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Swarnima Bahadur
University of Lucknow

Sariya Bano
University of Lucknow

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On Weighted Lacunary Interpolation

[1]Swarnima Bahadur and [2]Sariya Bano

Department of Mathematics and Astronomy
University of Lucknow
226007, Lucknow
Uttar Pradesh, India
[1]swarnimabahadur@ymail.com; [2]sariya2406@gmail.com

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Abstract

In this paper, we considered the non-uniformly distributed zeros on the unit circle, which are obtained by projecting vertically the zeros of the derivative of Legendre polynomial together with $x = 1$ and $x = -1$ onto the unit circle. An interpolating polynomial is considered by prescribing the function on the above said nodes and its second derivative with suitable weight function at all nodes except at $x = 1$ and $x = -1$. We obtained the existence, explicit representations, estimation and convergence theorem of that interpolatory polynomial. Such type of interpolation is known as weighted Lacunary interpolation on the unit circle.

Keywords: Weight function; Legendre polynomial; Explicit representation; Convergence

MSC 2010 No.: 41A05, 30E10

1. Introduction

Kiš (1960) considered the problem of (0,2) and $(0, 1, ..., r-2, r)$ – interpolation on the n^{th} roots of unity. For any integer $r \geq 2$, he established regularity, fundamental polynomials as well as convergence theorem for the same. The study of weighted $(0, 2)$ interpolation was initiated by Balázs (1961) on the real line, taking nodes as the zeros of an Ultraspherical polynomial by using the suitable weight function. A few years later, results for the cases $(0, m)$ and $(0, m_1, m_2)$ – interpolations were established by Sharma (1964 and 1966).

The modification of Hermite – Fejér interpolation was introduced by Pál (1975), in which the function values and the first derivatives were prescribed on two sets of nodes. The study of Lacunary interpolation was initiated by Turán (1979), considering (0,2) – interpolation. Later on, several mathematicians have considered the Lacunary interpolation problem on the unit circle.

The general Lacunary interpolation problem on the unit circle was solved by Sharma and Riemenschneider (1981) using the nodes as the n^{th} roots of unity. Szilli (1983) studied the problem, in which the first derivatives were interpolated at the zeros of n^{th} Legendre polynomial $P_n(x)$, whereas the function values were interpolated at the zeros of $P_n^{'}(x)$.

A decade later, Kasana and Kumar (1994) considered an approximation and interpolation of entire functions with index – pair (p,q). After that Xie (1995) established the regularity, explicit representation and convergence behaviour for $(0, 1, 3)^*$ – interpolation on the nodes, which were obtained by projecting vertically the zeros of n^{th} Legendre polynomial $P_n(x)$ together with $x = 1$ and $x = -1$ onto the unit circle. After one year, Xie (1996) came out with another paper established the regularity of $(0, 1, ..., r-2, r)^*-$ interpolation problem by projecting vertically the zeros of Jacobi polynomial onto the unit circle.

Kumar (2007) considered the L^p - convergence of Lagrange interpolation in the finite disc. In another paper, Kumar (2008) considered Ultraspherical expansion of generalized biaxially symmetric potentials and Pseudo-analytic functions. Bahadur and Mathur (2011) provided the convergence theorem for the weighted $(0, 2)^*$ - interpolation on the set of nodes similar to Xie (1995). Bahadur and Shukla (2014) considered (0,2)-interpolation problem on the unit circle and established the convergence theorem for the nodes considered by Xie (1996).

Srivastava and Singh (2018) considered an interpolation on the zeros of the Ultraspherical polynomial. Powar et al. (2020, 2021) considered linear and higher degree approximation by various operators. For more details one can refer to Gandhi et al. (2017) and also Mishra et al. (2013). Recently, Kumar et al. (2021 (a) and 2021 (b)) considered interpolation and quadrature formula in rational space and with Chebyshev-Markov function.

