifunctions, J. Math. Anal. Appl., Vol. 109, pp. 397-425.

Ramadan, A.A., Abbas, S.E. and Coker, D. (1992). Fuzzy γ-continuity in Šostak's fuzzy topology, The Journal of Fuzzy Sets and Systems, Vol. 48, pp. 371-375.

Šostak, A.P. (1985). On a fuzzy topological structure, Suppl. Rend. Circ. Matem. Palermo Ser II, Vol. 11, pp. 89–103.

Tsiporkova, E., De Baets, B. and Kerre, E. (1997). A fuzzy inclusion based approach to upper inverse images under fuzzy multivalued mappings, Fuzzy Sets and Systems, Vol. 85, pp. 93–108.

Vijayalakshmi, B., Bamini, S., Saraswathi, M. and Vadivel, A. (2019). Fuzzy M-open sets in Šostak's fuzzy topological spaces, Malaya Journal of Matematik, Vol. S, No. 1, pp. 234-242.

Wong, C.K. (1974). Fuzzy topology: Product and quotient theorems, J. Math. Anal. Appl, Vol. 45, pp. 512-521.