In this paper, we consider the weighted Lacunary interpolation on the zeros of the first derivative of n^{th} Legendre polynomial together with $x = \pm 1$ and establish the convergence of such interpolatory polynomial. In Section 2, we give some preliminaries. In Section 3, we describe the problem and its existence whereas Section 4 consists of explicit forms of the interpolatory polynomials. In Section 5 and Section 6, estimation and convergence of the interpolatory polynomials are given, respectively.

2. Preliminaries

Let z_k be the zeros of $S\left(z\right)$, defined as

$$Z_n = \begin{cases} z_k = \cos\varphi_k + i\,\sin\varphi_k, \\ z_{(n-1)+k} = \overline{z_k}\ , \end{cases} \tag{1}$$

where

$$S(z)=\prod_{k=1}^{2n-2}(z-z_k),$$

with the help of the following well known equalities for the n^{th} Legendre polynomial $P_n(x)$ (Xie (1995)),

$$P_n(x)=\frac{(2n-1)}{n}x^n+\ldots,$$

$$(1-2xt+t^2)^{-\frac{1}{2}}=\sum_{m=0}^{\infty}P_m(x)\,t^m.$$

We have

$$S(z)=K_nP_n'\left(\frac{1+z^2}{2z}\right)z^{n-1}, \tag{2}$$

where

$$K_n=2^{2n-1}(n-1)!\frac{\Gamma(n+1)}{\Gamma(2\mathrm{n}+1)}.$$

Moreover, using the equation

$$\left(1-x^2\right)P_n''(x)-2xP_n'(x)+n(n+1)P_n(x)=0, \tag{3}$$

we obtain

$$S'(z_k)\;=\frac{K_n}{2}\left(z_k^2-1\right)P_n^{''}(u_k)\,z_k^{n-3}, \tag{4}$$

$$S''(z_k)\;=K_n\left[(n-3)\left(z_k^2-1\right)-3\right]P_n^{''}(u_k)\,z_k^{n-4}\;. \tag{5}$$

Let

$$T(z)=\left(z^2-1\right)S(z)\,. \tag{6}$$

Then, we have

$$T'(z_k)=\left(z_k^2{-}1\right)S'(z_k)\,, \tag{7}$$

and

$$T''(z_k)=4z_kS^{'}(z_k)+\left(z_k^2-1\right)S''(z_k)\,. \tag{8}$$

We shall require the fundamental polynomial of Lagrange interpolation based on the zeros of $S(z)$ and $T(z)$, respectively, given as:

$$l_k(z)=\frac{S(z)}{(z-z_k)\,S'(z_k)},\quad k=1\,(1)\,2n-2, \tag{9}$$

$$L_k(z)=\frac{T(z)}{(z-z_k)\,T'(z_k)},\quad k=0\,(1)\,2n-1. \tag{10}$$

We will also use the following well known inequalities (Szegö (1959)), for $-1<x<1$,

$$\left(1-x^2\right)^{3/4}|P_n'(x)|\sim n^{1/2}, \tag{11}$$

$$\left(1-x^2\right)\left|P_n^{''}(x)\right| \sim n^2, \tag{12}$$

$$\left|P_n^{(r)}(x)\right| \sim \quad n^{2r}\ , \ \ r=1,2,3\dots \tag{13}$$

Let

$$x_k = \cos\theta_k, \ \ k = 1\,(1)\,n,$$

are the zeros of n^{th} Legendre polynomial $P_n(x)$, with

$$1 > x_1 > x_2 > \cdots > x_n > -1,$$

then,

$$\left(1-x_k^2\right)^{-1} \sim \left(\frac{k}{n}\right)^{-2}, \tag{14}$$

$$\left|P_n^{(s)}(x_k)\right| \sim \ k^{-s-\frac{1}{2}} \ \ n^{2s}\ , \ \ s=0,1,2,3\ . \tag{15}$$

3. The Problem and Regularity

Let $Z_n \cup \{-1,1\}$ be the vertical projection of $(1-x^2)\,P_n'(x)$ on the unit circle, where Z_n is defined in (1) with $z_0 = 1, z_{2n-1} = -1$ and $P_n'(x)$ stands for the first derivative of n^{th} Legendre polynomial having the zeros

$$u_k = \cos\varphi_k, \ \ k = 1\,(1)\,n-1,$$

such that

$$1 > u_1 > u_2 > \cdots > u_{n-1} > -1.$$

We determine the interpolatory polynomial $R_{4n-3}(z)$ of degree $\leq\ 4n-3$, such that

$$\begin{cases} R_{4n-3}(z_k) & = \alpha_k, \quad k = 0(1)2n-1, \\ \left[(z^2-1)^{3/2} R_{4n-3}(z)\right]^{''}_{z=z_k} & = \beta_k, \ \ k = 1(1)2n-2, \end{cases} \tag{16}$$

where $\alpha_k' s$ and $\beta_k' s$ are arbitrary complex constants. We establish a convergence theorem for the same.

Theorem 3.1.

$R_{4n-3}(z)$ is regular on $Z_n \cup \{-1,1\}$.

Proof:

It is sufficient if we show that the unique solution of (16) is

$$R_{4n-3}(z) \equiv 0,$$

when all data $\alpha_k = \ \beta_k = 0$. In this case, we have

$$R_{4n-3}(z) = S(z)\ \ q(z)\,,$$

where $q(z)$ is a polynomial of degree $\leq 2n-1$ and $S(z)$ is given in (2).

Obviously

$$R_{4n-3}(z_k) = 0, \quad k = 1\,(1)\,2n-2.$$

From

$$\left[\left(z^2-1\right)^{3/2}\ R_{4n-3}(z)\right]''_{z=z_k} = 0, \ \text{ for } \ k = 1\,(1)\,2n-2,$$

using (4) – (5), we obtained

$$z_k\, q'(z_k) + n\, q(z_k) = 0. \tag{17}$$

Therefore, we have

$$z\, q'(z) + n\, q(z) = (a+bz)\, S(z), \tag{18}$$

where a and b are arbitrary constants. Solving (18), we get

$$z^n\, q(z) = a\ J_{20}(z) + b\ J_{21}(z) + c, \tag{19}$$

where

$$J_{2j}(z) = \int_0^z t^{n+j-1}\ S(t)\ dt, \quad (\,j = 0,1)\,. \tag{20}$$

Puting $z=0$ in (19), we get $c=0$. Now, for $z=\pm 1$, we get

$$\begin{cases} a\ J_{21}(1) + b\ J_{20}(1) = 0, \\ a\ J_{21}(-1) + b\ J_{20}(-1) = 0. \end{cases} \tag{21}$$

Since

$$J_{2j}(-1) = (-1)^{n+j}\ J_{2j}(1)\,, \tag{22}$$

using (22) in (21) we get $a=b=0$.

Hence the theorem follows. ■

4. Explicit Representation of Interpolatory Polynomials

We shall write $R_{4n-3}(z)$ satisfying (16) as

$$R_{4n-3}(z) = \sum_{k=0}^{2n-1} \alpha_k U_k(z) + \sum_{k=1}^{2n-2} \beta_k V_k(z)\,, \tag{23}$$

where $U_k(z)$ and $V_k(z)$ are unique polynomials, each of degree at most $4n-3$ satisfying the conditions:
For $k = 0\,(1)\,2n-1$,

$$\begin{cases} U_k(z_j) = \delta_{jk}, & j = 0\,(1)\,2n-1, \\ \left[(z^2-1)^{3/2} U_k(z)\right]''_{z=z_j} = 0, & j = 1\,(1)\,2n-2. \end{cases} \tag{24}$$

For $k = 1\,(1)\,2n-2$,

$$\begin{cases} V_k\left(z_j\right) = 0, & j = 0\,(1)\,2n-1, \\ \left[\left(z^2-1\right)^{3/2} V_k\left(z\right)\right]''_{z=z_j} = \delta_{jk}, & j = 1\,(1)\,2n-2. \end{cases} \tag{25}$$

Theorem 4.1.

For $k = 1\,(1)\,2n-2$, we have

$$V_k\left(z\right) \;=\; z^{-n} S\left(z\right)\left\{b_k\; J_k^*\left(z\right) + b_{0k}\; J_{20}\left(z\right) + \; b_{1k}\; J_{21}\left(z\right)\right\}, \tag{26}$$

where

$$J_k^*\left(z\right) \;=\; \int_0^z t^{n+1}\; l_k\left(t\right)\; dt, \tag{27}$$

$$J_{2j}\left(z\right) \;=\; \int_0^z t^{n+j-1}\; S\left(t\right)\; dt, \tag{28}$$

$$b_k \;=\; \frac{1}{2 z_k \left(z_k^2 - 1\right)^{3/2}\; S'\left(z_k\right)}\,, \tag{29}$$

$$b_{0k} \;=\; -\,\frac{b_k}{2\; J_{20}\left(1\right)}\left\{J_k^*\left(1\right) + \left(-1\right)^n J_k^*\left(-1\right)\right\}, \tag{30}$$

$$b_{1k} \;=-\,\frac{b_k}{2\; J_{21}\left(1\right)}\left\{J_k^*\left(1\right) + \left(-1\right)^{n+1} J_k^*\left(-1\right)\right\}. \tag{31}$$

Proof:

From (26), obviously

$$V_k\left(z_j\right) = 0, \qquad j = 1\,(1)\,2n-2,$$

$$\left[\left(z^2-1\right)^{3/2} V_k\left(z\right)\right]''_{z=z_j} = 0, \;\; j \neq k,$$

and for $j = k$, we get (29).

From

$$V_k\left(z_j\right) = 0, \qquad j = 0 \text{ and } 2n-1,$$

we get (30) – (31). ∎

Theorem 4.2.

For $k = 1\,(1)\,2n-2$, we have

$$U_k\left(z\right) = L_k\left(z\right) l_k\left(z\right) + z^{-n}\frac{S\left(z\right)}{S'\left(z_k\right)}\left\{M_k\left(z\right) + a_{0k} J_{20}\left(z\right) + a_{1k} J_{21}\left(z\right)\right\} + a_k V_k\left(z\right), \tag{32}$$

where

$$M_k(z) = -\int_0^z t^n \frac{\left[L_k'(t) - L_k'(z_k)\, L_k(t)\right]}{(t-z_k)}\, dt, \tag{33}$$

$$\begin{aligned} a_k = &\left(z_k^2-1\right)^{3/2}\left\{L_k''(z_k) - l_k''(z_k)\right\} - 2\left(z_k^2-1\right)^{3/2} L_k'(z_k)\left\{L_k'(z_k) + l_k'(z_k)\right\} \\ &- 6z_k\left(z_k^2-1\right)^{1/2}\left\{L_k'(z_k)+l_k'(z_k)\right\} - 3\left(z_k^2-1\right)^{-1/2}\left(2\,z_k^2-1\right), \end{aligned} \tag{34}$$

$$a_{0k} = -\frac{\left\{M_k(1) + (-1)^n M_k(-1)\right\}}{2\, J_{20}(1)}, \tag{35}$$

$$a_{1k} = -\frac{\left\{M_k(1) + (-1)^{n+1} M_k(-1)\right\}}{2\, J_{21}(1)}\ . \tag{36}$$

For $k = 0$ and $2n-1$

$$U_k(z) = \frac{z^{-n}\, S(z)}{z_k^{-n} S'(z_k)}\left\{a_{0k}^* J_{20}(z) + a_{1k}^* J_{21}(z)\right\}, \tag{37}$$

where

$$a_{0,0}^* = (-1)^n a_{0,2n-1}^* = \frac{1}{2\, J_{20}(1)}, \tag{38}$$

$$a_{1,0}^* = (-1)^{n+1} a_{1,2n-1}^* = \frac{1}{2\, J_{21}(1)}. \tag{39}$$

Proof:

From (32), one can check that

$$U_k(z_j) = \delta_{jk}, \quad j = 1\,(1)\,2n-2.$$

From

$$\left[\left(z^2-1\right)^{3/2} U_k(z)\right]''_{z=z_j} = 0, \quad \text{for} \quad j \neq k,$$

we get

$$M_k'(z_j) = -\frac{z_j^n}{(z_j - z_k)} L_k'(z_j),$$

owing to the second condition of (24), and we get

$$M_k'(z) = -\, z^n \frac{\left[L_k'(z) - L_k'(z_k)\, L_k(z)\right]}{(z-z_k)},$$

on solving, we get (33).

From

$$\left[\left(z^2-1\right)^{3/2} U_k(z)\right]''_{z=z_k} = 0, \quad \text{for} \quad j = k,$$

we get (34).

For

$$U_k(z_j) = 0, \qquad \text{for } j = 0 \text{ and } 2n-1,$$

we get (35) – (36).

Similarly, one can obtain (38) – (39), owing to the condition (24). ■

5. Estimation of Fundamental Polynomials

Let $\lambda_n(z)$ and λ_n denote the Lebegue function and Lebegue constant for L_k, i.e.,

$$\lambda_n(z) = \sum_{k=0}^{2n-1} |L_k(z)|, \quad \lambda_n = max_{|z| \leq 1} \lambda_n(z). \tag{40}$$

Lemma 5.1.

For $|z| \leq 1$, we have

$$max_{|z|=1} \sum_{k=0}^{2n-1} |L_k(z)| \leq c \sum_{k=1}^{n-1} \frac{1}{k}, \tag{41}$$

where $L_k(z)$ is defined in (10) and c is a constant and independent of n and z.

Proof:

From maximal principle, we know

$$\lambda_n = max_{|z|=1} \lambda_n(z). \tag{42}$$

Using (3) in (2), we get

$$|S(z)| \leq max_{|z|=1} |S(z)| = S(1) = \frac{K_n}{2} n(n+1). \tag{43}$$

Using (43) in (10), we get

$$|L_0(z)| \leq 1, \quad |L_{2n-1}(z)| \leq 1, \quad \text{for } |z| \leq 1. \tag{44}$$

Let $z = x + iy$ and $|z| = 1$, then for $0 \leq \arg z < \pi$ and $k = 1(1)n-1$, we have

$$\begin{cases} z_k = u_k + iv_k, \\ |z^2 - 1| = 2\sqrt{1-x^2}, \\ |z_k^2 - 1| = 2\sqrt{1-u_k^2}, \\ |z - z_k| = \sqrt{2}\sqrt{1 - xu_k - \sqrt{1-x^2}\sqrt{1-u_k^2}}, \end{cases} \tag{45}$$

$$|L_k(z)| \leq c \frac{\sqrt{1-x^2}\,|P_n'(x)|}{2\sqrt{2}(1-u_k^2)\sqrt{1 - xu_k - \sqrt{1-x^2}\sqrt{1-u_k^2}}\,|P_n''(u_k)|}$$

$$\leq\ c\frac{\sqrt{1-x^2}\,|P_n'(x)|\,(1-xu_k)^{\frac{1}{2}}}{2\sqrt{2}\,(1-u_k^2)\,|P_n''(u_k)|\,(x-u_k)}\ =\ G_k(x)\,.$$

Also,

$$\left|L_{(n-1)+k}(z)\right| \leq G_k(x)\,.$$

Similarly, for $\pi \leq \arg z < 2\pi$ and $k = 1\,(1)\,n-1$, $|L_k(z)| \leq G_k(x)$ and $\left|L_{(n-1)+k}(z)\right| \leq G_{(n-1)+k}(x)$.

Therefore, for a fixed $z = x+iy,\ \ |z| = 1$ and $-1 < x < 1$,

$$\lambda_n\ \leq\ 2\sum_{k=1}^{n-1} G_k(x)\ +|L_0(z)|\ +\left|L_{(2n-1)}(z)\right|$$

$$=\ 2\sum_{|u_k-x|\geq\frac{1}{2}(1-u_k^2)} G_k(x)\ +\ 2\sum_{|u_k-x|<\frac{1}{2}(1-u_k^2)} G_k(x)\ +\ 2, \tag{46}$$

using (14) - (15), we get

$$\sum_{|u_k-x|\geq\frac{1}{2}(1-u_k^2)} G_k(x) \leq\ c\,n^{1/2}\sum_{k=1}^{n-1}\frac{1}{(1-u_k^2)^{3/2}\,|P_n''(u_k)|} \leq\ c\sum_{k=1}^{n-1}\frac{1}{k}. \tag{47}$$

Similarly, we can obtain

$$\sum_{|u_k-x|<\frac{1}{2}(1-u_k^2)} G_k(x)\ \leq\ c\sum_{k=1}^{n-1}\frac{1}{k}.$$

Hence, the lemma is established. ∎

Lemma 5.2.

For $|z|\ \leq 1$, we have

$$\max_{|z|=1}\sum_{k=1}^{2n-2}|l_k(z)|\ \ \leq\ \ c\ \sum_{k=1}^{n-1}\frac{1}{k^{-1/2}}\ , \tag{48}$$

where $l_{1k}(z)$ is defined in (9) and c is a constant independent of n and z.

Proof:

Using (2), (4) and (9), we get

$$|l_k(z)|\ \leq c\,\frac{|P_n'(x)|}{|z-z_k|\,|z_k^2{-}1|\,|P_n''(u_k)|}.$$

Following the same steps as in Lemma 5.1 and using (13) – (15), we get the result. ∎

Lemma 5.3.

For $|z| \leq 1$, we have

$$\sum_{k=1}^{2n-2} \left| \left(z^2-1\right)^{3/2} V_k\left(z\right) \right| \leq c n^{-1/2} \log n \ ,$$

where $V_k\left(z\right)$ be defined in Theorem 4.1 and c is a constant independent of n and z.

Proof:

It is sufficient, if we prove the result is true for $|z| = 1$.

Let $z = \ e^{i\theta} \ \left(0 \leq \theta < 2\pi\right)$. Then using (27) – (31) in (26), we get

$$\sum_{k=1}^{2n-2} \left| \left(z^2-1\right)^{3/2} V_k\left(z\right) \right| \ \leq c \sum_{k=1}^{2n-2} \frac{\left(1-x^2\right)^{3/4} \left|P_n^{'}\left(x\right)\right|}{\left(1-u_k^2\right)^{5/4} \left|P_n^{''}\left(u_k\right)\right|} \int_0^1 t^{n+1} \left| l_k\left(te^{i\theta}\right)\right| \ dt \ ,$$

using (11), (14) – (15) and Lemma 5.2, we get the result. ∎

Lemma 5.4.

For $z = e^{i\theta} \left(0 \leq \theta < 2\pi\right)$, we have

$$\sum_{k=1}^{2n-2} \left| z^{-n} (z^2-1)^{3/2} \frac{S\left(z\right)}{S'\left(z_k\right)} M_k\left(z\right) \right| \leq c n^{1/2} \log n \ , \tag{49}$$

where $M_k\left(z\right)$ is given by (33) .

Proof:

Differentiating (10), we get

$$\begin{cases} L_k\left(z\right) = \frac{T^{'}(z)}{T^{'}(z_k)} - (z-z_k) L_k^{'}(z), \\ L_k^{'}\left(z\right) \ = \frac{T^{''}(z)}{2T^{'}(z_k)} - \frac{1}{2}(z-z_k) L_k^{''}(z). \end{cases} \tag{50}$$

Using (50) in (33), we get

$$M_k\left(z\right) = \frac{1}{\left(1-z_k^2\right) T^{'}\left(z_k\right)} \int_0^z t^n\left(1+z_k t\right) S^{'}\left(t\right) dt + \frac{1}{2} \int_0^z t^n L_k^{''}\left(t\right) dt - \ L_k^{'}\left(z_k\right) \int_0^z t^n L_k^{'}\left(t\right) dt$$

$$+ \frac{3(n-1)}{\left(z_k^2-1\right)} \int_0^z t^n l_k\left(t\right) dt + \frac{n}{z_k} \int_0^{\boldsymbol{z}} t^{n-1} \left\{ L_k\left(t\right) + (t-z_k) L_k^{'}(t) \right\} dt$$

$$+ \frac{2z_k}{\left(z_k^2-1\right)^2} \int_0^{\boldsymbol{z}} t^{n+1} l_k\left(t\right) dt + \ \frac{2n}{\left(z_k^2-1\right)} \int_0^{\boldsymbol{z}} t^n l_k\left(t\right) dt.$$

Thus, we may write as

$$\sum_{k=1}^{2n-2} \left| z^{-n} \ \left(z^2-1\right)^{3/2} \frac{S\left(z\right)}{S'\left(z_k\right)} \ M_k\left(z\right) \right| \ \leq I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \tag{51}$$

Using (2), (4), (7), (11) and (13) – (15), we get

$$|I_1| \leq \; c\, n^{-1/2} \log n.$$

Using (2), (4) – (5), (11), (14) – (15), Lemma 5.1 and Bernstein inequality, we get

$$|I_2| + |I_3| + |I_5| \leq \; c \log n.$$

Using (2), (4), (11), (14) – (15) and Lemma 5.2, we get

$$|I_4| \leq \; c\, n^{1/2} \log n.$$

Using (2), (4), (11), (14) – (15) and Lemma 5.2, we get

$$|I_6| + \; |I_7| \leq \; c\, n^{1/2} \log n.$$

Therefore, combining all these, we get (49). ■

Lemma 5.5.

For $z = e^{i\theta}\,(0 \leq \theta < 2\pi)$, we have

$$\sum_{k=0}^{2n-1} \left| \left(z^2 - 1\right)^{3/2} U_k\left(z\right) \right| \leq c\, n^{3/2} \log n, \tag{52}$$

where $U_k\left(z\right)$ is given in Theorem 4.2 and c is a constant independent of n and z.

Proof:

From (32), we have

$$\sum_{k=1}^{2n-2} \left| \left(z^2 - 1\right)^{3/2} U_k\left(z\right) \right| \leq \sum_{k=1}^{2n-2} \left| \left(\mathrm{z}^2 - 1\right)^{3/2} \; L_k\left(\mathrm{z}\right) \right| \left| l_k\left(\mathrm{z}\right) \right| + \sum_{k=1}^{2n-2} \left| z^{-n} \; \left(z^2 - 1\right)^{3/2} \frac{S\left(z\right)}{S'\left(z_k\right)} \; M_k\left(z\right) \right|$$

$$+ \sum_{k=1}^{2n-2} |a_k| \left| \left(\mathrm{z}^2 - 1\right)^{3/2} V_k\left(\mathrm{z}\right) \right|,$$

$$\sum_{k=1}^{2n-2} \left| \left(z^2 - 1\right)^{3/2} U_k\left(z\right) \right| \; = I_1 + I_2 + I_3. \tag{53}$$

Using (14) – (15) in (34), we get

$$|a_k| \leq c\, n^2. \tag{54}$$

Using (54) and Lemma 5.3, we get

$$I_3 = \sum_{k=1}^{2n-2} |\mathrm{a_k}| \left| \left(\mathrm{z}^2 - 1\right)^{3/2} V_k\left(\mathrm{z}\right) \right| \leq c\, n^{3/2} \log n. \tag{55}$$

Using Lemma 5.2, (11) and (14) – (15), we get

$$I_1 \leq c \sum_{k=1}^{2n-2} \frac{\left(1 - x^2\right)^{3/4} \left| P_n'\left(x\right) \right|}{\left(1 - u_k^2\right) \left| P_n''\left(u_k\right) \right|} \left| l_k\left(\mathrm{z}\right) \right| \leq c\, n^{1/2} \log n. \tag{56}$$

Similarly, one can obtain for $k = 0$ and $2n - 1$.

On combining (55) – (56) and Lemma 5.4, we get the result. Hence, the Lemma follows. ■

6. Convergence

Theorem 6.1.

Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Let the arbitrary numbers $\beta_k' s$ be such that

$$|\beta_k| = O\left(n^2\, \omega_2\left(f, n^{-1}\right)\right), \qquad k = 1\,(1)\,2n-2. \tag{57}$$

Then, $\{R_{4n-3}(z)\}$ defined by

$$R_{4n-3}(z) = \sum_{k=0}^{2n-1} f(z_k)\, U_k(z) + \sum_{k=1}^{2n-2} \beta_k\, V_k(z), \tag{58}$$

satisfies the relation

$$\left|\left(z^2-1\right)^{3/2} \{R_{4n-3}(z) - f(z)\}\right| = O\left(n^{3/2}\, \omega_2\left(f, n^{-1}\right) \log n\right), \tag{59}$$

where $\omega_2(f, n^{-1})$ be the second modulus of continuity of $f(z)$.

To prove Theorem 6.1, we shall need the following.

Remark: Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$, and $f' \in Lip\alpha,\ \alpha > \frac{1}{2}$. Then the sequence $\{R_{4n-3}(z)\}$ converges uniformly to $f(z)$ in $|z| \leq 1$, follows from (59) provided

$$\omega_2\left(f, n^{-1}\right) = O\left(n^{-1-\alpha}\right). \tag{60}$$

Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Then, there exists a polynomial $F_n(z)$ of degree $\leq 4n-3$ satisfying Jackson's inequality,

$$|f(z) - F_n(z)| \leq c\, \omega_2\left(f, n^{-1}\right), \quad z = e^{i\theta}\,(0 \leq \theta < 2\pi), \tag{61}$$

and an inequality (Kiš (1960))

$$\left|F_n^{(m)}(z)\right| \leq c\, n^m\, \omega_2\left(f, n^{-1}\right), \quad m \in I^+. \tag{62}$$

Proof:

Since $R_{4n-3}(z)$ is a uniquely determined polynomial of degree $\leq 4n-3$ and the polynomial $F_n(z)$ of degree $\leq 4n-3$ satisfying (61) and (62) can be expressed as

$$F_n(z) = \sum_{k=0}^{2n-1} F_n(z_k)\, U_k(z) + \sum_{k=1}^{2n-2} F_n''(z_k)\, V_k(z),$$

then,

$$\left|\left(z^2-1\right)^{3/2} \{R_{4n-3}(z) - f(z)\}\right| \leq \left|z^2-1\right|^{3/2} |R_{4n-3}(z) - F_n(z)| + \left|z^2-1\right|^{3/2} |F_n(z) - f(z)|$$

$$\leq \sum_{k=0}^{2n-1} |f(z_k) - F_n(z_k)| \left|\left(z^2-1\right)^{3/2} U_k(z)\right|$$

$$+\sum_{k=1}^{2n-2}\left\{|\beta_k|+\left|F_n''(z_k)\right|\right\}\left|(z^2-1)^{3/2}V_k(z)\right|$$
$$+\left|z^2-1\right|^{3/2}|F_n(z)-f(z)|.$$

Using (57) – (58), (60) – (62), Lemma 5.3 and Lemma 5.5, we get (59). ∎

7. Conclusion

In this paper, we have considered the nodes on the unit circle, which are obtained by projecting vertically the zeros of the derivative of n^{th} Legendre polynomial together with $x=1$ and $x=-1$ and the interpolatory polynomial of degree at most $4n-3$, which is prescribed on above said nodes and its second derivative with a suitable weight function at all points except at $x=\pm 1$. Then, we proved the existence, uniqueness, explicit representation and the convergence of such interpolatory polynomial. If the function $f(z)$ continuous in the closed unit disk, analytic in the open unit disk and $f' \in Lip\alpha, (\alpha > \frac{1}{2})$, then the above said interpolatory polynomials converges uniformly to the function $f(z)$ in the closed unit disk.

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